Surface derivatives computation using Fourier Transform

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Résumé

We present a method for computing high order derivatives on a smooth surface $\mathscr{S}$ at a point $\mathbf{p}$ by analyzing the vibrations of the surface along circles in the tangent plane, centered at $\mathbf{p}$. By computing the Discrete Fourier Transform of the deviation of $\mathscr{S}$ from the tangent plane restricted to those circles, a linear relation between the Fourier coefficients and the derivatives can be expressed. Thus, given a smooth scalar field defined on the surface, all its derivatives at $\mathbf{p}$ can be computed simultaneously. The originality of this method is that no direct derivation process is applied to the data. Instead, integration is performed through the Discrete Fourier Transform, and the result is expressed as a one dimensional polynomial. We derive two applications of our framework namely normal correction and curvature estimation which we demonstrate on synthetic and real data.

Mots clé : Geometry Processing, Curvature Computation

1. Introduction

Local description of a surface gives important information that can help measure similarities between surfaces. To do so, one has to find a way to robustly represent the data. Such a representation should, for example, be rotational or scale invariant. One of the simplest representations is a height field over a tangent plane.

In this paper, we present a novel way to locally represent a shape, giving links between our representation and the high order derivatives of the function. The surface is represented locally as one-dimensional signals corresponding to the surface height over the tangent plane sampled on concentric circles. This construction shares some properties with Zernike polynomial introduced in [Zer34], and will allow a harmonic interpretation of higher order derivatives of two-dimensional functions.

Contributions To summarize, our contributions are the following:

- A method to estimate derivatives based on the Fourier Transform of geodesic circles.
- An application to correct normals from an input point set or estimate the curvatures.

2. Previous work

2.1. Using waves to describe a surface

Some work has already been done in looking on interpreting surfaces around a point as a vibrating function. [MT98] expressed the normal curvature as a periodic function depending on the coefficients of the first fundamental harmonic to design a measurement of how smooth second order derivatives field is (which they call second order smoothness). A similar processing applied to the variation of the normal curvature yielded a third order smoothness. [JS10] gave an intuitive interpretation of the behavior of the third order derivative of a function by splitting its height field as a sum of cosines and computing its Fourier Transform along a circle around a point. [JS10] claimed that the same process could be used to compute higher order derivatives. However, in order to do so, the Fourier Transform must be computed on multiple concentric circles to extract frequencies evolution with regards to the radius (which is polynomial). Zernike polynomials form an orthogonal basis of two-dimensional polynomials in polar coordinates. They were introduced by [Zer34] to efficiently correct optical aberrations of lenses. Each Zernike polynomial is formed as the multiplication of a one-dimensional polynomial of a certain order with the radius as variable, multiplied by the cosine of the phase a given frequency. Projecting the height field of the neighborhood of a given point on this basis gives a rotational invariant descriptor used by [MPVF11].

2.2. Curvature estimation

Curvature estimation by local analysis of neighborhoods has been studied extensively [CP03]. The major focus is to avoid the derivation of the surface, since derivation of a noisy measure leads to instabilities. Instead, the derivatives are estimated by local integration. This process is usually limited to second order derivatives, yielding the curvature. Integral invariants, [PWY*07] and [PWHY09], use local area and volume computation to estimate the curvature. [PGK02] uses the local covariance of points to estimate the *surface variation*, a curvature-like measure of the surface. [DMSL11] demonstrated the link between projection on the regression

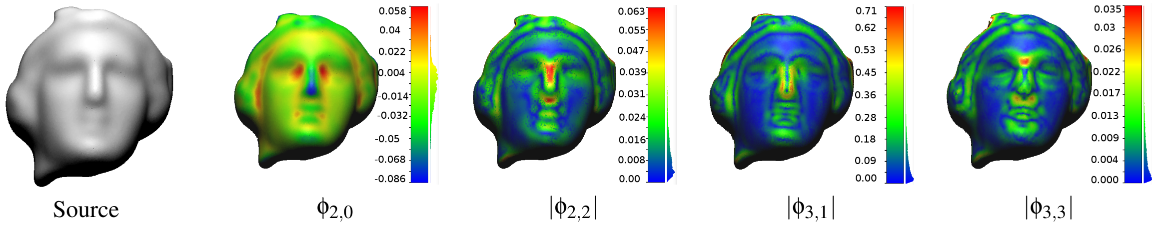


Figure 1: *Computation of some values of $\phi_{n,k}$ on a blurred face model of Caesar. $|\phi_{2,2}|$ is globally smooth but there remains some outliers.*

plane and mean curvature. However, none of these methods directly allowed for higher order derivatives computation.

3. Derivative Computation

3.1. Formulation

Let $\mathbf{p}$ be a point from a smooth surface $\mathscr{S}$. Given a parameterization $(\mathbf{x}, \mathbf{y})$ in the tangent plane at point $\mathbf{p}$, the deviation of $\mathscr{S}$ has a Taylor expansion $f_{\mathbf{p}}(x,y)$ from the tangent plane that depends on all the derivatives of $f_{\mathbf{p}}$. For simplification, let us remove the subscript $\mathbf{p}$ in the rest of the article:

$$f(x,y) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{f_{\mathbf{x}^{k-j}\mathbf{y}^j}}{(k-j)!\,j!} x^{k-j} y^j \tag{1}$$

Where $f_{\mathbf{x}^{k-j}\mathbf{y}^j}$ is the j^{th} cross derivative of order k of f. Let us use polar coordinates such that $(x,y) = (r\cos\theta, r\sin\theta)$. Let $f_r(\theta)$ be the restriction of f along a circle of radius r. Using trigonometric properties, the periodic function $f_r(\theta)$ can be written as follows:

$$f_r(\theta) = \sum_{n=0}^{\infty} |s_{r,n}(f)| \cos(n\theta + \theta_{r,n}(f)) \tag{2}$$

Where $s_{r,n}(f) \in \mathbb{C}$ is related to the n^{th} frequency of the Discrete Fourier Transform of $f_r(\theta)$, and where $\theta_{r,n}(f)$ is its phase. One can prove that the expression of $s_{2n}(r,f)$ (resp. $s_{2n+1}(r,f)$) is an even (resp. odd) polynomial of r.

$$\begin{aligned} s_{r,2n}(f) &= \sum_{k=0}^{\infty} r^{2k} \phi_{2k,2n}(f) \\ s_{r,2n+1}(f) &= \sum_{k=0}^{\infty} r^{2k+1} \phi_{2k+1,2n+1}(f) \end{aligned} \tag{3}$$

Where $\phi_{k,n}(f) \in \mathbb{C}$ represents the part of $s_{r,n}(f)$ belonging to the k^{th} derivative of f. $\mathrm{Re}(\phi_{k,n}(f))$ (resp. $\mathrm{Im}(\phi_{k,n}(f))$ is a linear combinaison of the even (resp. odd) k^{th} order cross-derivatives of f which can be determined for any k.

Let us evaluate $s_{r,n}(f)$ for $N+1$ different radius r_i, limiting the maximum order to $2k+1 = N_{\text{order}} \leq N$ and the highest frequency to $2n+1 = N_{\text{freq}} \leq N_{\text{order}}$. Let us write the following matrices:

- $V = \left(r_i^{2k}\right)_{i,k} \in \mathcal{M}_{N,N_{\text{order}}}(\mathbb{R})$
- $D = \mathrm{diag}\,(r_i) \in \mathcal{M}_{N,N}(\mathbb{R})$
- $S = (s_{r_i,n})_{i,n} \in \mathcal{M}_{N,2N_{\text{freq}}}(\mathbb{C}$
- $S_{\text{even}} = \left(s_{r_i,2n}\right)_{i,n} \in \mathcal{M}_{N,N_{\text{freq}}}(\mathbb{C})$
- $S_{\text{odd}} = \left(s_{r_i,2n+1}\right)_{i,n} \in \mathcal{M}_{N,N_{\text{freq}}}(\mathbb{C})$
- $\Phi = \left(\phi_{k,n}\right)_{k,n} \in \mathcal{M}_{2N_{\text{order}},2N_{\text{freq}}}(\mathbb{C})$
- $\Phi_{\text{even}} = \left(\phi_{2k,2n}\right)_{k,n} \in \mathcal{M}_{N_{\text{order}},N_{\text{freq}}}(\mathbb{C})$
- $\Phi_{\text{odd}} = \left(\phi_{2k+1,2n+1}\right)_{k,n} \in \mathcal{M}_{N_{\text{order}},N_{\text{freq}}}(\mathbb{C})$

Where V is a Vandermonde matrix of the squared radius and $\mathcal{M}_{m,n}(\mathbb{K})$ is the space of matrices of dimension $m \times n$ on a body $\mathbb{K}$. Equation (3) computed with different radius r_i can now be written as the following system of equations:

$$S_{\text{even}} = V\Phi_{\text{even}}, \quad S_{\text{odd}} = DV\Phi_{\text{odd}} \tag{4}$$

Since V and D are invertible by construction if $N = N_{\text{order}} = N_{\text{freq}}$, Φ can be recovered from S. If the equality $N = N_{\text{order}} = N_{\text{freq}}$ does not hold, the system is over determined and the pseudo-inverse of V is computed. Additionally, one can show that $\phi_{k,n} = 0$ if $k < n$ because the linearization of the power of k of a cosine only contains frequencies n lower than k, thus Φ is an upper triangular matrix. Some values of $\phi_{k,n}$ computed on a shape are shown on figure 1. Let Φ_k be the column of Φ representing the k^{th} order of f, restricted only to nonzero coefficients (it is then of dimension $\mathrm{Int}(k/2)+1$). Let us define the vector $X_k(f)$ as follows:

$$X_k(f) = \left(\frac{f_{\mathbf{x}^{k-2j}\mathbf{y}^{2j}}}{(k\text{-}2j)!(2j)!} + \boldsymbol{i} \frac{f_{\mathbf{x}^{k-2j-1}\mathbf{y}^{2j+1}}}{(k\text{-}2j\text{-}1)!(2j\text{+}1)!} \right)_{2j+1 \leq k}$$

$X_k(f)$ contains all the cross-derivatives of order k of f. Let A_k be the complex matrix in which the real (resp. imaginary) part represents the linear transform between the real (resp. imaginary) part of X_k and Φ_k. We can prove that A_k exists, so we have:

$$\begin{aligned} \mathrm{Re}(\Phi_k) &= \mathrm{Re}(A_k)\mathrm{Re}(X_k) \\ \mathrm{Im}(\Phi_k) &= \mathrm{Im}(A_k)\mathrm{Im}(X_k) \end{aligned} \tag{5}$$

A_k has a closed form [†] . It is also a matrix that is independant of f. We have not proved yet that its real and imaginary parts are invertible for any order k, but in our experiments, we could successfully compute those inverses for the first terms.

3.2. Computing matrix S

First of all, the input data treated in this paper is a point cloud. Neighborhoods are computed using a kd-tree representation of the data.

[†] https://liris.cnrs.fr/ybearzi/appendix.pdf

In order to compute Φ at point $\mathbf{p}$ on a surface $\mathscr{S}$, the matrix S is needed. The tangent plane at $\mathbf{p}$ is estimated and points lying on circles centered at $\mathbf{p}$ with growing radius are computed. For those points the height function of the surface $\mathscr{S}$ over the tangent plane must be interpolated.

Let $\mathcal{C}_{r_i}$ be the sample points to be interpolated on the circle of radius r_i and $\mathcal{P} = (\mathbf{p}_\ell)$ be the set of points near $\mathbf{p}$ used for interpolating the circles. Let $\mathcal{I}$ be the interpolator such that $\mathcal{I}(\mathcal{P}, \mathcal{C}_{r_i})$ gives the interpolated points along the circle $\mathcal{C}_{r_i}$. Let $\mathcal{F}$ be the Discrete Fourier Transform operator extracting only the coefficient of frequency zero and the double of strictly positive frequency coefficients. The computation of the i^{th} line S_i of S can be computed as follows:

$$S_i = \mathcal{F}(\mathcal{I}(\mathcal{P}, \mathcal{C}_{r_i})) \tag{6}$$

In the experimental results, the interpolation method used is Kriging, which is a geostatistical method developed in [Mat60] to interpolate the concentration of precious rocks in an area given known concentration on certain locations. Kriging interpolation computes the height of a point at position (x,y) in the tangent plane as a linear combination of the heights of the sample data. Importantly enough, the weights in the linear combination sum to 1 but are not necessarily nonnegative. Kriging extracts the weights so the variance of the estimated values with regards to the sample data is minimal. It needs a semivariogram model that we set to be quadratic.

4. Interpretation

In the previous section, we showed a relationship between the coefficients of the Discrete Fourier Transform applied to restrictions of f along concentric circles and its k^{th} ordered derivatives. Each k^{th} order derivative of f can be interpreted as a vibration of the surface around $\mathbf{p}$ which has at most k zeros. Each k^{th} coefficient, supported by a monomial of order k, is linearly blended in the coefficients of the Fourier series of the height field at constant radius from $\mathbf{p}$ due to De Moivre's formulas. The decomposition to Φ_{even} and Φ_{odd} respectively splits the signal around $\mathbf{p}$ into symmetric and antisymmetric parts.

Hence, we can now interpret the k^{th} order cross derivatives as the sum of cosines of frequencies sharing the same parity as k. The linear combination of those cosines will define the principal directions of the k^{th} order derivative, which are at most k and located at the maxima of the vibrating signal. The coefficients of high order derivatives can be put into symmetric supermatrices like proposed by [Qi07]. Supermatrices are an extension of matrices to higher dimensions. The principal directions of order k are equal to the Z-eigenvectors (for real eigenvectors) of the symmetric supermatrix coefficient of order k which are discussed in [Qi07]. Principal directions of high order then have some properties linked to the behavior of Z-eigenvectors, and more generally E-eigenvectors (including complex eigenvectors called recession vectors). Finding the maxima of the linear combination of cosines related to order k is equivalent to finding the Z-eigenvectors of order k.

5. Applications

5.1. Estimation of the k^{th} order derivatives of a scalar function g on a surface $\mathscr{S}$

The computation of Φ was given in the context of estimating the geometric properties of $\mathscr{S}$ at point $\mathbf{p}$, studying the deviation f from the tangent plane of $\mathscr{S}$ around $\mathbf{p}$. Given a smooth scalar function g lying on $\mathscr{S}$, $\Phi(g)$ can be computed as well, giving an estimation of all the k^{th} order derivatives of g at point $\mathbf{p}$.

5.2. Curvature computation

The Gaussian curvature K and the mean curvature H can be computed directly from Φ. For this purpose, the relationship between Φ_1 and $(g_{\mathbf{x}}, g_{\mathbf{y}})$ and the relationship between Φ_2 and $(g_{\mathbf{xx}}, g_{\mathbf{xy}}, g_{\mathbf{yy}})$ must be known:

$$g_{\mathbf{x}} = \mathrm{Re}(\phi_{1,1}(g)), \quad g_{\mathbf{y}} = \mathrm{Im}(\phi_{1,1}(g)) \tag{7}$$

$$\begin{aligned} g_{\mathbf{xx}} &= 2(\phi_{2,0}(g) + \mathrm{Re}(\phi_{2,2}(g))) \\ g_{\mathbf{xy}} &= 2\mathrm{Im}(\phi_{2,2}(g)) \\ g_{\mathbf{yy}} &= 2(\phi_{2,0}(g) - \mathrm{Re}(\phi_{2,2}(g))) \end{aligned} \tag{8}$$

The computation of H and K is then straightforward:

$$\begin{aligned} K(g) &= \frac{g_{\mathbf{xx}} g_{\mathbf{yy}} - g_{\mathbf{xy}}^2}{(1 + g_{\mathbf{x}}^2 + g_{\mathbf{y}}^2)^2} \\ K &= 4 \times \frac{\phi_{2,0}^2 - |\phi_{2,2}|^2}{(1 + |\phi_{1,1}|)^2} \end{aligned} \tag{9}$$

Since $\phi_{1,1}(f) = 0$:

$$K(f) = 4\left(\phi_{2,0}(f)^2 - |\phi_{2,2}(f)|^2\right) \tag{10}$$

Similarly, the mean curvature has the following form:

$$\begin{aligned} H(g) &= \frac{\left(1 + g_{\mathbf{x}}^2\right) g_{\mathbf{xx}} + \left(1 + g_{\mathbf{y}}^2\right) g_{\mathbf{yy}} - 2 g_{\mathbf{x}} g_{\mathbf{y}} g_{\mathbf{yy}}}{2\left(1 + g_{\mathbf{x}}^2 + g_{\mathbf{y}}^2\right)^{\frac{3}{2}}} \\ H &= 2 \times \frac{\phi_{2,0}\left(1 + |\phi_{1,1}|^2\right) + \mathrm{Re}\left(\overline{\phi_{2,2}} \phi_{1,1}\right)}{\left(1 + |\phi_{1,1}|^2\right)^{\frac{3}{2}}} \\ &= 2 \times \frac{\phi_{2,0} + |\phi_{1,1}|^2 \left(\phi_{2,0} + |\phi_{2,2}| \cos(2\varphi_{1,1} - \varphi_{2,2})\right)}{\left(1 + |\phi_{1,1}|^2\right)^{\frac{3}{2}}} \end{aligned} \tag{11}$$

Where $\varphi_{k,n}$ is the phase of $\phi_{k,n}$. In the case $g = f$, equation (11) becomes:

$$H(f) = 2\phi_{2,0}(f) \tag{12}$$

5.3. Normal correction

Since $\phi_{1,1}$ gives an estimate of the first derivatives of f, it should be equal to zero since f lies in the tangent plane. Having $\phi_{1,1} \neq 0$ implies that the current normal $\boldsymbol{n}$ at point $\mathbf{p}$ is incorrect. Figure 2 illustrates the fact that noisy normals mainly influence $\phi_{1,1}$ opposed to $\phi_{k,n}$, showing that our computation separates different orders well. Using the localization of the noise in $\phi_{1,1}$, we propose a normal correction algorithm. The frame used to describe $f(x,y)$ is $\mathcal{R} = (\mathbf{x}, \mathbf{y}, \boldsymbol{n})$. To correct the false normal $\boldsymbol{n}$ to the correct normal $\tilde{\boldsymbol{n}}$, one needs a rotation matrix $R_{\beta,\boldsymbol{u}}$ of axis $\boldsymbol{u}$ and angle β such that:

$$\tilde{\boldsymbol{n}} = R_{\beta,\boldsymbol{u}} \boldsymbol{n} \tag{13}$$

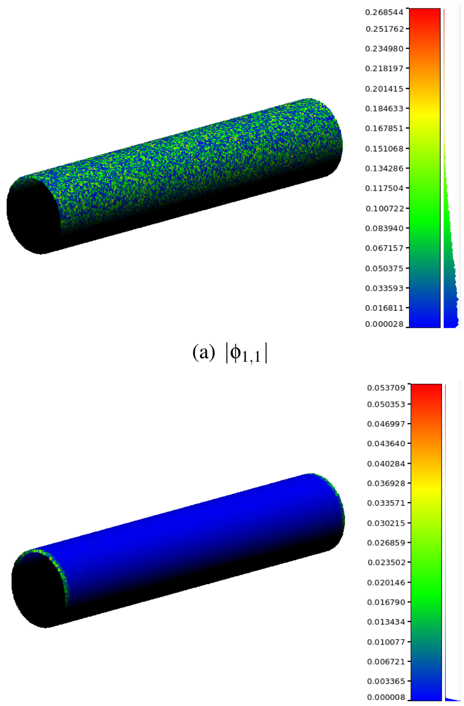
Since $\phi_{1,1} = f_\mathbf{x} + \boldsymbol{i} f_\mathbf{y}$, the module of $\phi_{1,1}$ gives information about the angle error and its phase gives information about the axis of rotation. $\boldsymbol{u}$ and β have the following relationship with $\phi_{1,1}$:

$$\begin{aligned} \beta &= \tan^{-1} |\phi_{1,1}| \\ \boldsymbol{u} &= \frac{1}{|\phi_{1,1}|} \mathcal{R} \times \left(-\mathrm{Im}(\phi_{1,1}), \mathrm{Re}(\phi_{1,1}), 0\right)^T \end{aligned} \tag{14}$$

In table 1 is displayed the mean error of the corrected normal on a cylinder with its theoritical normal.

noise std	$\frac{\pi}{45}$	$\frac{\pi}{18}$	$\frac{\pi}{9}$
error	0.067 deg	0.076 deg	0.1 deg

Table 1: *Mean error after normal correction with different input normal noises.*



(a) $|\phi_{1,1}|$

(b) $|\phi_{3,1}|$

Figure 2: $|\phi_{1,1}|$ *and* $|\phi_{3,1}|$ *computed on a cylinder of radius* 10 *with added gaussian noise on the normal direction (standard deviation of* $\frac{\pi}{45}$.*)*

6. Conclusion

We presented a method to simultaneously extract all k^{th} order derivatives of a surface $\mathscr{S}$ or any smooth scalar field defined on $\mathscr{S}$ at a point $\mathbf{p}$. To do so, we showed a link between a radial vibration interpretation of $\mathscr{S}$ around a point $\mathbf{p}$ and its derivatives at $\mathbf{p}$. Each k^{th} order derivative of $\mathscr{S}$ can be seen as a sum of vibrating modes of frequencies $n \leq k$ sharing the parity of k. A direct application of this work is to estimate Gaussian and Mean curvatures and correct a poor normal estimation.

In the future, we plan to extend this result to position noise instead of pure normal noise.

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