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Anisotropic random wave models

Anne Estrade* and Julie Fournier†

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Abstract

Let d be an integer greater or equal to 2 and let $\mathbf{k}$ be a d-dimensional random vector. We call random wave model with random wavevector $\mathbf{k}$ any stationary random field defined on $\mathbb{R}^d$ with covariance function $t \in \mathbb{R}^d \mapsto \mathbb{E}[\cos(\mathbf{k}.t)]$. The purpose of the present paper is to link properties that concern the geometry and the anisotropy of the random wave with the distribution of the random wavevector. For instance, when $\mathbf{k}$ almost surely belongs to the unit sphere in $\mathbb{R}^2$ and the random wave model is nothing but the anisotropic version of Berry's planar waves, we prove that the expected length of the nodal lines is decreasing as the anisotropy of the random wavevector is increasing. Also, when $\mathbf{k}$ almost surely belongs to the Airy surface in $\mathbb{R}^3$ and the associated random wave serves as a model for the sea waves, we prove that the direction that maximises the expected length of the static crests is not always orthogonal to what we call favorite direction of the random wavevector.

Keywords: Gaussian field; random wave; nodal statistics; level set; crossing theory; anisotropy

2010 Mathematics Subject Classification: primary 60G60; secondary 60G15, 60K40, 62H11, 86A05

1 Introduction

For many centuries, physicists have been using wave models defined on a multi-dimensional space in various domains as different as acoustics, electronics, geophysics, oceanography or seismology. In order to take into account variability or uncertainty, it is useful to consider random wave models. It is the exact purpose of a pioneer exhaustive study by Longuet and Higgins [17] that was concerned by sea waves modelized as a random moving surface. Another mathematical pioneer study was raised by Berry in several papers, [8] or [9] for instance. These seminal works opened a wide area of research in the last decades, either for

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statistical purposes ([4], [16], [1], [5], [7], [20]), or more recently for topological purposes in link with number theory ([25], [13], [19]). Ten years ago, the interest for nodal sets or level sets also met the theory of crossings developed by Rice for one-dimensional stochastic processes fifty years before, yielding two inspiring books by Adler and Taylor [2] and by Azaïs and Wschebor [6]. The present paper is clearly inspired by all the above references but to the best of our knowledge it is the first time that the different models are gathered in the same work and are studied under the same focus, the influence of anisotropy.

A big demand for anisotropic models is nowadays observed, in particular by practitioners in geostatistics, offshore engineering, heterogeneous material or medical imaging (see for instance [24], [12], [3]), but also for more theoretical studies dedicated to image synthesis and analysis, cosmology or arithmetic ([21], [10], [22], [14]).

In the present paper, we aim at exploring the anisotropy of anisotropic random waves that are defined on a d-dimensional space with $d \geq 1$. We start with a single random wave given by $t \in \mathbb{R}^d \mapsto a\ \cos(\mathbf{k} \cdot t + \eta)$, whose directional structure is given by a d-dimensional random wavevector $\mathbf{k}$, random phase η is uniformly distributed on $[0, 2\pi]$ and independent of $\mathbf{k}$, and amplitude a is kept constant. Since our focus is only dedicated to anisotropy, the latter assumption will remain all along the paper. We also study the stationary Gaussian counterpart, *i.e.* a stationary Gaussian random field on $\mathbb{R}^d$ with the same covariance function $t \in \mathbb{R}^d \mapsto a\,\mathbb{E}[\cos(\mathbf{k} \cdot t)]$. Our purpose is to link the geometric and anisotropic behaviour properties of the random wave with the distribution of its random wavevector, in particular its moments of finite order and its directional statistics. In particular, considering Berry's anisotropic planar waves, we prove that the expected length of the nodal lines is a decreasing function of the (properly quantified) anisotropy of the random wavevector. At the opposite, considering random sea waves, we prove that no general statement can be established: the direction that maximises the expected length of the static crests may be orthogonal to the favorite direction (properly defined) of the random wavevector as it may not. We like to mention that when $\mathbf{k}$ is equal to $A\mathbf{u}$ with A a matrix and $\mathbf{u}$ a random vector in $\mathbb{R}^d$ whose distribution is invariant under rotations, the associated random wave has the same distribution as an isotropic random wave deformed by the linear transformation A^T. In that case, the study of anisotropy, either in the spectral domain, or in the parameter domain, is equivalent. In the general case where no linear deformation is involved, studing anisotropy in those two domains are two different approaches. The latter point of view is adopted in [3] for instance, whereas our paper definitively belongs to the former type as did [11] or [24].

The paper is organised as follows. General facts are presented in Section 2, in particular the key point of spectral representation. Another important point is the link with partial differential equations that are solved by the random waves. Section 3 deals with the study of planar waves through specific tools that are used in directional statistical studies in dimension two, such as most probable direction, favorite or principal directions. In the fourth section, we introduce anisotropic versions of Berry's random waves, which are anisotropic solutions of

Helmholtz equation. We focus on the nodal ets, their Hausdorff measure and their directional statistics. Section 5 is devoted to a model for sea waves, that is to say a space-time model indexed by $\mathbb{R}^2 \times \mathbb{R}$. We study the mean length of static crests from a directional point of view. All over the paper, two specific distributions for the random wavevector in dimension two are examined. One is called "elementary model". It is described by a main direction and a bandwidth that quantifies the anisotropy. We call the other one "toy model". It is given by a positive probability density function only depending on a single parameter that carries out the whole quantified information on anisotropy. The technical computations are detailed in the Appendix Section.

Notations.
Let d be a positive integer. For $z \in \mathbb{R}^d$, zz^T stands for the $d \times d$ matrix $(z_i z_j)_{1\leq i,j\leq d}$, $||z||$ for the Euclidean norm of z and $z \cdot z'$ for the usual scalar product of z with $z' \in \mathbb{R}^d$.
For $\mathbf{k}$ being a random vector in $\mathbb{R}^d$, we respectively denote by $\mathbb{E}[\mathbf{k}]$ and $\mathbb{V}[\mathbf{k}]$ the expectation (d-dimensional vector) of $\mathbf{k}$ and the variance ($d \times d$ matrix) of $\mathbf{k}$.
For $\varphi \in [0, 2\pi]$, u_φ denotes the vector $(\cos\varphi, \sin\varphi)$ in $\mathbb{R}^2$.
We use $\mathbb{N}_0$ for the set $\{0, 1, 2, \cdots\}$ of all non-negative integers and for $\mathbf{j} = (j_1, \cdots, j_d) \in {\mathbb{N}_0}^d$, we write $|\mathbf{j}| = \sum_{l=1}^d j_l$. Moreover, if $\lambda \in \mathbb{R}^d$ and if F is a smooth map from $\mathbb{R}^d$ to $\mathbb{R}$, we write

$$\lambda^{\mathbf{j}} = \prod_{l=1}^{d} \lambda_l^{j_l} \quad \text{and} \quad \partial^{\mathbf{j}} F = \frac{\partial^{|\mathbf{j}|} F}{\partial^{j_1}\lambda_1 \cdots \partial^{j_d}\lambda_d}.$$

For s any positive integer, $\mathcal{H}_s$ denotes the Hausdorff measure of dimension s.

2 General setting

2.1 Anisotropic elementary random wave

Let d be a positive integer. We consider a random multi-dimensional model of elementary wave defined by,

$$\forall t \in \mathbb{R}^d, \quad X_{\mathbf{k}}(t) = \sqrt{2}\,\cos(\mathbf{k} \cdot t + \eta), \tag{1}$$

where $\mathbf{k}$ is a d-dimensional random vector called the random wavevector and where the random phase η is uniformly distributed on $[0, 2\pi]$.

The random field $X_{\mathbf{k}}$ is clearly not isotropic and the kind of anisotropy depends on the law of $\mathbf{k}$. As it will be stated in Proposition 2.1, isotropy occurs if and only if $\mathbf{k}$ is isotropically distributed. If $||\mathbf{k}||$ is almost surely constant, we write $\kappa = ||\mathbf{k}||$ the wavenumber of $X_{\mathbf{k}}$.

We will be particularly interested in examples where the random wavevector $\mathbf{k}$ is supported by $\{\lambda \in \mathbb{R}^d \,:\, P(\lambda) = 0\}$, the zero set of a multivariate polynomial P.

Example 1 (Toy model) A particular case with $d = 2$ is studied in [12]. The random wavevector is prescribed by $\mathbf{k} = (\cos\Theta, \sin\Theta)$ with Θ a random variable with support in $\mathbb{R}/2\pi\mathbb{Z}$ such that, for a fixed $\alpha \geq 0$, the density of Θ with respect to Lebesgue measure on $[0, 2\pi]$ is given by

$$\theta \mapsto C_\alpha \,|\cos\theta|^\alpha, \text{ with } C_\alpha = \frac{\Gamma(1+\alpha/2)}{2\sqrt{\pi}\Gamma(1/2+\alpha/2)}, \tag{2}$$

where Γ is the usual Gamma function. Parameter α is considered as an anisotropy parameter. Indeed, taking $\alpha = 0$, one gets the isotropic version of model (1), whereas, at the opposite, the case $\alpha \to +\infty$ corresponds with a totally anisotropic version of model (1) where $\mathbf{k}$ is *a.s.* along the x-axis.

Example 2 (Elementary model) Another particular case with $d = 2$ is studied in [10] and [23]. The random wavevector is prescribed by $\mathbf{k} = (\cos\Theta, \sin\Theta)$ with Θ a random variable uniformly distributed on $[\alpha_0 - \delta, \alpha_0 + \delta]$ with $0 \leq \delta \leq \pi$. Parameter α_0 indicates the main direction whereas parameter δ quantifies anisotropy. Actually, $\delta = 0$ corresponds with a totally anisotropic model, $\delta \approx 0$ corresponds with a model that could be named narrow spectrum, and $\delta = \pi$ corresponds with the isotropic model. If one wishes a symmetric model, one can also consider Θ uniformly distributed on $[\alpha_0 - \delta, \alpha_0 + \delta] \cup [\alpha_0 + \pi - \delta, \alpha_0 + \pi + \delta]$ with $0 \leq \delta \leq \pi/2$.

Example 3 (Berry random wave) We assume that the wavevector $\mathbf{k}$ satisfies $||\mathbf{k}|| = \kappa$, *a.s.* for some constant $\kappa > 0$ and that it is not necessarily isotropically distributed. The associated elementary wave is an anisotropic generalization of Berry's random wave model, an isotropic model that has originally been presented in [8] and intensively studied in the last years. Parameter κ is called the wavenumber. This model is the purpose of Section 4.

Example 4 (Sea waves) We will also examine the case where the random wavevector $\mathbf{k}$ is supported by the Airy surface in $\mathbb{R}^3$ ($d = 3$), namely $\{(x, y, z) \in \mathbb{R}^3\,;\, x^2 + y^2 - z^4 = 0\}$. The associated elementary wave is related to the space-time model used for the modelization of sea waves, assuming that the depth of the sea is infinite (see [17] for the original idea, [4] or [6] for more recent developments). Section 5 is devoted to the study of this model.

In the following proposition, we give some basic properties of the covariance function of $X_{\mathbf{k}}$.

Proposition 2.1 *1. The random field $X_{\mathbf{k}}$ is centred and second-order stationary with covariance function*

$$r(t) := \mathrm{Cov}[X_{\mathbf{k}}(0), X_{\mathbf{k}}(t)] = \mathbb{E}[\cos(\mathbf{k} \cdot t)], \quad t \in \mathbb{R}^d. \tag{3}$$

In particular, $\mathrm{Var}(X_{\mathbf{k}}(0)) = 1$.

2. *Let* $\mathbf{k}^s$ *be the symmetrized random variable associated to* $\mathbf{k}$ *and let* F *be its probability measure*[1]. *Then*

$$r(t) = \mathbb{E}[\exp(i\mathbf{k}^s \cdot t)] = \int_{\mathbb{R}^d} \exp(iu \cdot t)\, dF(u), \tag{4}$$

which means that r *is the characteristic function of the random variable* $\mathbf{k}^s$ *and that* F *is the spectral measure of* $X_{\mathbf{k}}$.
Moreover, $X_{\mathbf{k}}$ *is second-order isotropic if and only if the law of* $\mathbf{k}^s$ *is invariant under rotations.*

3. *The covariance function* r *admits derivatives up to order* m ($m \in \mathbb{N}_0$) *if and only if* $\mathbf{k}$ *admits moments of order* m. *In this case, for any* $\mathbf{j} \in {\mathbb{N}_0}^d$ *such that* $|\mathbf{j}| \leq m$, *we have*

$$\partial^{\mathbf{j}} r(0) = 0 \quad \text{if } |\mathbf{j}| \text{ is odd} \;\; ; \; \partial^{\mathbf{j}} r(0) = (-1)^{|\mathbf{j}|/2}\, \mathbb{E}[\mathbf{k}^{\mathbf{j}}] \quad \text{if } |\mathbf{j}| \text{ is even.}$$

In particular, $r''(0) = -\,\mathbb{E}[\mathbf{k}\mathbf{k}^T]$.

2.2 Anisotropic Gaussian wave model

We are still given a random vector $\mathbf{k}$ in $\mathbb{R}^d$ and we now consider a Gaussian, stationary and centred random field $G_{\mathbf{k}}$ with the same covariance function as the elementary random wave $X_{\mathbf{k}}$ introduced in the previous section. Such a field exists, consequently to Kolmogorov's extension theorem (see [6] Sections 1.1 and 1.2 for instance).

Note that such a Gaussian field can be obtained as a limit by considering independent and identically distributed versions of η and of $\mathbf{k}$, denoted respectively by $(\eta_j)_{j\in\mathbb{N}}$ and by $(\mathbf{k}_j)_{j\in\mathbb{N}}$. According to the central limit theorem applied to finite-dimensional distributions, the distribution of

$$\left(\sqrt{\frac{2}{N}} \sum_{j=1}^{N} \cos(\mathbf{k}_j \cdot t + \eta_j) \right)_{t\in\mathbb{R}^d}$$

converges as N tends to ∞ towards a Gaussian random field with the appropriate covariance function.

The covariance function of $G_{\mathbf{k}}$, being equal to the covariance function of $X_{\mathbf{k}}$, is given according to (4) in Proposition 2.1: $r(t) = \int_{\mathbb{R}^d} \exp(iu \cdot t)\, dF(u)$, $t \in \mathbb{R}^d$, where F is the distribution of $\mathbf{k}^s$. From this, we deduce a spectral representation of the field $G_{\mathbf{k}}$.
Let W_F be a complex Gaussian F-noise on $\mathbb{R}^d$, *i.e.* a $\mathbb{C}$-valued process defined on the set $\mathcal{B}(\mathbb{R}^d)$ of Borelians such that

[1] If $F_{\mathbf{k}}$ and $F_{-\mathbf{k}}$ are respectively the probability measures of $\mathbf{k}$ and $-\mathbf{k}$, then the symmetrized random variable associated with $\mathbf{k}$ is defined as the random variable with probability measure $F = \frac{1}{2}(F_{\mathbf{k}} + F_{-\mathbf{k}})$.

- *a.s.* W_F is a complex-valued measure on $\mathcal{B}(\mathbb{R}^d)$,
- $\forall A \in \mathcal{B}(\mathbb{R}^d)$, $W_F(A)$ is a complex-valued Gaussian variable with $\mathbb{E}[W_F(A)] = 0$ and $\mathbb{E}[W_F(A)\overline{W_F(A)}] = F(A)$, where $\overline{\cdot}$ denotes the complex conjugation,
- for any sequence $(A_n)_n$ of pairwise disjoint Borel sets, $(W_F(A_n))_n$ are independent random variables.

Moreover, we add the property that for any $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\overline{W_F(A)} = W_F(-A).$$

Then, it is easy to check that the Gaussian stationary random field prescribed by

$$\left(\int_{\mathbb{R}^d} e^{it\cdot u}\, dW_F(u) \right)_{t\in\mathbb{R}^d} \tag{5}$$

is real-valued, centred and that its covariance function is given by (4).

Reciprocally, if $Y : \mathbb{R}^d \to \mathbb{R}$ is a centred and stationary Gaussian random field with unit variance, according to Bochner's theorem, there exists a symmetric probability measure on $\mathbb{R}^d$, denoted by F, such that the covariance function r of Y is given by (4). It follows that we can associate with Y a symmetric random variable in $\mathbb{R}^d$ of probability measure F, denoted by $\mathbf{k}_Y$ and referred to in the following as *the random wavevector of* Y.

2.3 Link with partial differential equation

We will show that both $X_{\mathbf{k}}$ and $G_{\mathbf{k}}$ satisfy a specific partial differential equation if and only if the random wavevector $\mathbf{k}$ is supported by a specific hypersurface of $\mathbb{R}^d$.

Let P be an even d-multivariate polynomial. Then there exists a sequence of real numbers $(\alpha_{\mathbf{j}})_{\mathbf{j}\in{\mathbb{N}_0}^d}$ with only finitely many non-zero terms, such that

$$\forall \lambda \in \mathbb{R}^d, \qquad P(\lambda) = \sum_{\mathbf{j}\in{\mathbb{N}_0}^d;\ |\mathbf{j}|\ \text{even}} \alpha_{\mathbf{j}}\, \lambda^{\mathbf{j}}. \tag{6}$$

We associate with P the following differential operator:

$$\mathcal{L}_P(X) = \sum_{\mathbf{j}\in{\mathbb{N}_0}^d;\ |\mathbf{j}|\ \text{even}} (-1)^{|\mathbf{j}|/2} \alpha_{\mathbf{j}}\, \partial^{\mathbf{j}} X,$$

Let us remark that the random field $X_{\mathbf{k}}$ defined by (1) is obviously almost surely of class $\mathcal{C}^\infty$.

Proposition 2.2 *Let P be an even multivariate polynomial given by (6). Then $X_{\mathbf{k}}$ almost surely satisfies the partial differential equation*

$$\forall t \in \mathbb{R}^d, \qquad \mathcal{L}_P(X)(t) = 0 \tag{7}$$

if and only if $P(\mathbf{k}) = 0$ a.s.

Proof. For any $\mathbf{j} \in {\mathbb{N}_0}^d$ such that $|\mathbf{j}|$ is even, we have $\partial^{\mathbf{j}} X_{\mathbf{k}}(t) = (-1)^{|\mathbf{j}|/2} \mathbf{k}^{\mathbf{j}} \cos(\mathbf{k} \cdot t + \eta)$. Hence, we get $\mathcal{L}_P(X_{\mathbf{k}})(t) = P(\mathbf{k})\, X_{\mathbf{k}}(t)$ and the proof follows immediately. ■

Applying Proposition 2.2 to the examples given at the beginning of the section provides random anisotropic solutions of some famous partial differential equations. Indeed, we recall that the Laplacian operator Δ on $\mathbb{R}^d$ is defined by $\Delta = \sum_{1 \le j \le d} \frac{\partial^2}{\partial t_j^2}$. Then, the elementary random wave associated with Example 1 (case $d = 2$ and $\kappa = 1$) and Example 3 (any d and any κ) is an almost sure solution of Helmholtz equation $\Delta X + \kappa^2 X = 0$. In the same vein, the elementary random wave associated with Example 4 is an almost sure solution of the partial differential equation $\frac{\partial^2}{\partial x^2} X + \frac{\partial^2}{\partial y^2} X + \frac{\partial^4}{\partial z^4} X = 0$.

Let us now be concerned with $G_{\mathbf{k}}$. We assume that the random wavevector $\mathbf{k}$ admits moments of any order. Hence, the covariance function r of $G_{\mathbf{k}}$ is of class $\mathcal{C}^\infty$ and consequently there exists a version of $G_{\mathbf{k}}$ whose almost every realization is of class $\mathcal{C}^\infty$; it is given by representation (5) for instance. First, let us point out that $G_{\mathbf{k}}$ satisfies Proposition 2.2 as well as $X_{\mathbf{k}}$. Indeed, $G_{\mathbf{k}}$ is centred and admits the same covariance function as $X_{\mathbf{k}}$; therefore for any multivariate polynomial P given by (6), for any $t \in \mathbb{R}^d$, $\mathrm{Var}\,(\mathcal{L}_P(G_{\mathbf{k}})(t)) = \mathrm{Var}\,(\mathcal{L}_P(X_{\mathbf{k}})(t))$. However, the following theorem is a more general result: it provides a sufficient and necessary condition for any stationary Gaussian random field to satisfy Equation (7).

Theorem 2.3 *Let P be an even multivariate polynomial defined by (6) and let Y be Gaussian random field defined on $\mathbb{R}^d$ that is centred, stationary, with unit variance and almost surely of class $\mathcal{C}^\infty$. The following properties are equivalent.*

1. *The Gaussian random field Y almost surely satisfies the partial differential equation*
$$\forall t \in \mathbb{R}^d, \qquad \mathcal{L}_P(Y)(t) = 0.$$
2. *The Gaussian random field Y admits a spectral representation given by* (5)*, where F is a probability measure supported by $\{\lambda \in \mathbb{R}^d : P(\lambda) = 0\}$ and W_F is a complex Gaussian F-noise on $\mathbb{R}^d$.*
3. *The random wavevector $\mathbf{k}_Y$ associated with Y almost surely satisfies $P(\mathbf{k}_Y) = 0$.*

We insist on the fact that the above theorem provides all the Gaussian *a*.s. solutions, isotropic or not, of the partial differential equation $\mathcal{L}_P(Y) = 0$ (Helmholtz equation in the case of Example 1). Moreover, the equation gives information on the localization of the random variable $\mathbf{k}$.

Proof. Items 2 and 3 in Theorem 2.3 are clearly equivalent as F is the distribution of $\mathbf{k}_Y$. Since Y is centred, so are all its derivatives and the stationary random field $\mathcal{L}_P(Y)$. Therefore, $\mathcal{L}_P(Y)$ is almost surely identically zero if and only if its variance at each point is zero. But $\mathrm{Var}(\mathcal{L}_P(Y)(t))$ can be expressed as a linear combination of derivatives of the covariance function r_Y of Y. Hence Y is an a.s. solution of the partial differential equation $\mathcal{L}_P(Y)=0$ if and only if its covariance function r_Y satisfies

$$\sum_{\mathbf{j},\mathbf{k}\in\mathbb{N}_0{}^d;\,|\mathbf{j}|,|\mathbf{k}|\,\text{even}} (-1)^{(|\mathbf{j}|+|\mathbf{k}|)/2}\,\alpha_{\mathbf{j}}\,\alpha_{\mathbf{k}}\,\partial^{(\mathbf{j}+\mathbf{k})} r_Y(0)=0. \tag{8}$$

On the other hand, as it is the covariance function of a stationary centred field, r_Y satisfies Bochner's Theorem: there exists a Radon finite measure F on $\mathbb{R}^d$ such that $r_Y(t)=\hat{F}(t)$, where $\hat{F}$ denotes the Fourier transform, i.e. $\hat{F}(t)=\int_{\mathbb{R}^d} e^{it\cdot\lambda}\,dF(\lambda)$. Then r_Y satisfies (8) if and only if

$$0=\int_{\mathbb{R}^d}\Big(\sum_{\mathbf{j},\mathbf{k}\in\mathbb{N}_0{}^d;\,|\mathbf{j}|,|\mathbf{k}|\,\text{even}} (-1)^{|\mathbf{j}|+|\mathbf{k}|}\,\alpha_{\mathbf{j}}\,\alpha_{\mathbf{k}}\,\lambda^{\mathbf{j}}\,\lambda^{\mathbf{k}}\Big)\,dF(\lambda)=\int_{\mathbb{R}^d} P(\lambda)^2\,dF(\lambda).$$

The above integral vanishes if and only if the measure F is supported by $\{\lambda\in\mathbb{R}^d\,:\,P(\lambda)=0\}$. ■

3 Remarkable directions in the planar case

We introduce some definitions related to planar models. One can see [18] or [15] for more details in the domain of directional statistics.

3.1 Most probable and favorite directions

When Z is a two-dimensional random vector, one can write it out either using Euclidean coordinates $Z=(Z_1,Z_2)$ or, if $Z\neq 0$, $a.s.$, using polar coordinates $Z=Ru_\Theta$, where u_Θ is the vector $(\cos\Theta,\sin\Theta)$. Hence, we introduce two remarkable directions.

Definition 3.1 *Let Z be a random vector in $\mathbb{R}^2$ such that $Z\neq 0$, $a.s.$*
If the mode of the random variable Θ exists and is unique, we call it the most probable direction of Z. If there exists a mode that is not unique, we define the set of most probable directions of Z in $\mathbb{R}/2\pi\mathbb{Z}$ as the set of all modes of Θ.

If Θ is a discrete random variable then (at least) one most probable direction exists. If Θ is a continuous random variable with a probability density function (p.d.f.) h admitting a maximum on $\mathbb{R}/2\pi\mathbb{Z}$ (which is ensured if h is continuous), the set of the most probable directions can be expressed as the direction(s) in the set

$$\operatorname*{Argmax}_{\theta\in\mathbb{R}/2\pi\mathbb{Z}} h(\theta).$$

Note that if the distribution of the random vector (R,Θ) admits a probability density function $(r,\theta)\mapsto\tilde{f}(r,\theta)$ with respect to Lebesgue measure on $\mathbb{R}^+\times\mathbb{R}/2\pi\mathbb{Z}$, then for any $\theta\in\mathbb{R}/2\pi\mathbb{Z}$, $h(\theta)=\int_{\mathbb{R}^+}\tilde{f}(r,\theta)\,dr$.

Definition 3.2 *We assume that the matrix $\mathbb{E}[ZZ^T]$ does not belong to the set $\{\alpha I_2,\, \alpha \geq 0\}$. Then the favorite direction of Z is defined as the only element in*

$$\underset{\varphi \in \mathbb{R}/\pi\mathbb{Z}}{Argmax} \left(\mathbb{E}[(Z \cdot u_\varphi)^2] \right) = \underset{\varphi \in \mathbb{R}/\pi\mathbb{Z}}{Argmax} \left(u_\varphi \cdot \mathbb{E}[ZZ^T] u_\varphi \right),$$

where $u_\varphi = (\cos\varphi, \sin\varphi)$. Consequently, the favorite direction is nothing but the direction in $\mathbb{R}/\pi\mathbb{Z}$ of the eigensubspace of $\mathbb{R}^2$ associated with the largest eigenvalue of the symmetric positive matrix $\mathbb{E}[ZZ^T]$.

If $\mathbb{E}[ZZ^T] = \alpha I_2$ with $\alpha \geq 0$, then $\underset{\varphi \in \mathbb{R}/\pi\mathbb{Z}}{Argmax} \left(\mathbb{E}[(Z \cdot u_\varphi)^2] \right) = \mathbb{R}/\pi\mathbb{Z}$.

In some cases, such as in the following Examples 1, 3, 5 and 6, the most probable direction(s) modulo π coincides with the favorite direction(s). Nevertheless, in the general case, they don't.

Examples Let $Z = Ru_{\theta_0}$ a two-dimensional random vector such that $R \in (0, +\infty)$, *a.s.* and $\Theta \in [0, 2\pi)$, *a.s.*

1. If Θ almost surely takes a fixed value $\theta_0 \in [0, 2\pi)$, that is $Z = Ru_{\theta_0}$, then the most probable direction of Z is θ_0. On the other hand, $Z \cdot u_\varphi = R\cos(\theta_0 - \varphi)$ and hence the favorite direction of Z is θ_0 modulo π.

2. If (R, Θ) is distributed as $F_R \otimes \frac{1}{2}(\delta_0 + \delta_{\pi/2})$ on $(0, +\infty) \times [0, 2\pi)$, where δ stands for the Dirac distribution, then the most probable directions are 0 and $\pi/2$ modulo 2π whereas there is no favorite direction. In the same vein, with the distribution $F_R \otimes \frac{1}{2}(\delta_0 + \delta_{\pi/4})$, the most probable directions are 0 and $\pi/4$ modulo 2π whereas the favorite direction is $\pi/8$ modulo π.

3. If Θ and R are independent and if Θ is uniformly distributed on $[0, 2\pi]$, then Z admits $\mathbb{R}/2\pi\mathbb{Z}$ as its set of most probable directions. Moreover, Z is centred and $\mathbb{E}[ZZ^T] = \frac{1}{2}\mathbb{E}[R^2] I_2$, thus the set of favorite directions of Z is $\mathbb{R}/\pi\mathbb{Z}$.

4. If Θ and R are independent and if Θ is uniformly distributed on $[\alpha_0 - \delta, \alpha_0 + \delta]$ (see Example 2 in Section 2) , then the set of most probable directions is the whole interval $[\alpha_0 - \delta, \alpha_0 + \delta]$, whereas the favorite direction is reduced to the value α_0 modulo π.

5. If Θ admits a p.d.f. given by (2) (see Example 1 in Section 2), for a given $\alpha > 0$, and if Θ and R are independent, then the most probable direction of Z is clearly 0. On the other hand, Z is centred and $\mathbb{V}[Z] = \frac{\mathbb{E}[R^2]}{\alpha+2} \begin{pmatrix} \alpha + 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence, the favorite direction of Z is 0 as well. We refer to Lemma A.1 in Appendix section for the detailed computation of the moments.

6. Let Z be a 2-dimensional centred Gaussian vector with a variance matrix $\mathbb{V}[Z]$ that does not belong to $\{\alpha I_2,\, \alpha \in \mathbb{R}\}$. Then, the most probable direction of Z is $\text{Argmin}_\varphi \left(u_\varphi \cdot \mathbb{V}[Z]^{-1} u_\varphi \right) = \text{Argmax}_\varphi \left(u_\varphi \cdot \mathbb{V}[Z] u_\varphi \right)$, thus it is equal modulo π to the favorite direction of Z.

3.2 Principal direction

We now introduce a remarkable direction for real-valued planar random fields. Let $X : \mathbb{R}^2 \to \mathbb{R}$ be a stationary random field that is *a.s.* differentiable and satisfies $\mathbb{E}[||X'(0)||^2] < +\infty$. For φ a direction in $\mathbb{R}/\pi\mathbb{Z}$, we denote by $X^\varphi = (X(xu_\varphi))_{x\in\mathbb{R}}$ the one-dimensional stationary process obtained by restricting the X to the line $\mathbb{R}u_\varphi$.

Definition 3.3 *The principal direction of X is defined as*

$$\underset{\varphi\in\mathbb{R}/\pi\mathbb{Z}}{Argmax}(m_2(\varphi)), \quad where \;\; m_2(\varphi) = \mathbb{E}[(X^\varphi)'(0)^2],$$

understood as a certain value if the maximum is unique and as a set of values if it is not.

The latter notion has been introduced by Longuet-Higgins in [17] in his study of a planar random wave model for sea waves. Note that $m_2(\varphi)$ is nothing but the second spectral moment of X^φ and that restricting X to a certain line of the plane or to any parallel line does not change the law of the obtained process because X is stationnary. We have also,

$$m_2(\varphi) = \mathbb{E}[(X^\varphi)'(0)^2] = \mathbb{E}[(X'(0) \cdot u_\varphi)^2] = \mathbb{E}[(X'(t) \cdot u_\varphi)^2],$$

for any $t \in \mathbb{R}^2$ by stationarity. It yields the following remark.

Remark 3.4 *For any $t \in \mathbb{R}^2$, the principal direction of X coincides with the favorite direction of $X'(t)$.*

3.3 Random planar waves

Let $\mathbf{k}$ be a random vector in $\mathbb{R}^2$ and let us consider the associated planar single random wave $X_\mathbf{k}$ and its Gaussian counterpart $G_\mathbf{k}$ as defined in Section 2. We now study these random fields from a directional point of view.

Proposition 3.5 *Let $\mathbf{k}$ be a random vector in $\mathbb{R}^2$ and let $Y : \mathbb{R}^2 \to \mathbb{R}$ be a stationary and centred random field with covariance function given by (3). We assume that $\mathbb{E}[\mathbf{k}\mathbf{k}^T] \notin \{\alpha I_2, \alpha \in \mathbb{R}^+\}$.*
Then, the next three remarkable directions in $\mathbb{R}/\pi\mathbb{Z}$ coincide

- *the favorite direction of $\mathbf{k}$*
- *the principal direction of Y*
- *the favorite direction of $Y'(t)$ for any $t \in \mathbb{R}^2$.*

They are given by the direction of the eigensubspace of $\mathbb{R}^2$ associated with the largest eigenvalue of matrix $\mathbb{E}[\mathbf{k}\mathbf{k}^T]$.

Proof. It is enough to prove that the principal direction of Y is the favorite direction of $\mathbf{k}$ and then to apply Remark 3.4.
Let φ be fixed, the covariance function of the univariate process along a line of direction φ is given for any $x \in \mathbb{R}$ by $r_\varphi(x) = r(xu_\varphi) = \mathbb{E}[\cos(x\,\mathbf{k}.u_\varphi)]$. Hence $m_2(\varphi) = -r''_\varphi(0) = \mathbb{E}[(\mathbf{k}.u_\varphi)^2]$, which clearly yields the equality between the principal direction of $X_\mathbf{k}$ and the favorite direction of $\mathbf{k}$. ∎

We now turn to the directional study of the level sets of the Gaussian planar random waves $G_\mathbf{k}$ $\mathbb{E}[\mathbf{k}$. We assume that $\mathbf{k}^T] \notin \{\alpha I_2, \alpha \in \mathbb{R}^+\}$ and we fix $a \in \mathbb{R}$. The level set

$$G_\mathbf{k}^{-1}(a) = \{t \in \mathbb{R}^2 :\ G_\mathbf{k}(t) = a\}$$

is a finite union of curves whose direction at point $t \in G_\mathbf{k}^{-1}(a)$ is orthogonal to the vector $G'_\mathbf{k}(t)$. Applying Proposition 3.5 yields the next statement, that sounds physically intuitive.

Proposition 3.6 *Let $a \in \mathbb{R}$. Let τ_a be a two-dimensional vector field defined on the level set $G_\mathbf{k}^{-1}(a)$ such that, at any point t, $\tau_a(t)$ is tangent to $G_\mathbf{k}^{-1}(a)$ at t. Then, for any $t \in G_\mathbf{k}^{-1}(a)$, the favorite direction of $\tau_a(t)$ is orthogonal to the favorite direction of* $\mathbf{k}$.

Let us mention that the above proposition still holds in dimension $d > 2$ once the favorite direction of a d-dimensional random vector is defined as the direction of the eigenspace associated with the largest eigenvalue of $\mathbb{V}[Z]$.

4 Berry's anisotropic random waves

In this section, we focus on Example 3 of Section 2, *i.e.* on the case where the random wavevector $\mathbf{k}$ is such that, for some $\kappa > 0$,

$$\kappa^{-1}\mathbf{k} \in \mathbb{S}^{d-1} \ a.s.$$

As previously, we consider the (unique in distribution) associated stationary centred Gaussian random field $G_\mathbf{k}$ on $\mathbb{R}^d$ whose covariance function r is given by (3). Since $||\mathbf{k}||$ is *a.s.* bounded, it is clear that $G_\mathbf{k}$ is *a.s.* smooth then, rephrasing Theorem 2.3, we get that $G_\mathbf{k}$ is the generic Gaussian solution of Helmholtz equation

$$\Delta Y + \kappa^2 Y = 0.$$

Equivalently, $G_\mathbf{k}$ is an eigenfunction of the operator $-\Delta$, for the eigenvalue κ^2. Therefore, extending the definition introduced by Berry in [8], we refer to $G_\mathbf{k}$ as a Berry's anisotropic wave with random wavenumber κ.

Applying the appropriate change of variables $t \mapsto \kappa t$ yields the scaling property that $(G_\mathbf{k}(t))_{t\in\mathbb{R}^d}$ and $(G_{\kappa^{-1}\mathbf{k}}(\kappa t))_{t\in\mathbb{R}^d}$ have the same distribution, where we recall that the random vector $\kappa^{-1}\mathbf{k}$ takes its values in $\mathbb{S}^{d-1}$. We also remark that, if the distribution of $\kappa^{-1}\mathbf{k}^s$ admits a density $\tilde{f}$ with respect to the surface measure σ on $\mathbb{S}^{d-1}$, we can deduce from (4) that the covariance function of $G_{\mathbf{k}}$ is given by

$$r(t) = \int_{\mathbb{S}^{d-1}} e^{i\kappa u\cdot t} \tilde{f}(u)\, d\sigma(u).$$

4.1 Expected measure of level sets

We are now interested in the random level sets: for any $a \in \mathbb{R}$,

$$G_{\mathbf{k}}^{-1}(a) = \{t \in \mathbb{R}^d \,/\, G_{\mathbf{k}}(t) = a\},$$

which has Hausdorff dimension $d-1$ *a.s.*. If $a = 0$, this is exactly the nodal set of $G_{\mathbf{k}}$ and more precisely in the case $d = 2$, it is the nodal line of a Berry's anisotropic planar wave.

Let Q be a compact set in $\mathbb{R}^d$ with non empty interior and let $a \in \mathbb{R}$. We focus on the $(d-1)$-dimensional Hausdorff measure of the a-level set of $G_{\mathbf{k}}$ restricted to Q, namely

$$\ell(a, \mathbf{k}, Q) = \mathcal{H}_{d-1}\left(G_{\mathbf{k}}^{-1}(a) \cap Q\right) = \mathcal{H}_{d-1}\left(\{t \in Q \,/\, G_{\mathbf{k}}(t) = a\}\right).$$

Proposition 4.1 *Let $\kappa > 0$ and assume that $\mathbf{k}$ is a random vector in $\mathbb{R}^d$ such that $\widetilde{\mathbf{k}} := \kappa^{-1}\mathbf{k} \in \mathbb{S}^{d-1}$ a.s.*
Let Φ_d stand for the standard Gaussian probability density function on $\mathbb{R}^d$. Then,

$$\mathbb{E}[\ell(a, \mathbf{k}, Q)] = \mathcal{H}_d(Q)\, \frac{e^{-a^2/2}}{\sqrt{2\pi}}\, \kappa \int_{\mathbb{R}^d} (\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T]x \cdot x)^{1/2} \Phi_d(x)\, dx. \tag{9}$$

Proof. Kac-Rice formula (see [6] Theorem 6.8 for instance) yields

$$\mathbb{E}[\ell(a, \mathbf{k}, Q)] = \int_Q \mathbb{E}[\| G'_{\mathbf{k}}(t) \| \mid G_{\mathbf{k}}(t) = a]\, p_{G_{\mathbf{k}}(t)}(a)\, dt,$$

where $p_{G_{\mathbf{k}}(t)}$, the probability density function of $G_{\mathbf{k}}(t)$, is actually given by the standard Gaussian distribution. Using the stationarity of $G_{\mathbf{k}}$ and the fact that for a fixed point t, $G_{\mathbf{k}}(t)$ and $G'_{\mathbf{k}}(t)$ are independent random variables, we have

$$\mathbb{E}[\ell(a, \mathbf{k}, Q)] = \mathcal{H}_d(Q)\, \frac{e^{-a^2/2}}{\sqrt{2\pi}}\, \mathbb{E}[\| G'_{\mathbf{k}}(0) \|].$$

In order to conclude, it only remains to state that $\| G'_{\mathbf{k}}(0) \|$ is the Euclidean norm of a d-dimensional centred Gaussian vector with variance matrix $-r''(0) = \mathbb{E}[\mathbf{k}\mathbf{k}^T] = \kappa^2\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T]$. ■

We remark that the same proof (except last equality) can be applied to any random wavevector $\mathbf{k}$ with finite moments, even if $||\mathbf{k}||$ is not constant. It yields the next identity that is valid when dropping the condition $\widetilde{\mathbf{k}} := \kappa^{-1}\mathbf{k} \in \mathbb{S}^{d-1}$ *a.s.* in Proposition 4.1,

$$\mathbb{E}[\ell(a,\mathbf{k},Q)] = \mathcal{H}_d(Q)\,\frac{e^{-a^2/2}}{\sqrt{2\pi}}\int_{\mathbb{R}^d} (\mathbb{E}[\mathbf{k}\mathbf{k}^T]x\cdot x)^{1/2}\Phi_d(x)\,dx.$$

Let us come back to Berry's random waves. In the isotropic case, that is to say when $\widetilde{\mathbf{k}}$ is uniformly distributed on $\mathbb{S}^{d-1}$, $\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T] = \mathbb{V}[\widetilde{\mathbf{k}}] = (1/d)\,I_d$. Hence,

$$\mathbb{E}[\ell(a,\mathbf{k},Q)] = \mathcal{H}_d(Q)\,\frac{e^{-a^2/2}}{\sqrt{2\pi d}}\,\kappa\int_{\mathbb{R}^d} \parallel x \parallel\ \Phi_d(x)\,dx,$$

where the above integral is the mean of a χ-distributed random variable with d degrees of freedom and is known to be equal to $\sqrt{2}\,\frac{\Gamma((d+1)/2)}{\Gamma(d/2)}$.

4.2 Expected length of level curves

In the planar case, *i.e.* $d = 2$, the level sets $G_{\mathbf{k}}^{-1}(a)$ are one-dimensional and Formula (9) can be made much more precise. In particular, the following proposition states that the level curves mean length is decreasing as anisotropy is increasing.

Proposition 4.2 *Let* $\mathbf{k}$ *be a random vector in* $\mathbb{R}^2$ *such that* $\mathbf{k} = \kappa\,\widetilde{\mathbf{k}}$ *with* κ *a positive constant and* $\widetilde{\mathbf{k}} \in \mathbb{S}^1$ *a.s. Let us denote by* $c(\widetilde{\mathbf{k}})$ *the difference between the eigenvalues of* $\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T]$ *(*$0 \le c(\widetilde{\mathbf{k}}) \le 1$*). Let* $\mathcal{E}$ *be the elliptic integral given by* $\mathcal{E}(x) = \int_0^{\pi/2}(1-x^2\,\sin^2\theta)^{1/2}d\theta$, *for* $x \in [0,1]$. *Then,*

$$\mathbb{E}[\ell(a,\mathbf{k},Q)] = \mathcal{H}_2(Q)\,\frac{e^{-a^2/2}}{\pi\sqrt{2}}\,\kappa\,\mathcal{F}\big(c(\widetilde{\mathbf{k}})\big),$$

where the map $\mathcal{F} : c \in [0,1] \mapsto (1+c)^{1/2}\,\mathcal{E}\left(\left(\frac{2c}{1+c}\right)^{1/2}\right)$ *is strictly decreasing.*

Remark 4.3 *In the isotropic case,* $c(\widetilde{\mathbf{k}}) = 0$ *and hence we recover the following result concerning the nodal line of the isotropic Berry's planar wave (see [8]):* $\mathbb{E}[\ell(0,\mathbf{k},Q)] = \mathcal{H}_2(Q)\,\frac{\kappa}{\pi\sqrt{2}}\mathcal{E}(0) = \mathcal{H}_2(Q)\,\frac{\kappa}{2\sqrt{2}}$.

Remark 4.4 *In directional statistics, it is usual to introduce a parameter termed coherency index and defined as the ratio between the difference of eigenvalues and the sum of eigenvalues of a certain positive symmetric matrix* M*, see [18]. This index is performed in [23] (see also [12]) with* M *given by the so-named structure tensor in order to quantify the anisotropy of an anisotropic Gaussian planar field. In our context, we like to remark that the trace of matrix* $M = \mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T]$ *is equal to one, since* $||\widetilde{\mathbf{k}}|| = 1$, *a.s.. Parameter* $c(\widetilde{\mathbf{k}})$ *actually coincides with the coherency index of our model and hence quantifies its anisotropy.*

Proof. We use Proposition 4.1 in the case $d = 2$. For computing the integral in the right-hand side of (9), we use the next well known fact, that can be proven with simple algebra.
If M is a symmetric definite positive matrix with eigenvalues γ_- and γ_+ ($0 \leq \gamma_- \leq \gamma_+$ and $\gamma_+ > 0$), then

$$\int_{\mathbb{R}^2} (Mx \cdot x)^{1/2} \Phi_2(x)\, dx = \left(\frac{2\gamma_+}{\pi}\right)^{1/2} \mathcal{E}\left((1 - \gamma_-/\gamma_+)^{1/2}\right). \tag{10}$$

In our case, $M = \mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T]$ and $\gamma_- + \gamma_+ = 1$. Hence, $2\gamma_+ = 1 + c$ and $1 - \gamma_-/\gamma_+ = \frac{2c}{1+c}$.
The proof of the decreasing of mapping $\mathcal{F}$ is postponed to the Appendix section, see Lemma A.2. ■

We end the section applying Proposition 4.2 to our two favorite examples.

Example 1 (Toy model) Take $\widetilde{\mathbf{k}}$ distributed on $\mathbb{S}^1$ with probability density function given by (2) for some positive α (see Example 1 Section 2). The moments of $\widetilde{\mathbf{k}}$ are computed in the Appendix section, Lemma A.1. In particular, it holds $\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T] = \frac{1}{\alpha+2}\begin{pmatrix} \alpha+1 & 0 \\ 0 & 1 \end{pmatrix}$. Consequently, $c(\widetilde{\mathbf{k}}) = \frac{\alpha}{\alpha+2}$, which is an increasing function of parameter α. Thus, the more anisotropic the model is, the smaller the expected length of level sets is.

Example 2 (Elementary model) We choose the random wavevector $\mathbf{k} = \kappa\, \widetilde{\mathbf{k}}$ with $\widetilde{\mathbf{k}}$ uniformly distributed on $[\alpha_0 - \delta, \alpha_0 + \delta] \cup [\alpha_0 + \pi - \delta, \alpha_0 + \pi + \delta]$ for some $0 < \delta \leq \pi/2$, see Example 2 of Section 2. In order to simplify the computation, let us assume that $\alpha_0 = 0$. In that case, $\mathbb{E}[\widetilde{\mathbf{k}}\widetilde{\mathbf{k}}^T] = \frac{1}{2}\begin{pmatrix} 1 + \frac{\sin(2\delta)}{2\delta} & 0 \\ 0 & 1 - \frac{\sin(2\delta)}{2\delta} \end{pmatrix}$ and hence $c(\widetilde{\mathbf{k}}) = \frac{\sin(2\delta)}{2\delta}$, which is decreasing on $[0, \pi/2]$. Again, the mean length of level sets is decreasing with anisotropy, *i.e.* as δ is growing.

5 Gaussian sea waves

In this section, we now concentrate on Example 4 of Section 2 that considers the case where the random wavevector is 3-dimensional and *a.s.* belongs to Airy surface, *i.e.*

$$\mathbf{k} \in \Lambda = \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3\, ;\, (\lambda_1)^2 + (\lambda_2)^2 = (\lambda_3)^4\} \ a.s.$$

Then, following Proposition (2.1), the associated stationary Gaussian field $G_{\mathbf{k}}$ admits as covariance function

$$r(t) = \int_\Lambda \cos(t \cdot \lambda)\, dF(\lambda)\ ,\ t \in \mathbb{R}^3,$$

where F is the probability distribution of $\mathbf{k}$.
The field $G_{\mathbf{k}}$ coincides with the spatio-temporal Gaussian random fields that are used for the modelization of sea waves [17, 4, 6]. Indeed, for $(x, y, s) \in \mathbb{R}^2 \times \mathbb{R}$, $G_{\mathbf{k}}(x, y, s)$ can be seen as the algebraic height of a wave at point (x, y) and time s.

We use the following parametrization of Λ,

$$(\theta, z) \in [0, 2\pi) \times \mathbb{R} \mapsto (z^2 \cos\theta, z^2 \sin\theta, z),$$

which provides a bijection ϕ from $[0, 2\pi) \times \mathbb{R} \backslash \{0\}$ onto $\Lambda \backslash \{(0, 0, 0)\}$. Performing the appropriate change of variables yields

$$r(x, y, s) = \int_{(0,2\pi)\times\mathbb{R}} \cos(xz^2 \cos\theta + yz^2 \sin\theta + sz)\, d\widetilde{F}(\theta, z),$$

where $\widetilde{F}$ is the image of measure F by the map ϕ^{-1}. When $\mathbf{k}$ admits f as probability density function with respect to the surface measure on Λ, we get

$$r(x, y, s) = \int_{(0,2\pi)\times\mathbb{R}} \cos(xz^2 \cos\theta + yz^2 \sin\theta + sz) \widetilde{f}(\theta, z)\, d\theta dz,$$

where the map $\widetilde{f}$ is given by

$$\widetilde{f}(\theta, z) = f(z^2 \cos\theta, z^2 \sin\theta, z)\, z^2 (1 + 4z^2)^{1/2}.$$

Following the literature, $\widetilde{f}$ is called *directional power spectrum* of $G_{\mathbf{k}}$ (see [4] and [6] Chapter 11). Experimental directional power spectra are exhibited in [4], derived from sea data provided by Ifremer.

In order to avoid heavy notations, from now on we assume that the random wavevector $\mathbf{k}$ is symmetrically distributed. Hence, until the end of the present section we deal with the following covariance function

$$r(x, y, s) = \int_{\Lambda} e^{i((x,y,s)\cdot\lambda)}\, dF(\lambda),$$

where F is a probability measure on Λ satisfying $F(-A) = F(A)$ for any Borelian set $A \subset \Lambda$. In other words,

$$r(x, y, s) = \int_{(0,2\pi)\times\mathbb{R}} e^{i(xz^2 \cos\theta + yz^2 \sin\theta + sz)}\, d\widetilde{F}(\theta, z)$$

with $\widetilde{F}$ a probability measure on $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R} \setminus \{0\}$ that is invariant under the mapping $(\theta, z) \mapsto (\theta + \pi, -z)$. If $\mathbf{k}$ is not symmetrically distributed, the key to get the above expressions is to use the symmetrized probability measure of $\mathbf{k}$ instead of its probability measure.

Let us fix time $s = s_0$ and look at the random field defined on $\mathbb{R}^2$,

$$Z_{\mathbf{k}}(x,y) = G_{\mathbf{k}}(x,y,s_0) \qquad (x,y) \in \mathbb{R}^2,$$

as a picture of the sea height at time s_0. It is a two-dimensional stationary centred Gaussian random field, whose covariance function is given by

$$\Gamma(x,y) = r(x,y,0) = \int_{(0,2\pi)\times\mathbb{R}} e^{i(xz^2\cos\theta + yz^2\sin\theta)}\, d\widetilde{F}(\theta,z).$$

Actually, the random wavevector associated with $Z_{\mathbf{k}}$ is nothing but the projection of the Λ-valued random wavevector $\mathbf{k}$ onto the first two coordinates. We call it $\pi(\mathbf{k})$ in what follows. We will also need the spectral moments of $Z_{\mathbf{k}}$, namely for any integers j and k in $\mathbb{N}_0$

$$\begin{aligned} m_{j,k} &:= (-i)^{j+k}\, \partial^{(j,k)}\Gamma(0,0) \\ &= \int_{(0,2\pi)\times\mathbb{R}} (z^2\cos\theta)^j (z^2\sin\theta)^k\, d\widetilde{F}(\theta,z). \end{aligned} \tag{11}$$

5.1 Mean length of static crests

We are now interested in the (static) crest in direction $\varphi \in \mathbb{R}/\pi\mathbb{Z}$. More precisely, we introduce the random set

$$\{(x,y) \in \mathbb{R}^2 \,;\, Z'_{\mathbf{k}}(x,y)\cdot u_\varphi = 0\},$$

which contains all points (x,y) in $\mathbb{R}^2$ such that the gradient of $Z_{\mathbf{k}}$ at point (x,y) is orthogonal to direction φ. One can also say that the derivative of $Z_{\mathbf{k}}$ in direction φ at those points is zero. Hence, the crest in direction φ is a special case of a *specular points* set as defined in [17]. Its Hausdorff dimension is clearly equal to one and one can compute its length within a compact domain $Q \subset \mathbb{R}^2$ such that $\mathcal{H}_1(Q) > 0$,

$$l(\mathbf{k},Q,\varphi) := \mathcal{H}_1\left(\{(x,y)\in Q\,;\, Z'_{\mathbf{k}}(x,y)\cdot u_\varphi = 0\}\right).$$

Using the same arguments as for the proof of Proposition 4.1 and Formula (10), we get the following result that is also stated in [6] (Proposition 11.4) or in [4] (Assertion 3).

$$\mathbb{E}[l(\mathbf{k},Q,\varphi)] = \mathcal{H}_2(Q)\,\frac{1}{\pi}\left(\frac{\gamma_+(\varphi)}{v(\varphi)}\right)^{1/2} \mathcal{E}\big((1-\gamma_-(\varphi)/\gamma_+(\varphi))^{1/2}\big), \tag{12}$$

where $v(\varphi) = \mathrm{Var}(Z'_{\mathbf{k}}(0)\cdot u_\varphi)$ and $\gamma_-(\varphi) \le \gamma_+(\varphi)$ are the eigenvalues of the variance matrix $\mathbb{V}[Z''_{\mathbf{k}}(0)\,u_\varphi]$. The expressions for these quantities in terms of the spectral moments $m_{j,k}$ of $Z_{\mathbf{k}}$ are recalled in the Appendix section, see Lemma A.3.

The end of this section is dedicated at showing that the direction that maximises the expected crest length may be orthogonal to the most probable direction of the wavevector $\pi(\mathbf{k})$, as it may not (in the case where such directions exist). It is a clear consequence of (12), which depends on both the second order and the fourth order moments of $\mathbf{k}$, while the most probable direction depends on the mode of $\mathbf{k}$. Nevertheless, a rule of thumb is suggested in [17] or in [4] for instance, claiming that *the direction [that maximises the expected length of crests] is orthogonal to the direction for the maximum integral of the spectrum, i.e. is the most probable direction for the waves.* In this statement, the "most probable direction for the waves" has to be understood as the most probable direction of the random wavevector $\pi(\mathbf{k})$, as defined in Section 3.1.

Example 5.1 (Elementary wave) We take a random wavevector $\mathbf{k}$ with values in Λ *a.s.* and with a deterministic orientation. Precisely, the distribution of $\mathbf{k}$ is prescribed on $[0,2\pi)\times\mathbb{R}$ by $\widetilde{F}=\frac{1}{2}(\delta_{\alpha_0}+\delta_{\alpha_0+\pi})\otimes h$, where δ_{α_0} stands for the Dirac measure at $\alpha_0\in[0,\pi)$ and with h any symmetric probability measure on $\mathbb{R}\setminus\{0\}$.
On the one hand, the most probable direction of $\pi(\mathbf{k})$ is clearly α_0 modulo π.
On the other hand, the spectral moments of $Z_{\mathbf{k}}$ are easy to compute from (11). In the simplest case where $\alpha_0=0$, we get that $m_{2,0}=M_2$ and $m_{4,0}=M_4$ with $M_k:=\int_R z^{2k}\,dh(z)$, and that all the other moments up to order 4 are vanishing. Hence, following Lemma A.3, $v(\varphi)=M_2\cos^2\varphi$ and

$$\mathbb{V}[Z''_{\mathbf{k}}(0)\,u_\varphi]=M_4\cos^2\varphi\begin{pmatrix}\cos^2\varphi & \cos\varphi\sin\varphi\\ \cos\varphi\sin\varphi & \sin^2\varphi\end{pmatrix}.$$

Hence $\gamma_-(\varphi)=0$ and $\gamma_+(\varphi)=M_4\cos^2\varphi$, and Formula (12) allows us to state that the expected length of the crests of $Z_{\mathbf{k}}$ in direction φ does not depend on φ. Therefore, for this model, no link can be established between the direction that maximises the expected length of crests and the most probable direction of $\pi(\mathbf{k})$.

Example 5.2 (Counter-example) We now take a random vector $\mathbf{k}$ whose distribution is prescribed on $[0,2\pi)\times\mathbb{R}$ by $\widetilde{F}=\left(\frac{1}{4}\sum_{j=0}^{3}d\delta_{j\pi/2}\right)\otimes h$, with h any symmetric probability measure on $\mathbb{R}\setminus\{0\}$. Then, the set of most probable directions of $\mathbf{k}$ is $\{j\pi/2\,:\,j=0,1,2,3\}$.
Computing the spectral moments of $Z_{\mathbf{k}}$ from (11), we get $m_{1,1}=m_{2,2}=m_{3,1}=0$ whereas $m_{2,0}=m_{0,2}=\frac{M_2}{2}$ and $m_{4,0}=m_{0,4}=\frac{M_4}{2}$, where we have introduced $M_k:=\int_{\mathbb{R}}z^{2k}\,dh(z)$. Consequently, using Lemma A.3, for any $\varphi\in[0,2\pi]$, $v(\varphi)=\mathrm{Var}(Z'_{\mathbf{k}}(0)\cdot u_\varphi)=\frac{M_2}{2}$ and

$$\mathbb{V}[Z''_{\mathbf{k}}(0)\,u_\varphi]=\frac{M_4}{2}\begin{pmatrix}\cos^4\varphi+\sin^4\varphi & \cos\varphi\sin\varphi(\sin^2\varphi-\cos^2\varphi)\\ \cos\varphi\sin\varphi(\sin^2\varphi-\cos^2\varphi) & 2\cos^2\varphi\sin^2\varphi\end{pmatrix}.$$

The eigenvalues of the later matrix being $\frac{M_4}{4}(1\pm|\cos(2\varphi)|)$, we apply (12) to get the expected length of the crests of $Z_{\mathbf{k}}$ in direction φ. Up to a positive multiplicative constant that does not depend on φ, $\mathbb{E}[l(\mathbf{k},Q,\varphi)]$ is equal to

$$\big(1+|\cos(2\varphi)|\big)^{1/2}\,\epsilon\big(\big(\frac{2\,|\cos(2\varphi)|}{1+|\cos(2\varphi)|}\big)^{1/2}\big)=\mathcal{F}(|\cos(2\varphi)|),$$

where the function $\mathcal{F}$ is defined in Proposition 4.2. Since $\mathcal{F}$ is strictly decreasing on $[0,1]$ (see Lemma A.2), the mean length of crests is maximal when $\cos(2\varphi)=0$, *i.e.* for $\varphi=\pi/4$ or $3\pi/4$ modulo π. These directions are not orthogonal to the most probable directions of $\pi(\mathbf{k})$.
We also like to mention that any other distribution of the wavevector could serve as a counter-example, as soon as it is invariant by the transformation $\theta\mapsto\pi/2-\theta$. Indeed, this invariance yields the same values of the spectral moments.

5.2 Mean length of static crests with the toy model

We choose as a particular Gaussian wave the one whose directional power spectrum $\widetilde{f}$ is given by

$$\widetilde{f}(\theta,z)=C_\alpha\,|\cos\theta|^\alpha\,h(z), \tag{13}$$

where α is a positive real number (see Equation (2)) and h is an even probability density function on $\mathbb{R}$. As already mentioned, the most probable direction of $\pi(\mathbf{k})$ is 0 in that case.

The spectral moments (see (11)) of this particular Gaussian wave are given, for any j,k in $\mathbb{N}_0$, by

$$\begin{aligned} m_{j,k} &= \Big(\int_{\mathbb{R}} z^{2j+2k}\,h(z)\,dz\Big)\left(C_\alpha\int_{(0,2\pi)}(\cos\theta)^j\,(\sin\theta)^k\,|\cos\theta|^\alpha\,d\theta\right)\\ &:= M_{j+k}\,\mu_{j,k}. \end{aligned}$$

The first integral M_{j+k} equals the moment of order $2j+2k$ of h. Note that it does not contain any information on the anisotropy of the model. The second integral, named as $\mu_{j,k}$, is computed in Lemma A.1 in Appendix section.

Hence, the expected length of crests in a given direction φ can be evaluated through Formula (12) applied to this specific model. An asymptotic expansion of $\varphi\mapsto\mathbb{E}[\ell(\mathbf{k},Q,\varphi)]$ near $\varphi=\pi/2$ is performed in Lemma A.4. It shows that the expected length of crests admits a local maximum at $\varphi=\pi/2$, which is precisely orthogonal to the most probable direction of $\pi(\mathbf{k})$ and to its favorite direction as well (see the fifth example in Section 3.1).

A Appendix

A.1 Moments of a random wavevector given by the toy model

We perform some computations related to our toy model given by Example 1 in Section 2. We fix $\alpha \geq 0$ and we consider a two-dimensional random wavevector $\mathbf{k} = (\cos\Theta, \sin\Theta)$, with Θ that takes value in $[0, 2\pi]$ with a probability density function given by

$$\theta \mapsto C_\alpha |\cos\theta|^\alpha \quad \textit{with} \quad C_\alpha = \frac{\Gamma(1+\alpha/2)}{2\sqrt{\pi}\Gamma(1/2+\alpha/2)}.$$

Lemma A.1 *For any non negative integers j and k, let $\mu_{j,k}$ be the (j,k)-moment of $\mathbf{k}$, i.e.*

$$\mu_{j,k} = \mathbb{E}[(\cos\Theta)^j\,(\sin\Theta)^k] = C_\alpha \int_{[0,2\pi]} (\cos\theta)^j (\sin\theta)^k\,|\cos\theta|^\alpha\,d\theta.$$

Then

- $\mu_{0,0} = 1$
- $\mu_{j,k} = 0$ *whenever j or k is odd*
- $\mu_{j,0} = \frac{C_\alpha}{C_{\alpha+j}} = \frac{(\alpha+1)(\alpha+3)\cdots(\alpha+j-1)}{(\alpha+2)(\alpha+4)\cdots(\alpha+j)}$ *for j even ≥ 2*
- *for any even integers j and k, $\mu_{j,k} = \sum_{i=0}^{k/2}(-1)^i\binom{k/2}{i}\mu_{j+2i,0}$.*

In particular, it yields the non-zero second and fourth order moments of $\mathbf{k}$:

$$\mu_{2,0} = \frac{\alpha+1}{\alpha+2};\ \mu_{0,2} = \frac{1}{\alpha+2} \quad \text{and hence} \quad \mathbb{E}[\mathbf{k}\mathbf{k}^T] = \frac{1}{\alpha+2}\begin{pmatrix}\alpha+1 & 0\\ 0 & 1\end{pmatrix};$$
$$\mu_{4,0} = \frac{(\alpha+1)(\alpha+3)}{(\alpha+2)(\alpha+4)};\ \mu_{0,4} = \frac{3}{(\alpha+2)(\alpha+4)};\ \mu_{2,2} = \frac{\alpha+1}{(\alpha+2)(\alpha+4)}.$$

Proof. It is clear that $\mu_{0,0} = 1$, $\mu_{j,k} = 0$ whenever j or k is odd and that $\mu_{j,0} = C_\alpha/C_{\alpha+j}$ for any even integer j. Using the explicit value of C_α yields the value of $\mu_{j,0}$. Finally, for any even integers j and k, writing $\sin^2\theta = 1 - \cos^2\theta$ yields the formula for $\mu_{j,k}$. ■

A.2 Variations of map $\mathcal{F}$

Lemma A.2 *The map $\mathcal{F} : c \mapsto (1+c)^{1/2}\,\mathcal{E}\left(\left(\frac{2c}{1+c}\right)^{1/2}\right)$ is strictly decreasing on $[0,1]$.*

Proof. Recall that $\mathcal{E}(k) = \int_0^{\pi/2} (1-k^2\sin^2\theta)^{1/2}\,d\theta$ for $k\in[0,1]$. Then, for any $k\in[0,1)$, $\mathcal{E}'(k) = -k\int_0^{\pi/2} \frac{\sin^2\theta}{(1-k^2\sin^2\theta)^{1/2}}\,d\theta$. Therefore, for any $c\in[0,1)$,

$$\begin{aligned}
\mathcal{F}'(c) &= \frac{1}{2}(1+c)^{-1/2}\mathcal{E}\left((\frac{2c}{1+c})^{1/2}\right) + (1+c)^{1/2}\frac{(2c)^{-1/2}}{(1+c)^{3/2}}\,\mathcal{E}'\left((\frac{2c}{1+c})^{1/2}\right)\\
&= \frac{1}{2}(1+c)^{-1/2}\int_0^{\pi/2}\left[(1-\frac{2c}{1+c}\sin^2\theta)^{1/2} - \frac{\frac{2}{1+c}\sin^2\theta}{(1-\frac{2c}{1+c}\sin^2\theta)^{1/2}}\right]d\theta\\
&= \frac{1}{2}(1+c)^{-1/2}\int_0^{\pi/2}\frac{\cos(2\theta)}{(1-\frac{2c}{1+c}\sin^2\theta)^{1/2}}\,d\theta.
\end{aligned}$$

It remains to show that the above integral, which we call $J(k)$ with $k=(\frac{2c}{1+c})^{1/2}$, is negative. Splitting the integral $J(k) := \int_0^{\pi/2}\frac{\cos(2\theta)}{(1-k^2\sin^2\theta)^{1/2}}\,d\theta$ into two parts, on $[0,\pi/4]$ and on $[\pi/4,\pi/2]$, and performing the change of variables $\theta'=\pi/2-\theta$ within the second part, we get

$$J(k) = \int_0^{\pi/4}\cos(2\theta)\left[\frac{1}{(1-k^2\sin^2\theta)^{1/2}} - \frac{1}{(1-k^2\cos^2\theta)^{1/2}}\right]d\theta, \tag{14}$$

which is negative since $\cos\theta > \sin\theta$ for $\theta\in(0,\pi/4)$. ■

A.3 Second moments of $Z'(0)\cdot u_\varphi$ and of $Z''(0)u_\varphi$

Let Z be a two-dimensional stationary Gaussian field that is centred and that admits a spectral density f on $\mathbb{R}^2$. We assume that Z admits spectral moments of all orders and we denote them by $(m_{j,k})_{(j,k)\in\mathbb{N}_0^2}$, *i.e.*

$$m_{j,k} = \int_{\mathbb{R}^2}(\lambda_1)^j\,(\lambda_2)^k\,f(\lambda)\,d\lambda.$$

The following statements are borrowed from [4] page 412. Recall that for any $\varphi\in[0,2\pi]$, $u_\varphi = (\cos\varphi,\sin\varphi)$.

Lemma A.3 *For any* $\varphi\in[0,2\pi]$*, we have*

$$v(\varphi) = \mathrm{Var}\big(Z'(0)\cdot u_\varphi\big) = m_{2,0}\cos^2\varphi + 2m_{1,1}\cos\varphi\sin\varphi + m_{0,2}\sin^2\varphi$$

and

$$\mathbb{V}[Z''(0)u_\varphi] = \begin{pmatrix} a_{22}(\varphi) & a_{23}(\varphi)\\ a_{23}(\varphi) & a_{33}(\varphi)\end{pmatrix}$$

where

$$
\begin{aligned}
a_{22}(\varphi) &= m_{4,0}\cos^4\varphi + m_{0,4}\sin^4\varphi + 6m_{2,2}\cos^2\varphi\sin^2\varphi \\
&\quad +4m_{3,1}\cos^3\varphi\sin\varphi + 4m_{1,3}\cos\varphi\sin^3\varphi, \\
a_{33}(\varphi) &= (m_{4,0}+m_{0,4})\cos^2\varphi\sin^2\varphi + m_{2,2}\left((\cos^2\varphi-\sin^2\varphi)^2 - 2\cos^2\varphi\sin^2\varphi\right) \\
&\quad +2(m_{1,3}+m_{3,1})\cos\varphi\sin\varphi(\cos^2\varphi-\sin^2\varphi), \\
a_{23}(\varphi) &= -m_{4,0}\cos^3\varphi\sin\varphi + m_{3,1}\cos^2\varphi(\cos^2\varphi - 3\sin^2\varphi) + 3m_{2,2}\cos\varphi\sin\varphi(\cos^2\varphi-\sin^2\varphi) \\
&\quad +m_{1,3}\sin^2\varphi(3\cos^2\varphi-\sin^2\varphi) + m_{0,4}\cos\varphi\sin^3\varphi.
\end{aligned}
$$

Moreover the eigenvalues $\gamma_+(\varphi)$ and $\gamma_-(\varphi)$ of matrix $\mathbb{V}[Z''(0)u_\varphi]$ are equal to

$$\gamma_\pm(\varphi) = \frac{1}{2}\big(T(\varphi) \pm \sqrt{\Delta(\varphi)}\big),$$

where $T(\varphi) = Trace(\mathbb{V}[Z''(0)u_\varphi]) = a_{22}(\varphi) + a_{33}(\varphi)$ and $\Delta(\varphi) = (a_{22}(\varphi) - a_{33}(\varphi))^2 + 4a_{23}(\varphi)^2$.

A.4 Length of crests with the toy model

Considering Formula (12) prescribing the expected length of crests in a given direction and a given domain, we focus on the case where the direction of $\mathbf{k}$ is given by the toy model (see Example 1 in Section 2). As φ tends to 0, we write $g(\varphi) = \mathcal{O}(h(\varphi))$ if there exists $\varphi_0 \in (0, 2\pi)$ and $M > 0$ such that

$$\forall\varphi \in [0, 2\pi],\ |\varphi| < |\varphi_0| \ \Rightarrow\ |g(\varphi)| \leq M|h(\varphi)|.$$

Lemma A.4 *Let Q be a compact set in $\mathbb{R}^2$ and let $\mathbf{k}$ be a random wavevector in $\mathbb{R}^2$ prescribed by its directional spectral density $\tilde{f}(\theta, z)$ given by (13) for a fixed $\alpha > 0$.*
Let $f(\varphi) = \mathbb{E}[\ell(\mathbf{k}, Q, \pi/2 + \varphi)]$ where $\mathbb{E}[\ell(\mathbf{k}, Q, \varphi)]$ is given by (12). Then, as φ tends to 0,

$$f(\varphi) = f(0) - K\varphi^2 + \mathcal{O}(\varphi^4), \qquad \textit{with}\ \ K > 0.$$

Proof. From (12), we get

$$f(\varphi) = \left(\frac{\gamma_+(\pi/2+\varphi)}{v(\pi/2+\varphi)}\right)^{1/2} \mathcal{E}\big((1 - \frac{\gamma_-(\pi/2+\varphi)}{\gamma_+(\pi/2+\varphi)})^{1/2}\big),$$

where $\gamma_-(\varphi)$, $\gamma_+(\varphi)$ and $v(\varphi)$ are given in Lemma A.3. Moreover, the spectral moments $m_{j,k}$ are prescribed by (11) with $\mu_{j,k}$ given by Lemma A.1.

Since $\cos(\pi/2 + \varphi) = -\varphi + \frac{\varphi^3}{6} + \mathcal{O}(\varphi^4)$ and $\sin(\pi/2 + \varphi) = 1 - \frac{\varphi^2}{2} + \mathcal{O}(\varphi^4)$, we get

$$\begin{aligned}
v(\pi/2+\varphi) &= \mu_{20}\sin^2(\varphi)+\mu_{02}\cos^2(\varphi) = \mu_{02}+(\mu_{20}-\mu_{02})\varphi^2+\mathcal{O}(\varphi^4)\\
&= \frac{1}{\alpha+2}(1+\alpha\varphi^2)+\mathcal{O}(\varphi^4)\\
a_{22}(\pi/2+\varphi) &= \mu_{40}\sin^4(\varphi)+\mu_{04}\cos^4(\varphi)+6m_{22}\sin^2(\varphi)\cos^2(\varphi)\\
&= \mu_{04}+2(3\mu_{22}-\mu_{04})\varphi^2+\mathcal{O}(\varphi^4)\\
&= \frac{3}{(\alpha+2)(\alpha+4)}(1+2\alpha\varphi^2)+\mathcal{O}(\varphi^4)\\
a_{33}(\pi/2+\varphi) &= (\mu_{40}+\mu_{04})\sin^2(\varphi)\cos^2(\varphi)+\mu_{22}((\sin^2(\varphi)-\cos^2(\varphi))^2-2\sin^2(\varphi)\cos^2(\varphi))\\
&= \mu_{22}+(\mu_{40}+\mu_{04}-6\mu_{22})\varphi^2+\mathcal{O}(\varphi^4)\\
&= \frac{1}{(\alpha+2)(\alpha+4)}(\alpha+1+\alpha(\alpha-2)\varphi^2)+\mathcal{O}(\varphi^4)\\
a_{23}(\pi/2+\varphi) &= \mu_{40}\sin^3(\varphi)\cos(\varphi)-3\mu_{22}\sin(\varphi)\cos(\varphi)(\sin^2(\varphi)-\cos^2(\varphi))-\mu_{04}\sin(\varphi)\cos^3(\varphi)\\
&= (3\mu_{22}-\mu_{04})\varphi+(\mu_{40}+\frac{5}{3}\mu_{04}-8\mu_{22})\varphi^3+\mathcal{O}(\varphi^4)\\
&= \frac{1}{(\alpha+2)(\alpha+4)}[3\alpha\varphi+\alpha(\alpha-4)\varphi^3]+\mathcal{O}(\varphi^4).
\end{aligned}$$

The eigenvalues $\gamma_\pm(\pi/2+\varphi)$ are given by

$$\gamma_\pm(\pi/2+\varphi) = \frac{1}{2}(a_{22}(\pi/2+\varphi)+a_{33}(\pi/2+\varphi)) \pm \sqrt{\Delta(\pi/2+\varphi)},$$

with discriminant

$$\begin{aligned}
\Delta(\pi/2+\varphi) &= a_{22}(\pi/2+\varphi)-a_{33}(\pi/2+\varphi))^2+4\,a^2_{23(\pi/2+\varphi)}\\
&= \frac{1}{(\alpha+2)^2(\alpha+4)^2}[(2-\alpha)^2+2\alpha(\alpha+4)^2\varphi^2]+\mathcal{O}(\varphi^4).
\end{aligned}$$

<u>For $\alpha \neq 2$</u>, it yields the following expansions

$$\begin{aligned}
\sqrt{\Delta(\pi/2+\varphi)} &= \frac{|2-\alpha|}{(\alpha+2)(\alpha+4)}(1+\frac{\alpha(\alpha+4)^2}{(2-\alpha)^2}\varphi^2)+\mathcal{O}(\varphi^4)\\
\gamma_-(\pi/2+\varphi) &= \frac{\min(3,\alpha+1)}{(\alpha+2)(\alpha+4)}\left(1-\frac{\alpha(\alpha+4)}{|\alpha-2|}\varphi^2\right)+\mathcal{O}(\varphi^4),\\
\gamma_+(\pi/2+\varphi) &= \frac{\max(3,\alpha+1)}{(\alpha+2)(\alpha+4)}\left(1+\frac{\alpha(\alpha+4)}{|\alpha-2|}\varphi^2\right)+\mathcal{O}(\varphi^4).
\end{aligned}$$

The quantity $k(\pi/2+\varphi) := \left(1-\frac{\gamma_-(\pi/2+\varphi)}{\gamma_+(\pi/2+\varphi)}\right)^{1/2} \in [0,1]$ admits the following expansion

$$k(\pi/2+\varphi) = k_0\left(1+\frac{m\alpha(\alpha+4)}{(\alpha-2)^2}\right)\varphi^2+\mathcal{O}(\varphi^4),$$

where $m = \min(3, \alpha+1)$, $M = \max(3, \alpha+1)$ and $k_0 = \left(1 - \frac{m}{M}\right)^{1/2} = (\frac{|\alpha-2|}{M})^{1/2}$. Introducing the derivating of the map $\mathcal{E} : k \mapsto \int_0^{\pi/2} (1 - k^2 \sin^2\theta)^{1/2}\, d\theta$, we get

$$\mathcal{E}(k(\pi/2+\varphi)) = \mathcal{E}(k_0) + \frac{m\alpha(\alpha+4)}{(\alpha-2)^2} k_0\, \mathcal{E}'(k_0)\, \varphi^2 + \mathcal{O}(\varphi^4),$$

where $\mathcal{E}'(k) = -k \int_0^{\pi/2} \frac{\sin^2\theta}{(1-k^2\sin^2\theta)^{1/2}}\, d\theta$ and $\mathcal{E}'(0) = 0$.

It remains to expand $\left(\frac{\gamma_+(\pi/2+\varphi)}{v(\pi/2+\varphi)}\right)^{1/2}$:

$$\left(\frac{\gamma_+(\pi/2+\varphi)}{v(\pi/2+\varphi)}\right)^{1/2} = \left(\frac{M}{\alpha+4}\right)^{1/2} \left[1 + \frac{m\alpha}{|\alpha-2|}\varphi^2\right] + \mathcal{O}(\varphi^4).$$

Finally, as claimed, we get the asymptotic expansion of function f as φ tends to 0:

$$f(\varphi) = f(0) - K\varphi^2 + \mathcal{O}(\varphi^4),$$

with $f(0) = \mathcal{H}_2(Q)\, \frac{1}{\pi} \left(\frac{M}{\alpha+4}\right)^{1/2} \mathcal{E}((1 - m/M)^{1/2})$ and

$$K = -\,\mathcal{H}_2(Q)\, \frac{1}{\pi} \left(\frac{M}{\alpha+4}\right)^{1/2} \frac{m\alpha}{|\alpha-2|}\, J(k_0),$$

where $J(k)$ is introduced within the proof of Lemma A.2. As $J(k)$ is proven to be negative for any k, see (14), we obtain that $K > 0$ and the lemma is established in the case $\alpha \neq 2$.

For $\alpha = 2$, we get $\sqrt{\Delta(\pi/2+\varphi)} = \frac{1}{2}\,|\varphi|(1 + \mathcal{O}(\varphi^2))$, $\gamma_\pm(\pi/2+\varphi) = \frac{1}{8}\,(1 \pm 4|\varphi| + 2\varphi^2) + \mathcal{O}(\varphi^3)$ and hence $k(\pi/2+\varphi) = \sqrt{8|\varphi|}(1 - 2|\varphi| + \mathcal{O}(\varphi^2))$. Then, performing a Taylor expansion at order 4 of function $\mathcal{E}$ and using $\mathcal{E}''(0) = -\frac{\pi}{4}$, $\mathcal{E}^{(3)}(0) = 0$, $\mathcal{E}^{(4)}(0) = -\frac{9\pi}{16}$, we obtain

$$f(\varphi) = f(0)(1 - \varphi^2) + \mathcal{O}(\varphi^3),$$

with $f(0) = \frac{\mathcal{H}_2(Q)}{2\sqrt{2}}$. Lemma A.4 is proven. ∎

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