Formation-Tracking Control of Autonomous Vehicles Under Relaxed Persistency of Excitation Conditions

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***Abstract*—We present a smooth nonlinear time-varying formation-tracking controller for autonomous vehicles modeled as a non-holonomic unicycle. Our first result consists in a leader-follower tracking controller that guarantees uniform global asymptotic stability under the standing assumption that *either* the rotational *or* the translational reference velocity is persistently exciting. Then, we extend this result to the case of formation control of a swarm of vehicles. We show that this problem may be solved via decentralized tracking control, under the assumption that each robot communicates with one leader and one follower.**

I. Introduction

Tracking control of non-holonomic mobile robots was a very popular research problem in the control community in the last decades –see *e.g.*, the seminal work [1] where global stabilization is established via Lyapunov's second method. Articles considering parametric uncertainty and constrained inputs, include [2], [3], and [4]. In [5] a simple cascaded-based *linear* time-varying controller was proposed and uniform global asymptotic stabilization under the condition that the angular velocity is persistently exciting, was established. This approach was recently generalized in [6] to solve the problem of formation tracking control.

In [7] an adaptive controller for simultaneous stabilization and tracking for force-controlled vehicles was proposed. See also [8] where a high-gain observer is incorporated to address the problem via output feedback. In [9] a finite-time tracking controller was proposed and, based on analysis methods for cascaded-systems, finite-time stability is concluded for the overall system.

Some of the previous results have been extended to the case of formation tracking. See, *e.g.*, [10] where the problem of reaching a certain geometric configuration using a distributed control was addressed and necessary and sufficient conditions were deduced. In [11], [12], [13], and [14], the virtual-structure and leader-follower approaches were investigated. See the first for a comparison of the two methods.

In [15], the authors solve the formation tracking problem using a combination of the virtual structure and path-tracking approaches to generate the reference for each agent, then an output feedback control law is designed to track each agent toward its reference, using an asymptotic observer to estimate the velocities. This work was extended in [16] and [17], where the problem formation tracking with collision avoidance was considered. In [17] the consensus tracking problem was considered under parameter uncertainty. In [18] two different scenarios of *rendez-vous* cooperative control are considered under conditions of persistency of excitation on the reference velocities –see also [19] for the problem of distributed tracking control.

In this paper we address the problem of formation-tracking control for a swarm of mobile robots under the assumption that each of them communicates only with two neighbors hence, only the leader robot has the reference path information. In the particular case of tracking control we recover the control laws proposed in [20] and [21], in which asymptotic convergence of the tracking errors is ensured under the condition that the limit of the velocities (angular and forward) in norm is different from zero. Our contribution with respect to the latter references is to establish uniform global asymptotic stability of the origin both for the leader-follower tracking set-up and the general case of swarms of mobile robots advancing in formation, under the relaxed condition of persistency of excitation of the velocities' norm. This includes the scenario of straight-path tracking [6], but not of tracking-agreement control, in which the reference velocities vanish [33]. In addition, as a corollary of our main results, we also establish exponential convergence on any ball (some times called $\mathcal{K}$-exponential stability –[22]). Furthermore, we give an explicit way to compute the rate of convergence on each compact.

Our proofs are relatively direct. For the case of leader-follower tracking control, we invoke Matrosov's theorem, which is a generalization of Barbashin-Krasovskĭi's theorem to the case of time-varying systems. The proof relies on the construction of a function with sign-definite derivative for systems with persistency of excitation. The use of Matrosov's theorem leads to a very simple controller with unrestricted gains and, in our opinion, to a clear and concise stability proof of uniform global asymptotic stability, as opposed to *convergence of the error trajectories*, as often established in related literature. Then, for the general case of formation tracking we employ a recursive cascades argument which, we believe, is fairly intuitive.

The rest of the paper is organized as follows. In Section II we present our original solution to the leader-follower tracking control problem that is, for one robot following a virtual vehicle. In Section III we present our second result, the extension to the case of formation tracking under a spanning-tree interconnection topology. Numerical simulations that illustrate our main results are presented in Section IV, before concluding with some remarks in Section V.

II. A single agent case

Consider the kinematic model of a mobile robot, *i.e.*,

$$\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega, \end{aligned}$$

where $x, y \in \mathbb{R}$ are the Cartesian coordinates of a fixed point in the robot, $\theta \in \mathbb{R}$ denotes its orientation with respect to a fixed frame, and (v, ω) denote the forward and angular velocities; they also correspond to the two control inputs. The tracking control problem consists in following a fictitious reference vehicle

$$\dot{x}_r = v_r\cos\theta_r \tag{1a}$$

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$$\dot{y}_r = v_r \sin\theta_r \quad (1b)$$
$$\dot{\theta}_r = \omega_r \quad (1c)$$

which moves about with reference velocities $v_r(t)$ and $\omega_r(t)$. From a control viewpoint, the goal is to steer the differences between the Cartesian coordinates of the two robots to some constant values d_x, d_y —*cf.* [6], and the orientation angles to zero, *i.e.*,

$$\begin{aligned} p_x &= x_r - x + d_x \\ p_y &= y_r - y + d_y \\ p_\theta &= \theta_r - \theta. \end{aligned}$$

Then, as in [23] and many other succeeding works, we transform the error coordinates $[p_x,\ p_y,\ p_\theta]$ of the leader robot from the global coordinate frame to local coordinates fixed on the robot that is,

$$\begin{bmatrix} e_x \\ e_y \\ e_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_\theta \end{bmatrix}. \quad (2)$$

In the new coordinates, the error dynamics between the virtual reference vehicle and the follower becomes

$$\dot{e}_x = \omega e_y - v + v_r(t)\cos(e_\theta) \quad (3a)$$
$$\dot{e}_y = -\omega e_x + v_r(t)\sin(e_\theta) \quad (3b)$$
$$\dot{e}_\theta = \omega_r(t) - \omega. \quad (3c)$$

Therefore, the follow-the-leader tracking control problem of mobile robots amounts to a stabilization problem, at the origin, for the system (3).

In [5] persistency of excitation was explicitly used for the first time as a necessary and sufficient condition to guarantee global exponential stabilization of the origin for the tracking-error dynamics. In that reference the following simple linear time-varying controller was proposed:

$$v^* = v_r(t) + K_x e_x, \quad K_x > 0 \quad (4a)$$
$$\omega^* = \omega_r(t) + K_\theta e_\theta, \quad K_\theta > 0. \quad (4b)$$

Besides the simplicity of this controller, it is to be remarked that the closed-loop system (with $v = v^*$, $\omega = \omega^*$) has the convenient cascaded form

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \end{bmatrix} = \begin{bmatrix} -K_x & \omega_r(t) \\ -\omega_r(t) & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \end{bmatrix} + g(t,e) \quad (5a)$$
$$\dot{e}_\theta = -K_\theta e_\theta \quad (5b)$$

where $e = [e_x\ e_y\ e_\theta]^\top$ and

$$g(t,e) := \begin{bmatrix} v_0(t)\big[\cos(e_\theta) - 1\big] + K_\theta e_\theta e_y \\ v_0(t)\sin(e_\theta) - K_\theta e_\theta e_x \end{bmatrix}. \quad (6)$$

As it is showed in [5], for this system, uniform global asymptotic stability may be established via the following cascades argument (see *e.g.*, [24]): first, we observe that because $K_\theta > 0$, e_θ converges exponentially fast; then, we recognize that $g(t,e)$ has linear growth in e_x and e_y and it is uniformly bounded in t. Finally, for the equations (5a) with $g \equiv 0$, the origin is exponentially stable provided that the reference angular velocity is persistently exciting that is, assuming that there exist μ, $T > 0$ such that

$$\int_t^{t+T} \omega_r(s)^2 ds \geq \mu, \qquad \forall\, t \geq 0. \quad (7)$$

The latter argument for system (5a) relies on a large bulk of literature on stability of linear time-varying (adaptive) control systems. Notice, indeed, that the system (5a) with $g \equiv 0$ has the structure of model-reference-adaptive-control systems. However, for this to hold it is necessary for the angular velocity to satisfy $\omega_r \not\equiv 0$; such condition excludes straight-path trajectories. In [6] a modified version of this controller, using a condition of persistency of excitation tailored for nonlinear systems, was proposed for the case of tracking on straight-line paths.

Here, for the particular case of leader-follower tracking control problem we use the nonlinear time-varying controller

$$v = v_r(t)\cos(e_\theta) + K_x e_x \quad (8a)$$
$$\omega = \omega_r(t) + K_\theta e_\theta + v_r(t) K_y e_y \phi(e_\theta) \quad (8b)$$

where ϕ is the so-called 'sync' function defined by

$$\phi(e_\theta) := \frac{\sin(e_\theta)}{e_\theta}. \quad (9)$$

This function has several useful properties: it is smooth, bounded and locally positive, actually, $|\phi(s)| > 0$ for any $|s| < \pi$. For this controller, it was established in [21], [25] that the tracking errors converge to zero asymptotically under the condition that

$$\lim_{t\to\infty} v_r(t) \neq 0 \quad \text{or} \quad \lim_{t\to\infty} \omega_r(t) \neq 0. \quad (10)$$

Our standing assumption here is that either the angular or the forward reference velocity is persistently exciting. In particular, straight-line reference paths, in which case $w_r(t) \equiv 0$, are admissible if v_r is persistently exciting. This allows to establish uniform global asymptotic stability, in contrast to asymptotic convergence of the tracking errors.

The control design approach is motivated by the resulting structure of the closed-loop system, which includes a persistently-excited matrix with a convenient structure:

$$\dot{e} = A(t,e)e, \qquad e^\top := [e_x\ e_y\ e_\theta] \quad (11)$$
$$A(t,e) := \begin{bmatrix} -K_x & \omega(t,e) & 0 \\ -\omega(t,e) & 0 & v_r(t)\phi(e_\theta) \\ 0 & -v_r(t)K_y\phi(e_\theta) & -K_\theta \end{bmatrix}.$$

Theorem 1 *Assume that v_r, ω_r, $\dot{v}_r$ and $\dot{\omega}_r$ are bounded. If, moreover, $\sqrt{v_r^2 + \omega_r^2}$ is persistently exciting, that is, if there exist μ and $T > 0$ such that*

$$\int_t^{t+T} [\omega_r(s)^2 + v_r(s)^2] ds \geq \mu \qquad \forall\, t \geq 0, \quad (12)$$

the origin of (11) *in closed loop with the controller* (8) *is uniformly globally asymptotically stable, for any positive gains K_x, K_y and K_θ.*

Proof: Consider first the Lyapunov function candidate $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ defined as

$$V(t,e) = \frac{1}{2}\Big[e_x^2 + e_y^2 + \frac{1}{K_y}e_\theta^2\Big]. \quad (13)$$

This function satisfies

$$c_1|e|^2 \le V(t,e) \le c_2|e|^2$$

with $c_1 := \min\{1/2,\ 1/2K_y\}$ and $c_2 := \max\{1,\ 1/K_y\}$. Moreover, its total time derivative along trajectories of (11) is negative semidefinite, indeed,

$$\dot V(t,e) = -K_x e_x^2 - K_\theta e_\theta^2 \ \le\ 0. \tag{14}$$

Integrating on both sides of $\dot V(t,e(t)) \le 0$ and defining $c_3 := \sqrt{c_2/c_1}$ we obtain

$$|e(t)| \le c_3|e(t_\circ)| \quad \forall\, t \ge t_\circ \ge 0. \tag{15}$$

That is, the origin is uniformly globally stable with linear growth. In particular, for any $r > 0$ the solutions generated by $|e_\circ| \le r$ remain in a ball $B_R := \{x \in \mathbb{R}^n : |x| \le R := c_3 r\}$. Therefore, it is only left to establish uniform global convergence to the origin. We do this using Matrosov's theorem which, for the purposes of this paper, is paraphrased below in a slightly more restrictive form than original, following [26] and [27].

Theorem 2 ("Matrosov") *Consider the system $\dot x = f(t,x)$. Let $B_\rho := \{x \in \mathbb{R}^n : |x| < \rho\}$. Suppose that, for each $\rho > 0$, there exist three functions, $V : \mathbb{R}_{\ge 0} \times B_\rho \to \mathbb{R}$, $V^* : B_\rho \to \mathbb{R}$ and $W : \mathbb{R}_{\ge 0} \times B_\rho \to \mathbb{R}$, which are continuously differentiable. Assume further that, for each $R \in (0,\rho)$,*

a) *there exist $L > 0$ such that*

$$\max\{|W(t,x)|,\ |f(t,x)|\} \le L$$

for all[1] $(t,x) \in \mathbb{R}_{\ge 0} \times \bar B_R$*;*

b) *there exist class $\mathcal{K}$ functions α_1 and α_2 such that, for all $(t,x) \in \mathbb{R}_{\ge 0} \times \bar B_\rho$:*

$$\alpha_1(|x|) \le V(t,x) \le \alpha_2(|x|)$$
$$\dot V(t,x) \le V^*(x) \le 0\,;$$

c) *the function $\dot W(t,x)$ is non-zero definite on $M := \{x \in B_R : V^*(x) = 0\}$ that is, there exists $\alpha_3 \in \mathcal{K}$ such that*

$$|\dot W(t,x)| \ge \alpha_3(|x|) \qquad \forall\,(t,x) \in \mathbb{R}_{\ge 0} \times M.$$

Then, every solution such that $x(t,t_\circ,x_\circ) \in B_R$ for all $t \ge t_\circ$ tends uniformly to 0 as $t \to \infty$.

We construct a function W satisfying the conditions of Theorem 2. For any locally integrable function $f : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ such that $\sup_{t \ge 0}|f(t)| \le \bar f$, following [28], let us define

$$Q_f(t) := 1 + 2\bar f T - \frac{2}{T}\int_t^{t+T}\int_t^m f(s)\,ds\,dm. \tag{16}$$

Note that this function satisfies

$$1 \le Q_f(t) < \bar Q_f := 1 + 2\bar f T$$
$$\dot Q_f(t) = -\frac{2}{T}\int_t^{t+T} f(s)ds + 2f(s).$$

Next, let the function $W : \mathbb{R}_{\ge 0} \times \mathbb{R}^3 \to \mathbb{R}_{\ge 0}$ be defined as

$$\begin{aligned} W(t,e) \ &:= \ -\omega_r(t)e_x e_y + \alpha v_r(t) e_\theta e_y \\ &\quad + \frac{1}{2}\left[K_y\phi(e_\theta)Q_{v_r^2}(t) + Q_{\omega_r^2}(t)\right]e_y^2 \end{aligned} \tag{17}$$

where α and γ are positive constants to be defined. In view of the boundedness of $v_r(t)$, $\omega_r(t)$, and $\phi(e_\theta)$ for any $\alpha > 0$, there exists $c_2' > 0$ such that

$$|W(t,e)| \le c_2'|e|^2$$

for all $t \ge 0$ and all $e \in \mathbb{R}^3$. In view of this, of the continuity of $A(t,e)$ with respect to e and uniform boundedness with respect to t, item a) of Theorem 2 holds —boundedness and continuity of A comes from the boundedness assumption on the reference trajectories. Furthermore, item b) holds with V as defined in (13) and $V^*(e) := -K_x e_x^2 - K_\theta e_\theta^2$.

To verify item c) we evaluate the total time derivative of W along the closed-loop trajectories of (11), to obtain

$$\begin{aligned} \dot W \ &= \ \omega_r^2 e_x^2 + v_r^2\alpha\phi(e_\theta)e_\theta^2 \\ &\quad + \Psi_{xy}(t,e)e_x e_y + \Psi_{\theta y}(t,e)e_y e_\theta + \Psi_{\theta x}(t,e)e_x e_\theta \\ &\quad - \left[\int_t^{t+T}\frac{1}{T}v_r^2(s)ds\right]\phi(e_\theta)K_y e_y^2 \\ &\quad - [\alpha-1]v_r^2 K_y e_y^2\phi(e_\theta) - \int_t^{t+T}\frac{1}{T}\omega_r^2(s)ds e_y^2 \\ &\quad - \left[\omega_r + \frac{\cos(e_\theta)-\phi(e_\theta)}{e_\theta}Q_{v_r^2}K_y\right]K_y v_r\phi(e_\theta)e_y^3 \end{aligned}$$

where

$$\begin{aligned} \Psi_{xy} \ &= \ -\dot\omega_r + \omega_r K_x + K_y v_r\omega_r e_x\phi(e_\theta) \\ &\quad - \left[K_y\phi(e_\theta)Q_{v_r^2} + Q_{\omega_r^2}\right] \\ &\qquad \times [\omega_r + K_\theta e_\theta + e_y K_y v_r\phi(e_\theta)] \\ \Psi_{\theta x} \ &= \ \omega_r e_x K_\theta - v_r\omega_r\phi(e_\theta) - v_r\alpha\omega_r \\ &\quad - v_r^2\alpha y K_y\phi(e_\theta) - v_r\alpha K_\theta e_\theta \\ \Psi_{\theta y} \ &= \ -\omega_r K_\theta e_y - \alpha v_r K_\theta + \alpha\dot v_r \\ &\quad + \left[K_y\phi(e_\theta)Q_{v_r^2} + Q_{\omega_r^2}\right]v_r\phi(e_\theta) \\ &\quad - \left(\frac{\cos(e_\theta)-\phi(e_\theta)}{e_\theta}\right)Q_{v_r^2}K_y K_\theta e_y. \end{aligned}$$

The functions Ψ_{xy}, $\Psi_{\theta x}$ and $\Psi_{\theta y}$ are uniformly bounded for any $R > 0$ and all $(t,e) \in \mathbb{R}_{\ge 0} \times B_R$. Let $R > 0$ be arbitrary but fixed and define

$$M := \{e \in \mathbb{R}^3 : |e| \le R,\ e_x = e_\theta = 0\}.$$

Then, since $\phi(0) = 1$ and

$$\lim_{e_\theta \to 0}\left(\frac{\cos(e_\theta)-\phi(e_\theta)}{e_\theta}\right) = 0$$

we have, for all $t \ge 0$ and all $e \in B_R$,

$$\begin{aligned} \dot W(t,e) \le &- \left[\int_t^{t+T}\frac{1}{T}\omega_r^2(s)ds + K_y\int_t^{t+T}\frac{1}{T}v_r^2(s)ds\right]e_y^2 \\ &- [\alpha-1]K_y v_r^2 e_y^2 + \omega_r K_y v_r e_y^3. \end{aligned} \tag{18}$$

Note, moreover, that the last term on the right hand side may be bounded, for any R and $\lambda > 0$, as

$$\omega_r K_y v_r e_y^3 \le \frac{\lambda}{2}[\omega_r K_y R]^2 e_y^2 v_r^2 + \frac{e_y^2}{2\lambda}.$$

[1] We denote by $\bar B_R$ a closed ball of radius R centered at the origin.

Then, let $\alpha \geq 1 + K_y[\omega_r R]^2\lambda/2$. It follows from this and the assumption of persistency of excitation on $\sqrt{v_r^2 + w_r^2}$, that there exist $\mu > 0$ and $T > 0$ such that

$$|\dot{W}(t,e)| \geq \left[\frac{\mu}{T}\min\{1,\, K_y\} - \frac{1}{2\lambda}\right] e_y^2 \quad \forall\, (t,e) \in \mathbb{R}_{\geq 0} \times M.$$

Note that the factor of e_y^2 is positive for a sufficiently large value of λ. Item c) of Matrosov's theorem follows. Since the previous arguments hold for any choice of $R > 0$ without restricting the control gains nor the parameters μ and T, the origin is uniformly globally attractive. The result follows. ■

Another original worthy statement may be established based on the auxiliary function introduced in (17). Indeed, the following theorem, which is proved in the preliminary version of this paper —[29], provides a strict Lyapunov function for the closed-loop system on each compact set in $\mathbb{R}^3$. This leads to the computation of an estimate of the convergence rate.

Theorem 3 *The origin of the closed-loop system* (8), (11) *is globally $\mathcal{K}$-exponentially stable, under the conditions of Theorem 1.*

Moreover, for each $r > 0$ one can compute explicitly positive constants γ_r and α_r depending only on r, such that

$$\begin{aligned} V_2(t,e) = \quad & \gamma_r V_1(t,e) - \omega_r(t) e_x e_y + \alpha_r v_r(t) e_\theta e_y \phi(e_\theta) \\ & + \tfrac{1}{2}\left[K_y \phi^2(e_\theta) Q_{v_r^2}(t) + Q_{\omega_r^2}(t)\right] e_y^2, \end{aligned}$$

where $Q_f(t)$ is defined in (16)*, is a strict Lyapunov function for the closed-loop system. In particular, for all $|e(t_0)| \leq r$ and $t \geq t_0 \geq 0$,*

$$\dot{V}_2(t,e(t)) \leq -\min\left\{K_x, K_\theta, \frac{\mu}{2T}\right\} |e(t)|^2 \quad .$$

III. Formation-tracking control

We address now the following formation-tracking control problem. Given a swarm of robots with models

$$\begin{aligned} \dot{x}_i &= v_i \cos(\theta_i) && (19a) \\ \dot{y}_i &= v_i \sin(\theta_i) && (19b) \\ \dot{\theta}_i &= w_i, \qquad i \in [1,\, n] && (19c) \end{aligned}$$

it is required that they assume a prescribed formation pattern, determined by constant relative distances $d_{xi-1,i}$ and $d_{yi-1,i}$. In contrast to a multi-agent tracking scenario in which each robot tracks the reference trajectory, we assume that only one robot, designated swarm leader, has access to the reference velocities v_r and ω_r, as well as to the states of the fictitious vehicle (1). Then, each robot follows one leader that is, we assume that the vehicles are interconnected according to a spanning-tree topology, which is necessary to achieve consensus.

Similarly to the case of one-leader-one-follower tracking-control scenario studied in the previous section we define

$$\begin{aligned} p_{xi} &= x_{i-1} - x_i - d_{xi-1,i} \\ p_{yi} &= y_{i-1} - y_i - d_{yi-1,i} \\ p_{\theta i} &= \theta_{i-1} - \theta_i \qquad i \in [1,n] \end{aligned}$$

and we apply a coordinate transformation to define the errors between any pair of vehicles as

$$\begin{bmatrix} e_{xi} \\ e_{yi} \\ e_{\theta i} \end{bmatrix} = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{xi} \\ p_{yi} \\ p_{\theta i} \end{bmatrix}$$

which satisfy the dynamics equations

$$\begin{aligned} \dot{e}_{xi} &= w_i e_{iy} - v_i + v_{i-1}\cos(e_{\theta i}) && (20a) \\ \dot{e}_{yi} &= -w_i e_{ix} + v_{i-1}\sin(e_{\theta i}) && (20b) \\ \dot{e}_{\theta i} &= w_{i-1} - w_i. && (20c) \end{aligned}$$

Remark 1 Note that many swarm-formations are possible by adequately defining the distance parameters $d_{(\cdot)}$, but this problem is out of the paper's scope.

The formation-tracking control problem for n robots reduces to the stabilization of the origin in the space of $e := [e_x^\top,\, e_y^\top,\, e_\theta^\top]^\top$ where we redefined $e_{(\cdot)} := [e_{(\cdot)1}, \cdots e_{(\cdot)n}]^\top$. To that end, for each $i \in [1,n]$ we propose the controller

$$\begin{aligned} v_i &= v_{i-1}\cos(e_{\theta i}) + K_{xi} e_{xi} && (21a) \\ \omega_i &= \omega_{i-1} + K_{\theta i} e_{\theta i} + v_{i-1} K_{y_i} e_{yi} \phi(e_{\theta i}). && (21b) \end{aligned}$$

Remark 2 Our control approach may also be used under a general time-varying uni-directional connected communication graph and using distributed exponentially-convergent estimators for the leader trajectories, as in [30] and [31]. In this case, the controllers (21) reduce to (8) in which we replace (v_r, ω_r) by the estimated leader's velocities and we compute the errors e_i with respect to the estimated leader's trajectories.

Theorem 4 *Under the conditions of Theorem 1 on v_r and ω_r, the origin of the closed-loop system* (20)–(21) *is $\mathcal{K}$-exponentially stable, for any positive gains K_{xi}, K_{y_i} and $K_{\theta i}$.*

Proof: The closed-loop dynamics is

$$\begin{bmatrix} \dot{e}_{xi} \\ \dot{e}_{yi} \\ \dot{e}_{\theta_i} \end{bmatrix} = \underbrace{\begin{bmatrix} -K_{xi} & \omega_i(t,e_i) & 0 \\ -\omega_i(t,e_i) & 0 & v_{i-1}\phi(e_{\theta i}) \\ 0 & -v_{i-1}K_{yi}\phi(e_{\theta i}) & -K_{\theta i} \end{bmatrix}}_{A_i(e_i, v_{i-1}, \omega_i)} \begin{bmatrix} e_{xi} \\ e_{yi} \\ e_{\theta i} \end{bmatrix}$$

which has exactly the same structure as (11). Notably, for each $i \leq n$, the Lyapunov function

$$V_i(t,e_i) = \frac{1}{2}\left[e_{xi}^2 + e_{\theta i}^2 + \frac{1}{K_{yi}} e_{yi}^2\right] \tag{22}$$

satisfies

$$\dot{V}_i(t,e_i) = -\left[K_{xi}|e_{xi}|^2 + K_{\theta i}|e_{\theta i}|^2\right] \tag{23}$$

hence, the function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^{3n} \to \mathbb{R}_{\geq 0}$, defined as

$$V(t,e) := \sum_{i=1}^{n} V_i(t,e_i),$$

is positive definite, radially unbounded and satisfies

$$\dot{V}(t,e) = -\sum_{i=1}^{n}\left[K_{xi}|e_{xi}|^2 + K_{\theta i}|e_{\theta i}|^2\right] \leq 0.$$

It follows that the origin is uniformly globally stable with linear gain that is, (15) holds with an appropriate redefinition of c_3. In particular, the solutions are uniformly globally bounded.

Next, we exhibit the important fact that, by design, the closed-loop dynamics has a triangular structure composed by a cascade of n dynamical systems. This, and uniform global boundedness of the solutions, allows to invoke an inductive cascades argument.

To that end, let us introduce the variables $\tilde{v}_i = v_i - v_{i-1}$ and $\tilde{\omega}_i = \omega_i - \omega_{i-1}$. Then, for each $i \geq 1$, we have

$$\begin{aligned} \tilde{v}_i &= v_{i-1}\left[\cos(e_{\theta i}) - 1\right] + K_{xi}e_{xi} && \text{(24a)} \\ \tilde{\omega}_i &= K_{\theta i}e_{\theta i} + v_{i-1}K_{y_i}\phi(e_{\theta_i})e_{yi}, && \text{(24b)} \end{aligned}$$

and the closed-loop system takes the form:

$$\begin{aligned} \dot{e}_{xi} &= \varpi_i e_{yi} - K_{xi}e_{xi} + \left[\sum_{k=1}^{i-1}\tilde{\omega}_k\right]e_{yi} \\ &\quad + \left[\sum_{k=1}^{i-1}\tilde{v}_k\right]K_{y_i}\phi(e_{\theta i})e_{yi}^2 \\ \dot{e}_{yi} &= -\varpi_i e_{xi} + v_r(t)\sin(e_{\theta i}) - \left[\sum_{k=1}^{i-1}\tilde{\omega}_k\right]e_{xi} \\ &\quad - \left[\sum_{k=1}^{i-1}\tilde{v}_k\right]\left[K_{y_i}\phi(e_{\theta i})e_{xi}e_{yi} + \sin(e_{\theta i})\right] \\ \dot{e}_{\theta_i} &= -\varpi_i + \omega_r(t) - \left[\sum_{k=1}^{i-1}\tilde{v}_k\right]K_{y_i}\phi(e_{\theta i})e_{yi} + \left[\sum_{k=1}^{i-1}\tilde{\omega}_k\right] \end{aligned}$$

where

$$\varpi_i = K_{\theta i}e_{\theta i} + \omega_r(t) + v_r K_{y_i}\phi(e_{\theta i})e_{yi}. \quad (25)$$

With these notations, the error dynamics takes the form

$$\dot{e}_i = \bar{A}_i(t,e_i)e_i + M_i(t,e_i)g_i(t,e_1,\cdots,e_{i-1}) \quad (26)$$

where

$$\bar{A}_i(t,e_i) := \begin{bmatrix} -K_{xi} & \varpi_i(t,e_i) & 0 \\ -\varpi_i(t,e_i) & 0 & v_r(t)\phi(e_{\theta i}) \\ 0 & -v_r(t)K_{yi}\phi(e_{\theta i}) & -K_{\theta i} \end{bmatrix}$$

$$M_i(t,e_i) := \begin{bmatrix} K_{y_i}\phi(e_{\theta i})e_{yi}^2 & e_{yi} \\ -K_{y_i}\phi(e_{\theta i})e_{xi}e_{yi} - \sin(e_{\theta i}) & -e_{xi} \\ -K_{y_i}\phi(e_{\theta i})e_{yi} & 1 \end{bmatrix}$$

and

$$g_i(t,e_1,\cdots,e_{i-1}) := \sum_{k=1}^{i-1}\begin{bmatrix}\tilde{v}_k \\ \tilde{\omega}_k\end{bmatrix}, \quad \begin{bmatrix}\tilde{v}_k \\ \tilde{\omega}_k\end{bmatrix} = B_k(t,e)\xi(e_k)$$

with

$$\begin{aligned} B_k(t,e) &:= \begin{bmatrix} K_{xk} & 0 & v_{k-1} & 0 \\ 0 & v_{k-1}K_{y_k}\phi(e_{\theta_k}) & 0 & K_{\theta k} \end{bmatrix} \\ \xi(e_k) &:= \begin{bmatrix} e_{xk} & e_{yk} & \cos(e_{\theta k}) - 1 & e_{\theta k} \end{bmatrix}^\top. \end{aligned}$$

Notice that for each $k \leq i-1$, B_k only depends on $[e_1 \cdots e_{i-1}]$ that is, $\tilde{v}_k$ and $\tilde{\omega}_k$ are independent of e_i hence, the closed-loop equations have the triangular structure:

$$\begin{aligned} \Sigma_n' : \dot{e}_n &= \bar{A}_n(t,e_n)e_n + M_n(t,e_n)g_n(t,e_1,\cdots,e_n) \\ &\;\vdots \\ \Sigma_i \left\{ \begin{aligned} \dot{e}_i &= \bar{A}_i(t,e_i)e_i + M_i(t,e_i)g_i(t,e_1,\cdots,e_{i-1}) \\ &\;\vdots \\ \dot{e}_2 &= \bar{A}_2(t,e_2)e_2 + M_2(t,e_2)g_1(t,e_1) \\ \dot{e}_1 &= \bar{A}_1(t,e_1)e_1 \end{aligned} \right. \end{aligned}$$

Now, for each $i \in [1,n]$, the equation Σ_i' forms a cascaded system with Σ_i hence, we can invoke [24, Lemma 2] and an inductive argument. To that end, we first note that the solutions are uniformly globally bounded for any $n \geq 1$. Firstly, let $n = 2$ and consider the two bottom equations. The two nominal systems $\dot{e}_i = \bar{A}_i(t,e_i)$ with $i \in \{1,2\}$ have exactly the form (11) —note that this is true for any i— hence, by Theorem 1 each of these systems is uniformly globally asymptotically stable at $\{e_i = 0\}$. By [24, Lemma 2] the same holds for the equilibrium $\{(e_1,e_2) = (0,0)\}$ of the system Σ_2. Next, let $n = 3$ and consider the cascaded system composed of Σ_3' with Σ_2. The solutions are uniformly globally bounded and, in view of Theorem 1, the origin for $\dot{e}_3 = \bar{A}_3(t,e_3)$ is uniformly globally asymptotically stable. Hence the same holds for the overall cascaded system. The argument applies for any value of n. ■

Corollary 1 *Under the conditions of Theorem 4 the origin of the system* (20) *in closed loop with* (21) *is exponentially stable in the large on any compact*[2].

IV. SIMULATIONS

To illustrate our results we have performed simulation tests under Simulink™ of Matlab™. We consider a group of four mobile robots following a virtual leader. In this simulation, the desired formation shape of the four mobile robots is a diamond configuration that tracks the trajectory of the virtual leader. See Figure 2. We define the reference velocities v_r and ω_r in a way that the condition on persistency of excitation (12) holds –see Figure 1.

The initial conditions are set to $[x_r(0), y_r(0), \theta_r(0)] = [0,0,0]$, $[x_1(0), y_1(0), \theta_1(0)] = [1,2,4]$, $[x_2(0), y_2(0), \theta_2(0)] = [0,2,2]$, $[x_3(0), y_3(0), \theta_3(0)] = [0,5,1]$ and $[x_4(0), y_4(0), \theta_4(0)] = [2,2,1]$, and we set the control gains to $K_{x_i} = K_{y_i} = K_{\theta_i} = 2$. The formation shape with a certain desired distance between the robots is obtained by setting all desired orientation offsets to zero and defining $[d_{x_{r,1}}, d_{y_{r,1}}] = [0,0]$, $[d_{x_{1,2}}, d_{y_{1,2}}] = [1,0]$ and $[d_{x_{2,3}}, d_{y_{2,3}}] = [-1,1]$ and $[d_{x_{3,4}}, d_{y_{3,4}}] = [0,1]$. See Figure 2.

V. CONCLUSIONS

We solved a specific problem of formation-tracking control of autonomous vehicles via fairly simple control laws and we established strong stability properties for the closed-loop system. Even though our results remain academic they may

[2] See [32] for a definition.

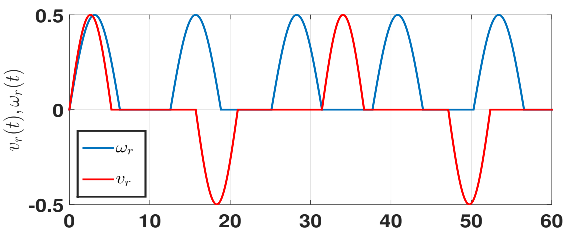

Fig. 1. Reference velocities v_r and ω_r

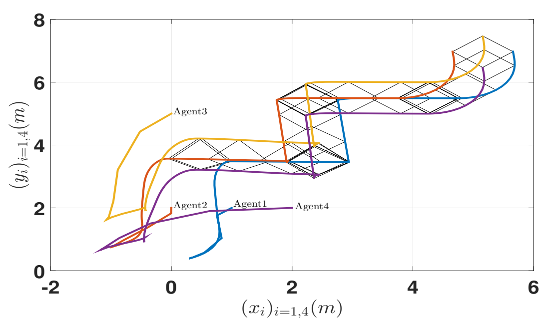

Fig. 2. Illustration of the path-tracking in formation

serve as a starting block to the solution of more realistic engineering problems such as assuming that the communications topology switches, the interconnections are faulty, and the path obstacles must be avoided. For instance, one may consider that robots communicate with different neighbors during distinct time intervals. This problem may be addressed as a problem of stability of switched systems, based upon the statement of Theorem 4. These topics are under current investigation.

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