Anisotropic Gaussian wave models

Anne Estrade, Julie Fournier

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Abstract

Let d be an integer greater or equal to 2 and let $\mathbf{k}$ be a d-dimensional random vector. We call random wave model with random wavevector $\mathbf{k}$ any stationary random field defined on $\mathbb{R}^d$ with covariance function $t \in \mathbb{R}^d \mapsto \mathbb{E}[\cos(\mathbf{k}.t)]$. The purpose of the present paper is to link properties that concern the geometry of the random wave with the distribution of the random wavevector. We focus on Gaussian random waves such that the distribution of the norm of the wavevector and the one of its direction are independent. We illustrate our results on two specific models: a generalization of Berry's planar waves and a spatiotemporal sea wave model whose random wavevector is supported by the Airy surface in $\mathbb{R}^3$. These two Gaussian fields are anisotropic almost sure solutions of partial differential equations that involve the Laplacian operator: $\Delta f + \kappa^2 f = 0$ (where $\kappa = \|\mathbf{k}\|$) for the former, $\Delta f + \partial_t^4 f = 0$ for the latter. In the planar case, we prove that the expected length of the nodal lines is decreasing as the anisotropy of the wavevector is increasing, and we study the direction that maximizes the expected length of the crest lines.

Keywords: Gaussian field; random wave; nodal statistics; level set; crossing theory; anisotropy

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1 Introduction

For many centuries, physicists have been using wave models defined on a multidimensional space in various domains as different as acoustics, electronics, geophysics, oceanography or seismology. In order to take into account variability or uncertainty, it is useful to consider random wave models. It is the exact purpose of a pioneer exhaustive study by Longuet and Higgins [19] that was concerned by sea waves modelized as a random moving surface. Another mathematical pioneer study was raised by Berry in several papers, [9] or [10] for instance.

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These seminal works opened a wide area of research in the last decades, either for statistical purposes ([5], [18], [1], [6], [8], [22]), or more recently for topological purposes in link with number theory ([27], [15], [21]). Ten years ago, the interest for nodal sets or level sets also met the theory of crossings developed by Rice for one-dimensional stochastic processes fifty years before, yielding two inspiring books by Adler and Taylor [2] and by Azaïs and Wschebor [7]. The present paper is clearly inspired by all the above references but to the best of our knowledge it is the first time that the different models are gathered in the same work and are studied under the same focus, the influence of anisotropy.

A big demand for anisotropic models is nowadays observed, in particular by practitioners in geostatistics, offshore engineering, heterogeneous material or medical imaging (see for instance [26], [14], [3]), but also for more theoretical studies dedicated to image synthesis and analysis, optics, cosmology or arithmetic ([11], [23], [4], [24], [16]).

In the present paper, we aim at exploring the anisotropy of anisotropic random waves that are defined on a d-dimensional space with $d \geq 1$. Our first model is a single random wave given by $t \mapsto a\ \cos(\mathbf{k}\cdot t+\eta)$, where the directional structure is given by a d-dimensional random wavevector $\mathbf{k}$, the random phase η is uniformly distributed on $[0, 2\pi]$ and independent of $\mathbf{k}$, and amplitude a is kept constant. Since we focus on anisotropy, the latter assumption will remain unchanged all along the paper. We also study its stationary Gaussian counterpart, *i.e.* a stationary Gaussian random field on $\mathbb{R}^d$ with the same covariance function $t \in \mathbb{R}^d \mapsto a\,\mathbb{E}[\cos(\mathbf{k}\cdot t)]$ that we call Gaussian random wave associated with $\mathbf{k}$. Our purpose is to link the geometric and anisotropic behaviour properties of the random wave with the distribution of its random wavevector.

We focus on Gaussian random waves associated with wavevectors whose norm and direction are independent random variables; we call them separable wavevectors. In order to illustrate our results, some models are specified. Precisely, we exhibit anisotropic versions of famous Berry's random waves (see [9]), which are solutions of Helmholtz equation and are prescribed by a random wavevector belonging to a sphere. Another famous model is studied in the present paper: a space-time model adjusted for sea waves (see [19]). It is a Gaussian random wave indexed by $\mathbb{R}^2 \times \mathbb{R}$ that indicates the sea height at each point and each time instant. Its three-dimensional random wavevector is forced to live in a two-dimensional surface known as Airy surface.

Our major contribution lies in proving that the expected length of the nodal lines of planar random waves is a decreasing function of the anisotropy of its random wavevector. For this purpose, we properly quantify the anisotropy and give a closed formula for the nodal lines mean length. We extend our study by presenting a formula for the expected length of crests in a fixed direction. It allows us to prove that the direction that maximises the expected length of the crest lines is not necessarily orthogonal to the mode of the random wavevector's direction.

We like to mention that when the random wavevector $\mathbf{k}$ is equal to $A\mathbf{u}$ with A a deterministic matrix and $\mathbf{u}$ a random vector in $\mathbb{R}^d$ whose distribution is invariant under rotations, the associated Gaussian random wave has the same distribution as an isotropic random wave deformed by the linear transformation A^T. In that case, the study of anisotropy, either in the spectral domain, or in the parameter domain, is equivalent. In the general case where no linear deformation is involved, studing anisotropy in those two domains falls under two different approaches. The latter point of view is adopted in [3] or in [13] for instance, whereas our paper definitively belongs to the former type as did [12] or [26].

The paper is organised as follows. General facts are presented in Section 2, in particular the key point of spectral representation. Another important point is the link with partial differential equations that are solved by the random waves. Section 3 deals with the presentation of various models of Gaussian random waves characterized by the distribution of their random wavevector. In Section 4, we focus on the nodal sets, their directional statistics and their Hausdorff measure. Section 5 is devoted to the study of the crest lines from a directional point of view. All over the paper, two specific distributions for the random wavevector in dimension two are examined. One is called "elementary model". It is described by a main direction and a bandwidth that quantifies the anisotropy. We call the other one "toy model". It is given by a positive probability density function only depending on a single parameter that carries out the whole quantified information on anisotropy. The technical computations are detailed in the Appendix section.

Notations.

We write $\mathbb{N}_0$ for the set $\{0, 1, 2, \cdots\}$ of the non-negative integers and $\mathbb{N}$ the set of positive integers.

Let $d \in \mathbb{N}$. We fix an orthonormal basis of $\mathbb{R}^2$ and we use the same notation for a vector z in $\mathbb{R}^d$ and the vector of its coordinates in this basis. For any z and z' in $\mathbb{R}^d$, we write $z \cdot z'$ the canonical Euclidian scalar product of z and z', $\|\cdot\|$ the associated norm and I_d the identity matrix of size d. For $\varphi \in [0, 2\pi]$, u_φ denotes the vector $(\cos\varphi, \sin\varphi)$ in $\mathbb{R}^2$.

For $\mathbf{j} = (j_1, \cdots, j_d) \in {\mathbb{N}_0}^d$, we write $|\mathbf{j}| = \sum_{l=1}^d j_l$. Moreover, if $\lambda \in \mathbb{R}^d$ and if F is a smooth map from $\mathbb{R}^d$ to $\mathbb{R}$, we write

$$\lambda^{\mathbf{j}} = \prod_{l=1}^d \lambda_l^{j_l} \quad \text{and} \quad \partial^{\mathbf{j}} F = \frac{\partial^{|\mathbf{j}|} F}{\partial^{j_1}\lambda_1 \cdots \partial^{j_d}\lambda_d}.$$

We also denote by $F'(t)$ and by $F''(t)$ the gradient vector and the Hessian matrix of F at point t, respectively.

For any positive integer s, $\mathcal{H}_s$ denotes the Hausdorff measure of dimension s.

Let $Z = (z_i)_{1\le i\le d}$ be a random vector in $\mathbb{R}^d$. We write $\mathbb{E}[Z]$ its d-dimensional expectation vector, $Z\,Z^T$ the $d \times d$ matrix $(z_i z_j)_{1\le i,j\le d}$ and $\mathbb{E}[Z\,Z^T]$ the matrix of the second moments of Z. The standard Gaussian probability density function on $\mathbb{R}^d$ is denoted by Φ_d.

2 General setting

2.1 Anisotropic single random wave

Let d be a positive integer. We consider a random multi-dimensional model of single wave defined by

$$\forall t \in \mathbb{R}^d, \quad X_{\mathbf{k}}(t) = \sqrt{2}\,\cos(\mathbf{k} \cdot t + \eta), \tag{1}$$

where $\mathbf{k}$ is a d-dimensional random vector called the random wavevector and where the random phase η is uniformly distributed on $[0, 2\pi]$ and independent of $\mathbf{k}$.

The random field $X_{\mathbf{k}}$ is clearly not isotropic. As it will be stated in Proposition 2.2, the kind of anisotropy of $X_{\mathbf{k}}$ depends on the law of $\mathbf{k}$ through its covariance function.

In this paper, we will focus on two families of law for $\mathbf{k}$. First, we are interested in cases where the random wavevector $\mathbf{k}$ is supported by the zero set of a multivariate polynomial P, $\{\lambda \in \mathbb{R}^d : P(\lambda) = 0\}$. In particular, the results of Section 2.3 are derived under this assumption. Besides, we will also concentrate on the case where $\mathbf{k}$ is separable, in the sense of the following definition.

Definition 2.1 *Let $\mathbf{k}$ be a random vector in $\mathbb{R}^d$. We say that $\mathbf{k}$ is separable if a.s. $\|\mathbf{k}\| \neq 0$ and if $\|\mathbf{k}\|$ and $\frac{1}{\|\mathbf{k}\|}\,\mathbf{k}$ are independent.*

If $\mathbf{k}$ is separable, we write $\mathbf{k} = \|\mathbf{k}\|\,\tilde{\mathbf{k}}$, where $\tilde{\mathbf{k}}$ is a random variable in $\mathbb{S}^{d-1}$. For instance, if $||\mathbf{k}||$ is almost surely constant equal to $\kappa > 0$, then $\mathbf{k}$ is separable and it is as well supported by the zero set of $P(x) = \sum_{i=1}^{d} x_i^2 - \kappa^2$. In this case, we call $\kappa = \|\mathbf{k}\|$ the wavenumber of $X_{\mathbf{k}}$. In Section 3.1, we introduce properly this particular model corresponding to what we call Berry's anisotropic random waves.

We refer to Section 3 for specific examples of distributions of the random wavevectors. The following proposition gathers some basic properties of the covariance function of $X_{\mathbf{k}}$.

Proposition 2.2

1. *The random field $X_{\mathbf{k}}$ is centred and second-order stationary with covariance function*

$$r(t) := \mathrm{Cov}[X_{\mathbf{k}}(0), X_{\mathbf{k}}(t)] = \mathbb{E}[\cos(\mathbf{k} \cdot t)], \quad t \in \mathbb{R}^d. \tag{2}$$

   In particular, $\mathrm{Var}(X_{\mathbf{k}}(0)) = 1$.

2. *Let $\mathbf{k}^s$ be the symmetrized random variable associated to $\mathbf{k}$ and let F be its probability measure*[1]*. Then*

$$r(t) = \mathbb{E}[\exp(i\mathbf{k}^s \cdot t)] = \int_{\mathbb{R}^d} \exp(iu \cdot t)\, dF(u), \tag{3}$$

[1] If $F_{\mathbf{k}}$ and $F_{-\mathbf{k}}$ are respectively the probability measures of $\mathbf{k}$ and $-\mathbf{k}$, then the symmetrized random variable associated with $\mathbf{k}$ is defined as the random variable with probability measure $F = \frac{1}{2}(F_{\mathbf{k}} + F_{-\mathbf{k}})$.

which means that r is the characteristic function of the random variable $\mathbf{k}^s$ and that F is the spectral measure of $X_\mathbf{k}$.
Moreover, $X_\mathbf{k}$ is second-order isotropic if and only if the law of $\mathbf{k}^s$ is invariant under rotations.

3. *The covariance function r admits derivatives up to order m ($m \in \mathbb{N}_0$) if and only if $\mathbf{k}$ admits moments of order m. In this case, for any $\mathbf{j} \in {\mathbb{N}_0}^d$ such that $|\mathbf{j}| \leq m$, we have*

$$\partial^\mathbf{j} r(0) = 0 \quad \text{if } |\mathbf{j}| \text{ is odd} \quad ; \ \partial^\mathbf{j} r(0) = (-1)^{|\mathbf{j}|/2}\, \mathbb{E}[\mathbf{k}^\mathbf{j}] \quad \text{if } |\mathbf{j}| \text{ is even.}$$

In particular, $r''(0) = -\,\mathbb{E}[\mathbf{k}\mathbf{k}^T]$.

If $\mathbf{k}$ is centred, which is for instance the case if $\mathbf{k}$ has a symmetric law, the coefficients of matrix $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$ are the covariances between the coordinates of $\mathbf{k}$ in our basis. In the planar case, this matrix is involved in the definition of the coherency index of $\mathbf{k}$.

Definition 2.3 (Coherency index) *Let $\mathbf{k}$ be a random vector in $\mathbb{R}^2$. We assume that the symmetric non-negative matrix $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$ is non zero. We write λ_- and λ_+ its eigenvalues, such that $0 \leq \lambda_- \leq \lambda_+$ and $\lambda_+ > 0$. The coherency index of $\mathbf{k}$ is a real number in $[0,1]$ defined as*

$$c(\mathbf{k}) = \frac{\lambda_+ - \lambda_-}{\lambda_+ + \lambda_-}.$$

Despite the fact that matrix $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$ depends on the choice of the basis, note that the coherency index does not depend on it. The coherency index is a common tool in spatial statistics and physics to characterize the anisotropy of a model, see [20] and [14] for instance. In [25], it is also computed for the so-named structure tensor in order to quantify the anisotropy of an anisotropic Gaussian self-similar planar field with stationary increments.
Note also that writing $\mathbb{E}[\mathbf{k}\mathbf{k}^T] = \begin{pmatrix} m_{2,0} & m_{1,1} \\ m_{1,1} & m_{0,2} \end{pmatrix}$ and denoting by $\lambda_\pm$ the eigenvalues of $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$, we have

$$\lambda_\pm = \frac{1}{2}(T \pm \sqrt{\Delta}) \quad \text{and} \quad c(\mathbf{k}) = \frac{\sqrt{\Delta}}{T}, \tag{4}$$

where $T = m_{2,0} + m_{0,2}$ and $\Delta = (m_{2,0} - m_{0,2})^2 + 4m_{1,1}^2$.
Let us consider two extreme cases. In the isotropic case, $\lambda_- = \lambda_+$ and $c(\mathbf{k}) = 0$. On the contrary, if $\mathbf{k}$ is totally anisotropic in the sense that $\mathbf{k}$ is *a.s.* directed along a single deterministic direction, then $\lambda_- = 0$ and $c(\mathbf{k}) = 1$.

Remark 2.4 *If $\mathbf{k}$ is separable then its coherency index only depends on the directional distribution of $\mathbf{k}$. Indeed, writing $\mathbf{k} = \|\mathbf{k}\|\widetilde{\mathbf{k}}$, we obtain $\mathbb{E}[\mathbf{k}\mathbf{k}^T] = (\mathbb{E}\|\mathbf{k}\|^2)\,\mathbb{E}[\tilde{\mathbf{k}}\tilde{\mathbf{k}}^T]$, so $c(\mathbf{k}) = c(\widetilde{\mathbf{k}})$ and the coherency index of $\tilde{\mathbf{k}}$ is simply the difference between the eigenvalues of $\mathbb{E}[\tilde{\mathbf{k}}\tilde{\mathbf{k}}^T]$ because the trace of this matrix is equal to one.*

2.2 Anisotropic Gaussian wave model

We are still given a random vector $\mathbf{k}$ in $\mathbb{R}^d$ and we now consider a Gaussian, stationary and centred random field with the same covariance function as the single random wave $X_{\mathbf{k}}$ defined by (1). Due to Kolmogorov extension theorem (see [7] Sections 1.1 and 1.2 for instance), such a field exists and its distribution is unique. Consequently, we call it the Gaussian random wave associated with the wavevector $\mathbf{k}$, and we name it $G_{\mathbf{k}}$.

Note that such a Gaussian field can be obtained as a limit by considering independent and identically distributed versions of η and of $\mathbf{k}$, denoted respectively by $(\eta_j)_{j\in\mathbb{N}}$ and by $(\mathbf{k}_j)_{j\in\mathbb{N}}$. According to the central limit theorem applied to finite-dimensional distributions, the distribution of the random field

$$\left(\sqrt{\frac{2}{N}}\sum_{j=1}^{N}\cos(\mathbf{k}_j\cdot t+\eta_j)\right)_{t\in\mathbb{R}^d}$$

converges as N tends to ∞ towards a Gaussian random field with the appropriate covariance function.

The covariance function of $G_{\mathbf{k}}$ is given by (3) in Proposition 2.2: $r(t) = \int_{\mathbb{R}^d}\exp(iu\cdot t)\,dF(u)$, $t\in\mathbb{R}^d$, where F is the distribution of $\mathbf{k}^s$. From this, we deduce a spectral representation of the field $G_{\mathbf{k}}$.
Let W_F be a complex Gaussian F-noise on $\mathbb{R}^d$, *i.e.* a $\mathbb{C}$-valued process defined on the set $\mathcal{B}(\mathbb{R}^d)$ of Borelians such that

- *a.s.* W_F is a complex-valued measure on $\mathcal{B}(\mathbb{R}^d)$,
- $\forall A\in\mathcal{B}(\mathbb{R}^d)$, $W_F(A)$ is a complex-valued Gaussian variable with $\mathbb{E}[W_F(A)] = 0$ and $\mathbb{E}[W_F(A)\overline{W_F(A)}] = F(A)$, where $\overline{\cdot}$ denotes the complex conjugation,
- for any sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint Borel sets, $(W_F(A_n))_{n\in\mathbb{N}}$ are independent random variables.

Moreover, we add the property that for any $A\in\mathcal{B}(\mathbb{R}^d)$,

$$\overline{W_F(A)} = W_F(-A).$$

Then, it is easy to check that the Gaussian stationary random field prescribed by

$$\left(\int_{\mathbb{R}^d} e^{it\cdot u}\,dW_F(u)\right)_{t\in\mathbb{R}^d} \tag{5}$$

is real-valued, centred and that its covariance function is given by (3).

Reciprocally, if $Y:\mathbb{R}^d\to\mathbb{R}$ is a centred and stationary Gaussian random field with unit variance, according to Bochner's theorem, there exists a symmetric probability measure on $\mathbb{R}^d$, denoted by F, such that the covariance function r of Y is given by (3). It follows that we can associate with Y a symmetric random variable in $\mathbb{R}^d$ of probability measure F, denoted by $\mathbf{k}_Y$ and referred to in the following as *the random wavevector of* Y.

2.3 Link with partial differential equation

We are going to prove that both $X_{\mathbf{k}}$ and $G_{\mathbf{k}}$ satisfy a specific partial differential equation if and only if the random wavevector $\mathbf{k}$ is supported by a specific hypersurface of $\mathbb{R}^d$.

Let P be an even d-multivariate polynomial. Then there exists a sequence of real numbers $(\alpha_{\mathbf{j}})_{\mathbf{j}\in{\mathbb{N}_0}^d}$ with only finitely many non-zero terms, such that

$$\forall \lambda \in \mathbb{R}^d, \qquad P(\lambda) = \sum_{\mathbf{j}\in{\mathbb{N}_0}^d;\ |\mathbf{j}|\ \text{even}} \alpha_{\mathbf{j}}\, \lambda^{\mathbf{j}}. \tag{6}$$

We associate with P the following differential operator:

$$\mathcal{L}_P(X) = \sum_{\mathbf{j}\in{\mathbb{N}_0}^d;\ |\mathbf{j}|\ \text{even}} (-1)^{|\mathbf{j}|/2} \alpha_{\mathbf{j}}\, \partial^{\mathbf{j}} X.$$

Let us remark that the random field $X_{\mathbf{k}}$ defined by (1) is obviously almost surely of class $\mathcal{C}^\infty$.

Proposition 2.5 *Let P be an even multivariate polynomial given by (6). Then $X_{\mathbf{k}}$ almost surely satisfies the partial differential equation*

$$\forall t \in \mathbb{R}^d, \qquad \mathcal{L}_P(X)(t) = 0 \tag{7}$$

if and only if $P(\mathbf{k}) = 0$ a.s.

Proof. For any $\mathbf{j} \in {\mathbb{N}_0}^d$ such that $|\mathbf{j}|$ is even, we have $\partial^{\mathbf{j}} X_{\mathbf{k}}(t) = (-1)^{|\mathbf{j}|/2} \mathbf{k}^{\mathbf{j}}$ $\cos(\mathbf{k}\cdot t + \eta)$. Hence, we get $\mathcal{L}_P(X_{\mathbf{k}})(t) = P(\mathbf{k})\, X_{\mathbf{k}}(t)$ and the proof follows immediately. ■

Proposition 2.5 allows us to exhibit random anisotropic solutions of some famous partial differential equations. Let us give an example, using the Laplacian operator Δ on $\mathbb{R}^d$ that is defined by $\Delta = \sum_{1\le j\le d} \frac{\partial^2}{\partial t_j^2}$. If for some positive κ, $\mathbf{k} \in \kappa\mathbb{S}^{d-1}$, then the single random wave $X_{\mathbf{k}}$ is an almost sure solution of Helmholtz equation $\Delta X + \kappa^2 X = 0$. In the same vein, the single random wave defined on $\mathbb{R}^3$, associated with a random wavevector with support in the Airy surface $\{(x,y,z)\in\mathbb{R}^3;\ x^2+y^2-z^4=0\}$ is an almost sure solution of the partial differential equation $\frac{\partial^2}{\partial x^2}X + \frac{\partial^2}{\partial y^2}X + \frac{\partial^4}{\partial z^4}X = 0$. The Gaussian counterpart of this single random wave is used as spatiotemporal random sea wave model and it is thoroughly presented in Section 3.3.

Let us now be concerned with $G_{\mathbf{k}}$. We assume that the random wavevector $\mathbf{k}$ admits moments of any order. Hence, the covariance function r of $G_{\mathbf{k}}$ is of class $\mathcal{C}^\infty$ and consequently there exists a version of $G_{\mathbf{k}}$ with almost every realization of class $\mathcal{C}^\infty$; it is given by representation (5) for instance. First, let us point out that $G_{\mathbf{k}}$ satisfies Proposition 2.5 as well as $X_{\mathbf{k}}$. Indeed, $G_{\mathbf{k}}$ is centred and admits the same covariance function as $X_{\mathbf{k}}$; therefore for any multivariate polynomial P

given by (6), for any $t \in \mathbb{R}^d$, $\mathrm{Var}\,(\mathcal{L}_P(G_{\mathbf{k}})(t)) = \mathrm{Var}\,(\mathcal{L}_P(X_{\mathbf{k}})(t))$. However, the following theorem is a more general result: it provides a sufficient and necessary condition for any stationary Gaussian random field to satisfy Equation (7).

Theorem 2.6 *Let P be an even multivariate polynomial defined by (6) and let Y be a Gaussian random field defined on $\mathbb{R}^d$ that is centred, stationary, with unit variance and almost surely of class $\mathcal{C}^\infty$. The following properties are equivalent.*

1. *The Gaussian random field Y almost surely satisfies the partial differential equation*
$$\forall t \in \mathbb{R}^d, \qquad \mathcal{L}_P(Y)(t) = 0.$$
2. *The Gaussian random field Y admits a spectral representation given by* (5)*, where F is a probability measure supported by $\{\lambda \in \mathbb{R}^d \,:\, P(\lambda) = 0\}$ and W_F is a complex Gaussian F-noise on $\mathbb{R}^d$.*
3. *The random wavevector $\mathbf{k}_Y$ associated with Y almost surely satisfies $P(\mathbf{k}_Y) =$ 0.*

We insist on the fact that the above theorem provides all the Gaussian *a*.s. solutions, isotropic or not, of the partial differential equation $\mathcal{L}_P(Y) = 0$. Moreover, the equation gives information on the localization of the associated random wavevector.

Proof. Items 2 and 3 in Theorem 2.6 are clearly equivalent as F is the distribution of $\mathbf{k}_Y$. Since Y is centred, so are all its derivatives and the stationary random field $\mathcal{L}_P(Y)$. Therefore, $\mathcal{L}_P(Y)$ is almost surely identically zero if and only if its variance at each point is zero. But $\mathrm{Var}(\mathcal{L}_P(Y)(t))$ can be expressed as a linear combination of derivatives of the covariance function r_Y of Y. Hence Y is an *a*.s. solution of the partial differential equation $\mathcal{L}_P(Y) = 0$ if and only if its covariance function r_Y satisfies

$$\sum_{\mathbf{j},\mathbf{k}\in\mathbb{N}_0{}^d;\,|\mathbf{j}|,|\mathbf{k}|\text{ even}} (-1)^{(|\mathbf{j}|+|\mathbf{k}|)/2}\,\alpha_{\mathbf{j}}\,\alpha_{\mathbf{k}}\,\partial^{(\mathbf{j}+\mathbf{k})} r_Y(0) = 0. \tag{8}$$

On the other hand, as it is the covariance function of a stationary centred field, r_Y satisfies Bochner's Theorem: there exists a Radon finite measure F on $\mathbb{R}^d$ such that $r_Y(t) = \hat{F}(t)$, where $\hat{F}$ denotes the Fourier transform, *i*.e. $\hat{F}(t) = \int_{\mathbb{R}^d} e^{it\cdot\lambda}\,dF(\lambda)$. Then r_Y satisfies (8) if and only if

$$0 = \int_{\mathbb{R}^d} \Big(\sum_{\mathbf{j},\mathbf{k}\in\mathbb{N}_0{}^d;\,|\mathbf{j}|,|\mathbf{k}|\text{ even}} (-1)^{|\mathbf{j}|+|\mathbf{k}|}\,\alpha_{\mathbf{j}}\,\alpha_{\mathbf{k}}\,\lambda^{\mathbf{j}}\,\lambda^{\mathbf{k}} \Big)\,dF(\lambda) = \int_{\mathbb{R}^d} P(\lambda)^2\,dF(\lambda).$$

The above integral vanishes if and only if the measure F is supported by $\{\lambda \in \mathbb{R}^d \,:\, P(\lambda) = 0\}$. ■

3 Presentation of the models

3.1 Berry's anisotropic random waves

In this section, we focus on the case where the random wavevector $\mathbf{k}$ is such that, for some deterministic wavenumber $\kappa > 0$,

$$\kappa^{-1}\mathbf{k} \in \mathbb{S}^{d-1} \ a.s..$$

Since $\mathbf{k}$ is not necessarily isotropically distributed, the associated single wave is anisotropic. We consider the (unique in distribution) associated stationary centred Gaussian random field $G_{\mathbf{k}}$ on $\mathbb{R}^d$ introduced in Section 2.2. Since $\|\mathbf{k}\|$ is *a.s.* bounded, it is clear that $G_{\mathbf{k}}$ is *a.s.* smooth. Hence, rephrasing Theorem 2.6, we get that $G_{\mathbf{k}}$ is the generic Gaussian solution of Helmholtz equation

$$\Delta Y + \kappa^2 Y = 0.$$

Equivalently, $G_{\mathbf{k}}$ is an eigenfunction of the operator $-\Delta$, for the eigenvalue κ^2. Therefore, extending the definition introduced by Berry in [9] and intensively studied in the last years, we refer to $G_{\mathbf{k}}$ as a Berry's anisotropic wave with random wavenumber κ.

Applying the appropriate change of variables $t \mapsto \kappa t$ yields the scaling property that $(G_{\mathbf{k}}(t))_{t\in\mathbb{R}^d}$ and $(G_{\kappa^{-1}\mathbf{k}}(\kappa t))_{t\in\mathbb{R}^d}$ have the same distribution, where we recall that the random vector $\kappa^{-1}\mathbf{k}$ takes its values in $\mathbb{S}^{d-1}$.

Besides, we remark that $\mathbf{k}^s$, the symmetrised random variable associated with $\mathbf{k}$, is such that $\kappa^{-1}\mathbf{k}^s$ is supported by $\mathbb{S}^{d-1}$. Hence, we can deduce from the second point of Proposition 2.2 that the covariance function of $G_{\mathbf{k}}$ is given by

$$r(t) = \int_{\mathbb{S}^{d-1}} e^{i\kappa u\cdot t}\, d\mu(u), \quad t \in \mathbb{R}^d,$$

where μ denotes the probability measure of $\kappa^{-1}\mathbf{k}^s$.

3.2 Planar and separable random waves

In this section, we set $d = 2$ and we assume that the wavevector $\mathbf{k}$ is separable, in the sense of Definition 2.1. The two following examples of parametric distribution for the unitary wavevector $\tilde{\mathbf{k}} = \frac{1}{\|k\|}\mathbf{k}$ will allow us to make computations to illustrate and comment the results of Sections 4 and 5. We write $\tilde{\mathbf{k}} = (\cos\Theta, \sin\Theta)$, where Θ is a random variable in $[0, 2\pi]$ and we fix θ_0 in $[0, 2\pi]$.

Example 1 (Toy model) Let $\alpha \geq 0$. The density of Θ with respect to Lebesgue measure on $[0, 2\pi]$ is given by

$$\theta \mapsto C_\alpha\, |\cos(\theta - \theta_0)|^\alpha, \text{ with } C_\alpha = \frac{\Gamma(1+\alpha/2)}{2\sqrt{\pi}\Gamma(1/2+\alpha/2)}, \tag{9}$$

where Γ is the usual Gamma function. Parameter α is considered as an anisotropy parameter: $\alpha = 0$ brings an isotropic model, whereas, at the opposite, $\alpha \to +\infty$

corresponds with a totally anisotropic random field since $\mathbf{k}$ is $a.s.$ oriented along the x-axis. Our toy model is inspired by [14] Section 2.1.2, where it is introduced to represent anisotropic spatial structures in physics. It is also used in [4] with $\alpha = 2$ and $\theta_0 = 0$ or $\theta_0 = \pi/2$ to modelize the two coordinates of a two-dimensional electromagnetic wave.

Example 2 (Elementary model) The random variable Θ is uniformly distributed on $[\theta_0 - \delta, \theta_0 + \delta]$ with $0 \leq \delta \leq \pi$. Parameter θ_0 indicates the main direction whereas parameter δ negatively quantifies anisotropy, in the sense that the more anisotropic the model is, the smaller parameter δ is. Actually, $\delta = 0$ corresponds with a totally anisotropic model, $\delta \approx 0$ corresponds with what is named narrow spectrum model in [19] Section I.6, and $\delta = \pi$ corresponds with the isotropic model. If one wishes a symmetric model, one can also consider Θ uniformly distributed on $[\theta_0 - \delta, \theta_0 + \delta] \cup [\theta_0 + \pi - \delta, \theta_0 + \pi + \delta]$ with $0 \leq \delta \leq \pi/2$. The elementary model is studied in [11] and [25].

Up to a rotation of the basis, we can assume that in both examples $\theta_0 = 0$. Thus we set $\theta_0 = 0$ in the following.

3.3 Gaussian sea waves

In this section, we now concentrate on the case where the random wavevector $\mathbf{k}$ is 3-dimensional and $a.s.$ belongs to Airy surface, $i.e.$

$$\mathbf{k} \in \mathcal{A} = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3\,;\, (\lambda_1)^2 + (\lambda_2)^2 = (\lambda_3)^4\}\ a.s..$$

We study the Gaussian random wave $G_{\mathbf{k}}$ associated with $\mathbf{k}$, as defined in Section 2.2. Its covariance function is

$$r(t) = \mathbb{E}[\cos(\mathbf{k} \cdot t)] = \int_{\mathcal{A}} \cos(\lambda \cdot t)\, dF(\lambda), \quad t \in \mathbb{R}^3,$$

where F is the probability distribution of $\mathbf{k}$ supported by $\mathcal{A}$.
The random field $G_{\mathbf{k}}$ coincides with the one used for the spatiotemporal random modelization of sea waves, assuming that the depth of the sea is infinite (see [19] for the original idea, [5] or [7] for more recent developments). More precisely, for $(x, y, s) \in \mathbb{R}^2 \times \mathbb{R}$, $G_{\mathbf{k}}(x, y, s)$ can be seen as the algebraic height of a wave at point (x, y) and time s. If the moments of $\mathbf{k}$ are finite up to order four, we recall that according to Theorem 2.6, $G_{\mathbf{k}}$ solves the partial differential equation

$$\Delta G_{\mathbf{k}} + \partial_t^4 G_{\mathbf{k}} = 0,$$

with Δ the two-dimensional spatial Laplacian operator and ∂_t^4 the fourth temporal derivative.

We use the following parametrization of $\mathcal{A}$,

$$(z, \theta) \in \mathbb{R} \times [0, 2\pi) \mapsto (z^2 \cos\theta, z^2 \sin\theta, z),$$

which provides a bijection ϕ from $\mathbb{R}\backslash\{0\}\times[0,2\pi)$ onto $\mathcal{A}\backslash\{(0,0,0)\}$. Performing the appropriate change of variables yields

$$r(x,y,s) = \int_{\mathbb{R}\times(0,2\pi)} \cos(xz^2\cos\theta + yz^2\sin\theta + sz)\, dF^{\phi}(z,\theta),$$

where F^{ϕ} is the image of measure F by the map ϕ^{-1}. When $\mathbf{k}$ admits f as probability density function with respect to the surface measure on $\mathcal{A}$, consequently to the coarea formula, we get

$$r(x,y,s) = \int_{\mathbb{R}\times(0,2\pi)} \cos(xz^2\cos\theta + yz^2\sin\theta + sz) g(z,\theta)\, dz d\theta,$$

where the map g is given by

$$g(z,\theta) = f(z^2\cos\theta, z^2\sin\theta, z)\, z^2(1+4z^2)^{1/2}.$$

Following the literature, g is called *directional power spectrum* of $G_{\mathbf{k}}$ (see [5] and [7] Chapter 11). Experimental directional power spectra are exhibited in [5], derived from sea data provided by Ifremer.

Let us fix time $s = s_0$ and look at the random field defined on $\mathbb{R}^2$,

$$Z_{\mathbf{k}}(x,y) = G_{\mathbf{k}}(x,y,s_0) \qquad (x,y)\in\mathbb{R}^2,$$

as a picture of the sea height at time s_0. It is a two-dimensional stationary centred Gaussian random field, whose covariance function is given by

$$\begin{aligned} r_0(x,y) &= r(x,y,0) \\ &= \int_{\mathbb{R}\times(0,2\pi)} \cos(xz^2\cos\theta + yz^2\sin\theta)\, dF^{\phi}(z,\theta) \\ &= \mathbb{E}[\cos((x,y)\cdot\pi(\mathbf{k}))], \end{aligned}$$

where the random vector $\pi(\mathbf{k})$ is described in polar coordinates by (R^2,Θ) with (R,Θ) distributed according to measure F^{ϕ}. Thus $\pi(\mathbf{k})$ is the random wavevector associated with $Z_{\mathbf{k}}$ and it is nothing but the projection of the $\mathcal{A}$-valued random wavevector $\mathbf{k}$ onto the plane of the first two coordinates. Consequently, the moments of $\pi(\mathbf{k})$ are given, for any integers j and k in $\mathbb{N}_0$, by

$$m_{j,k} = \int_{(0,2\pi)\times\mathbb{R}} (z^2\cos\theta)^j (z^2\sin\theta)^k\, dF^{\phi}(z,\theta),$$

assuming that they are finite.
Note that if F^{ϕ} can be written as a tensorial product measure: $dF^{\phi}(z,\theta) = d\,\Xi(z)\otimes d\Lambda(\theta)$ then, according to Definition 2.1, $\pi(\mathbf{k})$ is separable.

4 Level sets of Gaussian random waves

Let $\mathbf{k}$ be a random wavevector in $\mathbb{R}^d$ admitting moments up to order four and let $G_{\mathbf{k}}$ be the associated Gaussian random field defined on $\mathbb{R}^d$. From now on in this section, we assume that $G_{\mathbf{k}}$ is almost surely of class C^2.

Let $a \in \mathbb{R}$. We are interested in the level set

$$G_{\mathbf{k}}^{-1}(a) = \{t \in \mathbb{R}^d : G_{\mathbf{k}}(t) = a\},$$

which is *a.s.* a C^2-submanifold of $\mathbb{R}^d$ with dimension $d-1$, called nodal set in the case $a = 0$.

4.1 Favorite orientation

Definition 4.1 *We call favorite direction of a random vector V in $\mathbb{R}^d$ any $u \in \mathbb{S}^{d-1}$ that maximizes $\mathbb{E}[(V.u)^2]$.*

Since $\mathbb{E}[(V.u)^2] = u.\mathbb{E}[VV^T]u$, the favorite directions of V are the eigenvectors with norm one associated with the largest eigenvalue of the symmetric positive matrix $\mathbb{E}[VV^T]$. If the largest eigenvalue is simple, *i.e.* if the dimension of the associated eigenspace is one, then there are exactly two favorite directions which are opposite one another. Note that if a random wavevector $\mathbf{k}$ is separable then its favorite directions are exactly the ones of $\tilde{\mathbf{k}}$.

We turn to the directional study of the level set $G_{\mathbf{k}}^{-1}(a)$. For any $t \in G_{\mathbf{k}}^{-1}(a)$, the tangent space at point t, $T_t G_{\mathbf{k}}^{-1}(a)$, is a $(d-1)$-dimensional linear subspace that is orthogonal to the vector $G'_{\mathbf{k}}(t)$. Using the previous definition, the favorite directions of $G'_{\mathbf{k}}(t)$ are given by the unitary eigenvectors associated with the largest eigenvalue of $\mathbb{E}[G'_{\mathbf{k}}(t)G'_{\mathbf{k}}(t)^T]$. As the latter matrix is equal to $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$ according to the third point of Proposition 2.2, the favorite directions of $G'_{\mathbf{k}}(t)$ coincide with those of $\mathbf{k}$. Hence, we get the next statement that sounds physically intuitive: *the favorite orientations of the level sets $G_{\mathbf{k}}^{-1}(a)$ are orthogonal to the favorite directions of* $\mathbf{k}$. Actually, it can be written out as a precise proposition.

Proposition 4.2 *Let τ be a d-dimensional vector field defined on the level set $G_{\mathbf{k}}^{-1}(a)$ such that, at any point $t \in G_{\mathbf{k}}^{-1}(a)$, $\tau(t)$ is orthogonal to $T_t G_{\mathbf{k}}^{-1}(a)$. Then, at any point t, the favorite directions of $\tau(t)$ are given by the favorite directions of* $\mathbf{k}$.

This formalizes an assertion in [19] Section 2.3, according to which *the direction of the contour is near the principal direction.* In this statement, the principal direction corresponds to our favorite direction.

Let us illustrate the notion of favorite direction on two-dimensional separable random wavevectors. We write $\mathbf{k} = \|\mathbf{k}\| u_\Theta$, $\|\mathbf{k}\|$ and Θ being independent random variables. Note that for any $\varphi \in [0, 2\pi]$, $\mathbf{k} \cdot u_\varphi = \|k\| \cos(\theta - \varphi)$.
On the one hand, if Θ is uniformly distributed on $[0, 2\pi]$, then $\mathbb{E}[\mathbf{k}\mathbf{k}^T] =$

$\frac{1}{2}\mathbb{E}[\|k\|^2]I_2$, thus the set of favorite directions of $\mathbf{k}$ is $\mathbb{S}^1$. On the other hand, if Θ almost surely takes a fixed value $\theta_0 \in [0, 2\pi]$ then the favorite directions of $\mathbf{k}$ are $\pm u_{\theta_0}$. Let us now focus on our favorite examples. We refer to the appendix Section A.4 for the detailed computation of their moments.

Example 1 (Toy model) If Θ admits a probability density function given by (9), for a given $\alpha > 0$, then $\mathbb{E}[\mathbf{k}\mathbf{k}^T] = \mathbb{E}[\|\mathbf{k}\|^2]\frac{1}{\alpha+2}\begin{pmatrix}\alpha+1 & 0\\ 0 & 1\end{pmatrix}$. Hence, the favorite directions of $\mathbf{k}$ are $\pm u_0$.

Example 2 (Elementary model) If Θ is uniformly distributed on $[-\delta, \delta]\cup[\pi-\delta, \pi+\delta]$ for some $0 < \delta \le \pi/2$, then $\mathbb{E}[\mathbf{k}\mathbf{k}^T] = \mathbb{E}[\|\mathbf{k}\|^2]\,\frac{1}{2}\begin{pmatrix}1+\frac{\sin(2\delta)}{2\delta} & 0\\ 0 & 1-\frac{\sin(2\delta)}{2\delta}\end{pmatrix}$ and the favorite directions of $\mathbf{k}$ are again $\pm u_0$.

4.2 Expected measure

We are now interested in the expected measure of the level sets of $G_{\mathbf{k}}$. Let Q be a compact set in $\mathbb{R}^d$ with non empty interior. We focus on the $(d-1)$-dimensional Hausdorff measure of the a-level set restricted to Q, namely

$$\ell(a, \mathbf{k}, Q) := \mathcal{H}_{d-1}\left(G_{\mathbf{k}}^{-1}(a) \cap Q\right) = \mathcal{H}_{d-1}\left(\{t \in Q \,/\, G_{\mathbf{k}}(t) = a\}\right).$$

For now on in Section 4, we assume that $G'_{\mathbf{k}}(0)$ is a non-degenerate Gaussian vector or, equivalently, that $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$ is invertible. This allows us to apply Kac-Rice formula (see [7] Theorem 6.8 for instance). It yields

$$\mathbb{E}[\ell(a, \mathbf{k}, Q)] = \int_Q \mathbb{E}[\|G'_{\mathbf{k}}(t)\| \,|\, G_{\mathbf{k}}(t) = a]\, p_{G_{\mathbf{k}}(t)}(a)\, dt,$$

where $p_{G_{\mathbf{k}}(t)}$, the probability density function of $G_{\mathbf{k}}(t)$, is actually given by the standard Gaussian distribution. Using the stationarity of $G_{\mathbf{k}}$ and the fact that for a fixed point t, $G_{\mathbf{k}}(t)$ and $G'_{\mathbf{k}}(t)$ are independent random variables, we have

$$\mathbb{E}[\ell(a, \mathbf{k}, Q)] = \mathcal{H}_d(Q)\,\frac{e^{-a^2/2}}{\sqrt{2\pi}}\,\mathbb{E}[\|G'_{\mathbf{k}}(0)\|].$$

Consequently, recalling that $\|G'_{\mathbf{k}}(0)\|$ is the Euclidean norm of a d-dimensional centred Gaussian vector with variance matrix $-r''(0) = \mathbb{E}[\mathbf{k}\mathbf{k}^T]$ and that Φ_d stands for the standard Gaussian probability density function on $\mathbb{R}^d$, we obtain

$$\mathbb{E}[\ell(a, \mathbf{k}, Q)] = \mathcal{H}_d(Q)\,\frac{e^{-a^2/2}}{\sqrt{2\pi}}\,\int_{\mathbb{R}^d}(\mathbb{E}[\mathbf{k}\mathbf{k}^T]x\cdot x)^{1/2}\Phi_d(x)\,dx.$$

In the separable case, we deduce from the above formula the following straightforward lemma.

Lemma 4.3 *We assume that* $\mathbf{k}$ *is separable with* $\mathbf{k} = \|\mathbf{k}\|\widetilde{\mathbf{k}}$ *and that* $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$ *is invertible. Then*

$$\mathbb{E}[\ell(a, \mathbf{k}, Q)] = \mathcal{H}_d(Q)\, \frac{e^{-a^2/2}}{\sqrt{2\pi}}\, \mathbb{E}[\|\mathbf{k}\|^2]^{1/2} \int_{\mathbb{R}^d} (\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T]x \cdot x)^{1/2} \Phi_d(x)\, dx. \quad (10)$$

The above formula applies to Berry's isotropic random wave, *i.e.* to the case where $\|\mathbf{k}\|$ is *a.s.* constant equal to some positive constant κ and $\widetilde{\mathbf{k}}$ is uniformly distributed in $\mathbb{S}^{d-1}$. In this case, $\mathbb{E}[\|\mathbf{k}\|^2] = \kappa^2$ and $\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T] = (1/d)\, I_d$. Hence, the involved integral becomes $\int_{\mathbb{R}^d} \|x\|\, \Phi_d(x)\, dx$, which is the mean of a χ-distributed random variable with d degrees of freedom and is known to be equal to $\sqrt{2}\, \frac{\Gamma((d+1)/2)}{\Gamma(d/2)}$, with Γ the usual Gamma function. Finally, we recover the well known formula expressing the expected measure of the a-level set of a Berry isotropic random wave (see [9] in the planar case):

$$\mathbb{E}[\ell(a, \mathbf{k}, Q)] = \mathcal{H}_d(Q)\, \frac{e^{-a^2/2}}{\sqrt{d\pi}}\, \kappa\, \frac{\Gamma((d+1)/2)}{\Gamma(d/2)}.$$

4.3 Planar case

In the planar case, *i.e.* $d = 2$, the level sets $G_{\mathbf{k}}^{-1}(a)$ are one-dimensional. Assuming moreover that $\mathbf{k}$ is separable, we will establish that the level curves mean length decreases with anisotropy. Our formula involves the coherency index, introduced in Definition 2.3.

To compute the integral in the right-hand side of (10), we use the following well known fact, that can be proved with simple algebra.
If M is a symmetric positive definite matrix with eigenvalues γ_- and γ_+ such that $0 < \gamma_- \leq \gamma_+$, then

$$\int_{\mathbb{R}^2} (Mx \cdot x)^{1/2} \Phi_2(x)\, dx = \left(\frac{2\gamma_+}{\pi}\right)^{1/2} \mathcal{E}\left((1 - \gamma_-/\gamma_+)^{1/2}\right), \quad (11)$$

where $\mathcal{E}$ stands for the elliptic integral given by $\mathcal{E}(x) = \int_0^{\pi/2} (1 - x^2 \sin^2\theta)^{1/2} d\theta$, for $x \in [0, 1]$.
In our case, we set $M = \mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T]$ and $\gamma_- + \gamma_+ = 1$ since $\widetilde{\mathbf{k}}$ belongs to $\mathbb{S}^1$, *a.s.*. Hence, writing $c = c(\widetilde{\mathbf{k}})$, we have $2\gamma_+ = 1 + c$ and $1 - \gamma_-/\gamma_+ = \frac{2c}{1+c}$. Consequently, the following proposition holds.

Proposition 4.4 *Let* $\mathbf{k}$ *be a separable random wavevector in* $\mathbb{R}^2$ *such that* $\mathbf{k} = \|\mathbf{k}\|\widetilde{\mathbf{k}}$ *and* $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$ *is invertible. Let us denote by* $c(\widetilde{\mathbf{k}}) = c(\mathbf{k})$ *the coherency index of* $\mathbf{k}$. *Then,*

$$\mathbb{E}[\ell(a, \mathbf{k}, Q)] = \mathcal{H}_2(Q)\, \frac{e^{-a^2/2}}{\pi\sqrt{2}}\, \mathbb{E}[\|\mathbf{k}\|^2]^{1/2}\, \mathcal{F}\big(c(\widetilde{\mathbf{k}})\big),$$

where the map $\mathcal{F} : c \in [0, 1] \mapsto (1 + c)^{1/2}\, \mathcal{E}\left(\left(\frac{2c}{1+c}\right)^{1/2}\right)$ *is strictly decreasing.*

The proof of the decrease of mapping $\mathcal{F}$ is postponed to the Appendix section, see Lemma A.1. Another expression for the same expectation can be found in [19] Formula (2.3.13), however our formulation highlights the effect of the wavevector's distribution on the mean length of level sets.

Remark 4.5 *Regarding the coherency index $c(\widetilde{\mathbf{k}})$ as a parameter that positively quantifies anisotropy, the above formula clearly indicates that the mean length of level curves is decreasing as the anisotropy of $\mathbf{k}$ increases.*

We now apply Proposition 4.4 to our separable examples, prescribing the directional distribution of the wavevector $\mathbf{k}$.

Example 1 (Toy model) Take $\widetilde{\mathbf{k}}$ distributed on $\mathbb{S}^1$ with probability density function given by (9) for some positive α. We already mentioned that $\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T] = \frac{1}{\alpha+2}\begin{pmatrix}\alpha+1 & 0\\ 0 & 1\end{pmatrix}$. Consequently, $c(\widetilde{\mathbf{k}}) = \frac{\alpha}{\alpha+2}$, which is an increasing function of parameter α. As observed in Remark 4.5, the more anisotropic the model is, the smaller the expected length of level sets is.

Example 2 (Elementary model) We choose $\widetilde{\mathbf{k}}$ to be uniformly distributed on $[-\delta,\delta]\cup[\pi-\delta,\pi+\delta]$ for some $0<\delta\leq\pi/2$. In that case, $\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T] = \frac{1}{2}\begin{pmatrix}1+\frac{\sin(2\delta)}{2\delta} & 0\\ 0 & 1-\frac{\sin(2\delta)}{2\delta}\end{pmatrix}$. Hence, $c(\widetilde{\mathbf{k}}) = \frac{\sin(2\delta)}{2\delta}$, which is decreasing on $(0,\pi/2]$. Again, the mean length of level sets is decreasing with anisotropy, *i.e.* as δ is decreasing.

5 Crest lines of planar Gaussian waves

Let $\mathbf{k}$ be a two-dimensional random wavevector. We write F its probability law and $G_{\mathbf{k}}$ its associated planar Gaussian random wave that we assume to be almost surely of class $\mathcal{C}^3$. We assume that $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$ is invertible, which equivalently excludes the case of $\mathbf{k}$ oriented almost surely along a fixed direction.

We fix a direction $\varphi\in[0,\pi)$ and we consider the *crest line in direction* φ. More precisely, we introduce the random set

$$\{t\in\mathbb{R}^2\,;\,G'_{\mathbf{k}}(t)\cdot u_\varphi=0\},$$

which contains all the points x in $\mathbb{R}^2$ such that the gradient of $G_{\mathbf{k}}$ at point t is orthogonal to direction φ. Hence, the crest in direction φ is a special case of a *specular points* set as defined in [19].

5.1 Directional derivative as a random wave

The crest line can be considered as the nodal line of the random field $(G'_{\mathbf{k}}\cdot u_\varphi)$, which is the partial derivative of $G_{\mathbf{k}}$ along vector u_φ. This random field is

Gaussian, stationary, centred and we denote by $v_{0,0}(\varphi)$ its variance:

$$v_{0,0}(\varphi) = \mathbb{E}[(G'_{\mathbf{k}}(0) \cdot u_\varphi)^2] = u_\varphi \cdot \mathbb{E}[\mathbf{k}\mathbf{k}^T] u_\varphi = \mathbb{E}[(\mathbf{k} \cdot u_\varphi)^2].$$

Note that $v_{0,0}(\varphi)$ does not vanish since $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$ is invertible.

By deriving the integral representation of r resulting from (2), we obtain that the covariance function of $(G'_{\mathbf{k}} \cdot u_\varphi)$ is

$$\forall t \in \mathbb{R}^2, \quad \mathbb{E}[(G'_{\mathbf{k}}(t) \cdot u_\varphi)(G'_{\mathbf{k}}(0) \cdot u_\varphi)] = \int_{\mathbb{R}^2} \cos(t \cdot \lambda)(\lambda \cdot u_\varphi)^2 \, dF(\lambda). \tag{12}$$

Therefore, the following lemma follows. It allows to associate a random wavevector with the unit variance Gaussian random wave $\dfrac{G'_{\mathbf{k}} \cdot u_\varphi}{v_{0,0}(\varphi)^{1/2}}$.

Lemma 5.1 *The covariance function of* $\dfrac{G'_{\mathbf{k}} \cdot u_\varphi}{v_{0,0}(\varphi)^{1/2}}$ *is given by*

$$t \in \mathbb{R}^2 \mapsto \mathbb{E}[\cos(\mathbf{K}_\varphi \cdot t)],$$

where $\mathbf{K}_\varphi$ *is a two-dimensional random vector whose probability distribution is equal to* $\frac{(\lambda \cdot u_\varphi)^2}{v_{0,0}(\varphi)} \, dF(\lambda)$*, with* F *the probability measure of* $\mathbf{k}$*. In other words,* $\mathbf{K}_\varphi$ *is the random wavevector associated to* $\dfrac{G'_{\mathbf{k}} \cdot u_\varphi}{v_{0,0}(\varphi)^{1/2}}$.

We introduce the moments of measure F and the ones of measure $(\lambda \cdot u_\varphi)^2 \, dF(\lambda)$: for any $(i, j) \in \mathbb{N}_0^2$,

$$\begin{aligned} m_{i,j} &:= \int_{\mathbb{R}^2} (\lambda_1)^i (\lambda_2)^j \, dF(\lambda) = \mathbb{E}[(\mathbf{k} \cdot u_0)^i (\mathbf{k} \cdot u_{\pi/2})^j], \\ v_{i,j}(\varphi) &:= \int_{\mathbb{R}^2} (\lambda_1)^i (\lambda_2)^j \, (\lambda \cdot u_\varphi)^2 \, dF(\lambda). \end{aligned} \tag{13}$$

By a mere development, the moments $(v_{i,j}(\varphi))$ can be expressed as polynomial functions of $(\cos\varphi, \sin\varphi)$, the coefficients depending linearly on the moments $(m_{i,j})$ (see (19) in the Appendix section).

Until the end of this section and in the following one, we focus on the case of a separable random wavevector $\mathbf{k}$, that we write $\mathbf{k} = \|\mathbf{k}\| u_\Theta$. We denote by Ξ the probability distribution of $\|\mathbf{k}\|$ on $(0, +\infty)$ and Λ the one of Θ on $[0, 2\pi]$, assuming that $\|\mathbf{k}\|$ and Θ are independent. We introduce

$$\forall j \in \mathbb{N}_0, \quad M_j = \mathbb{E}[\|\mathbf{k}\|^j] = \int_{\mathbb{R}^+} \rho^j d\Xi(\rho)$$

the moments of the random variable $\|\mathbf{k}\|$ and $\mu_{j,k}$ the ones of Θ, that is for any $(j, k) \in \mathbb{N}_0^2$,

$$\mu_{i,j} = \int_{[0,2\pi]} \cos^i \theta \sin^j \theta \, d\Lambda(\theta).$$

For any integrable function h,

$$\mathbb{E}[h(\mathbf{k})] = \int_{[0,+\infty)\times[0,2\pi]} h(\rho\cos\theta, \rho\sin\theta) d\Xi \otimes d\Lambda(\rho,\theta).$$

Therefore,

$$\mathbb{E}[h(\mathbf{K}_\varphi)] = \frac{1}{v_{0,0}(\varphi)} \int_{(0,+\infty)\times[0,2\pi]} h(\rho\cos\theta, \rho\sin\theta)\rho^2\cos^2(\theta-\varphi) d\Xi \otimes d\Lambda(\rho,\theta).$$

To end with, we write

$$\nu_{i,j}(\varphi) = \int_0^{2\pi} \cos^i(\theta)\sin^j(\theta)\cos^2(\theta-\varphi)\, d\Lambda(\theta). \tag{14}$$

We can directly state the following lemma.

Lemma 5.2 *If $\mathbf{k}$ is separable, then $\mathbf{K}_\varphi$ is also separable and*

$$\forall (i,j) \in {\mathbb{N}_0}^2, \quad m_{i,j} = M_{i+j}\,\mu_{i,j} \quad \textit{and} \quad v_{i,j}(\varphi) = M_{2+i+j}\nu_{i,j}(\varphi),$$

assuming that the above moments exist.

5.2 Mean length of a crest line in a fixed direction

The Hausdorff dimension of a crest line is clearly equal to one. We consider its length within a compact domain $Q \subset \mathbb{R}^2$ such that $\mathcal{H}_2(Q) > 0$,

$$l(\mathbf{k}, Q, \varphi) := \mathcal{H}_1\left(\{t \in Q\,;\, G'_{\mathbf{k}}(t)\cdot u_\varphi = 0\}\right).$$

Since the crest line of $G_{\mathbf{k}}$ in direction φ is nothing but the nodal line of $\frac{G'_{\mathbf{k}}\cdot u_\varphi}{v_{0,0}(\varphi)^{1/2}}$, Lemma 5.1 allows us to apply Proposition 4.4 with the random wavevector $\mathbf{K}_\varphi$. It yields the following proposition.

Proposition 5.3 *We assume that $\mathbf{k}$ is separable and that $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$ is invertible. Then for any $\varphi \in [0,\pi)$ such that $\mathbb{E}[\mathbf{K}_\varphi\mathbf{K}_\varphi^T]$ is invertible,*

$$\mathbb{E}[l(\mathbf{k}, Q, \varphi)] = \mathcal{H}_2(Q)\,\frac{1}{\pi\sqrt{2}}\,\mathbb{E}[\|\mathbf{K}_\varphi\|^2]^{1/2}\,\mathcal{F}(c(\widetilde{\mathbf{K}}_\varphi)),$$

where function $\mathcal{F}$ has been introduced in Proposition 4.4,

$$\mathbb{E}(\|\mathbf{K}_\varphi\|^2) = \frac{M_4}{M_2}, \tag{15}$$

$$c(\widetilde{\mathbf{K}}_\varphi) = \frac{\left((\nu_{2,0}(\varphi)-\nu_{0,2}(\varphi))^2 + 4\nu_{1,1}(\varphi)^2\right)^{1/2}}{\nu_{0,0}(\varphi)}, \tag{16}$$

$$= \frac{\left((v_{2,0}(\varphi)-v_{0,2}(\varphi))^2 + 4v_{1,1}(\varphi)^2\right)^{1/2}}{v_{2,0}(\varphi)+v_{0,2}(\varphi)} \tag{17}$$

and according to Formula (19), $c(\widetilde{\mathbf{K}}_\varphi)$ can be expressed in terms of the fourth-order moments $(m_{i,j})_{(i,j)\in\mathbb{N}_0^2,\, i+j=4}$.

We refer to the appendix Section A.2 for the proof of (15), (16) and (17). According to theses Formulas, $\mathbb{E}[l(\mathbf{k}, Q, \varphi)]$ depends on the directional distribution of $\mathbf{k}$ through the factor $\mathcal{F}(c(\widetilde{\mathbf{K}}_\varphi))$ (involving the fourth-order moments of $\mathbf{k}$), and on its radial distribution through the factor $\mathbb{E}[\|\mathbf{K}_\varphi\|^2]$ (involving the second-order and the fourth-order moment of Ξ). Consequently to Proposition 5.3, the mean length of crest lines is a decreasing function of the anisotropy of $\mathbf{K}_\varphi$. The variations of the expected length of the crests with respect to the direction can be studied through the variations of the map $\varphi \mapsto c(\widetilde{\mathbf{K}}_\varphi)$. For more detailed formulas, see the beginning of the appendix Section A.5.

A similar formula is also derived in [7] (Proposition 11.4) and in [5] (Assertion 3). However, it is apparently different from ours, since the gradient of the random field $G'_{\mathbf{k}} \cdot u_\varphi$ is not computed according to the directions of the canonical basis but according to the ones of $(u_\varphi, u_{\varphi+\pi/2})$. The link between both formulas is detailed in the appendix Section A.3.

Now we apply Proposition 5.3 to our separable examples. All is about computing the coherency index of $\widetilde{\mathbf{K}}_\varphi$. For each case, we examine the crest direction(s) that maximize(s) the expectation of the crest length. In [19] or in [5], a rule of thumb is suggested claiming that *the direction [that maximises the expected length of crests] is orthogonal to the direction for the maximum integral of the spectrum, i.e. is the most probable direction for the waves.* In this statement, the "most probable direction for the waves" refers to the mode of the random wavevector $\widetilde{\mathbf{k}}$. However, according to our examples, such a rule is not necessarily satisfied. This can be explained by the expectation formula of Proposition 5.3 itself, which shows a dependency on both the second-order and the fourth-order moments of $\mathbf{k}$, and not on the mode of $\widetilde{\mathbf{k}}$.

Examples We consider a random wavevector $\mathbf{k}$ that we write $\mathbf{k} = \|\mathbf{k}\| u_\Theta$ as before and we use the formulas of the appendix Section A.2 and A.5.

- **Toy Model.** The probability density function of Θ is given by (9), for some fixed $\alpha > 0$ and θ_0 equal to zero. The mode of $\widetilde{\mathbf{k}}$ is 0 in that case. An asymptotic expansion of $\varphi \mapsto c(\widetilde{\mathbf{K}}_\varphi)^2$ near $\varphi = \pi/2$ is performed in Lemma A.6 in the Appendix section. It shows that the expected length of crests admits a local maximum at $\varphi = \pi/2$, which is precisely orthogonal to the most probable direction of $\mathbf{k}$ and to its favorite direction as well.

- **Elementary wave.** We assume that Θ is uniformely distributed on $[-\delta, \delta] \cup [\pi - \delta, \pi + \delta]$. The moments coherency index of $\widetilde{\mathbf{K}}_\varphi$ can be computed thanks to Formula (21) and Lemma A.5. Figure 2 shows the graph of $\varphi \mapsto c(\widetilde{\mathbf{K}}_\varphi)^2$. The expected length of crests appears to admit a maximum at $\varphi = \pi/2$, thus it is orthogonal to the favorite direction of $\mathbf{k}$.

- **A very special example.** Let $\frac{1}{4}\sum_{j=0}^{3} d\delta_{j\pi/2}$ be the distribution of Θ on $[0, 2\pi]$. The modes of $\widetilde{\mathbf{k}}$ are $\{u_{j\pi/2} : j = 0, 1, 2, 3\}$. (Also note that it does not admit any favorite direction.) Computing the moments of F, we get $m_{1,1} = m_{2,2} = m_{3,1} = m_{3,1} = 0$ whereas $m_{2,0} = m_{0,2} = \frac{M_2}{2}$ and

$m_{4,0} = m_{0,4} = \frac{M_4}{2}$. Since

$$\mathbb{E}[\widetilde{\mathbf{K}}_\varphi \widetilde{\mathbf{K}}_\varphi^T] = \begin{pmatrix} \cos^2\varphi & 0 \\ 0 & \sin^2\varphi \end{pmatrix},$$

$c(\widetilde{\mathbf{K}}_\varphi) = |\cos(2\varphi)|$. Since $\mathcal{F}$ is strictly decreasing on $[0,1]$, the mean length of crests is maximal when $\cos(2\varphi) = 0$, *i.e.* for $\varphi = \pi/4$ or $3\pi/4$ modulo π. These directions are not orthogonal to the modes of $\widetilde{\mathbf{k}}$.

A Appendix

A.1 Variations of map $\mathcal{F}$

Lemma A.1 *Let $\mathcal{E}$ be the elliptic integral given by $\mathcal{E}(x) = \int_0^{\pi/2}(1 - x^2\sin^2\theta)^{1/2}d\theta$, for $x \in [0,1]$. Then, the map $\mathcal{F} : c \mapsto (1+c)^{1/2}\,\mathcal{E}\left((\frac{2c}{1+c})^{1/2}\right)$ is strictly decreasing on $[0,1]$.*

Proof. For any $k \in [0,1)$, $\mathcal{E}'(k) = -k\int_0^{\pi/2}\frac{\sin^2\theta}{(1-k^2\sin^2\theta)^{1/2}}\,d\theta$. Therefore, for any $c \in [0,1)$,

$$\begin{aligned}
\mathcal{F}'(c) &= \frac{1}{2}(1+c)^{-1/2}\mathcal{E}\left((\frac{2c}{1+c})^{1/2}\right) + (1+c)^{1/2}\frac{(2c)^{-1/2}}{(1+c)^{3/2}}\,\mathcal{E}'\left((\frac{2c}{1+c})^{1/2}\right) \\
&= \frac{1}{2}(1+c)^{-1/2}\int_0^{\pi/2}\left[(1-\frac{2c}{1+c}\sin^2\theta)^{1/2} - \frac{\frac{2}{1+c}\sin^2\theta}{(1-\frac{2c}{1+c}\sin^2\theta)^{1/2}}\right]d\theta \\
&= \frac{1}{2}(1+c)^{-1/2}\int_0^{\pi/2}\frac{\cos(2\theta)}{(1-\frac{2c}{1+c}\sin^2\theta)^{1/2}}\,d\theta.
\end{aligned}$$

It remains to show that the above integral, which we call $J(k)$ with $k = (\frac{2c}{1+c})^{1/2}$, is negative. Splitting the integral $J(k) := \int_0^{\pi/2}\frac{\cos(2\theta)}{(1-k^2\sin^2\theta)^{1/2}}\,d\theta$ into two parts, on $[0,\pi/4]$ and on $[\pi/4,\pi/2]$, and performing the change of variables $\theta' = \pi/2-\theta$ within the second part, we get

$$J(k) = \int_0^{\pi/4}\cos(2\theta)\left[\frac{1}{(1-k^2\sin^2\theta)^{1/2}} - \frac{1}{(1-k^2\cos^2\theta)^{1/2}}\right]d\theta, \qquad (18)$$

which is negative since $\cos\theta > \sin\theta$ for $\theta \in (0,\pi/4)$. ∎

A.2 Coherency index of $\mathbf{K}_\varphi$

The results in this section allow us to compute the coherency index of $\mathbf{K}_\varphi$ in the separable case. More precisely, it contains the proofs of Formulas (15), (16) and (17). As a consequence of Lemma 5.1 and the last point of Proposition

2.2, $\mathbb{E}[\mathbf{K}_\varphi \mathbf{K}_\varphi^T]$ can be seen as the covariance matrix of the gradient vector of $\frac{G'_{\mathbf{k}} \cdot u_\varphi}{v_{0,0}(\varphi)^{1/2}}$. The gradient vector of $G'_{\mathbf{k}} \cdot u_\varphi$ is actually the vectorial random field $(G''_{\mathbf{k}}(t)\, u_\varphi)_{t\in\mathbb{R}^2}$, with covariance matrix $\mathbb{V}[G''_{\mathbf{k}}(0)u_\varphi]$.

Lemma A.2 *1.* $\mathbb{E}[\mathbf{K}_\varphi \mathbf{K}_\varphi^T] = \frac{1}{v_{0,0}(\varphi)} \mathbb{V}[G''_{\mathbf{k}}(0)u_\varphi]$, *where*

$$v_{0,0}(\varphi) = m_{2,0}\cos^2\varphi + 2m_{1,1}\cos\varphi\sin\varphi + m_{0,2}\sin^2\varphi$$

$$\textit{and} \quad \mathbb{V}[G''_{\mathbf{k}}(0)u_\varphi] = \begin{pmatrix} v_{2,0}(\varphi) & v_{1,1}(\varphi) \\ v_{1,1}(\varphi) & v_{0,2}(\varphi) \end{pmatrix}$$

$$\textit{with} \begin{cases} v_{2,0}(\varphi) = \cos^2\varphi\, m_{4,0} + 2\cos\varphi\sin\varphi\, m_{3,1} + \sin^2\varphi\, m_{2,2}, \\ v_{0,2}(\varphi) = \cos^2\varphi\, m_{2,2} + 2\cos\varphi\sin\varphi\, m_{1,3} + \sin^2\varphi\, m_{0,4}, \\ v_{1,1}(\varphi) = \cos^2\varphi\, m_{3,1} + 2\cos\varphi\sin\varphi\, m_{2,2} + \sin^2\varphi\, m_{1,3}. \end{cases} \tag{19}$$

2. $\mathbb{E}[\|\mathbf{K}_\varphi\|^2] = \frac{v_{2,0}(\varphi)+v_{0,2}(\varphi)}{v_{0,0}(\varphi)}$.

3. $c(\mathbf{K}_\varphi) = \frac{\left((v_{2,0}(\varphi)-v_{0,2}(\varphi))^2+4v_{1,1}(\varphi)^2\right)^{1/2}}{v_{2,0}(\varphi)+v_{0,2}(\varphi)}$.

Proof. Point 1 results from Lemma 5.1. Point 2 stems from the equality

$$\mathbb{E}[\|\mathbf{K}_\varphi\|^2] = \operatorname{Trace}\left(\mathbb{E}[\mathbf{K}_\varphi \mathbf{K}_\varphi^T]\right)$$

and Point 3 derives from Formula (4). ■

Now let us assume that $\mathbf{k}$ is separable. Lemma 5.1 ensures that $\mathbf{K}_\varphi$ is also separable. Then

$$\mathbb{E}[\widetilde{\mathbf{K}}_\varphi \widetilde{\mathbf{K}}_\varphi^T] = \frac{1}{\nu_{0,0}(\varphi)} \begin{pmatrix} \nu_{2,0}(\varphi) & \nu_{1,1}(\varphi) \\ \nu_{1,1}(\varphi) & \nu_{0,2}(\varphi) \end{pmatrix},$$

where the moments $(\nu_{i,j}(\varphi))$ have been defined by (14). It results from Lemma 5.2 and from $\nu_{2,0}(\varphi)+\nu_{0,2}(\varphi) = \nu_{0,0}(\varphi)$ that $\mathbb{E}[\|\mathbf{K}_\varphi\|^2] = \frac{M_4}{M_2}$ and that the trace of $\mathbb{E}[\widetilde{\mathbf{K}}_\varphi \widetilde{\mathbf{K}}_\varphi^T]$ equals $\mathbb{E}[\|\widetilde{\mathbf{K}}_\varphi\|^2] = 1$ (this can also be seen as a consequence of $\widetilde{\mathbf{K}}_\varphi \in \mathbb{S}^1$). Consequently, its coherency index is

$$c(\widetilde{\mathbf{K}}_\varphi) = \frac{\left((\nu_{2,0}(\varphi) - \nu_{0,2}(\varphi))^2 + 4\nu_{1,1}(\varphi)^2\right)^{1/2}}{\nu_{0,0}(\varphi)}.$$

According to Remark 2.4, it coincides with the one of $\mathbf{K}_\varphi$.

A.3 Link with a preexisting result

In this section, we refer to a formula for the expected length of crests given in [7] (Proposition 11.4) and in [5] (Assertion 3), which is different from the one of Propostion 5.3, although also based on an application of Rice formula. We explain the reason of the difference between both formulas and why they actually coincide in the separable case.

The key point is that the authors of [7] and [5] compute the gradient vector of $G_{\mathbf{k}} \cdot u_\varphi$ by derivating it in the directions of the vector of the orthonormal basis $(u_\varphi, u_{\varphi+\pi/2})$ instead of the ones of the canonical basis of $\mathbb{R}^2$.

Let us denote by $R_{-\varphi}$ the matrix of the rotation with angle $-\varphi$ in $\mathbb{R}^2$. We introduce the moments of the image of mesure F through $R_{-\varphi}$, which we write F_φ.

$$\begin{aligned} \forall (i,j) \in \mathbb{N}_0^2, \quad m_{i,j}(\varphi) &= \int_{\mathbb{R}^2} \cos^i(\theta) \sin^j(\theta)\, dF_\varphi(\theta) \\ &= \int_{\mathbb{R}^2} \cos^i(\theta-\varphi) \cos^j(\theta-\varphi)\, dF(\theta) \\ &= \mathbb{E}[(\mathbf{k}\cdot u_\theta)^i (\mathbf{k}\cdot u_{\theta+\pi/2})^j]. \end{aligned}$$

Note that $m_{2,0}(\varphi) = v_{0,0}(\varphi)$ and that the moments $(m_{i,j}(\varphi))$ can be expressed as polynomial functions in $(\cos\varphi, \sin\varphi)$, with the coefficients depending linearly on the moments $(m_{i,j})$. For any $t \in \mathbb{R}^2$, we also write, for any $t \in \mathbb{R}^2$, $\nabla_\varphi(G'_{\mathbf{k}}(t)\cdot u_\varphi)$ the gradient vector of the random field $G'_{\mathbf{k}}\cdot u_\varphi$ at point t, computed by derivating it in the directions of the basis $(u_\varphi, u_{\varphi+\pi/2})$. The following lemma expresses the covariance matrix of this vector.

Lemma A.3

$$\mathbb{V}[\nabla_\varphi(G'_{\mathbf{k}}(0)\cdot u_\varphi)] = \begin{pmatrix} m_{4,0}(\varphi) & m_{3,1}(\varphi) \\ m_{3,1}(\varphi) & m_{2,2}(\varphi) \end{pmatrix},$$

$$\begin{aligned} \textit{where} \quad m_{4,0}(\varphi) &= m_{4,0}\cos^4\varphi + m_{0,4}\sin^4\varphi + 6m_{2,2}\cos^2\varphi\sin^2\varphi \\ &\quad +4m_{3,1}\cos^3\varphi\sin\varphi + 4m_{1,3}\cos\varphi\sin^3\varphi, \\ m_{2,2}(\varphi) &= (m_{4,0}+m_{0,4})\cos^2\varphi\sin^2\varphi \\ &\quad +m_{2,2}\left((\cos^2\varphi-\sin^2\varphi)^2 - 2\cos^2\varphi\sin^2\varphi\right) \\ &\quad +2(m_{1,3}+m_{3,1})\cos\varphi\sin\varphi(\cos^2\varphi-\sin^2\varphi), \\ m_{3,1}(\varphi) &= -m_{4,0}\cos^3\varphi\sin\varphi + m_{3,1}\cos^2\varphi(\cos^2\varphi - 3\sin^2\varphi) \\ &\quad +3m_{2,2}\cos\varphi\sin\varphi(\cos^2\varphi-\sin^2\varphi) \\ &\quad +m_{1,3}\sin^2\varphi(3\cos^2\varphi-\sin^2\varphi) + m_{0,4}\cos\varphi\sin^3\varphi. \end{aligned}$$

Proof. We write C the covariance function of $G'_{\mathbf{k}}\cdot u_\varphi$ given by (12). The covariance matrix of $\nabla_\varphi(G'_{\mathbf{k}}(0)\cdot u_\varphi)$ is minus the Hessian matrix of C computed by derivating twice C in the directions u_φ and $u_{\varphi+\pi/2}$. ■

The formula stated in [7] (Proposition 11.4) or in [5] (Assertion 3) is the following.
For any $\varphi \in [0, 2\pi]$,

$$\mathbb{E}[l(\mathbf{k}, Q, \varphi)] = \mathcal{H}_2(Q)\,\frac{1}{2\pi}\left(\frac{\gamma_2}{m_{2,0}(\varphi)}\right)^{1/2} \mathcal{E}((1-\gamma_1/\gamma_2)^{1/2}), \tag{20}$$

with

$$m_{2,0}(\varphi) = m_{20}\cos^2\varphi + 2m_{11}\cos\varphi\sin\varphi + m_{02}\sin^2\varphi$$

and γ_1 and γ_2 ($\gamma_1 \leq \gamma_2$) are the eigenvalues of the matrix

$$\begin{pmatrix} m_{4,0}(\varphi) & m_{3,1}(\varphi) \\ m_{3,1}(\varphi) & m_{2,2}(\varphi) \end{pmatrix}.$$

The equality $\mathbb{V}[\nabla_\varphi(G'_{\mathbf{k}}(0) \cdot u_\varphi)] = R_{-\varphi}\mathbb{V}[G''_{\mathbf{k}}(0)u_\varphi)]R_\varphi$ holds since $\nabla_\varphi(G'_{\mathbf{k}}(0) \cdot u_\varphi) = R_{-\varphi}\, G''_{\mathbf{k}}(0)u_\varphi$. Therefore both covariance matrices have the same eigenvalues and coherency index. Moreover, if $\mathbf{k}$ is separable, using a trigonometric identity, we can express the trace of the above matrix as

$$\frac{M_4}{M_2} m_{2,0}(\varphi).$$

This allows to deduce Proposition 5.3 from Formula (20).

According to Point 1 in Lemma A.2, note that we can also write

$$\mathbb{E}[R_{-\varphi}\mathbf{K}_\varphi(R_{-\varphi}\mathbf{K}_\varphi)^T] = \frac{1}{m_{2,0}(\varphi)}\mathbb{V}[\nabla_\varphi(G'_{\mathbf{k}}(0) \cdot u_\varphi)].$$

The vector $R_{-\varphi}\mathbf{K}_\varphi$ is simply vector $\mathbf{K}_\varphi$ expressed in the basis $(u_\varphi, u_\varphi + \pi/2)$.

A.4 Moments of two specific planar random wavevectors

Let $\mathbf{k}$ be a planar separable wavevector that we write $\mathbf{k} = \|\mathbf{k}\|\, \widetilde{\mathbf{k}}$ with $\|\mathbf{k}\|$ and $\widetilde{\mathbf{k}}$ independent and $\widetilde{\mathbf{k}} = (\cos\Theta, \sin\Theta)$. In the two following lemmas, we compute the moments of $\widetilde{\mathbf{k}}$, *i.e.*

$$\mu_{j,k} = \mathbb{E}[(\cos\Theta)^j\, (\sin\Theta)^k],$$

for any non negative integers j and k, in two specific cases, namely the toy model and the elementary model as introduced in Section 3.2.

Lemma A.4 *If Θ has a probability density function given by*

$$\theta \mapsto C_\alpha |\cos\theta|^\alpha \quad \text{with} \quad C_\alpha = \frac{\Gamma(1+\alpha/2)}{2\sqrt{\pi}\Gamma(1/2+\alpha/2)}.$$

for some nonnegative constant α, then the following formulas hold:

- $\mu_{0,0} = 1$
- $\mu_{j,k} = 0$ *whenever j or k is odd*
- $\mu_{j,0} = \frac{C_\alpha}{C_{\alpha+j}} = \frac{(\alpha+1)(\alpha+3)\cdots(\alpha+j-1)}{(\alpha+2)(\alpha+4)\cdots(\alpha+j)}$ *for j even ≥ 2*
- *for any even integers j and k, $\mu_{j,k} = \sum_{i=0}^{k/2}(-1)^i \binom{k/2}{i} \mu_{j+2i,0}$.*

In particular, it yields the non-zero second and fourth-order moments of $\widetilde{\mathbf{k}}$:

$$\mu_{2,0} = \frac{\alpha+1}{\alpha+2};\ \mu_{0,2} = \frac{1}{\alpha+2} \quad \text{and hence} \quad \mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T] = \frac{1}{\alpha+2}\begin{pmatrix}\alpha+1 & 0\\ 0 & 1\end{pmatrix};$$
$$\mu_{4,0} = \frac{(\alpha+1)(\alpha+3)}{(\alpha+2)(\alpha+4)};\ \mu_{0,4} = \frac{3}{(\alpha+2)(\alpha+4)};\ \mu_{2,2} = \frac{\alpha+1}{(\alpha+2)(\alpha+4)}.$$

Proof. It is clear that $\mu_{0,0} = 1$, $\mu_{j,k} = 0$ whenever j or k is odd and that $\mu_{j,0} = C_\alpha/C_{\alpha+j}$ for any even integer j. Using the explicit value of C_α yields the value of $\mu_{j,0}$. Finally, for any even integers j and k, writing $\sin^2\theta = 1-\cos^2\theta$ yields the formula for $\mu_{j,k}$. ■

Lemma A.5 *If Θ is uniformly distributed on $[-\delta,\delta]\cup[\pi-\delta,\pi+\delta]$ for some constant $\delta\in(0,\pi/2)$, then the following formulas hold:*

- $\mu_{0,0} = 1$,
- $\mu_{j,k} = 0$ *whenever j or k is odd,*
- $\mu_{2,0} = \frac{1}{2}(1+\mathrm{sinc}(2\delta))$; $\mu_{0,2} = \frac{1}{2}(1-\mathrm{sinc}(2\delta))$; $\mu_{2,2} = \frac{1}{8}(1-\mathrm{sinc}(4\delta))$,
- $\mu_{4,0} = \frac{1}{8}(3+4\mathrm{sinc}(2\delta)+\mathrm{sinc}(4\delta))$; $\mu_{0,4} = \frac{1}{8}(3-4\mathrm{sinc}(2\delta)+\mathrm{sinc}(4\delta))$,

where $\mathrm{sinc}(\theta) = \frac{\sin(\theta)}{\theta}$ *for any $\theta\neq 0$. In particular, this implies*

$$\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T] = \frac{1}{2}\begin{pmatrix}1+\mathrm{sinc}(2\delta) & 0\\ 0 & 1-\mathrm{sinc}(2\delta)\end{pmatrix}.$$

Proof. Symmetry arguments explain the vanishing moments, while the non-zero ones can be evaluated by linearizing trigonometric functions. ■

A.5 Variations of the mean length of crests

Thanks to Proposition 5.3, we know that the mean length of a crest of direction φ is given as a decreasing function of $c(\widetilde{\mathbf{K}}_\varphi)$. Recall that Formula (16) allows us to write out $c(\widetilde{\mathbf{K}}_\varphi)^2$ as

$$c(\widetilde{\mathbf{K}}_\varphi)^2 = \frac{1}{\nu_{0,0}(\varphi)^2}\left((\nu_{2,0}(\varphi)-\nu_{0,2}(\varphi))^2 + 4\nu_{1,1}(\varphi)^2\right),$$

where the $\nu_{i,j}(\varphi)$'s are defined by (14). By Lemma 5.2, the moments $\nu_{i,j}(\varphi)$ are such that $\nu_{i,j}(\varphi) = \frac{1}{M_{2+i+j}}\, v_{i,j}(\varphi)$. We use (19) to express $v_{i,j}(\varphi)$ in terms of the moments of $\mathbf{k}$ denoted by $m_{i,j}$. Since $m_{i,j} = M_{i+j}\,\mu_{i,j}$, Lemma 5.2 also yields

$$\begin{cases} \nu_{0,0}(\varphi) = \mu_{2,0}\cos^2\varphi + 2\mu_{1,1}\cos\varphi\sin\varphi + \mu_{0,2}\sin^2\varphi, \\ \nu_{2,0}(\varphi) = \mu_{4,0}\cos^2\varphi + 2\mu_{3,1}\cos\varphi\sin\varphi + \mu_{2,2}\sin^2\varphi, \\ \nu_{0,2}(\varphi) = \mu_{2,2}\cos^2\varphi + 2\mu_{1,3}\cos\varphi\sin\varphi + \mu_{0,4}\sin^2\varphi, \\ \nu_{1,1}(\varphi) = \mu_{3,1}\cos^2\varphi + 2\mu_{2,2}\cos\varphi\sin\varphi + \mu_{1,3}\sin^2\varphi. \end{cases} \tag{21}$$

In the case where the moments $\mu_{1,1}, \mu_{1,3}$ and $\mu_{3,1}$ simultaneously vanish, which is the case in our two favorite examples (see Lemmas A.4 and A.5), we like to remark that there exist two polynomials of degree two, say P and Q, such that

$$\forall \varphi \in [0, 2\pi], \quad c(\widetilde{\mathbf{K}}_\varphi)^2 = \frac{P(\cos(2\varphi))}{Q(\cos(2\varphi))}.$$

Consequently, $\varphi = 0$ and $\varphi = \pi/2$ appear as stationary points of the π-periodic map $\varphi \mapsto c(\widetilde{\mathbf{K}}_\varphi)^2$, whose graph is symmetric with respect to $\pi/2$.

A.5.1 The toy model case

We assume that the direction of $\mathbf{k}$ is given by the toy model (see Example 1 in Section 3.2). Using (21) and the expression for the $\mu_{i,j}$'s given by Lemma A.4, we can compute $c(\widetilde{\mathbf{K}}_\varphi)^2$. Figure 1 shows the graph of $\varphi \mapsto c(\widetilde{\mathbf{K}}_\varphi)^2$ for various values of α. We observe that $\varphi \mapsto c(\widetilde{\mathbf{K}}_\varphi)^2$ is minimal at $\varphi = \pi/2$, which is substantiated by the next lemma.

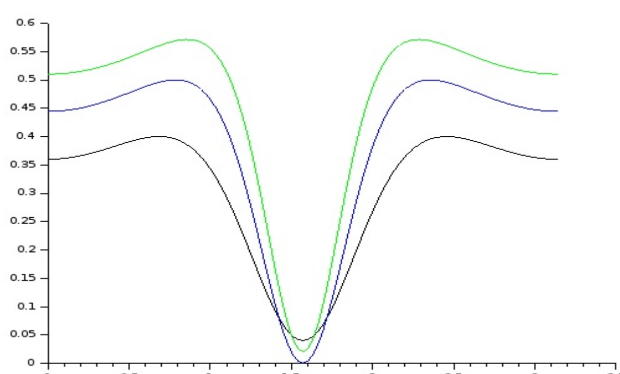

Figure 1: Graph of $\varphi \mapsto c(\widetilde{\mathbf{K}}_\varphi)^2$ for $\alpha = 1$ (black), $\alpha = 2$ (blue) and $\alpha = 3$ (green).

Lemma A.6 *The map $\varphi \mapsto c(\widetilde{\mathbf{K}}_\varphi)$ admits a local minimum at $\varphi = \pi/2$.*

Proof. We will perform an asymptotic expansion of $c(\widetilde{\mathbf{K}}_{\pi/2+\varphi})^2$ near $\varphi = 0$ in order to show that $\varphi \mapsto c(\widetilde{\mathbf{K}}_\varphi)$ admits a local minimum at $\pi/2$.

In the following lines, we write $g(\varphi) = \mathcal{O}(h(\varphi))$ as φ tends to 0, if there exists $\varphi_0 \in (0, 2\pi)$ and $M > 0$ such that

$$\forall \varphi \in [0, 2\pi],\ |\varphi| < |\varphi_0| \Rightarrow |g(\varphi)| \leq M|h(\varphi)|.$$

In particular, we write $\begin{cases} \cos(\pi/2 + \varphi) = -\varphi + \dfrac{\varphi^3}{6} + \mathcal{O}(\varphi^4), \\ \sin(\pi/2 + \varphi) = 1 - \dfrac{\varphi^2}{2} + \mathcal{O}(\varphi^4). \end{cases}$

From (21) and Lemma A.4, we get

$$\begin{aligned}
\nu_{0,0}(\pi/2 + \varphi) &= \mu_{0,2} + (\mu_{2,0} - \mu_{0,2})\varphi^2 + \mathcal{O}(\varphi^4) \\
&= \frac{1}{\alpha + 2}(1 + \alpha\varphi^2) + \mathcal{O}(\varphi^4) \\
\nu_{2,0}(\pi/2 + \varphi) &= \mu_{2,2} + (\mu_{4,0} - \mu_{2,2})\varphi^2 + \mathcal{O}(\varphi^4) \\
&= \frac{\alpha + 1}{(\alpha + 2)(\alpha + 4)}(1 + (\alpha + 2)\varphi^2 + \mathcal{O}(\varphi^4) \\
\nu_{0,2}(\pi/2 + \varphi) &= \mu_{0,4} + (\mu_{2,2} - \mu_{0,4})\varphi^2 + \mathcal{O}(\varphi^4) \\
&= \frac{1}{(\alpha + 2)(\alpha + 4)}(3 + (\alpha - 2)\varphi^2) + \mathcal{O}(\varphi^4) \\
\nu_{1,1}(\pi/2 + \varphi) &= 2\mu_{2,2}\varphi(1 - \frac{1}{2}\varphi^2) + \mathcal{O}(\varphi^4) \\
&= \frac{\alpha + 1}{(\alpha + 2)(\alpha + 4)}\varphi(2 - \varphi^2) + \mathcal{O}(\varphi^4).
\end{aligned}$$

After some algebra, it gives

$$c(\widetilde{\mathbf{K}}_{\pi/2+\varphi})^2 = (\alpha - 2)^2 + 24\alpha(\alpha + 1)\varphi^2 + \mathcal{O}(\varphi^4),$$

which clearly admits a minimum at $\varphi = 0$. ■

A.5.2 The elementary model case

We assume that the direction of $\mathbf{k}$ is given by the elementary model (see Example 2 in Section 3.2). Using (21) and the expression for the $\mu_{i,j}$'s given by Lemma A.5, one can compute $c(\widetilde{\mathbf{K}}_\varphi)^2$. Figure 2 shows the graph of $\varphi \mapsto c(\widetilde{\mathbf{K}}_\varphi)^2$ for various values of δ. We again observe that it is minimal at $\varphi = \pi/2$.

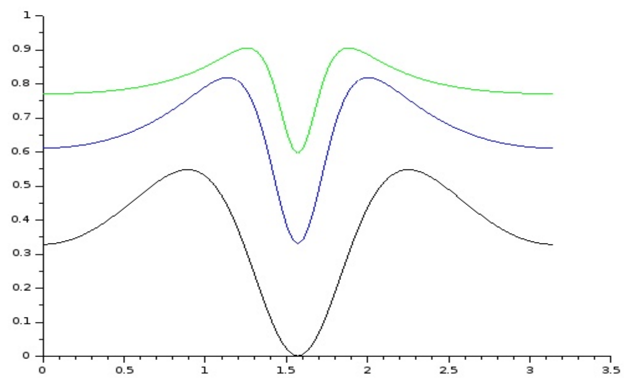

Figure 2: Graph of $\varphi \mapsto c(\widetilde{\mathbf{K}}_\varphi)^2$ for $\delta = \pi/3$ (black), $\delta = \pi/5$ (blue) and $\delta = \pi/7$ (green).

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