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<hal-01736228>

HAL Id: hal-01736228
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Submitted on 16 Mar 2018

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Rational solutions to the KPI equation of order 7 depending on 12 parameters

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January 27, 2018

Abstract
We construct in this paper, rational solutions as a quotient of two determinants of order $2N = 14$ and we obtain what we call solutions of order $N = 7$ to the Kadomtsev-Petviashvili equation (KPI) as a quotient of 2 polynomials of degree 112 in $x$, $y$ and $t$ depending on 12 parameters. The maximum of modulus of these solutions at order 7 is equal to $2(2N + 1)^2 = 450$. We make the study of the patterns of their modulus in the plane $(x, y)$ and their evolution according to time and parameters $a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6$. When all these parameters grow, triangle and ring structures are obtained.

Keywords: KPI equation; Fredholm determinants; Wronskians; rogue waves; lumps.

PACS numbers : 33Q55, 37K10, 47.10A-, 47.35.Fg, 47.54.Bd

1 Introduction
We consider the Kadomtsev-Petviashvili equation (KPI), first proposed in 1970 [1] in the following normalization

$$\left(4u_t - 6uu_x + u_{xxx}\right)_x - 3u_{yy} = 0. \tag{1}$$

As usual, subscripts $x$, $y$ and $t$ denote partial derivatives.
The first rational solutions were constructed in 1977 by Manakov, Zakharov, Bordag and Matveev [2]. Other more general rational solutions of the KPI
equation were found by Krichever in 1978 [3, 4], Satsuma and Ablowitz in 1979 [5], Matveev in 1979 [6], in particular among many works on this subject.

We construct rational solutions of order \( N \) depending on \( 2N - 2 \) parameters which can be written as a ratio of two polynomials in \( x, y \) and \( t \) of degree \( 2N(N + 1) \).

The maximum of the modulus of these solutions at order \( N \) is a rational solution of the KPI equation (1), where

\[
\text{The rational solutions to the KPI equation are given by the following result [38, 40]:}
\]

The rational solutions to the KPI equation were found by Krichever in 1978 [3, 4], Satsuma and Ablowitz in 1979 [5], Matveev in 1979 [6], in particular among many works on this subject.

\[
\text{We construct rational solutions of order } N \text{ depending on } 2N - 2 \text{ parameters, and the representations of their modulus in the plane of the coordinates } (x, y) \text{ according to the real parameters } a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, a_5, b_5, a_6, b_6 \text{ and time } t. \text{ When the parameters grow, we obtain } \\
\frac{N(N+1)}{2} \text{ peaks in particular structures, such as triangles, rings, or concentric rings.}
\]

\[\text{2 Rational solutions to KPI equation of order } N \text{ depending on } 2N - 2 \text{ parameters}\]

The rational solutions to the KPI equation are given by the following result [38, 40]:

\[
\text{The function } v \text{ defined by}
\]

\[v(x, y, t) = -2 \frac{\det((n_{j,k}))_{j,k \in [1, 2N]}}{\det((d_{j,k}))_{j,k \in [1, 2N]}}^2 \quad (2)
\]

\[\text{is a rational solution of the KPI equation } (1), \text{ where}
\]

\[
n_{j,1} = \varphi_{j,1}(x, y, t, 0), \quad 1 \leq j \leq 2N
\]

\[
n_{j,N+1} = \varphi_{j,N+1}(x, y, t, 0), \quad 1 \leq j \leq 2N
\]

\[
d_{j,1} = \psi_{j,1}(x, y, t, 0), \quad 1 \leq j \leq 2N
\]

\[
d_{j,N+1} = \psi_{j,N+1}(x, y, t, 0), \quad 1 \leq j \leq 2N
\]

The functions \( \varphi \) and \( \psi \) are defined in (4), (5), (6), (7).

\[
\varphi_{4j+1,k} = \gamma_k^{4j-1} \sin X_k, \quad \varphi_{4j+2,k} = \gamma_k^{4j} \cos X_k, \\
\varphi_{4j+3,k} = -\gamma_k^{4j+1} \sin X_k, \quad \varphi_{4j+4,k} = -\gamma_k^{4j+2} \cos X_k,
\]

\[\text{for } 1 \leq k \leq N, \text{ and}
\]

\[
\varphi_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos X_{N+k}, \quad \varphi_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin X_{N+k}, \\
\varphi_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos X_{N+k}, \quad \varphi_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin X_{N+k},
\]

\[\text{for } 1 \leq k \leq N.
\]

The functions \( \psi_{j,k} \) for \( 1 \leq j \leq 2N, \ 1 \leq k \leq 2N \) are defined in the same way, the term \( X_k \) is only replaced by \( Y_k \).

\[X_v = \frac{\kappa_v x}{2} + i\delta_v y - i\frac{x_3, v}{2} - i\frac{\tau_v}{2} t - i\frac{\nu}{2}
\]
Explicit expression of rational solutions of order 7 depending on 12 parameters.

We cannot give the complete analytic expressions of the solutions to the KPI equation of order 7 depending on 12 parameters. In the following, we explicitly construct rational solutions to the KPI equation as

\[ v \]

The rational solutions to the KPI equation can be written as

\[ v \text{ defined by the formulas:} \]

\[ \psi_{4j+1,k} = \gamma_{k}^{4j-1} \sin Y_{k}, \quad \psi_{4j+2,k} = \gamma_{k}^{4j} \cos Y_{k}, \]

\[ \psi_{4j+3,k} = -\gamma_{k}^{4j+1} \sin Y_{k}, \quad \psi_{4j+4,k} = -\gamma_{k}^{4j+2} \cos Y_{k}, \]

for \( 1 \leq k \leq N \), and

\[ \psi_{4j+1,N+k} = \gamma_{k}^{2N-4j-2} \cos Y_{N+k}, \quad \psi_{4j+2,N+k} = -\gamma_{k}^{2N-4j-3} \sin Y_{N+k}, \]

\[ \psi_{4j+3,N+k} = -\gamma_{k}^{2N-4j-4} \cos Y_{N+k}, \quad \psi_{4j+4,N+k} = \gamma_{k}^{2N-4j-5} \sin Y_{N+k}. \]

Real numbers \( \lambda_j \) are such that \(-1 < \lambda_\nu < 1, \nu = 1, \ldots, 2N\) depending on a parameter \( \epsilon \) which will be intended to tend towards 0; they can be written as

\[ \lambda_j = 1 - 2\epsilon_j^2, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \leq j \leq N. \]

The terms \( \kappa_\nu, \delta_\nu, \gamma_\nu, \tau_\nu \) and \( x_{r,\nu} \) are functions of \( \lambda_\nu, 1 \leq \nu \leq 2N \); they are defined by the formulas:

\[ \kappa_j = 2\sqrt{1 - \lambda_j^2}, \quad \delta_j = \kappa_j \lambda_j, \quad \gamma_j = \sqrt{\frac{1 - \lambda_j}{1 + \lambda_j}}, \]

\[ x_{r,j} = (r - 1) \ln \frac{\sqrt{1 - \lambda_j}}{\gamma_j}, \quad r = 1, 3, \quad \tau_j = -12i\lambda_j^2 \sqrt{1 - \lambda_j^2} - 4i(1 - \lambda_j^2) \sqrt{1 - \lambda_j^2}, \]

\[ \kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = \gamma_j^{-1}, \]

\[ x_{r,N+j} = -x_{r,j}, \quad \tau_{N+j} = \tau_j, \quad j = 1, \ldots, N. \]

\[ \epsilon_\nu, 1 \leq \nu \leq 2N \] are defined in the following way:

\[ \epsilon_j = 2i \left( \sum_{k=1}^{1/2M-1} a_k(\epsilon_j)^{2k-1} - i \sum_{k=1}^{1/2M-1} b_k(\epsilon_j)^{2k-1} \right), \]

\[ \epsilon_{N+j} = 2i \left( \sum_{k=1}^{1/2M-1} a_k(\epsilon_j)^{2k-1} + i \sum_{k=1}^{1/2M-1} b_k(\epsilon_j)^{2k-1} \right), \quad 1 \leq j \leq N, \quad 1 \leq k \leq N - 1. \]

\[ \epsilon_\nu, 1 \leq \nu \leq 2N \] are real numbers defined by:

\[ \epsilon_j = 1, \quad \epsilon_{N+j} = 0, \quad 1 \leq j \leq N. \]

\section{Explicit expression of rational solutions of order 7 depending on 12 parameters}

In the following, we explicitly construct rational solutions to the KPI equation of order 7 depending on 12 parameters.

We cannot give the complete analytic expressions of the solutions to the KPI equation of order 7 with twelve parameters because of their lengths.

The rational solutions to the KPI equation can be written as

\[ v(x, y, t) = -2 \frac{|d_3(x, y, t)|^2}{d_1(x, y, t)^2}, \]
with $d_3$ and $d_1$ polynomials of degree 112 in $x$, $y$ and $t$. The number of terms of the polynomials of the numerator $d_3$ and denominator $d_1$ of the solutions are shown in the table below (Table 1) when only one of the parameters $a_i$ and $b_i$ are set non equal to 0.

<table>
<thead>
<tr>
<th>$a_i$</th>
<th>$d_3$</th>
<th>$b_i$</th>
<th>$d_3$</th>
<th>$d_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>86 927</td>
<td>1</td>
<td>86 926</td>
<td>45 036</td>
</tr>
<tr>
<td>2</td>
<td>55 509</td>
<td>2</td>
<td>55 509</td>
<td>28 790</td>
</tr>
<tr>
<td>3</td>
<td>42 219</td>
<td>3</td>
<td>42 309</td>
<td>21 962</td>
</tr>
<tr>
<td>4</td>
<td>34 968</td>
<td>4</td>
<td>34 968</td>
<td>18 167</td>
</tr>
<tr>
<td>5</td>
<td>30 342</td>
<td>5</td>
<td>30 342</td>
<td>15 778</td>
</tr>
<tr>
<td>6</td>
<td>26 595</td>
<td>6</td>
<td>26 595</td>
<td>13 846</td>
</tr>
</tbody>
</table>

Table 1: Number of terms for the polynomials $d_3$ and $d_1$ of the solutions to the KPI equation in the case $N = 7$.

We give patterns of the modulus of the solutions in the plane $(x, y)$ of coordinates in functions of parameters $a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6$ and time $t$. The maximum of modulus of these solutions is checked equal in this case $N = 7$ to $2(2N + 1)^2 = 2 \times 15^2 = 450$.

When all the parameters are equal to 0, we obtain the lump $L_7$ with a highest amplitude of the modulus equal to 450.
Figure 0. Solution of order 7 to KPI, for $t = 0$ when all parameters equal to 0.

Figure 1. Solution of order 7 to KPI, for $a_1 \neq 0$ and all other parameters equal to 0; on the left for $t = 0$ and $a_1 = 1$; in the center for $t = 0$ and $a_1 = 10^3$; on the right for $t = 1$ and $a_1 = 10^5$.

Figure 2. Solution of order 7 to KPI, for $b_1 \neq 0$ and all other parameters equal to 0; on the left for $t = 0$ and $b_1 = 1$; in the center for $t = 0$ and $b_1 = 10^3$; on the right for $t = 10$ and $b_1 = 10^6$. 

5
Figure 3. Solution of order 7 to KPI, for $a_2 \neq 0$ and all other parameters equal to 0; on the left for $t = 0$ and $a_2 = 10^6$; in the center for $t = 0.1$ and $a_2 = 10^6$; on the right for $t = 10$ and $a_2 = 10^3$.

Figure 4. Solution of order 7 to KPI, for $b_2 \neq 0$ and all other parameters equal to 0; on the left for $t = 0$ and $b_2 = 10^5$; in the center for $t = 0.1$ and $b_2 = 10^3$; on the right for $t = 10$ and $b_2 = 10^5$.

Figure 5. Solution of order 7 to KPI, for $a_3 \neq 0$ and all other parameters equal to 0; on the left for $t = 0$ and $a_3 = 10^8$; in the center for $t = 0.1$ and $a_3 = 10^5$; on the right for $t = 10$ and $a_3 = 10^3$. 
Figure 6. Solution of order 7 to KPI, for $b_3 \neq 0$ and all other parameters equal to 0; on the left for $t = 0$ and $b_3 = 10^7$; in the center for $t = 0.1$ and $b_3 = 10^4$; on the right for $t = 10$ and $b_3 = 10^3$.

Figure 7. Solution of order 7 to KPI, for $a_4 \neq 0$ and all other parameters equal to 0; on the left for $t = 0$ and $a_4 = 10^9$; in the center for $t = 0.1$ and $a_4 = 10^5$; on the right for $t = 10$ and $a_4 = 10^3$.

Figure 8. Solution of order 7 to KPI, for $b_4 \neq 0$ and all other parameters equal to 0; on the left for $t = 0$ and $b_4 = 10^9$; in the center for $t = 0.1$ and $b_4 = 10^5$; on the right for $t = 10$ and $b_4 = 10^3$. 
Figure 9. Solution of order 7 to KPI, for \( a_5 \neq 0 \) and all other parameters equal to 0; on the left for \( t = 0 \) and \( a_5 = 10^{11} \); in the center for \( t = 0.1 \) and \( a_5 = 10^5 \); on the right for \( t = 20 \) and \( a_5 = 10^{11} \).

Figure 10. Solution of order 7 to KPI, for \( b_5 \neq 0 \) and all other parameters equal to 0; on the left for \( t = 0 \) and \( b_5 = 10^{12} \); in the center for \( t = 0.1 \) and \( b_5 = 10^5 \); on the right for \( t = 50 \) and \( b_5 = 10^{11} \).

Figure 11. Solution of order 7 to KPI, for \( a_6 \neq 0 \) and all other parameters equal to 0; on the left for \( t = 0 \) and \( a_6 = 10^{14} \); in the center for \( t = 0.1 \) and \( a_6 = 10^5 \); on the right for \( t = 20 \) and \( a_6 = 10^{11} \).
4 Conclusion

We construct 7-th order rational solutions to the KPI equation depending on 12 real parameters. These solutions can be expressed in terms of a ratio of two polynomials of degree $2N(N+1) = 112$ in $x$, $y$ and $t$. The maximum of the modulus of these solutions is equal to $2(2N+1)^2 = 450$; this solution which can be called lump $L_7$ is obtained when all parameters are equal to 0 at the instant $t = 0$. Here we have given a complete description of rational solutions of order 7 with 12 parameters by constructing explicit expressions of polynomials of these solutions.

We deduce the construction of the modulus of solutions in the $(x,y)$ plane of coordinates; different structures appear. For a given $t$ close to 0, when one parameter grows and the other ones are equal to 0 we obtain triangles, rings or concentric rings. There are six types of patterns. In the cases $a_1 \neq 0$ or $b_1 \neq 0$ we obtain triangles with a maximum of 28 peaks (figures 1 and 2); for $a_2 \neq 0$ or $b_2 \neq 0$, we have 3 concentric rings with two of them with 10 peaks and another with 5 peaks (figures 3 and 4). For $a_3 \neq 0$ or $b_3 \neq 0$, we obtain 4 concentric rings without central peak with 7 peaks on each of them (figures 5 and 6). For $a_4 \neq 0$ or $b_4 \neq 0$, we have 3 concentric rings with 9 peaks, with a peak in the center(figures 7 and 8). For $a_5 \neq 0$ or $b_5 \neq 0$, we obtain 2 concentric rings without central peak with 11 peaks on each of them (figures 9 and 10). For $a_6 \neq 0$ or $b_6 \neq 0$, we have only one ring with 13 peaks (figures 11 and 12). But, when $t$ grows, all the structures disappear very quickly and the heights of the peaks decrease even more quickly.

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