A robust δ-persistently exciting controller for formation-agreement stabilization of multiple mobile robots

Mohamed Maghenem, Antonio Loria, Elena Panteley

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***Abstract*— We propose a δ-persistently exciting controller [17] for leader-follower agreement control of a group of non-holonomic mobile robots, under the assumption that the virtual-leader velocities converge to zero. We assume that each of the vehicles in the formation communicates only with one leader and one, or several followers hence, that is, they form a spanning-tree communication topology rooted at the virtual leader. The control is decentralized and guarantees the convergence of the error coordinate of each agent, relatively to its neighbor. More significantly, our proofs are based on Lyapunov's direct method that is, we provide strict Lyapunov functions to guarantee strong integral input-to-state stability with respect to the reference velocities and, hence, uniform global asymptotic stability of the closed-loop system provided that the reference velocities are integrable.**

I. INTRODUCTION

Over the turn of last century there was a considerable bulk of literature on tracking and stabilization of non-holonomic mobile robots. Remarkable examples include the landmark papers [12] on the tracking-control problem and [24] on the more difficult problem of set-point stabilization, via smooth time-varying feedback. Indeed, it is well known that nonholonomic systems cannot be stabilized to a point via static smooth feedback. In [8] and [11] a backstepping controller is proposed for the full-model, *i.e.*, including the kinematics and force dynamics; uniformly global asymptotic stability is established for both, the tracking and the set-point stabilization problems. In [6] is presented an adaptive controller for simultaneous stabilization and tracking control.

In [22] a cascades-based *linear* time-varying tracking controller was proposed and uniform global asymptotic stability was established under a condition of persistency of excitation on the angular reference velocity. This was, to the best of our knowledge, the first time that persistency of excitation was used in control design for nonholonomic systems. Then, strongly inspired by the seminal paper [25], the approach was extended to the stabilization problem in [17], [16] where the so-called δ-persistently exciting controllers were proposed. Roughly, these are controllers which use persistency of excitation as stabilization mechanism, but the controller ceases to be persistently exciting as the stabilization errors converge. This control method is also effective, for instance, in the case of tracking control over straight paths [15].

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Some of the control approaches to the tracking problem have been extended to the case of formation control for swarms of vehicles. For instance, in [14] the problem of reaching a certain geometric configuration using a distributed control was addressed; necessary and sufficient graphical conditions were established. In [26], [3], and [9], a virtual-structure and a leader-follower-based approaches were investigated. A comparison between the two methods can be found in [26]. In [7] the formation-tracking control problem is solved using a combination of the virtual structure and path-tracking approaches to generate the reference for each agent. Then, an output-feedback observer-based controller is designed.

In this paper we solve the distributed leader follower agreement control for a group of mobile robots, when the leader velocities converge to zero. Our control approach is decentralized; we design a local controller for each robot, relying on its own state measurements and the values of a leader robot. Our control laws are smooth time-varying and possess the so-called property of δ-persistency of excitation [17], [23]. In contrast to most similar results, we establish uniform global asymptotic stability for the closed-loop, via Lyapunov's direct method. To construct our Lyapunov functions we follow the guidelines of [18], [19], [20]. In addition, we establish *strong integral input-to-state stability* (strong iISS) –see [1] and the Appendix. The importance of strong iISS is that not only this property guarantees robustness with respect to measurement noise, but it renders the solution to the formation control problem straightforward.

In section II we provide a problem formulation. In Section III we present our main result on leader-follower tracking-agreement control. Based on the latter, we present a statement for swarms of vehicles, in Section IV. We provide some illustrative simulation results in Section V, before concluding with some remarks.

II. PROBLEM FORMULATION

Consider the kinematic model of a mobile robot, given by

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta, \end{aligned}$$

where v denotes the forward velocity, ω corresponds to the angular velocity, (x, y) denote the Cartesian coordinates, and θ its orientation with respect to a fixed frame. It is assumed that the robot is velocity-controlled that is, v and ω also correspond to the control inputs.

The tracking control problem consists in making the robot follow a reference vehicle

$$\dot{\theta}_r = \omega_r \quad (1a)$$
$$\dot{x}_r = v_r \cos\theta_r \quad (1b)$$
$$\dot{y}_r = v_r \sin\theta_r . \quad (1c)$$

In addition, according to the tracking-*agreement* control goal, it is assumed that

$$\lim_{t\to\infty} |v_r(t)| + |\omega_r(t)| = 0. \quad (2)$$

Note that this excludes conditions of persistency of excitation or, even more restrictive, that the the references are always separated from zero –*cf.* [17], [4], [5].

In other words, from a control viewpoint, the goal is to steer to zero the differences between the Cartesian coordinates of the two robots, as well as orientation angles, that is,

$$\begin{aligned} p_\theta &= \theta_r - \theta \\ p_x &= x_r - x - d_x \\ p_y &= y_r - y - d_y \end{aligned}$$

where d_x, d_y are design parameters. These are distances to define the position and posture of the robot with respect to the (virtual) leader. In general, these may be functions of time and state or may be assumed constant, depending on the desired path to be followed. For simplicity, here, we consider them constant –*cf.* [15]. Then, according to the approach in [12] we transform the error coordinates $[p_x,\ p_y,\ p_\theta]$ from the global coordinate frame to local coordinates fixed on the robot, to obtain

$$\begin{bmatrix} e_\theta \\ e_x \\ e_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} p_\theta \\ p_x \\ p_y \end{bmatrix} \quad (3)$$

In the new coordinates, the error dynamics between the virtual reference vehicle and the follower becomes

$$\dot{e}_\theta = \omega_r(t) - \omega. \quad (4a)$$
$$\dot{e}_x = \omega e_y - v + v_r(t)\cos(e_\theta) \quad (4b)$$
$$\dot{e}_y = -\omega e_x + v_r(t)\sin(e_\theta) \quad (4c)$$

Therefore, the follow-the-leader tracking-agreement control problem comes to stabilizing (4) at the origin, under the assumption that (2) holds.

III. Leader-follower agreement control

Consider the controller given by

$$v = K_x e_x + v_r \cos e_\theta \quad (5a)$$
$$\omega = \omega_r + K_\theta e_\theta + K_y \left[e_y^2 + e_x^2\right] P(t) \quad (5b)$$

under the standing assumption that $\dot{P}$ is persistently exciting that is, there exist $\mu > 0$ and $T > 0$ such that

$$\int_t^{t+T} \dot{P}(s) ds \geq \mu \quad \forall t \geq 0. \quad (6)$$

This type of controller is called δ-persistently exciting; roughly speaking, a function $\phi(t,x)$ is called δ-persistently exciting (with respect to x) if, for any $\delta > 0$, there exist $\mu > 0$ and $T > 0$ such that

$$|x| \geq \delta \implies \int_t^{t+T} |\phi(s,x)| ds \geq \mu \quad \forall t \geq 0. \quad (7)$$

For instance, $\phi(t,x) := [e_y^2 + e_x^2]\varphi(t)$ satisfies (7) with $x = [e_x,\ e_y]^\top$ and φ persistently exciting, such as $\sin(t)$, white noise, a chaotic signal, etc.

Via our main result (Proposition 1 below), we establish, under the action of the controller (5), strong integral input-to-state stability with respect to the reference trajectories v_r and ω_r. In particular, we state that the tracking errors converge to zero for *any* reference velocities satisfying (2), even slowly-converging references.

Proposition 1 Consider the system (4) in closed loop with the controller (5). Let K_x, K_θ, and $K_y > 0$ and let P and $\dot{P}$ be bounded and persistently exciting. Then, the closed-loop system (4), (5) is strongly integral input-to-state stable with respect to the reference trajectories v_r and ω_r.

Corollary 1 Under the conditions of Proposition 1 the following hold:

1) under the action of <u>any</u> converging reference velocities v_r and ω_r, that is, satisfying (2), we have
$$\lim_{t\to\infty} |e(t)| = 0; \quad (8)$$
2) the origin is uniformly globally asymptotically stable if, moreover, v_r and ω_r are integrable, that is, there exists $\alpha > 0$ such that
$$\int_0^\infty |v_r(s)| + |\omega_r(s)| ds \leq \alpha. \quad (9)$$

Remark 1 The integrability assumption on the reference velocities holds, for instance, if they converge exponentially fast in a neighbourhood of zero.

<u>Proof of Proposition 1</u>

We start writing the closed-loop system (4) with (5) in the output-injection form

$$\dot{e} = A(t,e)e + K(t,e) \quad (10)$$

where $e := [e_\theta\ e_x\ e_y]^\top$,

$$A(t,e) := \begin{bmatrix} -K_\theta & -K_y P(t) e_x & -K_y P(t) e_y \\ 0 & -K_x & \psi(t,e) \\ 0 & -\psi(t,e) & 0 \end{bmatrix},$$

$$K(t,e) = \begin{bmatrix} 0 \\ w_r(t) e_y \\ -w_r(t) e_x + v_r(t)\sin(e_\theta) \end{bmatrix},$$

$$\psi(t,e) := K_\theta e_\theta + K_y P(t) \left[e_y^2 + e_x^2\right].$$

Then, we establish the proof is in three steps:

1) we construct a strong Lyapunov function for the nominal system $\dot{e} = A(t, e)e$;
2) we use this Lyapunov function to establish the small input-to-state stability property with respect to the velocity references ω_r and v_r;
3) we establish integral input-to-state-stability of (10), (11) with respect to ω_r and v_r.

Stability of the nominal system $\dot{e} = A(t, e)e$

Let f_m and $f_M > 0$ and consider the positive differentiable function $f : \mathbb{R}_+ \to [f_m, f_M]$ satisfying

$$\dot{f} = -K_\theta f + k_y P. \tag{11}$$

Then, consider the new coordinate

$$e_z = e_\theta + f(t)\left[e_y^2 + e_x^2\right],$$

which, by direct calculation and using (10), satisfies

$$\dot{e}_z = -K_\theta e_z - 2fK_x e_x^2. \tag{12}$$

Then, in the new coordinates, the nominal system becomes

$$\dot{e}_z = -K_\theta e_z - 2fK_x e_x^2 \tag{13a}$$

$$\begin{bmatrix}\dot{e}_x\\ \dot{e}_y\end{bmatrix} = \begin{bmatrix}-K_x & \dot{f}[e_y^2+e_x^2]\\ -\dot{f}[e_y^2+e_x^2] & 0\end{bmatrix}\begin{bmatrix}e_x\\ e_y\end{bmatrix} + e_z\begin{bmatrix}0 & K_\theta\\ -K_\theta & 0\end{bmatrix}\begin{bmatrix}e_x\\ e_y\end{bmatrix} \tag{13b}$$

Now, since $\dot{P}$ persistently exciting and f satisfies

$$\ddot{f} = -K_\theta \dot{f} + \dot{P} \tag{14}$$

we conclude that $\dot{f}$ is also persistently exciting [21]. Based on these properties, following the methodology of [18], we proceed to construct a Lyapunov function for (13).

Lemma 1 Let $f : \mathbb{R}_+ \to \mathbb{R}$ and $\dot{f}$ be persistently exciting (this holds if so are P and $\dot{P}$, which is an assumption of Proposition 1). Then, the system (13) admits the following strict Lyapunov function:

$$V_3(t, e) := \gamma_2(V_1)V_1 + V_2 + \gamma_3(V_1)e_z^2 \tag{15}$$

where $V_1 := \frac{1}{2}\left(e_x^2 + e_y^2\right)$,

$$V_2 := \gamma_1(V_1)V_1 + Q_{\dot{f}^2}V_1^3 - \dot{f}V_1 e_x e_y + P_1(V_1)V_1, \tag{16}$$

$$Q_{\dot{f}^2}(t) := 1 + 2\bar{\dot{f}}^2 T - \frac{2}{T}\int_t^{t+T}\int_t^m \dot{f}(s)^2 ds dm \tag{17}$$

and $\bar{(\cdot)}$ denotes an upper bound on $(\cdot)$.

Furthermore, let γ_1, γ_2 and γ_3 be continuous maps $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\begin{aligned} K_x\gamma_1(V_1) \ \geq\ & 12\bar{\dot{f}}^2 V_1^2 + K_x\bar{\dot{f}}\frac{T}{2\mu}\left[K_x\bar{\dot{f}} + \bar{\ddot{f}}\right] + K_x\bar{\dot{f}}V_1 \\ & + 2\bar{\dot{f}}^2 V_1 + \bar{\ddot{f}}\frac{T}{2\mu}\left[K_x\bar{\dot{f}} + \bar{\ddot{f}}\right] + \bar{\ddot{f}}V_1 \\ & + \bar{\dot{f}}V_1 K_x + K_x + 28\bar{\dot{f}}^2 V_1^2, \end{aligned} \tag{18}$$

$$\gamma_3(V_1) \ \geq\ 2\left[K_\theta^2\bar{\dot{f}}^2\frac{2T}{\mu}V_1 + K_\theta\bar{\dot{f}}V_1 + 2V_1 + 1\right], \tag{19}$$

$$K_x\gamma_2(V_1) \ \geq\ \bar{f}^2 K_x^2\gamma_3(V_1)V_1 + K_\theta\bar{\dot{f}}V_1^2 + V_1^2, \tag{20}$$

and let $P_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a first-order polynomial function of V_1, such that $P_1(V_1) \geq \frac{\bar{\dot{f}}}{2}V_1$.

Under these conditions,

$$\dot{V}_3 \leq -\frac{\mu}{4T}V_1^3 - \frac{\gamma_3(V_1)}{4}e_z^2. \qquad \square$$

The proof follows by direct calculation but it is ommitted due to space constraints. In particular, the choice of the functions γ_i guarantees the negative-definiteness of $\dot{V}_3$. Positivity of V_3 follows from a simple inspection, considering that $1 \leq Q_{\dot{f}^2}(t) \leq 1 + 2\bar{\dot{f}}^2 T$ for all $t \geq 0$.

Small ISS property

We recall that a system $\dot{x} = f(t, x, u)$ is said to be "small ISS" if it is input-to-state stable for sufficiently small values of u.

The proof of this property for the system (10) relies on the function V_3 and, especially on its order of growth in V_1. Note that the function V_3 in (15) satisfies $V_3(t, e) \equiv \mathcal{V}_3(t, e, V_1)$ where

$$\mathcal{V}_3(t, e, V_1) = P_2(t, V_1)V_1 - \dot{f}V_1 e_x e_y + P_1(V_1)e_z^2 \tag{21}$$

and $P_2 : \mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a smooth function, uniformly bounded in t and $P_2(t, \cdot)$ is a polynomial of degree 2 with strictly positive coefficients. In particular,

$$\frac{\partial P_2}{\partial V_1} \geq 0 \quad \forall\,(t, V_1) \in \mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}$$

By Lemma 1 the time derivative of V_3 along the nominal system (13) satisfies

$$\dot{V}_3(t, e) \leq -\frac{\mu}{4T}V_1^3 - \frac{P_1(V_1)}{4}e_z^2 \tag{22}$$

hence, the time derivative of V_3 along trajectories of (10) satisfies

$$\dot{V}_3 \leq -\frac{\mu}{4T}V_1^3 - \frac{P_1(V_1)}{4}e_z^2 + \frac{\partial V_3}{\partial e}K(t, e). \tag{23}$$

Now, setting

$$K = K_1 + K_2,\ K_1 = \omega_r\begin{bmatrix}0\\ e_y\\ -e_x\end{bmatrix},\ K_2 = v_r\begin{bmatrix}0\\ 0\\ \sin e_\theta\end{bmatrix}$$

and, using the fact that

$$\frac{\partial V_1}{\partial e}K_1(t, e) = 0,$$

it follows that

$$\begin{aligned} \dot{V}_3 \ \leq\ & -\frac{\mu}{4T}V_1^3 - \frac{P_1(V_1)}{4}e_z^2 \\ & - 2\dot{f}\omega_r V_1\left[e_y^2 - e_x^2\right] + \frac{\partial V_3}{\partial e}K_2(t, e) \\ \leq & -\frac{\mu}{4T}V_1^3 - \frac{P_1(V_1)}{4}e_z^2 + 2\bar{\dot{f}}\,|\omega_r|\,V_1^2 + \left|\frac{\partial V_3}{\partial e}\right||K_2|. \end{aligned}$$

On the other hand:

$$\left|\frac{\partial V_3}{\partial e}\right| \ \leq\ \left[\frac{\partial P_2}{\partial V_1}V_1 + P_2(V_1) + 2\bar{\dot{f}}V_1\right]\left[\,|e_y| + |e_x|\,\right]$$

$$+ \frac{\partial P_1}{\partial V_1}\big[\,|e_y| + |e_x|\,\big] e_z^2$$
$$+ 4P_1(V_1)\bar{f}\,|e_z|\,\big[\,|e_y| + |e_x|\,\big]. \tag{24}$$

Thus, defining

$$q_2(V_1) := \frac{\partial P_2}{\partial V_1} V_1 + P_2(V_1) + 2\bar{\bar{f}} V_1$$

as a positive polynomial of degree 2, and

$$\alpha := \frac{\partial P_1}{\partial V_1}$$

as a positive constant, and using them in (24), we obtain

$$\begin{aligned}\dot{V}_3 \leq &- \frac{\mu}{4T} V_1^3 - \frac{P_1(V_1)}{4} e_z^2 + 2\bar{\bar{f}}\,|\omega_r|\,V_1^2 \\ &+ q_2(V_1)\,|v_r|\,\big[|e_y| + |e_x|\big] + \alpha |v_r| \big[|e_y| + |e_x|\big] e_z^2 \\ &+ 4P_1(V_1)\bar{f}|v_r||e_z|\big[|e_y| + |e_x|\big].\end{aligned} \tag{25}$$

Using the inequality $|e_z|\,(|e_y| + |e_x|) \leq e_z^2 + 2V_1$ in (25) yields:

$$\begin{aligned}\dot{V}_3 \leq &- \frac{\mu}{4T} V_1^3 + 2\bar{\bar{f}}\,|\omega_r|\,V_1^2 + q_2(V_1)|v_r|\big[|e_y| + |e_x|\big] \\ &+ 8P_1(V_1)\bar{f}\,|v_r|\,V_1 \\ &- \left[\Big[\frac{1}{4} - 4\bar{f}|v_r|\Big] P_1(V_1) - \alpha\,|v_r|\,\Big[|e_y| + |e_x|\Big]\right] e_z^2 \\ \leq &\; \Phi_1(e_x, e_y, v_r, \omega_r) + \Phi_2(e_x, e_y, v_r) e_z^2.\end{aligned}$$

where,

$$\begin{aligned}\Phi_1 := &- \frac{\mu}{4T} V_1^3 + 2\bar{\bar{f}}|\omega_r| V_1^2 + q_2(V_1)|v_r|\big[|e_y| + |e_x|\big] \\ &+ 8P_1(V_1)\bar{f}|v_r|V_1,\end{aligned}$$

and,

$$\Phi_2 := - \Big[\frac{1}{4} - 4\bar{f}|v_r|\Big] P_1(V_1) + \alpha|v_r|\big[|e_y| + |e_x|\big].$$

It can be noticed that the negative term of Φ_1 dominates for sufficiently high values of e_x and e_y. With a similar analysis, for a sufficiently small values of v_r, the function Φ_2 is negative since all the coefficients of $P_1(V_1)$ are strictly positive. So the system is input-to-state stable for sufficiently small values of v_r and ω_r.

The iISS property

The proof of Proposition 1 is finalized by establishing integral input-to-state stability of the system (10) with respect to v_r, ω_r. To that end, consider the proper positive-definite Lyapunov function

$$W_3(t, e) = \ln\left(1 + V_3(t, e)\right) \tag{26}$$

whose time derivative along (10) satisfies

$$\dot{W}_3 \;=\; \frac{\dot{V}_3(t, e)}{1 + V_3(t, e)} \leq - \frac{\frac{\mu}{4T} V_1^3 + \frac{P_1(V_1)}{4} e_z^2}{1 + V_3(t, e)} + \frac{\frac{\partial V_3}{\partial e} K}{1 + V_3(t, e)}.$$

Now, $V_3(t, e)$ is positive definite and radially unbounded. Actually, from Lemma 1 and (21), it follows that there exist polynomials $g_1(V_1)$ and $g_2(V_1)$, with strictly positive coefficients and of degrees 1 and 2 respectively, such that

$$V_3(t, e) \geq g_2(V_1) V_1 + g_1(V_1) e_z^2. \tag{27}$$

Therefore, there exists a positive definite function α_1 such that

$$\alpha_1(|e|) \geq \frac{\frac{\mu}{4T} V_1(e)^3 + \frac{P_1(V_1(e))}{4} e_z^2}{1 + V_3(t, e)}. \tag{28}$$

Then,

$$\begin{aligned}\dot{W}_3 \leq &- \alpha_1(|e|) + \frac{2\bar{\bar{f}}\,|\omega_r|\,V_1^2}{1 + g_2(V_1)V_1} + \frac{q_2(V_1)|v_r|\big[|e_y| + |e_x|\big]}{1 + g_2(V_1)V_1} \\ &+ \frac{\alpha|v_r|\big[|e_y| + |e_x|\big] e_z^2 + 4P_1(V_1)\bar{f}|v_r||e_z|\big[|e_y| + |e_x|\big]}{1 + g_1(V_1)e_z^2 + g_2(V_1)V_1}.\end{aligned} \tag{29}$$

Note that $V_1 = \mathcal{O}(|e|^2)$ hence, the second, third, and forth terms in (29) are bounded functions of e. It follows that there exists a constant $k > 0$ such that:

$$\dot{W}_3 \leq - \alpha_1(|e|) + k\big|[v_r \;\omega_r]\big|. \tag{30}$$

Therefore, the system (10) is integral input-to-state stable. ■

IV. Multi-agent formation control

We extend the previous results on tracking control to the case of leader-follower formation-agreement control, under a spanning tree communication topology. That is, we consider a group of n mobile robots with kinematic models:

$$\dot{\theta}_i \;=\; w_i, \qquad i \in [1,\, n] \tag{31a}$$
$$\dot{x}_i \;=\; v_i \cos(\theta_i) \tag{31b}$$
$$\dot{y}_i \;=\; v_i \sin(\theta_i) \tag{31c}$$

where, for the i^{th} robot, x_i and y_i determine the position with respect to a globally-fixed frame, θ_i defines the heading angle, and the linear and angular velocities are denoted by v_i and w_i respectively.

The control objective is to make the n robots take specific postures and to make the swarm follow a path determined by a virtual reference vehicle; as before, the reference velocities are assumed to converge to zero. Any physically feasible geometrical configuration may be achieved and one can choose any point in the Cartesian plane to follow the virtual reference vehicle. We solve this problem using a slightly modified recursive implementation of the tracking leader-follower controller of the previous section. For each vehicle the local control law depends on the reference trajectory generated by the virtual leader. From a configuration viewpoint, the robots are interconnected in a spanning-tree topology, that is, the minimal configuration to achieve consensus. Accordingly, each robot has only one leader and may have one or several followers.

The fictitious vehicle, which serves as reference to the swarm, describes a reference trajectory defined by the desired linear and angular velocities v_r and w_r which are communicated to the swarm leader robot only. According to this communication topology, and following the setting for tracking control, the formation-agreement control problem reduces to stabilizing the origin of the error systems,

$$\dot{e}_{\theta_i} = \omega_{i-1} - \omega_i \tag{32a}$$
$$\dot{e}_{x_i} = \omega_i e_{y_i} - v_i + v_{i-1}\cos e_{\theta_i} \tag{32b}$$
$$\dot{e}_{y_i} = -\omega_i + e_{x_i} + v_{i-1}\sin e_{\theta_i} \tag{32c}$$

for each $i \in [1, n]$ –by definition, $\omega_0 := \omega_r$ and $v_0 := v_r$. Similarly to the controller proposed previously, we define

$$v_i = v_{i-1}\cos(e_{\theta_i}) + K_{xi}e_{x_i} \tag{33a}$$
$$\omega_i = \omega_{i-1} + K_{\theta_i}e_{\theta_i} + K_{y_i}P(t)\left[e_{y_i}^2 + e_{x_i}^2\right] \tag{33b}$$

where $P : \mathbb{R}_+ \to [P_m, P_M]$, with $P_m > 0$, is bounded and smooth. Moreover, we assume that this function, and its first time derivative, $\dot{P}(t)$, are persistently exciting.

Proposition 2 For each $i \in [1, n]$, consider the systems (32) in closed loop with the controller (33). Let K_{xi}, K_{yi}, $K_{\theta i} > 0$ and let P and $\dot{P}$ be bounded and persistently exciting. Then, under (2), (8) holds for $e := [e_{xi},\ e_{yi},\ e_{\theta i}]$. If, in addition (9) is satisfied, then the origin is uniformly globally asymptotically stable. □

Sketch of proof. To compact the notation, let us define

$$\psi_i(t, e_i) := K_{\theta i}e_{\theta_i} + K_{y_i}P(t)\left[e_{y_i}^2 + e_{x_i}^2\right]$$

Then, the closed-loop system under (33a) and (33b) may be written in the form

$$\begin{bmatrix}\dot{e}_{\theta_i}\\ \dot{e}_{x_i}\\ \dot{e}_{y_i}\end{bmatrix} = \begin{bmatrix}-K_{\theta_i} & -K_{y_i}P(t)e_{x_i} & -K_{y_i}P(t)e_{y_i}\\ 0 & -K_{xi} & \psi_i(t, e_i)\\ 0 & -\psi_i(t, e_i) & 0\end{bmatrix}\begin{bmatrix}e_{\theta_i}\\ e_{x_i}\\ e_{y_i}\end{bmatrix} + \begin{bmatrix}0\\ \omega_{i-1}e_{y_i}\\ -\omega_{i-1}e_{x_i} + v_{i-1}\sin e_{\theta_i}\end{bmatrix}. \tag{34}$$

The rationale of the proof follows a recursive cascades argument. From Proposition 1 the system (34) is strongly integral input-to-state stable with respect to v_{i-1}, ω_{i-1}, and for all $i \in [2, n]$. For $i = 1$, the system (34) is strongly iISS with respect to v_r and ω_r, and uniformly globally asymptotically stable under the integrability condition on v_r and ω_r. The proof is completed from the preservation of the strong iISS property under a cascaded interconnection –see [2]. This implies that the system composed of n agents is strongly iISS with respect to v_r and ω_r.

V. Simulations

In order to illustrate our results we have performed some simulation tests under Simulink™ of Matlab™. We consider a group of four mobile robots following a virtual leader. In this simulation, the desired formation shape of the four mobile robots is a diamond configuration that tracks the converging trajectory of the virtual leader. See Figure 3. We define the reference velocities v_r and ω_r in a way that they converge to zero asymptotically but relatively slowly, which is a hard-case scenario –see Figure 1.

The initial conditions are set to $[x_r(0), y_r(0), \theta_r(0)] = [0, 0, 0]$, $[x_1(0), y_1(0), \theta_1(0)] = [1, 3, 4]$, $[x_2(0), y_2(0), \theta_2(0)] = [0, 2, 2]$, $[x_3(0), y_3(0), \theta_3(0)] = [0, 4, 1]$ and $[x_4(0), y_4(0), \theta_4(0)] = [2, 2, 1]$; the control gains were set to $K_{x_i} = K_{y_i} = K_{\theta_i} = 4$ and the function $P(t) = 13 + 12\sin(\pi t)$, which is persistently exciting. The formation shape with a certain desired distance between the robots is obtained by setting all desired orientation offsets to zero and defining $[d_{x_{r,1}}, d_{y_{r,1}}] = [0, 0]$, $[d_{x_{1,2}}, d_{y_{1,2}}] = [-1, 0]$ and $[d_{x_{2,3}}, d_{y_{2,3}}] = [1/2, 1/2]$ and $[d_{x_{3,4}}, d_{y_{3,4}}] = [0, 1]$. See Figure 3. The results of the simulation are shown in Figures 2–3. In Figure 2 we show the convergence of the tracking errors between the agent and its neighborhood.

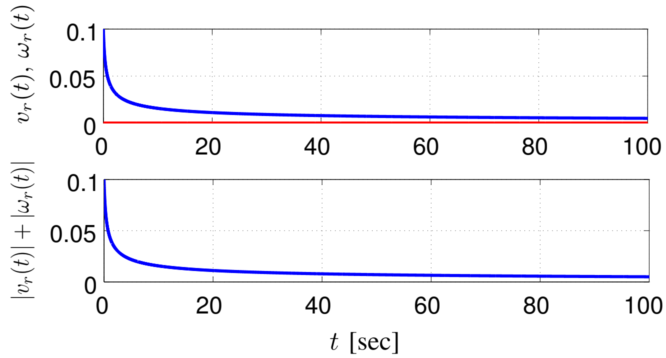

Fig. 1. Reference velocities v_r and ω_r

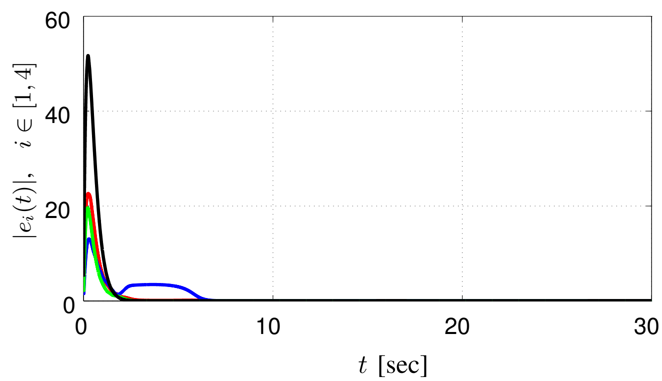

Fig. 2. Exponential convergence of the relative errors (in norm) for each pair leader-follower

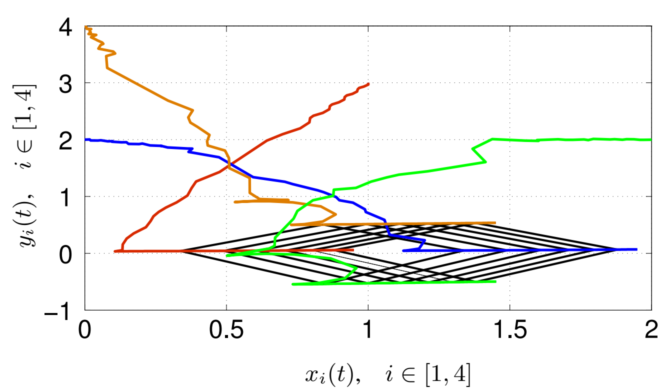

Fig. 3. Illustration of the path-tracking in formation

VI. CONCLUSIONS

We presented a simple decentralized controller for leader-follower agreement that is, we consider that the leader velocities converge. Further research is being carried out to incorporate the dynamics at the force-level and to consider more realistic cases, such as that of parametric uncertainty and output feedback control.

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APPENDIX

On input-to-state stability

Paraphrasing [1], we say that the dynamical system $\dot{x} = f(t,x,u)$ is strongly integral input-to-state stable (strongly iISS) with respect to u, if it is:

1) integral input-to-state stable (iISS), *i.e.*, there exists a class $\mathcal{KL}$ function β and class $\mathcal{K}_\infty$ functions μ_1, μ_2 such that, for all $x_\circ \in \mathbb{R}^n$, $t \geq t_\circ \geq 0$,

$$|x(t,t_\circ,x_\circ)| \leq \beta(|x_\circ|, t-t_\circ) + \mu_1\left(\int_{t_\circ}^{t} \mu_2(|u(\tau)|)d\tau\right);$$

2) small-input-to-state stable (small-ISS), *i.e.*, there exist $R > 0$ and functions $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$, such that, for all $x_\circ \in \mathbb{R}^n$ and all $t \geq t_\circ \geq 0$,

$$|u| < R \implies |x(t,t_\circ,x_\circ)| \leq \beta(|x_\circ|, t-t_\circ) + \mu(|u|).$$

Lemma 2 (Lyapunov characterization of ISS [13]) Let $V : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable Lyapunov function such that:

$$\underline{\alpha}(|x|) \leq V(t,x) \leq \overline{\alpha}(|x|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) \leq -W_3(x), \quad \forall |x| \geq \rho(|u|) > 0$$

where $\underline{\alpha}$, $\overline{\alpha}$ are $\mathcal{K}_\infty$ functions, ρ a class K function, and W_3 a continuous positive definite function. Then, the system $\dot{x} = f(t,x,u)$ is ISS with respect to the input u.

Lemma 3 (Lyapunov characterization of iISS [10]) Let $V : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable Lyapunov function such that

$$\underline{\alpha}(|x|) \leq V(t,x) \leq \overline{\alpha}(|x|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) \leq -\alpha_1(|x|) + \rho(|u|)$$

where $\underline{\alpha}$, $\overline{\alpha}$, and ρ are class $\mathcal{K}_\infty$ functions and α_1 is positive definite. Then, the system $\dot{x} = f(t,x,u)$ is integral ISS with respect to u.