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Abstract: We give a probabilistic interpretation for the weak Sobolev solution of obstacle problem for semilinear parabolic partial integro-differential equations (PIDE). The results of Léandre [29] about the homeomorphic property for the solution of SDE with jumps are used to construct random test functions for the variational equation for such PIDE. This yields to the natural connection with the associated Reflected Backward Stochastic Differential Equations with jumps (RBSDE), namely the Feynman Kac's formula for the solution of the PIDE.

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1. Introduction

Our main interest is to study the following partial integro-differential equations (in short PIDEs) of parabolic type:

$$(\partial_t + \mathcal{L})u(t,x) + f(t,x,u(t,x),\sigma^*\nabla u(t,x),u(t,x+\beta(x,\cdot)) - u(t,x)) = 0 \qquad (1)$$

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over the time interval $[0,T]$, with a given final condition $u_T = g$, f is a nonlinear function and $\mathcal{L} = \mathcal{K}_1 + \mathcal{K}_2$ is the second order integro-differential operator associated with a jump diffusion which is defined component by component with

$$\begin{aligned} \mathcal{K}_1\varphi(x) &= \sum_{i=1}^d b^i(x)\frac{\partial}{\partial x_i}\varphi(x) + \frac{1}{2}\sum_{i,j=1}^d a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}\varphi(x) \text{ and} \\ \mathcal{K}_2\varphi(x) &= \int_{I\!E}\Big(\varphi(x+\beta(x,e)) - \varphi(x) - \sum_{i=1}^d \beta^i(x,e)\frac{\partial}{\partial x_i}\varphi(x)\Big)\lambda(de), \quad \varphi \in C^2(\mathbb{R}^d). \end{aligned} \tag{2}$$

This class of PIDEs appears in the pricing and hedging contigent claims in financial market including jumps. Matache, von Petersdorff and Schwab [30] have studied a particular case where f is linear in (y,z) and not depends on v (the jump size variable). They have shown the existence and uniqueness of the Sobolev solution of the variational form of some kind of PIDE, arisen from the pricing problem in Lévy market. They used an analytic method in order to derive a numerical schema based on wavelet Galerkin method.

Our nonlinear PIDE includes the case of pricing of contingent claims with constraints in the wealth or portfolio processes. As an example, we may consider the hedging claims with higher interest rate for borrowing in a financial market with jumps. El Karoui, Peng and Quenez [16] have studied this example in a continuous financial market where the non linear source function f is given by $f(t,x,y,\tilde{z}) = r_t y + \theta_t\sigma_t\tilde{z} - (R_t - r_t)(y - \sum_{i=1}^n \tilde{z}^i)$.

In the classical literature, the obstacle problem is related to the variational inequalities which were first studied by Mignot-Puel [36], and then by Michel Pierre [43, 44] (see also Bensoussan-Lions [7]). More recently, the pricing and hedging of american option in the markovian case and the related the obstacle problem for PDEs, was studied by El Karoui et al [13], [?].

In the case where f do not depend of u and ∇u, , the equation (1) becomes a linear parabolic PIDE. If $h : [0,T]\times\mathbb{R}^d \longrightarrow \mathbb{R}$ is a given function such that $h(T,x) \leqslant g(x)$, we may roughly say that the solution of the obstacle problem for (1) is a function $u \in \mathbf{L}^2\big([0,T]; H^1(\mathbb{R}^d)\big)$ such that the following conditions are satisfied in $(0,T)\times\mathbb{R}^d$:

$$\begin{aligned} &(i)\ \ u \geqslant h, \quad dt\otimes dx - \text{a.e.}, \\ &(ii)\ \ \partial_t u + \mathcal{L}u + f \leqslant 0 \\ &(iii)\ \ \big(u-h\big)\big(\partial_t u + \mathcal{L}u + f\big) = 0. \\ &(iv)\ \ u_T = g, \quad dx - \text{a.e.} \end{aligned} \tag{3}$$

The relation (ii) means that the distribution appearing in the LHS of the inequality is a non-positive measure. The relation (iii) is not rigourously stated. We may roughly say that one has $\partial_t u + \mathcal{L}u + f = 0$ on the set $\{u > h\}$.

In the case of obstacle problem for PDE's (when the non local term operator $\mathcal{K}_2$ =0), if one expresses the obstacle problem in terms of variational inequalities one should also ask that the solution has a minimality property (see Mignot-Puel [36] or Bensoussan-Lions [7] p.250). The work of El Karoui et al [13] treats

the obstacle problem for (1) within the framework of backward stochastic differential equations (BSDE in short). Namely the equation (1) is considered with f depending of u and ∇u , $\lambda = 0$ and $\beta = 0$ and the obstacle h is continuous. The solution is represented stochastically as a process and the main new object of this BSDE framework is a continuous increasing process that controls the set $\{u = h\}$. This increasing process determines in fact the measure from the relation (ii). Bally et al [3] (see also Matoussi and Xu [32]) point out that the continuity of this process allows one to extend the classical notion of strong variational solution (see Theorem 2.2 of [7] p.238) and express the solution to the obstacle as a pair (u, ν) where ν equals the LHS of (ii) and is supported by the set $\{u = h\}$. Barles, Buckdahn and Pardoux [4] have provided a probabilistic interpretation for the viscosity solution of (1) by using a forward BSDE with jumps. Situ [47] has studied the Sobolev solution of (1) via an appropriate BSDE with jump, whose method is mainly based on Sobolev Embedding theorem.
More recently, Matoussi and Stoica [34] studied obstacle problem for parabolic quasilinear PDE's and gave a probabilistic interpretation of the reflected measure ν in term of the associated increasing process a component of solution of a reflected BSDE's. They call such potentials and measures, regular potentials, respectively regular measures or Revuz measure. Their method is based on probabilistic quasi-sure analysis.

Michel Pierre [43, 44] has studied a parabolic PDE with obstacle using the parabolic potential as a tool. He proved that the solution uniquely exists and is quasi-continuous with respect to so called analytical capacity. Moreover, he gave a representation of the reflected measure ν in term of the associated regular potential and the approach used is based on analytical quasi-sure analysis. More recently, Denis, Matoussi and Zhang [12] have extended the approach of Michel Pierre [43, 44] for the obstacle problem of quasilinear SPDE.

More precisely, let us consider the final condition to be a fixed function $g \in \mathbf{L}^2(\mathbb{R}^d)$ and the obstacle h be a continuous function $h : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}$. Then the obstacle problem for the equation (1) is defined as a pair (u, ν), where ν is a regular measure concentrated on $\{u = h\}$ and $u \in \mathbf{L}^2\big([0,T] \times R^d; R)\big)$ satisfies the following relations :

$$\begin{aligned}
&(i')\ \ u \geqslant h, \quad d\mathbb{P} \otimes dt \otimes dx - \text{a.e.},\\
&(ii')\ \ \partial_t u(t,x) + \ \mathcal{L}u(t,x) + f(t,x,u(t,x),\sigma^* \nabla u(t,x), u(t,x+\beta(x,\cdot)) - u(t,x)) = -\nu(dt,dx),\\
&(iii')\ \ \nu\big(u > h\big) = 0, \quad a.s.,\\
&(iv')\ \ u_T = g, \quad dx - \text{a.e.}.
\end{aligned} \tag{4}$$

ν represents the quantity which permits to pass from inequality (ii) to equality (ii').
In Section 4, we explain the rigorous sense of the relation (iii') which is based on the probabilistic representation of the measure ν and plays the role of quasi-continuity of u in our context. The main result of our paper is Theorem 2 which ensures the existence and uniqueness of the solution (u, ν) of the obstacle problem for (1) using probabilistic method based on reflected BSDE with jumps. The proof is based on the penalization procedure. We note that the quasi-sure approaches for the PIDE (probabilistic [34] or analytical one [12]) are unsuccessful. **It's not clear for us, until now, how to define the anlytical potential associated to the operator $\mathcal{L}$ specially for the non local operator $\mathcal{K}_2$. Therefore, it's not obvious to define the associated analytical capacity. Thus**, we use the stochastic flow method developed by Bally and Matoussi in [2] for a class of parabolic semilinear SPDE's.
As a preliminary work, we present first the existence and uniqueness of Sobolev solution of PIDE (1) (with-

out obstacle) and provide a probabilistic interpretation by using solution of BSDEs driven by a Brownian motion and an independent random measure. We are concerned with solving our problem by developing a stochastic flow method based on the results of Léandre [29] about the homeomorphic property for the solution of SDE with jumps. The key element in [2] is to use the inversion of stochastic flow which transforms the variational formulation of the PDEs to the associated BSDEs. Thus it plays the same role as Itô's formula in the case of classical solution of PDEs. Note that more recently, in [23] based on stochastic flow arguments, the author shows that the probabilistic equivalent formulation of Dupire's PDE is the Put-Call duality equality in local volatility models including exponential Lévy jumps. Also in [37] and [14], the inversion of stochastic flow technics are used for building a family of forward utilities for a given optimal portfolio.

Our paper is organized as following: in section 2, we first present the basic assumptions and the definitions of the solutions for PIDEs. We provide useful results on stochastic flow associated to the forward SDEs with jumps, then we introduce in this setting a class of random test functions and their semimartingale decomposition. Finally, an equivalence norm result is given in the jump diffusion case. In section 3, we prove the existence and uniqueness results of the solution of our PIDE and give the associated probabilistic interpretation via the FBSDE with jumps. The uniqueness is a consequence of the variational formulation of the PIDE written with a random test functions and the uniqueness of the solution of the FBSDE. The existence of the solution is established by an approximation penalization procedure, a priori estimates and the equivalence norm results. In section 4, we prove existence and uniqueness of the solution of the obstacle problem for the PIDEs. In the Appendix, we first give the proof of the equivalence norm results, then we prove a regularity result for the BSDE solution with respect to the time-state variable (t, x), in order to relate the solution of BSDE to the classical solution of our PIDE. Finally, we give a proof of a technical lemma which is crucial for the existence of the regular measure part of the solution of our obstacle problem for PIDE.

2. Hypotheses and preliminaries

Let $T > 0$ be a finite time horizon and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which is defined two independent processes:

- a d-dimensional Brownian motion $W_t = (W_t^1, \cdots, W_t^d)$;

- a Poisson random measure $\mu = \mu(dt, de)$ on $([0,T] \times I\!\!E, \mathcal{B}([0,T]) \otimes \mathcal{B}_{I\!\!E})$, where $I\!\!E = \mathbb{R}^l \setminus \{0\}$ is equipped with its Borel field $\mathcal{B}_{I\!\!E}$, with compensator $\upsilon(dt, de) = \lambda(de)dt$, such that $\{\tilde{\mu}([0,T] \times A) = (\mu - \upsilon)([0,T] \times A)\}_{t \geqslant 0}$ is a martingale for all $A \in \mathcal{B}_{I\!\!E}$ satisfying $\lambda(A) < \infty$. λ is assumed to be a $\sigma-$ finite measure on $(I\!\!E, \mathcal{B}_{I\!\!E})$ satisfying

$$\int_{I\!\!E} (1 \wedge |e|^2)\lambda(de) < +\infty$$

Denote $\widetilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{B}_{I\!\!E}$ where $\mathcal{P}$ is the predictable σ-field on $\Omega \times [0,T]$.

Let $(\mathcal{F}_t)_{t \geqslant 0}$ be the filtration generated by the above two processes and augmented by the P-null sets of $\mathcal{F}$. Besides let us define:

- $|X|$ the Euclidean norm of a vector X;

- $\mathbf{L}^2(\mathbb{E},\mathcal{B}_{\mathbb{E}},\lambda;\mathbb{R}^n)$ (noted as $\mathbf{L}^2_\lambda$ for convenience) the set of measurable functions from $(\mathbb{E},\mathcal{B}_{\mathbb{E}},\lambda)$ to $\mathbb{R}^n$ endowed with the topology of convergence in measure and for $v\in\mathbf{L}^2(\mathbb{E},\mathcal{B}_{\mathbb{E}},\lambda;\mathbb{R}^n)$

$$\|v\|^2=\int_{\mathbb{E}}|v(e)|^2\lambda(de)\in\mathbb{R}^+\cup\{+\infty\};$$

-$\mathbf{L}^p_n(\mathcal{F}_T)$ the space of n-dimensional $\mathcal{F}_T$-measurable random variables ξ such that

$$\|\xi\|^p_{L^p}:=E(|\xi|^p)<+\infty;$$

-$\mathcal{H}^p_{n\times d}([0,T])$ the space of $\mathbb{R}^{n\times d}$-valued $\mathcal{P}$-measurable process $Z=(Z_t)_{t\leqslant T}$ such that

$$\|Z\|^p_{\mathcal{H}^p}:=E[(\int_0^T|Z_t|^2dt)^{p/2}]<+\infty;$$

- $\mathcal{S}^p_n([0,T])$ the space of n-dimensional $\mathcal{F}_t$-adapted càdlàg processes $Y=(Y_t)_{t\leqslant T}$ such that

$$\|Y\|^p_{\mathcal{S}^p}:=E[\sup_{t\leqslant T}|Y_t|^p]<+\infty;$$

- $\mathcal{A}^p_n(t,T)$ the space of n-dimensional $\mathcal{F}_t$-adapted non-decreasing càdlàg processes $K=(K_t)_{t\leqslant T}$ such that

$$\|K\|^p_{\mathcal{A}^p}:=E[|K_T|^p]<+\infty;$$

-$\mathcal{L}^p_n([0,T])$ the space of $\mathbb{R}^n$-valued $\widetilde{\mathcal{P}}$-measurable mappings $V(\omega,t,e)$ such that

$$\|V\|^p_{\mathcal{L}^p}:=E[(\int_0^T\|V_t\|^2dt)^{p/2}]=E[(\int_0^T\int_{\mathbb{E}}|V_t(e)|^2\lambda(de)dt)^{p/2}]<+\infty.$$

- $C^k_{l,b}(\mathbb{R}^p,\mathbb{R}^q)$ the set of C^k-functions which grow at most linearly at infinity and whose partial derivatives of order less than or equal to k are bounded.

- $\mathbf{L}^2_\rho\left(\mathbb{R}^d\right)$ will be the basic Hilbert space of our framework. We employ the usual notation for its scalar product and its norm,

$$(u,v)_\rho=\int_{\mathbb{R}^d}u\left(x\right)v\left(x\right)\rho(x)dx,\ \|u\|_2=\left(\int_{\mathbb{R}^d}u^2\left(x\right)\rho(x)dx\right)^{\frac{1}{2}}.$$

where ρ is a continuous positive and integrable weight function. We assume additionally that $\frac{1}{\rho}$ is locally integrable.

In general, we shall use the notation

$$(u,v)=\int_{\mathbb{R}^d}u(x)v(x)\,dx,$$

where u, v are measurable functions defined in $\mathbb{R}^d$ and $uv\in\mathbf{L}^1(\mathbb{R}^d)$.

Our evolution problem will be considered over a fixed time interval $[0,T]$ and the norm for an element of $\mathbf{L}^2_\rho\left([0,T]\times\mathbb{R}^d\right)$ will be denoted by

$$\|u\|_{2,2}=\left(\int_0^T\int_{\mathbb{R}^d}|u(t,x)|^2\rho(x)dxdt\right)^{\frac{1}{2}}.$$

We usually omit the subscript when $n=1$. We assume the following hypotheses :

(A1) g belongs to $\mathbf{L}^2_\rho(\mathbb{R}^d)$;

(A2) $f:[0,T]\times\mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbf{L}^2(\mathbb{E},\mathcal{B}_{\mathbb{E}},\lambda;\mathbb{R}^m)\to\mathbb{R}^m$ is measurable in (t,x,y,z,v) and satisfies

$f^0 \in \mathbf{L}^2_\rho\left([0,T]\times\mathbb{R}^d\right)$ where $f^0 := f(\cdot,\cdot,0,0,0)$.

(A3) f satisfies Lipschitz condition in (y,z,v), i.e., there exists a constant C such that for any $(t,x)\in[0,T]\times\mathbb{R}^d$ and $(y,z,v),(y',z',v')\in\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbf{L}^2(I\!E,\mathcal{B}_{I\!E},\lambda;\mathbb{R}^m)$:

$$|f(t,x,y,z,v)-f(t,x,y',z',v')|\leqslant C(|y-y'|+|z-z'|+\|v-v'\|);$$

(A4) $b\in C^3_{l,b}(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma\in C^3_{l,b}(\mathbb{R}^d;\mathbb{R}^{d\times d})$, $\beta:\mathbb{R}^d\times I\!E\to\mathbb{R}^d$ be measurable and for all $e\in I\!E$, $\beta(\cdot,e)\in C^3_{l,b}(\mathbb{R}^d;\mathbb{R}^d)$, and for some $K>0$ and for all $x\in\mathbb{R}^d$, $e\in I\!E$,

$$|\beta(x,e)|\leqslant K(1\wedge|e|),\quad |D^\alpha\beta(x,e)|\leqslant K(1\wedge|e|)\text{ for }1\leqslant|\alpha|\leqslant 3,$$

where $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_d)$ is a multi-index and $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_d$. D^α is the differential operator $D^\alpha=\dfrac{\partial^{|\alpha|}}{(\partial^{\alpha_1}x_1)(\partial^{\alpha_2}x_2)\cdots(\partial^{\alpha_d}x_d)}$.

2.1. Weak formulation for the partial differential-integral equations

The space of test functions which we employ in the definition of weak solutions of the evolution equations (1) is $\mathcal{D}_T=\mathcal{C}^\infty([0,T])\otimes\mathcal{C}^\infty_c\left(\mathbb{R}^d\right)$, where $\mathcal{C}^\infty([0,T])$ denotes the space of real functions which can be extended as infinite differentiable functions in the neighborhood of $[0,T]$ and $\mathcal{C}^\infty_c\left(\mathbb{R}^d\right)$ is the space of infinite differentiable functions with compact support in $\mathbb{R}^d$. We denote the space of solutions by

$$\mathcal{H}_T:=\{u\in\mathbf{L}^2_\rho([0,T]\times\mathbb{R}^d)\;\big|\;\sigma^*\nabla u\in\mathbf{L}^2_\rho([0,T]\times\mathbb{R}^d)\}$$

endowed with the norm

$$\|u\|_{\mathcal{H}_T}=\Big(\int_{\mathbb{R}^d}\int_0^T[|u(s,x)|^2+|(\sigma^*\nabla u)(s,x)|^2]ds\rho(x)dx\Big)^{1/2},$$

where we denote the gradient by $\nabla u(t,x)=\big(\partial_1 u(t,x),\cdots,\partial_d u(t,x)\big)$.

Definition 1. *We say that $u\in\mathcal{H}_T$ is a Sobolev solution of PIDE (1) if the following relation holds, for each $\phi\in\mathcal{D}_T$,*

$$\begin{aligned}&\int_t^T(u(s,x),\partial_s\phi(s,x))ds+(u(t,x),\phi(t,x))-(g(x),\phi(T,x))-\int_t^T(u(s,x),\mathcal{L}^*\phi(s,x))ds\\&=\int_t^T(f(s,x,u(s,x),\sigma^*\nabla u(s,x),u(s,x+\beta(x,\cdot))-u(s,x)),\phi(s,x))ds.\end{aligned}\tag{5}$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$. We denote by $u:=\mathcal{U}(g,f)$ such a solution.

2.2. Stochastic flow of diffeomorphism and random test functions

In this section, we shall study the stochastic flow associated to the forward jump diffusion component. The main motivation is to generalize in the jump setting the flow technics which was first introduced in [2] for the study of semilinear PDE's. Let $(X_{t,s}(x))_{t\leqslant s\leqslant T}$ be the strong solution of the equation:

$$X_{t,s}(x)=x+\int_t^s b(X_{t,r}(x))dr+\int_t^s\sigma(X_{t,r}(x))dW_r+\int_t^s\int_{I\!E}\beta(X_{t,r-}(x),e)\widetilde{\mu}(dr,de).\tag{6}$$

The existence and uniqueness of this solution was proved in Fujiwara and Kunita [17]. Moreover, we have the following properties (see Theorem 2.2 and Theorem 2.3 in [17]):

Proposition 1. *For each $t > 0$, there exists a version of $\{X_{t,s}(x);\, x \in \mathbb{R}^d,\, s \geqslant t\}$ such that $X_{t,s}(\cdot)$ is a $C^2(\mathbb{R}^d)$-valued càdlàg process. Moreover:*

(i) $X_{t,s}(\cdot)$ and $X_{0,s-t}(\cdot)$ have the same distribution, $0 \leqslant t \leqslant s$;

(ii) $X_{t_0,t_1}, X_{t_1,t_2}, \ldots, X_{t_{n-1},t_n}$ are independent, for all $n \in \mathbb{N}$, $0 \leqslant t_0 < t_1 < \cdots < t_n$;

(iii) $X_{t,r}(x) = X_{s,r} \circ X_{t,s}(x)$, $0 \leqslant t < s < r$.

Furthermore, for all $p \geqslant 2$, there exists M_p such that for all $0 \leqslant t < s$, $x, x' \in \mathbb{R}^d$, $h, h' \in \mathbb{R}\backslash\{0\}$,

$$
\begin{aligned}
&E(\sup_{t \leqslant r \leqslant s} |X_{t,r}(x) - x|^p) \leqslant M_p(s-t)(1+|x|^p),\\
&E(\sup_{t \leqslant r \leqslant s} |X_{t,r}(x) - X_{t,r}(x') - (x-x')|^p) \leqslant M_p(s-t)(|x-x'|^p),\\
&E(\sup_{t \leqslant r \leqslant s} |\Delta_h^i[X_{t,r}(x) - x]|^p) \leqslant M_p(s-t),\\
&E(\sup_{t \leqslant r \leqslant s} |\Delta_h^i X_{t,r}(x) - \Delta_{h'}^i X_{t,r}(x')|^p) \leqslant M_p(s-t)(|x-x'|^p + |h-h'|^p),
\end{aligned}
$$

where $\Delta_h^i g(x) = \frac{1}{h}(g(x+he_i) - g(x))$, and $(e_1, \cdots, e_d)$ is an orthonormal basis of $\mathbb{R}^d$.

It is also known that the stochastic flow solution of a continuous SDE satisfies the homeomorphic property (see Bismut [8], Kunita [26], [27]). But this property fails for the solution of SDE with jumps in general. P.-A. Meyer in [35] (Remark p.111), gave a counterexample with the following exponential equation:

$$
X_{0,t}(x) = x + \int_0^t X_{0,s-} dZ_s
$$

where Z is semimartingale, $Z_0 = 0$, such that Z has a jump of size -1 at some stopping time τ, $\tau > 0$ a.s. Then all trajectories of X, starting at any initial value x, become zero at τ and stay there after τ. This may be seen trivially by the explicit form of the solution given by the Doléans-Dade exponential:

$$
X_{0,t}(x) = x \exp\big(Z_t - \frac{1}{2}[Z,Z]_t^c\big) \prod_{0<s \leqslant t} (1 + \Delta Z_s) e^{-\Delta Z_s}.
$$

In the general setting of non-linear SDE, at the jump time τ, the solution jumps from $X_{0,\tau-}(x)$ to $X_{0,\tau-}(x) + \beta(X_{0,\tau-}(x))$. Léandre [29] gave a necessary and sufficient condition under which the homeomorphic property is preserved at the jump time, namely, for each $e \in I\!\!E$, the maps $H_e : x \mapsto x + \beta(x,e)$ should be one to one and onto. One can read also Fujiwara and Kunita [17] and Protter [45] for more details on the subject. Therefore, we assume additionally that, for each $e \in I\!\!E$, *the linkage operator*:

(A5) $\mathrm{H}_e : x \mapsto x + \beta(x,e)$ is a C^2-diffeomorphism.

We denote by H_e^{-1} the inverse map of H_e, and set $h(x,e) := x - \mathrm{H}_e^{-1}(x)$. We have the following result where the proof can be found in [28] (Theorem 3.13, p.359):

Proposition 2. *Assume the assumptions **(A4)** and **(A5)** hold. Then $\{X_{t,s}(x); x \in \mathbb{R}^d\}$ is a C^2-diffeomorphism a.s. stochastic flow. Moreover the inverse of the flow satisfies the following backward SDE*

$$
\begin{aligned}
X_{t,s}^{-1}(y) = y &- \int_t^s \widehat{b}(X_{r,s}^{-1}(y))dr - \int_t^s \sigma(X_{r,s}^{-1}(y))\overleftarrow{dW}_r - \int_t^s \int_{I\!\!E} \beta(X_{r,s}^{-1}(y), e)\widetilde{\mu}(\overleftarrow{dr}, de)\\
&+ \int_t^s \int_{I\!\!E} \widehat{\beta}(X_{r,s}^{-1}(y), e)\mu(\overleftarrow{dr}, de).
\end{aligned} \tag{7}
$$

for any $t < s$, where

$$
\widehat{b}(x) = b(x) - \sum_{i,j} \frac{\partial \sigma^j(x)}{\partial x_i}\sigma^{ij}(x) \quad \textit{and} \quad \widehat{\beta}(x,e) = \beta(x,e) - h(x,e). \tag{8}
$$

The explicit form (7) will be used in the proof of the equivalence of norms (Proposition 3).

Remark 1. *In (7), the three terms* $\int_t^s \sigma(X_{r,s}^{-1}(y))\overleftarrow{dW}_r$, $\int_t^s \int_{\mathbb{E}} \beta(X_{r,s}^{-1}(y),e)\widetilde{\mu}(\overleftarrow{dr},de)$ *and* $\int_t^s \int_{\mathbb{E}} \widehat{\beta}(X_{r,s}^{-1}(y),e)\mu(\overleftarrow{dr},de)$ *are backward Itô integrals. We refer the readers to literature [28] for the definition ([28] p. 358). For convenience, we give the definition of the backward Itô integral with respect to a Brownian motion. Let* $f(r)$ *be a right continuous backward adapted process, then the backward Itô integral is defined by*

$$\int_t^s f(r)\overleftarrow{dW}_r := \lim_{|\Pi|\to 0}\sum_k f(t_{k+1})(W_{t_{k+1}} - W_{t_k}),$$

where $\Pi = \{t = t_0 < t_1 < \cdots < t_n = s\}$ *are partitions of the interval* $[t,s]$. *The other two terms can be defined similarly. Note that the inverse flow* $X_{r,s}^{-1}$ *is backward adapted, so we may define the backward integrals such as* $\int_t^s \sigma(X_{r,s}^{-1}(y))\overleftarrow{dW}_r$ *etc..*

Remark 2. *In the paper of Ouknine and Turpin [38], the authors have weakened the regularity of the coefficients* b *and* σ *of the diffusion, but added additional boundedness on them. Since this improvement is not essential and the same discussion is also valid for our case, we omit it.*

We denote by $J(X_{t,s}^{-1}(x))$ the determinant of the Jacobian matrix of $X_{t,s}^{-1}(x)$, which is positive and $J(X_{t,t}^{-1}(x)) = 1$. For $\phi \in C_c^\infty(\mathbb{R}^d)$, we define a process $\phi_t : \Omega \times [t,T] \times \mathbb{R}^d \to \mathbb{R}$ by

$$\phi_t(s,x) := \phi(X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x)). \tag{9}$$

We know that for $v \in \mathbf{L}^2(\mathbb{R}^d)$, the composition of v with the stochastic flow is

$$(v \circ X_{t,s}(\cdot), \phi) := (v, \phi_t(s,\cdot)).$$

In fact, by a change of variable, we have

$$(v \circ X_{t,s}(\cdot), \phi) = \int_{\mathbb{R}^d} v(X_{t,s}(x))\phi(x)dx = \int_{\mathbb{R}^d} v(y)\phi(X_{t,s}^{-1}(y))J(X_{t,s}^{-1}(y))dy = (v, \phi_t(s,\cdot)).$$

Since $(\phi_t(s,x))_{t\leqslant s}$ is a process, we may not use it directly as a test function because $\int_t^T (u(s,\cdot), \partial_s\phi_t(s,\cdot))$ has no sense. However $\phi_t(s,x)$ is a semimartingale and we have the following decomposition of $\phi_t(s,x)$:

Lemma 1. *For every function* $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\phi_t(s,x) &= \phi(x) + \int_t^s \mathcal{L}^*\phi_t(r,x)dr - \sum_{j=1}^d \int_t^s \left(\sum_{i=1}^d \frac{\partial}{\partial x_i}(\sigma^{ij}(x)\phi_t(r,x))\right) dW_r^j \\ &\quad + \int_t^s \int_{\mathbb{E}} \mathcal{A}_e^*\phi_t(r-,x)\widetilde{\mu}(dr,de),\end{aligned} \tag{10}$$

where $\mathcal{A}_e u(t,x) = u(t,\mathrm{H}_e(x)) - u(t,x)$, $\mathcal{A}_e^* u(t,x) = u(t,\mathrm{H}_e^{-1}(x))J(\mathrm{H}_e^{-1}(x)) - u(t,x)$ *and* $\mathcal{L}^*$ *is the adjoint operator of* $\mathcal{L}$.

Proof : Assume that $v \in C_c^\infty(\mathbb{R}^d)$. Applying the change of variable $y = X_{t,s}^{-1}(x)$, we can get

$$\begin{aligned}\int_{\mathbb{R}^d} v(x)(\phi_t(s,x) - \phi(x))dx &= \int_{\mathbb{R}^d} v(x)(\phi(X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x)) - \phi(X_{t,t}^{-1}(x))J(X_{t,t}^{-1}(x))dx \\ &= \int_{\mathbb{R}^d} (v(X_{t,s}(y))\phi(y) - v(y)\phi(y))dy \\ &= \int_{\mathbb{R}^d} \phi(y)(v(X_{t,s}(y)) - v(y))dy.\end{aligned}$$

As v is smooth enough, using Itô's formula for $v(X_{t,s}(y))$, we have

$$\begin{aligned} v(X_{t,s}(y)) - v(y) &= \int_t^s \mathcal{L}v(X_{t,r-}(y))dr + \int_t^s \sum_{i=1}^d \frac{\partial v}{\partial x_i}(X_{t,r}(y)) \sum_{j=1}^d \sigma^{ij}(X_{t,r}(y))dW_r^j \\ &+ \int_t^s \int_{\mathbb{E}} \mathcal{A}_e v(X_{t,r-}(y))\widetilde{\mu}(dr, de). \end{aligned}$$

Therefore,

$$\begin{aligned} & \int_{\mathbb{R}^d} v(x)(\phi_t(s,x) - \phi(x))dx \\ = & \int_{\mathbb{R}^d} \phi(y)\Big\{ \int_t^s \mathcal{L}v(X_{t,r}(y))dr + \int_t^s \sum_{i=1}^d \frac{\partial v}{\partial x_i}(X_{t,r}(y)) \sum_{j=1}^d \sigma^{ij}(X_{t,r}(y))dW_r^j \Big\}dy \\ + & \int_{\mathbb{R}^d} \phi(y) \int_t^s \int_{\mathbb{E}} \mathcal{A}_e v(X_{t,r-}(y))\widetilde{\mu}(dr, de)dy. \end{aligned}$$

Since the first term has been dealt in [2], and the adjoint operator of $\mathcal{L}$ exists thanks to [30], we focus only on the second term. Using the stochastic Fubini theorem and the change of variable $x = X_{t,r}(y)$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \phi(y) \int_t^s \int_{\mathbb{E}} \mathcal{A}_e v(X_{t,r-}(y))\widetilde{\mu}(dr, de)dy &= \int_t^s \int_{\mathbb{E}} \int_{\mathbb{R}^d} \phi(y)\mathcal{A}_e v(X_{t,r-}(y))dy\widetilde{\mu}(dr, de) \\ &= \int_t^s \int_{\mathbb{E}} \int_{\mathbb{R}^d} \phi_t(r-, x)\mathcal{A}_e v(x)dx\widetilde{\mu}(dr, de). \end{aligned}$$

Finally, we use the change of variable $y = \mathrm{H}_e^{-1}(x)$ in the right hand side of the previous expression

$$\begin{aligned} & \int_t^s \int_{\mathbb{E}} \int_{\mathbb{R}^d} \phi_t(r-, x)\mathcal{A}_e v(x)dx\widetilde{\mu}(dr, de) \\ = & \int_t^s \int_{\mathbb{E}} \int_{\mathbb{R}^d} \phi_t(r-, x)\big(v(\mathrm{H}_e(x)) - v(x)\big)\, dx\widetilde{\mu}(dr, de) \\ = & \int_t^s \int_{\mathbb{E}} \int_{\mathbb{R}^d} v(x)\mathcal{A}_e^* \phi_t(r-, x)dx\widetilde{\mu}(dr, de). \end{aligned}$$

Since v is an arbitrary function, the lemma is proved. □

We also need equivalence of norms result which plays an important role in the proof of the existence of the solution for PIDE as a connection between the functional norms and random norms. For continuous SDEs, this result was first proved by Barles and Lesigne [5] by using an analytic method. In [2], the authors have proved the result with a probabilistic method. Note that Klimisiak [24] have extended this estimes for Markov process associated to a non-homogenious divergence operator. The following result generalize Proposition 5.1 in [2] (see also [5]) in the case of a diffusion process with jumps, and the proof will be given in Appendix 5.1.

Proposition 3. *There exists two constants $c > 0$ and $C > 0$ such that for every $t \leqslant s \leqslant T$ and $\varphi \in L^1(\mathbb{R}^d, dx)$,*

$$c\int_{\mathbb{R}^d} |\varphi(x)|\rho(x)dx \leqslant \int_{\mathbb{R}^d} E(|\varphi(X_{t,s}(x))|)\rho(x)dx \leqslant C\int_{\mathbb{R}^d} |\varphi(x)|\rho(x)dx. \tag{11}$$

Moreover, for every $\Psi \in L^1([0,T] \times \mathbb{R}^d, dt \otimes dx)$,

$$c\int_{\mathbb{R}^d} \int_t^T |\Psi(s,x)|ds\rho(x)dx \leqslant \int_{\mathbb{R}^d} \int_t^T E(|\Psi(s, X_{t,s}(x))|)ds\rho(x)dx \leqslant C\int_{\mathbb{R}^d} \int_t^T |\Psi(s,x)|ds\rho(x)dx. \tag{12}$$

We give now the following result which allows us to link by a natural way the solution of PIDE with the associated BSDE. Roughly speaking, if we choose in the variational formulation (5) the random functions

$\phi_t(\cdot,\cdot)$ defined by (9), as a test functions, then we obtain the associated BSDE. In fact, this result plays the same role as Itô's formula used in [39] and [42] (see [39], Theorem 3.1, p. 20) to relate the solution of some semilinear PDE's with the associated BSDE:

Proposition 4. *Assume that all the previous assumptions hold. Let $u \in \mathcal{H}_T$ be a weak solution of PIDE(1), then for $s \in [t,T]$ and $\phi \in C_c^\infty(\mathbb{R}^d)$,*

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_s^T u(r,x)d\phi_t(r,x)dx + (u(s,x),\phi_t(s,x)) - (g(x),\phi_t(T,x)) - \int_s^T (u(r,\cdot),\mathcal{L}^*\phi_t(r,\cdot))dr \\ &= \int_{\mathbb{R}^d}\int_s^T f(r,x,u(r,x),\sigma^*\nabla u(r,x),u(r,x+\beta(x,\cdot))-u(r,x))\phi_t(r,x)drdx. \quad a.s.\end{aligned} \tag{13}$$

where $\int_{\mathbb{R}^d}\int_s^T u(r,x)d\phi_t(r,x)dx$ is well defined thanks to the semimartingale decomposition result (Lemma 1).

Remark 3. *Note that $\phi_t(r,x)$ is $\mathbb{R}$-valued. We consider that in (13), the equality holds for each component of u.*

Remark 4. *Under Brownian framework, this proposition is first proved by Bally and Matoussi in [2] for linear case via the polygonal approximation for Brownian motion (see Appendix A in [2] for more details). In fact, thanks especially to the fact that λ is finite, we can make the similar approximation for Itô-Lévy processes only by approximating polygonally the Brownian motion, the proof of this proposition follows step by step the proof of Proposition 2.3 in [2] (pp. 156), so we omit it.*

3. Sobolev solutions for parabolic semilinear PIDEs

In this section, we consider the PIDE (1) under assumptions (A1)-(A5). Moreover, we consider the following decoupled forward backward stochastic differential equation (FBSDE in short) :

$$\begin{cases} X_{t,s}(x) = x + \int_t^s b(X_{t,r}(x))dr + \int_t^s \sigma(X_{t,r}(x))dW_r + \int_t^s\int_{\mathbb{E}} \beta(X_{t,r-}(x),e)\widetilde{\mu}(dr,de); \\ Y_s^{t,x} = g(X_{t,T}(x)) + \int_s^T f(r,X_{t,r}(x),Y_r^{t,x},Z_r^{t,x},V_r^{t,x})dr \\ \qquad - \int_s^T Z_r^{t,x}dW_r - \int_s^T\int_{\mathbb{E}} V_r^{t,x}(e)\widetilde{\mu}(dr,de). \end{cases} \tag{14}$$

According to Proposition 5.4 in [11] which deals with of reflected BSDE, we know that (14) has a unique solution. Moreover, we have the following estimate of the solution.

Proposition 5. *There exists a constant $c > 0$ such that, for any $s \in [t,T]$:*

$$\sup_{s\in[t,T]} E\big[\|Y_s^{t,\cdot}\|_2^2\big] + E\Big[\int_t^T \|Z_s^{t,\cdot}\|_2^2 ds + \int_t^T\int_{\mathbb{E}} |V_s^{t,\cdot}(e)|_2^2\lambda(de)ds\Big] \leqslant c\big[\|g\|_2^2 + \int_t^T \|f_s^0\|_2^2 ds\big]. \tag{15}$$

Our main result in this section is the following where the proof will be given in Appendix 5.3:

Theorem 1. *Assume that **(A1)-(A5)** hold. There exists a unique solution $u \in \mathcal{H}_T$ of the PIDE (1). Moreover, we have the probabilistic representation of the solution: $u(t,x) = Y_t^{t,x}$, where $(Y_s^{t,x}, Z_s^{t,x}, V_s^{t,x})$ is the solution of BSDE (14) and, we have, $ds \otimes dP \otimes \rho(x)dx - a.e.$,*

$$\begin{aligned}&Y_s^{t,x} = u(s,X_{t,s}(x)), \quad Z_s^{t,x} = (\sigma^*\nabla u)(s,X_{t,s}(x)), \\ &V_s^{t,x}(\cdot) = u(s,X_{t,s-}(x)+\beta(X_{t,s-}(x),\cdot)) - u(s,X_{t,s-}(x)).\end{aligned} \tag{16}$$

Remark 5. *Since $u \in \mathcal{H}_T$, u and $v = (\sigma^* \nabla u)$ are elements in $L^2_\rho([0,T] \times \mathbb{R}^d)$ and they are determined $\rho(x)dx$ a.e., but because of the equivalence of norms, there is no ambiguity in the definition of $u(s, X_{t,s}(x))$ and the others terms of* (16).

Remark 6. *This stochastic flow method can be generalized to the study of Sobolev solution of stochastic partial integro-differential equations (SPIDEs for short) without essential difficulties (see e.g. [2] for Brownian framewrok). More precisely, as the authors have done in [2, 40], by introducing an appropriate backward doubly stochastic differential equation (BDSDE for short) with jump, we can provide a probabilistic interpretation for Sobolev solution of an SPIDE by the solution of the BDSDE with jump.*

4. Obstacle problem for PIDEs

In this part, we will study the obstacle problem (4) with obstacle function h, where we restrict our study in the one dimensional case ($n = 1$). We shall assume the following hypothesis on the obstacle:

(A6) $h \in C([0,T] \times \mathbb{R}^d; \mathbb{R})$ and there exit $\iota,\ \kappa > 0$ such that $|h(t,x)| \leqslant \iota(1 + |x|^\kappa)$, for all $x \in \mathbb{R}^d$.

We first introduce the reflected BSDE with jumps (RBSDE with jumps for short) associated with (g, f, h) which has been studied by Hamadène and Ouknine [19]:

$$\begin{cases} Y^{t,x}_s = g(X_{t,T}(x)) + \displaystyle\int_s^T f(r, X_{t,r}(x), Y^{t,x}_r, Z^{t,x}_r, V^{t,x}_r)dr + K^{t,x}_T - K^{t,x}_s \\ \qquad\qquad - \displaystyle\int_s^T Z^{t,x}_r dW_r - \int_s^T \int_{\mathbb{E}} V^{t,x}_r(e)\widetilde{\mu}(dr, de),\ P\text{-}a.s.,\ \forall\, s \in [t,T] \\ Y^{t,x}_s \geqslant L^{t,x}_s, \quad \displaystyle\int_t^T (Y^{t,x}_s - L^{t,x}_s) dK^{t,x}_s = 0,\ P\text{-}a.s. \end{cases} \tag{17}$$

The obstacle process $L^{t,x}_s = h(s, X_{t,s}(x))$ is a càdlàg process which has only inaccessible jumps since h is continuous and $(X_{t,s}(x))_{t \leqslant s \leqslant T}$ admits inaccessible jumps. Moreover, using assumption **(A1)** and **(A2)** and equivalence of norm results (11) and (12), we get

$$g(X_{t,T}(x)) \in \mathbf{L}^2(\mathcal{F}_T), \text{ and } f(s, X_{t,s}(x), 0, 0, 0) \in \mathcal{H}^2_d(t,T).$$

Therefore according to [19], there exists a unique quadruple $(Y^{t,x}, Z^{t,x}, V^{t,x}, K^{t,x}) \in \mathcal{S}^2(t,T) \times \mathcal{H}^2_d(t,T) \times \mathcal{L}^2(t,T) \times \mathcal{A}^2(t,T)$ solution of the RBSDE with jumps (17).

More precisely, we consider the following definition of weak solutions for the obstacle problem (4):

Definition 2. *We say that (u, ν) is the weak solution of the PIDE with obstacle associated to (g, f, h), if*
(i) $\|u\|^2_{\mathcal{H}_T} < \infty$, $u \geqslant h$, and $u(T,x) = g(x)$,
(ii) ν is a positive Radon regular measure in the following sense, i.e. for every measurable bounded and positive functions ϕ and ψ,

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_t^T \phi(s, X^{-1}_{t,s}(x)) J(X^{-1}_{t,s}(x)) \psi(s,x) 1_{\{u=h\}}(s,x) \nu(ds, dx) \\ &= \int_{\mathbb{R}^d} \int_t^T \phi(s,x) \psi(s, X_{t,s}(x)) dK^{t,x}_s dx,\ a.s.. \end{aligned} \tag{18}$$

where $(Y_s^{t,x}, Z_s^{t,x}, V_s^{t,x}, K_s^{t,x})_{t\leqslant s\leqslant T}$ *is the solution of RBSDE with jumps (17) and such that* $\int_0^T \int_{\mathbb{R}^d} \rho(x)\nu(dt,dx) < \infty$,

(iii) for every $\phi \in \mathcal{D}_T$

$$\begin{aligned}
&\int_t^T \int_{\mathbb{R}^d} u(s,x)\partial_s\phi(s,x)dxds + \int_{\mathbb{R}^d} (u(t,x)\phi(t,x) - g(x)\phi(T,x))dx + \int_t^T \int_{\mathbb{R}^d} (u, \mathcal{L}^*\phi)dxds \\
&= \int_t^T \int_{\mathbb{R}^d} f(s,x,u,\sigma^*\nabla u, u(s,x+\beta(x,\cdot)) - u(s,x))\phi(s,x)dxds \\
&+ \int_t^T \int_{\mathbb{R}^d} \phi(s,x)1_{\{u=h\}}(s,x)\nu(ds,dx). \qquad (19)
\end{aligned}$$

First, we give a weak Itô's formula similar to the one given in Proposition 4. This result is essential to show the link between a Sobolev solution to the obstacle problem and the associated reflected BSDE with jumps, which in turn insures the uniqueness of the solution. The proof of this proposition is the same as Proposition 4.

Proposition 6. *Assume that conditions **(A1)-(A6)** hold and* $\rho(x) = (1+|x|)^{-p}$ *with* $p \geqslant \gamma$ *where* $\gamma = \kappa + d + 1$. *Let* $u \in \mathcal{H}_T$ *be a weak solution of PIDE with obstacle associated to* (g,f,h), *then for* $s \in [t,T]$ *and* $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_s^T u(r,x)d\phi_t(r,x)dx + (u(s,\cdot), \phi_t(s,\cdot)) - (g(\cdot), \phi_t(T,\cdot)) - \int_s^T (u(r,\cdot), \mathcal{L}^*\phi_t(r,\cdot))dr \\
&= \int_{\mathbb{R}^d} \int_s^T f(r,x,u(r,x),\sigma^*\nabla u(r,x), u(r,x+\beta(x,\cdot)) - u(r,x))\phi_t(r,x)drdx \qquad (20) \\
&+ \int_{\mathbb{R}^d} \int_s^T \phi_t(r,x)1_{\{u=h\}}(r,x)\nu(dr,dx). \quad a.s.
\end{aligned}$$

where $\int_{\mathbb{R}^d} \int_s^T u(r,x)d\phi_t(r,x)dx$ *is well defined thanks to the semimartingale decomposition result (Lemma 1).*

The main result of this section is the following

Theorem 2. *Assume that conditions **(A1)-(A6)** hold and* $\rho(x) = (1+|x|)^{-p}$ *with* $p \geqslant \gamma$ *where* $\gamma = \kappa + d + 1$. *There exists a weak solution* (u,ν) *of the PIDE with obstacle (4) associated to* (g,f,h) *such that,* $ds \otimes dP \otimes \rho(x)dx - a.e.$,

$$\begin{aligned}
&Y_s^{t,x} = u(s, X_{t,s}(x)), Z_s^{t,x} = (\sigma^*\nabla u)(s, X_{t,s}(x)), \\
&V_s^{t,x}(\cdot) = u(s, X_{t,s-}(x) + \beta(X_{t,s-}(x),\cdot)) - u(s, X_{t,s-}(x)),\ a.s.. \qquad (21)
\end{aligned}$$

Moreover, the reflected measure ν *is a regular measure in the sense of the definition (ii) and satisfying the probabilistic interpretation* (18).

If $(\overline{u}, \overline{\nu})$ *is another solution of the PIDE with obstacle(4) such that* $\overline{\nu}$ *satisfies (18) with some* $\overline{K}$ *instead of* K, *where* $\overline{K}$ *is a continuous process in* $\mathcal{A}^2(t,T)$, *then* $\overline{u} = u$ *and* $\overline{\nu} = \nu$.

In other words, there is a unique Randon regular measure with support $\{u = h\}$ *which satisfies (18).*

Remark 7. *The expression (18) gives us the probabilistic interpretation (Feymamn-Kac's formula) for the measure* ν *via the nondecreasing process* $K^{t,x}$ *of the RBSDE with jumps. This formula was first introduced in Bally et al. [3] (see also [32]). Here we generalize their results to the case of PIDE's.*

From Lemma 3.1 in [11], we know that if we have more information on the obstacle L, we can give a more explicit representation for the processes K. Then as a result of the above theorem, we have when h

is smooth enough, the reflected measure ν is Lebesgue absolute continuous, moreover there exist a unique $\widetilde{\nu}$ and a measurable function $(\alpha_s)_{s\,\geqslant\,0}$ such that $\nu(ds,dx)=\alpha_s\widetilde{\nu}_s(dx)ds$.

Proof. **a) Existence**: The existence of a solution will be proved in two steps. For the first step, we suppose that f does not depend on y,z,w, then we are able to apply the usual penalization method. In the second step, we study the case when f depends on y,z,w with the result obtained in the first step.

Step 1 : We will use the penalization method. For $n\in\mathbb{N}$, we consider for all $s\in[t,T]$,

$$\begin{aligned}Y_s^{n,t,x}&=g(X_{t,T}(x))+\int_s^T f(r,X_{t,r}(x))dr+n\int_s^T(Y_r^{n,t,x}-h(r,X_{t,r}(x)))^-dr\\&\quad-\int_s^T Z_r^{n,t,x}dW_r-\int_s^T\int_{\mathbb{E}}V_r^{n,t,x}(e)\widetilde{\mu}(dr,de).\end{aligned}$$

From Theorem (1) in section 3, we know that $u_n(t,x):=Y_t^{n,t,x}$, is solution of the PIDE(g,f_n), where $f_n(t,x,y)=f(t,x)+n(y-h(t,x))^-$, i.e. for every $\phi\in\mathcal{D}_T$

$$\begin{aligned}&\int_t^T(u^n(s,\cdot),\partial_s\phi(s,\cdot))ds+(u^n(t,\cdot),\phi(t,\cdot))-(g(\cdot),\phi(T,\cdot))+\int_t^T(u^n(s,\cdot),\mathcal{L}^*\phi(s,\cdot))ds\\&\qquad=\int_t^T(f(s,\cdot),\phi(s,\cdot))ds+n\int_t^T((u^n-h)^-(s,\cdot),\phi(s,\cdot))ds.\end{aligned}\tag{22}$$

Moreover

$$\begin{aligned}&Y_s^{n,t,x}=u_n(s,X_{t,s}(x)),\,Z_s^{n,t,x}=\sigma^*\nabla u_n(s,X_{t,s}(x)),\\&V_s^{n,t,x}(\cdot)=u_n(s,X_{t,s-}(x)+\beta(X_{t,s-}(x),\cdot))-u_n(s,X_{t,s-}(x))\end{aligned}\tag{23}$$

Set $K_s^{n,t,x}=n\int_t^s(Y_r^{n,t,x}-h(r,X_{t,r}(x)))^-dr$. Then by (23), we have that $K_s^{n,t,x}=n\int_t^s(u_n-h)^-(r,X_{t,r}(x))dr$.

Following the estimates and convergence results for $(Y^{n,t,x},Z^{n,t,x},V^{n,t,x},K^{n,t,x})$ in the step 3 and step 5 of the proof of Theorem 1.2. in [19], we get as m, n tend to infinity :

$$\begin{aligned}&E\sup_{t\,\leqslant\,s\,\leqslant\,T}\left|Y_s^{n,t,x}-Y_s^{m,t,x}\right|^2+E\int_t^T\left|Z_s^{n,t,x}-Z_s^{m,t,x}\right|^2ds\\&+E\int_t^T\int_{\mathbb{E}}\left|V_s^{n,t,x}(e)-V_s^{m,t,x}(e)\right|^2\lambda(de)ds+E\sup_{t\,\leqslant\,s\,\leqslant\,T}\left|K_s^{n,t,x}-K_s^{m,t,x}\right|^2\longrightarrow 0,\end{aligned}$$

and

$$\begin{aligned}&\sup_n E\left[\sup_{t\,\leqslant\,s\,\leqslant\,T}\left|Y_s^{n,t,x}\right|^2+\int_t^T(\left|Z_s^{n,t,x}\right|^2ds)+\int_t^T\int_{\mathbb{E}}\left|V_s^{n,t,x}(e)\right|^2\lambda(de)ds+(K_T^{n,t,x})^2\right]\\&\leqslant C\left(1+|x|^{2\kappa}\right).\end{aligned}\tag{24}$$

By the equivalence of norms (12), we get

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_t^T\rho(x)(|u_n(s,x)-u_m(s,x)|^2+|\sigma^*\nabla u_n(s,x)-\sigma^*\nabla u_m(s,x)|^2)dsdx\\\leqslant\quad&\frac{1}{k_2}\int_{\mathbb{R}^d}\rho(x)E\int_t^T(\left|Y_s^{n,t,x}-Y_s^{m,t,x}\right|^2+\left|Z_s^{n,t,x}-Z_s^{m,t,x}\right|^2)dsdx\to 0.\end{aligned}$$

Thus $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_T$, and the limit $u=\lim_{n\to\infty} u_n$ belongs to $\mathcal{H}_T$. Denote $\nu_n(dt,dx)=n(u_n-h)^-(t,x)dtdx$ and $\pi_n(dt,dx)=\rho(x)\nu_n(dt,dx)$, then by (12)

$$\begin{aligned}\pi_n([0,T]\times\mathbb{R}^d) &= \int_{\mathbb{R}^d}\int_0^T \rho(x)\nu_n(dt,dx)=\int_{\mathbb{R}^d}\int_0^T \rho(x)n(u_n-h)^-(t,x)dtdx\\ &\leqslant \frac{1}{k_2}\int_{\mathbb{R}^d}\rho(x)E\left|K_T^{n,0,x}\right|dx\leqslant C\int_{\mathbb{R}^d}\rho(x)\left(1+|x|^\kappa\right)dx<\infty.\end{aligned}$$

It follows that

$$\sup_n \pi_n([0,T]\times\mathbb{R}^d)<\infty. \tag{25}$$

Moreover by Lemma 2 (see Appendix 5.4), the sequence of measures $(\pi_n)_{n\in\mathbb{N}}$ is tight. Therefore, there exits a subsequence such that $(\pi_n)_{n\in\mathbb{N}}$ converges weakly to a positive measure π. Define $\nu=\rho^{-1}\pi$; ν is a positive measure such that $\int_0^T\int_{\mathbb{R}^d}\rho(x)\nu(dt,dx)<\infty$, and so we have for $\phi\in\mathcal{D}_T$ with compact support in x,

$$\int_{\mathbb{R}^d}\int_t^T\phi d\nu_n=\int_{\mathbb{R}^d}\int_t^T\frac{\phi}{\rho}d\pi_n\to\int_{\mathbb{R}^d}\int_t^T\frac{\phi}{\rho}d\pi=\int_{\mathbb{R}^d}\int_t^T\phi d\nu.$$

Now passing to the limit in the PIDE(g,f_n) (22), we get that that (u,ν) satisfies the PIDE with obstacle associated to (g,f,h), i.e. for every $\phi\in\mathcal{D}_T$, we have

$$\begin{aligned}&\int_t^T(u(s,\cdot),\partial_s\phi(s,\cdot))ds+(u(t,\cdot),\phi(t,\cdot))-(g(\cdot),\phi(T,\cdot))+\int_t^T(u(s,\cdot),\mathcal{L}^*\phi(s,\cdot))ds\\ =&\int_t^T(f(s,\cdot),\phi(s,\cdot))ds+\int_t^T\int_{\mathbb{R}^d}\phi(s,x)\nu(ds,dx).\end{aligned} \tag{26}$$

The last point is to prove that ν satisfies the probabilistic interpretation (18). Since $K^{n,t,x}$ converges to $K^{t,x}$ uniformly in t, the measure $dK^{n,t,x}\to dK^{t,x}$ weakly in probability.

Fix two continuous functions $\phi,\psi:[0,T]\times\mathbb{R}^d\to\mathbb{R}^+$ which have compact support in x and a continuous function with compact support $\theta:\mathbb{R}^d\to\mathbb{R}^+$, we have

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_t^T\phi(s,X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))\psi(s,x)\theta(x)\nu(ds,dx)\\ =&\lim_{n\to\infty}\int_{\mathbb{R}^d}\int_t^T\phi(s,X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))\psi(s,x)\theta(x)n(u_n-h)^-(s,x)dsdx\\ =&\lim_{n\to\infty}\int_{\mathbb{R}^d}\int_t^T\phi(s,x)\psi(s,X_{t,s}(x))\theta(X_{t,s}(x))n(u_n-h)^-(t,X_{t,s}(x))dtdx\\ =&\lim_{n\to\infty}\int_{\mathbb{R}^d}\int_t^T\phi(s,x)\psi(s,X_{t,s}(x))\theta(X_{t,s}(x))dK_s^{n,t,x}dx\\ =&\int_{\mathbb{R}^d}\int_t^T\phi(s,x)\psi(s,X_{t,s}(x))\theta(X_{t,s}(x))dK_s^{t,x}dx.\end{aligned}$$

We take $\theta=\theta_R$ to be the regularization of the indicator function of the ball of radius R and pass to the limit with $R\to\infty$, it follows that

$$\int_{\mathbb{R}^d}\int_t^T\phi(s,X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))\psi(s,x)\nu(ds,dx)=\int_{\mathbb{R}^d}\int_t^T\phi(s,x)\psi(s,X_{t,s}(x))dK_s^{t,x}dx. \tag{27}$$

Since $(Y_s^{n,t,x},Z_s^{n,t,x},V_s^{n,t,x},K_s^{n,t,x})$ converges to $(Y_s^{t,x},Z_s^{t,x},V_s^{t,x},K_s^{t,x})$ as $n\to\infty$ in $\mathcal{S}^2(t,T)\times\mathcal{H}_d^2(t,T)\times\mathcal{L}^2(t,T)\times\mathcal{A}^2(t,T)$, and $(Y_s^{t,x},Z_s^{t,x},V_s^{t,x},K_s^{t,x})$ is the solution of RBSDE with jumps $(g(X_{t,T}(x)),\ f,\ h)$, then we have

$$\int_t^T(Y_s^{t,x}-L_s^{t,x})dK_s^{t,x}=\int_t^T(u-h)(s,X_{t,s}(x))dK_s^{t,x}=0,\text{a.s.}$$

it follows that $dK_s^{t,x} = 1_{\{u=h\}}(s, X_{t,s}(x))dK_s^{t,x}$. In (27), setting $\psi = 1_{\{u=h\}}$ yields

$$\int_{\mathbb{R}^d}\int_t^T \phi(s, X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))1_{\{u=h\}}(s,x)\nu(ds,dx)$$
$$= \int_{\mathbb{R}^d}\int_t^T \phi(s, X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))\nu(ds,dx), \text{ a.s.}$$

Note that the family of functions $A(\omega) = \{(s,x) \to \phi(s, X_{t,s}^{-1}(x)) : \phi \in C_c^\infty\}$ is an algebra which separates the points (because $x \to X_{t,s}^{-1}(x)$ is a bijection). Given a compact set G, $A(\omega)$ is dense in $C([0,T] \times G)$. It follows that $J(X_{t,s}^{-1}(x))1_{\{u=h\}}(s,x)\nu(ds,dx) = J(X_{t,s}^{-1}(x))\nu(ds,dx)$ for almost every ω. While $J(X_{t,s}^{-1}(x)) > 0$ for almost every ω, we get $\nu(ds,dx) = 1_{\{u=h\}}(s,x)\nu(ds,dx)$, and (18) follows.

Then we get easily that $Y_s^{t,x} = u(s, X_{t,s}(x))$, $Z_s^{t,x} = \sigma^*\nabla u(s, X_{t,s}(x))$ and $V_s^{t,x}(\cdot) = u(s, X_{t,s-}(x) + \beta(X_{t,s-}(x),\cdot)) - u(s, X_{t,s-}(x))$, in view of the convergence results for $(Y_s^{n,t,x}, Z_s^{n,t,x}, V_s^{n,t,x})$ and the equivalence of norms. So $u(s, X_{t,s}(x)) = Y_s^{t,x} \geqslant h(s, X_{t,s}(x))$. Specially for $s = t$, we have $u(t,x) \geqslant h(t,x)$.

Step 2 : The nonlinear case where f depends on y, z and w.
Let define $F(s,x) \triangleq f(s,x,Y_s^{s,x}, Z_s^{s,x}, V_s^{s,x})$. By plugging into the facts that $f^0 \in \mathbf{L}_\rho^2([0,T] \times \mathbb{R}^d)$ and f is Lipschitz with respect to (y,z,v), then thanks to Proposition 5 we have $F(s,x) \in \mathbf{L}_\rho^2([0,T] \times \mathbb{R}^d)$. Since F is independent of y, z, w, by applying the result of Step 1 yields that there exists (u,ν) satisfying the PIDE with obstacle (g, F, h), i.e. for every $\phi \in \mathcal{D}_T$, we have

$$\int_t^T (u(s,\cdot), \partial_s\phi(s,\cdot))ds + (u(t,\cdot), \phi(t,\cdot)) - (g(\cdot), \phi(T,\cdot)) + \int_t^T (u(s,\cdot), \mathcal{L}^*\phi(s,\cdot))ds$$
$$= \int_t^T (F(s,\cdot), \phi(s,\cdot))ds + \int_t^T \int_{\mathbb{R}^d} \phi(s,x)1_{\{u=h\}}(s,x)\nu(ds,dx). \quad (28)$$

Then by the uniqueness of the solution to the RBSDE with jumps $(g(X_{t,T}(x)),\ f,\ h(X_{t,s}(x)))$, we get easily that $Y_s^{t,x} = u(s, X_{t,s}(x))$, $Z_s^{t,x} = \sigma^*\nabla u(s, X_{t,s}(x))$, $V_s^{t,x}(\cdot) = u(s, X_{t,s-}(x) + \beta(X_{t,s-}(x),\cdot)) - u(s, X_{t,s-}(x))$, and ν satisfies the probabilistic interpretation (18). So $u(s, X_{t,s}(x)) = Y_s^{t,x} \geqslant h(s, X_{t,s}(x))$. Specially for $s = t$, we have $u(t,x) \geqslant h(t,x)$, which is the desired result.

b) Uniqueness : Set $(\overline{u}, \overline{\nu})$ to be another weak solution of the PIDE with obstacle (19) associated to (g,f,h); with $\overline{\nu}$ verifies (18) for a nondecreasing process $\overline{K}$. We fix $\phi : \mathbb{R}^d \to \mathbb{R}$, a smooth function in $C_c^2(\mathbb{R}^d)$ with compact support and denote $\phi_t(s,x) = \phi(X_{t,s}^{-1}(x))J(X_{t,s}^{-1}(x))$. From Proposition 6, one may use $\phi_t(s,x)$ as a test function in the PIDE(g,f,h) with $\partial_s\phi(s,x)ds$ replaced by a stochastic integral with respect to the semimartingale $\phi_t(s,x)$. Then we get, for $t \leqslant s \leqslant T$

$$\int_{\mathbb{R}^d}\int_s^T \overline{u}(r,x)d\phi_t(r,x)dx + (\overline{u}(s,\cdot), \phi_t(s,\cdot)) - (g(\cdot), \phi_t(T,\cdot)) - \int_s^T (\overline{u}_r, \mathcal{L}^*\phi_r)dr$$
$$= \int_s^T \int_{\mathbb{R}^d} f(r,x,\overline{u}(r,x), \sigma^*\nabla\overline{u}(r,x), \overline{u}(r, x+\beta(x,\cdot)) - \overline{u}(r,x))\phi_t(r,x)drdx$$
$$+ \int_s^T \int_{\mathbb{R}^d} \phi_t(r,x)1_{\{\overline{u}=h\}}(r,x)\overline{\nu}(dr,dx). \quad (29)$$

By (10) in Lemma 1, we have

$$\int_{\mathbb{R}^d}\int_s^T \overline{u}d_r\phi_t(r,x)dx = \int_s^T (\int_{\mathbb{R}^d} (\sigma^*\nabla\overline{u})(r,x)\phi_t(r,x)dx)dW_r$$
$$+ \int_s^T \int_{\mathbb{E}}\int_{\mathbb{R}^d} \overline{u}(r,x)\mathcal{A}_e^*\phi_t(r-,x)dx\widetilde{\mu}(dr,de) + \int_s^T (\overline{u}_r, \mathcal{L}^*\phi_r)dr.$$

Substitute this equality in (29), we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} \overline{u}(s,x)\phi_t(s,x)dx = (g(\cdot),\phi_t(T,\cdot)) - \int_s^T (\int_{\mathbb{R}^d} (\sigma^*\nabla\overline{u})(r,x)\phi_t(r,x)dx)dW_r \\
&+ \int_s^T \int_{I\!E} \int_{\mathbb{R}^d} \mathcal{A}_e\overline{u}(r-,x)\phi_t(r,x)dx\widetilde{\mu}(dr,de) \\
&+ \int_{\mathbb{R}^d}\int_s^T f(r,x,\overline{u}(r,x),\sigma^*\nabla\overline{u}(r,x),\overline{u}(r,x+\beta(x,\cdot))-\overline{u}(r,x))\phi_t(r,x)drdx \\
&+ \int_s^T \int_{\mathbb{R}^d} \phi_t(r,x)1_{\{\overline{u}=h\}}(r,x)\overline{\nu}(dr,dx).
\end{aligned}$$

Then by changing of variable $y = X_{t,r}^{-1}(x)$ and applying (18) for $\overline{\nu}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \overline{u}(s,X_{t,s}(y))\phi(y)dy = \int_{\mathbb{R}^d} g(X_{t,T}(y))\phi(y)dy \\
&+ \int_{\mathbb{R}^d}\int_s^T \phi(y)f(r,X_{t,r}(y),\overline{u}(r,X_{t,r}(y)),\sigma^*\nabla\overline{u}(r,X_{t,r}(y)),\mathcal{A}_e\overline{u}(r,X_{t,r-}(y)))drdy \\
&+ \int_s^T \int_{\mathbb{R}^d} \phi(y)1_{\{\overline{u}=h\}}(r,X_{t,s}(y))d\overline{K}_r^{t,y}dy - \int_s^T (\int_{\mathbb{R}^d} (\sigma^*\nabla\overline{u})(r,X_{t,r}(y))\phi(y)dy)dW_r \\
&+ \int_s^T \int_{\mathbb{E}} \left(\int_{\mathbb{R}^d} \mathcal{A}_e\overline{u}(r,X_{t,r-}(y))\phi(y)dy\right)\tilde{\mu}(dr,de).
\end{aligned}$$

Since ϕ is arbitrary, we can prove that for $\rho(y)dy$ almost every y, $(\overline{u}(s,X_{t,s}(y)),$ $(\sigma^*\nabla\overline{u})(s,X_{t,s}(y)),\ \overline{u}(s,X_{t,s-}(y)+\beta(X_{t,s-}(y),\cdot))-\overline{u}(s,X_{t,s-}(y)),\widehat{K}_s^{t,y})$ solves the RBSDE with jumps $(g(X_{t,T}(y)),f,h)$. Here $\widehat{K}_s^{t,y}{=}\int_t^s 1_{\{\overline{u}=h\}}(r,X_{t,r}(y))d\overline{K}_r^{t,y}$. Then by the uniqueness of the solution of the RBSDE with jumps, we know $\overline{u}(s,X_{t,s}(y)) = Y_s^{t,y} = u(s,X_{t,s}(y))$, $(\sigma^*\nabla\overline{u})(s,X_{t,s}(y)) = Z_s^{t,y} = (\sigma^*\nabla u)(s,X_{t,s}(y))$, $\overline{u}(s,X_{t,s-}(y)+\beta(X_{t,s-}(y),\cdot))-\overline{u}(s,X_{t,s-}(y)) = V_s^{t,y}(\cdot) = u(s,X_{t,s-}(y)+\beta(X_{t,s-}(y),\cdot))-u(s,X_{t,s-}(y))$ and $\widehat{K}_s^{t,y} = K_s^{t,y}$. Taking $s=t$ we deduce that $\overline{u}(t,y) = u(t,y)$, $\rho(y)dy$-a.s. and by the probabilistic interpretation (18), we obtain

$$\int_s^T \int \phi_t(r,x)1_{\{\overline{u}=h\}}(r,x)\overline{\nu}(dr,dx) = \int_s^T \int \phi_t(r,x)1_{\{u=h\}}(r,x)\nu(dr,dx).$$

So $1_{\{\overline{u}=h\}}(r,x)\overline{\nu}(dr,dx) = 1_{\{u=h\}}(r,x)\nu(dr,dx)$. □

5. Appendix

5.1. Proof of Proposition 3

In order to prove (11), it is sufficient to prove that

$$c \leqslant E[\frac{J(X_{t,s}^{-1}(x))\rho(X_{t,s}^{-1}(x))}{\rho(x)}] \leqslant C.$$

In fact, making the change of variable $y = X_{t,s}(x)$, we can get the following relation:

$$\int_{\mathbb{R}^d} E(|\varphi(X_{t,s}(x))|)\rho(x)dx = \int_{\mathbb{R}^d} |\varphi(y)|E[\frac{J(X_{t,s}^{-1}(y))\rho(X_{t,s}^{-1}(y))}{\rho(y)}]\rho(y)dy.$$

We differentiate with respect to y in (7) in order to get:

$$\begin{aligned}\nabla X^{-1}_{t,s}(y) &= I-\int_t^s \nabla\widehat{b}(X^{-1}_{r,s}(y)\nabla X^{-1}_{t,r}(y)dr-\int_t^s \nabla\sigma(X^{-1}_{r,s}(y))\nabla X^{-1}_{t,r}(y)\overleftarrow{dW}_r\\ &\quad -\int_t^s\int_{\mathbb{E}} \nabla\beta(X^{-1}_{r,s}(y),e)\nabla X^{-1}_{t,r}(y)\widetilde{\mu}(\overleftarrow{dr},de)\\ &\quad +\int_t^s\int_{\mathbb{E}} \nabla\widehat{\beta}(X^{-1}_{r,s}(y),e)\nabla X^{-1}_{t,r}(y)\mu(\overleftarrow{dr},de)\\ &:= I+\Gamma_{t,s}(y)\end{aligned} \tag{30}$$

where ∇b, $\nabla\sigma$, $\nabla\beta$ and $\nabla\widehat{\beta}$ are the gradient of b, σ,β and $\widehat{\beta}$, respectively and I is the identity matrix. Since $J(X^{-1}_{t,s}(y)) := \det\nabla X^{-1}_{t,s}(y) = \inf_{\|\xi\|=1} < \nabla X^{-1}_{t,s}(y)\xi,\xi >$, we obtain

$$1-\|\Gamma_{t,s}(y)\| \leqslant J(X^{-1}_{t,s}(y)) \leqslant 1+\|\Gamma_{t,s}(y)\|.$$

Writing $\Gamma_{t,s}(y) := C_{t,s}(y)+D_{t,s}(y)$, where $D_{t,s}(y)$ denotes the integration with respect to the random measure and $C_{t,s}(y)$ denotes the others.
According to Bally and Matoussi [2], we know that for any y, $E[|C_{t,s}(y)|^2] \leqslant K(s-t)$. Now we are going to prove the similar relation for $D_{t,s}(y)$. We only deal with the first term of $D_{t,s}(y)$ because another one can be treated similarly without any difficulty. In fact, by Burkholder-David-Gundy inequality (of course in the backward sense), we have

$$\begin{aligned}&E[|\int_t^s\int_{\mathbb{E}} \nabla\beta(X^{-1}_{r,s}(y),e)\nabla X^{-1}_{t,r}(y)\widetilde{\mu}(\overleftarrow{dr},de)|^2]\\ \leqslant\ & CE[\int_t^s\int_{\mathbb{E}} |\nabla\beta(X^{-1}_{r,s}(y),e)\nabla X^{-1}_{r,s}(y)|^2\lambda(de)dr]\\ \leqslant\ & C_1E[\int_t^s\int_{\mathbb{E}} (1\wedge|e|)^2|\nabla X^{-1}_{r,s}(y)|^2\lambda(de)dr]\\ =\ & C_1E[\int_t^s |\nabla X^{-1}_{r,s}(y)|^2\int_{\mathbb{E}}(1\wedge|e|^2)\lambda(de)dr]\\ \leqslant\ & K(s-t),\end{aligned}$$

since $E[|\nabla X^{-1}_{r,s}(y)|^2]<\infty$. Therefore, $E[|\Gamma^{t,y}_s|^2] \leqslant 4K(s-t)$. Hence,

$$1-2\sqrt{K(s-t)} \leqslant E[J(X^{-1}_{t,s}(y))] \leqslant 1+2\sqrt{K(s-t)}$$

and the desired result follows.

5.2. Regularity of the solution of BSDE with jumps

In this section, we are going to prove regularity results for the solution of BSDE's with jumps with respect to the parameter (t,x) in order to relate the solution of BSDE to the classic solution of PIDE. We note that some part of the results given in this section were established in a preprint of Buckdahn and Pardoux (1994) [9]. However, for convenience of the reader and for completeness of the paper, we give the whole proofs. We first start by giving the L^p-estimates for the solution of the following BSDE's with jumps:

$$Y_t = \xi+\int_t^T f(s,Y_s,Z_s,V_s)ds-\int_t^T Z_s dW_s-\int_t^T\int_{\mathbb{E}} V_s(e)\widetilde{\mu}(ds,de). \tag{31}$$

Theorem 3. *Assume that f is uniformly Lipschitz with respect to (y,z,v) and additionally that for some $p\geqslant 2$, $\xi\in L^p_m(\mathcal{F}_T)$ and*

$$E\int_0^T |f(t,0,0,0)|^p dt<\infty. \tag{32}$$

Then

$$E\Big[\sup_t |Y_t|^p + \Big(\int_0^T |Z_t|^2 dt\Big)^{p/2} + \Big(\int_0^T \Big(\int_{\mathbb{E}} |V_t(e)|^2 \lambda(de)\Big) dt\Big)^{p/2}\Big] < \infty.$$

Proof. We follow the idea of Buckdahn and Pardoux [9]. The proof is divided into 3 steps.

Step 1: From (31),

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s, V_s) ds + \int_0^t Z_s dW_s + \int_0^t \int_{\mathbb{E}} V_s(e) \widetilde{\mu}(ds, de).$$

then by Itô's formula,

$$\begin{aligned} |Y_t|^2 &= |Y_0|^2 - 2\int_0^t (f(s, Y_s, Z_s, V_s), Y_s) ds + \int_0^t (|Z_s|^2 + \|V_s\|^2) ds \\ &+ 2\int_0^t (Y_s, Z_s dW_s) + \int_0^t \int_{\mathbb{E}} (|Y_{s-} + V_s(e)|^2 - |Y_{s-}|^2) \widetilde{\mu}(ds, de) \end{aligned}$$

Let $\phi_{n,p}(x) = (x \wedge n)^p + pn^{p-1}(x-n)^+$ for all $p \geqslant 1$. Then $\phi_{n,p} \in C^1(\mathbb{R}_+)$, $\phi'_{n,p}(x) = p(x \wedge n)^{p-1}$ bounded and absolutely continuous with

$$\phi''_{n,p}(x) = p(p-1)(x \wedge n)^{p-2} \mathbf{1}_{[0,n]}(x).$$

Again by applying Itô's formula to $\phi_{n,p}(|Y_t|^2)$,

$$\begin{aligned} &\phi_{n,p}(|Y_T|^2) = \phi_{n,p}(|Y_t|^2) - 2\int_t^T \phi'_{n,p}(|Y_s|^2)(f(s, Y_s, Z_s, V_s), Y_s) ds \\ &+2\int_t^T \phi'_{n,p}(|Y_s|^2)(Y_s, Z_s dW_s) + \int_t^T \phi''_{n,p}(|Y_s|^2)(Z_s Z_s^* Y_s, Y_s) ds \\ &+\int_t^T \phi'_{n,p}(|Y_s|^2)|Z_s|^2 ds + \int_t^T \phi'_{n,p}(|Y_s|^2)\|V_s\|^2 ds \\ &+\int_t^T \int_{\mathbb{E}} [\phi_{n,p}(|Y_{s-} + V_s(e)|^2) - \phi_{n,p}(|Y_{s-}|^2)] \widetilde{\mu}(ds, de) \\ &+\int_t^T \int_{\mathbb{E}} [\phi_{n,p}(|Y_{s-} + V_s(e)|^2) - \phi_{n,p}(|Y_{s-}|^2)] - (|Y_{s-} + V_s(e)|^2 - |Y_{s-}|^2)\phi'_{n,p}(|Y_{s-}|^2)\lambda(de) ds \end{aligned}$$

Since $(Y, Z, V) \in \mathcal{B}^2$, it follows from Burkholder-Davis-Gundy's inequality,

$$E\left[\sup_{0 \leqslant t \leqslant T} \left|\int_0^t \phi'_{n,p}(|Y_s|^2)(Y_s, Z_s dW_s)\right|\right] \leqslant Cpn^{p-1} \|Y\|_{\mathcal{S}^2} \|Z\|_{\mathcal{H}^2},$$

that the dW integral above is uniformly integrable, hence it is a martingale with zero expectation. From the boundedness of $\phi'_{n,p}$, the integrand in the $\widetilde{\mu}$-integral can be written as follows

$$\phi_{n,p}(|Y_{s-} + V_s(e)|^2) - \phi_{n,p}(|Y_{s-}|^2) = \psi_s(e)(2(Y_{s-}, V_s(e)) + |V_s(e)|^2),$$

where $\psi_s(e)$ is a bounded and predictable process. By BDG's inequality

$$\begin{aligned} E\left[\sup_{0 \leqslant t \leqslant T} \left|\int_0^t \int_{\mathbb{E}} \psi_s(e)(Y_{s-}, V_s(e)) \widetilde{\mu}(ds, de)\right|\right] &\leqslant cE\left[\Big(\int_0^T \int_{\mathbb{E}} |\psi_s(e)(Y_{s-}, V_s(e))|^2 \mu(ds, de)\Big)^{1/2}\right] \\ &\leqslant cpn^{p-1} \|Y\|_{\mathcal{S}^2} \|V\|_{\mathcal{L}^2}, \end{aligned}$$

and by the decomposition $\widetilde{\mu}(ds, de) = \mu(ds, de) - ds\lambda(de)$,

$$\begin{aligned} E\left[\sup_{0 \leqslant t \leqslant T} \left|\int_0^t \int_{\mathbb{E}} \psi_s(e)|V_s(e)|^2 \widetilde{\mu}(ds, de)\right|\right] &\leqslant 2pn^{p-1} E\left[\int_0^T \int_{\mathbb{E}} |V_s(e)|^2 \lambda(de) ds\right] \\ &= 2pn^{p-1} \|V\|^2_{\mathcal{L}^2}, \end{aligned}$$

we have also

$$E\left[\int_0^t \int_{\mathbb{E}} (\phi_{n,p}(|Y_{s-}+V_s(e)|^2) - \phi_{n,p}(|Y_{s-}|^2))\widetilde{\mu}(ds,de)\right] = 0.$$

From Taylor's expansion of $\phi_{n,p}$ and the positivity of $\phi''_{n,p}$, we conclude

$$\int_t^T \int_{\mathbb{E}} [\phi_{n,p}(|Y_{s-}+V_s(e)|^2) - \phi_{n,p}(|Y_{s-}|^2) - (|Y_{s-}+V_s(e)|^2 - |Y_{s-}|^2)\phi'_{n,p}(|Y_{s-}|^2)]\lambda(de)ds \geqslant 0$$

Using again $\phi''_{n,p} \geqslant 0$ we conclude that:

$$\begin{aligned} E\phi_{n,p}(|Y_t|^2) \quad &+ \quad E\int_t^T \phi'_{n,p}(|Y_s|^2)(|Z_s|^2 + ||V_s||^2)ds \\ &\leqslant \quad E\phi_{n,p}(|\xi|^2) + 2E\int_t^T \phi'_{n,p}(|Y_s|^2)(f(s,Y_s,Z_s,V_s),Y_s)ds \\ &\leqslant \quad E\phi_{n,p}(|\xi|^2) + CE\int_t^T \phi'_{n,p}(|Y_s|^2)|Y_s|(|f(s,0,0,0)| + |Y_s| + |Z_s| + \|V_s\|)ds. \end{aligned}$$

Hence we deduce that

$$\begin{aligned} E\phi_{n,p}(|Y_t|^2) \quad &+ \quad \frac{1}{2}E\int_t^T \phi'_{n,p}(|Y_s|^2)(|Z_s|^2 + \|V_s\|^2)ds \\ &\leqslant \quad E\phi_{n,p}(|\xi|^2) + CE\int_t^T \phi'_{n,p}(|Y_s|^2)(|f(s,0,0,0)|^2 + |Y_s|^2)ds \\ &\leqslant \quad E\phi_{n,p}(|\xi|^2) + C'E\int_t^T (|f(s,0,0,0)|^{2p} + \phi'_{n,p}(|Y_s|^2)^{\frac{p}{p-1}} + \phi'_{n,p}(|Y_s|^2)|Y_s|^2)ds \\ &\leqslant \quad E\phi_{n,p}(|\xi|^2) + \overline{C}E\int_t^T (|f(s,0,0,0)|^{2p} + \phi_{n,p}(|Y_s|^2))ds. \end{aligned}$$

Then it follows from Gronwall's lemma that there exists a constant $C(p,T)$ independent of n such that

$$\sup_{0\leqslant t\leqslant T} E\phi_{n,p}(|Y_t|^2) \leqslant C(p,T)E[|\xi|^{2p} + \int_0^T |f(t,0,0,0)|^{2p}dt],$$

hence from Fatou's lemma

$$\sup_{0\leqslant t\leqslant T} E[|Y_t|^{2p}] < \infty, \tag{33}$$

and also

$$E\int_0^T |Y_s|^{2(p-1)}(|Z_s|^2 + \|V_s\|^2)ds < \infty, \tag{34}$$

$$E\int_0^T \int_{\mathbb{E}} [|Y_{s-}+V_s(e)|^{2p} - |Y_{s-}|^{2p} - p(|Y_{s-}+V_s(e)|^2 - |Y_{s-}|^2)|Y_{s-}|^{2(p-1)}]\lambda(de)ds < \infty \tag{35}$$

and this holds for any $p \geqslant 1$.

Step 2: Now, again from Itô's formula,

$$\begin{aligned} |Y_T|^{2p} \quad &\geqslant \quad |Y_t|^{2p} - 2p\int_t^T |Y_s|^{2(p-1)}(f(s,Y_s,Z_s,V_s),Y_s)ds \\ &+ \quad p\int_t^T |Y_s|^{2(p-1)}(|Z_s|^2 + \|V_s\|^2)ds + 2\int_t^T |Y_s|^{2(p-1)}(Y_s, Z_s dW_s) \\ &+ \quad \int_t^T \int_{\mathbb{E}} (|Y_{s-}+V_s(e)|^{2p} - |Y_{s-}|^{2p})\widetilde{\mu}(ds,de). \end{aligned}$$

It follows from (33) and (34) that the above dW-integral is a uniformly integral martingale, and from (34) and (35) that the $\widetilde{\mu}$-integral is a uniformly integrable martingale. It is then easy to conclude that

$$E[\sup_{0\leqslant t\leqslant T}|Y_t|^{2p}]<\infty,\ p\geqslant 1.$$

Step 3: Finally,

$$\int_s^t Z_r dW_r+\int_s^t\int_{\mathbb{E}} V_r(e)\widetilde{\mu}(dr,de)=Y_t-Y_s+\int_s^t f(r,Y_r,Z_r,V_r)dr,$$

and from BDG inequality, (33) and (32), for any $p\geqslant 2$, there exists C_p such that for all $0\leqslant s\leqslant t\leqslant T$, $n\geqslant 1$ if $\tau_n=\inf\{u\geqslant s,\int_s^u(|Z_r|^2+\|V_r\|^2)dr\geqslant n\}\wedge t$,

$$\begin{aligned}E\left[(\int_s^{\tau_n}(|Z_r|^2+\|V_r\|^2)dr)^{p/2}\right] &\leqslant C_pE\left[1+(\int_s^{\tau_n}(|Z_r|+\|V_r\|)dr)^p\right]\\ &\leqslant C_p(1+(t-s)^{p/2}E\left[(\int_s^{\tau_n}(|Z_r|^2+\|V_r\|^2)dr)^{p/2}\right].\end{aligned}$$

Hence if $C_p(t-s)^{p/2}<1,\ n\geqslant 1,\ E\left[(\int_s^{\tau_n}(|Z_r|^2+\|V_r\|^2)dr)^{p/2}\right]\leqslant\frac{C_p}{1-C_p(t-s)^{p/2}}$. It clearly follows that

$$E\left[\left(\int_0^T|Z_t|^2dt\right)^{p/2}+\left(\int_0^T\|V_t\|^2dt\right)^{p/2}\right)\right]<\infty,\ p\geqslant 2.$$

□

From now on, we denote by $\Sigma=(Y,Z,V)$ and $\mathcal{B}^p$ the space of solutions, *i.e.*,

$$||\Sigma||^p_{\mathcal{B}^p}\triangleq E\left[\sup_t|Y_t|^p+\left(\int_0^T|Z_t|^2dt\right)^{p/2}+\left(\int_0^T||V_t||^2dt\right)^{p/2}\right].$$

In the sequel, we will consider a specific class of BSDE where

$$\xi=g(X_{t,T}(x))\text{ and }f(s,y,z,v)=f(s,X_{t,s}(x),y,z,v)$$

and we assume

$$(\mathbf{H})\begin{cases}g\in C^3_p(\mathbb{R}^d;\mathbb{R}^m),\\ \forall s\in[0,T],(x,y,z,v)\mapsto f(s,x,y,z,v)\in C^3\\ \text{and all their derivatives are bounded.}\end{cases}$$

Note that f is differentiable w.r.t. v in the sens of Fréchet and its Fréchet differential is bounded with the norm in $L^2(\mathbb{E},\lambda;\mathbb{R}^k)$.

Let $(Y^{t,x}_s,Z^{t,x}_s,V^{t,x}_s)_{t\leqslant s\leqslant T}$ denote the unique solution of the following BSDE:

$$Y^{t,x}_s=g(X_{t,T}(x))+\int_s^T f(r,X_{t,r}(x),Y^{t,x}_r,Z^{t,x}_r,V^{t,x}_r)dr-\int_s^T Z^{t,x}_r dW_r-\int_s^T\int_{\mathbb{E}}V^{t,x}_r(e)\widetilde{\mu}(dr,de).\tag{36}$$

It follows easily from the existence result in [4]:

Corollary 1. *For each $t\in[0,T]$, $x\in\mathbb{R}^d$, the BSDE(36) has a unique solution*

$$\Sigma^{t,x}=(Y^{t,x},Z^{t,x},V^{t,x})\in\mathcal{B}^2,$$

and $Y^{t,x}_t$ defines a deterministic mapping from $[0,T]\times\mathbb{R}^d$ into $\mathbb{R}^m$.

Now we are going to deal with the regularity of the solution with respect to the parameter x. Let us establish the following proposition:

Proposition 7. *Under the assumption in the previous theorem and assume moreover that (H) holds. Then, for any $p \geqslant 2$, there exists C_p, q such that for any $0 \leqslant t \leqslant T$, $x, x' \in \mathbb{R}^d$, $h, h' \in \mathbb{R} \setminus \{0\}$, $1 \leqslant i \leqslant d$,*

(i) $\|\Sigma^{t,x} - \Sigma^{t,x'}\|^p_{\mathcal{B}^p} \leqslant C_p(1+|x|+|x'|)^q|x-x'|^p$;

(ii) $\|\Delta^i_h \Sigma^{t,x} - \Delta^i_{h'}\Sigma^{t,x'}\|^p_{\mathcal{B}^p} \leqslant C_p(1+|x|+|x'|+|h|+|h'|)^q(|x-x'|^p+|h-h'|^p)$

where $\Delta^i_h \Sigma^{t,x}_s = \frac{1}{h}(Y^{t,x+he_i}_s - Y^{t,x}_s, Z^{t,x+he_i}_s - Z^{t,x}_s, V^{t,x+he_i}_s - V^{t,x}_s)$, *and* $(e_1, \cdots, e_d)$ *is an orthonormal basis of* $\mathbb{R}^d$.

Proof. Note that after applying L^p-estimation of the solution to the present situation, we can deduce that $\forall p \geqslant 2$, there exist C_p, q such that

$$E\bigg[\sup_s |Y^{t,x}_s|^p + \bigg(\int_t^T |Z^{t,x}_s|^2 ds\bigg)^{p/2} + \bigg(\int_t^T \|V^{t,x}_s\|^2 ds\bigg)^{p/2}\bigg] \leqslant C_p(1+|x|^q).$$

For $t \leqslant s \leqslant T$,

$$\begin{aligned}
& Y^{t,x}_s - Y^{t,x'}_s \\
= \; & g(X_{t,T}(x)) - g(X_{t,T}(x')) + \int_s^T (f(r, X_{t,r}(x), Y^{t,x}_r, Z^{t,x}_r, V^{t,x}_r) - f(r, X_{t,r}(x'), Y^{t,x'}_r, Z^{t,x'}_r, V^{t,x'}_r))dr \\
& - \int_s^T (Z^{t,x}_r - Z^{t,x'}_r)dW_r - \int_s^T \int_{I\!\!E} (V^{t,x}_r(e) - V^{t,x'}_r(e))\widetilde{\mu}(dr, de) \\
= \; & \int_0^1 g'(\theta X_{t,T}(x) + (1-\theta)X_{t,T}(x'))(X_{t,T}(x) - X_{t,T}(x'))d\theta + \int_s^T \Big[\varphi_r(t,x,x')(X_{t,r}(x) - X_{t,r}(x')) \\
& + \psi_r(t,x,x')(Y^{t,x}_r - Y^{t,x'}_r) + \xi_r(t,x,x')(Z^{t,x}_r - Z^{t,x'}_r) + \langle \eta_r(t,x,x'), (V^{t,x}_r - V^{t,x'}_r)\rangle\Big]dr \\
& - \int_s^T (Z^{t,x}_r - Z^{t,x'}_r)dW_r - \int_s^T \int_{I\!\!E} (V^{t,x}_r(e) - V^{t,x'}_r(e))\widetilde{\mu}(dr, de),
\end{aligned}$$

where

$$\begin{gathered}
\varphi_r(t,x,x') = \int_0^1 \frac{\partial f}{\partial x}(\Xi^{t,x,x'}_{r,\theta})d\theta, \quad \psi_r(t,x,x') = \int_0^1 \frac{\partial f}{\partial y}(\Xi^{t,x,x'}_{r,\theta})d\theta \\
\xi_r(t,x,x') = \int_0^1 \frac{\partial f}{\partial z}(\Xi^{t,x,x'}_{r,\theta})d\theta, \quad \eta_r(t,x,x') = \int_0^1 \frac{\partial f}{\partial v}(\Xi^{t,x,x'}_{r,\theta})d\theta, \\
\Xi^{t,x,x'}_{r,\theta} = \theta\Sigma^{t,x}_r + (1-\theta)\Sigma^{t,x'}_r,
\end{gathered}$$

and $\langle \cdot, \cdot \rangle$ stands for the scalar product in $L^2(I\!\!E, \lambda; \mathbb{R}^m)$.

Then given the boundedness of the derivatives, the previous proposition and

$$|\varphi_r(t,x,x')| \leqslant C(1+|X_{t,r}(x)| + |X_{t,r}(x')|)^q,$$

we know that (i) holds true after applying the L^p-estimates of the solution.

Now we turn to (ii). In fact, we have

$$\begin{aligned}\Delta_h^i Y_s^{t,x} &= \frac{1}{h}(Y_s^{t,x+he_i} - Y_s^{t,x}) \\ &= \int_0^1 g'(\theta X_{t,T}(x+he_i) + (1-\theta)X_{t,T}(x))\Delta_h^i X_{t,T}(x)d\theta \\ &\quad + \int_s^T \Big[\varphi_r(t,x+he_i,x)\Delta_h^i X_{t,r}(x) + \psi_r(t,x+he_i,x)\Delta_h^i Y_r^{t,x} \\ &\qquad + \xi_r(t,x+he_i,x)\Delta_h^i Z_r^{t,x} + \langle \eta_r(t,x+he_i,x), \Delta_h^i V_r^{t,x}\rangle\Big] dr \\ &\quad - \int_s^T \Delta_h^i Z_r^{t,x} dW_r - \int_s^T \int_{\mathbb{E}} \Delta_h^i V_r^{t,x}(e)\widetilde{\mu}(dr,de).\end{aligned}$$

Note that for each $p \geqslant 2$, there exists C_p such that $E[\sup_s |\Delta_h^i X_{t,s}(x)|^p] \leqslant C_p$. Then same calculation as in (i) implies that

$$\|\Delta_h^i \Sigma^{t,x}\|_{\mathcal{B}^p}^p \leqslant C_p(1 + |x|^q + |h|^q).$$

Finally, we consider

$$\begin{aligned}&\Delta_h^i Y_s^{t,x} - \Delta_{h'}^i Y_s^{t,x'} \\ = \;&\Gamma^{i,t,s}(h,x;h',x') \\ &+ \int_s^T \Big[\varphi_r(t,x+he_i,x)(\Delta_h^i X_{t,r}(x) - \Delta_{h'}^i X_{t,r}(x')) + \psi_r(t,x+he_i,x)(\Delta_h^i Y_r^{t,x} - \Delta_{h'}^i Y_r^{t,x'}) \\ &\qquad + \xi_r(t,x+he_i,x)(\Delta_h^i Z_r^{t,x} - \Delta_{h'}^i Z_r^{t,x'}) + \langle \eta_r(t,x+he_i,x), \Delta_h^i V_r^{t,x} - \Delta_{h'}^i V_r^{t,x'}\rangle\Big] dr \\ &- \int_s^T (\Delta_h^i Z_r^{t,x} - \Delta_{h'}^i Z_r^{t,x'})dW_r - \int_s^T \int_{\mathbb{E}} (\Delta_h^i V_r^{t,x}(e) - \Delta_{h'}^i V_r^{t,x'}(e))\widetilde{\mu}(dr,de),\end{aligned}$$

where

$$\begin{aligned}&\Gamma^{i,t,s}(h,x;h',x') \\ = \;&\int_0^1 \Big(g'(\theta X_{t,T}(x+he_i) + (1-\theta)X_{t,T}(x))\Delta_h^i X_{t,T}(x) \\ &- g'(\theta X_{t,T}(x'+h'e_i) + (1-\theta)X_{t,T}(x'))\Delta_{h'}^i X_{t,T}(x')\Big) d\theta \\ &+ \int_s^T \Big[(\varphi_r(t,x+he_i,x) - \varphi_r(t,x'+h'e_i,x'))\Delta_{h'}^i X_{t,r}(x') \\ &\qquad + (\psi_r(t,x+he_i,x) - \psi_r(t,x'+h'e_i,x'))\Delta_{h'}^i Y_r^{t,x'} \\ &\qquad + (\xi_r(t,x+he_i,x) - \xi_r(t,x'+h'e_i,x'))\Delta_{h'}^i Z_r^{t,x'} \\ &\qquad + \langle \eta_r(t,x+he_i,x) - \eta_r(t,x'+h'e_i,x'), \Delta_{h'}^i V_r^{t,x'}\rangle\Big] dr.\end{aligned}$$

Since f has bounded derivatives, we have

$$\begin{aligned}&|\psi_r(t,x+he_i,x) - \psi_r(t,x'+h'e_i,x')| + |\xi_r(t,x+he_i,x) - \xi_r(t,x'+h'e_i,x')| \\ &\quad + |\eta_r(t,x+he_i,x) - \eta_r(t,x'+h'e_i,x')| \\ \leqslant \;& C(|\Sigma_r^{t,x+he_i} - \Sigma_r^{t,x'+h'e_i}| + |\Sigma_r^{t,x} - \Sigma_r^{t,x'}|)\end{aligned}$$

and

$$\begin{aligned} & |\varphi_r(t,x+he_i,x)-\varphi_r(t,x'+h'e_i,x')| \\ \leqslant \quad & C\Big(1+|\Sigma_r^{t,x+he_i}|+|\Sigma_r^{t,x'+h'e_i}|+|\Sigma_r^{t,x}|+|\Sigma_r^{t,x'}|\Big)^q\Big(|\Sigma_r^{t,x+he_i}-\Sigma_r^{t,x'+h'e_i}|+|\Sigma_r^{t,x}-\Sigma_r^{t,x'}|\Big), \end{aligned}$$

it follows from the previous proposition that there exist a constant C_p and some α_p such that

$$E[\sup_{t\leqslant s\leqslant T}|\Gamma^{i,t,s}(h,x;h',x')|^p]\leqslant C_p(1+|x|+|x'|+|h|+|h'|)^{\alpha_p}(|x-x'|^p+|h-h'|^p).$$

Then, statement (ii) follows from the same estimate, which ends the proof. □

Therefore, using Kolmogorov's criterion, we can get that Y, Z, V are a.s. differentiable w.r.t. x. Iterating the same argument, we get in fact:

Theorem 4. *For all $0\leqslant t\leqslant T$, there exists a version of $\Sigma^{t,x}=(Y^{t,x},Z^{t,x},V^{t,x})$ such that $x\mapsto\Sigma^{t,x}$ is a.e. of class C^2 from $\mathbb{R}^d$ into $D([t,T];\mathbb{R}^m)\times L^2([t,T];\mathbb{R}^{m\times d})\times L^2([t,T]\times I\!\!E,ds\lambda(de);\mathbb{R}^m)$, where $D([t,T];\mathbb{R}^m)$ denotes the set of $\mathbb{R}^m$-valued càdlàg functions on $[t,T]$. Moreover, for any $p\geqslant 2$, there exist C_p and q such that*

$$\begin{aligned} &\|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\Sigma^{t,x}\|^p_{\mathcal{B}^p}\leqslant C_p(1+|x|^q),\quad 0\leqslant|\alpha|\leqslant 2,\\ &\|\frac{\partial^2}{\partial x_i\partial x_j}\Sigma^{t,x}-\frac{\partial^2}{\partial x_i\partial x_j}\Sigma^{t,x'}\|^p_{\mathcal{B}^p}\leqslant C_p(1+|x|+|x'|)^q|x-x'|^p,\quad 0\leqslant t\leqslant T. \end{aligned}$$

Additionally, we have the following regularity in t:

Proposition 8. *For all $p\geqslant 2$ and $s\geqslant t\vee t'$, there exist C_p and q such that*

$$\begin{aligned} &\|\Sigma_s^{t,x}-\Sigma_s^{t',x'}\|^p_{\mathcal{B}^p}\leqslant C_p(1+|x|+|x'|)^q(|t-t'|^{p/2}+|x-x'|^p),\\ &\|\frac{\partial^{|\alpha|}}{\partial x^\alpha}\Sigma_s^{t,x}-\frac{\partial^{|\alpha|}}{\partial x^\alpha}\Sigma_s^{t',x'}\|^p_{\mathcal{B}^p}\leqslant C_p(1+|x|+|x'|)^q(|t-t'|^{p/2}+|x-x'|^p),\quad 0\leqslant|\alpha|\leqslant 2. \end{aligned}$$

Hence, we have

Corollary 2. *The function $(t,x)\mapsto Y_t^{t,x}\in C_p^{1,2}([0,T]\times\mathbb{R}^d;\mathbb{R}^m)$.*

5.3. Proof of Theorem 1: existence and uniqueness of solution of PIDE (1)

(i) *Uniqueness* : Let $u^1,u^2\in\mathcal{H}_T$ be two weak solutions of (1), then Prop. 4 implies that for $i=1,2$

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_s^T u^i(r,x)d\phi_t(r,x)dx+(u^i(s,x),\phi_t(s,x))-(g(x),\phi_t(T,x))-\int_s^T(u^i(r,\cdot),\mathcal{L}^*\phi_t(r,\cdot))dr\\ =&\int_{\mathbb{R}^d}\int_s^T f(r,x,u^i(r,x),\sigma^*\nabla u^i(r,x),u^i(r,x+\beta(x,\cdot))-u^i(r,x))\phi_t(r,x)drdx. \end{aligned}$$

By the decomposition of the semimartingale $\phi_t(s,x)$, we have

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_s^T u^i(r,x)d\phi_t(r,x)dx\\ =&\int_{\mathbb{R}^d}\int_s^T u^i(r,x)\mathcal{L}^*\phi_t(r,x)dr-\sum_{j=1}^d\int_{\mathbb{R}^d}\int_s^T\left(\sum_{i=1}^d\frac{\partial}{\partial x_i}(\sigma^{ij}(x)u^i(r,x)\phi_t(r,x))\right)dW_r^j \qquad (37)\\ &+\int_{\mathbb{R}^d}\int_s^T\int_{I\!\!E}u^i(r,x)\mathcal{A}_e^*\phi_t(r-,x)\widetilde{\mu}(dr,de). \end{aligned}$$

We substitute this in the above equation and get

$$\begin{aligned}
&\int_{\mathbb{R}^d} u^i(s,x)\phi_t(s,x)dx \\
= \quad &\int_{\mathbb{R}^d} g(x)\phi_t(T,x)dx - \int_s^T \int_{\mathbb{R}^d} (\sigma^*\nabla u^i)(r,x)\phi_t(r,x)dxdW_r \\
&-\int_s^T \int_{\mathbb{E}} \int_{\mathbb{R}^d} u^i(r,x)\mathcal{A}_e^*\phi_t(r-,x)dx\widetilde{\mu}(dr,de) \\
&+\int_s^T \int_{\mathbb{R}^d} f(r,x,u^i(r,x),\sigma^*\nabla u^i(r,x),u^i(r,x+\beta(x,\cdot))-u^i(r,x))\phi_t(r,x)drdx.
\end{aligned}$$

Then by the change of variable, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} u^i(s,X_{t,s}(x))\phi(x)dx \\
= \quad &\int_{\mathbb{R}^d} g(X_{t,T}(x))\phi(x)dx - \int_{\mathbb{R}^d} \int_s^T \phi(x)(\sigma^*\nabla u^i)(r,X_{t,r}(x))dxdW_r \\
&-\int_{\mathbb{R}^d} \int_s^T \int_{\mathbb{E}} \phi(x)(u^i(s,X_{t,r-}(x)+\beta(X_{t,r-}(x),e))-u(s,X_{t,r-}(x)))\widetilde{\mu}(dr,de)dx \\
&+\int_{\mathbb{R}^d} \int_s^T \phi(x)f(r,X_{t,r}(x),u^i(r,X_{t,r}(x)),\sigma^*\nabla u^i(r,X_{t,r}(x)),\mathcal{A}_e u^i(r,X_{t,r-}(x)))drdx.
\end{aligned}$$

Since ϕ is arbitrary, we deduce that $(Y_s^{i,t,x}, Z_s^{i,t,x}, V_s^{i,t,x})$ solve the BSDE associated with $(g(X_{t,T}(x)), f)$, $\rho(x)dx$-a.e., where

$$\begin{aligned}
Y_s^{i,t,x} &= u^i(s,X_{t,s}(x)), Z_s^{i,t,x} = (\sigma^*\nabla u^i)(s,X_{t,s}(x)) \text{ and} \\
V_s^{i,t,x} &= u^i(s,X_{t,s-}(x)+\beta(X_{t,s-}(x),\cdot)) - u^i(s,X_{t,s-}(x)).
\end{aligned}$$

This means $\rho(x)dx$-a.e., we have

$$Y_s^{i,t,x} = g(X_{t,T}(x)) + \int_s^T f(r,X_{t,r}(x),Y_r^{i,t,x},Z_r^{i,t,x},V_r^{i,t,x})dr - \int_s^T Z_r^{i,t,x}dW_r - \int_s^T V_r^{i,t,x}(e)\widetilde{\mu}(dr,de).$$

Then the uniqueness follows from the uniqueness of the BSDE.

(ii) Existence : Let us set

$$F(s,x) \triangleq f(s,x,Y_s^{s,x},Z_s^{s,x},V_s^{s,x}).$$

Plugging into the facts that $f^0 \in \mathbf{L}^2_\rho([0,T]\times\mathbb{R}^d)$ and f is Lipschitz with respect to (y,z,v), then thanks to Prop. 5 we have $F(s,x) \in \mathbf{L}^2_\rho([0,T]\times\mathbb{R}^d)$. Since $g \in \mathbf{L}^2_\rho(\mathbb{R}^d)$, we can approximate them by a sequence of smooth functions with compact support (g_n, F_n) such that

$$\begin{cases} g_n \to g \text{ in } \mathbf{L}^2_\rho(\mathbb{R}^d), \\ F_n(s,x) \to F(s,x) \text{ in } \mathbf{L}^2_\rho([0,T]\times\mathbb{R}^d). \end{cases} \tag{38}$$

Denote $(Y_s^{n,t,x}, Z_s^{n,t,x}, V_s^{n,t,x}) \in \mathcal{S}^2\times\mathcal{H}^2\times\mathcal{L}^2$ the solution of the BSDE associated with (ξ_n, F_n), where $\xi_n = g_n(X_{t,T}(x))$, i.e.

$$Y_s^{n,t,x} = g_n(X_{t,T}(x)) + \int_s^T F_n(r,X_{t,r}(x))dr - \int_s^T Z_r^{n,t,x}dW_r - \int_s^T \int_{\mathbb{E}} V_r^{n,t,x}(e)\widetilde{\mu}(dr,de). \tag{39}$$

Since F_n and g_n are smooth enough, from the regularity result of the solution with respect to (t,x) (see the proof in Appendix), we know that $u_n(t,x) := Y_t^{n,t,x} \in C^{1,2}([0,T]\times\mathbb{R}^d)$ is the classic solution for the following PIDE:

$$\begin{cases} (\partial_t + \mathcal{L})u(t,x) + F_n(t,x) = 0 \\ u(T,x) = g_n(x). \end{cases} \tag{40}$$

Moreover, we know that $v_n(t,x) := Z_t^{n,t,x} = \sigma^* \nabla u_n(t,x)$. Besides, from the flow property, we can deduce that $Y_s^{n,t,x} = u_n(s, X_{t,s}(x))$ and $Z_s^{n,t,x} = \sigma^* \nabla u_n(s, X_{t,s}(x))$, as well as the representation of the jump part

$$V_s^{n,t,x} := w_n(s, X_{t,s}(x)) := u_n(s, X_{t,s-}(x) + \beta(X_{t,s-}(x), \cdot)) - u_n(s, X_{t,s-}(x)),\ a.s..$$

Applying the equivalence of norms result and the Proposition 5, we have

$$\begin{aligned}
& \int_{\mathbb{R}^d} \int_t^T (|u_n(s,x)|^2 + |\sigma^* \nabla u_n(s,x)|^2) ds \rho(x) dx \\
\leqslant\ & \frac{1}{c} \int_{\mathbb{R}^d} \int_t^T E(|u_n(s, X_{t,s}(x))|^2 + |\sigma^* \nabla u_n(s, X_{t,s}(x))|^2) ds \rho(x) dx \\
=\ & \frac{1}{c} \int_{\mathbb{R}^d} \int_t^T E(|Y_s^{n,t,x}|^2 + |Z_s^{n,t,x}|^2) ds \rho(x) dx \\
\leqslant\ & c' (\int_{\mathbb{R}^d} E|g_n(X_{t,T}(x))|^2 \rho(x) dx + \int_{\mathbb{R}^d} \int_t^T E|F_n(s, X_{t,s}(x))|^2 ds \rho(x) dx) \\
\leqslant\ & c'' (\int_{\mathbb{R}^d} |g_n(x)|^2 \rho(x) dx + \int_{\mathbb{R}^d} \int_t^T |F_n(s,x)|^2 ds \rho(x) dx) < +\infty.
\end{aligned}$$

Therefore, we have shown that $\forall n$, $u_n \in \mathcal{H}_T$ solves the PIDE associated with (g_n, f_n), and for every $\phi \in C_c^{1,\infty}([0,T] \times \mathbb{R}^d)$,

$$\begin{aligned}
& \int_t^T (u_n(s,\cdot), \partial_s \phi(s,\cdot)) ds + (u_n(t,\cdot), \phi(t,\cdot)) - (g_n(\cdot), \phi(T,\cdot)) - \int_t^T (\mathcal{L} u_n(s,\cdot), \phi(s,\cdot)) ds \\
& = \int_t^T (F_n(s,\cdot), \phi(s,\cdot)) ds.
\end{aligned} \tag{41}$$

Now for $m, n \in N$, applying Itô's formula to $|Y_s^{m,t,x} - Y_s^{n,t,x}|^2$, we get for any $s \in [t,T]$,

$$\begin{aligned}
& E|Y_s^{m,t,x} - Y_s^{n,t,x}|^2 + E\int_s^T |Z_r^{m,t,x} - Z_r^{n,t,x}|^2 dr + E\int_s^T \int_{\mathbb{E}} |V_r^{m,t,x}(e) - V_r^{n,t,x}(e)|^2 \lambda(de) dr \\
=\ & E|g_m(X_{t,T}(x)) - g_n(X_{t,T}(x))|^2 \\
& +2E\int_s^T (Y_r^{m,t,x} - Y_r^{n,t,x})(F_m(r, X_{t,r}(x)) - F_n(r, X_{t,r}(x))) dr.
\end{aligned}$$

Combining the properties of f_m and f_n with the basic inequality $2ab \leqslant \varepsilon a^2 + \varepsilon^{-1} b^2$, we have:

$$\begin{aligned}
& E|Y_s^{m,t,x} - Y_s^{n,t,x}|^2 + E\int_s^T |Z_r^{m,t,x} - Z_r^{n,t,x}|^2 dr \\
\leqslant\ & E|g_m(X_{t,T}(x)) - g_n(X_{t,T}(x))|^2 + \varepsilon E \int_s^T |Y_r^{m,t,x} - Y_r^{n,t,x}|^2 dr \\
& + \frac{1}{\varepsilon} E \int_s^T |F_m(r, X_{t,r}(x)) - F_n(r, X_{t,r}(x))|^2 dr.
\end{aligned}$$

It follows again from the equivalence of norms that

$$\begin{aligned}
& \int_{\mathbb{R}^d} E|Y_s^{m,t,x} - Y_s^{n,t,x}|^2 \rho(x) dx \\
\leqslant\ & \int_{\mathbb{R}^d} E|g_m(X_{t,T}(x)) - g_n(X_{t,T}(x))|^2 \rho(x) dx + \varepsilon \int_{\mathbb{R}^d} E \int_s^T |Y_r^{m,t,x} - Y_r^{n,t,x}|^2 dr \rho(x) dx \\
& + \frac{1}{\varepsilon} \int_{\mathbb{R}^d} E \int_s^T |F_m(r, X_{t,r}(x)) - F_n(r, X_{t,r}(x))|^2 dr \rho(x) dx \\
\leqslant\ & \varepsilon C \int_{\mathbb{R}^d} E \int_s^T |Y_r^{m,t,x} - Y_r^{n,t,x}|^2 dr \rho(x) dx + C \int_{\mathbb{R}^d} |g_m(x) - g_n(x)|^2 \rho(x) dx \\
& + \frac{C}{\varepsilon} \int_{\mathbb{R}^d} \int_s^T |F_m(r,x) - F_n(r,x)|^2 dr \rho(x) dx.
\end{aligned}$$

Choosing ε appropriately, by Gronwall's inequality and the convergence of F_n and g_n, we get as $m, n \to \infty$,

$$\sup_{t \leqslant s \leqslant T} \int_{\mathbb{R}^d} E|Y_s^{m,t,x} - Y_s^{n,t,x}|^2 \rho(x) dx \to 0,$$

which implies immediately as $m, n \to \infty$,

$$\int_{\mathbb{R}^d} E\int_s^T |Y_r^{m,t,x} - Y_r^{n,t,x}|^2 dr\rho(x)dx + \int_{\mathbb{R}^d} E\int_s^T |Z_r^{m,t,x} - Z_r^{n,t,x}|^2 dr\rho(x)dx \to 0.$$

Thanks again to the equivalence of norms (12), we get as $m, n \to \infty$:

$$\begin{aligned} & \int_t^T\int_{\mathbb{R}^d} (|u_m(s,x) - u_n(s,x)|^2 + |\sigma^*\nabla u_m(s,x) - \sigma^*\nabla u_n(s,x)|^2)\rho(x)dxds \\ \leqslant\quad & \frac{1}{C}\int_t^T\int_{\mathbb{R}^d} E(|(u_m - u_n)(s, X_{t,s}(x))|^2 + |(\sigma^*\nabla u_m - \sigma^*\nabla u_n)(s, X_{t,s}(x))|^2)\rho(x)dxds \\ =\quad & \frac{1}{C}\int_t^T\int_{\mathbb{R}^d} E(|Y_s^{m,t,x} - Y_s^{n,t,x}|^2 + |Z_s^{m,t,x} - Z_s^{n,t,x}|^2)\rho(x)dxds \to 0, \end{aligned}$$

which means that u_n is Cauchy sequence in $\mathcal{H}_T$. Denote its limit as u, then $u \in \mathcal{H}_T$. Moreover, since $X_{t,s}(x)$ has at most countable jumps, using again the equivalence of norms, we can deduce that

$$\int_t^T\int_{\mathbb{R}^d} |w_m(s,x) - w_n(s,x)|^2\rho(x)dxds \to 0,$$

henceforth there exists at least a subsequence $\{u_{n_k}\}$, such that $dt \otimes dP \otimes \rho(x)dx$-a.e.,

$$\begin{cases} Y_s^{n_k,t,x} &= u_{n_k}(s, X_{t,s}(x)) \to u(s, X_{t,s}(x)) =: Y_s^{t,x}, \\ Z_s^{n_k,t,x} &= \sigma^*\nabla u_{n_k}(s, X_{t,s}(x)) \to \sigma^*\nabla u(s, X_{t,s}(x)) =: Z_s^{t,x}, \\ V_s^{n_k,t,x} &= w_{n_k}(s, X_{t,s}(x)) \to u(s, X_{t,s-}(x) + \beta(X_{t,s-}(x), \cdot)) - u(s, X_{t,s-}(x)) := V_s^{t,x}, \end{cases}$$

Then we get the desired probabilistic representations (16). Passing limit in (41), we have for every ϕ,

$$\begin{aligned} & \int_t^T (u(s,\cdot), \partial_s\phi(s,\cdot))ds + (u(t,\cdot), \phi(t,\cdot)) - (g(\cdot), \phi(T,\cdot)) - \int_t^T (u(s,\cdot), \mathcal{L}^*\phi(s,\cdot))ds \\ & = \int_t^T (f(s,\cdot, u(s,\cdot), \sigma^*\nabla u(s,\cdot), u(s, x + \beta(x,\cdot)) - u(s,x)), \phi(s,\cdot))ds, \end{aligned} \tag{42}$$

which means that $u \in \mathcal{H}_T$ is the weak solution of (1). □

5.4. Proof of the tightness of the sequence $(\pi_n)_{n\in\mathbb{N}}$

Recall first that $\nu_n(dt,dx) = n(u_n - h)^-(t,x)dtdx$ and $\pi_n(dt,dx) = \rho(x)\nu_n(dt,dx)$ where u_n is the solution of the PIDE's (22).

Lemma 2. *The sequence of measure $(\pi_n)_{n\in\mathbb{N}}$ is tight.*

Proof. Since here we need to deal with the additional jump part, we adapt the proof of Theorem 4 in [3]. We shall prove that for every $\epsilon > 0$, there exists some constant K such that

$$\int_0^T\int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geqslant 2K\}}\pi_n(ds,dx) \leqslant \epsilon,\ \forall n \in N. \tag{43}$$

We first write

$$\begin{aligned} & \int_0^T\int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geqslant 2K\}}\pi_n(ds,dx) \\ & = \int_0^T\int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geqslant 2K\}} \left(\mathbf{1}_{\{|X_{0,s}^{-1}(x) - x| \leqslant K\}} + \mathbf{1}_{\{|X_{0,s}^{-1}(x) - x| \geqslant K\}}\right)\pi_n(ds,dx) \\ & := I_K^n + L_K^n, \quad P - a.s. \end{aligned}$$

Taking expectation yields

$$\int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|x| \geqslant 2K\}} \pi_n(ds, dx) = EI_K^n + EL_K^n.$$

By (25) and for $K \geqslant 2\|b\|_\infty T$, we get

$$\begin{aligned} EL_K^n &\leqslant \int_0^T \int_{\mathbb{R}^d} \mathbb{P}\left(\sup_{0 \leqslant r \leqslant T} \left| X_{0,r}^{-1}(x) - x \right| \geqslant K \right) \pi_n(ds, dx) \\ &\leqslant \left(C_1 \exp(-C_2 K^2) + C_3 \exp(-C_4 K) \right) \pi_n \left([0,T] \times \mathbb{R}^d \right) \\ &\leqslant C_1' \exp(-C_2 K^2) + C_3' \exp(-C_4 K), \end{aligned}$$

so $EL_K^n \leqslant \epsilon$ for K sufficiently large.

On the other hand, if $|x| \geqslant 2K$ and $\left| X_{0,s}^{-1}(x) - x \right| \leqslant K$ then $\left| X_{0,s}^{-1}(x) \right| \geqslant K$. Therefore

$$\begin{aligned} EI_K^n &\leqslant E \int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\left\{ \left| X_{0,s}^{-1}(x) \right| \geqslant K \right\}} \rho(x) \nu_n(ds, dx) \\ &= E \int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\left\{ \left| X_{0,s}^{-1}(x) \right| \geqslant K \right\}} \rho(x) n(u_n - h)^-(s,x) ds dx \end{aligned}$$

which, by the change of variable $y = X_{0,s}^{-1}(x)$, becomes

$$\begin{aligned} & E \int_0^T \int_{\mathbb{R}^d} \mathbf{1}_{\{|y| \geqslant K\}} \rho(X_{0,s}(y)) J(X_{0,s}(y)) n(u_n - h)^-(s, X_{0,s}(y)) ds dy \\ & \leqslant E \int_{\mathbb{R}^d} \rho(x) \left(\rho(x)^{-1} \mathbf{1}_{\{|x| \geqslant K\}} \sup_{0 \leqslant r \leqslant T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right) K_T^{n,0,x} dx \\ & \leqslant \left(E \int_{\mathbb{R}^d} \left(K_T^{n,0,x} \right)^2 \rho(x) dx \right)^{1/2} \\ & \quad \left(E \int_{\mathbb{R}^d} \left(\rho(x)^{-1} \mathbf{1}_{\{|x| \geqslant K\}} \sup_{0 \leqslant r \leqslant T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right)^2 \rho(x) dx \right)^{1/2} \\ & \leqslant C \left(E \int_{\mathbb{R}^d} \left(\rho(x)^{-1} \mathbf{1}_{\{|x| \geqslant K\}} \sup_{0 \leqslant r \leqslant T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right)^2 \rho(x) dx \right)^{1/2} \end{aligned}$$

where the last inequality is a consequence of (24). It is now sufficient to prove that

$$\int_{\mathbb{R}^d} \rho(x)^{-1} E \left[\left(\sup_{0 \leqslant r \leqslant T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right)^2 \right] dx < \infty. \tag{44}$$

Note that

$$\begin{aligned} & E \left[\left(\sup_{0 \leqslant r \leqslant T} \rho(X_{0,r}(x)) J(X_{0,r}(x)) \right)^2 \right] \\ & \leqslant \left[E \left(\sup_{0 \leqslant r \leqslant T} |\rho(X_{0,r}(x))| \right)^4 \right]^{1/2} \left[E \left(\sup_{0 \leqslant r \leqslant T} |J(X_{0,r}(x))| \right)^4 \right]^{1/2} \\ & \leqslant C \left[E \left(\sup_{0 \leqslant r \leqslant T} |\rho(X_{0,r}(x))| \right)^4 \right]^{1/2}. \end{aligned}$$

Therefore it is sufficient to prove that:

$$\int_{\mathbb{R}^d} \frac{1}{\rho(x)} \left(E \left[\sup_{t \leqslant r \leqslant T} |\rho(X_{t,r}(x))|^4 \right] \right)^{1/2} dx < \infty.$$

Since $\rho(x) \leqslant 1$, we have

$$
\begin{aligned}
E\left[\sup_{t \leqslant r \leqslant T} |\rho(X_{t,r}(x))|^4\right] \quad &\leqslant \quad E\left[\sup_{t \leqslant r \leqslant T} |\rho(X_{t,r}(x))|^4 \mathbf{1}_{\left\{\sup_{t \leqslant r \leqslant T} \left|X_{t,r}(x)-x\right| \leqslant \frac{|x|}{2}\right\}}\right] \\
&\quad +\mathbb{P}\left(\sup_{t \leqslant r \leqslant T} |X_{t,r}(x) - x| \geqslant \frac{|x|}{2}\right) \\
&= \quad : A(x) + B(x).
\end{aligned}
$$

If $\sup_{t \leqslant r \leqslant T} |X_{t,r}(x) - x| \leqslant \frac{|x|}{2}$ then $|X_{t,r}(x)| \geqslant \frac{|x|}{2}$ and so $|\rho(X_{t,r}(x))| \leqslant \left(1 + \frac{|x|}{2}\right)^{-p}$. Thus we have that $A(x) \leqslant \left(1 + \frac{|x|}{2}\right)^{-4p}$ and so $\int_{\mathbb{R}^d} (1 + |x|)^p \, A(x)^{1/2} dx < \infty$. On the other hand, if $|x| \geqslant 4\|b\|_\infty T$, then (the same argument as in the existence proof step 2 of Theorem 4 in [3] for the Itô integral with respect to the Brownian motion; and see e.g. Theorem 5.2.9 in [1] for the integral with respect to the compensated Poisson random measure)

$$
\begin{aligned}
B(x) &\leqslant \mathbb{P}\left(\sup_{t \leqslant s \leqslant T} \left|\int_0^s \sigma(X_{0,r}(x)) dW_r\right| \geqslant \frac{|x|}{8}\right) \\
&\quad + \mathbb{P}\left(\sup_{t \leqslant s \leqslant T} \left|\int_0^s \int_{\mathbb{E}} \beta(X_{0,r-}(x), e)\tilde{\mu}(dr, de)\right| \geqslant \frac{|x|}{8}\right) \\
&\leqslant C_1 \, \exp(-C_2|x|^2) + C_3 \, \exp(-C_4|x|)
\end{aligned}
$$

and so $\int_{\mathbb{R}^d} (1 + |x|)^p \, B(x)^{1/2} dx < \infty$.

□

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