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Sweeping Processes Perturbed by Rough Signals

Charles CASTAING, Nicolas MARIE and Paul RAYNAUD DE FITTE

Abstract This paper deals with the existence, the uniqueness and an approximation scheme of the solution to sweeping processes perturbed by a continuous signal of finite p-variation with $p \in [1,3[$. It covers pathwise stochastic noises directed by a fractional Brownian motion of Hurst parameter greater than $1/3$.

1 Introduction

Consider a multifunction $C : [0,T] \rightrightarrows \mathbb{R}^e$ with $e \in \mathbb{N}^*$. Roughly speaking, the Moreau sweeping process (see Moreau [20]) associated to C is the path X, living in C, such that when it hits the frontier of C, a minimal force is applied to X in order to keep it inside of C. Precisely, X is a solution to the following differential inclusion:

$$\begin{cases} -\dfrac{dDX}{d|DX|}(t) \in N_{C(t)}(X(t)) \quad |DX|\text{-a.e.} \\ X(0) = a \in C(0) \end{cases} \tag{1}$$

where DX is the differential measure associated with the continuous function of bounded variation X, $|DX|$ is its variation measure, and $N_{C(t)}(X(t))$ is the normal cone of $C(t)$ at $X(t)$. This problem has been deeply studied by many authors. For instance, the reader can refer to Moreau [20], Valadier [27] or Monteiro Marques [19].

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Several authors studied some perturbed versions of Problem (1), in particular by a stochastic multiplicative noise in Itô's calculus framework (see Revuz and Yor [22]). For instance, the reader can refer to Bernicot and Venel [3] or Castaing et al. [5]. On reflected diffusion processes, which are perturbed sweeping processes with constant constraint set, the reader can refer to Kang and Ramanan [13].

The general way to formulate the perturbed Problem (1) when the perturbation has unbounded variation is to split the unknown X in two parts, following ideas of Skorokhod [24, 25]: one part has bounded variation and represents the "pure" sweeping process, and the other one may have unbounded variation and represents the perturbed part. In this line, we consider the perturbed sweeping process

$$\begin{cases} X(t) = H(t) + Y(t) \\ H(t) = \displaystyle\int_0^t f(X(s))dZ(s) \\ -\dfrac{dY}{d|DY|}(t) \in N_{C_H(t)}(Y(t)) \ |DY|\text{-a.e. with } Y(0) = a \end{cases} \tag{2}$$

where X, Y and H are unknown, $C_H(t) = C(t) - H(t)$, $t \in [0,T]$ (thus $N_{C_H(t)}(Y(t)) = N_{C(t)}(X(t))$), $Z : [0,T] \to \mathbb{R}^d$ is a continuous signal of finite p-variation with $d \in \mathbb{N}^*$ and $p \in [1,\infty[$, $f \in \text{Lip}^\gamma(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$ with $\gamma > p$, and the integral against Z is taken in the sense of rough paths (see Section 2 for precise definitions)). On the rough integral, the reader can refer to Lyons [16], Friz and Victoir [11] or Friz and Hairer [9]. Throughout the paper, the multifunction C satisfies the following assumption.

Assumption 1 *C is a convex compact valued multifunction, continuous for the Hausdorff distance, and there exists a continuous selection $\gamma : [0,T] \to \mathbb{R}^e$ satisfying*

$$\overline{B}_e(\gamma(t), r) \subset \text{int}(C(t)) \ ; \forall t \in [0,T],$$

where $\overline{B}_e(\gamma(t), r)$ denotes the closed ball of radius r centered at $\gamma(t)$.

This assumption is equivalent to saying that $C(t)$ has nonempty interior for every $t \in [0,T]$, see [5, Lemma 2.2].

In Falkowski and Słomiński [8], when $p \in [1,2[$ and $C(t)$ is a cuboid of $\mathbb{R}^e$ for every $t \in [0,T]$, the authors proved the existence and uniqueness of the solution of Problem (2). Furthermore, several authors studied the existence and uniqueness of the solution for reflected rough differential equations. In [1], M. Besalú et al. proved the existence and uniqueness of the solution for delayed rough differential equations with non-negativity constraints. Recently, S. Aida gets the existence of solutions for a large class of reflected rough differential equations in [2] and [1]. Finally, in [6], A. Deya et al. proved the existence and uniqueness of the solution for 1-dimensional reflected rough differential equations. An interesting remark related to these references is that when C is not a cuboid, moving or not, it is a challenge to get the uniqueness of the solution for reflected rough differential equations and sweeping processes.

For $p \in [1,3[$, the purpose of this paper is to prove the existence of solutions to Problem (2) when C satisfies Assumption 1, and a necessary and sufficient condition for uniqueness, close to the monotonicity of the normal cone which allows to prove the uniqueness in the case when $p = 1$ and there is an additive continuous signal of finite q-variation with $q \in [1,3[$. In that, the convergence of an approximation scheme is also proved.

Section 2 deals with some preliminaries on sweeping processes and the rough integral. Section 3 is devoted to the existence of solutions to Problem (2) when Z is a moderately irregular signal (i.e. $p \in [1,2[$) and when Z is a rough signal (i.e. $p \in [2,3[$). Section 4 deals with some uniqueness results. The convergence of an approximation scheme based on Moreau's catching up algorithm is proved in Section 5 in the case when $p = 1$ and there is an additive continuous signal of finite q-variation with $q \in [1,3[$. Finally, Section 6 deals with sweeping processes perturbed by a pathwise stochastic noise directed by a fractional Brownian motion of Hurst parameter greater than $1/3$.

The following notations, definitions and properties are used throughout the paper.

Notations and elementary properties:

1. $C_h(t) := C(t) - h(t)$ for every function $h : [0,T] \to \mathbb{R}^e$.
2. $N_C(x)$ is the normal cone of C at x, for any closed convex subset C of $\mathbb{R}^e$ and any $x \in \mathbb{R}^e$ (recall that $N_C(x) = \emptyset$ if $x \notin C$).
3. $\Delta_T := \{(s,t) \in [0,T]^2 : s < t\}$ and $\Delta_{s,t} := \{(u,v) \in [s,t]^2 : u < v\}$ for every $(s,t) \in \Delta_T$.
4. For every function x from $[0,T]$ into $\mathbb{R}^d$ and $(s,t) \in \Delta_T$, $x(s,t) := x(t) - x(s)$.
5. Consider $(s,t) \in \Delta_T$. The vector space of continuous functions from $[s,t]$ into $\mathbb{R}^d$ is denoted by $C^0([s,t],\mathbb{R}^d)$ and equipped with the uniform norm $\|.\|_{\infty,s,t}$ defined by
$$\|x\|_{\infty,s,t} := \sup_{u\in[s,t]} \|x(u)\|$$
for every $x \in C^0([s,t],\mathbb{R}^d)$, or the semi-norm $\|.\|_{0,s,t}$ defined by
$$\|x\|_{0,s,t} := \sup_{u,v\in[s,t]} \|x(v) - x(u)\|$$
for every $x \in C^0([s,t],\mathbb{R}^d)$. Moreover, $\|.\|_{\infty,T} := \|.\|_{\infty,0,T}$, $\|.\|_{0,T} := \|.\|_{0,0,T}$ and
$$C_0^0([s,t],\mathbb{R}^d) := \{x \in C^0([s,t],\mathbb{R}^d) : x(0) = 0\}.$$
6. Consider $(s,t) \in \Delta_T$. The set of all dissections of $[s,t]$ is denoted by $\mathfrak{D}_{[s,t]}$ and the set of all strictly increasing sequences $(s_n)_{n\in\mathbb{N}}$ of $[s,t]$ such that $s_0 = s$ and $\lim_\infty s_n = t$ is a denoted by $\mathfrak{D}_{\infty,[s,t]}$.

7. Consider $(s,t)\in\Delta_T$. A function $x:[s,t]\rightarrow\mathbb{R}^d$ has finite p-variation if and only if,

$$\|x\|_{p\text{-var},s,t} := \sup\left\{\left|\sum_{k=1}^{n-1}\|x(t_k,t_{k+1})\|^p\right|^{1/p} ; n\in\mathbb{N}^* \text{ and } (t_k)_{k\in[\![1,n]\!]}\in\mathfrak{D}_{[s,t]}\right\}$$
$$< \infty.$$

Consider the vector space

$$C^{p\text{-var}}([s,t],\mathbb{R}^d) := \{x\in C^0([s,t],\mathbb{R}^d) : \|x\|_{p\text{-var},s,t}<\infty\}.$$

The map $\|.\|_{p\text{-var},s,t}$ is a semi-norm on $C^{p\text{-var}}([s,t],\mathbb{R}^d)$.

Moreover, $\|.\|_{p\text{-var},T} := \|.\|_{p\text{-var},0,T}$.

Remarks:

a. For every $q,r\in[1,\infty[$ such that $q\geqslant r$,

$$\forall x\in C^{r\text{-var}}([s,t],\mathbb{R}^d),\ \|x\|_{q\text{-var},s,t}\leqslant\|x\|_{r\text{-var},s,t}.$$

In particular, any continuous function of bounded variation on $[s,t]$ belongs to $C^{q\text{-var}}([s,t],\mathbb{R}^d)$ for every $q\in[1,\infty[$.

b. For every $(s,t)\in\Delta_T$ and $x\in C^{1\text{-var}}([s,t],\mathbb{R})$,

$$\|x\|_{1\text{-var},s,t} = \int_s^t |Dx|,$$

where $|Dx|$ is the variation measure of the differential measure Dx associated with x.

8. The vector space of Lipschitz continuous maps from $\mathbb{R}^e$ into $\mathcal{M}_{e,d}(\mathbb{R})$ is denoted by $\text{Lip}(\mathbb{R}^e,\mathcal{M}_{e,d}(\mathbb{R}))$ and equipped with the Lipschitz semi-norm $\|.\|_{\text{Lip}}$ defined by

$$\|\varphi\|_{\text{Lip}} := \sup\left\{\frac{\|\varphi(y)-\varphi(x)\|}{\|y-x\|} ; x,y\in\mathbb{R}^e \text{ and } x\neq y\right\}$$

for every $\varphi\in\text{Lip}(\mathbb{R}^e,\mathcal{M}_{e,d}(\mathbb{R}))$.

9. For every $\lambda\in\mathbb{R}$,

$$\lfloor\lambda\rfloor := \max\{n\in\mathbb{Z} : n<\lambda\}$$

and $\{\lambda\} := \lambda-\lfloor\lambda\rfloor$.

10. Consider $\gamma\in[1,\infty[$. A continuous map $\varphi:\mathbb{R}^e\rightarrow\mathcal{M}_{d,e}(\mathbb{R})$ is γ-Lipschitz in the sense of Stein if and only if,

$$\|\varphi\|_{\text{Lip}^\gamma} := \|D^{\lfloor\gamma\rfloor}\varphi\|_{\{\gamma\}\text{-Höl}}\vee\max\{\|D^k\varphi\|_\infty ; k\in[\![0,\lfloor\gamma\rfloor]\!]\}$$
$$< \infty.$$

Consider the vector space

$$\text{Lip}^{\gamma}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R})) := \{\varphi \in C^0(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R})) : \|\varphi\|_{\text{Lip}^{\gamma}} < \infty\}.$$

The map $\|.\|_{\text{Lip}^{\gamma}}$ is a norm on $\text{Lip}^{\gamma}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$.

Remarks:

a. If $\varphi \in \text{Lip}^{\gamma}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$, then $\varphi \in \text{Lip}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$.
b. If $\varphi \in C^{\lfloor\gamma\rfloor+1}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$ is bounded with bounded derivatives, then $\varphi \in \text{Lip}^{\gamma}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$.

2 Preliminaries

This section deals with some preliminaries on sweeping processes and the rough integral. The first subsection states some fundamental results on unperturbed sweeping processes coming from Moreau [20], Valadier [27] and Monteiro Marques [19]. A continuity result of Castaing et al. [5], which is the cornerstone of the proofs of Theorem 3 and Theorem 4, is also stated. The second subsection deals with the integration along rough paths. In this paper, definitions and propositions are stated as in Friz and Hairer [9], in accordance with M. Gubinelli's approach (see Gubinelli [12]).

2.1 Sweeping processes

The following theorem, due to Monteiro Marques [17, 18, 19] using an estimation due to Valadier (see [4, 27]), states a sufficient condition of existence and uniqueness of the solution of the unperturbed sweeping process defined by Problem (1).

Proposition 1 *Assume that C is a convex compact valued multifunction, continuous for the Hausdorff distance, and such that there exists $(x,r) \in \mathbb{R}^e \times]0,\infty[$ satisfying*

$$\overline{B}_e(x,r) \subset C(t) \; ; \forall t \in [0,T].$$

Then Problem (1) has a unique continuous solution of finite 1-variation $y : [0,T] \to \mathbb{R}^e$ such that

$$\|y\|_{1\text{-var},T} \leqslant l(r, \|a - x\|),$$

where $l : \mathbb{R}_+^2 \to \mathbb{R}_+$ is the map defined by

$$l(s,S) := \begin{cases} \max\left\{0, \dfrac{S^2 - s^2}{2s}\right\} & \textit{if } e > 1 \\ \max\{0, S - s\} & \textit{if } e = 1 \end{cases} \; ; \forall s, S \in \mathbb{R}_+.$$

This proposition is a consequence of the two following ones. These two propositions are also used in Section 5.

Proposition 2 *Under Assumption 1, a map $y : [0,T] \to \mathbb{R}^e$ is a solution of Problem (1) if it satisfies the two following conditions:*

1. *For every $t \in [0,T]$, $y(t) \in C(t)$.*
2. *For every $(s,t) \in \Delta_T$ and $z \in \cap_{\tau\in[s,t]} C(\tau)$,*

$$\langle z, y(t) - y(s)\rangle \geqslant \frac{1}{2}(\|y(t)\|^2 - \|y(s)\|^2).$$

Let us denote by $\text{proj}_A(.)$ the orthogonal projection on a convex set $A \subset \mathbb{R}^e$.

Proposition 3 *Consider $n \in \mathbb{N}^*$, $(t_0^n, \dots, t_n^n)$ the dissection of $[0,T]$ of constant mesh T/n and the step function Y^n defined by*

$$\begin{cases} Y_0^n := a \\ Y_{k+1}^n = \text{proj}_{C(t_{k+1}^n)}(Y_k^n)\ ;\ k \in [\![0, n-1]\!] \\ Y^n(t) := Y_k^n\ ;\ t \in [t_k^n, t_{k+1}^n[,\ k \in [\![0, n-1]\!]. \end{cases}$$

1. *Under the conditions of Proposition 1 on C, $\|Y^n\|_{1\text{-var},T} \leqslant l(r, \|a - x\|)$.*
2. *Under Assumption 1, for every $m \in n\mathbb{N}^*$ and $t \in [0,T]$, there exist $i \in [\![1,n]\!]$ and $j \in [\![1,m]\!]$ such that $t \in [t_{i-1}^n, t_i^n[$, $t \in [t_{j-1}^m, t_j^m[$ and*

$$\|Y^n(t) - Y^m(t)\|^2 \leqslant 2 d_H(C(t_i^n), C(t_j^m))(\|Y^n\|_{1\text{-var},t} + \|Y^m\|_{1\text{-var},t}).$$

See Monteiro Marques [19], Chapter 2 for the proofs of the three previous propositions.

Let h be a continuous function from $[0,T]$ into $\mathbb{R}^e$ such that $h(0) = 0$. If it exists, a Skorokhod decomposition of (C, a, h) is a couple (v_h, w_h) such that:

$$\begin{cases} v_h(t) = h(t) + w_h(t) \\ -\dfrac{dDw_h}{d|Dw_h|}(t) \in N_{C_h(t)}(w_h(t))\ |Dw_h|\text{-a.e. with } w_h(0) = a \end{cases} \tag{3}$$

where v_h and w_h are continuous, and w_h has bounded variation. Since $N_{C_h(t)}(x) = \emptyset$ when $x \notin C_h(t)$, the system (3) implies that, $|Dw_h|$-a.e., $w_h(t) \in C_h(t)$, that is, $v_h(t) \in C(t)$. Under Assumption 1, by Proposition 1 together with Castaing et al. [5, Lemma 2.2], (C, a, h) has a unique Skorokhod decomposition (v_h, w_h).

Theorem 1 *Under Assumption 1, if $(h_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions from $[0,T]$ into $\mathbb{R}^e$ which converges uniformly to $h \in C^0([0,T], \mathbb{R}^e)$, then*

$$\sup_{n\in\mathbb{N}} \|w_{h_n}\|_{1\text{-var},T} < \infty$$

and

$$(v_{h_n}, w_{h_n}) \xrightarrow[n\to\infty]{\|.\|_{\infty,T}} (v_h, w_h).$$

See Castaing et al. [5, Theorem 2.3].

Under Assumption 1, note that there exist $R > 0$, $N \in \mathbb{N}^*$ and a dissection $(t_0, \dots, t_N)$ of $[0,T]$ such that

$$\overline{B}_e(\gamma(t_k), R) \subset C(u) \tag{4}$$

for every $k \in [\![0, N-1]\!]$ and $u \in [t_k, t_{k+1}]$.

Proposition 4 *Under Assumption 1:*

1. *The map $(v_., w_.)$ is continuous from*

$$C_0^0([0,T], \mathbb{R}^e) \text{ to } C^0([0,T], \mathbb{R}^e) \times C^{1\text{-var}}([0,T], \mathbb{R}^e).$$

2. *Consider $(s,t) \in \Delta_T$ and $\rho \in]0, R/2]$ where R is defined in (4). For every $h \in C_0^0([s,t], \mathbb{R}^e)$ such that $\|h\|_{0,s,t} \leqslant \rho$,*

$$\|w_h\|_{1\text{-var},s,t} \leqslant \mathfrak{M}(\rho)$$

with

$$\mathfrak{M}(\rho) := \frac{N}{2\rho} \sup_{u\in[0,T]} \sup_{x,y\in C(u)} \|x-y\|^2.$$

Proof Refer to Castaing et al. [5, Lemma 5.3] for a proof of the first point.

Let us insert s and t in the dissection $(t_0, \dots, t_N)$ of $[0,T]$ and define $k(s), k(t) \in [\![0, N+2]\!]$ by

$$t_{k(s)} := s \text{ and } t_{k(t)} := t.$$

Consider $k \in [\![k(s), k(t)-1]\!]$ and $u \in [t_k, t_{k+1}]$.

On the one hand,

$$\overline{B}_e(\gamma(t_k) - h(t_k), \rho) \subset B_e(\gamma(t_k) - h(u), R) \subset C_h(u).$$

So,

$$\overline{B}_e(\gamma(t_k) - v_h(t_k), \rho) \subset C(u) - h(t_k, u) - v_h(t_k).$$

On the other hand,

$$v_h(t_k, u) = h(t_k, u) + w_{h,t_k}(u)$$

with

$$w_{h,t_k}(u) := w_h(u) - w_h(t_k).$$

Moreover,

$$-\frac{dDw_h}{d|Dw_h|}(u) \in N_{C(u)-h(u)}(w_h(u)) \ |Dw_h|\text{-a.e.}$$

and then,

$$\begin{cases} -\dfrac{dDw_{h,t_k}}{d|Dw_{h,t_k}|}(u) \in N_{C(u)-h(t_k,u)-v_h(t_k)}(w_{h,t_k}(u)) \; |Dw_{h,t_k}|\text{-a.e.} \\ w_{h,t_k}(t_k) = 0. \end{cases}$$

So, by Proposition 1:

$$\begin{aligned} \|w_h\|_{1\text{-var},t_k,t_{k+1}} &= \|w_{h,t_k}\|_{1\text{-var},t_k,t_{k+1}} \\ &\leqslant l(\rho, \|\gamma(t_k) - v_h(t_k)\|). \end{aligned}$$

Therefore,

$$\begin{aligned} \|w_h\|_{1\text{-var},s,t} &= \sum_{k=k(s)}^{k(t)-1} \|w_h\|_{1\text{-var},t_k,t_{k+1}} \\ &\leqslant N \sup_{u\in[s,t]} l(\rho, \|\gamma(u) - v_h(u)\|) \\ &\leqslant \mathfrak{M}(\rho). \end{aligned}$$

□

2.2 Young's integral, rough integral

The first part of the subsection deals with the definition and some basic properties of Young's integral which allow to integrate a map $y \in C^{r\text{-var}}([0,T], \mathcal{M}_{e,d}(\mathbb{R}))$ with respect to $z \in C^{q\text{-var}}([0,T], \mathbb{R}^d)$ when $q, r \in [1, \infty[$ and $1/q + 1/r > 1$. The second part of the subsection deals with the rough integral which extends Young's integral when the condition $1/q + 1/r > 1$ is not satisfied anymore. The signal z has to be enhanced as a rough path.

Definition 1 A map $\omega : \Delta_T \longrightarrow \mathbb{R}_+$ is a control function if and only if,

1. ω is continuous.
2. $\omega(s,s) = 0$ for every $s \in [0,T]$.
3. ω is super-additive:

$$\omega(s,u) + \omega(u,t) \leqslant \omega(s,t)$$

for every $s,t,u \in [0,T]$ such that $s \leqslant u \leqslant t$.

Example. Let $p \geqslant 1$. For every $z \in C^{p\text{-var}}([0,T], \mathbb{R}^d)$, the map

$$\omega_{p,z} : (s,t) \in \Delta_T \longmapsto \omega_{p,z}(s,t) := \|z\|^p_{p\text{-var},s,t}$$

is a control function.

Proposition 5 *Let $p \geqslant 1$. Consider $x \in C^0([0,T], \mathbb{R}^d)$ and a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $C^{p\text{-var}}([0,T], \mathbb{R}^d)$ such that*

$$\lim_{n\to\infty} \|x_n - x\|_{\infty,T} = 0 \text{ and } \sup_{n\in\mathbb{N}} \|x_n\|_{p\text{-var},T} < \infty.$$

Then $x \in C^{p\text{-var}}([0,T],\mathbb{R}^d)$ *and*

$$\lim_{n\to\infty} \|x_n - x\|_{(p+\varepsilon)\text{-var},T} = 0 \,;\, \forall \varepsilon > 0.$$

See Friz and Victoir [11, Lemma 5.12 and Lemma 5.27] for a proof.

Proposition 6 *(Young's integral) Consider* $q,r \in [1,\infty[$ *such that* $1/q + 1/r > 1$*, and two maps* $y \in C^{r\text{-var}}([0,T],\mathcal{M}_{e,d}(\mathbb{R}))$ *and* $z \in C^{q\text{-var}}([0,T],\mathbb{R}^d)$*. For every* $n \in \mathbb{N}^*$ *and* $(t_k^n)_{k\in[\![1,n]\!]} \in \mathfrak{D}_{[0,T]}$*, the limit*

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} y(t_k^n) z(t_k^n, t_{k+1}^n)$$

exists and does not depend on the dissection $(t_k^n)_{k\in[\![1,n]\!]}$*. That limit is denoted by*

$$\int_0^T y(s)dz(s)$$

and called Young's integral of y *with respect to* z *on* $[0,T]$*. Moreover, there exists a constant* $c(q,r) > 0$*, depending only on* q *and* r*, such that for every* $(s,t) \in \Delta_T$*,*

$$\left\| \int_0^{\cdot} y(s)dz(s) \right\|_{r\text{-var},s,t} \leqslant c(q,r)\|z\|_{q\text{-var},s,t}(\|y\|_{r\text{-var},s,t} + \|y\|_{\infty,s,t}).$$

See Lyons [16, Theorem 1.16], Lejay [14, Theorem 1] or Friz and Victoir [11, Theorem 6.8] for a proof.

Proposition 7 *Consider* $q,r \in [1,\infty[$ *such that* $1/q + 1/r > 1$*, two maps* $y \in C^{r\text{-var}}([0,T],\mathcal{M}_{e,d}(\mathbb{R}))$ *and* $z \in C^{q\text{-var}}([0,T],\mathbb{R}^d)$*, and a sequence* $(y_n)_{n\in\mathbb{N}}$ *of elements of* $C^{r\text{-var}}([0,T],\mathcal{M}_{e,d}(\mathbb{R}))$ *such that:*

$$\lim_{n\to\infty} \|y_n - y\|_{\infty,T} = 0 \text{ and } \sup_{n\in\mathbb{N}} \|y_n\|_{r\text{-var},T} < \infty.$$

Then,

$$\lim_{n\to\infty} \left\| \int_0^{\cdot} y_n(s)dz(s) - \int_0^{\cdot} y(s)dz(s) \right\|_{\infty,T} = 0.$$

See Friz and Victoir [11, Proposition 6.12] for a proof.

Consider $p \in [2,3[$ and let us define the rough integral for continuous functions of finite p-variation.

Remark. In the sequel, the reader has to keep in mind that:

1. $\mathcal{M}_{e,d}(\mathbb{R}) \cong \mathbb{R}^e \otimes \mathbb{R}^d$.
2. $\mathcal{M}_{d,1}(\mathbb{R}) \cong \mathcal{M}_{1,d}(\mathbb{R}) \cong \mathbb{R}^d$.
3. $\mathcal{L}(\mathbb{R}^d, \mathcal{M}_{e,d}(\mathbb{R})) \cong \mathcal{L}(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d,\mathbb{R}^e)) \cong \mathcal{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^e)$.

Definition 2 Consider $z \in C^{1\text{-var}}([0,T],\mathbb{R}^d)$. The step-2 signature of z is the map $S_2(z) : \Delta_T \to \mathbb{R}^d \times \mathcal{M}_d(\mathbb{R})$ defined by

$$S_2(z)(s,t) := \left(z(s,t), \int_{s<u<v<t} dz(v) \otimes dz(u)\right)$$

for every $(s,t) \in \Delta_T$.

Notation. $\mathfrak{S}_T(\mathbb{R}^d) := \{S_2(z)(0,.) \ ; \ z \in C^{1\text{-var}}([0,T],\mathbb{R}^d)\}$.

Definition 3 The geometric p-rough paths metric space $G\Omega_{p,T}(\mathbb{R}^d)$ is the closure of $\mathfrak{S}_T(\mathbb{R}^d)$ in $C^{p\text{-var}}([0,T],\mathbb{R}^d) \times C^{p/2\text{-var}}([0,T],\mathcal{M}_d(\mathbb{R}))$.

Definition 4 For $z \in C^{p\text{-var}}([0,T],\mathbb{R}^d)$, a map $y \in C^{p\text{-var}}([0,T],\mathcal{M}_{e,d}(\mathbb{R}))$ is controlled by z if and only if there exists $y' \in C^{p\text{-var}}([0,T],\mathcal{L}(\mathbb{R}^d,\mathcal{M}_{e,d}(\mathbb{R})))$ such that

$$y(s,t) = y'(s)z(s,t) + R_y(s,t) \ ; \ \forall (s,t) \in \Delta_T$$

with $\|R_y\|_{p/2\text{-var},T} < \infty$. For fixed z, the pairs (y,y') as above define a vector space denoted by $\mathfrak{D}_z^{p/2}([0,T],\mathcal{M}_{e,d}(\mathbb{R}))$ and equipped with the semi-norm $\|.\|_{z,p/2,T}$ such that

$$\|(y,y')\|_{z,p/2,T} := \|y'\|_{p\text{-var},T} + \|R_y\|_{p/2\text{-var},T}$$

for every $(y,y') \in \mathfrak{D}_z^{p/2}([0,T],\mathcal{M}_{e,d}(\mathbb{R}))$.

Theorem 2 *(Rough integral) Consider* $\mathbf{z} := (z,\mathbb{Z}) \in G\Omega_{p,T}(\mathbb{R}^d)$ *and* $(y,y') \in \mathfrak{D}_z^{p/2}([0,T],\mathcal{M}_{e,d}(\mathbb{R}))$. *For every* $n \in \mathbb{N}^*$ *and* $(t_k^n)_{k\in[\![1,n]\!]} \in \mathfrak{D}_{[0,T]}$, *the limit*

$$\lim_{n\to\infty} \sum_{k=1}^{n-1} (y(t_k^n)z(t_k^n,t_{k+1}^n) + y'(t_k^n)\mathbb{Z}(t_k^n,t_{k+1}^n))$$

exists and does not depend on the dissection $(t_k^n)_{k\in[\![1,n]\!]}$. *That limit is denoted by*

$$\int_0^T y(s)d\mathbf{z}(s)$$

and called rough integral of y *with respect to* $\mathbf{z}$ *on* $[0,T]$. *Moreover,*

1. *There exists a constant* $c(p) > 0$, *depending only on* p, *such that for every* $(s,t) \in \Delta_T$,

$$\left\|\int_s^t y(u)d\mathbf{z}(u) - y(s)z(s,t) - y'(s)\mathbb{Z}(s,t)\right\| \leqslant c(p)(\|z\|_{p\text{-var},s,t}\|R_y\|_{p/2\text{-var},s,t} + \|\mathbb{Z}\|_{p/2\text{-var},s,t}\|y'\|_{p\text{-var},s,t}).$$

2. *The map*

$$(y,y') \longmapsto \left(\int_0^{.} y(s)d\mathbf{z}(s), y\right)$$

is continuous from $\mathfrak{D}_z^{p/2}([0,T],\mathcal{M}_{e,d}(\mathbb{R}))$ *into* $\mathfrak{D}_z^{p/2}([0,T],\mathbb{R}^e)$.

See Friz and Shekhar [10, Theorem 34] for a proof with the p-variation topology, and see Gubinelli [12, Theorem 1] or Friz and Hairer [9, Theorem 4.10] for a proof with the $1/p$-Hölder topology.

Proposition 8 *Consider* $\mathbf{z} := (z,\mathbb{Z}) \in G\Omega_{p,T}(\mathbb{R}^d)$*, a continuous map*

$$(y,y') : [0,T] \longrightarrow \mathcal{M}_{e,d}(\mathbb{R}) \times \mathcal{L}(\mathbb{R}^d,\mathcal{M}_{e,d}(\mathbb{R})),$$

and a sequence $(y_n,y_n')_{n\in\mathbb{N}}$ *of elements of* $\mathfrak{D}_z^{p/2}([0,T],\mathcal{M}_{e,d}(\mathbb{R}))$ *such that*

$$(y_n',R_{y_n}) \xrightarrow[n\to\infty]{d_{\infty,T}} (y',R_y) \text{ and } \sup_{n\in\mathbb{N}} \|(y_n,y_n')\|_{z,p/2,T} < \infty.$$

Then, $(y,y') \in \mathfrak{D}_z^{p/2}([0,T],\mathcal{M}_{e,d}(\mathbb{R}))$ *and*

$$\lim_{n\to\infty} \left\| \int_0^{.} y_n(s)d\mathbf{z}(s) - \int_0^{.} y(s)d\mathbf{z}(s) \right\|_{\infty,T} = 0.$$

Proof On the one hand, since

$$(y_n',R_{y_n}) \xrightarrow[n\to\infty]{d_{\infty,T}} (y',R_y),$$

the function y is the uniform limit of the sequence $(y_n)_{n\in\mathbb{N}}$. Moreover, since

$$\sup_{n\in\mathbb{N}} \|(y_n,y_n')\|_{z,p/2,T} < \infty,$$

by Proposition 5,

$$y' \in C^{p\text{-var}}([0,T],\mathcal{L}(\mathbb{R}^d,\mathcal{M}_{e,d}(\mathbb{R}))) \text{ and } R_y \in C^{p/2\text{-var}}([0,T],\mathcal{M}_{e,d}(\mathbb{R})).$$

So, $(y,y') \in \mathfrak{D}_z^{p/2}([0,T],\mathcal{M}_{e,d}(\mathbb{R}))$.

On the other hand, also by Proposition 5, for any $\varepsilon > 0$ such that $p+\varepsilon \in [2,3[$,

$$\begin{aligned}\lim_{n\to\infty} \|(y_n,y_n') - (y,y')\|_{z,(p+\varepsilon)/2,T} &= \lim_{n\to\infty} \|y_n' - y'\|_{(p+\varepsilon)\text{-var},T} \\ &\quad + \lim_{n\to\infty} \|R_{y_n} - R_y\|_{(p+\varepsilon)/2\text{-var},T} \\ &= 0.\end{aligned}$$

□

So, by continuity of the rough integral (see Theorem 2),

$$\lim_{n\to\infty} \left\| \left(\int_0^{.} y_n(s)d\mathbf{z}(s), y_n \right) - \left(\int_0^{.} y(s)d\mathbf{z}(s), y \right) \right\|_{z,(p+\varepsilon)/2,T} = 0.$$

Therefore, in particular:

$$\lim_{n\to\infty}\left\|\int_0^{\cdot} y_n(s)d\mathbf{z}(s)-\int_0^{\cdot} y(s)d\mathbf{z}(s)\right\|_{\infty,T}=0.$$

Proposition 9 *Consider* $\mathbf{z} := (z,\mathbb{Z}) \in G\Omega_{p,T}(\mathbb{R}^d)$, $(x,x') \in \mathfrak{D}_z^{p/2}([0,T],\mathbb{R}^e)$ *and* $\varphi \in \mathrm{Lip}^{\gamma-1}(\mathbb{R}^e,\mathbb{R}^d)$. *The couple of maps* $(\varphi(x),\varphi(x)')$, *defined by*

$$\varphi(x)(t) := \varphi(x(t)) \text{ and } \varphi(x)'(t) := D\varphi(x(t))x'(t)$$

for every $t\in[0,T]$, *belongs to* $\mathfrak{D}_z^{p/2}([0,T],\mathcal{M}_{e,d}(\mathbb{R}))$.

Remark. By Theorem 2 and Proposition 9 together,

$$\int_0^{\cdot}\varphi(x(u))d\mathbf{z}(u)$$

is defined. For every $(s,t)\in\Delta_T$, consider

$$\mathfrak{I}_{\varphi,\mathbf{z},x}(s,t) := \left\|\int_s^t\varphi(x(u))d\mathbf{z}(u)-\varphi(x(s))z(s,t)-D\varphi(x(s))x'(s)\mathbb{Z}(s,t)\right\|.$$

For every $(s,t)\in\Delta_T$, since

$$\begin{aligned}
\|\varphi(x)\|_{p\text{-var},s,t} &\leqslant \|\varphi\|_{\mathrm{Lip}^{\gamma-1}}\|x\|_{p\text{-var},s,t},\\
\|\varphi(x)'\|_{p\text{-var},s,t} &\leqslant \|\varphi\|_{\mathrm{Lip}^{\gamma-1}}(\|x'\|_{p\text{-var},s,t}+\|x'\|_{\infty,s,t}\|x\|_{p\text{-var},s,t}) \text{ and}\\
\|R_{\varphi(x)}\|_{p/2\text{-var},s,t} &\leqslant \|\varphi\|_{\mathrm{Lip}^{\gamma-1}}(\|x\|_{p\text{-var},s,t}^2+\|R_x\|_{p/2\text{-var},s,t}),
\end{aligned}$$

by Theorem 2,

$$\begin{aligned}
\mathfrak{I}_{\varphi,\mathbf{z},x}(s,t) \leqslant\; & c(p)\|\varphi\|_{\mathrm{Lip}^{\gamma-1}}(\|x'\|_{p\text{-var},s,t}+\|x'\|_{\infty,s,t}\|x\|_{p\text{-var},s,t}\\
&+\|x\|_{p\text{-var},s,t}^2+\|R_x\|_{p/2\text{-var},s,t})\omega_{p,\mathbf{z}}(s,t)^{1/p},
\end{aligned}$$

where $\omega_{p,\mathbf{z}}:\Delta_T\longrightarrow\mathbb{R}_+$ is the control function defined by

$$\omega_{p,\mathbf{z}}(u,v) := 2^{p-1}(\|z\|_{p\text{-var},u,v}^p+\|\mathbb{Z}\|_{p/2\text{-var},u,v}^p)\ ;\ \forall(u,v)\in\Delta_T.$$

Proposition 10 *Consider* $\mathbf{z} := (z,\mathbb{Z}) \in G\Omega_{p,T}(\mathbb{R}^d)$, $\varphi\in\mathrm{Lip}^{\gamma-1}(\mathbb{R}^e,\mathbb{R}^d)$, *a continuous map*

$$(x,x') : [0,T]\longrightarrow\mathbb{R}^e\times\mathcal{L}(\mathbb{R}^d,\mathbb{R}^e),$$

and a sequence $(x_n,x_n')_{n\in\mathbb{N}}$ *of elements of* $\mathfrak{D}_z^{p/2}([0,T],\mathbb{R}^e)$ *such that*

$$(x_n',R_{x_n})\xrightarrow[n\to\infty]{d_{\infty,T}}(x',R_x) \text{ and } \sup_{n\in\mathbb{N}}\|(x_n,x_n')\|_{z,p/2,T}<\infty.$$

Then, $(\varphi(x),\varphi(x)')\in\mathfrak{D}_z^{p/2}([0,T],\mathcal{M}_{e,d}(\mathbb{R}))$ *and*

$$\lim_{n\to\infty}\left\|\int_0^{\cdot}\varphi(x_n(s))d\mathbf{z}(s)-\int_0^{\cdot}\varphi(x(s))d\mathbf{z}(s)\right\|_{\infty,T}=0.$$

Proof Since

$$(x_n', R_{x_n})\xrightarrow[n\to\infty]{d_{\infty,T}}(x', R_x)\text{ and }\sup_{n\in\mathbb{N}}\|(x_n, x_n')\|_{z,p/2,T}<\infty,$$

by Friz and Hairer [9, Theorem 7.5] together with Proposition 5,

$$(\varphi(x_n)', R_{\varphi(x_n)})\xrightarrow[n\to\infty]{d_{\infty,T}}(\varphi(x)', R_{\varphi(x)})\text{ and }\sup_{n\in\mathbb{N}}\|(\varphi(x_n),\varphi(x_n)')\|_{z,p/2,T}<\infty.$$

So, by Proposition 8,

$$\lim_{n\to\infty}\left\|\int_0^{\cdot}\varphi(x_n(s))d\mathbf{z}(s)-\int_0^{\cdot}\varphi(x(s))d\mathbf{z}(s)\right\|_{\infty,T}=0.$$

3 Existence of solutions

The existence of a solution to Problem (2) is established in Theorem 3 when $p\in[1,2[$, and in Theorem 4 when $p\in[2,3[$.

Theorem 3 *Under Assumption 1, if* $p\in[1,2[$ *and* $f\in\text{Lip}^{\gamma}(\mathbb{R}^e,\mathcal{M}_{e,d}(\mathbb{R}))$*, then Problem (2) has at least one solution which belongs to* $C^{p\text{-var}}([0,T],\mathbb{R}^e)$.

Proof Consider the discrete scheme

$$\begin{cases} X_n(t) = H_n(t)+Y_n(t)\\ H_n(t) = \displaystyle\int_0^t f(X_{n-1}(s))dZ(s)\\ -\dfrac{dDY_n}{d|DY_n|}(t)\in N_{C_{H_n}(t)}(Y_n(t))\ |DY_n|\text{-a.e. with }Y_n(0)=a\end{cases}\tag{5}$$

for Problem (2), initialized by

$$\begin{cases} -\dfrac{dDX_0}{d|DX_0|}(t)\in N_{C(t)}(X_0(t))\ |DX_0|\text{-a.e.}\\ X_0(0) = a\end{cases}\tag{6}$$

Since the map $\|Z\|_{p\text{-var},0,.}$ is continuous from $[0,T]$ into $\mathbb{R}_+$, and since $\|Z\|_{p\text{-var},0,0}=0$, there exists $\tau_0\in[0,T]$ such that

$$\|Z\|_{p\text{-var},\tau_0}\leqslant\mu:=\frac{m}{c(p,p)\|f\|_{\text{Lip}^{\gamma}}(m+M+1)},$$

where $m := R/2$ and $M := \mathfrak{M}(R/2)$ (see Proposition 4.(2)). Let us show that for every $n \in \mathbb{N}$,

$$\begin{cases} \|X_n\|_{p\text{-var},\tau_0} \leqslant m + M \\ \|H_n\|_{p\text{-var},\tau_0} \leqslant m \\ \|Y_n\|_{1\text{-var},\tau_0} \leqslant M \end{cases} \tag{7}$$

By (6) together with Proposition 4,

$$\|X_0\|_{p\text{-var},\tau_0} \leqslant M.$$

Assume that Condition (7) is satisfied for $n \in \mathbb{N}$ arbitrarily chosen. By Proposition 6, and since $\|Z\|_{p\text{-var},0,.}$ is an increasing map,

$$\begin{aligned} \|H_{n+1}\|_{p\text{-var},\tau_0} &\leqslant c(p,p)\|Z\|_{p\text{-var},\tau_0}(\|Df\|_\infty\|X_n\|_{p\text{-var},\tau_0} + \|f \circ X_n\|_{\infty,\tau_0}) \\ &\leqslant \mu c(p,p)\|f\|_{\mathrm{Lip}^\gamma}(m + M + 1) \\ &\leqslant m. \end{aligned}$$

Since $Y_{n+1} = w_{H_{n+1}}$, by Proposition 4,

$$\|Y_{n+1}\|_{1\text{-var},\tau_0} \leqslant M.$$

Therefore,

$$\|X_{n+1}\|_{p\text{-var},\tau_0} \leqslant \|H_{n+1}\|_{p\text{-var},\tau_0} + \|Y_{n+1}\|_{p\text{-var},\tau_0} \leqslant m + M.$$

By induction, (7) is satisfied for every $n \in \mathbb{N}$.

For every $t \in [0,T]$, the map $\|Z\|_{p\text{-var},t,.}$ is continuous from $[t,T]$ into $\mathbb{R}_+$ and $\|Z\|_{p\text{-var},t,t} = 0$. Moreover, the constant μ depends only on p, m, M and $\|f\|_{\mathrm{Lip}^\gamma}$. So, since $[0,T]$ is compact, there exist $N \in \mathbb{N}^*$ and $(\tau_k)_{k\in[\![0,N]\!]} \in \mathfrak{D}_{[\tau_0,T]}$ such that

$$\|Z\|_{p\text{-var},\tau_k,\tau_{k+1}} \leqslant \mu \ ; \forall k \in [\![0, N-1]\!].$$

Since for every $n \in \mathbb{N}^*$ the maps

$$\begin{aligned} (s,t) \in \Delta_T &\longmapsto \|X_n\|_{p\text{-var},s,t}^p, \\ (s,t) \in \Delta_T &\longmapsto \|H_n\|_{p\text{-var},s,t}^p \text{ and} \\ (s,t) \in \Delta_T &\longmapsto \|Y_n\|_{1\text{-var},s,t} \end{aligned}$$

□

are control functions, recursively, the sequence $(H_n, X_n, Y_n)_{n\in\mathbb{N}^*}$ is bounded in

$$\mathfrak{C}_T^{p,1} := C^{p\text{-var}}([0,T],\mathbb{R}^e) \times C^{p\text{-var}}([0,T],\mathbb{R}^e) \times C^{1\text{-var}}([0,T],\mathbb{R}^e).$$

By Proposition 6, for every $n \in \mathbb{N}^*$ and $(s,t) \in \Delta_T$,

$$\|H_n(t) - H_n(s)\| \leqslant c(p,p)\left(\|Df\|_\infty \sup_{n\in\mathbb{N}} \|X_n\|_{p\text{-var},T} + \|f\|_\infty\right)\|Z\|_{p\text{-var},s,t}.$$

Since $(s,t) \in \Delta_T \mapsto \|Z\|_{p\text{-var},s,t}$ is a continuous map such that $\|Z\|_{p\text{-var},t,t} = 0$ for every $t \in [0,T]$, $(H_n)_{n\in\mathbb{N}^*}$ is equicontinuous. Therefore, by Arzelà-Ascoli's theorem together with Proposition 5, there exists an extraction $\varphi : \mathbb{N}^* \to \mathbb{N}^*$ such that $(H_{\varphi(n)})_{n\in\mathbb{N}^*}$ converges uniformly to an element H of $C^{p\text{-var}}([0,T],\mathbb{R}^e)$.

Since $(H_{\varphi(n)})_{n\in\mathbb{N}^*}$ converges uniformly to H, by Theorem 1, $(X_{\varphi(n)}, Y_{\varphi(n)})_{n\in\mathbb{N}^*}$ converges uniformly to $(X,Y) := (v_H, w_H)$. So, for every $t \in [0,T]$,

$$\begin{cases} X(t) = H(t) + Y(t) \\ -\dfrac{dDY}{d|DY|}(t) \in N_{C_H(t)}(Y(t)) \ |DY|\text{-a.e. with } Y(0) = a, \end{cases}$$

and by Proposition 5,

$$X \in C^{p\text{-var}}([0,T],\mathbb{R}^e) \text{ and } Y \in C^{1\text{-var}}([0,T],\mathbb{R}^e).$$

Moreover, since $(X_{\varphi(n)})_{n\in\mathbb{N}^*}$ converges uniformly to X, by Proposition 7,

$$\lim_{n\to\infty} \left\| H_{\varphi(n)} - \int_0^{\cdot} f(X(s))dZ(s) \right\|_{\infty,T} = 0.$$

Therefore, since $(H_{\varphi(n)})_{n\in\mathbb{N}^*}$ converges also to H in $C^0([0,T],\mathbb{R}^e)$,

$$H(t) = \int_0^t f(X(s))dZ(s) \ ; \forall t \in [0,T].$$

In the sequel, assume that there exists $\mathbb{Z} : [0,T] \to \mathcal{M}_d(\mathbb{R})$ such that $\mathbf{Z} := (Z,\mathbb{Z}) \in G\Omega_{p,T}(\mathbb{R}^d)$.

Theorem 4 *Under Assumption 1, if $p \in [2,3[$ and $f \in \text{Lip}^\gamma(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$, then Problem (2) has at least one solution which belongs to $C^{p\text{-var}}([0,T],\mathbb{R}^e)$.*

Proof Consider the discrete scheme

$$\begin{cases} X_n(t) = H_n(t) + Y_n(t) \\ H_n(t) = \displaystyle\int_0^t f(X_{n-1}(s))d\mathbf{Z}(s) \\ -\dfrac{dDY_n}{d|DY_n|}(t) \in N_{C_{H_n}(t)}(Y_n(t)) \ |DY_n|\text{-a.e. with } Y_n(0) = a \end{cases} \tag{8}$$

for Problem (2), initialized by

$$\begin{cases} -\dfrac{dDX_0}{d|DX_0|}(t) \in N_{C(t)}(X_0(t)) \ |DX_0|\text{-a.e.} \\ X_0(0) = a \end{cases} \tag{9}$$

At step $n \in \mathbb{N}^*$ of the scheme, the integral involved in the definition of $H_n(t)$, $t \in [0,T]$, is the rough integral on $[0,T]$ of $f(X_{n-1}(\cdot))$ with respect to the geometric rough path $\mathbf{Z}$ over Z (see Definition 2).

Since the map $\omega_{p,\mathbf{Z}}(0,.)$ is continuous from $[0,T]$ into $\mathbb{R}_+$, and since $\omega_{p,\mathbf{Z}}(0,0) = 0$, there exists $\tau_0 \in [0,T]$ such that

$$\omega_{p,\mathbf{Z}}(0,\tau_0) \leqslant \frac{m_C}{c(p,1)^p \|f\|_{\mathrm{Lip}^\gamma}^p (M_C+1)^p} \wedge \frac{m_C}{(c_2 \vee c_6)^p (1+\mu_C+M_R+\mu_C^2)^p} \wedge \frac{1}{1+\mu_C^p+M_R^p+\mu_C^{2p}},$$

where $m_C := R/2$, $M_C := \mathfrak{M}(R/2)$, $\mu_C := m_C + M_C$,

$$M_R := (c_1 M_C 4^{1/p}) \vee (c_5(\mu_C^p + 1)^{1/p})$$

and the positive constants c_1, c_2, c_5 and c_6, depending only on p and $\|f\|_{\mathrm{Lip}^\gamma}$, are defined in the sequel.

First of all, let us control the solution of the discrete scheme for $n \in \{0,1\}$:

- ($n = 0$) By (9) together with Proposition 4:

$$\|X_0\|_{1\text{-var},\tau_0} \leqslant M_C.$$

- ($n = 1$) Since $X_0 \in C^{1\text{-var}}([0,T],\mathbb{R}^e)$, by Proposition 6:

$$\begin{aligned}\|H_1\|_{p\text{-var},\tau_0} &\leqslant c(p,1)\omega_{p,\mathbf{Z}}(0,\tau_0)^{1/p}\|f\|_{\mathrm{Lip}^\gamma}(M_C+1)\\ &\leqslant m_C.\end{aligned}$$

  Since $Y_1 = w_{H_1}$, by Proposition 4:

$$\|Y_1\|_{1\text{-var},\tau_0} \leqslant M_C.$$

  Therefore,

$$\|X_1\|_{p\text{-var},\tau_0} \leqslant \|H_1\|_{p\text{-var},\tau_0} + \|Y_1\|_{p\text{-var},\tau_0} \leqslant \mu_C.$$

Let us show that for every $n \in \mathbb{N}\backslash\{0,1\}$,

$$(X_{n-1}, f(X_{n-2})) \in \mathfrak{D}_Z^{p/2}([0,\tau_0],\mathbb{R}^e) \tag{10}$$

and

$$\begin{cases} \|X_n\|_{p\text{-var},\tau_0} \leqslant \mu_C \\ \|H_n\|_{p\text{-var},\tau_0} \leqslant m_C \\ \|Y_n\|_{1\text{-var},\tau_0} \leqslant M_C \\ \|R_{X_n}\|_{p/2\text{-var},\tau_0} \leqslant M_R \end{cases} \tag{11}$$

Set $X_1' := f(X_0)$. For every $(s,t) \in \Delta_{\tau_0}$,

$$\begin{aligned} R_{X_1}(s,t) &= X_1(s,t) - X_1'(s)Z(s,t) \\ &= Y_1(s,t) + \int_s^t f(X_0(u))dZ(u) - f(X_0(s))Z(s,t). \end{aligned}$$

By Young-Love estimate (see Friz and Victoir [11, Theorem 6.8], or [7, Section 3.6 and the interesting historical notes pages 212-213]), for every $(s,t) \in \Delta_{\tau_0}$,

$$\|R_{X_1}(s,t)\| \leqslant \|Y_1\|_{p/2\text{-var},s,t} + \frac{1}{1-2^{1-3/p}}\|f\|_{\text{Lip}^\gamma}\|Z\|_{p\text{-var},\tau_0}\|X_0\|_{p/2\text{-var},s,t}.$$

By super-additivity of the control functions $\|Y_1\|_{p/2\text{-var},.}^{p/2}$ and $\|X_0\|_{p/2\text{-var},.}^{p/2}$, there exists a constant $c_1 > 0$, depending only on p and $\|f\|_{\text{Lip}^\gamma}$, such that

$$\begin{aligned} \|R_{X_1}\|_{p/2\text{-var},\tau_0} &\leqslant c_1(\|Y_1\|_{p/2\text{-var},\tau_0}^{p/2} + \|Z\|_{p\text{-var},\tau_0}^{p/2}\|X_0\|_{p/2\text{-var},\tau_0}^{p/2})^{2/p} \\ &\leqslant c_1 M_C(1 + \omega_{p,\mathbf{Z}}(0,\tau_0)^{1/2})^{2/p}. \end{aligned}$$

Then, $\|R_{X_1}\|_{p/2\text{-var},\tau_0} \leqslant c_1 M_C 4^{1/p} \leqslant M_R$ and

$$(X_1, f(X_0)) \in \mathfrak{D}_Z^{p/2}([0,\tau_0],\mathbb{R}^e).$$

So, the rough integral

$$H_2 := \int_0^{\cdot} f(X_1(s))d\mathbf{Z}(s)$$

is well defined. For every $(s,t) \in \Delta_T$,

$$\begin{aligned} \|H_2(s,t)\| &\leqslant \|f\|_{\text{Lip}^\gamma}\|Z\|_{p\text{-var},s,t} + \|f\|_{\text{Lip}^\gamma}^2\|\mathbb{Z}\|_{p/2\text{-var},s,t} + \mathfrak{I}_{f,\mathbf{Z},X_n}(s,t) \\ &\leqslant (\|f\|_{\text{Lip}^\gamma} \vee \|f\|_{\text{Lip}^\gamma}^2)\omega_{p,\mathbf{Z}}(s,t)^{1/p} \\ &\quad + c(p)\|f\|_{\text{Lip}^\gamma}(1 \vee \|f\|_{\text{Lip}^\gamma})(\|X_0\|_{p\text{-var},s,t} + \|X_1\|_{p\text{-var},s,t} \\ &\quad + \|X_1\|_{p\text{-var},s,t}^2 + \|R_{X_1}\|_{p/2\text{-var},s,t})\omega_{p,\mathbf{Z}}(s,t)^{1/p} \\ &\leqslant c_2(1 + \mu_C + M_R + \mu_C^2)\omega_{p,\mathbf{Z}}(s,t)^{1/p}, \end{aligned}$$

where $c_2 > 0$ is a constant depending only on p and $\|f\|_{\text{Lip}^\gamma}$. By super-additivity of the control function $\omega_{p,\mathbf{Z}}$:

$$\begin{aligned} \|H_2\|_{p\text{-var},\tau_0} &\leqslant c_2(1 + \mu_C + M_R + \mu_C^2)\omega_{p,\mathbf{Z}}(0,\tau_0)^{1/p} \\ &\leqslant m_C. \end{aligned}$$

So, by Proposition 4,

$$\|Y_2\|_{p\text{-var},\tau_0} \leqslant M_C$$

and

$$\|X_2\|_{p\text{-var},\tau_0} \leqslant \|H_2\|_{p\text{-var},\tau_0} + \|Y_2\|_{p\text{-var},\tau_0} \leqslant \mu_C.$$

Therefore, Conditions (10)-(11) hold true for $n = 2$.

Assume that Conditions (10)-(11) hold true until $n \in \mathbb{N}\backslash\{0,1\}$ arbitrarily chosen. Set $X'_n := f(X_{n-1})$. For every $(s,t) \in \Delta_{\tau_0}$,

$$\begin{aligned} R_{X_n}(s,t) &= X_n(s,t) - X'_n(s)Z(s,t) \\ &= Y_n(s,t) + \int_s^t f(X_{n-1}(u))d\mathbf{Z}(u) - f(X_{n-1}(s))Z(s,t). \end{aligned}$$

So, for every $(s,t) \in \Delta_{\tau_0}$,

$$\begin{aligned} \|R_{X_n}(s,t)\| &\leqslant \|Y_n(s,t)\| + \|Df(X_{n-1}(s))f(X_{n-2}(s))\mathbb{Z}(s,t)\| + \mathfrak{I}_{f,\mathbf{Z},X_{n-1}}(s,t) \\ &\leqslant \|Y_n\|_{p/2\text{-var},s,t} + \|f\|_{\text{Lip}^\gamma}^2\|\mathbb{Z}\|_{p/2\text{-var},s,t} \\ &\quad + c(p)(\|Z\|_{p\text{-var},s,t}\|R_{f(X_{n-1})}\|_{p/2\text{-var},s,t} \\ &\quad + \|\mathbb{Z}\|_{p/2\text{-var},s,t}\|Df(X_{n-1}(.))f(X_{n-2})\|_{p\text{-var},s,t}) \\ &\leqslant \|Y_n\|_{p/2\text{-var},s,t} + \|f\|_{\text{Lip}^\gamma}^2\|\mathbb{Z}\|_{p/2\text{-var},s,t} \\ &\quad + c(p)\|f\|_{\text{Lip}^\gamma}(\|Z\|_{p\text{-var},\tau_0}\omega_n(s,t)^{2/p} + M(n,\tau_0)\|\mathbb{Z}\|_{p/2\text{-var},s,t}), \end{aligned}$$

where

$$\begin{aligned} M(n,\tau_0) &:= \|f(X_{n-2})\|_{p\text{-var},\tau_0} + \|f(X_{n-2})\|_{\infty,\tau_0}\|X_{n-1}\|_{p\text{-var},\tau_0} \\ &\leqslant \|f\|_{\text{Lip}^\gamma}(\|X_{n-2}\|_{p\text{-var},\tau_0} + \|X_{n-1}\|_{p\text{-var},\tau_0}) \end{aligned}$$

and $\omega_n : \Delta_{\tau_0} \to \mathbb{R}_+$ is the control function defined by

$$\omega_n(u,v) := 2^{p/2-1}(\|X_{n-1}\|_{p\text{-var},u,v}^p + \|R_{X_{n-1}}\|_{p/2\text{-var},u,v}^{p/2})$$

for every $(u,v) \in \Delta_{\tau_0}$. By super-additivity of the control functions

$$\|Y_n\|_{p/2\text{-var},.}^{p/2},\ \|\mathbb{Z}\|_{p/2\text{-var},.}^{p/2} \text{ and } \omega_n,$$

there exist three constants $c_3, c_4, c_5 > 0$, depending only on p and $\|f\|_{\text{Lip}^\gamma}$, such that

$$\begin{aligned} \|R_{X_n}\|_{p/2\text{-var},\tau_0} &\leqslant c_3(\|Y_n\|_{p/2\text{-var},\tau_0}^{p/2} + \|\mathbb{Z}\|_{p/2\text{-var},\tau_0}^{p/2} \\ &\quad + \|Z\|_{p\text{-var},\tau_0}^{p/2}\omega_n(0,\tau_0) + M(n,\tau_0)^{p/2}\|\mathbb{Z}\|_{p/2\text{-var},\tau_0}^{p/2})^{2/p} \\ &\leqslant c_4(\|Y_n\|_{p/2\text{-var},\tau_0}^p + (1 + \omega_n(0,\tau_0) + M(n,\tau_0)^{p/2})^2\omega_{p,\mathbf{Z}}(0,\tau_0))^{1/p} \\ &\leqslant c_5(\mu_C^p + (1 + \mu_C^p + \|R_{X_{n-1}}\|_{p/2\text{-var},\tau_0}^p + \mu_C^{2p})\omega_{p,\mathbf{Z}}(0,\tau_0))^{1/p} \\ &\leqslant c_5(\mu_C^p + (1 + \mu_C^p + M_R^p + \mu_C^{2p})\omega_{p,\mathbf{Z}}(0,\tau_0))^{1/p}. \end{aligned}$$

Then, $\|R_{X_n}\|_{p/2\text{-var},\tau_0} \leqslant c_5(\mu_C^p + 1)^{1/p} \leqslant M_R$ and

$$(X_n, f(X_{n-1})) \in \mathfrak{D}_Z^{p/2}([0,\tau_0],\mathbb{R}^e).$$

So, the rough integral

$$H_{n+1} := \int_0^{\cdot} f(X_n(s))d\mathbf{Z}(s)$$

is well defined. For every $(s,t) \in \Delta_T$,

$$\begin{aligned}\|H_{n+1}(s,t)\| &\leqslant \|f\|_{\text{Lip}^\gamma}\|Z\|_{p\text{-var},s,t} + \|f\|_{\text{Lip}^\gamma}^2\|\mathbb{Z}\|_{p/2\text{-var},s,t} + \mathfrak{I}_{f,\mathbf{Z},X_n}(s,t)\\ &\leqslant (\|f\|_{\text{Lip}^\gamma} \vee \|f\|_{\text{Lip}^\gamma}^2)\omega_{p,\mathbf{Z}}(s,t)^{1/p}\\ &\quad + c(p)\|f\|_{\text{Lip}^\gamma}(1 \vee \|f\|_{\text{Lip}^\gamma})(\|X_{n-1}\|_{p\text{-var},s,t} + \|X_n\|_{p\text{-var},s,t}\\ &\quad + \|X_n\|_{p\text{-var},s,t}^2 + \|R_{X_n}\|_{p/2\text{-var},s,t})\omega_{p,\mathbf{Z}}(s,t)^{1/p}\\ &\leqslant c_6(1 + \mu_C + M_R + \mu_C^2)\omega_{p,\mathbf{Z}}(s,t)^{1/p},\end{aligned}$$

where $c_6 > 0$ is a constant depending only on p and $\|f\|_{\text{Lip}^\gamma}$. By super-additivity of the control function $\omega_{p,\mathbf{Z}}$:

$$\begin{aligned}\|H_{n+1}\|_{p\text{-var},\tau_0} &\leqslant c_6(1 + \mu_C + M_R + \mu_C^2)\omega_{p,\mathbf{Z}}(0,\tau_0)^{1/p}\\ &\leqslant m_C.\end{aligned}$$

So, by Proposition 4,

$$\|Y_{n+1}\|_{p\text{-var},\tau_0} \leqslant M_C$$

and

$$\|X_{n+1}\|_{p\text{-var},\tau_0} \leqslant \|H_{n+1}\|_{p\text{-var},\tau_0} + \|Y_{n+1}\|_{p\text{-var},\tau_0} \leqslant \mu_C.$$

By induction, Conditions (10)-(11) are satisfied for every $n \in \mathbb{N}\backslash\{0,1\}$. As in the proof of Theorem 3, the sequence $(H_n, X_n, Y_n)_{n\in\mathbb{N}\backslash\{0,1\}}$ is bounded in $\mathfrak{C}_T^{p,1}$. In addition, the sequence $(R_{X_n})_{n\in\mathbb{N}\backslash\{0,1\}}$ is bounded in $C^{p/2\text{-var}}([0,T],\mathbb{R}^e)$.

For every $n \in \mathbb{N}\backslash\{0,1\}$ and $(s,t) \in \Delta_T$,

$$\begin{aligned}\|H_n(s,t)\| &\leqslant (\|f\|_{\text{Lip}^\gamma} \vee \|f\|_{\text{Lip}^\gamma}^2\\ &\quad + c(p)\|f\|_{\text{Lip}^\gamma}(1 \vee \|f\|_{\text{Lip}^\gamma})(\sup_{n\in\mathbb{N}}\|X_{n-2}\|_{p\text{-var},T} + \sup_{n\in\mathbb{N}}\|X_{n-1}\|_{p\text{-var},T}\\ &\quad + \sup_{n\in\mathbb{N}}\|X_{n-1}\|_{p\text{-var},T}^2 + \sup_{n\in\mathbb{N}}\|R_{X_{n-1}}\|_{p/2\text{-var},T}))\omega_{p,\mathbf{Z}}(s,t)^{1/p}. \qquad \square\end{aligned}$$

Since $\omega_{p,\mathbf{Z}}$ is a control function, $(H_n)_{n\in\mathbb{N}\backslash\{0,1\}}$ is equicontinuous. Therefore, by Arzelà-Ascoli's theorem together with Proposition 5, there exists an extraction $\varphi : \mathbb{N}\backslash\{0,1\} \to \mathbb{N}\backslash\{0,1\}$ such that $(H_{\varphi(n)})_{n\in\mathbb{N}\backslash\{0,1\}}$ converges uniformly to an element H of $C^{p\text{-var}}([0,T],\mathbb{R}^e)$.

Since $(H_{\varphi(n)})_{n\in\mathbb{N}\backslash\{0,1\}}$ converges uniformly to H, by Theorem 1, the sequence $(X_{\varphi(n)}, Y_{\varphi(n)})_{n\in\mathbb{N}\backslash\{0,1\}}$ converges uniformly to $(X,Y) := (v_H, w_H)$. So, for every $t \in [0,T]$,

$$\begin{cases} X(t) &= H(t) + Y(t) \\ -\dfrac{dY}{d|DY|}(t) &\in N_{C_{H(t)}}(Y(t)) \ |DY|\text{-a.e. with } Y(0) = a, \end{cases}$$

and by Proposition 5,

$$X \in C^{p\text{-var}}([0,T],\mathbb{R}^e) \text{ and } Y \in C^{1\text{-var}}([0,T],\mathbb{R}^e).$$

Denoting $X' := f(X)$, X' (resp. R_X) is the uniform limit of $(X'_{\varphi(n)})_{n\in\mathbb{N}\setminus\{0,1\}}$ (resp. $(R_{X_{\varphi(n)}})_{n\in\mathbb{N}\setminus\{0,1\}}$). So, by Proposition 10:

$$\lim_{n\to\infty}\left\|H_{\varphi(n)} - \int_0^{\cdot} f(X(s))d\mathbf{Z}(s)\right\|_{\infty,T} = 0.$$

Therefore, since $(H_{\varphi(n)})_{n\in\mathbb{N}^*}$ converges also to H in $C^0([0,T],\mathbb{R}^e)$,

$$H(t) = \int_0^t f(X(s))d\mathbf{Z}(s) \ ; \forall t \in [0,T].$$

4 Some uniqueness results

When $p = 1$ and there is an additive continuous signal of finite q-variation with $q \in [1,3[$, the uniqueness of the solution to Problem (2) is established in Proposition 11 below. Proposition 12 and Proposition 13 provide necessary and sufficient conditions for uniqueness of the solution when $p \in [1,2[$ and $p \in [2,3[$ respectively. These conditions are close to the monotonicity of the normal cone which allows to prove the uniqueness when $p = 1$ (see Proposition 11). The criteria of Propositions 12 and 13 seem difficult to apply.

Proposition 11 *Assume that $p = 1$ and $f \in \text{Lip}^1(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$. Consider the Skorokhod problem*

$$\begin{cases} X(t) &= H(t) + Y(t) \\ H(t) &= \displaystyle\int_0^t f(X(s))dZ(s) + W(t) \\ -\dfrac{dY}{d|DY|}(t) &\in N_{C_{H(t)}}(Y(t)) \ |DY|\text{-a.e. with } Y(0) = a \end{cases} \tag{12}$$

where $W \in C^{q\text{-var}}([0,T],\mathbb{R}^e)$ with $q \in [1,3[$. Under Assumption 1, Problem (12) has a unique solution which belongs to $C^{q\text{-var}}([0,T],\mathbb{R}^e)$.

Proof Consider two solutions (X,Y) and (X^*,Y^*) of Problem (2) on $[0,T]$. Since $(s,t) \in \Delta_T \mapsto \|Z\|_{1\text{-var},s,t}$ is a control function, there exists $n \in \mathbb{N}^*$ and $(\tau_k)_{k\in[\![0,n]\!]} \in \mathfrak{D}_{[0,T]}$ such that

$$\|Z\|_{1\text{-var},\tau_k,\tau_{k+1}} \leqslant M := \frac{1}{4\|f\|_{\text{Lip}^\gamma}}\ ;\ \forall k \in [\![0, n-1]\!] \tag{13}$$

For every $t \in [0, \tau_1]$,

$$\begin{aligned}\|X(t) - X^*(t)\|^2 &= \|H(t) - H^*(t)\|^2 + 2\int_0^t \langle Y(s) - Y^*(s), d(Y - Y^*)(s)\rangle \\ &\quad +2\int_0^t \langle H(t) - H^*(t), d(Y - Y^*)(s)\rangle \\ &\leqslant m_1(\tau_1)^2 + 2m_2(t) + 2m_3(t),\end{aligned}$$

with $m_1(\tau_1) := \|H - H^*\|_{\infty,\tau_1}$,

$$m_2(t) := \int_0^t \langle X(s) - X^*(s), d(Y - Y^*)(s)\rangle,$$

and

$$m_3(t) := \int_0^t \langle H(t) - H^*(t) - (H(s) - H^*(s)), d(Y - Y^*)(s)\rangle.$$

Consider $t \in [0, \tau_1]$. By Friz and Victoir [11], Proposition 2.2:

$$\begin{aligned}\|H(t) - H^*(t)\| &= \left\|\int_0^t (f(X(s)) - f(X^*(s)))dZ(s)\right\| \\ &\leqslant \|f\|_{\text{Lip}^\gamma}\|X - X^*\|_{\infty,\tau_1}\|Z\|_{1\text{-var},\tau_1}.\end{aligned}$$

So,

$$m_1(\tau_1) \leqslant \frac{1}{4}\|X - X^*\|_{\infty,\tau_1} \tag{14}$$

Since the map $x \in C(t) \mapsto N_{C(t)}(x)$ is monotone, $m_2(t) \leqslant 0$. By the integration by parts formula,

$$\begin{aligned}m_3(t) &= \int_0^t \langle Y(s) - Y^*(s), d(H - H^*)(s)\rangle \\ &= \int_0^t \langle X(s) - X^*(s) - (H(s) - H^*(s)), (f(X(s)) - f(X^*(s)))dZ(s)\rangle.\end{aligned}$$

So, by Friz and Victoir [11], Proposition 2.2 and Inequality (14),

$$\begin{aligned}m_3(t) &\leqslant \|Df\|_\infty\|X - X^*\|_{\infty,\tau_1}(\|X - X^*\|_{\infty,\tau_1} + \|H - H^*\|_{\infty,\tau_1})\|Z\|_{1\text{-var},\tau_1} \\ &\leqslant \|f\|_{\text{Lip}^\gamma}\|Z\|_{1\text{-var},\tau_1}(1 + \|f\|_{\text{Lip}^\gamma}\|Z\|_{1\text{-var},\tau_1})\|X - X^*\|^2_{\infty,\tau_1} \\ &\leqslant 5/16\|X - X^*\|^2_{\infty,\tau_1}.\end{aligned}$$

Therefore,

$$\|X - X^*\|^2_{\infty,\tau_1} \leqslant \frac{11}{16}\|X - X^*\|^2_{\infty,\tau_1}.$$

Necessarily, $(X,Y) = (X^*,Y^*)$ on $[0,\tau_1]$.

For $k \in [\![0,n-1]\!]$, assume that $(X,Y) = (X^*,Y^*)$ on $[0,\tau_k]$. By Equation (13) and exactly the same ideas as on $[0,\tau_1]$:

$$\|X - X^*\|^2_{\infty,\tau_k,\tau_{k+1}} \leqslant \frac{11}{16}\|X - X^*\|^2_{\infty,\tau_k,\tau_{k+1}}.$$

So, $(X,Y) = (X^*,Y^*)$ on $[0,\tau_{k+1}]$. Recursively, Problem (2) has a unique solution on $[0,T]$. □

Remark. The cornerstone of the proof of Proposition 11 is that

$$\int_0^t \langle X(s) - X^*(s), d(Y - Y^*)(s)\rangle \leqslant 0 \ ; \forall t \in [0,T] \tag{15}$$

Thanks to the monotonicity of the map $x \in C(t) \mapsto N_{C(t)}(x)$ ($t \in [0,T]$), Inequality (15) is true. When $p \in]1,3[$, it is not possible to get inequalities involving only the uniform norm of $X - X^*$. In that case, the construction of the Young/rough integral suggests to use ideas similar to those of the proof of Proposition 11, but using the p-variation norm of $X - X^*$.

In a probabilistic setting, uniqueness up to equality almost everywhere can be obtained for Brownian motion, with $p > 2$, in the frame of Itô calculus, using the martingale property of stochastic integrals and Doob's inequality, see [26, 15, 23] for a fixed convex set C and [3, 5] for a moving set.

The two following propositions show that when $p \in]1,3[$, there exist some conditions close to Inequality (15), ensuring the uniqueness of the solution to Problem (2). These criteria seem quite difficult to apply, and we have no example where they do. However we think they are interesting for themselves because they shed a light on the open problem of uniqueness of the solution to Problem (2) when $p > 1$.

Proposition 12 *Consider* $(s,t) \in \Delta_T$, $p \in [1,2[$, $f \in \text{Lip}^\gamma(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$ *and two solutions* (X,Y) *and* (X^*,Y^*) *to Problem (2) under Assumption 1. On* $[s,t]$, $(X,Y) = (X^*,Y^*)$ *if and only if* $X(s) = X^*(s)$ *and*

$$\int_u^v \langle X(u,r) - X^*(u,r), d(Y - Y^*)(r)\rangle \leqslant 0 \ ; \forall (u,v) \in \Delta_{s,t} \tag{16}$$

Proof For the sake of simplicity, the proposition is proved on $[0,T]$ instead of $[s,t]$, with $(s,t) \in \Delta_T$.

First of all, if $(X,Y) = (X^*,Y^*)$ on $[s,t]$, then

$$\int_u^v \langle X(u,r) - X^*(u,r), d(Y - Y^*)(r)\rangle = 0 \ ; \forall (u,v) \in \Delta_{s,t}.$$

Now, let us prove that if $X(s) = X^*(s)$ and Inequality (16) is true, then $(X,Y) = (X^*,Y^*)$.

For every $(s,t) \in \Delta_T$,

$$\begin{aligned}
\|X(s,t) - X^*(s,t)\|^2 &= \|H(s,t) - H^*(s,t)\|^2 \\
&\quad +2\int_s^t \langle Y(s,u) - Y^*(s,u), d(Y - Y^*)(u)\rangle \\
&\quad +2\int_s^t \langle H(s,t) - H^*(s,t), d(Y - Y^*)(u)\rangle \\
&= \|H(s,t) - H^*(s,t)\|^2 \\
&\quad +2\int_s^t \langle X(s,u) - X^*(s,u), d(Y - Y^*)(u)\rangle \\
&\quad +2\int_s^t \langle H(s,t) - H(s,u) - (H^*(s,t) - H^*(s,u)), d(Y - Y^*)(u)\rangle \\
&\leqslant \|H - H^*\|^2_{p\text{-var},s,t} + 2m(s,t)
\end{aligned}$$

with

$$m(s,t) := \int_s^t \langle H(u,t) - H^*(u,t), d(Y - Y^*)(u)\rangle.$$

Let $(s,t) \in \Delta_T$ be arbitrarily chosen.

On the one hand,

$$m(s,t) \leqslant 2e \cdot c(p,p)\|H - H^*\|_{p\text{-var},s,t}\|Y - Y^*\|_{p\text{-var},s,t}.$$

So, there exists a constant $c_1 > 0$, not depending on s and t, such that

$$\begin{aligned}
\|X(s,t) - X^*(s,t)\|^p &\leqslant (\|H - H^*\|^2_{p\text{-var},s,t} + 2m(s,t))^{p/2} \\
&\leqslant c_1(\|H - H^*\|^p_{p\text{-var},s,t} \\
&\quad +(\|H - H^*\|^p_{p\text{-var},s,t})^{1/2}(\|Y - Y^*\|^p_{p\text{-var},s,t})^{1/2}).
\end{aligned}$$

Since $1/2 + 1/2 = 1$, the right-hand side of the previous inequality defines a control function (see Friz and Victoir [11], Exercice 1.9), and then there exists a constant $c_2 > 0$, not depending on s and t, such that

$$\begin{aligned}
\|X - X^*\|^p_{p\text{-var},s,t} &\leqslant c_1(\|H - H^*\|^p_{p\text{-var},s,t} + \|H - H^*\|^{p/2}_{p\text{-var},s,t}\|Y - Y^*\|^{p/2}_{p\text{-var},s,t}) \\
&\leqslant c_2(\|H - H^*\|^p_{p\text{-var},s,t} \\
&\quad +\|H - H^*\|^{p/2}_{p\text{-var},s,t}\|X - X^*\|^{p/2}_{p\text{-var},s,t}) \qquad (17)
\end{aligned}$$

The right-hand side of the previous inequality defines a control function.

On the other hand, since $X(0) = X^*(0)$:

$$\begin{aligned}\|H-H^*\|_{p\text{-var},s,t} &\leqslant c(p,p)\|Z\|_{p\text{-var},s,t}(\|f\circ X-f\circ X^*\|_{p\text{-var},s,t}\\ &\quad +\|f(X(s))-f(X(0))-(f(X^*(s))-f(X^*(0)))\|)\\ &\leqslant 2c(p,p)\|Z\|_{p\text{-var},s,t}\|f\circ X-f\circ X^*\|_{p\text{-var},t}.\end{aligned}$$

Consider $(u,v)\in\Delta_t$ and

$$\delta(u,v):=\|(f\circ X)(u,v)-(f\circ X^*)(u,v)\|.$$

Applying Taylor's formula to the map f between $X(u,v)$ and $X^*(u,v)$:

$$\begin{aligned}\delta(u,v) &\leqslant \left\|\int_0^1 Df(X(u)+\theta X(u,v))(X(u,v)-X^*(u,v))d\theta\right\|\\ &\quad +\left\|\int_0^1(Df(X(u)+\theta X(u,v))-Df(X^*(u)+\theta X^*(u,v)))X^*(u,v)d\theta\right\|\\ &\leqslant \|f\|_{\mathrm{Lip}^\gamma}(\|X-X^*\|_{p\text{-var},u,v}+2\|X^*\|_{p\text{-var},u,v}\|X-X^*\|_{p\text{-var},t}).\end{aligned}$$

So, there exists a constant $c_3>0$, not depending on t, such that

$$\|f\circ X-f\circ X^*\|_{p\text{-var},t}\leqslant c_3\|X-X^*\|_{p\text{-var},t},$$

and then there exists a constant $c_4>0$, not depending on s and t, such that

$$\|H-H^*\|_{p\text{-var},s,t}\leqslant c_4\|Z\|_{p\text{-var},s,t}\|X-X^*\|_{p\text{-var},t} \tag{18}$$

By Equation (17) and Equation (18) together, there exists a constant $c_5>0$, not depending on s and t, such that

$$\|X-X^*\|_{p\text{-var},s,t}\leqslant c_5\|Z\|_{p\text{-var},s,t}^{1/2}\|X-X^*\|_{p\text{-var},t}.$$

Since $(u,v)\in\Delta_T\mapsto\|Z\|_{p\text{-var},u,v}^p$ is a control function, there exists $N\in\mathbb{N}^*$ and $(\tau_k)_{k\in[\![0,N]\!]}\in\mathfrak{D}_{[0,T]}$ such that

$$\|Z\|_{p\text{-var},\tau_k,\tau_{k+1}}\leqslant\frac{1}{4c_5^2}\ ;\ \forall k\in[\![0,N-1]\!].$$

First,

$$\begin{aligned}\|X-X^*\|_{p\text{-var},\tau_1} &\leqslant c_5\|Z\|_{p\text{-var},\tau_1}^{1/2}\|X-X^*\|_{p\text{-var},\tau_1}\\ &\leqslant \frac{1}{2}\|X-X^*\|_{p\text{-var},\tau_1}.\end{aligned}$$

So, $X=X^*$ on $[0,\tau_1]$. For $k\in[\![1,N-1]\!]$, assume that $X=X^*$ on $[0,\tau_k]$. Then,

$$\begin{aligned}\|X-X^*\|_{p\text{-var},\tau_{k+1}} &= \|X-X^*\|_{p\text{-var},\tau_k,\tau_{k+1}}\\ &\leqslant \frac{1}{2}\|X-X^*\|_{p\text{-var},\tau_{k+1}}.\end{aligned}$$

So, $X = X^*$ on $[0,\tau_{k+1}]$. Recursively, $X = X^*$ on $[0,T]$. □

Proposition 13 *Consider* $(s,t) \in \Delta_T$, $p \in [2,3[$, $f \in \text{Lip}^\gamma(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$ *and two solutions* (X,Y) *and* (X^*,Y^*) *to Problem (2) under Assumption 1. On* $[s,t]$, $(X,Y) = (X^*,Y^*)$ *if and only if* $X(s) = X^*(s)$ *and*

$$\int_u^v \langle R_X(u,r) - R_{X^*}(u,r), d(Y-Y^*)(r)\rangle \leqslant 0 \ ; \forall (u,v) \in \Delta_{s,t} \tag{19}$$

Proof For the sake of simplicity, the proposition is proved on $[0,T]$ instead of $[s,t]$ with $(s,t) \in \Delta_T$.

First of all, if $(X,Y) = (X^*,Y^*)$ on $[s,t]$, then

$$\int_u^v \langle R_X(u,r) - R_{X^*}(u,r), d(Y-Y^*)(r)\rangle = 0 \ ; \forall (u,v) \in \Delta_{s,t}.$$

Now, let us prove that if $X(s) = X^*(s)$ and Inequality (19) is true, then $(X,Y) = (X^*,Y^*)$.

There exists a constant $c_1 > 0$ such that for every $(s,t) \in \Delta_T$,

$$\begin{aligned}
&\|(X - X^*, (X - X^*)')\|_{Z,p/2,s,t}\\
&\quad= \|f(X) - f(X^*)\|_{p\text{-var},s,t} + \|R_X - R_{X^*}\|_{p/2\text{-var},s,t}\\
&\quad\leqslant c_1(\|R_X - R_{X^*}\|_{p/2\text{-var},s,t} + \|Z\|_{p\text{-var},s,t}\|(X - X^*, (X - X^*)')\|_{Z,p/2,t})
\end{aligned} \tag{20}$$

Let us find a suitable control function dominating

$$(s,t) \in \Delta_T \longmapsto \|R_X - R_{X^*}\|^{p/2}_{p/2\text{-var},s,t}.$$

For every $(s,t) \in \Delta_T$,

$$\begin{aligned}
\|R_X(s,t) - R_{X^*}(s,t)\|^2 &= \|R_H(s,t) - R_{H^*}(s,t)\|^2\\
&\quad +2\int_s^t \langle Y(s,u) - Y^*(s,u), d(Y - Y^*)(u)\rangle\\
&\quad +2\int_s^t \langle R_H(s,t) - R_{H^*}(s,t), d(Y - Y^*)(u)\rangle\\
&= \|R_H(s,t) - R_{H^*}(s,t)\|^2\\
&\quad +2\int_s^t \langle R_X(s,u) - R_{X^*}(s,u), d(Y - Y^*)(u)\rangle\\
&\quad +2\int_s^t \langle R_H(s,t) - R_H(s,u)\\
&\quad -(R_{H^*}(s,t) - R_{H^*}(s,u)), d(Y - Y^*)(u)\rangle\\
&\leqslant \|R_H - R_{H^*}\|^2_{p/2\text{-var},s,t} + 2m(s,t)
\end{aligned}$$

with

$$m(s,t) := \int_s^t \langle R_H(u,t) - R_{H^*}(u,t), d(Y - Y^*)(u)\rangle.$$

Let $(s,t) \in \Delta_T$ be arbitrarily chosen.

On the one hand,

$$m(s,t) \leqslant 2e \cdot c(p,p)\|R_H - R_{H^*}\|_{p/2\text{-var},s,t}\|Y - Y^*\|_{p/2\text{-var},s,t}.$$

So, there exists a constant $c_2 > 0$, not depending on s and t, such that

$$\begin{aligned}\|R_X(s,t) - R_{X^*}(s,t)\|^{p/2} &\leqslant (\|R_H - R_{H^*}\|^2_{p/2\text{-var},s,t} + 2m(s,t))^{p/4}\\ &\leqslant c_2(\|R_H - R_{H^*}\|^{p/2}_{p/2\text{-var},s,t}\\ &\quad +(\|R_H - R_{H^*}\|^{p/2}_{p/2\text{-var},s,t})^{1/2}(\|Y - Y^*\|^{p/2}_{p/2\text{-var},s,t})^{1/2}).\end{aligned}$$

Since $1/2 + 1/2 = 1$, the right-hand side of the previous inequality defines a control function (see Friz and Victoir [11], Exercice 1.9), and then there exists a constant $c_3 > 0$, not depending on s and t, such that

$$\begin{aligned}\|R_X - R_{X^*}\|^{p/2}_{p/2\text{-var},s,t} &\leqslant c_2(\|R_H - R_{H^*}\|^{p/2}_{p/2\text{-var},s,t}\\ &\quad +\|R_H - R_{H^*}\|^{p/4}_{p/2\text{-var},s,t}\|Y - Y^*\|^{p/4}_{p/2\text{-var},s,t})\\ &\leqslant c_3(\|R_H - R_{H^*}\|^{p/2}_{p/2\text{-var},s,t}\\ &\quad +\|R_H - R_{H^*}\|^{p/4}_{p/2\text{-var},s,t}\|R_X - R_{X^*}\|^{p/4}_{p/2\text{-var},s,t}) \quad (21)\end{aligned}$$

On the other hand, since $X(0) = X^*(0)$:

$$\begin{aligned}\|f(X)'(s) - f(X^*)'(s)\| &= \|Df(X(s))f(X(s)) - Df(X(0))f(X(0))\\ &\quad -(Df(X^*(s))f(X^*(s)) - Df(X^*(0))f(X^*(0)))\|\\ &\leqslant \|f(X)' - f(X^*)'\|_{p\text{-var},t}.\end{aligned}$$

Then,

$$\begin{aligned}\|R_H(s,t) - R_{H^*}(s,t)\| &\leqslant \Im_{\mathbf{Z},f(X)-f(X^*)}(s,t) + \|(f(X)'(s) - f(X^*)'(s))\mathbb{Z}(s,t)\|\\ &\leqslant c(p)(\|R_{f(X)} - R_{f(X^*)}\|_{p/2\text{-var},s,t}\|Z\|_{p\text{-var},s,t}\\ &\quad +\|f(X)' - f(X^*)'\|_{p\text{-var},s,t}\|\mathbb{Z}\|_{p/2\text{-var},s,t})\\ &\quad +\|f(X)' - f(X^*)'\|_{p\text{-var},t}\|\mathbb{Z}\|_{p/2\text{-var},s,t}.\end{aligned}$$

So, with the same ideas as in P. Friz and M. Hairer [9, Theorem 8.4 p. 115], there exists a constant $c_4 > 0$, not depending on s and t, such that

$$\|R_H - R_{H^*}\|_{p/2\text{-var},s,t} \leqslant c_4\|(X - X^*, (X - X^*)')\|_{Z,p/2,t}\,\omega_{p,\mathbf{Z}}(s,t)^{1/p} \quad (22)$$

By Equations (20), (21) and (22) together, there exists a constant $c_5 > 0$, not depending on s and t, such that

$$\|(X - X^*, (X - X^*)')\|_{Z,p/2,s,t} \leqslant c_5 \|(X - X^*, (X - X^*)')\|_{Z,p/2,t}\,\omega_{p,\mathbf{Z}}(s,t)^{1/(2p)}.$$

The conclusion of the proof is the same as in Proposition 12. □

5 Approximation scheme

In Proposition 11, it has been proved that, under Assumption 1, Problem (2) has a unique solution (X, Y) if $p = 1$ and if, moreover, there is an additive continuous signal of finite q-variation W with $q \in [1, 3[$. This section deals with the convergence of the following approximation scheme for X:

$$\begin{cases} X_0^n := a \\ X_{k+1}^n = \text{proj}_{C(t_{k+1}^n)}(X_k^n + f(X_k^n)(t_{k+1}^n - t_k^n) + W(t_k^n, t_{k+1}^n)) \; ; \; k \in [\![0, n-1]\!] \end{cases} \tag{23}$$

where $n \in \mathbb{N}^*$ and $(t_0^n, \dots, t_n^n)$ is the dissection of $[0, T]$ of constant mesh T/n.

Consider the maps X^n, H^n and Y^n from $[0, T]$ into $\mathbb{R}^e$ defined by $X^n(t) := X_k^n$,

$$H^n(t) := \sum_{i=0}^{k-1} f(X_i^n)(t_{i+1}^n - t_i^n) + f(X_k^n)(t - t_k^n) + W(t) \tag{24}$$

and

$$Y^n(t) := X^n(t) - H^n(t_k^n) \tag{25}$$

for every $k \in [\![0, n-1]\!]$ and $t \in [t_k^n, t_{k+1}^n[$.

Lemma 1 *Under Assumption 1, if $f \in C^0(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$, one can extract a uniformly converging subsequence from any subsequence of $(H^n)_{n \in \mathbb{N}^*}$.*

Proof On the one hand, since $C(t)$ is a bounded set for every $t \in [0, T]$, C is continuous on $[0, T]$ for the Hausdorff distance and $X^n([0, T]) \subset \cup_{t \in [0,T]} C(t)$ for every $n \in \mathbb{N}^*$ by construction,

$$\sup_{n \in \mathbb{N}^*} \|X^n\|_{\infty,T} < \infty.$$

On the other hand, consider $(s, t) \in \Delta_T$ and $j, k \in [\![0, n]\!]$ such that $s < t_i^n \leqslant t$ for every $i \in [\![j, k]\!]$. Then,

$$\begin{aligned}\|H^n(t) - H^n(s)\| &= \left\|\sum_{i=0}^{k-1} f(X_i^n)(t_{i+1}^n - t_i^n) + f(X_k^n)(t - t_k^n)\right.\\ &\quad \left. - \sum_{i=0}^{j-2} f(X_i^n)(t_{i+1}^n - t_i^n) - f(X_{j-1}^n)(s - t_j^n) + W(s,t)\right\|\\ &= \left\|\int_s^t f(X^n(u))du + W(s,t)\right\|\\ &\leqslant \varphi(s,t) := |t-s| \sup_{n\in\mathbb{N}^*} \|f\circ X^n\|_{\infty,T} + \|W\|_{q\text{-var},s,t}.\end{aligned}$$

Since $(s,t) \in \Delta_T \mapsto \varphi(s,t)$ is a continuous map such that $\varphi(t,t) = 0$ for every $t \in [0,T]$, $(H^n)_{n\in\mathbb{N}^*}$ is equicontinuous. Therefore, by Arzelà-Ascoli's theorem, one can extract a uniformly converging subsequence from any subsequence of $(H^n)_{n\in\mathbb{N}^*}$. □

Lemma 2 *Under Assumption 1, if* $f \in C^0(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$*, there exist* $R > 0$ *and* $N \in \mathbb{N}^*$ *such that*

$$\sup_{n\in\mathbb{N}^*} \|Y^n\|_{1\text{-var},T} \leqslant M(N,R)$$

with

$$M(N,R) := \frac{N}{R}\left(\|\gamma\|_{\infty,T} + \sup_{n\in\mathbb{N}^*}\|X^n\|_{\infty,T} + \varphi(0,T)\right)^2.$$

Proof On the one hand, since the map $(s,t) \in \Delta_T \mapsto \varphi(s,t)$ defined in the proof of Lemma 1 is continuous and satisfies $\varphi(t,t) = 0$ for every $t \in [0,T]$, by Assumption 1, there exist $R > 0$, $N \in \mathbb{N}^*$ and a dissection $(\tau_0, \dots, \tau_N)$ of $[0,T]$ such that

$$\overline{B}_e(\gamma(\tau_i), R) \subset C(u) \text{ and } \varphi(\tau_i, \tau_{i+1}) \leqslant R/2$$

for every $i \in [\![0, N-1]\!]$ and $u \in [\tau_i, \tau_{i+1}[$. Then,

$$\overline{B}_e(\gamma(\tau_i) - H^n(\tau_i), R/2) \subset \overline{B}_e(\gamma(\tau_i) - H^n(u), R) \subset C(u) - H^n(u)$$

for every $i \in [\![0, N-1]\!]$ and $u \in [\tau_i, \tau_{i+1}[$.

On the other hand, for every $k \in [\![1, n]\!]$,

$$\begin{aligned}Y^n(t_k^n) &= \text{proj}_{C(t_k^n)}(X_{k-1}^n + H^n(t_{k-1}^n, t_k^n)) - H^n(t_k^n)\\ &= \text{proj}_{C(t_k^n) - H^n(t_k^n)}(Y^n(t_{k-1}^n)).\end{aligned}$$

So, for any $i \in [\![0, N-1]\!]$, by applying Proposition 3.(1) to Y^n on $[\tau_i, \tau_{i+1}[$:

$$\begin{aligned}\|Y^n\|_{1\text{-var},\tau_i,\tau_{i+1}} &\leqslant l(R/2, \|\gamma(\tau_i) - H^n(\tau_i) - Y^n(\tau_i)\|)\\ &\leqslant R^{-1}\|\gamma(\tau_i) - H^n(\tau_i) - Y^n(\tau_i)\|^2.\end{aligned}$$

Since there exists $j \in [\![0, n]\!]$ such that $Y^n(\tau_i) = Y^n(t_j^n)$,

$$\begin{aligned}\|Y^n\|_{1\text{-var},\tau_i,\tau_{i+1}} &\leqslant R^{-1}(\|\gamma(\tau_i)\| + \|H^n(\tau_i) - H^n(t_j^n)\| + \|H^n(t_j^n) + Y^n(t_j^n)\|)^2 \\ &\leqslant R^{-1}(\|\gamma(\tau_i)\| + \varphi(t_j^n, \tau_i) + \|X_j^n\|)^2 \\ &\leqslant N^{-1}M(N, R).\end{aligned}$$

□

Therefore,

$$\|Y^n\|_{1\text{-var},T} = \sum_{i=0}^{N-1} \|Y^n\|_{1\text{-var},\tau_i,\tau_{i+1}} \leqslant M(N, R).$$

Lemma 3 *Under Assumption 1, if* $f \in C^0(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$*, for every* $(s,t) \in \Delta_T$ *and* $z \in \cap_{\tau\in[s,t]}(C(\tau) - H^n(\tau))$*,*

$$\langle z, Y^n(t) - Y^n(s)\rangle \geqslant \frac{1}{2}(\|Y^n(t)\|^2 - \|Y^n(s)\|^2).$$

Proof Consider $(s,t) \in \Delta_T$. There exists a maximal interval $[\![j,k]\!] \subset [\![0,n]\!]$ such that

$$s < t_i^n \leqslant t \ ; \forall i \in [\![j,k]\!].$$

Consider $z \in \cap_{\tau\in[s,t]}(C(\tau) - H^n(\tau))$. In particular, for every $i \in [\![j,k]\!]$, there exists $y_i \in C(t_i^n)$ such that $z = y_i - H^n(t_i^n)$. For every $i \in [\![j,k]\!]$,

$$\begin{aligned}\langle z - Y^n(t_i^n), Y^n(t_i^n) - Y^n(t_{i-1}^n)\rangle &= \\ \langle y_i - H^n(t_i^n) - Y^n(t_i^n), Y^n(t_i^n) - Y^n(t_{i-1}^n)\rangle &= \\ \langle y_i - X_i^n, X_i^n - (X_{i-1}^n + H^n(t_{i-1}^n, t_i^n))\rangle &\geqslant 0\end{aligned}$$

because

$$X_i^n = \mathrm{proj}_{C(t_i^n)}(X_{i-1}^n + H^n(t_{i-1}^n, t_i^n)).$$

Then,

$$\begin{aligned}\langle z, Y^n(t) - Y^n(s)\rangle &= \langle z, Y^n(t_k^n) - Y^n(t_{j-1}^n)\rangle = \sum_{i=j}^{k}\langle z, Y^n(t_i^n) - Y^n(t_{i-1}^n)\rangle \\ &\geqslant \sum_{i=j}^{k}\langle Y^n(t_i^n), Y^n(t_i^n) - Y^n(t_{i-1}^n)\rangle \\ &\geqslant \frac{1}{2}\sum_{i=j}^{k}(\|Y^n(t_i^n)\|^2 - \|Y^n(t_{i-1}^n)\|^2) = \frac{1}{2}(\|Y^n(t)\|^2 - \|Y^n(s)\|^2).\end{aligned}$$

□

Now, W is $1/q$-Hölder continuous from $[0,T]$ into $\mathbb{R}^e$. Then, there exists a constant $C_\varphi > 0$ such that

$$\varphi(s,t) \leqslant C_\varphi|t - s|^{1/q} \ ; \forall (s,t) \in \Delta_T.$$

Theorem 5 *Assume that* C *fulfills Assumption 1,* $f \in \mathrm{Lip}^1(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$*, and that there exists* $(K,\alpha) \in]0,\infty[\times]0,1[$ *such that*

$$d_H(C(s),C(t)) \leqslant K|t-s|^{\alpha} \text{ ; } \forall (s,t) \in \Delta_T.$$

Then, $(X^n,Y^n)_{n\in\mathbb{N}^*}$ converges uniformly to the unique solution (X,Y) to Problem (2).

Proof Consider an extraction $\psi : \mathbb{N}^* \to \mathbb{N}^*$ such that $(H^{\psi(n)})_{n\in\mathbb{N}^*}$ is uniformly converging to a limit H^*.

On the one hand, consider $n \in \mathbb{N}^*$ such that $T/n \in]0,1]$, $m \in n\mathbb{N}^*$ and $t \in [0,T]$. By Proposition 3.(2) together with Lemma 2, there exist $R > 0$, $N \in \mathbb{N}^*$, $i \in [\![1,n]\!]$ and $j \in [\![1,m]\!]$ such that $t \in [t_{i-1}^n, t_i^n[$, $t \in [t_{j-1}^m, t_j^m[$ and

$$\begin{aligned}
\|Y^n(t)-Y^m(t)\|^2 &\leqslant 2d_H(C(t_i^n)-H^n(t_i^n), C(t_j^m)-H^m(t_j^m)) \\
&\quad \times(\|Y^n\|_{1\text{-var},T}+\|Y^m\|_{1\text{-var},T}) \\
&\leqslant 4M(N,R)(d_H(C(t_i^n),C(t_j^m))+\|H^n(t_i^n)-H^m(t_j^m)\|) \\
&\leqslant 4M(N,R)(K|t_i^n-t_j^m|^{\alpha} \\
&\quad +\|H^n(t_i^n)-H^n(t_j^m)\|+\|H^n(t_j^m)-H^m(t_j^m)\|) \\
&\leqslant 4M(N,R)((C_{\varphi}+K)|T/n|^{\alpha\wedge 1/q}+\|H^n-H^m\|_{\infty,T}) \quad (26)
\end{aligned}$$

Consider $\varepsilon > 0$. There exists $N_{\varepsilon} \in \mathbb{N}^*$ such that for every $n,m \in \mathbb{N}^* \cap [N_{\varepsilon},\infty[$,

$$|T/\psi(n)|^{\alpha\wedge 1/q}, |T/\psi(m)|^{\alpha\wedge 1/q} \leqslant \frac{\varepsilon}{16M(N,R)(C_{\varphi}+K)} \wedge 1 \qquad (27)$$

and

$$\|H^{\psi(n)}-H^{\psi(m)}\|_{\infty,T} \leqslant \frac{\varepsilon}{16M(N,R)} \qquad (28)$$

Consider $n,m \in \mathbb{N}^* \cap [N_{\varepsilon},\infty[$ and let p be the least common multiple of $\psi(n)$ and $\psi(m)$. By Inequality (26):

$$\begin{aligned}
\|Y^{\psi(n)}-Y^{\psi(m)}\|_{\infty,T}^2 &\leqslant 2(\|Y^{\psi(n)}-Y^p\|_{\infty,T}^2+\|Y^{\psi(m)}-Y^p\|_{\infty,T}^2) \\
&\leqslant 8M(N,R)((C_{\varphi}+K)|T/\psi(n)|^{\alpha}+\|H^{\psi(n)}-H^{\psi(\psi^{-1}(p))}\|_{\infty,T}) \\
&\quad +8M(N,R)((C_{\varphi}+K)|T/\psi(m)|^{\alpha}+\|H^{\psi(m)}-H^{\psi(\psi^{-1}(p))}\|_{\infty,T}).
\end{aligned}$$

Since $p \geqslant \psi(n)$ and $p \geqslant \psi(m)$, $\psi^{-1}(p) \geqslant n \vee m \geqslant N_{\varepsilon}$. Then, by (27) and (28) together:

$$\|Y^{\psi(n)}-Y^{\psi(m)}\|_{\infty,T}^2 \leqslant \varepsilon.$$

Therefore, $(Y^{\psi(n)})_{n\in\mathbb{N}^*}$ is a uniformly converging sequence and by Equation (25), $(X^{\psi(n)})_{n\in\mathbb{N}^*}$ also. In the sequel, the limit of $(Y^{\psi(n)})_{n\in\mathbb{N}^*}$ (resp. $(X^{\psi(n)})_{n\in\mathbb{N}^*}$) is denoted by Y^* (resp. X^*).

On the other hand, consider $(s,t) \in \Delta_T$, $z \in \cap_{\tau\in[s,t]} C(\tau)$ and $\tau \in [s,t]$. By Lemma 3:

$$\langle z-H^{\psi(n)}(\tau), Y^{\psi(n)}(t)-Y^{\psi(n)}(s)\rangle \geqslant \frac{1}{2}(\|Y^{\psi(n)}(t)\|^2-\|Y^{\psi(n)}(s)\|^2).$$

So, when n goes to infinity:

$$\langle z - H^*(\tau), Y^*(t) - Y^*(s)\rangle \geqslant \frac{1}{2}(\|Y^*(t)\|^2 - \|Y^*(s)\|^2).$$

Therefore, by Proposition 2:

$$-\frac{dDY^*}{d|DY^*|}(t) \in N_{C(t)-H^*(t)}(Y^*(t)) \ |DY^*|\text{-a.e.}$$

Moreover, since $(X^{\psi(n)})_{n\in\mathbb{N}^*}$ is a sequence of step functions uniformly converging to X^*, the definition of $(H^{\psi(n)})_{n\in\mathbb{N}^*}$ given by Equality (24) ensures that:

$$H^*(t) = \int_0^t f(X^*(s))ds + W(t) \ ; \forall t \in [0,T].$$

Since the solution (X,Y) to (2) is unique by Proposition 11, $(X^*,Y^*) = (X,Y)$ and $H^* = X - Y$.

We have proved that, for each subsequence of $(X^n,Y^n)_{n\in\mathbb{N}}$, we can extract a further subsequence which converges uniformly to the solution (X,Y). Thus $(X^n,Y^n)_{n\in\mathbb{N}^*}$ converges uniformly to (X,Y). □

6 Sweeping processes perturbed by a stochastic noise directed by a fBm

First of all, let us recall the definition of fractional Brownian motion.

Definition 5 Let $(\mathrm{B}(t))_{t\in[0,T]}$ be a d-dimensional centered Gaussian process. It is a fractional Brownian motion of Hurst parameter $H \in]0,1[$ if and only if,

$$\mathrm{cov}(B_i(s),B_j(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})\delta_{i,j}$$

for every $(i,j) \in [\![1,d]\!]^2$ and $(s,t) \in [0,T]^2$.

Fore more details on fractional Brownian motion, we refer the reader to Nualart [21, Chapter 5].

Let $B := (B(t))_{t\in[0,T]}$ be a d-dimensional fractional Brownian motion of Hurst parameter $H \in]1/3,1[$, defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$.

By Garcia-Rodemich-Rumsey's lemma (see Nualart [21, Lemma A.3.1]), the paths of B are α-Hölder continuous for every $\alpha \in]0,H[$. So, in particular, the paths of B are continuous and of finite p-variation for every $p \in]1/H,\infty[$. By Friz and Victoir [11, Proposition 15.5 and Theorem 15.33], there exists an enhanced Gaussian process $\mathbf{B}$

such that $\mathbf{B}^{(1)} = B$.

Consider $b \in C^{[p]+1}(\mathbb{R}^e)$, $\sigma \in C^{[p]+1}(\mathbb{R}^e, \mathcal{M}_{e,d}(\mathbb{R}))$ and the following sweeping process, perturbed by a pathwise stochastic noise directed by $\mathbf{B}$:

$$\begin{cases} X(t) = H(t) + Y(t) \\ H(t) = \int_0^t b(X(s))ds + \int_0^t \sigma(X(s))d\mathbf{B}(s) \\ -\frac{dDY}{d|DY|}(t) \in N_{C_{H(t)}}(Y(t)) \ |DY|\text{-a.e. with } Y(0) = a \end{cases} \tag{29}$$

In the following, since H can be deduced from X, and Y from X and H, we say that X is a solution to Problem (29) if the corresponding triple (X, H, Y) satisfies (29).

Let $W := (W(t))_{t\in[0,T]}$ be the stochastic process defined by

$$W(t) := te_1 + \sum_{k=1}^{d} B_k(t)e_{k+1} \ ; \forall t \in [0, T].$$

By Friz and Victoir [11, Theorem 9.26], there exists a $G\Omega_{p,T}(\mathbb{R}^{d+1})$-valued enhanced stochastic process $\mathbf{W}$ such that $\mathbf{W}^{(1)} := W$. Consider also the map $f : \mathbb{R}^e \to \mathcal{M}_{e,d+1}(\mathbb{R})$ defined by:

$$f(x)(u, v) := b(x)u + \sigma(x)v \ ; \forall x \in \mathbb{R}^e, \forall (u, v) \in \mathbb{R}^{d+1}.$$

So, Problem (29) can be reformulated as follow:

$$\begin{cases} X(t) = H(t) + Y(t) \\ H(t) = \int_0^t f(X(s))d\mathbf{W}(s) \\ -\frac{dDY}{d|DY|}(t) \in N_{C_{H(t)}}(Y(t)) \ |DY|\text{-a.e. with } Y(0) = a. \end{cases}$$

Therefore, the previous results of this paper apply to Problem (29):

Theorem 6 *(Existence) Assume that, for every $t \in [0, T]$, $C(t)$ is a random set with convex compact values with nonempty interior, and that the paths of C are continuous for the Hausdorff distance. Then Problem (29) has at least one solution, whose paths belong to $C^{p\text{-var}}([0, T], \mathbb{R}^e)$, for $p \in]1/H, \infty[$.*

Proof This is a direct pathwise application of Theorems 3 and 4. □

Proposition 14 *(Existence and uniqueness for an additive fractional noise) Assume that, for every $t \in [0, T]$, $C(t)$ is a random set with convex compact values with nonempty interior, and that the paths of C are continuous for the Hausdorff distance. If σ is a constant map, then Problem (29) has a unique solution, whose paths belong to $C^{p\text{-var}}([0, T], \mathbb{R}^e)$, for $p \in]1/H, \infty[$.*

Proof This is a direct pathwise application of Theorem 3, Theorem 4 and Proposition 11. □

Remark. For instance, Proposition 14 ensures the existence and uniqueness of the solution to a multidimensional reflected fractional Ornstein-Uhlenbeck process.

Proposition 15 *Assume that, for every $t \in [0,T]$, $C(t)$ is a random set with convex compact values with nonempty interior, and that the paths of C are α-Hölder continuous for the Hausdorff distance with $\alpha \in]0,1[$. If σ is a constant map, then the sequence of processes $(X^n)_{n\in\mathbb{N}^*}$ defined by*

$$\begin{cases} X_0^n & := a \\ X_{k+1}^n & = \text{proj}_{C((k+1)T/n)}(X_k^n + b(X_k^n)T/n + \sigma B(kT/n,(k+1)T/n)) \text{ ; } k \in [\![0, n-1]\!] \\ X^n(t) & := X_k^n \text{ ; } t \in [kT/n,(k+1)T/n[, \ k \in [\![0, n-1]\!] \end{cases}$$

for every $n \in \mathbb{N}^*$ converges pathwise uniformly to the unique solution X to Problem (29).

Proof This is a direct pathwise application of Theorem 5. □

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