A short note on the complexity of computing strong pathbreadth

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Abstract

The strong pathbreadth of a given graph G is the minimum ρ such that G admits a Robertson and Seymour's path decomposition where every bag is the complete ρ-neighbourhood of some vertex in G. We prove that deciding whether a given graph has strong pathbreadth at most one is NP-complete. The latter answers negatively to a conjecture of [Leitert and Dragan, COCOA'16].

Keywords: strong pathbreadth; path decomposition; acyclic clustering; graph theory; NP-complete.

1. Introduction

We refer to [4] for graph terminology. All graphs considered are finite and simple (hence, with neither loops nor multiple edges). In this paper, we solve for the first time the complexity of computing the *strong pathbreadth* of a given graph — that is a pathlike invariant first introduced in [13]. Roughly, a graph $G = (V, E)$ has strong pathbreadth at most ρ if there exists a sequence $\mathcal{X}$ of balls of radius *exactly* ρ in G so that, for any vertex $v \in V$ (resp., for any edge $e = \{u, v\} \in E$), the balls in $\mathcal{X}$ that contain v (resp. that both contain u and v) induce a nonempty consecutive subsequence. This concept can be seen as a broad generalization of *convex bipartite graphs*: a well-known class of graphs in algorithmic graph theory with strong pathbreadth equal to one [11].

Related work.. Strong pathbreadth is part of the parameters expressible in the rich framework of *acyclic clustering* [9]. We first survey the known results about this framework. Formally, a Robertson and Seymour's *tree decomposition* $(T, \mathcal{X})$ of G is a pair consisting of a tree T and of a family $\mathcal{X} = (X_t)_{t \in V(T)}$ of subsets of V indexed by the nodes of T and satisfying:

- $\bigcup_{t \in V(T)} X_t = V$;
- for any edge $e = \{u, v\} \in E$, there exists a $t \in V(T)$ such that $u, v \in X_t$;
- for any $v \in V$, $\{t \in V(T) \mid v \in X_t\}$ induces a subtree, denoted by T_v, of T.

The sets X_t are called *the bags* of the decomposition [16]. Furthermore, the *length* of $(T, \mathcal{X})$ is the minimum D such that every bag $X_t \in \mathcal{X}$ has weak diameter $\max_{u,v \in X_t} dist_G(u, v) \leq D$. The *breadth* of $(T, \mathcal{X})$ is the minimum R such that every bag $X_t \in \mathcal{X}$ is included in the complete R-neighbourhood $N_G^R[v_t] = \{v \in V \mid dist_G(v, v_t) \leq R\}$ of some vertex $v_t \in V$ (v_t may not be in the bag). An *acyclic* (R, D)*-clustering* of G is a tree decomposition of G with respective length and breadth no more than D and R. Such decompositions can be used in order to obtain compact routing schemes and to solve approximately some distance related problems on graphs (*e.g.*, see [1] and the papers cited therein). More recently, nontrivial relationships between the existence of acyclic (R, D)-clustering and the *treewidth* of a graph have been proved in [5]. The *treewidth* of a graph G is the minimum ω such that G admits a tree decomposition where all the bags have size at most $\omega + 1$; many NP-hard problems on general graphs can be solved in quasi linear-time on bounded-treewidth graphs [2].

The complexity of computing acyclic (R, D)-clusterings has been studied recently. In particular, the *treelength* of a graph G is the minimum D such that it admits an acyclic (D, D)-clustering [6]. The *treebreadth* of G is the minimum R such that it admits an acyclic $(R, 2R)$-clustering [7]. Similarly, the *pathlength* and the *pathbreadth* of G are, respectively, the minimum length and breadth of a *path decomposition* of G — with a path decomposition being any tree decomposition $(T, \mathcal{X})$ where T is prescribed to be a path [8]. Computing any of these four above parameters is NP-hard [10, 15]. These negative results have motivated the study of a new optimization criterion, hopefully easier to compute. Namely, a tree decomposition $(T, \mathcal{X})$ of G has *strong breadth* ρ if every bag $X_t \in \mathcal{X}$ is the complete ρ-neighbourhood $X_t = N_G^\rho[v_t] = \{u \in V \mid dist_G(u, v_t) \leq \rho\}$ for some vertex $v_t \in X_t$. The *strong treebreadth* of G (resp., the *strong pathbreadth* of G) is the minimum ρ such that G admits a tree decomposition (resp., a path decomposition) of strong breadth ρ. Computing the strong treebreadth of a given graph G is still NP-hard [13]. However, it was conjectured by the authors in [13] that

the strong pathbreadth of a given graph can be computed in polynomial time. In this note, we disprove their conjecture.

More precisely, our main result is that deciding whether a graph has strong pathbreadth equal to one is NP-complete (Theorem 1). We do so by reducing from a relative of the Chordal Sandwich problem [12]. Similar reductions have been used for proving the hardness of computing treelength, treebreadth and their path counterparts. However, we use different gadgets in our reduction than those presented in [10, 13, 15]. Furthermore, we use some properties that are specific to strong pathbreadth for proving the correctness of our reduction.

2. The reduction

Let $spb(G)$ denote the strong pathbreadth of a given graph G. This whole section is devoted to prove the following result.

Theorem 1. *Deciding whether $spb(G) = 1$ for a given graph G is NP-complete.*

We need to introduce the following additional terminology and notations. First, we identify any path decomposition $(T, \mathcal{X})$ of G, where $T = (t_1, t_2, \ldots, t_k)$ and $\mathcal{X} = (X_i)_{1 \leq i \leq k}$, with the sequence $(X_1, X_2, \ldots, X_k)$. A *clique-path* is a path decomposition where all the bags are cliques (inducing complete subgraphs), and a graph G is an *interval graph* if and only if it has a clique-path [14]. In order to prove Theorem 1, we reduce from the Intervalizing coloured Graph problem (ICG), defined as follows:

Problem 1 (Intervalizing coloured Graph (ICG))**.**

Input: *A graph $G = (V, E)$; a colouring $c : V \to \{1, \ldots, k\}$.*

Question: *Is there an interval supergraph H of G which is properly coloured by c ?*

ICG is proved to be NP-complete already for $k = 4$ (*i.e.*, c is 4-colouring) [3]. This constant upper-bound on the number of colours used will be a key element in our reduction.

Construction of the graph.. We now describe the reduction from ICG to the recognition of graphs with unit strong pathbreadth. Let $G = (V, E)$ be a graph and $c : V \to \{1, \ldots 4\}$ a proper 4-colouring of G. The graph $G' = (V', E')$ is constructed from G as follows.

- Let $\mathcal{P}$ be the family of all vertex subsets $W \subseteq V$ such that, for every $u, v \in W$, we have $c(u) \neq c(v)$. Note that, since c is a 4-colouring, any such subset W has size $|W| \leq 4$. Hence, we have $|\mathcal{P}| = \mathcal{O}(n^4)$. Furthermore, let K be a set containing a vertex x_W for every subset $W \in \mathcal{P}$. We set $V' = V \cup E \cup K$.

- We make of K a clique of G'. The other edges in E' are:

  1. the edges $\{v, e\}$ for every $v \in V$ and for every $e \in E$ that is incident to v;
  2. the edges $\{v, x_W\}$ for every $W \in \mathcal{P}$ and for every $v \in W$;
  3. finally, the edges $\{e, x_W\}$ for every $W \in \mathcal{P}$ and for every $e = \{u, v\} \in E$ such that $u, v \in W$.

The resulting graph G' can be constructed from G in $\mathcal{O}(n^4)$-time.

Correctness.. The correctness of our reduction follows from the two lemmas that are presented next.

Lemma 2. *For every $G = (V, E)$ and $c : V \to \{1, \ldots, 4\}$ being a yes-instance of* ICG*, the graph $G' = (V', E')$ of the reduction satisfies $spb(G') = 1$.*

Proof. Let H be an interval supergraph of G which is properly coloured by c (such a supergraph exists by the hypothesis). Furthermore, let $(W_1, W_2, \ldots, W_t)$ be a clique-path of H. Observe that, since H is properly coloured by c, we have that, for every $1 \leq i \leq t$, $W_i \in \mathcal{P}$. We claim that $(N_{G'}[x_{W_1}], N_{G'}[x_{W_2}], \ldots, N_{G'}[x_{W_t}])$ is a path decomposition of G'. Since the vertices of K are contained in all the neighbour sets $N_{G'}[x_{W_i}]$, $1 \leq i \leq t$, we only need to verify that $(N_{G'}[x_{W_1}] \setminus K, N_{G'}[x_{W_2}] \setminus K, \ldots, N_{G'}[x_{W_t}] \setminus K)$ is a path decomposition of $G' \setminus K$. For that, let us first observe that, for every $1 \leq i \leq t$, $N_{G'}[x_{W_i}] \cap V = W_i$. Since by the hypothesis $(W_1, W_2, \ldots, W_t)$ is a path decomposition of G, every vertex of V is contained in a nonempty consecutive subsequence of bags. Furthermore, for every $e = \{u, v\} \in E$, there is a nonempty consecutive subsequence of bags that both contain u and v, which is by construction the set of bags $N_{G'}[x_{W_i}]$ containing vertex e. Then, we have that e and its two neighbours u and v are contained in a common bag $N_{G'}[x_{W_i}]$, for some $1 \leq i \leq t$. There are no other vertices and edges in $G' \setminus K$. So, the claim is proved. It directly implies $spb(G') = 1$. □

Lemma 3. *If, given $G = (V, E)$ and $c : V \to \{1, \ldots, 4\}$, the graph G' of the reduction satisfies $spb(G') = 1$, then G and c is a yes-instance of* ICG*.*

Proof. Let $(X_1, X_2, \ldots, X_t)$ be a path decomposition of G' such that: every bag X_i, $1 \leq i \leq t$, is the closed neighbourhood of some vertex of G'; and the number of pairs $u, v \in V$ such that $\{u, v\} \in E$ and u and v are contained in a common bag is maximized. We claim that $(X_1 \cap V, X_2 \cap V, \ldots, X_t \cap V)$ is a path decomposition of G. In order to prove it, it suffices to prove that, for every $e = \{u, v\} \in E$, there is bag that contains both u and v. By contradiction, let $e = \{u, v\} \in E$ be such that u and v are not contained in a common bag. Since $(X_1, X_2, \ldots, X_t)$ is a path decomposition of G', vertex e must be contained in a bag. Furthermore, we have $N_{G'}[e] = \{u, v, e\} \cup \{x_W \in K \mid u, v \in W\}$. In particular, every vertex of $N_{G'}[e] \setminus \{u, v\}$ is adjacent to u and v. Hence, since every bag containing e must be the closed neighbourhood of some vertex in $N_{G'}[e]$, there must be two consecutive bags X_i, X_{i+1} such that $\{X_i, X_{i+1}\} = \{N_{G'}[u], N_{G'}[v]\}$. However, in this situation $X_i \cap X_{i+1} = \{e\} \cup \{x_W \in K \mid u, v \in W\} = N_{G'}[e] \setminus \{u, v\}$. Thus, we could add a new bag $Y = N_{G'}[e]$ between X_i and X_{i+1}, that would increase the number of pairs $u', v' \in V$ such that $\{u', v'\} \in E$ and u' and v' are contained in a common bag. The latter contradicts the maximality of the path decomposition with respect to this property, so, u and v must be contained in a common bag. It proves, as claimed, that $(X_1 \cap V, X_2 \cap V, \ldots, X_t \cap V)$ is a path decomposition of G. Furthermore, by construction we have that for every two vertices $u, v \in V$ such that $c(u) = c(v)$, there can be no common neighbour of u and v in G'. As a result, $X_i \cap V \in \mathcal{P}$ for every $1 \leq i \leq t$.

Finally, let $H = (V, \bigcup_{i=1}^{t} \{\{u, v\} \mid u, v \in X_i\})$. It follows from the above that H is an interval supergraph of G and that c is a proper colouring of H. Thus, G and c is a yes-instance of ICG. □

For every graph $G = (V, E)$ and $c : V \to \{1, \ldots, 4\}$, let $G' = (V', E')$ the graph obtained with our reduction. Combining Lemmas 2 and 3, we have $spb(G') = 1$ if and only if G and c is a yes-instance of ICG. Since ICG is NP-complete and that the graph G' of the reduction can be constructed in polynomial time, it proves Theorem 1.

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