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Reasoning about negligibility and proximity in the set of all hyperreals

Philippe Balbiani

Institut de Recherche en Informatique de Toulouse, Toulouse University, 118 Route de Narbonne, 31062 Toulouse Cedex 9, France

A B S T R A C T

We consider the binary relations of negligibility, comparability and proximity in the set of all hyperreals. Associating with negligibility, comparability and proximity the binary predicates N, C and P and the connectives $[N]$, $[C]$ and $[P]$, we consider a first-order theory based on these predicates and a modal logic based on these connectives. We investigate the axiomatization/completeness and the decidability/complexity of this first-order theory and this modal logic.

Keywords:
Qualitative reasoning
First-order theory
Modal logic
Axiomatization/completeness
Decidability/complexity
Definability

1. Introduction

Within the context of the modeling of the behavior of a complex system, when numeric information is useless or when available information is unprecise, the use of qualitative reasoning is often required [20,22]. It is a fact that engineering practice usually induces the experts to handle the symbols $\ll$ ("is negligible with respect to") and $\simeq$ ("is in the proximity of") while simplifying complex equations. Nevertheless, this rule of thumb has to be formalized if one intends to mechanically reproduce by means of algorithms the engineers ability to reason about the behavior of a complex system. This formalization task is at the heart of the qualitative reasoning enterprise.

Restricting his discussion to the relative orders of magnitude paradigm, Raiman [19] introduced a formal system, FOG, based on the binary relations Ne ("is negligible with respect to"), Co ("is comparable to") and Vo ("is in the proximity of"). Without studying its completeness, he justified the use of FOG by showing the soundness of the inference rules of FOG with respect to nonstandard analysis, i.e. by interpreting Ne, Co and Vo as follows: $Ne(a,b)$ iff a/b is infinitesimal, $Co(a,b)$ iff a/b is appreciable and $Vo(a,b)$ iff $a/b-1$ is infinitesimal for each hyperreals a, b.

E-mail address: philippe.balbiani@irit.fr.

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Variants of *FOG* have been later introduced in order, for example, to incorporate numeric information [10] or to relate together different types of order-of-magnitude knowledge [21]. See also [11,17]. Nevertheless, there is something wrong with them: if the soundness or the complexity of the proposed formal systems are sometimes examined, their completeness with respect to such-and-such semantics is never studied. The first purpose of the present paper is to investigate the axiomatization/completeness and the decidability/complexity of the first-order theory the binary relations of negligibility, comparability and proximity give rise to in the set of all positive hyperreals.

Recently, modal languages for qualitative order-of-magnitude reasoning have been considered [4,6]. See also [5,16]. In these modal languages, connectives are associated to the binary relations of negligibility and comparability. Nevertheless, the first-order conditions put on these binary relations in the Kripke frames used to interpret these modal languages do not constitute a complete axiomatization of their first-order theory in the set of all positive hyperreals. The second purpose of the present paper is to investigate the axiomatization/completeness and the decidability/complexity of the modal logic the binary relations of negligibility, comparability and proximity give rise to in the set of all positive hyperreals.

The binary relations of negligibility, comparability and proximity in the set of all positive hyperreals will be presented in section 2. Section 3 will associate with negligibility, comparability and proximity the binary predicates N, C and P and will study the first-order theory based on these predicates. Section 4 will associate with negligibility, comparability and proximity the connectives $[N]$, $[C]$ and $[P]$ and will study the modal logic based on these connectives. Variants of our first-order and modal languages based, in the set of all positive hyperreals, on the relation of precedence and the operation of addition will be presented in section 5.

2. Hyperreals

2.1. *What are the hyperreals?*

In the set of all reals, there are no such things as infinitely small and infinitely large numbers. While reals all belong to the same order of magnitude, it is the fact that hyperreals are either infinitesimal, appreciable or unlimited which sets them apart. The thing is that hyperreals contains the reals as a subset, but also contains infinitely small (infinitesimal) numbers and infinitely large (unlimited) numbers. In mathematics, these new entities offer new definitions of familiar concepts like convergence and continuity [15]. In other areas of science and technology, they justify the algebraic processing of small numbers and large numbers that researchers and engineers often do—witness their use in multifarious domains like market models [9] for modeling option pricing and in electrical networks [23] for modeling infinite networks. In computer science and artificial intelligence, infinitesimal numbers and unlimited numbers have been used for analyzing texts [2] and reasoning about time in deductive databases [14].

2.2. *Ultrapower construction of the hyperreals*

Following the introduction to non-standard analysis proposed in [15], let us introduce a number of basic concepts. Let I be the set of all positive integers. We use $\mathbb{R}^I$ to denote the set of all real-valued sequences, $\mathcal{P}(I)$ to denote the power set of I and $\mathcal{P}(\mathcal{P}(I))$ to denote the power set of $\mathcal{P}(I)$. For a start, suppose that the notion of a large set of positive integers, in a sense that is to be determined, is at our disposal. Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^I$, we shall say that $\mathbf{a}$ agrees with $\mathbf{b}$ iff $\{n \in I\colon \mathbf{a}(n) = \mathbf{b}(n)\}$ is large. The set $\{n \in I\colon \mathbf{a}(n) = \mathbf{b}(n)\}$ may be thought of as a measure of the extent to which the statement "$\mathbf{a}$ agrees with $\mathbf{b}$" is true. In order to ensure that agreement between real-valued sequences is a non-trivial equivalence relation, the following conditions must be satisfied:

- I is large,
- $\emptyset$ is not large,
- for all $X, Y \in \mathcal{P}(I)$, if X is large and Y is large then $X \cap Y$ is large.

Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^I$, we shall say that $\mathbf{a}$ precedes $\mathbf{b}$ iff $\{n \in I\colon \mathbf{a}(n) < \mathbf{b}(n)\}$ is large. The set $\{n \in I\colon \mathbf{a}(n) < \mathbf{b}(n)\}$ may be thought of as a measure of the extent to which the statement "**a** precedes **b**" is true. In order to ensure that precedence between real-valued sequences is a total relation modulo agreement, the following condition must be satisfied:

- for all $X, Y \in \mathcal{P}(I)$, if $X \cup Y$ is large then X is large or Y is large.

The above conditions suggest to determine the notion of a large set of positive integers by means of ultrafilters on I. A set $U \in \mathcal{P}(\mathcal{P}(I))$ is said to have the finite intersection property iff the intersection of any finite number of elements of U is non-empty. A set $U \in \mathcal{P}(\mathcal{P}(I))$ is said to be an ultrafilter on I iff

- $I \in U$,
- $\emptyset \notin U$,
- for all $X, Y \in \mathcal{P}(I)$, $X \cap Y \in U$ iff $X \in U$ and $Y \in U$,
- for all $X, Y \in \mathcal{P}(I)$, $X \cup Y \in U$ iff $X \in U$ or $Y \in U$.

These requirements imply that for all $X \in \mathcal{P}(I)$, $I \setminus X \in U$ iff $X \notin U$. A large supply of ultrafilters on I is provided by the ultrafilter theorem.

Proposition 1. *Let $U \in \mathcal{P}(\mathcal{P}(I))$. If U has the finite intersection property then there exists a set $U' \in \mathcal{P}(\mathcal{P}(I))$ such that $U \subseteq U'$ and U' is an ultrafilter on I.*

Let $n \in I$ be a positive integer. Consider the set $U_n = \{X \in \mathcal{P}(I)\colon n \in X\}$. Clearly, U_n is an ultrafilter on I. We call such ultrafilters the principal ultrafilters on I. Let $U_\omega = \{X \in \mathcal{P}(I)\colon I \setminus X \text{ is finite}\}$. As the reader can easily ascertain, U_ω has the finite intersection property. Hence, by the ultrafilter theorem, there exists a set $U'_\omega \in \mathcal{P}(\mathcal{P}(I))$ such that $U_\omega \subseteq U'_\omega$ and U'_ω is an ultrafilter on I. Such ultrafilters are called the non-principal ultrafilters on I. It is a well-known fact that principal ultrafilters and non-principal ultrafilters constitute a partition of the set of all ultrafilters on I. Let U be an ultrafilter on I. We define a binary relation $\equiv_U$ on $\mathbb{R}^I$ by putting

- $\mathbf{a} \equiv_U \mathbf{b}$ iff $\{n \in I\colon \mathbf{a}(n) = \mathbf{b}(n)\} \in U$

for each $\mathbf{a}, \mathbf{b} \in \mathbb{R}^I$. Note that $\equiv_U$ is an equivalence relation on $\mathbb{R}^I$. Given $\mathbf{a} \in \mathbb{R}^I$, we call the set of all $\mathbf{b} \in \mathbb{R}^I$ such that $\mathbf{a} \equiv_U \mathbf{b}$, denoted by $\mid \mathbf{a} \mid_{|\equiv_U}$, the equivalence class with $\mathbf{a}$ as its representative modulo $\equiv_U$. The set of all equivalence classes modulo $\equiv_U$, denoted by $\mathbb{R}^I_{|\equiv_U}$, is called the quotient set of $\mathbb{R}^I$ modulo $\equiv_U$. We call the elements of $\mathbb{R}$ the real numbers while the elements of $\mathbb{R}^I_{|\equiv_U}$ are called the hyperreal numbers modulo $\equiv_U$. On $\mathbb{R}^I_{|\equiv_U}$, we define the binary relation $\prec_{|\equiv_U}$ and the binary operations $\oplus_{|\equiv_U}$ and $\otimes_{|\equiv_U}$ by putting

- $\mid \mathbf{a} \mid_{\equiv_U} \prec_{|\equiv_U} \mid \mathbf{b} \mid_{\equiv_U}$ iff $\{n \in I\colon \mathbf{a}(n) < \mathbf{b}(n)\} \in U$,
- $\mid \mathbf{a} \mid_{\equiv_U} \oplus_{|\equiv_U} \mid \mathbf{b} \mid_{\equiv_U}$ is $\mid \mathbf{a} + \mathbf{b} \mid_{\equiv_U}$,
- $\mid \mathbf{a} \mid_{\equiv_U} \otimes |\equiv_U \mid \mathbf{b} \mid_{\equiv_U}$ is $\mid \mathbf{a} \times \mathbf{b} \mid_{\equiv_U}$

for each $\mathbf{a}, \mathbf{b} \in \mathbb{R}^I$. The binary relation $\prec_{|\equiv_U}$ and the binary operations $\oplus_{|\equiv_U}$ and $\otimes_{|\equiv_U}$ are well-defined seeing that for all $\mathbf{a}', \mathbf{b}' \in \mathbb{R}^I$ and for all $\mathbf{a}'', \mathbf{b}'' \in \mathbb{R}^I$, if $\mathbf{a}' \equiv_U \mathbf{a}''$ and $\mathbf{b}' \equiv_U \mathbf{b}''$ then $\{n \in I\colon \mathbf{a}'(n) < \mathbf{b}'(n)\} \in U$ iff $\{n \in I\colon \mathbf{a}''(n) < \mathbf{b}''(n)\} \in U$, $\mid \mathbf{a}' + \mathbf{b}' \mid_{\equiv_U} = \mid \mathbf{a}'' + \mathbf{b}'' \mid_{\equiv_U}$ and $\mid \mathbf{a}' \times \mathbf{b}' \mid_{\equiv_U} = \mid \mathbf{a}'' \times \mathbf{b}'' \mid_{\equiv_U}$.

Proposition 2. *The structure* $\langle \mathbb{R}^I_{|\equiv_U}, \prec_{|\equiv_U}, \oplus_{|\equiv_U}, \otimes_{|\equiv_U} \rangle$ *is an ordered field.*

For all reals $r \in \mathbb{R}$, we define the real-valued sequence $\mathbf{r} \in \mathbb{R}^I$ by putting

- $\mathbf{r}(n) = r$

for each $n \in I$.

Proposition 3. *The map* $r \in \mathbb{R} \mapsto \mid \mathbf{r} \mid_{|\equiv_U} \in \mathbb{R}^I_{|\equiv_U}$ *is an ordered-preserving field isomorphism from* $\mathbb{R}$ *into* $\mathbb{R}^I_{|\equiv_U}$.

The construction of $\mathbb{R}^I_{|\equiv_U}$ as the quotient set of $\mathbb{R}^I$ modulo $\equiv_U$ depends on the choice of the ultrafilter U on I. It has been shown that

- for all principal ultrafilters U on I, $\langle \mathbb{R}^I_{|\equiv_U}, \prec_{|\equiv_U}, \oplus_{|\equiv_U}, \otimes_{|\equiv_U} \rangle$ is isomorphic to $\langle \mathbb{R}, <, +, \times \rangle$,
- for all non-principal ultrafilters U', U'' on I, $\langle \mathbb{R}^I_{|\equiv_{U'}}, \prec_{|\equiv_{U'}}, \oplus_{|\equiv_{U'}}, \otimes_{|\equiv_{U'}} \rangle$ is isomorphic to $\langle \mathbb{R}^I_{|\equiv_{U''}}, \prec_{|\equiv_{U''}}, \oplus_{|\equiv_{U''}}, \otimes_{|\equiv_{U''}} \rangle$.

Let U be a fixed non-principal ultrafilter on I. Given $\mathbf{a} \in \mathbb{R}^I$, we will denote more briefly as $\mid \mathbf{a} \mid$ the equivalence class $\mid \mathbf{a} \mid_{|\equiv_U}$ with $\mathbf{a}$ as its representative modulo $\equiv_U$. The quotient set $\mathbb{R}^I_{|\equiv_U}$ of $\mathbb{R}^I$ modulo $\equiv_U$ will be denoted more briefly by ${}^\star\mathbb{R}$. We will denote more briefly as $\prec^\star$ the binary relation $\prec_{|\equiv_U}$ on $\mathbb{R}^I_{|\equiv_U}$. The binary operations $\oplus_{|\equiv_U}$ and $\otimes_{|\equiv_U}$ on $\mathbb{R}^I_{|\equiv_U}$ will be denoted more briefly by $\oplus^\star$ and $\otimes^\star$. We shall say that the hyperreal $\mid \mathbf{a} \mid \in {}^\star\mathbb{R}$ is infinitesimal iff $\mid -\mathbf{r} \mid \prec^\star \mid \mathbf{a} \mid$ and $\mid \mathbf{a} \mid \prec^\star \mid \mathbf{r} \mid$ for each real $r \in \mathbb{R}$ such that $r > 0$. For example, if $\epsilon \in \mathbb{R}^I$ is the real-valued sequence defined by putting

- $\epsilon(n) = 1/n$

for each $n \in I$ then $\mid \epsilon \mid$ is infinitesimal. The hyperreal $\mid \mathbf{a} \mid \in {}^\star\mathbb{R}$ is said to be unlimited iff $\mid \mathbf{a} \mid \prec^\star \mid -\mathbf{r} \mid$ or $\mid \mathbf{r} \mid \prec^\star \mid \mathbf{a} \mid$ for each real $r \in \mathbb{R}$ such that $r > 0$. For example, if $\omega \in \mathbb{R}^I$ is the real-valued sequence defined by putting

- $\omega(n) = n$

for each $n \in I$ then $\mid \omega \mid$ is unlimited. We shall say that the hyperreal $\mid \mathbf{a} \mid \in {}^\star\mathbb{R}$ is appreciable iff $\mid \mathbf{a} \mid$ is neither infinitesimal nor unlimited. Hence, on ${}^\star\mathbb{R}$, we define the binary relations $\prec^\star_\epsilon$, $\prec^\star_\omega$ and $\prec^\star_1$ by putting

- $\mid \mathbf{a} \mid \prec^\star_\epsilon \mid \mathbf{b} \mid$ iff $\mid \mathbf{a} \mid \prec^\star \mid \mathbf{b} \mid$ and $\mid \mathbf{b} - \mathbf{a} \mid$ is infinitesimal,
- $\mid \mathbf{a} \mid \prec^\star_\omega \mid \mathbf{b} \mid$ iff $\mid \mathbf{a} \mid \prec^\star \mid \mathbf{b} \mid$ and $\mid \mathbf{b} - \mathbf{a} \mid$ is unlimited,
- $\mid \mathbf{a} \mid \prec^\star_1 \mid \mathbf{b} \mid$ iff $\mid \mathbf{a} \mid \prec^\star \mid \mathbf{b} \mid$ and $\mid \mathbf{b} - \mathbf{a} \mid$ is appreciable

for each $\mathbf{a}, \mathbf{b} \in \mathbb{R}^I$.

2.3. Primitive relations

Restricting our discussion to the set of all positive hyperreals (with typical members now denoted by a, b, etc.), let us examine the primitive relations that may be involved. For obvious reasons, the identity relation (denoted by $\equiv$) will be fundamental. But several additional relations arise of which the following three has been especially studied: negligibility (denoted by N), comparability (denoted by C) and proximity (denoted by P). According to Raiman [19] and his followers, a is negligible with respect to b iff a/b is infinitesimal, a is comparable to b iff a/b is appreciable and a is in the proximity of b iff $a/b-1$ is infinitesimal. Conditions on these three relations may be formulated in a first-order way. To begin with, an important aspect of negligibility is the absence of stops, seeing that for all infinitesimal numbers ϵ, $\epsilon \times a$ is negligible with respect to a and a is negligible with respect to a/ϵ. This aspect is expressed by the condition of seriality:

Ser(N): for all a, there exist b, c such that b is negligible with respect to a and a is negligible with respect to c.

Another important aspect of negligibility is flow, seeing that if a/c and c/b are infinitesimal then a/b is infinitesimal too. This aspect is expressed by the condition of transitivity:

Tra(N): for all a, b, if there exists c such that a is negligible with respect to c and c is negligible with respect to b then a is negligible with respect to b.

Reciprocally, seeing that the negligibility of a with respect to b implies the negligibility of a with respect to $\sqrt{a \times b}$ and the negligibility of $\sqrt{a \times b}$ with respect to b, the condition of density is called for:

Den(N): for all a, b, if a is negligible with respect to b then there exists c such that a is negligible with respect to c and c is negligible with respect to b.

The conditions on comparability and proximity are more evident than the preceding ones:

Ref(C): for all a, a is comparable to a.
Sym(C): for all a, b, if a is comparable to b then b is comparable to a.
Tra(C): for all a, b, if there exists c such that a is comparable to c and c is comparable to b then a is comparable to b.
Ref(P): for all a, a is in the proximity of a.
Sym(P): for all a, b, if a is in the proximity of b then b is in the proximity of a.
Tra(P): for all a, b, if there exists c such that a is in the proximity of c and c is in the proximity of b then a is in the proximity of b.

There are also mixed conditions connecting negligibility, comparability and proximity. To begin with, seeing that if a/b is infinitesimal then a/b is not appreciable, one has to consider the following condition of disjointness:

Dis(N, C): for all a, b, if a is negligible with respect to b then a is not comparable to b.

Moreover, seeing that if $a/b - 1$ is infinitesimal then a/b is appreciable, one has to consider the following condition of inclusion:

Inc(C, P): for all a, b, if a is in the proximity of b then a is comparable to b.

Finally, seeing that a/b is infinitesimal or a/b is appreciable or b/a is infinitesimal, one has to consider the following condition of universality:

$Uni(N, C)$: for all a, b, a is negligible with respect to b or a is comparable to b or b is negligible with respect to a.

What plausible conditions could be added? By $Ref(C)$, $Sym(C)$, $Tra(C)$, $Ref(P)$, $Sym(P)$ and $Tra(P)$, comparability and proximity are equivalence relations. By $Inc(C, P)$, every equivalence class modulo proximity is contained in exactly one equivalence class modulo comparability. Nevertheless, the above first-order conditions do not prevent equivalence classes modulo comparability and equivalence classes modulo proximity to be finite. This leads us to the following conditions of infinity where n denotes an arbitrary nonnegative integer:

$Inf_n(P, \equiv)$: every equivalence class modulo proximity contains at least n elements.
$Inf_n(C, P)$: every equivalence class modulo comparability contains at least n equivalence classes modulo proximity.

3. First-order theory

3.1. Syntax

It is now time to meet the first-order language we will be working with. We assume some familiarity with model theory. Readers wanting more details may refer, for example, to [8] or [12]. Our first-order theory is based on the idea of associating with negligibility, comparability and proximity the binary predicates N, C and P. The *formulas* are given by the rule:

- $\phi ::= N(x, y) \mid C(x, y) \mid P(x, y) \mid x \equiv y \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \forall x.\phi$

where x and y range over a countable set of *variables.* The *size of* ϕ*,* denoted by $\mid \phi \mid$, is defined as the number of symbols occurring in ϕ. We adopt the standard definitions for the remaining Boolean operations and for the existential quantifier. It is usual to omit parentheses if this does not lead to any ambiguity. We define the following abbreviations:

- $\bar{N}(x, y) := \neg N(x, y)$,
- $\bar{C}(x, y) := \neg C(x, y)$,
- $\bar{P}(x, y) := \neg P(x, y)$,
- $x \not\equiv y := \neg x \equiv y$.

$N(x, y)$, $C(x, y)$, $P(x, y)$ will be respectively read "x is negligible with respect to y", "x is comparable to y", "x is in the proximity of y".

3.2. Semantics

Formulas will be interpreted in *frames,* i.e. relational structures of the form $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ where $H_\mathcal{S}$ is a nonempty set and $N_\mathcal{S}$, $C_\mathcal{S}$ and $P_\mathcal{S}$ are binary relations on $H_\mathcal{S}$. We shall say that a in $H_\mathcal{S}$ is *reflexive* iff $N_\mathcal{S}(a, a)$. A frame $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ is said to be *normal* iff the following sentences hold in $\mathcal{S}$:

$Ser(N)$: $\forall x.\exists y.\exists z.(N(y, x) \wedge N(x, z))$.

$Tra(N)$: $\forall x.\forall y.(\exists z.(N(x,z) \wedge N(z,y)) \rightarrow N(x,y))$.
$Den(N)$: $\forall x.\forall y.(N(x,y) \rightarrow \exists z.(N(x,z) \wedge N(z,y)))$.
$Ref(C)$: $\forall x.C(x,x)$.
$Sym(C)$: $\forall x.\forall y.(C(x,y) \rightarrow C(y,x))$.
$Tra(C)$: $\forall x.\forall y.(\exists z.(C(x,z) \wedge C(z,y)) \rightarrow C(x,y))$.
$Ref(P)$: $\forall x.P(x,x)$.
$Sym(P)$: $\forall x.\forall y.(P(x,y) \rightarrow P(y,x))$.
$Tra(P)$: $\forall x.\forall y.(\exists z.(P(x,z) \wedge P(z,y)) \rightarrow P(x,y))$.
$Dis(N,C)$: $\forall x.\forall y.(N(x,y) \rightarrow \bar{C}(x,y))$.
$Inc(C,P)$: $\forall x.\forall y.(P(x,y) \rightarrow C(x,y))$.
$Uni(N,C)$: $\forall x.\forall y.(N(x,y) \vee C(x,y) \vee N(y,x))$.
$Inf_n(P,\equiv)$: $\forall x_1.\ldots\forall x_n.(\bigwedge\{P(x_i,x_j) \wedge x_i \not\equiv x_j\colon 1 \leq i < j \leq n\} \rightarrow \exists y.\bigwedge\{P(y,x_i) \wedge y \not\equiv x_i\colon 1 \leq i \leq n\})$.
$Inf_n(C,P)$: $\forall x_1.\ldots\forall x_n.(\bigwedge\{C(x_i,x_j) \wedge \bar{P}(x_i,x_j)\colon 1 \leq i < j \leq n\} \rightarrow \exists y.\bigwedge\{C(y,x_i) \wedge \bar{P}(y,x_i)\colon 1 \leq i \leq n\})$.

In the sentences $Inf_n(P,\equiv)$ and $Inf_n(C,P)$, n denotes an arbitrary nonnegative integer. Let $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ be a normal frame. By $Ref(C)$, $Sym(C)$ and $Tra(C)$, obviously, $C_\mathcal{S}$ is an equivalence relation on $H_\mathcal{S}$. In the sequel, the set of all elements equivalent to a in $H_\mathcal{S}$ modulo $C_\mathcal{S}$, denoted by $[a]_{C_\mathcal{S}}$, is called the *equivalence class modulo $C_\mathcal{S}$ with a as its representative.* The set of all equivalence classes of $H_\mathcal{S}$ modulo $C_\mathcal{S}$, denoted by $H_\mathcal{S}/C_\mathcal{S}$, is called the *quotient set of $H_\mathcal{S}$ modulo $C_\mathcal{S}$*. Let $\prec_\mathcal{S}$ be the binary relation on $H_\mathcal{S}/C_\mathcal{S}$ defined by:

- $[a]_{C_\mathcal{S}} \prec_\mathcal{S} [b]_{C_\mathcal{S}}$ iff there exist c, d in $H_\mathcal{S}$ such that $C_\mathcal{S}(a,c)$, $C_\mathcal{S}(b,d)$ and $N_\mathcal{S}(c,d)$.

It is a rather remarkable fact that

Lemma 4. *$(H_\mathcal{S}/C_\mathcal{S}, \prec_\mathcal{S})$ is a dense linear proper order without endpoints.*

Proof. Density follows from $Den(N)$, irreflexivity and transitivity follow from $Tra(N)$, $Dis(N,C)$ and $Uni(N,C)$, linearity follows from $Uni(N,C)$ and absence of endpoints follows from $Ser(N)$. $\square$

By $Ref(P)$, $Sym(P)$ and $Tra(P)$, obviously, $P_\mathcal{S}$ is an equivalence relation on $H_\mathcal{S}$. In the sequel, the set of all elements equivalent to a in $H_\mathcal{S}$ modulo $P_\mathcal{S}$, denoted by $[a]_{P_\mathcal{S}}$, is called the *equivalence class modulo $P_\mathcal{S}$ with a as its representative.* By the sentences $Inf_n(P,\equiv)$, it is a simple matter to check that every equivalence class in $H_\mathcal{S}$ modulo $P_\mathcal{S}$ is made up of infinitely many elements. The set of all equivalence classes of $H_\mathcal{S}$ modulo $P_\mathcal{S}$, denoted by $H_\mathcal{S}/P_\mathcal{S}$, is called the *quotient set of $H_\mathcal{S}$ modulo $P_\mathcal{S}$*. By $Inc(C,P)$ and the sentences $Inf_n(C,P)$, it is worth noting at this point the following: every equivalence class in $H_\mathcal{S}$ modulo $C_\mathcal{S}$ is made up of infinitely many equivalence classes in $H_\mathcal{S}$ modulo $P_\mathcal{S}$. The truth is that there exist normal frames in each infinite power. We should consider, for instance, the frame $\mathcal{S}_{PH} = (H_{\mathcal{S}_{PH}}, N_{\mathcal{S}_{PH}}, C_{\mathcal{S}_{PH}}, P_{\mathcal{S}_{PH}})$. Its set $H_{\mathcal{S}_{PH}}$ of elements consists of all positive hyperreals whereas:

- $N_{\mathcal{S}_{PH}}(a,b)$ iff a/b is infinitesimal,
- $C_{\mathcal{S}_{PH}}(a,b)$ iff a/b is appreciable,
- $P_{\mathcal{S}_{PH}}(a,b)$ iff $a/b-1$ is infinitesimal.

Clearly, $\mathcal{S}_{PH}$ is uncountable and normal. A countable structure approximating $\mathcal{S}_{PH}$ is $\mathcal{S}_{QQ} = (H_{\mathcal{S}_{QQ}}, N_{\mathcal{S}_{QQ}}, C_{\mathcal{S}_{QQ}}, P_{\mathcal{S}_{QQ}})$. Its set $H_{\mathcal{S}_{QQ}}$ of elements consists of all triples of positive rationals whereas:

- $N_{\mathcal{S}_{QQ}}((q_1,q_2,q_3),(r_1,r_2,r_3))$ iff $q_1 < r_1$,

- $C_{\mathcal{S}_{QQ}}((q_1, q_2, q_3), (r_1, r_2, r_3))$ iff $q_1 = r_1$,
- $P_{\mathcal{S}_{QQ}}((q_1, q_2, q_3), (r_1, r_2, r_3))$ iff $q_1 = r_1$ and $q_2 = r_2$.

Clearly, $\mathcal{S}_{QQ}$ is countable and normal. Now, let us compare normal frames together. The proof of the next result necessitates the use of pebble games over $(\mathcal{S}, [a_1, \ldots, a_n])$ and $(\mathcal{S}', [a'_1, \ldots, a'_n])$. See [12] for details. Let m, n be nonnegative integers. In the n-pebble m-game over $(\mathcal{S}, [a_1, \ldots, a_n])$ and $(\mathcal{S}', [a'_1, \ldots, a'_n])$, we have n pebbles $\alpha_1, \ldots, \alpha_n$ for $\mathcal{S}$ and n pebbles $\alpha'_1, \ldots, \alpha'_n$ for $\mathcal{S}'$. Initially, each α_i is placed on a_i and each α'_i is placed on a'_i. Each play consists of a finite sequence of m moves. In its j-th move, the first player selects a normal frame, either $\mathcal{S}$ or $\mathcal{S}'$, and a pebble for this structure. If it selects $\mathcal{S}$ and α_i then the first player places α_i on some element of $\mathcal{S}$ and the second player places α'_i on some element of $\mathcal{S}'$. If it selects $\mathcal{S}'$ and α'_i then the first player places α'_i on some element of $\mathcal{S}'$ and the second player places α_i on some element of $\mathcal{S}$. The second player wins the game if for each $j \leq m$, the elements of $\mathcal{S}$ marked by $\alpha_1, \ldots, \alpha_n$ and the elements of $\mathcal{S}'$ marked by $\alpha'_1, \ldots, \alpha'_n$ constitute a partial isomorphism between the two normal frames.

Proposition 5. *Let $\mathcal{S}$, $\mathcal{S}'$ be normal frames. If $\mathcal{S}$ is countable then $\mathcal{S}$ is elementary embeddable in $\mathcal{S}'$.*

Proof. Let $\mathcal{S} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}})$, $\mathcal{S}' = (H_{\mathcal{S}'}, N_{\mathcal{S}'}, C_{\mathcal{S}'}, P_{\mathcal{S}'})$ be normal frames. Suppose $\mathcal{S}$ is countable. Let g be an injective homomorphism on $(H_{\mathcal{S}}/C_{\mathcal{S}}, \prec_{\mathcal{S}})$ to $(H_{\mathcal{S}'}/C_{\mathcal{S}'}, \prec_{\mathcal{S}'})$. Since $\mathcal{S}$ is countable, such an injective homomorphism exists. For each equivalence class $[a]_{C_{\mathcal{S}}}$ in $H_{\mathcal{S}}$ modulo $C_{\mathcal{S}}$, let $h_{[a]_{C_{\mathcal{S}}}}$ be an injective homomorphism on $([a]_{C_{\mathcal{S}}}, P_{\mathcal{S}|[a]_{C_{\mathcal{S}}}})$ to $(g([a]_{C_{\mathcal{S}}}), P_{\mathcal{S}'g([a]_{C_{\mathcal{S}}})})$. Since $\mathcal{S}$ is countable, such an injective homomorphism exists. Let f be the mapping on $H_{\mathcal{S}}$ to $H_{\mathcal{S}'}$ defined by:

- $f(a) = h_{[a]_{C_{\mathcal{S}}}}(a)$.

Obviously, for all nonnegative integers m, n and for all $a_1, \ldots, a_n$ in $H_{\mathcal{S}}$, the second player wins all n-pebble m-games over $(\mathcal{S}, [a_1, \ldots, a_n])$ and $(\mathcal{S}', [f(a_1), \ldots, f(a_n)])$. Hence, by [12, theorem 3.3.5], for all nonnegative integers n, for all $a_1, \ldots, a_n$ in $H_{\mathcal{S}}$ and for all formulas $\phi(x_1, \ldots, x_n)$ with variables among $x_1, \ldots, x_n$, $\mathcal{S} \models \phi(x_1, \ldots, x_n)\ [a_1, \ldots, a_n]$ iff $\mathcal{S}' \models \phi(x_1, \ldots, x_n)\ [f(a_1), \ldots, f(a_n)]$. Thus, f is an elementary embedding of $\mathcal{S}$ to $\mathcal{S}'$. $\square$

As a result, any two normal frames are elementary equivalent. In particular,

Corollary 6. *Let ϕ be a formula. The following conditions are equivalent:*

(1) *ϕ holds in every normal frame.*
(2) *ϕ holds in $\mathcal{S}_{PH}$.*
(3) *ϕ holds in $\mathcal{S}_{QQ}$.*

Proof. By Proposition 5, since $\mathcal{S}_{PH}$ and $\mathcal{S}_{QQ}$ are normal frames. $\square$

3.3. Axiomatization

Let *SQS* be the *first-order theory of N, C, P and* $\equiv$ that contains $Ser(N)$, $Tra(N)$, $Den(N)$, $Ref(C)$, $Sym(C)$, $Tra(C)$, $Ref(P)$, $Sym(P)$, $Tra(P)$, $Dis(N, C)$, $Inc(C, P)$, $Uni(N, C)$, $Inf_n(P)$ and $Inf_n(C, P)$ as proper axioms. The following proposition sums up all the simple properties that we can prove with the machinery available to us at present.

Proposition 7.

(1) *SQS is ω-categorical.*
(2) *SQS is not categorical in any uncountable power.*
(3) *SQS is maximal consistent.*

Proof. (1) Let $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$, $\mathcal{S}' = (H_{\mathcal{S}'}, N_{\mathcal{S}'}, C_{\mathcal{S}'}, P_{\mathcal{S}'})$ be countable normal frames. Let g be a bijective homomorphism on $(H_\mathcal{S}/C_\mathcal{S}, \prec_\mathcal{S})$ to $(H_{\mathcal{S}'}/C_{\mathcal{S}'}, \prec_{\mathcal{S}'})$. Since $\mathcal{S}$ and $\mathcal{S}'$ are countable, such a bijective homomorphism exists. For each equivalence class $[a]_{C_\mathcal{S}}$ in $H_\mathcal{S}$ modulo $C_\mathcal{S}$, let $h_{[a]_{C_\mathcal{S}}}$ be a bijective homomorphism on $([a]_{C_\mathcal{S}}, P_{\mathcal{S}|[a]_{C_\mathcal{S}}})$ to $(g([a]_{C_\mathcal{S}}), P_{\mathcal{S}'g([a]_{C_\mathcal{S}})})$. Since $\mathcal{S}$ and $\mathcal{S}'$ are countable, such a bijective homomorphism exists. Let f be the mapping on $H_\mathcal{S}$ to $H_{\mathcal{S}'}$ defined by:

- $f(a) = h_{[a]_{C_\mathcal{S}}}(a)$.

Obviously, f is an isomorphism on $\mathcal{S}$ to $\mathcal{S}'$.

(2) Let α, α' be uncountable powers. Let S, S' be respectively sets of power α, α'. Let $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S})$, $\mathcal{S}' = (H_{\mathcal{S}'}, N_{\mathcal{S}'}, C_{\mathcal{S}'})$ be respectively the normal frames of power α, α' such that $H_\mathcal{S} = \mathbb{Q}^{+*} \times \mathbb{Q}^{+*} \times S$, $H_{\mathcal{S}'} = \mathbb{Q}^{+*} \times \mathbb{Q}^{+*} \times S'$ whereas:

- $N_\mathcal{S}((q_1, q_2, q_3), (r_1, r_2, r_3))$ iff $q_1 < r_1$,
- $C_\mathcal{S}((q_1, q_2, q_3), (r_1, r_2, r_3))$ iff $q_1 = r_1$,
- $P_\mathcal{S}((q_1, q_2, q_3), (r_1, r_2, r_3))$ iff $q_1 = r_1$ and $q_2 = r_2$,
- $N_{\mathcal{S}'}((q'_1, q'_2, q'_3), (r'_1, r'_2, r'_3))$ iff $q'_1 < r'_1$,
- $C_{\mathcal{S}'}((q'_1, q'_2, q'_3), (r'_1, r'_2, r'_3))$ iff $q'_1 = r'_1$,
- $P_{\mathcal{S}'}((q'_1, q'_2, q'_3), (r'_1, r'_2, r'_3))$ iff $q'_1 = r'_1$ and $q'_2 = r'_2$.

A cardinality argument immediately gives that if $\alpha \neq \alpha'$ then $\mathcal{S}$ and $\mathcal{S}'$ are not isomorphic.

(3) Obviously, every formula is either true in $\mathcal{S}_{QQ}$ or false in $\mathcal{S}_{QQ}$. Hence, by Corollary 6, for all formulas ϕ, either ϕ is in SQS or $\neg\phi$ is in SQS. Thus, SQS is maximal consistent. □

3.4. Completeness

Now, we turn to the completeness of SQS.

Proposition 8.

(1) *SQS is complete with respect to $\mathcal{S}_{PH}$.*
(2) *SQS is complete with respect to $\mathcal{S}_{QQ}$.*
(3) *SQS is not axiomatizable with finitely many variables.*

Proof. (1) Immediately follows from item (3) of Proposition 7, since $\mathcal{S}_{PH}$ is a model of SQS.

(2) Immediately follows from item (3) of Proposition 7, since $\mathcal{S}_{QQ}$ is a model of SQS.

(3) Suppose that SQS is axiomatizable with finitely many variables. Hence, there exists a positive integer n and there exists a set Γ of sentences with variables among $x_1, \ldots, x_n$ such that SQS is equal to the set of all consequences of Γ. Since $\mathcal{S}_{QQ}$ is a model of SQS, $\mathcal{S}_{QQ} \models \Gamma$. Let $\mathcal{S}_n = (H_{\mathcal{S}_n}, N_{\mathcal{S}_n}, C_{\mathcal{S}_n}, P_{\mathcal{S}_n})$ be the frame such that $H_{\mathcal{S}_n} = \mathbb{Q}^{+*} \times \mathbb{Q}^{+*} \times \{1, \ldots, n\}$ whereas:

- $N_{\mathcal{S}_n}((q_1, q_2, q_3), (r_1, r_2, r_3))$ iff $q_1 < r_1$,

- $C_{\mathcal{S}_n}((q_1,q_2,q_3),(r_1,r_2,r_3))$ iff $q_1 = r_1$,
- $P_{\mathcal{S}_n}((q_1,q_2,q_3),(r_1,r_2,r_3))$ iff $q_1 = r_1$ and $q_2 = r_2$.

Obviously, $\mathcal{S}_n \not\models \mathit{Inf}_n(P)$. Moreover, for all $a_1,\ldots,a_n$ in $H_{\mathcal{S}_n}$, the second player wins all n-pebble games over $(\mathcal{S}_n,[a_1,\ldots,a_n])$ and $(\mathcal{S}_{QQ},[a_1,\ldots,a_n])$. Thus, by [12, theorem 3.3.5], for all $a_1,\ldots,a_n$ in $H_{\mathcal{S}_n}$ and for all formulas $\phi(x_1,\ldots,x_n)$ with variables among $x_1,\ldots,x_n$, $\mathcal{S}_n \models \phi(x_1,\ldots,x_n)$ $[a_1,\ldots,a_n]$ iff $\mathcal{S}_{QQ} \models \phi(x_1,\ldots,x_n)$ $[a_1,\ldots,a_n]$. Since the variables occurring in Γ are among $x_1,\ldots,x_n$ and $\mathcal{S}_{QQ} \models \Gamma$, $\mathcal{S}_n \models \Gamma$. Since SQS is equal to the set of all Γ's consequences, $\mathcal{S}_n \models \mathit{Inf}_n(P)$: a contradiction. $\square$

3.5. Complexity

In this section, we investigate the decidability/complexity of the membership problem in SQS.

Proposition 9.

(1) *The membership problem in SQS is decidable.*
(2) *The membership problem in SQS is PSPACE-hard.*
(3) *The membership problem in SQS is in PSPACE.*

Proof. (1) Immediately follows from item (3) of Proposition 7.

(2) Let EQ^∞ be the first-order theory of $\equiv$ in all infinite sets. Obviously, SQS is a conservative extension of EQ^∞. Since the membership problem in EQ^∞ is PSPACE-hard [1], the membership problem in SQS is PSPACE-hard.

(3) To every variable x^i, we associate a triple (x^i_1,x^i_2,x^i_3) of variables. The function $\tau(\cdot)$ assigning to each formula ϕ in N, C, P and $\equiv$ a formula $\tau(\phi)$ in $<$ and $\equiv$ is given by:

- $\tau(N(x^i,x^j)) = x^i_1 < x^j_1$,
- $\tau(C(x^i,x^j)) = x^i_1 \equiv x^j_1$,
- $\tau(P(x^i,x^j)) = x^i_1 \equiv x^j_1 \wedge x^i_2 \equiv x^j_2$,
- $\tau(x^i \equiv x^j) = x^i_1 \equiv x^j_1 \wedge x^i_2 \equiv x^j_2 \wedge x^i_3 \equiv x^j_3$,
- $\tau(\bot) = \bot$,
- $\tau(\neg\phi) = \neg\tau(\phi)$,
- $\tau(\phi \vee \psi) = \tau(\phi) \vee \tau(\psi)$,
- $\tau(\forall x^i.\phi) = \forall x^i_1.\forall x^i_2.\forall x^i_3.\tau(\phi)$.

Obviously, $\tau(\phi)$ can be computed in space $\log \mid \phi \mid$. Moreover, for all nonnegative integers n, for all q^1_1, q^1_2, $q^1_3,\ldots,q^n_1$, q^n_2, q^n_3 in $\mathbb{Q}^{+*}$ and for all formulas $\phi(x^1,\ldots,x^n)$ with variables among $x^1,\ldots,x^n$ in N, C, P and $\equiv$,

- if $\mathcal{S}_{QQ} \models \phi(x^1,\ldots,x^n)$ $[(q^1_1,q^1_2,q^1_3),\ldots,(q^n_1,q^n_2,q^n_3)]$ then $(\mathbb{Q}^{+*},<) \models \tau(\phi(x^1,\ldots,x^n))$ $[q^1_1,q^1_2,q^1_3,\ldots,q^n_1,q^n_2,q^n_3]$,
- if $\mathcal{S}_{QQ} \not\models \phi(x^1,\ldots,x^n)$ $[(q^1_1,q^1_2,q^1_3),\ldots,(q^n_1,q^n_2,q^n_3)]$ then $(\mathbb{Q}^{+*},<) \not\models \tau(\phi(x^1,\ldots,x^n))$ $[q^1_1,q^1_2,q^1_3,\ldots,q^n_1,q^n_2,q^n_3]$.

The two above items can be proved by induction on the complexity of ϕ. Hence, for all formulas ϕ in N, C, P and $\equiv$, $\mathcal{S}_{QQ} \models \phi$ iff $(\mathbb{Q}^{+*},<) \models \tau(\phi)$. Thus, by item (2) of Proposition 8, for all formulas ϕ in N, C, P and $\equiv$, ϕ is in SQS iff $\tau(\phi)$ is in the first-order theory of $<$ and $\equiv$ in all dense linear proper orders without endpoints. Since the first-order theory of $<$ and $\equiv$ in all dense linear propers order without endpoints is in PSPACE [13], the membership problem in SQS is in PSPACE. $\square$

3.6. Definability

We tackle the problem of the definability of N, C, P and $\equiv$ in the class of all normal frames. The following result implies that C is the only binary predicate in our language that can be eliminated. Its proof necessitates the use of Ehrenfeucht games over $(\mathcal{S}, [a_1, \ldots, a_n])$ and $(\mathcal{S}', [a'_1, \ldots, a'_n])$. See [12] for details. Let m, n be nonnegative integers. Each play of the n-Ehrenfeucht m-game over $(\mathcal{S}, [a_1, \ldots, a_n])$ and $(\mathcal{S}', [a'_1, \ldots, a'_n])$ consists of a finite sequence of m moves. In its j-th move, the first player selects a normal frame, either $\mathcal{S}$ or $\mathcal{S}'$. If it selects $\mathcal{S}$ then the first player chooses an element b_j of $\mathcal{S}$ and the second player chooses an element b'_j of $\mathcal{S}'$. If it selects $\mathcal{S}'$ then the first player chooses an element b'_j of $\mathcal{S}'$ and the second player chooses an element b_j of $\mathcal{S}$. The second player wins the game if for each $j \leq m$, the elements $a_1, \ldots, a_n, b_1, \ldots, b_j$ of $\mathcal{S}$ and the elements $a'_1, \ldots, a'_n, b'_1, \ldots, b'_j$ of $\mathcal{S}'$ constitute a partial isomorphism between the two normal frames.

Proposition 10.

(1) *N is not definable with C, P and $\equiv$ in the class of all normal frames.*
(2) *C is definable with N in the class of all normal frames.*
(3) *C is not definable with P and $\equiv$ in the class of all normal frames.*
(4) *P is not definable with N, C and $\equiv$ in the class of all normal frames.*
(5) *$\equiv$ is not definable with N, C and P in the class of all normal frames.*

Proof. (1) Suppose N is definable with C, P and $\equiv$ in the class of all normal frames. Hence, there exists a formula $\phi(x, y)$ in C, P and $\equiv$ such that $(*)$ for all normal frames $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ and for all a, b in $H_\mathcal{S}$, $N_\mathcal{S}(a, b)$ iff $\mathcal{S} \models \phi(x, y)$ $[a, b]$. Let $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ be a normal frame. By $Ser(N)$, there exist a, b in $H_\mathcal{S}$ such that $N_\mathcal{S}(a, b)$. By $Tra(N)$, $Ref(C)$ and $Dis(N, C)$, not $N_\mathcal{S}(b, a)$. Moreover, since $(*)$, $\mathcal{S} \models \phi(x, y)$ $[a, b]$. Obviously, the frame $(H_\mathcal{S}, N_\mathcal{S}^{-1}, C_\mathcal{S}, P_\mathcal{S})$ is normal. Moreover, the second player wins all Ehrenfeucht games over $(\mathcal{S}, [a, b])$ and $((H_\mathcal{S}, N_\mathcal{S}^{-1}, C_\mathcal{S}, P_\mathcal{S}), [a, b])$ with respect to C, P and $\equiv$. Thus, by [12, theorem 2.2.8], for all formulas $\psi(x, y)$ in C, P and $\equiv$, $\mathcal{S} \models \psi(x, y)$ $[a, b]$ iff $(H_\mathcal{S}, N_\mathcal{S}^{-1}, C_\mathcal{S}, P_\mathcal{S}) \models \psi(x, y)$ $[a, b]$. Since $\mathcal{S} \models \phi(x, y)$ $[a, b]$, $(H_\mathcal{S}, N_\mathcal{S}^{-1}, C_\mathcal{S}, P_\mathcal{S}) \models \phi(x, y)$ $[a, b]$. Since $(*)$, $N_\mathcal{S}^{-1}(a, b)$. Therefore, $N_\mathcal{S}(b, a)$: a contradiction.

(2) It suffices to observe that for all normal frames $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ and for all a, b in $H_\mathcal{S}$, $C_\mathcal{S}(a, b)$ iff $\mathcal{S} \models \bar{N}(x, y) \wedge \bar{N}(y, x)$ $[a, b]$. For the verification, use $Sym(C)$, $Dis(N, C)$ and $Uni(N, C)$.

(3) Suppose C is definable with P and $\equiv$ in the class of all normal frames. Hence, there exists a formula $\phi(x, y)$ in P and $\equiv$ such that $(*)$ for all normal frames $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ and for all a, b in $H_\mathcal{S}$, $C_\mathcal{S}(a, b)$ iff $\mathcal{S} \models \phi(x, y)$ $[a, b]$. Let $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ be a normal frame. By $Sym(C)$, $Sym(P)$ and $Inf_1(C, P)$, there exist a, b in $H_\mathcal{S}$ such that $C_\mathcal{S}(a, b)$ and not $P_\mathcal{S}(a, b)$. Since $(*)$, $\mathcal{S} \models \phi(x, y)$ $[a, b]$. Let $\mathcal{S}' = (H_{\mathcal{S}'}, N_{\mathcal{S}'}, C_{\mathcal{S}'}, P_{\mathcal{S}'})$ be a normal frame. By $Ser(N)$, there exist a', b' in $H_{\mathcal{S}'}$ such that $N_{\mathcal{S}'}(a', b')$. By $Dis(N, C)$ and $Inc(C, P)$, neither $C_{\mathcal{S}'}(a', b')$ nor $P_{\mathcal{S}'}(a', b')$. Obviously, the second player wins all Ehrenfeucht games over $(\mathcal{S}, [a, b])$ and $(\mathcal{S}', [a', b'])$ with respect to P and $\equiv$. Thus by [12, theorem 2.2.8], for all formulas $\psi(x, y)$ in P and $\equiv$, $\mathcal{S} \models \psi(x, y)$ $[a, b]$ iff $\mathcal{S}' \models \psi(x, y)$ $[a', b']$. Since $\mathcal{S} \models \phi(x, y)$ $[a, b]$, $\mathcal{S}' \models \phi(x, y)$ $[a', b']$. Since $(*)$, $C_{\mathcal{S}'}(a', b')$: a contradiction.

(4) Suppose P is definable with N, C and $\equiv$ in the class of all normal frames. Hence, there exists a formula $\phi(x, y)$ in N, C and $\equiv$ such that $(*)$ for all normal frames $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ and for all a, b in $H_\mathcal{S}$, $P_\mathcal{S}(a, b)$ iff $\mathcal{S} \models \phi(x, y)$ $[a, b]$. Let $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ be a normal frame. By $Sym(P)$ and $Inf_1(P, \equiv)$, there exist a, b in $H_\mathcal{S}$ such that $P_\mathcal{S}(a, b)$ and $a \neq b$. Since $(*)$, $\mathcal{S} \models \phi(x, y)$ $[a, b]$. Let $\mathcal{S}' = (H_{\mathcal{S}'}, N_{\mathcal{S}'}, C_{\mathcal{S}'}, P_{\mathcal{S}'})$

be a normal frame. By $Sym(C)$, $Sym(P)$ and $Inf_1(C,P)$, there exist a', b' in $H_{\mathcal{S}'}$ such that $C_{\mathcal{S}'}(a',b')$ and not $P_{\mathcal{S}'}(a',b')$. Obviously, the second player wins all Ehrenfeucht games over $(\mathcal{S},[a,b])$ and $(\mathcal{S}',[a',b'])$ with respect to N, C and $\equiv$. Thus, by [12, theorem 2.2.8], for all formulas $\psi(x,y)$ in N, C and $\equiv$, $\mathcal{S} \models \psi(x,y)\ [a,b]$ iff $\mathcal{S}' \models \psi(x,y)\ [a',b']$. Since $\mathcal{S} \models \phi(x,y)\ [a,b]$, $\mathcal{S}' \models \phi(x,y)\ [a',b']$. Since $(*)$, $P_{\mathcal{S}'}(a',b')$: a contradiction.

(5) Suppose $\equiv$ is definable with N, C and P in the class of all normal frames. Hence, there exists a formula $\phi(x,y)$ in N, C and P such that $(*)$ for all normal frames $\mathcal{S} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}})$ and for all a, b in $H_{\mathcal{S}}$, $a = b$ iff $\mathcal{S} \models \phi(x,y)\ [a,b]$. Let $\mathcal{S} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}})$ be a normal frame. Let a in $H_{\mathcal{S}}$. Since $(*)$, $\mathcal{S} \models \phi(x,y)\ [a,a]$. Let $\mathcal{S}' = (H_{\mathcal{S}'}, N_{\mathcal{S}'}, C_{\mathcal{S}'}, P_{\mathcal{S}'})$ be a normal frame. By $Sym(P)$ and $Inf_1(P,\equiv)$, there exist a', b' in $H_{\mathcal{S}'}$ such that $P_{\mathcal{S}'}(a',b')$ and $a' \neq b'$. Obviously, the second player wins all Ehrenfeucht games over $(\mathcal{S},[a,a])$ and $(\mathcal{S}',[a',b'])$ with respect to N, C and P. Thus, by [12, theorem 2.2.8], for all formulas $\psi(x,y)$ in N, C and P, $\mathcal{S} \models \psi(x,y)\ [a,a]$ iff $\mathcal{S}' \models \psi(x,y)\ [a',b']$. Since $\mathcal{S} \models \phi(x,y)\ [a,a]$, $\mathcal{S}' \models \phi(x,y)\ [a',b']$. Since $(*)$, $a' = b'$: a contradiction. $\square$

4. Modal logic

4.1. Syntax

It is now time to meet the modal language we will be working with. We assume some familiarity with modal logic. Readers wanting more details may refer, for example, to [3] or [7]. Our modal logic is based on the idea of associating with negligibility, comparability and proximity the connectives $[N]$, $[C]$ and $[P]$. The *formulas* are given by the rule:

- $\phi ::= p \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid [N]\phi \mid [C]\phi \mid [P]\phi$

where p ranges over a countable set of *atoms.* Let the *size of* ϕ, denoted by $\mid \phi \mid$, be the number of symbols occurring in ϕ. $SF(\phi)$ will denote the set of all ϕ's subformulas. We adopt the standard definitions for the remaining Boolean operations. It is usual to omit parentheses if this does not lead to any ambiguity. We define $\langle N\rangle\phi := \neg[N]\neg\phi$, $\langle C\rangle\phi := \neg[C]\neg\phi$ and $\langle P\rangle\phi := \neg[P]\neg\phi$. $[N]\phi$, $[C]\phi$, $[P]\phi$ will be respectively read "at all points with respect to which the current point is negligible with respect to, ϕ", "at all points with respect to which the current point is comparable to, ϕ", "at all points with respect to which the current point is in the proximity of, ϕ".

4.2. Semantics

A *model* is a pair $\mathcal{M} = (\mathcal{S}, V)$, where $\mathcal{S} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}})$ is a frame and V is a *valuation on* $\mathcal{S}$, i.e. a function assigning to each atom p a subset $V(p)$ of $H_{\mathcal{S}}$. For all a in $H_{\mathcal{S}}$, let $V(a)$ be the set of all atoms p such that a is in $V(p)$, $\theta_V([a]_{P_{\mathcal{S}}}) = \{V(b)\colon P_{\mathcal{S}}(a,b)\}$ and $\Theta_V([a]_{C_{\mathcal{S}}}) = \{\theta_V([b]_{P_{\mathcal{S}}})\colon C_{\mathcal{S}}(a,b)\}$. If $\mathcal{M} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}}, V)$ is a model and a is in $H_{\mathcal{S}}$ then the formula ϕ is *true in* $\mathcal{M}$ *at* a, denoted by $\mathcal{M}, a \models \phi$, is defined inductively on the complexity of formulas ϕ as usual. In particular:

- $\mathcal{M}, a \models [N]\phi$ iff for all b in $H_{\mathcal{S}}$, if $N_{\mathcal{S}}(a,b)$ then $\mathcal{M}, b \models \phi$,
- $\mathcal{M}, a \models [C]\phi$ iff for all b in $H_{\mathcal{S}}$, if $C_{\mathcal{S}}(a,b)$ then $\mathcal{M}, b \models \phi$,
- $\mathcal{M}, a \models [P]\phi$ iff for all b in $H_{\mathcal{S}}$, if $P_{\mathcal{S}}(a,b)$ then $\mathcal{M}, b \models \phi$.

We shall say that ϕ is *true in the model* $\mathcal{M} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}}, V)$, denoted by $\mathcal{M} \models \phi$, iff $\mathcal{M}, a \models \phi$ for all a in $H_{\mathcal{S}}$. ϕ is said to be *valid in the frame* $\mathcal{S} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}})$, denoted by $\mathcal{S} \models \phi$, iff $\mathcal{M} \models \phi$ for all models $\mathcal{M} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}}, V)$ based on $\mathcal{S}$. It is a simple exercise in modal logic to check that the following formulas are valid in all normal frames:

$ML(weakSer(N))$: $[N]\phi \to \langle N\rangle\phi$.
$ML(Tra(N))$: $[N]\phi \to [N][N]\phi$.
$ML(Den(N))$: $[N][N]\phi \to [N]\phi$.
$ML(Ref(C))$: $[C]\phi \to \phi$.
$ML(Sym(C))$: $\phi \to [C]\langle C\rangle\phi$.
$ML(Tra(C))$: $[C]\phi \to [C][C]\phi$.
$ML(Ref(P))$: $[P]\phi \to \phi$.
$ML(Sym(P))$: $\phi \to [P]\langle P\rangle\phi$.
$ML(Tra(P))$: $[P]\phi \to [P][P]\phi$.
$ML(Tra_1(N,C))$: $[N]\phi \to [N][C]\phi$.
$ML(Tra_2(N,C))$: $[N]\phi \to [C][N]\phi$.
$ML(Inc(C,P))$: $[C]\phi \to [P]\phi$.
$ML(weakUni(N,C))$: $\langle N\rangle\phi \wedge \langle N\rangle\psi \to \langle N\rangle(\phi \wedge \langle N\rangle\psi) \vee \langle N\rangle(\phi \wedge \langle C\rangle\psi) \vee \langle N\rangle(\psi \wedge \langle N\rangle\phi)$.

Now, we will show that

Proposition 11. *Let $\mathcal{S}$, $\mathcal{S}'$ be normal frames. If $\mathcal{S}$ is countable then $\mathcal{S}$ and $\mathcal{S}'$ are modally equivalent.*

Proof. Let $\mathcal{S} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}})$, $\mathcal{S}' = (H_{\mathcal{S}'}, N_{\mathcal{S}'}, C_{\mathcal{S}'}, P_{\mathcal{S}'})$ be normal frames. Suppose $\mathcal{S}$ is countable. If $\mathcal{S}$ and $\mathcal{S}'$ are not modally equivalent then there exists a formula ϕ such that $\mathcal{S} \models \phi$ and $\mathcal{S}' \not\models \phi$ or there exists a formula ϕ such that $\mathcal{S} \not\models \phi$ and $\mathcal{S}' \models \phi$. In the former case, $\mathcal{S} \models \phi$ and $\mathcal{S}' \not\models \phi$. Since $\mathcal{S}$ is countable, by item (1) of Proposition 7, ϕ is valid in every countable normal frame. By [3, proposition 2.47], the downward Löwenheim–Skolem property and the first-order definability of the class of all normal frames, ϕ is valid in every normal frame. Hence, $\mathcal{S}' \models \phi$: a contradiction. In the latter case, $\mathcal{S} \not\models \phi$ and $\mathcal{S}' \models \phi$. Thus, there exists a model $\mathcal{M} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}}, V)$ based on $\mathcal{S}$ such that $\mathcal{M} \not\models \phi$. Therefore, there exists a_0 in $H_{\mathcal{S}}$ such that $\mathcal{M}, a_0 \not\models \phi$. Restricting our discussion to the atoms actually occurring in ϕ, the reader may easily verify that there exists a_ω in $H_{\mathcal{S}}$ such that for all a in $H_{\mathcal{S}}$, there exists b in $H_{\mathcal{S}}$ such that $N_{\mathcal{S}}(a,b)$ and $\Theta_V([b]_{C_{\mathcal{S}}}) = \Theta_V([a_\omega]_{C_{\mathcal{S}}})$. Let f be the mapping on $H_{\mathcal{S}}$ to $H_{\mathcal{S}'}$ defined as in the proof of Proposition 5. Obviously, there exists a valuation V' on $\mathcal{S}'$ such that for all b' in $H_{\mathcal{S}'}$,

- if there exists b in $H_{\mathcal{S}}$ such that $b' = f(b)$ then $V'(b') = V(b)$,
- if for all b in $H_{\mathcal{S}}$, $b' \neq f(b)$ and there exists b in $H_{\mathcal{S}}$ such that $P_{\mathcal{S}'}(b', f(b))$ then $\theta_{V'}([b']_{P_{\mathcal{S}'}}) = \theta_V([b]_{P_{\mathcal{S}}})$,
- if for all b in $H_{\mathcal{S}}$, not $P_{\mathcal{S}'}(b', f(b))$ and there exists b in $H_{\mathcal{S}}$ such that $C_{\mathcal{S}'}(b', f(b))$ then $\Theta_{V'}([b']_{C_{\mathcal{S}'}}) = \Theta_V([b]_{C_{\mathcal{S}}})$,
- if for all b in $H_{\mathcal{S}}$, not $C_{\mathcal{S}'}(b', f(b))$ then $\Theta_{V'}([b']_{C_{\mathcal{S}'}}) = \Theta_V([a_\omega]_{C_{\mathcal{S}}})$.

The reader may easily verify that for all formulas ψ in $SF(\phi)$ and for all a in $H_{\mathcal{S}}$, $\mathcal{M}, a \models \psi$ iff $(\mathcal{S}', V'), f(a) \models \psi$. Since $\mathcal{M}, a_0 \not\models \phi$, $(\mathcal{S}', V'), f(a_0) \not\models \phi$. Consequently, $(\mathcal{S}', V') \not\models \phi$. Hence, $\mathcal{S}' \not\models \phi$: a contradiction. $\square$

As a result, any two normal frames are modally equivalent. In particular,

Corollary 12. *Let ϕ be a formula. The following conditions are equivalent:*

(1) *ϕ is valid in every normal frame.*
(2) $\mathcal{S}_{PH} \models \phi$.
(3) $\mathcal{S}_{QQ} \models \phi$.

Proof. By Proposition 11, since $\mathcal{S}_{PH}$ and $\mathcal{S}_{QQ}$ are normal frames. $\square$

4.3. Axiomatization

Let $ML(SQS)$ be the *least normal modal logic of* $[N]$*,* $[C]$ *and* $[P]$ that contains $ML(weakSer(N))$, $ML(Tra(N))$, $ML(Den(N))$, $ML(Ref(C))$, $ML(Sym(C))$, $ML(Tra(C))$, $ML(Ref(P))$, $ML(Sym(P))$, $ML(Tra(P))$, $ML(Tra_1(N,C))$, $ML(Tra_2(N,C))$, $ML(Inc(C,P))$ and $ML(weakUni(N,C))$ as proper axioms. Let $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ be the subframe of the canonical frame for $ML(SQS)$ generated by one of its elements. Seeing that $ML(weakSer(N))$ is a Sahlqvist formula, the following sentence holds in $\mathcal{S}$:

$weakSer(N)$: $\forall x.\exists y.N(x,y)$.

Seeing that $ML(Tra(N))$, $ML(Den(N))$, $ML(Ref(C))$, $ML(Sym(C))$, $ML(Tra(C))$, $ML(Ref(P))$, $ML(Sym(P))$ and $ML(Tra(P))$ are Sahlqvist formulas, $Tra(N)$, $Den(N)$, $Ref(C)$, $Sym(C)$, $Tra(C)$, $Ref(P)$, $Sym(P)$ and $Tra(P)$ hold in $\mathcal{S}$. Seeing that $ML(Tra_1(N,C))$ and $ML(Tra_2(N,C))$ are Sahlqvist formulas, the following sentences hold in $\mathcal{S}$:

$Tra_1(N,C)$: $\forall x.\forall y.(\exists z.(N(x,z) \wedge C(z,y)) \rightarrow N(x,y))$.
$Tra_2(N,C)$: $\forall x.\forall y.(\exists z.(C(x,z) \wedge N(z,y)) \rightarrow N(x,y))$.

Seeing that $ML(Inc(C,P))$ is a Sahlqvist formula, $Inc(C,P)$ holds in $\mathcal{S}$. Seeing that $ML(weakUni(N,C))$ is a Sahlqvist formula, the following sentence holds in $\mathcal{S}$:

$weakUni(N,C)$: $\forall x.\forall y.\forall z.(N(x,y) \wedge N(x,z) \rightarrow N(y,z) \vee C(y,z) \vee N(z,y))$.

Since $\mathcal{S}$ is point-generated, by $Tra(N)$, $Ref(C)$, $Sym(C)$, $Tra(C)$, $Tra_1(N,C)$, $Tra_2(N,C)$ and $Inc(C,P)$, $Uni(N,C)$ holds in $\mathcal{S}$. This motivates the following definition. A frame $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ is said to be *prenormal* iff it satisfies $weakSer(N)$, $Tra(N)$, $Den(N)$, $Ref(C)$, $Sym(C)$, $Tra(C)$, $Ref(P)$, $Sym(P)$, $Tra(P)$, $Tra_1(N,C)$, $Tra_2(N,C)$, $Inc(C,P)$ and $Uni(N,C)$. Obviously, every normal frame is prenormal. The importance of prenormal frames lies in the following proposition.

Proposition 13. *Let ϕ be a formula. The following conditions are equivalent:*

(1) *ϕ is in $ML(SQS)$.*
(2) *ϕ is valid in every prenormal frame.*
(3) *ϕ is valid in every countable prenormal frame.*

Proof. By Sahlqvist completeness theorem [3, chapter 5], [3, proposition 2.47], the downward Löwenheim–Skolem property and the first-order definability of the class of all prenormal frames. □

4.4. Completeness

Now, we turn to the completeness of $ML(SQS)$. We shall say that a countable prenormal frame $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ is *unlimited* iff every equivalence class in $H_\mathcal{S}$ modulo $P_\mathcal{S}$ is made up of infinitely many elements and every equivalence class in $H_\mathcal{S}$ modulo $C_\mathcal{S}$ is made up of infinitely many equivalence classes in $H_\mathcal{S}$ modulo $P_\mathcal{S}$. We first prove a simple result.

Proposition 14. *Let $\mathcal{S}$ be a prenormal frame. If $\mathcal{S}$ is countable then there exists a countable unlimited prenormal frame $\mathcal{S}'$ such that $\mathcal{S}$ is a bounded morphic image of $\mathcal{S}'$.*

Proof. Let $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ be a prenormal frame. Suppose $\mathcal{S}$ is countable. Let $\mathcal{S}' = (H_{\mathcal{S}'}, N_{\mathcal{S}'}, C_{\mathcal{S}'}, P_{\mathcal{S}'})$ be the countable unlimited prenormal frame such that $H_{\mathcal{S}'} = H_\mathcal{S} \times \mathbb{Q}^{+*} \times \mathbb{Q}^{+*}$ whereas:

- $N_{\mathcal{S}'}((a, q_2, q_3), (b, r_2, r_3))$ iff $N_\mathcal{S}(a, b)$,
- $C_{\mathcal{S}'}((a, q_2, q_3), (b, r_2, r_3))$ iff $C_\mathcal{S}(a, b)$,
- $P_{\mathcal{S}'}((a, q_2, q_3), (b, r_2, r_3))$ iff $P_\mathcal{S}(a, b)$ and $q_2 = r_2$.

Obviously, $\mathcal{S}$ is a bounded morphic image of $\mathcal{S}'$. $\square$

Let $\mathcal{S}^g_{QQ} = (H^g_{QQ}, N^g_{QQ}, C^g_{QQ}, P^g_{QQ})$ be the subframe of $\mathcal{S}_{QQ}$ generated by one of its elements, say (q^0_1, q^0_2, q^0_3). Clearly, $\mathcal{S}^g_{QQ}$ is countable, unlimited and normal. Remark that for all (q_1, q_2, q_3) in H^g_{QQ}, $q^0_1 \leq q_1$. The importance of $\mathcal{S}^g_{QQ}$ lies in the following proposition.

Proposition 15. *Let ϕ be a formula. The following conditions are equivalent:*

(1) *ϕ is valid in every countable prenormal frame.*
(2) *$\mathcal{S}_{QQ} \models \phi$.*
(3) *$\mathcal{S}^g_{QQ} \models \phi$.*

We defer proving Proposition 15 till the end of this section. In the meantime, we demonstrate some useful results. Let $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$ be a countable prenormal frame. Hence, $\mathcal{S}$ is generated by one of its elements, say a_0. Let f: $H^g_{QQ} \mapsto H_\mathcal{S}$ be a partial function. Its *domain* will be denoted by $dom(f)$ whereas its *range* will be denoted by $ran(f)$. We shall say that f is *finite* iff $dom(f)$ is finite. f will be called *homomorphism* iff for all (q_1, q_2, q_3), (r_1, r_2, r_3) in H^g_{QQ}, if $N^g_{QQ}((q_1, q_2, q_3), (r_1, r_2, r_3))$ then $N_\mathcal{S}(f(q_1, q_2, q_3), f(r_1, r_2, r_3))$, if $C^g_{QQ}((q_1, q_2, q_3), (r_1, r_2, r_3))$ then $C_\mathcal{S}(f(q_1, q_2, q_3), f(r_1, r_2, r_3))$ and if $P^g_{QQ}((q_1, q_2, q_3), (r_1, r_2, r_3))$ then $P_\mathcal{S}(f(q_1, q_2, q_3), f(r_1, r_2, r_3))$. Let f_0: $H^g_{QQ} \mapsto H_\mathcal{S}$ be the partial function defined by $dom(f) = \{(q^0_1, q^0_2, q^0_3)\}$ and $f(q^0_1, q^0_2, q^0_3) = a_0$. Suppose $\mathcal{S}$ is unlimited. The following lemmas constitute the heart of our method.

Lemma 16. *Let (q_1, q_2, q_3) in H^g_{QQ} and f: $H^g_{QQ} \mapsto H_\mathcal{S}$ be a finite homomorphism containing f_0. There exists a finite homomorphism g: $H^g_{QQ} \mapsto H_\mathcal{S}$ containing f and such that (q_1, q_2, q_3) is in $dom(g)$.*

Proof. Since f is a finite homomorphism containing f_0, there exists a positive integer k and there exist $(q^1_1, q^1_2, q^1_3), \ldots, (q^k_1, q^k_2, q^k_3)$ in H^g_{QQ} such that $dom(f) = \{(q^1_1, q^1_2, q^1_3), \ldots, (q^k_1, q^k_2, q^k_3)\}$ and $q^1_1 \leq \ldots \leq q^k_1$. Since f contains f_0, $q^1_1 = q^0_1$. Now, consider the following cases.

(1) Suppose there exists a positive integer l such that $l \leq k$ and $q^l_1 = q_1$, $q^l_2 = q_2$ and $q^l_3 = q_3$. Let g: $H^g_{QQ} \mapsto H_\mathcal{S}$ be the partial function f.
(2) Suppose for all positive integers l, if $l \leq k$ then $q^l_1 \neq q_1$ or $q^l_2 \neq q_2$ or $q^l_3 \neq q_3$ and there exists a positive integer l such that $l \leq k$ and $q^l_1 = q_1$ and $q^l_2 = q_2$. Since f is finite and $\mathcal{S}$ is unlimited, there exists a in $[f(q^l_1, q^l_2, q^l_3)]_{P_\mathcal{S}} \setminus ran(f)$. Let g: $H^g_{QQ} \mapsto H_\mathcal{S}$ be the least partial function containing f and such that $g(q_1, q_2, q_3) = a$.
(3) Suppose for all positive integers l, if $l \leq k$ then $q^l_1 \neq q_1$ or $q^l_2 \neq q_2$ and there exists a positive integer l such that $l \leq k$ and $q^l_1 = q_1$. Since f is finite and $\mathcal{S}$ is unlimited, there exists a in $[f(q^l_1, q^l_2, q^l_3)]_{C_\mathcal{S}} \setminus [f(q^l_1, q^l_2, q^l_3)]_{P_\mathcal{S}} \setminus ran(f)$. Let g: $H^g_{QQ} \mapsto H_\mathcal{S}$ be the least partial function containing f and such that $g(q_1, q_2, q_3) = a$.
(4) Suppose for all positive integers l, if $l \leq k$ then $q^l_1 \neq q_1$. Now, consider the following cases.
    (a) Suppose there exists a positive integer l such that $1 \leq l-1$, $l \leq k$ and $q^{l-1}_1 < q_1 < q^l_1$. Hence, $N^g_{QQ}((q^{l-1}_1, q^{l-1}_2, q^{l-1}_3), (q^l_1, q^l_2, q^l_3))$. Since f is a homomorphism, $N_\mathcal{S}(f(q^{l-1}_1, q^{l-1}_2, q^{l-1}_3), f(q^l_1, q^l_2, q^l_3))$.

Since $Den(N)$ holds in $\mathcal{S}$, there exists a in $H_{\mathcal{S}}$ such that $N_{\mathcal{S}}(f(q_1^{l-1}, q_2^{l-1}, q_3^{l-1}), a)$ and $N_{\mathcal{S}}(a, f(q_1^l, q_2^l, q_3^l))$. Let g: $H_{QQ}^g \mapsto H_{\mathcal{S}}$ be the least partial function containing f and such that $g(q_1, q_2, q_3) = a$.

(b) Suppose $q_1^k < q_1$. Hence, $N_{QQ}^g((q_1^k, q_2^k, q_3^k), (q_1, q_2, q_3))$. Since $weakSer(N)$ holds in $\mathcal{S}$, there exists a in $H_{\mathcal{S}}$ such that $N_{\mathcal{S}}(f(q_1^k, q_2^k, q_3^k), a)$. Let g: $H_{QQ}^g \mapsto H_{\mathcal{S}}$ be the least partial function containing f and such that $g(q_1, q_2, q_3) = a$.

The reader may easily verify that g is a homomorphism. $\square$

The partial function g defined by Lemma 16 is called *forward completion of f with respect to* (q_1, q_2, q_3).

Lemma 17. *Let (q_1, q_2, q_3) in H_{QQ}^g, a in $H_{\mathcal{S}}$ and f: $H_{QQ}^g \mapsto H_{\mathcal{S}}$ be a finite homomorphism containing f_0. Suppose (q_1, q_2, q_3) is in $dom(f)$.*

(1) *If $N_{\mathcal{S}}(f(q_1, q_2, q_3), a)$ then there exists (r_1, r_2, r_3) in H_{QQ}^g and there exists a finite homomorphism g: $H_{QQ}^g \mapsto H_{\mathcal{S}}$ containing f and such that $N_{QQ}^g((q_1, q_2, q_3), (r_1, r_2, r_3))$, (r_1, r_2, r_3) is in $dom(g)$ and $g(r_1, r_2, r_3) = a$.*

(2) *If $C_{\mathcal{S}}(f(q_1, q_2, q_3), a)$ then there exists (r_1, r_2, r_3) in H_{QQ}^g and there exists a finite homomorphism g: $H_{QQ}^g \mapsto H_{\mathcal{S}}$ containing f and such that $C_{QQ}^g((q_1, q_2, q_3), (r_1, r_2, r_3))$, (r_1, r_2, r_3) is in $dom(g)$ and $g(r_1, r_2, r_3) = a$.*

(3) *If $P_{\mathcal{S}}(f(q_1, q_2, q_3), a)$ then there exists (r_1, r_2, r_3) in H_{QQ}^g and there exists a finite homomorphism g: $H_{QQ}^g \mapsto H_{\mathcal{S}}$ containing f and such that $P_{QQ}^g((q_1, q_2, q_3), (r_1, r_2, r_3))$, (r_1, r_2, r_3) is in $dom(g)$ and $g(r_1, r_2, r_3) = a$.*

Proof. (1) Suppose $N_{\mathcal{S}}(f(q_1, q_2, q_3), a)$. Since f is a finite homomorphism, there exists a nonnegative integer k and there exist $(r_1^1, r_2^1, r_3^1), \ldots, (r_1^k, r_2^k, r_3^k)$ in H_{QQ}^g such that $dom(f) \cap \{(r_1, r_2, r_3)$ in H_{QQ}^g: $N_{QQ}^g((q_1, q_2, q_3), (r_1, r_2, r_3))\} = \{(r_1^1, r_2^1, r_3^1), \ldots, (r_1^k, r_2^k, r_3^k)\}$ and $r_1^1 \leq \ldots \leq r_1^k$. Firstly, suppose $k = 0$. Since $weakSer(N)$ holds in $\mathcal{S}_{QQ}^g$, there exists (r_1, r_2, r_3) in H_{QQ}^g such that $N_{QQ}^g((q_1, q_2, q_3), (r_1, r_2, r_3))$. In this case, let g: $H_{QQ}^g \mapsto H_{\mathcal{S}}$ be the least partial function containing f and such that $g(r_1, r_2, r_3) = a$. Secondly, suppose $k \geq 1$. Now, consider the following cases.

(1) Suppose there exists a positive integer l such that $l \leq k$ and $f(r_1^l, r_2^l, r_3^l) = a$. Let (r_1, r_2, r_3) be (r_1^l, r_2^l, r_3^l) and g: $H_{QQ}^g \mapsto H_{\mathcal{S}}$ be the partial function f.

(2) Suppose for all positive integers l, if $l \leq k$ then $f(r_1^l, r_2^l, r_3^l) \neq a$ and there exists a positive integer l such that $l \leq k$ and $P_{\mathcal{S}}(f(r_1^l, r_2^l, r_3^l), a)$. Since f is finite and $\mathcal{S}_{QQ}^g$ is unlimited, there exists (r_1, r_2, r_3) in $[(r_1^l, r_2^l, r_3^l)]_{P_{QQ}^g} \setminus dom(f)$. Let g: $H_{QQ}^g \mapsto H_{\mathcal{S}}$ be the least partial function containing f and such that $g(r_1, r_2, r_3) = a$.

(3) Suppose for all positive integers l, if $l \leq k$ then not $P_{\mathcal{S}}(f(r_1^l, r_2^l, r_3^l), a)$ and there exists a positive integer l such that $l \leq k$ and $C_{\mathcal{S}}(f(r_1^l, r_2^l, r_3^l), a)$. Since f is finite and $\mathcal{S}_{QQ}^g$ is unlimited, there exists (r_1, r_2, r_3) in $[(r_1^l, r_2^l, r_3^l)]_{C_{QQ}^g} \setminus [(r_1^l, r_2^l, r_3^l)]_{P_{QQ}^g} \setminus dom(f)$. Let g: $H_{QQ}^g \mapsto H_{\mathcal{S}}$ be the least partial function containing f and such that $g(r_1, r_2, r_3) = a$.

(4) Suppose for all positive integers l, if $l \leq k$ then not $C_{\mathcal{S}}(f(r_1^l, r_2^l, r_3^l), a)$. Hence, for all positive integers l, if $l \leq k$ then $N_{\mathcal{S}}(f(r_1^l, r_2^l, r_3^l), a)$ or $N_{\mathcal{S}}(a, f(r_1^l, r_2^l, r_3^l))$. Now, consider the following cases.

(a) Suppose $N_{\mathcal{S}}(a, f(r_1^1, r_2^1, r_3^1))$. Since $Den(N)$ hold in $\mathcal{S}_{QQ}^g$, there exists (r_1, r_2, r_3) in H_{QQ}^g such that $N_{QQ}^g((q_1, q_2, q_3), (r_1, r_2, r_3))$ and $N_{QQ}^g((r_1, r_2, r_3), (r_1^1, r_2^1, r_3^1))$. Let g: $H_{QQ}^g \mapsto H_{\mathcal{S}}$ be the least partial function containing f and such that $g(r_1, r_2, r_3) = a$.

(b) Suppose there exists a positive integer l such that $1 \leq l - 1$, $l \leq k$, $N_{\mathcal{S}}(f(r_1^{l-1}, r_2^{l-1}, r_3^{l-1}), a)$ and $N_{\mathcal{S}}(a, f(r_1^l, r_2^l, r_3^l))$. Since $Den(N)$ holds in $\mathcal{S}_{QQ}^g$, there exists (r_1, r_2, r_3) in H_{QQ}^g such that

$N^g_{QQ}((r_1^{l-1}, r_2^{l-1}, r_3^{l-1}), (r_1, r_2, r_3))$ and $N^g_{QQ}((r_1, r_2, r_3), (r_1^l, r_2^l, r_3^l))$. Let g: $H^g_{QQ} \mapsto H_{\mathcal{S}}$ be the least partial function containing f and such that $g(r_1, r_2, r_3) = a$.

(c) Suppose $N_{\mathcal{S}}(f(r_1^k, r_2^k, r_3^k), a)$. Since *weakSer*$(N)$ holds in $\mathcal{S}^g_{QQ}$, there exists (r_1, r_2, r_3) in H^g_{QQ} such that $N^g_{QQ}((r_1^k, r_2^k, r_3^k), (r_1, r_2, r_3))$. Let g: $H^g_{QQ} \mapsto H_{\mathcal{S}}$ be the least partial function containing f and such that $g(r_1, r_2, r_3) = a$.

The reader may easily verify that g is a homomorphism.

(2) Suppose $C_{\mathcal{S}}(f(q_1, q_2, q_3), a)$. Since f is a finite homomorphism, there exists a positive integer k and there exist $(r_1^1, r_2^1, r_3^1), \ldots, (r_1^k, r_2^k, r_3^k)$ in H^g_{QQ} such that $dom(f) \cap \{(r_1, r_2, r_3)$ in H^g_{QQ}: $C^g_{QQ}((q_1, q_2, q_3), (r_1, r_2, r_3))\} = \{(r_1^1, r_2^1, r_3^1), \ldots, (r_1^k, r_2^k, r_3^k)\}$. Now, consider the following cases.

(1) Suppose there exists a positive integer l such that $l \leq k$ and $f(r_1^l, r_2^l, r_3^l) = a$. Let (r_1, r_2, r_3) be (r_1^l, r_2^l, r_3^l) and g: $H^g_{QQ} \mapsto H_{\mathcal{S}}$ be the partial function f.

(2) Suppose for all positive integers l, if $l \leq k$ then $f(r_1^l, r_2^l, r_3^l) \neq a$ and there exists a positive integer l such that $l \leq k$ and $P^g_{QQ}(f(r_1^l, r_2^l, r_3^l), a)$. Since f is finite and $\mathcal{S}^g_{QQ}$ is unlimited, there exists (r_1, r_2, r_3) in $[(r_1^l, r_2^l, r_3^l)]_{P^g_{QQ}} \setminus dom(f)$. Let g: $H^g_{QQ} \mapsto H_{\mathcal{S}}$ be the least partial function containing f and such that $g(r_1, r_2, r_3) = a$.

(3) Suppose for all positive integers l, if $l \leq k$ then not $P^g_{QQ}(f(r_1^l, r_2^l, r_3^l), a)$. Since f is finite and $\mathcal{S}^g_{QQ}$ is unlimited, there exists (r_1, r_2, r_3) in $[(q_1, q_2, q_3)]_{C^g_{QQ}} \setminus [(q_1, q_2, q_3)]_{P^g_{QQ}} \setminus dom(f)$. Let g: $H^g_{QQ} \mapsto H_{\mathcal{S}}$ be the least partial function containing f and such that $g(r_1, r_2, r_3) = a$.

The reader may easily verify that g is a finite homomorphism.

(3) Suppose $P_{\mathcal{S}}(f(q_1, q_2, q_3), a)$. Since f is a finite homomorphism, there exists a positive integer k and there exist $(r_1^1, r_2^1, r_3^1), \ldots, (r_1^k, r_2^k, r_3^k)$ in H^g_{QQ} such that $dom(f) \cap \{(r_1, r_2, r_3)$ in H^g_{QQ}: $P^g_{QQ}((q_1, q_2, q_3), (r_1, r_2, r_3))\} = \{(r_1^1, r_2^1, r_3^1), \ldots, (r_1^k, r_2^k, r_3^k)\}$. Now, consider the following cases.

(1) Suppose there exists a positive integer l such that $l \leq k$ and $f(r_1^l, r_2^l, r_3^l) = a$. Let (r_1, r_2, r_3) be (r_1^l, r_2^l, r_3^l) and g: $H^g_{QQ} \mapsto H_{\mathcal{S}}$ be the partial function f.

(2) Suppose for all positive integers l, if $l \leq k$ then $f(r_1^l, r_2^l, r_3^l) \neq a$. Since f is finite and $\mathcal{S}^g_{QQ}$ is unlimited, there exists (r_1, r_2, r_3) in $[(q_1, q_2, q_3)]_{P^g_{QQ}} \setminus dom(f)$. Let g: $H^g_{QQ} \mapsto H_{\mathcal{S}}$ be the least partial function containing f and such that $g(r_1, r_2, r_3) = a$.

The reader may easily verify that g is a homomorphism. $\square$

The partial function g defined by Lemma 17 in (1) (respectively (2), (3)) is called *backward completion of f with respect to* (q_1, q_2, q_3)*, a and N (respectively C, P)*. We can now prove the following result.

Proposition 18. *Let $\mathcal{S}$ be a countable prenormal frame. If $\mathcal{S}$ is unlimited then $\mathcal{S}$ is a bounded morphic image of $\mathcal{S}^g_{QQ}$.*

Proof. Let $\mathcal{S} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}})$ be a countable prenormal frame generated by one of its elements, say a_0. Suppose $\mathcal{S}$ is unlimited. We think of the construction of the surjective bounded morphism from $\mathcal{S}^g_{QQ}$ to $\mathcal{S}$ as a process approaching a limit via a sequence g_0: $H^g_{QQ} \mapsto H_{\mathcal{S}}$, g_1: $H^g_{QQ} \mapsto H_{\mathcal{S}}, \ldots$ of finite homomorphisms containing f_0. The partial function f_0 is used to initiate the construction whereas Lemmas 16 and 17 are used to make improvements at each step of the construction. Consider an enumeration $((r_1^0, r_2^0, r_3^0), b_0, \alpha_0)$, $((r_1^1, r_2^1, r_3^1), b_1, \alpha_1), \ldots$ of $H^g_{QQ} \times H_{\mathcal{S}} \times \{N, C, P\}$ where each item appears infinitely often. We inductively define a sequence g_0: $H^g_{QQ} \mapsto H_{\mathcal{S}}$, g_1: $H^g_{QQ} \mapsto H_{\mathcal{S}}, \ldots$ of finite homomorphisms containing f_0 in the following way.

Basis. Let $g_0 \colon H^g_{QQ} \mapsto H_{\mathcal{S}}$ be the partial function f_0.
Step. Let h_n be the forward completion of g_n with respect to (r^n_1, r^n_2, r^n_3) and g_{n+1} be the backward completion of h_n with respect to (r^n_1, r^n_2, r^n_3), b_n and α_n.
The reader may easily verify that the sequence g_0, $g_1, \ldots$ of finite homomorphisms containing f_0 is such that $dom(g_0) \subseteq dom(g_1) \subseteq \ldots$, $\bigcup\{dom(g_n)\colon n$ is a nonnegative integer$\} = H^g_{QQ}$ and for all nonnegative integers n, $g_{n+1}(r_1, r_2, r_3) = g_n(r_1, r_2, r_3)$ for each (r_1, r_2, r_3) in $dom(g_n)$. Let $f\colon H^g_{QQ} \mapsto H_{\mathcal{S}}$ be the function defined by $dom(f) = H^g_{QQ}$ and $f(r_1, r_2, r_3) = g_n(r_1, r_2, r_3)$ for each (r_1, r_2, r_3) in H^g_{QQ}, n being a nonnegative integer such that (r_1, r_2, r_3) is in $dom(g_n)$. The reader may easily verify that f is a surjective bounded morphism. $\square$

The result that emerges from the discussion above is the following

Proof of Proposition 15. (1) $\Rightarrow$ (2): Obvious, since $\mathcal{S}_{QQ}$ is a countable prenormal frame.
(2) $\Rightarrow$ (3): By [3, theorem 3.14], since $\mathcal{S}^g_{QQ}$ is a generated subframe of $\mathcal{S}_{QQ}$.
(3) $\Rightarrow$ (1): By [3, theorem 3.14] and Propositions 14 and 18. $\square$

As a result,

Corollary 19. *The following conditions are equivalent:*

(1) *ϕ is in $ML(SQS)$.*
(2) *ϕ is valid in every normal frame.*
(3) *ϕ is valid in every prenormal frame.*

Proof. By Corollary 12 and Propositions 13 and 15. $\square$

4.5. Complexity

In this section, we investigate the decidability/complexity of the membership problem in $ML(SQS)$. Let ϕ be a formula.

Lemma 20. *If there exists a prenormal frame $\mathcal{S} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}})$ such that $\mathcal{S} \not\models \phi$ then there exists a finite prenormal frame $\mathcal{S}' = (H_{\mathcal{S}'}, N_{\mathcal{S}'}, C_{\mathcal{S}'}, P_{\mathcal{S}'})$ such that $\mathcal{S}' \not\models \phi$.*

Proof. Suppose there exists a prenormal frame $\mathcal{S} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}})$ such that $\mathcal{S} \not\models \phi$. Hence, there exists a valuation V on $\mathcal{S}$ such that $(\mathcal{S}, V) \not\models \phi$. Thus, there exists a_0 in $H_{\mathcal{S}}$ such that $(\mathcal{S}, V), a_0 \not\models \phi$. Let Γ_ϕ be the least set of formulas such that

- ϕ is in Γ_ϕ,
- Γ_ϕ is closed under subformulas,
- for all formulas ψ, if there exists α in $\{N, C, P\}$ such that $[\alpha]\psi$ is in Γ_ϕ then for all α in $\{N, C, P\}$, $[\alpha]\psi$ is in Γ_ϕ.

Let $\equiv_{\Gamma_\phi}$ be the equivalence relation on $H_{\mathcal{S}}$ defined by

- $a \equiv_{\Gamma_\phi} b$ iff for all formulas ψ in Γ_ϕ, $\mathcal{M}, a \models \psi$ iff $\mathcal{M}, b \models \psi$.

For all a in $H_{\mathcal{S}}$, the equivalence class of a modulo $\equiv_{\Gamma_\phi}$ is denoted by $[a]_{\equiv_{\Gamma_\phi}}$. The quotient set of $H_{\mathcal{S}}$ modulo $\equiv_{\Gamma_\phi}$ is denoted by $H_{\mathcal{S}}/\equiv_{\Gamma_\phi}$. Let $\mathcal{S}' = (H_{\mathcal{S}'}, N_{\mathcal{S}'}, C_{\mathcal{S}'}, P_{\mathcal{S}'})$ be the frame such that $H_{\mathcal{S}'} = H_{\mathcal{S}}/\equiv_{\Gamma_\phi}$ whereas

- $N_{\mathcal{S}'}([a]_{\equiv_{\Gamma_\phi}}, [b]_{\equiv_{\Gamma_\phi}})$ iff for all formulas ψ, if $[N]\psi$ is in Γ_ϕ then
  - if $(\mathcal{S}, V), a \models [N]\psi$ then $(\mathcal{S}, V), b \models [N]\psi$ and $(\mathcal{S}, V), b \models [C]\psi$,
- $C_{\mathcal{S}'}([a]_{\equiv_{\Gamma_\phi}}, [b]_{\equiv_{\Gamma_\phi}})$ iff for all formulas ψ, if $[C]\psi$ is in Γ_ϕ then
  - $(\mathcal{S}, V), a \models [N]\psi$ iff $(\mathcal{S}, V), b \models [N]\psi$,
  - $(\mathcal{S}, V), a \models [C]\psi$ iff $(\mathcal{S}, V), b \models [C]\psi$,
- $P_{\mathcal{S}'}([a]_{\equiv_{\Gamma_\phi}}, [b]_{\equiv_{\Gamma_\phi}})$ iff for all formulas ψ, if $[P]\psi$ is in Γ_ϕ then
  - $(\mathcal{S}, V), a \models [N]\psi$ iff $(\mathcal{S}, V), b \models [N]\psi$,
  - $(\mathcal{S}, V), a \models [C]\psi$ iff $(\mathcal{S}, V), b \models [C]\psi$,
  - $(\mathcal{S}, V), a \models [P]\psi$ iff $(\mathcal{S}, V), b \models [P]\psi$.

Obviously, $\mathcal{S}'$ is finite and prenormal. Now, let V' be the valuation on $\mathcal{S}'$ defined by

- $V'(p) = \{[a]_{\equiv_{\Gamma_\phi}} \colon a \in V(p)\}$.

The reader may easily verify that $(\mathcal{S}', V')$ is a filtration of $(\mathcal{S}, V)$ through Γ_ϕ. Therefore, by [3, theorem 2.39], for all formulas ψ, if ψ is in Γ_ϕ then for all a in $H_{\mathcal{S}}$, $(\mathcal{S}, V), a \models \psi$ iff $(\mathcal{S}', V'), [a]_{\equiv_{\Gamma_\phi}} \models \psi$. Since $(\mathcal{S}, V), a_0 \not\models \phi$, $(\mathcal{S}', V'), [a_0]_{\equiv_{\Gamma_\phi}} \not\models \phi$. Consequently, $(\mathcal{S}', V') \not\models \phi$. Hence, $\mathcal{S}' \not\models \phi$. ◻

Lemma 21. *If there exists a finite prenormal frame $\mathcal{S} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}})$ such that $\mathcal{S} \not\models \phi$ then there exists a prenormal frame $\mathcal{S}' = (H_{\mathcal{S}'}, N_{\mathcal{S}'}, C_{\mathcal{S}'}, P_{\mathcal{S}'})$ such that $Card(H_{\mathcal{S}'}) \leq\mid \phi \mid^3$ and $\mathcal{S}' \not\models \phi$.*

Proof. Suppose there exists a finite prenormal frame $\mathcal{S} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}})$ such that $\mathcal{S} \not\models \phi$. Hence, there exists a valuation V on $\mathcal{S}$ such that $(\mathcal{S}, V) \not\models \phi$. Thus, there exists a_0 in $H_{\mathcal{S}}$ such that $(\mathcal{S}, V), a_0 \not\models \phi$. Let $\simeq$ be the equivalence relation on $H_{\mathcal{S}}$ defined by

- $a \simeq b$ iff $C_{\mathcal{S}}(a, b)$ or both $N_{\mathcal{S}}(a, b)$ and $N_{\mathcal{S}}(b, a)$.

For all a in $H_{\mathcal{S}}$, the equivalence class of a modulo $\simeq$ is denoted by $[a]_\simeq$. The quotient set of $H_{\mathcal{S}}$ modulo $\simeq$ is denoted by $H_{\mathcal{S}}/\simeq$. Next, let $\prec$ be the binary relation on $H_{\mathcal{S}}/\simeq$ defined by

- $[a]_\simeq \prec [b]_\simeq$ iff $N_{\mathcal{S}}(a, b)$ and not $N_{\mathcal{S}}(b, a)$.

Obviously, $\prec$ is a strict linear ordering on $H_{\mathcal{S}}/\simeq$. Let $[N]\psi_1, \ldots, [N]\psi_n$ be an enumeration of the set of all ϕ's subformulas which are of the form $[N]\psi$ and such that $(\mathcal{S}, V), a_0 \not\models [N]\psi$. For all positive integers i, if $i \leq n$ then let b_i in $H_{\mathcal{S}}$ be such that $N_{\mathcal{S}}(a_0, b_i)$, $(\mathcal{S}, V), b_i \not\models \psi_i$ and for all b in $H_{\mathcal{S}}$, if $[b_i]_\simeq \prec [b]_\simeq$ then $(\mathcal{S}, V), b \models \psi_i$. Let i be a positive integer such that $i \leq n$. Let $[C]\psi'_{i,1}, \ldots, [C]\psi'_{i,n_i}$ be an enumeration of the set of all ϕ's subformulas which are of the form $[C]\psi'$ and such that $(\mathcal{S}, V), b_i \not\models [C]\psi'$. For all positive integers j, if $j \leq n_i$ then let $c_{i,j}$ in $H_{\mathcal{S}}$ be such that $C_{\mathcal{S}}(b_i, c_{i,j})$ and $(\mathcal{S}, V), c_{i,j} \not\models \psi'_{i,j}$. Let j be a positive integer such that $j \leq n_i$. Let $[P]\psi''_{i,j,1}, \ldots, [P]\psi''_{i,j,n_{i,j}}$ be an enumeration of the set of all ϕ's subformulas which are of the form $[P]\psi''$ and such that $(\mathcal{S}, V), c_{i,j} \not\models [P]\psi''$. For all positive integers k, if $k \leq n_{i,j}$ then let $d_{i,j,k}$ in $H_{\mathcal{S}}$ be such that $P_{\mathcal{S}}(c_{i,j}, d_{i,j,k})$ and $(\mathcal{S}, V), d_{i,j,k} \not\models \psi''_{i,j,k}$. Let $\mathcal{S}' = (H_{\mathcal{S}'}, N_{\mathcal{S}'}, C_{\mathcal{S}'}, P_{\mathcal{S}'})$ be the frame such that $H_{\mathcal{S}'} = \{a_0\} \cup \{b_i \colon 1 \leq i \leq n\} \cup \{c_{i,j} \colon 1 \leq i \leq n$ and $1 \leq j \leq n_i\} \cup \{d_{i,j,k} \colon 1 \leq i \leq n$, $1 \leq j \leq n_i$ and $1 \leq k \leq n_{i,j}\}$ whereas

- $N_{\mathcal{S}'}$ is the restriction of $N_{\mathcal{S}}$ to $H_{\mathcal{S}'}$,
- $C_{\mathcal{S}'}$ is the restriction of $C_{\mathcal{S}}$ to $H_{\mathcal{S}'}$,
- $P_{\mathcal{S}'}$ is the restriction of $P_{\mathcal{S}}$ to $H_{\mathcal{S}'}$.

Obviously, $\mathcal{S}'$ is prenormal. Moreover, $Card(H_{\mathcal{S}'}) \leq\mid \phi \mid^3$. Now, let V' be the valuation on $\mathcal{S}'$ defined by

- $V'(p) = V(p) \cap H_{\mathcal{S}'}$.

The reader may easily verify that for all formulas ψ, if ψ is in Γ_ϕ then for all a in $H_{\mathcal{S}'}$, $(\mathcal{S}, V), a \models \psi$ iff $(\mathcal{S}', V'), a \models \psi$. Since $(\mathcal{S}, V), a_0 \not\models \phi$, $(\mathcal{S}', V'), a_0 \not\models \phi$. Consequently, $(\mathcal{S}', V') \not\models \phi$. Hence, $\mathcal{S}' \not\models \phi$. $\square$

As a result,

Proposition 22.

(1) *The membership problem in $ML(SQS)$ is decidable.*
(2) *The membership problem in $ML(SQS)$ is co-NP-hard.*
(3) *The membership problem in $ML(SQS)$ is in co-NP.*

Proof. (1) By [3, theorem 6.13], Corollary 19, Lemma 20 and the recursive enumerability of the class of all prenormal frames.

(2) Let PL be the set of all valid formulas of propositional logic. Obviously, $ML(SQS)$ is a conservative extension of PL. Since the membership problem in PL is co-NP-hard [18], the membership problem in $ML(SQS)$ is co-NP-hard.

(3) By [3, lemma 6.35], Corollary 19, Lemmas 20 and 21 and the tractability of the problem of deciding whether a given finite frame is prenormal. $\square$

4.6. Definability

We tackle the problem of the definability of $[N]$, $[C]$ and $[P]$ in the class of all normal frames. The following results imply that the connectives $[N]$, $[C]$ and $[P]$ cannot be eliminated from our language.

Proposition 23.

(1) *$[N]$ is not definable with $[C]$ and $[P]$ in the class of all normal frames.*
(2) *$[C]$ is not definable with $[N]$ and $[P]$ in the class of all normal frames.*
(3) *$[P]$ is not definable with $[N]$ and $[C]$ in the class of all normal frames.*

Proof. (1) Suppose $[N]$ is definable with $[C]$ and $[P]$ in the class of all normal frames. Hence, there exists a formula $\phi(p)$ in $[C]$ and $[P]$ such that $(*)$ for all normal frames $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$, for all valuations V on $\mathcal{S}$ and for all a in $H_\mathcal{S}$, $(\mathcal{S}, V), a \models [N]p$ iff $(\mathcal{S}, V), a \models \phi(p)$. Let a in $H_\mathcal{S}$. Let V be the valuation on $\mathcal{S}$ defined by $V(p) = N_\mathcal{S}(a) \cup [a]_{C_\mathcal{S}}$. Obviously, $(\mathcal{S}, V), a \models [N]p$. Thus, by $(*)$, $(\mathcal{S}, V), a \models \phi(p)$. Let V' be the valuation on $\mathcal{S}$ defined by $V'(p) = [a]_{C_\mathcal{S}}$. Obviously, $(\mathcal{S}, V'), a \not\models [N]p$. Moreover, for all formulas $\psi(p)$ in $[C]$ and $[P]$, $(\mathcal{S}, V), a \models \psi(p)$ iff $(\mathcal{S}, V'), a \models \psi(p)$. Since $(\mathcal{S}, V), a \models \phi(p)$, $(\mathcal{S}, V'), a \models \phi(p)$. Therefore, by $(*)$, $(\mathcal{S}, V'), a \models [N]p$: a contradiction.

(2) Suppose $[C]$ is definable with $[N]$ and $[P]$ in the class of all normal frames. Hence, there exists a formula $\phi(p)$ in $[N]$ and $[P]$ such that $(*)$ for all normal frames $\mathcal{S} = (H_\mathcal{S}, N_\mathcal{S}, C_\mathcal{S}, P_\mathcal{S})$, for all valuations V on $\mathcal{S}$ and for all a in $H_\mathcal{S}$, $(\mathcal{S}, V), a \models [C]p$ iff $(\mathcal{S}, V), a \models \phi(p)$. Let a in $H_\mathcal{S}$. Let V be the valuation on $\mathcal{S}$ defined by $V(p) = [a]_{C_\mathcal{S}}$. Obviously, $(\mathcal{S}, V), a \models [C]p$. Thus, by $(*)$, $(\mathcal{S}, V), a \models \phi(p)$. Let V' be the valuation on $\mathcal{S}$ defined by $V'(p) = [a]_{P_\mathcal{S}}$. Obviously, $(\mathcal{S}, V'), a \not\models [C]p$. Moreover, for all formulas $\psi(p)$ in $[N]$ and $[P]$, $(\mathcal{S}, V), a \models \psi(p)$ iff $(\mathcal{S}, V'), a \models \psi(p)$. Since $(\mathcal{S}, V), a \models \phi(p)$, $(\mathcal{S}, V'), a \models \phi(p)$. Therefore, by $(*)$, $(\mathcal{S}, V'), a \models [C]p$: a contradiction.

(3) Suppose $[P]$ is definable with $[N]$ and $[C]$ in the class of all normal frames. Hence, there exists a formula $\phi(p)$ in $[N]$ and $[C]$ such that $(*)$ for all standard frames $\mathcal{S} = (H_{\mathcal{S}}, N_{\mathcal{S}}, C_{\mathcal{S}}, P_{\mathcal{S}})$, for all valuations V on $\mathcal{S}$ and for all a in $H_{\mathcal{S}}$, $(\mathcal{S}, V), a \models [P]p$ iff $(\mathcal{S}, V), a \models \phi(p)$. Let a in $H_{\mathcal{S}}$. By $Sym(C)$, $Sym(P)$ and $Inf_1(C, P)$, there exists b in $H_{\mathcal{S}}$ such that $C_{\mathcal{S}}(a, b)$ and not $P_{\mathcal{S}}(a, b)$. Let V be the valuation on $\mathcal{S}$ defined by $V(p) = [a]_{C_{\mathcal{S}}} \setminus \{b\}$. Obviously, $(\mathcal{S}, V), a \models [P]p$. Thus, by $(*)$, $(\mathcal{S}, V), a \models \phi(p)$. By $Sym(P)$ and $Inf_1(P, \equiv)$, there exists c in $H_{\mathcal{S}}$ such that $P_{\mathcal{S}}(a, c)$ and $a \neq c$. Let V' be the valuation on $\mathcal{S}$ defined by $V'(p) = [a]_{C_{\mathcal{S}}} \setminus \{c\}$. Obviously, $(\mathcal{S}, V'), a \not\models [P]p$. Moreover, for all formulas $\psi(p)$ in $[N]$ and $[C]$, $(\mathcal{S}, V), a \models \psi(p)$ iff $(\mathcal{S}, V'), a \models \psi(p)$. Since $(\mathcal{S}, V), a \models \phi(p)$, $(\mathcal{S}, V'), a \models \phi(p)$. Therefore, by $(*)$, $(\mathcal{S}, V'), a \models [P]p$: a contradiction. $\square$

5. Variants

Other primitives may be defined as well. In this section, we consider the predicate symbol $<$ of precedence between positive hyperreals and the function symbol $+$ of addition between positive hyperreals.

5.1. Adding precedence

Let us add a predicate symbol $<$ of arity 2 to our first-order language. The *formulas* are now given by the rule:

- $\phi ::= N(x, y) \mid C(x, y) \mid P(x, y) \mid x < y \mid x \equiv y \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \forall x.\phi$.

Let $<$ be interpreted in $\mathcal{S}_{PH}$ by means of the relation $<_{\mathcal{S}_{PH}}$ of precedence between positive hyperreals in nonstandard analysis. The following result implies that the predicate symbol $<$ really increases the expressivity of our first-order language.

Proposition 24. *$<$ is not definable with N, C, P and $\equiv$ in $\mathcal{S}_{PH}$.*

Proof. Suppose $<$ is definable with N, C, P and $\equiv$ in $\mathcal{S}_{PH}$. Hence, there exists a formula $\phi(x, y)$ in N, C, P and $\equiv$ such that $(*)$ for all a, b in $H_{\mathcal{S}_{PH}}$, $a <_{\mathcal{S}_{PH}} b$ iff $\mathcal{S}_{PH} \models \phi(x, y)$ $[a, b]$. Let a, b in $H_{\mathcal{S}_{PH}}$ such that $P_{\mathcal{S}_{PH}}(a, b)$ and $a <_{\mathcal{S}_{PH}} b$. Since $(*)$, $\mathcal{S}_{PH} \models \phi(x, y)$ $[a, b]$. Obviously, the second player wins all Ehrenfeucht games over $(\mathcal{S}_{PH}, [a, b])$ and $(\mathcal{S}_{PH}, [b, a])$ with respect to N, C, P and $\equiv$. Thus, by [12, theorem 2.2.8], for all formulas $\psi(x, y)$ in N, C, P and $\equiv$, $\mathcal{S}_{PH} \models \psi(x, y)$ $[a, b]$ iff $\mathcal{S}_{PH} \models \psi(x, y)$ $[b, a]$. Since $\mathcal{S}_{PH} \models \phi(x, y)$ $[a, b]$, $\mathcal{S}_{PH} \models \phi(x, y)$ $[b, a]$. Since $(*)$, $b <_{\mathcal{S}_{PH}} a$. Therefore, $a \not<_{\mathcal{S}_{PH}} b$: a contradiction. $\square$

What about the axiomatization/completeness or the decidability/complexity of the first-order theory based on the predicates N, C, P and $<$? As for the modal logic option, the obvious road consists in adding a connective $[<]$ interpreted in $\mathcal{S}_{PH}$ in the following way:

- $(\mathcal{S}_{PH}, V), a \models [<]\phi$ iff for all b in $H_{\mathcal{S}_{PH}}$, if $a <_{\mathcal{S}_{PH}} b$ then $(\mathcal{S}_{PH}, V), b \models \phi$.

What about the axiomatization/completeness or the decidability/complexity of the modal logic based on the connectives $[N]$, $[C]$, $[P]$ and $[<]$?

5.2. Adding addition

Let us add a function symbol $+$ of arity 2 to our first-order language. The *formulas* are now given by the rule:

- $\phi ::= N(s, t) \mid C(s, t) \mid P(s, t) \mid s \equiv t \mid \bot \mid \neg\phi \mid (\phi \vee \psi) \mid \forall x.\phi$

where s and t range over the set of *terms* defined by the rule

- $s ::= x \mid (s + t)$.

Let $+$ be interpreted in $\mathcal{S}_{PH}$ by means of the operation $+_{\mathcal{S}_{PH}}$ of addition between positive hyperreals in nonstandard analysis. In $\mathcal{S}_{PH}$, it appears that if we restrict the language to the predicate N or if we restrict the language to the predicates N and C then the function symbol $+$ can be eliminated. To see this, it suffices to observe that the following sentences hold in $\mathcal{S}_{PH}$:

- $\forall x.\forall y.\forall z.(N(x + y, z) \leftrightarrow N(x, z) \wedge N(y, z))$,
- $\forall x.\forall y.\forall z.(N(x, y + z) \leftrightarrow N(x, y) \vee N(x, z))$,
- $\forall x.\forall y.\forall z.(C(x + y, z) \leftrightarrow (C(x, z) \wedge \bar{N}(z, y)) \vee (C(y, z) \wedge \bar{N}(z, x)))$,
- $\forall x.\forall y.\forall z.(C(x, y + z) \leftrightarrow (C(x, y) \wedge \bar{N}(x, z)) \vee (C(x, z) \wedge \bar{N}(x, y)))$.

But this leaves open the possibility that the function symbol $+$ can be eliminated if we restrict the language to a different set of predicates.

Proposition 25.

(1) *In $\mathcal{S}_{PH}$, if we restrict the language to the predicate C then the function symbol $+$ cannot be eliminated.*
(2) *In $\mathcal{S}_{PH}$, if we restrict the language to a set of predicates containing P then the function symbol $+$ cannot be eliminated.*
(3) *In $\mathcal{S}_{PH}$, if we restrict the language to a set of predicates containing $\equiv$ then the function symbol $+$ cannot be eliminated.*

Proof. (1) Suppose there exists a formula $\phi(x, y, z)$ in C such that $(*)$ for all a, b, c in $H_{\mathcal{S}_{PH}}$, $C_{\mathcal{S}_{PH}}(a +_{\mathcal{S}_{PH}} b, c)$ iff $\mathcal{S}_{PH} \models \phi(x, y, z)\ [a, b, c]$. Let a, b, c in $H_{\mathcal{S}_{PH}}$ be such that $C_{\mathcal{S}_{PH}}(a +_{\mathcal{S}_{PH}} b, c)$ and not $C_{\mathcal{S}_{PH}}(a^{-1} +_{\mathcal{S}_{PH}} b^{-1}, c^{-1})$. Since $(*)$, $\mathcal{S} \models \phi(x, y, z)\ [a, b, c]$. Obviously, the second player wins all Ehrenfeucht games over $(\mathcal{S}_{PH}, [a, b, c])$ and $(\mathcal{S}_{PH}, [a^{-1}, b^{-1}, c^{-1}])$ with respect to C. Thus, by [12, theorem 2.2.8], for all formulas $\psi(x, y, z)$ in C, $\mathcal{S}_{PH} \models \psi(x, y, z)\ [a, b, c]$ iff $\mathcal{S}_{PH} \models \psi(x, y, z)\ [a^{-1}, b^{-1}, c^{-1}]$. Since $\mathcal{S}_{PH} \models \phi(x, y, z)\ [a, b, c]$, $\mathcal{S}_{PH} \models \phi(x, y, z)\ [a^{-1}, b^{-1}, c^{-1}]$. Since $(*)$, $C_{\mathcal{S}_{PH}}(a^{-1} +_{\mathcal{S}_{PH}} b^{-1}, c^{-1})$: a contradiction.

(2) Suppose there exists a formula $\phi(x, y, z)$ in N, C, P and $\equiv$ such that $(*)$ for all a, b, c in $H_{\mathcal{S}_{PH}}$, $P_{\mathcal{S}_{PH}}(a +_{\mathcal{S}_{PH}} b, c)$ iff $\mathcal{S}_{PH} \models \phi(x, y, z)\ [a, b, c]$. Let a, b, c in $H_{\mathcal{S}_{PH}}$ be such that $P_{\mathcal{S}_{PH}}(a +_{\mathcal{S}_{PH}} b, c)$ and not $P_{\mathcal{S}_{PH}}(a^2 +_{\mathcal{S}_{PH}} b^2, c^2)$. Since $(*)$, $\mathcal{S} \models \phi(x, y, z)\ [a, b, c]$. Obviously, the second player wins all Ehrenfeucht games over $(\mathcal{S}_{PH}, [a, b, c])$ and $(\mathcal{S}_{PH}, [a^2, b^2, c^2])$ with respect to N, C, P and $\equiv$. Thus, by [12, theorem 2.2.8], for all formulas $\psi(x, y, z)$ in N, C, P and $\equiv$, $\mathcal{S}_{PH} \models \psi(x, y, z)\ [a, b, c]$ iff $\mathcal{S}_{PH} \models \psi(x, y, z)\ [a^2, b^2, c^2]$. Since $\mathcal{S}_{PH} \models \phi(x, y, z)\ [a, b, c]$, $\mathcal{S}_{PH} \models \phi(x, y, z)\ [a^2, b^2, c^2]$. Since $(*)$, $P_{\mathcal{S}_{PH}}(a^2 +_{\mathcal{S}_{PH}} b^2, c^2)$: a contradiction.

(3) Suppose there exists a formula $\phi(x, y, z)$ in N, C, P and $\equiv$ such that $(*)$ for all a, b, c in $H_{\mathcal{S}_{PH}}$, $a + b = c$ iff $\mathcal{S}_{PH} \models \phi(x, y, z)\ [a, b, c]$. Let a, b, c in $H_{\mathcal{S}_{PH}}$ be such that $a +_{\mathcal{S}_{PH}} b = c$ and $a^2 +_{\mathcal{S}_{PH}} b^2 \neq c^2$. Since $(*)$, $\mathcal{S} \models \phi(x, y, z)\ [a, b, c]$. Obviously, the second player wins all Ehrenfeucht games over $(\mathcal{S}_{PH}, [a, b, c])$ and $(\mathcal{S}_{PH}, [a^2, b^2, c^2])$ with respect to N, C, P and $\equiv$. Thus, by [12, theorem 2.2.8], for all formulas $\psi(x, y, z)$ in N, C, P and $\equiv$, $\mathcal{S}_{PH} \models \psi(x, y, z)\ [a, b, c]$ iff $\mathcal{S}_{PH} \models \psi(x, y, z)\ [a^2, b^2, c^2]$. Since $\mathcal{S}_{PH} \models \phi(x, y, z)\ [a, b, c]$, $\mathcal{S}_{PH} \models \phi(x, y, z)\ [a^2, b^2, c^2]$. Since $(*)$, $a^2 +_{\mathcal{S}_{PH}} b^2 = c^2$: a contradiction. $\square$

What about the axiomatization/completeness or the decidability/complexity of the first-order theory based on the predicates N, C and P and the function $+$? As for the modal logic option, the obvious road consists in adding a connective $\oplus$ interpreted in $\mathcal{S}_{PH}$ in the following way:

- $(\mathcal{S}_{PH}, V), a \models \phi \oplus \psi$ iff there exist b, c in $H_{\mathcal{S}_{PH}}$ such that $a = b +_{\mathcal{S}_{PH}} c$, $(\mathcal{S}_{PH}, V), b \models \phi$ and $(\mathcal{S}_{PH}, V), c \models \psi$.

What about the axiomatization/completeness or the decidability/complexity of the modal logic based on the connectives $[N]$, $[C]$, $[P]$ and $\oplus$?

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