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SPECTRAL INEQUALITIES FOR THE CONTROL OF LINEAR PDES

by

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Abstract. — Some spectral inequalities were introduced in control theory by David Russell and George Weiss in 1994 [**21**] as an infinite-dimensional version of the Hautus test for controllability. They are an efficient tool for the control of the linear Schrödinger equation in arbitrary time from a localized source term as proved by Nicolas Burq and Maciej Zworski in 2004 [**1**] using the unitarity of the Fourier transform in Hilbert spaces. They also allow to analyze which filtering scale is sufficient to discretize this equation in space, as initiated by Sylvain Ervedoza in 2008 [**5**]. A parallel approach to the control of the linear heat equation in arbitrary time from a localized source term has developed. It starts from another type of spectral inequality introduced by Gilles Lebeau in the late 90s and follows the strategy devised in 1995 by Lebeau and Robbiano [**9**]. This paper will connect these two spectral approaches, compare the control of the Schrödinger group and the heat semigroup at the level of abstract functional analysis, and illustrate this with examples of PDE problems. It is mainly based on a joint work [**4**] with Thomas Duyckaerts (Université Paris 13).

Contents

1. Background on the interior control of linear PDEs

This paper mainly discusses elementary functional analytic tools for studying the controllability properties of basic linear evolutions generated in various ways from a partial differential operator. The main example of operator to keep in mind is Δ,

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the Laplacian on a smooth domain M in $\mathbb{R}^d$ *with Dirichlet boundary condition* (n.b. this operator must be independent of time, but may vary with the position, e.g. the Laplacian on a Riemannian manifold).

▶ *Beware*: Ω denotes both a non-empty open *subset* of M and the multiplication *operator* by the characteristic function of this subset, i.e. $(\Omega v)(x) = \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega. \end{cases}$

Since the norm on $L^2(M)$ is $\|v\|^2 = \int_M |v(x)|^2 dx$, we have $\|\Omega v\|^2 = \int_\Omega |v(x)|^2 dx$.

1.1. Duality between control and observation. — Our controllability properties are equivalent to a priori estimates on the free evolution, known as observability inequalities. The main example to keep in mind is the control of the temperature f in the domain M, with Dirichlet boundary condition, from a chosen source u acting in the non-empty open control set $\Omega \subset M$ during a time T.

In this example, writing the heat diffusion equation as an O.D.E. in the state space $\mathcal{E} = L^2(M)$ with input $u \in L^2(\mathbb{R}; \mathcal{E})$, $\partial_t f - \Delta f = \Omega u$, the classical duality theorem is:

Null-controllability in time T (at cost κ_T) of $\partial_t f - \Delta f = \Omega u$, i.e.

$$\forall f(0) \in \mathcal{E}, \quad \exists u, \quad f(T) = 0 \text{ and } \int_0^T \|u(t)\|^2 dt \leqslant \kappa_T \|f(0)\|^2.$$

$\Leftrightarrow$ *Final-observability* in time T (at cost κ_T):

$$\|e^{T\Delta} v\|^2 \leqslant \kappa_T \int_0^T \|\Omega e^{t\Delta} v\|^2 dt, \quad v \in \mathcal{E}.$$

Here $t \mapsto f(t) = e^{t\Delta} v$ is the solution of the free heat O.D.E.: $\partial_t f - \Delta f = 0$, $f(0) = v$. Together with the background uniqueness estimate $\|v\| \leqslant e^{TF(v)^2} \|e^{T\Delta} v\|$, for all $T \geqslant 0$, non-null $v \in H_0^1(M)$ and frequency function $F(v) = \|\nabla v\| / \|v\|$ (an easy consequence of the log-convexity method), this inequality quantifies unique continuation: $f = 0$ on the observed subset $[0, T] \times \Omega \Rightarrow f = 0$ on the whole space $[0, +\infty) \times M$.

1.2. Links between heat/Schrödinger/waves controllability. — From the observation point of view, a single PDE operator, Δ, defines various dynamics, which may be locally observed through the same operator, Ω.

The control of the Schrödinger equation is generally considered as "in between" the better understood cases of the heat and wave equations. The "big picture" is:

Controllabilty of		Restriction on Ω	Restriction on T
Heat eq.	$\partial_t f - \Delta f = \Omega u$	No	No
Schrödinger eq.	$i\partial_t \psi - \Delta \psi = \Omega u$	Yes	No
Wave eq.	$\partial_t^2 w - \Delta w = \Omega u$	Yes	Yes

Rigorous proofs of these implications have been given at various levels of generality:

1. $\exists T$, wave control $\Rightarrow \forall T$, heat control
   (by the control transmutation method, cf. Russell [**20**], Phung [**17**], Miller [**12**]).
2. $\exists T$, wave control $\Rightarrow \forall T$, Schrödinger control
   (by resolvent conditions, cf. Liu [**10**], Miller [**11**], Tucsnak-Weiss [**25**]).

3. $\exists T$, wave control $\Leftrightarrow$ $\exists T$, wave group (or half wave) control: $i\partial_t\psi + \sqrt{-\Delta}\psi = \Omega u$ (by resolvent conditions, cf. Miller [**13**])

This leads to the new question : Schrödinger control $\Rightarrow$ heat control ? We answer *No* (cf. §4.2), *But almost* (cf. the more precise theorem 2.5), in the following sense:

Schrödinger control $\Rightarrow$ control of fractional diffusion $\partial_t f + (-\Delta)^s f = \Omega u$, $s > 1$.

1.3. Abstract semigroup framework: $t \mapsto e^{-tA}$ observed by C.—

We state our results in the semigroup notations (cf. [**25**]) which best delineates their nature. Now $\mathcal{E}$ (states) and $\mathcal{F}$ (observations) are two Hilbert spaces, $-A$ is the generator of a strongly continuous semigroup $t \mapsto e^{-tA}$ on $\mathcal{E}$, $C \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ is the bounded operator which defines what is observed. Its adjoint C^* defines how the input $u : t \mapsto \mathcal{F}$ acts in order to control, in the duality theorem which now writes:

Null-controllability in time T (at cost κ_T) of $\partial_t f + A^* f = C^* u$, with $u \in L^2(\mathbb{R}; \mathcal{F})$:

$$\forall f(0) \in \mathcal{E}, \quad \exists u, \quad f(T) = 0 \text{ and } \int_0^T \|u(t)\|^2 dt \leqslant \kappa_T \|f(0)\|^2.$$

$\Leftrightarrow$ *Final-observability* in time T (at cost κ_T):

$$(FinalObs) \quad \|e^{-TA}v\|^2 \leqslant \kappa_T \int_0^T \|Ce^{-tA}v\|^2 dt, \quad v \in \mathcal{E}.$$

As before, this inequality is the continuous prediction of the final state from the observation of the dynamics through C between the initial and final times, and the function $t \mapsto f(t) = e^{-tA}v$ is the solution of the free O.D.E.: $\partial_t f + Af = 0$, $f(0) = v$.

The heat problem on the domain M observed on $\Omega \subset M$ considered in §1.1 was the special case $A = -\Delta$, $\mathcal{E} = \mathcal{F} = L^2(M)$, $\mathcal{D}(A) = H^2(M) \cap H^1_0(M)$, $C = \Omega$.

2. Resolvent conditions for parabolic equations (main results)

It is customary in the spectral theory of elliptic boundary value problems to ressort to the corresponding parabolic or hyperbolic dynamics (e.g. to obtain precise results on eigenvalues asymptotics using Tauberian theorems). Conversely, it is customary to characterize the well-posedness, regularity and growth properties of a semigroup by spectral conditions on its generator. The topic of this paper is a similar stationary approach to the study of observability properties through C of the various dynamics generated by A.

Going from spectral to dynamic inequalities, the key is often the unitarity of the Fourier transform in Hilbert spaces. E.g. in the Greiner-Huang-Prüss'84 [**6, 19**] test:

Uniform (hence exponential) stability of $t \mapsto e^{-tA}$, i.e. $\exists \varepsilon > 0$,

$$e^{\varepsilon t}\|e^{-At}\| \to 0, \quad \text{as } t \to +\infty,$$

$\Leftrightarrow$ *Uniform estimate of the resolvent* in the left half-space, i.e. $\exists m > 0$,

$$\|(A - \lambda)^{-1}\| \leqslant m, \quad \operatorname{Re}\lambda < 0.$$

2.1. Resolvent conditions for control (infinite dimensional Hautus tests). — Before addressing the new parabolic case [**4**] which applies to the heat equation, we recall a few facts about the unitary case [**16**] which applies to Schrödinger and waves equations. Here $A = A^*$ (self-adjoint), hence $(e^{itA})^* = e^{-itA}$ and $\|e^{itA}v\| = \|v\|$ (conservation law). Therefore (exact) observability in time T (at cost κ_T) writes:

$$(\mathit{ExactObs}) \quad \|v\|^2 = \|e^{iTA}v\|^2 \leqslant \kappa_T \int_0^T \|Ce^{itA}v\|^2 dt, \quad v \in \mathcal{E}.$$

E.g. the interior control problem for the Schrödinger equation on M observed on Ω is the special case $A = -\Delta$, $\mathcal{E} = \mathcal{F} = L^2(M)$, $\mathcal{D}(A) = H^2(M) \cap H^1_0(M)$, $C = \Omega$.

Theorem 2.1 **(M.[11][(2)]: resolvent condition to observe a unitary group)** *The above observability of $t \mapsto e^{itA}$, $A = A^*$, by C, for some T, is equivalent to*

$$(\mathit{Res}) \quad \|v\|^2 \leqslant m\|(A-\lambda)v\|^2 + \tilde{m}\|Cv\|^2, \quad v \in \mathcal{D}(A), \quad \lambda \in \mathbb{R}.$$

In particular, (Res) implies $(\mathit{ExactObs})$ for $T > \pi\sqrt{m}$ and $\kappa_T = 2\tilde{m}T/(T^2 - m\pi^2)$.

Our main result uses (cf. §4.1): if $m \sim \tilde{m}$, then $T \sim \kappa_T \sim \sqrt{m}$ in $(\mathit{ExactObs})$.

Similar inequalities first appeared as necessary conditions for infinite time controllability of stable semigroups in Russel-Weiss'94 [**21**]; in Zhou-Yamamoto'97, the detour through Huang-Prüss limits generality and forbids specifying a time and cost; the key idea for $\Rightarrow$ is using the unitarity of the Fourier transform as in Burq-Zworski'04 [**1**]. This theorem was extended to more general semigroups in [**7**]. We refer to [**5, 16**] for applications to discretization, but we envision applications to other approximation asymptotics such as homogenization and singular perturbation.

A similar theorem for second-order equations like the wave equation has been proved in increasing generality in Liu'97 [**10**] (by Huang-Prüss), M'05 [**11**] ($\Leftarrow$), Ramdani-Takahashi-Tenenbaum-Tucsnak'05 [**24**] (compact resolvent), M'12 [**12**]. Here $A > 0$ (positive self-adjoint). Control of $\partial_t^2 w + Aw = C^*u$ for some $T \Leftrightarrow$

$$\|v\|^2 \leqslant \frac{m}{\lambda}\|(A-\lambda)v\|^2 + \tilde{m}\|Cv\|^2, \quad v \in \mathcal{D}(A), \quad \lambda \in \mathbb{R}^*.$$

N.b. such a resolvent condition with variable coefficients is recovered, up to factor 4, from its restriction to the spectrum $\sigma(A)$: if $\lambda \notin \sigma(A)$, then $\|(A-\lambda)^{-1}\| \leqslant |\mu - \lambda|^{-1}$, with $\mu \in \sigma(A)$ closest to λ; hence $\|(A-\mu)v\| \leqslant \|(A-\lambda)v\| + \|(\lambda-\mu)v\| \leqslant 2\|(A-\lambda)v\|$. Thus, resolvent conditions such as (PowRes) below can be restricted equivalently to $\lambda \geqslant \inf A > 0$, where $\inf A = \inf \sigma(A)$ denotes the bottom of the spectrum of $A > 0$.

A final fact to be aware of in order to appreciate our next result is that $(\mathit{ExactObs})$ holds for all time $T > 0$ if the coefficient m in (Res) is a function which tends to 0 as $\lambda \to +\infty$ (as in (PowRes) below), and if $A > 0$ has discrete spectrum (cf. [**25, 16**]).

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(2) The proof has already been repeated in [**25, 16**] and the survey, relevant to our topic, by C. Laurent, *Internal control of the Schrödinger equation,* Math. Control Relat. Fields **4** (2014) 161–186.

2.2. Sufficient resolvent conditions for $t \mapsto e^{-tA}$, $A > 0$. — Here A is positive self-adjoint (parabolic evolution); C is bounded (or admissible to some degree, [**4**]).

Theorem 2.2 **(Duyckaerts-M. [4]: Main Result).** — *Let $A > 0$, $C \in \mathcal{L}(\mathcal{E}, \mathcal{F})$. If the resolvent condition with power-law factor: $\exists m > 0$,*

$$(PowRes) \quad \|v\|^2 \leqslant m\lambda^\delta \left(\frac{1}{\lambda} \|(A-\lambda)v\|^2 + \|Cv\|^2 \right), \quad v \in \mathcal{D}(A), \quad \lambda > 0,$$

holds for some $\delta \in [0,1)$, then observability of $t \mapsto e^{-tA}$ from C, $(FinalObs)$, holds for all $T > 0$ with the control cost estimate $\kappa_T \leqslant ce^{c/T^\beta}$ for $\beta = \frac{1+\delta}{1-\delta}$ and some $c > 0$.

The previous links can be restated as : $(ExactObs)$ for "Schrödinger" $\partial_t \psi - iA\psi = 0$ $\Rightarrow \delta = 1$ in $(PowRes) \Rightarrow \delta = 0$ in $(PowRes) \Leftrightarrow (ExactObs)$ for "wave" $\partial_t^2 w + Aw = 0$.

Thus the theorem deduces final-observability for the "heat" equation $\partial_t f + Af = 0$ from a scale of resolvent conditions ranging between the exact-observability of the "wave" equation $\partial_t^2 w + Aw = 0$ (equivalent to $\delta = 0$), down to and excluding exact-observability of the "Schrödinger" equation $\partial_t \psi - iA\psi = 0$ (since $\delta = 1$ is excluded). The counter-example in §4.2 proves that it cannot be included indeed.

The next corollary draws consequences of the observability of the "Schrödinger" equation, $\partial_t \psi - iA\psi = 0$, on the observability of diffusions generated by powers of A. The counter-example excludes the classic diffusion $\partial_t f + Af = 0$ (the power $\gamma = 1$).

Corollary 2.3 **(Duyckaerts-M. [4]: "Schrödinger" to "fractional diffusion")**
If $(ExactObs)$ for the unitary group $t \mapsto e^{itA}$, $A > 0$, holds for some T,
then $(FinalObs)$ for "higher-order diffusion" $t \mapsto e^{-tA^\gamma}$, $\gamma > 1$ holds for all T.

2.3. Improved sufficient resolvent conditions for $t \mapsto e^{-tA}$, $A > 0$. — Here A is positive self-adjoint and C is bounded (or admissible for "wave" $\partial_t^2 w + Aw = 0$).

The optimality of the theorems in the previous paragraph in the range of power-laws, excluding the critical linear law, leads to investigate their validity for asymptotics closer to the linear law. Thus, in our main theorem 2.2, we now improve the factor λ^δ, $\delta < 1$, into $\lambda/(\varphi(\lambda))^2$, with $\varphi(\lambda) = (\log(\lambda+1))^\alpha$, $\alpha > 1$.

Theorem 2.4 **(Duyckaerts-M. [4]: Main Result, log-improved)**
If the resolvent condition with logarithmic factor: $\exists m > 0$,

$$\|v\|^2 \leqslant \frac{m\lambda}{(\varphi(\lambda))^2} \left(\frac{1}{\lambda} \|(A-\lambda)v\|^2 + \|Cv\|^2 \right), \quad v \in \mathcal{D}(A), \quad \lambda > 0,$$

holds for some $\alpha > 1$, then observability $(FinalObs)$ holds for all $T > 0$.

Corollary 2.5 **(Duyckaerts-M. [4]: from "Schrödinger" to "*log*-diffusion")**
If $(ExactObs)$ for the unitary group $t \mapsto e^{itA}$ holds for some T,
then $(FinalObs)$ for "higher-order diffusion" $t \mapsto e^{-tA\varphi(1+A)}$, for all $\alpha > 1$, $T > 0$.

These logarithmic improvements of the theorems of the previous paragraph are obtained by a logarithmic improvement of the Lebeau-Robbiano strategy (cf. §3.4). Unfortunately, by weakening the assumptions we loose track of the control cost.

2.4. Application to the control of diffusions in a potential well. —

$$\text{Here } A = -\Delta + V \text{ on } \mathcal{E} = L^2(\mathbb{R}), \quad \mathcal{D}(A) = \left\{u \in H^2(\mathbb{R}) \mid Vu \in L^2(\mathbb{R})\right\};$$
$$V(x) = x^{2k},\ k \in \mathbb{N},\ k > 0; \quad C = \Omega = (-\infty, x_0),\ x_0 \in \mathbb{R}.$$

Theorem 2.6 **(M. [15]).** — *With the above notations, the resolvent condition*

$$\|v\|^2 \leqslant m\lambda^{1/k}\left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v \in \mathcal{D}(A), \quad \lambda > 0,$$

holds and the decay of the first coefficient $m/\lambda^{1-1/k}$ cannot be improved.

Applying our main theorem 2.2 yields the following corollary of the above theorem:

Corollary 2.7 **(Duyckaerts-M. [4]).** — *From the half-line $\Omega = (-\infty, x_0)$, $x_0 \in \mathbb{R}$, the diffusion in the potential well $V(x) = x^{2k}$, $k \in \mathbb{N}$, $k > 1$,*

$$\partial_t\phi - \partial_x^2\phi + V\phi = \Omega u, \quad \phi(0) = \phi_0 \in L^2(\mathbb{R}),\ u \in L^2([0,T] \times \mathbb{R}),$$

is null-controllable in any time, i.e. $\forall T > 0$, $\forall \phi_0$, $\exists u$ such that $\phi(T) = 0$.

2.5. Necessary resolvent conditions for any semigroup $t \mapsto e^{-tA}$. — The sufficient condition in theorem 2.2 is far from being necessary! Null-controllability of the heat diffusion with Dirichlet boundary condition in §1.1 holds at any time and from any subset Ω of a compact Riemannian manifold M. In this context, the following resolvent condition always hold (cf. the semiclassical generalization in [**14**]):

$$\|v\|^2 \leqslant me^{a\sqrt{\lambda}}\left(\|(-\Delta-\lambda)v\|^2 + \|\Omega v\|^2\right),\ v \in \mathcal{D}(A),\ \operatorname{Re}\lambda > 0.$$

And, in the following classical example, this exponential factor is optimal!

Example 2.1 **(worst observability for the Laplacian on a manifold)**
M is the sphere $S^2 = \left\{x^2 + y^2 + z^2 = 1\right\}$, $M \backslash \Omega$ is a neighborhood of the great circle $\{z = 0\}$, $e_n(x, y, z) = (x + iy)^n$: $(-\Delta - \lambda_n)e_n = 0$ and $\exists a > 0$, $\|e_n\| \geqslant ae^{a\sqrt{\lambda_n}}\|\Omega e_n\|$.

This leads to discussing such resolvent conditions with exponential factor in the most general abstract framework where $-A$ is any generator of a continous semigroup and $C \in \mathcal{L}(\mathcal{D}(A), \mathcal{F})$ is only admissible: $\exists m > 0$, $\exists \tau > 0$,

$$(ExpRes) \quad \|v\|^2 \leqslant me^{2\tau(\operatorname{Re}\lambda)^\alpha}\left(\|(A-\lambda)v\|^2 + \|Cv\|^2\right),\ v \in \mathcal{D}(A),\ \operatorname{Re}\lambda > 0.$$

Recall that final-observability of $t \mapsto e^{-tA}$ from C in time T at cost κ_T means:

$$(FinalObs) \quad \|e^{-TA}v\|^2 \leqslant \kappa_T \int_0^T \|Ce^{-tA}v\|^2 dt, \quad v \in \mathcal{E}.$$

Theorem 2.8 **(Duyckaerts-M. [4]).** — *Assume $C \in \mathcal{L}(\mathcal{D}(A), \mathcal{F})$ is admissible:*

$$\int_0^T \|Ce^{-tA}v\|^2 dt \leqslant k_T\|v\|^2, \quad v \in \mathcal{E}.$$

If $(FinalObs)$ holds for some $T > 0$, then $(ExpRes)$ holds with $\alpha = 1$ and $\tau = T$.

If $(FinalObs)$ holds for all $T \in (0, T_0]$ with the control cost $\kappa_T = ce^{c/T^\beta}$ for some $\beta > 0$, $c > 0$, $T_0 > 0$, then $(ExpRes)$ holds with $\alpha = \frac{\beta}{\beta+1} < 1$.

3. Improvements of the Lebeau-Robbiano strategy

There are two reasons for this subsidiary section. Although the hypothesis of our main theorem 2.2 is a resolvent condition, the proof catches up and clings to the established Lebeau-Robbiano strategy for proving the null-controllability of the heat equation (cf. §4.1). Moreover, one of the possible starting points of this strategy is a stationary condition (cf. §3.1) which suits comparison with the other kind of spectral inequality considered in the main section 2.

The original strategy provided an iterative construction of the input u working in spectral spaces of increasing dimension. It combined both sides of the duality between control and observation (cf. §1.1) and failed to provide the optimal cost estimate. This section concerns the *direct version of the Lebeau-Robbiano strategy* introduced in [**13**], which keeps on the observation side all the way, and allows either strengthening the conclusion (cost estimates in [**13**]) or weakening the assumption (logarithmically close to the insufficient limit case in [**4**]). Moreover, the general framework put forward in [**13**] envisions wider applications to systems and to dynamics where the dissipation is not measured with respect to the spectral spaces of the generator itself, cf. §3.3 and §3.5. This direct version also prompted the extension in [**18**] to parabolic equations observed from a set of times which is only measurable, instead of the interval $(0,T)$. In subsequent papers about measurable observation sets in both space and time, it has been referred to as the *telescoping series method* after the key telescoping lemma 2.1 in [**13**]. We refer to [**8**] for further references.

3.1. Observability on spectral spaces (estimate on sums of eigenfunctions). — This beautifully simple statement is published in papers by Lebeau and Jerison ('96), and Lebeau and Zuazua ('98), but it is based on the boundary Carleman estimates introduced in the earlier landmark paper by Lebeau and Robbiano [**9**]:

$$\int_M |w(x)|^2 dx \leqslant \tilde{c}e^{c\sqrt{\lambda}} \int_\Omega |w(x)|^2 dx, \quad \text{for all } \lambda > 0 \text{ and } w = \sum_{\mu \leqslant \lambda} e_\mu,$$

where $\begin{cases} -\Delta e_\mu = \mu e_\mu & \text{on } M, \\ e_\mu = 0 & \text{on } \partial M, \end{cases}$ are the spectral data of the Laplacian on M.

It quantifies and extends to sums the well-known unique continuation property of eigenfunctions of elliptic operators. As in section 1, it is still valid on a smooth compact Riemannian manifold. Already for single eigenfunctions instead of a sum, the exponential factor is optimal in example 2.1 (sphere observed off the equator). But it is always optimal for sums, even when the restriction of the inequality to single eigenfunctions holds with a much better factor, e.g. a constant factor in example 4.1 (where eigenfunctions are sine functions). Indeed, c is never lower than the (Riemannian) distance of any point of M to Ω (this is an easy consequence of the finite propagation speed for the wave equation, drawn in section 5 of [**13**]).

We may write the previous spectral observability estimate concisely with spectral subspaces $\mathcal{E}_\lambda = \mathrm{Span}_{\mu<\lambda}\, e_\mu$ of the Dirichlet Laplacian $-\Delta$

$$\|w\| \leqslant \tilde{a}e^{a\sqrt{\lambda}} \|\Omega w\|, \quad w \in \mathcal{E}_\lambda, \quad \lambda > 0.$$

More generally the spectral subspaces $\mathcal{E}_\lambda$ may be defined by some functional calculus, e.g. $\mathcal{E}_\lambda = \mathbf{1}_{A<\lambda}\,\mathcal{E}$ if $A>0$ (we started with the special case $A=-\Delta$, M compact).

3.2. Direct Lebeau-Robbiano strategy: simple proof.— This proof hopes to convince that the direct version is not less intuitive and even simpler than both the original version and the approximate controllability version of Seidman in [**22**]. To keep familiar notations, we only consider the framework of §1.1 where the observation operator is bounded by one, i.e. $\|\Omega\|\leqslant 1$. With no extra effort we obtain the optimal cost estimate already reached by [**22**] and do not use that the spectral spaces are invariant by the semigroup. For simplicity, we do not care about constants in the assumption and conclusion.

Assume observability on the spectral subspaces $\mathcal{E}_\lambda = \mathbf{1}_{-\Delta<\lambda}\,\mathcal{E}$:

$$(\mathit{BasicSpecObs})\quad \|w\|^2 \leqslant e^{\sqrt{\lambda}}\|\Omega w\|^2,\quad \lambda>0,\quad w\in\mathcal{E}_\lambda.$$

The goal is fast final-observability at cost $\kappa_T \leqslant c e^{c/T}$ for some $c>0$:

$$(\mathit{FastFinalObs})\quad \|e^{T\Delta}v\|^2 \leqslant \kappa_T \int_0^T \|\Omega e^{t\Delta}v\|^2 dt,\quad v\in\mathcal{E},\quad T>0.$$

We first aim at final-observability with approximation rate $\varepsilon_\tau\to 0$ as $\tau\to 0$:

$$(\mathit{ApproxObs})\quad \frac{1}{\kappa_\tau}\|e^{\tau\Delta}v\|^2 - \frac{\varepsilon_\tau}{\kappa_\tau}\|v\|^2 \leqslant \int_0^\tau \|\Omega e^{t\Delta}v\|^2 dt,\quad v\in\mathcal{E}.$$

(In terms of controllability, this is equivalent to: $\forall f(0)\in\mathcal{E}$, $\exists u\in L^2(\mathbb{R};\mathcal{E})$, such that the solution of $\partial_t f - \Delta f = \Omega u$ satisfies $\displaystyle\frac{1}{\kappa_\tau}\int_0^\tau \|u(t)\|^2 dt + \frac{1}{\varepsilon_\tau}\int_0^\tau \|f(\tau)\|^2 dt \leqslant \|f(0)\|^2$.)

Integrating $(\mathit{BasicSpecObs})$ yields an exponentially *bad spatial cost* on $\mathcal{E}_\lambda$, which we hope to compensate with the *good dissipation in time* on its orthogonal:

$$\begin{cases} \text{Spatial cost:} & \tau\|e^{\tau\Delta}w\|^2 \leqslant e^{\sqrt{\lambda}}\displaystyle\int_0^\tau \|\Omega e^{t\Delta}w\|^2 dt, & w\in\mathcal{E}_\lambda,\\ \text{Time decay:} & \|e^{\tau\Delta}r\| \leqslant e^{-\tau\lambda}\|r\|, & r\perp\mathcal{E}_\lambda,\end{cases}$$

This rationale for choosing the "frequency" of the same order as the inverse of the "infinitesimal time" $\sqrt{\lambda}\sim\frac{1}{\tau}$ turns out to be correct. But the most naive idea, decomposing $v\in\mathcal{E}$ as a sum of orthogonal projections $v=w+r$, does not succeed. The difficulty is: the observation of the remainder r is not negligible, since decay needs time. The cure is: *do not observe too early.* Thus we only observe on $(\tau/2,\tau)$.

In the current framework $e^{t\Delta}\mathcal{E}_\lambda=\mathcal{E}_\lambda$. But this *invariance is not required* in this proof: we illustrate that we only use $\mathcal{E}_\lambda$ as a scale of spaces which measures the dissipation of the semigroup. Therefore we introduce the orthogonal projection P_λ on the space orthogonal to $\mathcal{E}_\lambda$, i.e. for all of $v\in\mathcal{E}$, $v-P_\lambda v\in\mathcal{E}_\lambda$ and $P_\lambda v\perp\mathcal{E}_\lambda$, we write $(\mathit{BasicSpecObs})$ for $w=(1-P_\lambda)e^{t\Delta}v$ and we rewrite the time decay in this form:

$$\begin{cases} \text{Spatial cost:} & e^{-\sqrt{\lambda}}\|(1-P_\lambda)e^{t\Delta}v\|^2 \leqslant \|\Omega(1-P_\lambda)e^{t\Delta}v\|^2, & v\in\mathcal{E},\\ \text{Time decay:} & \|P_\lambda e^{t\Delta}v\| \leqslant e^{-t\lambda}\|v\|, & v\in\mathcal{E}.\end{cases}$$

In the spatial cost inequality, we use orthogonality on the left and the parallelogram inequality on the right: $\frac{1}{2}e^{-\sqrt{\lambda}}\left(\|e^{t\Delta}v\|^2 - \|P_\lambda e^{t\Delta}v\|^2\right) \leqslant \|\Omega e^{t\Delta}v\|^2 + \|\Omega P_\lambda e^{t\Delta}v\|^2$. Then we use $\|\Omega\| \leqslant 1$ and $\exp \leqslant 1$: $\frac{1}{2}e^{-\sqrt{\lambda}}\|e^{t\Delta}v\|^2 \leqslant \|\Omega e^{t\Delta}v\|^2 + 2\|P_\lambda e^{t\Delta}v\|^2$.

Integrating on $(\tau/2, \tau)$, using the contraction property $\|e^{t\Delta}\| \leqslant 1$ on the left and the time decay inequality on the right:

$$\tfrac{\tau}{4}e^{-\sqrt{\lambda}}\|e^{\tau\Delta}v\|^2 \leqslant \int_{\frac{\tau}{2}}^{\tau} \|\Omega e^{t\Delta}v\|^2 dt + \tau e^{-\tau\lambda}\|v\|^2.$$

There is no loss in bounding the polynomials by exponentials for small enough infinitesimal times τ. Thus, taking $\tau \in (0, \frac{1}{2})$, so that $e^{-1/\tau} < \tau < \frac{1}{4\tau} < \frac{1}{4}e^{1/\tau}$, yields:

$$\tfrac{1}{4}e^{-\sqrt{\lambda}-1/\tau}\|e^{\tau\Delta}v\|^2 - \tfrac{1}{4}e^{-\tau\lambda+1/\tau}\|v\|^2 \leqslant \int_{\frac{\tau}{2}}^{\tau} \|\Omega e^{t\Delta}v\|^2 dt.$$

We plug $\sqrt{\lambda} = \gamma/\tau$, set $f(\tau) = \frac{1}{4}e^{-\sqrt{\lambda}-1/\tau} = \frac{1}{4}e^{-(\gamma+1)/\tau}$, and take $q = \frac{1}{\gamma-1}$, so that:

$$f(\tau)\|e^{\tau\Delta}v\|^2 - f(q\tau)\|v\|^2 \leqslant \int_0^{\tau} \|\Omega e^{t\Delta}v\|^2 dt, \quad v \in \mathcal{E}, \ \tau \in (0, \tfrac{1}{2}).$$

For $\gamma > 2$, this reaches our infinitesimal aim (*ApproxObs*) with approximation rate $\varepsilon_\tau = 4f(\tau/(\gamma-2)) \to 0$ as $\tau \to 0$. Moreover, $q < 1$ allows a geometric partition of time. For simplicity we take a dyadic partition, i.e. $\gamma = 3$, $q = \frac{1}{2}$, $f(\tau) = \frac{1}{4}e^{-4/\tau}$.

As Lebeau and Robbiano in [**9**], we write T as a geometric sequence of times τ with ratio q, i.e. *partition* $(0, T] = \cup(T_{k+1}, T_k]$ with $T_k - T_{k+1} = \tau_k = q\tau_{k-1}$, $k \in \mathbb{N}$, hence $T = T_0 = \sum_k \tau_k = \tau_0/(1-q)$.

Applying (*ApproxObs*) on $(T_{k+1}, T_k]$ and adding the *telescoping series* yields:

$$f(\tau_0)\|e^{T_0\Delta}v\|^2 - 0 \times \|v\|^2 \leqslant \int_0^{T_0} \|\Omega e^{t\Delta}v\|^2 dt.$$

This achieves our goal ($FastFinalObs$): $\kappa_T \leqslant \frac{1}{f(\tau_0)} = \frac{1}{f((1-q)T)} = 4e^{\frac{\gamma^2+1}{\gamma-2}\frac{1}{T}} = 4e^{10/T}$.

3.3. Direct Lebeau-Robbiano strategy: cost improvement. — In this paragraph and the next, the generator $-A$ is self-adjoint, $\mathcal{E}_\lambda = \mathbf{1}_{A<\lambda}\,\mathcal{E}$, and the observation operator C is bounded or admissible.

The proof of the preceding paragraph is still valid if the exponential factor $e^{\sqrt{\lambda}}$ in the observability on spectral spaces is replaced by a range of such factors up-to and excluding $e^{a\lambda}$. This proof was fine-tuned in [**13**] to optimize the relation between the constants in the assumption and the conclusion. We only recall here the relation between exponents. Moreover, we emphasize the intermediate *dynamic observability on spectral subspaces* joined by the last part of the proof of our main theorem 2.2 (it says that, the null-control of the projection on $\mathcal{E}_\lambda$ in time T costs less than $e^{\lambda^\alpha + 1/T^\beta}$):

Observability on spectral subspaces with power $\alpha \in (0, 1)$:

$$(\textit{SpecObs}) \quad \|w\| \leqslant \tilde{a}e^{a\lambda^\alpha}\|Cw\|, \quad w \in \mathcal{E}_\lambda, \quad \lambda \geqslant \lambda_0 > 0.$$

⇒ *Dynamic observability on spectral subspaces*, with the same α and any $\beta > 0$:

$$\|e^{-TA}w\|^2 \leqslant \tilde{a}e^{a\lambda^\alpha + b/T^\beta}\int_0^T \|Ce^{-tA}w\|^2 dt, \quad w \in \mathcal{E}_\lambda, \quad T > 0, \quad \lambda \geqslant \lambda_0.$$

⇒ *Fast final-observability*, moreover with cost $\kappa_T \leqslant \tilde{c}e^{c/T^\beta}$ assuming $\beta = \frac{\alpha}{1-\alpha}$:

$$\|e^{-TA}w\|^2 \leqslant \kappa_T \int_0^T \|Ce^{-tA}w\|^2 dt, \quad w \in \mathcal{E}, \quad T > 0.$$

In this framework, the dissipation is spectrally determined: $\|e^{-tA}v\| \leqslant e^{-t\lambda}\|v\|$, $\lambda > 0$, for all $v \perp \mathcal{E}_\lambda$, where $\mathcal{E}_\lambda = \mathbf{1}_{A<\lambda}\mathcal{E}$ is the spectral space. But, if C is bounded, the proof of §3.2 extends to the dissipation assumption (weaker if $\nu > 0$ or $\delta > 1$)(3)

$$\|P_\lambda e^{-tA}v\| \leqslant m_0 e^{m\lambda^\nu - t^\delta\lambda}\|v\|, \quad v \in \mathcal{E}, \quad t > 0, \quad \lambda \geqslant \lambda_0,$$

where P_λ is the projection on the space orthogonal to $\mathcal{E}_\lambda \subset \mathcal{D}(A)$. Here the "growth spaces" $\mathcal{E}_\lambda$ are not required to be spectral spaces of A nor even satisfy the following invariance: $e^{-tA}\mathcal{E}_\lambda = \mathcal{E}_\lambda$. They could be the spectral spaces of a simpler form of the generator (e.g. if the generator is a perturbation of another). Now the cost of fast final-observability is $\kappa_T \leqslant \tilde{c}e^{c/T^\beta}$, $\beta = \frac{\delta\alpha}{1-\alpha}$ (the conclusion is weaker for $\delta > 1$).

3.4. Direct Lebeau-Robbiano Strategy: log**-improvement.** — The optimality of the theorem in the previous paragraph in the range of exponential of power-laws, excluding the critical linear law, leads to investigate its validity for asymptotics closer to the linear law. E.g. the factor λ^α, $\alpha < 1$, can be improved into $\lambda/\varphi(\lambda)$, with $\varphi(\lambda) = (\log(\log\lambda))^\alpha \log\lambda$, $\alpha > 2$. A more general result is stated in §3.5.

Theorem 3.1 **(Duyckaerts-M. [4]: logarithmic L.-R. strategy)**
Let $\varphi(\lambda) = (\log(\log\lambda))^\alpha \log\lambda$, $\alpha > 2$.
The logarithmic observability on spectral subspaces:

$$\|v\|^2 \leqslant ae^{a\lambda/\varphi(\lambda)}\|Cv\|^2, \quad v \in \mathcal{E}_\lambda, \quad \lambda \geqslant \lambda_0 > e.$$

implies fast final-observability (without estimate on the cost κ_T):

$$\|e^{-TA}v\|^2 \leqslant \kappa_T \int_0^T \|Ce^{-tA}v\|^2 dt, \quad v \in \mathcal{E}, \quad T > 0.$$

Combining this theorem with the estimate on sums of eigenfunctions in §3.1 yields:

Corollary 3.2 **(Duyckaerts-M. [4]: logarithmic anomalous diffusion)**
Let $\varphi(\lambda) = (\log\lambda)^\alpha$, $\alpha > 1$ or $\varphi(\lambda) = (\log(\log\lambda))^\alpha \log\lambda$, $\alpha > 2$.
The following anomalous diffusion is null-controllable in any time $T > 0$:

$$\partial_t\phi + \sqrt{-\Delta}\varphi(\sqrt{-\Delta})\phi = \Omega u, \quad \phi(0) = \phi_0 \in L^2(M), \ u \in L^2([0,T] \times M).$$

(3) This point, not explicit in [**13**], was first made (without cost estimate) in K. Beauchard, K. Pravda-Starov, *Null-controllability of hypoelliptic quadratic differential equations*, arXiv:1603.05367 (2016).

3.5. Direct Lebeau-Robbiano strategy: general framework. — Here is the more intricate framework put forward in [**13, 4**] to encompass more applications.

As in §1.3, $\mathcal{E}$ (states) and $\mathcal{F}$ (observations) are two Hilbert spaces, $-A$ is the generator of a strongly continuous semigroup $t \mapsto e^{-tA}$ on $\mathcal{E}$. In order to state the other data of this framework, let T_0 and λ_0 be positive constants, φ and ψ be positive increasing continuous functions defined on $(\lambda_0, +\infty)$ and $(1/T_0, +\infty)$ respectively such that φ, $\lambda \mapsto \lambda/\varphi(\lambda)$ and $\omega \mapsto \psi(\omega)/\omega$ tend to $+\infty$ at $+\infty$. The four other data are:

1. An *Observation operator* $C \in \mathcal{L}(\mathcal{D}(A), \mathcal{F})$ with admissibility condition
$$\int_0^T \|Ce^{-tA}v\|^2 dt \leqslant \mathrm{K}_T \|v\|^2, \quad v \in \mathcal{D}(A), \quad T > 0.$$
2. A *Reference observation operator* $C_0 \in \mathcal{L}(\mathcal{D}(A), \mathcal{F})$ with final-observability:
$$\|e^{-TA}v\|^2 \leqslant b_0 e^{2T\psi(1/T)} \int_0^T \|C_0 e^{-tA}v\|^2 dt, \quad v \in \mathcal{D}(A), \quad T \in (0, T_0).$$
The operator C_0 should correspond to full observation and the function ψ should measure the easily computed cost in this situation. Usually C_0 is the identity and $e^{2T\psi(1/T)}$ is a power of $\frac{1}{T}$, as if the dimension of the state space was finite. But in dynamics generated by a system of equations, C_0 could be the full observation of only one component.
3. A *Scale of semigroup invariant spaces*, $e^{-tA}\mathcal{E}_\mu \subset \mathcal{E}_\mu \subset \mathcal{E}_\lambda \subset \mathcal{D}(A)$, $\lambda \geqslant \mu \geqslant \lambda_0$, with the semigroup growth property:
$$\|e^{-tA}v\| \leqslant m_0 e^{m\lambda/\varphi(\lambda)} e^{-\lambda t}\|v\|, \quad v \perp \mathcal{E}_\lambda, \quad t \in (0, T_0), \quad \lambda \geqslant \lambda_0.$$
4. An inequality of *Observability on growth subspaces $\mathcal{E}_\lambda$ with respect to C_0*:
$$\|C_0 w\|^2 \leqslant a_0 e^{2\lambda/\varphi(\lambda)} \|Cw\|^2, \quad w \in \mathcal{E}_\lambda, \quad \lambda \geqslant \lambda_0.$$
This generalizes the estimate for sums of eigenfunctions of §3.1 and the function φ measures the discrepancy from the insufficient exponential factor $e^{a\lambda}$.

The stationary assumptions on C_0, 2 and 4, can be replaced by the single assumption:

4'. A *Dynamic observability on growth subspaces $\mathcal{E}_\lambda$*:
$$\|e^{-TA}w\|^2 \leqslant c_0 e^{2T\psi(1/T)\lambda/\varphi(\lambda)} \int_0^T \|Ce^{-tA}w\|^2 dt, \ w \in \mathcal{E}_\lambda, \ T \in (0, T_0), \ \lambda \geqslant \lambda_0.$$

Theorem 3.3 **(Duyckaerts-M. [4]: integrable L.-R. strategy)**

If $s \mapsto 1/\psi^{-1}\left(\dfrac{\varphi(q^s)}{p}\right) = \tau(q^s)$ *is integrable at* $+\infty$ *for some* $p > m+1$ *and* $q > 1$, *then final-observability (FinalObs) holds for all* $T > 0$.

The function τ in this theorem is like the infinitesimal time τ in the proof of §3.2. The integrability condition reflects the idea that the infinitesimal times should add up to a finite time T. It is so weak that we are unable to estimate the control cost. Indeed, the geometric partition is performed on the variable λ instead of the times.

The framework in §3.3 corresponds to $\psi(\omega) = \omega^{\beta+1}$ and $\varphi(\lambda) = \lambda^{1-\alpha}$, $\beta = \frac{\alpha}{1-\alpha}$. The framework in §3.4 corresponds to $\psi(\omega) = \omega \log \omega$ and $\varphi(\lambda) = (\log(\log \lambda))^\alpha \log \lambda$, $\alpha > 2$, which satisfy the integrability condition for all $p > 0$ and $q > 1$.

4. Proofs of the main result and counter-example, basic examples

4.1. Sketch of proof of the main result. — Now we are ready to sketch the proof of the sufficient resolvent conditions for $t \mapsto e^{-tA}$, $A > 0$ stated in theorem 2.2.

The proof combines three ingredients:

▶ some links between resolvent conditions for unitary groups build on A, cf. [**16**],

▶ the control transmutation method from "wave" to "heat" equations, cf. [**12**],

▶ the Lebeau-Robiano strategy from a cost estimate on spectral spaces, cf. §3.3.

More precisely, we enumerate the key arguments to sketch the proof. It keeps track of how the control time and cost (therefore the constants in the resolvent conditions) depend on the "size" of the spectral spaces used as state spaces (i.e. on the spectral cut-off threshold): recall the spectral space $\mathcal{E}_\lambda = \mathbf{1}_{A<\lambda}\,\mathcal{E}$, hence $\mathcal{E}_{\lambda^2} = \mathbf{1}_{\sqrt{A}<\lambda}\,\mathcal{E}$.

1. Resolvent Conditions: $i\partial_t\psi + A\psi = C^*u$ to $i\partial_t\psi + \sqrt{A}\psi = C^*u$ ([**16**] thm. 3.5). Resolvent conditions for A and C with coefficients depending on the spectral variable, are linked to resolvent conditions for $\sqrt{A}$ and C by a change of variable.
2. Resolvent Conditions: "half wave" $i\partial_t\psi+\sqrt{A}\psi = C^*u \Leftrightarrow$ wave $\partial_t^2 w+Aw = C^*u$. Writing the second order wave equation as a first order system, the generator $W = \left(\begin{smallmatrix} 0 & I \\ -A & 0\end{smallmatrix}\right)$ is isomorphic to $J^{-1}WJ = i\sqrt{A}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right)$, $J = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ i\sqrt{A} & -i\sqrt{A}\end{smallmatrix}\right)$ between suitable state spaces ([**16**] thm. 3.8 and thm. 3.13).
3. Control Transmutation Method: from $\partial_t^2 w + Aw = C^*\tilde{u}$ to $\partial_t f + Af = C^*u$. If the cost to control "waves" w in time L is $\tilde{\kappa}_L$, then the null-control of the "heat" f is less than a fixed multiple of $\tilde{\kappa}_L e^{kL^2/T}$ (we do not need the best k, which is an open problem related to heat control from the boundaries of a segment [**23**]).
4. Lebeau-Robiano strategy with cost estimate starting from (cf. §3.3):

$$\|e^{-TA}w\|^2 \leqslant \tilde{c}e^{\lambda^\alpha+1/T^\beta}\int_0^T \|Ce^{-tA}w\|^2 dt, \quad w\in\mathcal{E}_\lambda, \quad T>0, \quad \lambda\geqslant\lambda_0.$$

With these four key tools at hand, we are ready to sketch the proof.

Here C is bounded, and to simplify the computations, take $\delta = \frac{1}{3}$, hence $\beta = 2$.

The assumption of theorem 2.2 is the resolvent condition:

$$(PowRes) \quad \|v\|^2 \leqslant m\lambda^\delta\left(\frac{1}{\lambda}\|(A-\lambda)v\|^2 + \|Cv\|^2\right), \quad v\in\mathcal{D}(A),\ \lambda>0.$$

By point 1, it implies:

$$\|v\|^2 \leqslant m(\lambda^2)^\delta\left(\|(\sqrt{A}-\lambda)v\|^2 + \|Cv\|^2\right), \quad v\in\mathcal{D}(\sqrt{A}),\ \lambda>0.$$

By point 2, it implies controllability of waves on $\mathcal{E}_{\lambda^2}\times\mathcal{E}_{\lambda^2}$ for times and cost $\sim m^{\frac{1}{2}}\lambda^\delta$.
By point 3, it implies controllability of heat on $\mathcal{E}_{\lambda^2}$ for all $T>0$ at cost $\sim e^{(\lambda^\delta)^2/T}$.
This is equivalent to controllability of heat on $\mathcal{E}_\lambda$ for all $T>0$ at cost $\sim e^{\lambda^\delta/T}$.
By Cauchy-Schwarz, $\frac{\lambda^{1/3}}{T} \leqslant \lambda^\alpha + \frac{1}{T^\beta}$ where $\alpha = \frac{2}{3}$ and $\beta = 2$ satisfy $\beta = \frac{\alpha}{1-\alpha}$.
By point 4, this implies the conclusion of theorem 2.2, for all $T>0$,

$$(FinalObs) \quad \|e^{-TA}v\|^2 \leqslant ce^{c/T^\beta}\int_0^T \|Ce^{-tA}v\|^2 dt, \quad v\in\mathcal{E}.$$

4.2. Counter-example to the limit case of the main result. — The one dimensional example in this paragraph disproves that the exact controllability of the Schrödinger equation implies the null-controllability of the heat equation:

$$\partial_t f - \partial_x^2 f + x^2 f = \Omega u \quad \text{versus} \quad i\partial_t \psi - \partial_x^2 \psi + x^2 \psi = \Omega u.$$

Here we control on $\Omega = (-\infty, x_0)$, $x_0 \in \mathbb{R}$; and $A = -\partial_x^2 + x^2$ on $\mathcal{E} = L^2(\mathbb{R}) = \mathcal{F}$.

Theorem 4.1 **(M. [15]: harmonic oscillator observed from a half-line)**

1. *Observability (FinalObs) for "heat" $t \mapsto e^{-tA}$ does not hold for any time.*
2. *Observability (ExactObs) for "Schrödinger" $t \mapsto e^{itA}$ holds for some time.*

The proof does not use the spectral data although they are known very well (eigenvalues $\lambda_n = 2n+1$, and eigenfunctions $e_n(x) = c_n (\partial_x - x)^n e^{-x^2/2} = c_n H_n(x) e^{-x^2/2}$, where $c_n = (\sqrt{\pi} 2^n (n!))^{-1/2}$, $H_n = (-1)^n e^{x^2} \partial_x^n e^{-x^2}$ are the Hermite polynomials.) N.b. if we had $\dim \mathcal{F} = 1$, then $\sum \frac{1}{\lambda_n} = +\infty$ would imply the negative heat result 1.

The following proof is rather sketchy, but prop. 5.1 in [**4**] provides the full details, including the optimality of the resolvent condition (*Res*) below.

1. To disprove the inequality equivalent to "heat" null-controllability at time T,

$$(FinalObs) \quad \int_{-\infty}^{+\infty} |(e^{-TA}v)(x)|^2 dx \leqslant \kappa_T^2 \int_0^T \int_{-\infty}^{x_0} |(e^{-tA}v)(x)|^2 dx dt,$$

we take the Dirac mass at $y \notin \Omega$ as initial data v and let $y \to \infty$.

More precisely, $v(x) = e^{-\varepsilon A}(x, y)$, where ε is a small time, $e^{-tA}(x, y)$ is the kernel of the operator e^{tA}. Hence $(e^{-tA}v)(x) = e^{-(t+\varepsilon)A}(x, y)$.

We bound from below the fundamental state e_0, hence the final state:

$$\|e^{-TA}v\| \geqslant e^{-(T+\varepsilon)\lambda_0} |e_0(y)| \geqslant c_T \exp\left(-\frac{y^2}{2}\right).$$

We bound from above the kernel hence the observation, using *Mehler formula*:

$$e^{-tA}(x, y) = \frac{e^{-t}}{\sqrt{\pi(1 - e^{-4t})}} \exp\left(-\frac{1 + e^{-4t}}{1 - e^{-4t}} \frac{x^2 + y^2}{2} + \frac{2e^{-2t}}{1 - e^{-4t}} xy\right).$$

The contradiction comes from plugging these bounds in (*FinalObs*), and comparing the factors of y^2 inside the exponential: $-\dfrac{y^2}{2} \geqslant -\dfrac{1 + e^{-4t}}{1 - e^{-4t}} \dfrac{y^2}{2}$.

2. To prove the resolvent condition equivalent to "Schrödinger" controllability,

$$(Res) \quad \|v\|^2 \leqslant m\|(A - \lambda)v\|^2 + m\|\Omega v\|^2, \quad v \in \mathcal{D}(A), \quad \lambda \geqslant \lambda_0 = 1,$$

we use a semiclassical reduction and microlocal propagation, as in [**1**].

By the change of variable $u(y) = v(x)$, $y = \sqrt{h}x$, $h = \frac{1}{\lambda}$, (*Res*) reduces to the *semiclassical resolvent condition*: for all $u \in C_0^\infty(\mathbb{R})$ and $h \in (0, 1]$,

$$\int_{-\infty}^{+\infty} |u(y)|^2 dy \leqslant \frac{m}{h^2} \int_{-\infty}^{+\infty} |-h^2 u''(y) + (y^2 - 1)u(y)|^2 dy + m \int_{-\infty}^{\sqrt{h}x_0} |u(y)|^2 dy.$$

Arguing by contradiction, we introduce a semiclassical measure in phase space $(x, \xi) \in \mathbb{R}^2$ (also known as Wigner measure). Due to the generator term (first on the right), this measure is supported on the circle $\{x^2+\xi^2=1\}$ (the characteristic set) and it is invariant by rotation (the hamiltonian flow). Due to the observation term (second on the right), this measure is null on $\{x<0\}$. Hence this measure must be null, which yields a contradiction (due to the left term).

4.3. Basic examples of resolvent conditions. — To help the readers unfamiliar with resolvent conditions grasp their meaning, we provide the two simplest examples.

▶ Firstly, $M=[a,b]$, $C=\Omega$ and $A=-\Delta\geqslant\lambda_0=\pi/|M|>0$, where $|M|=b-a$.

Example 4.1 **(Dirichlet Laplacian on a segment observed on an interval Ω)** *The following resolvent condition (equivalent to the control of waves) holds*

$$\|v\|^2\leqslant\frac{m}{\lambda}\|(\partial_x^2+\lambda)v\|^2+\tilde{m}\|\Omega v\|^2,\quad v\in H^2(M)\cap H_0^1(M),\quad \lambda\geqslant\lambda_0>0.$$

But it fails if m is replaced by a function such that $m(\lambda)\to 0$ as $\lambda\to\infty$.

Assume for simplicity $0\in\Omega\subset M$. Let $\chi\in C^\infty(\mathbb{R})$, $\chi=1$ out of Ω, $\chi=0$ near 0. It is sufficient to prove: $\|\chi v\|^2\leqslant\frac{m}{\lambda}\|f\|^2+\tilde{m}\|\Omega v\|^2$, where $f=(\partial_x^2+\lambda)v$.

Solving the O.D.E. $u''+\lambda u=g$ with null initial conditions $u(0)=u'(0)=0$ yields: $u(x)=\frac{1}{\sqrt{\lambda}}\int_0^x\sin\big(\sqrt{\lambda}(x-y)\big)g(y)dy$. Hence $\|u\|^2\leqslant\frac{|M|^2}{\lambda}\|g\|^2$.

Applying this to $u=\chi v$ yields: $g=(\partial_x^2+\lambda)(\chi v)=\chi f+r$, with $r=2\chi'v'+\chi''v$, hence the resolvent condition above since

- χf contributes $\frac{m}{\lambda}\|f\|^2$,
- integrating by parts $\frac{2}{\sqrt{\lambda}}\sin\big(\sqrt{\lambda}(x-y)\big)\chi'(y)\times v'(y)$ and using $r=0$ out of Ω: the remainder r contributes $\tilde{m}\|\Omega v\|^2$.

Conversely, the *unobserved quasimode* $v_\lambda(x)=\chi(x)e^{i\sqrt{\lambda}x}$ (with phase $\varphi(x)=x$ satisfying the eikonal equation $|\varphi'(x)|^2=1$), with smooth amplitude $\chi\neq 0$ such that $\chi=0$ on Ω and out of M, disproves any better resolvent condition as $\lambda\to\infty$ since

- $\|v_\lambda\|=\|\chi\|>0$ does not depend on λ,
- $\frac{1}{\lambda}\|(\partial_x^2+\lambda)v_\lambda\|^2$ is bounded, due to $v_\lambda''+\lambda v_\lambda=2i\chi'\sqrt{\lambda}e^{i\sqrt{\lambda}x}+\chi''e^{i\sqrt{\lambda}x}$.

(It is sufficient to use the real part of $v_\lambda(x)$, but the computation is a little less nice).

N.b. as noted in §2.1, the resolvent condition above is equivalent to the controllability of the wave equation for some time, and it implies the controllability of the Schrödinger equation for all $T>0$, due to the limit of the factor: $\frac{m}{\lambda}\to 0$, as $\lambda\to\infty$.

N.b. the normalized eigenfunctions $e_n(x)=\sqrt{2}\sin(n\pi x)$ of the Dirichlet Laplacian $-\Delta$ on $M=[0,1]$ satisfy $\|\Omega e_n\|^2\to|\Omega|$ as $\lambda_n=n^2\pi\to\infty$, hence $\|e_n\|\leqslant\tilde{c}\|\Omega e_n\|^2$ (in fact, this inequality and the gap $|\lambda_n-\lambda_k|\geqslant\sqrt{2\lambda_n}$, $k\geqslant n$, imply the resolvent condition above by general functional analytic arguments discussed under the name *wavepacket conditions* in [**24, 16, 4**]). But the exponent of λ inside the exponential factor of the spectral inequality $\|w\|\leqslant\tilde{a}e^{a\sqrt{\lambda}}\|\Omega w\|$, for all $w=\sum_{\lambda_n\leqslant\lambda}e_n$ and $\lambda>0$, cannot be improved, as noted in §3.1.

This illustrates that the *resolvent condition is very sensitive to the geometrical configuration* (e.g. this example satisfies the geodesics condition of Bardos-Lebeau-Rauch), whereas (stationary) *observability on spectral spaces is impervious to it.*

▶ Secondly, $M = [0,a] \times [0,1]$, $C = \Omega = \omega \times (0,1)$, $\omega \subsetneq (0,a)$, $A = -\Delta$.

Example 4.2 **(Dirichlet Laplacian on a rectangle observed from a strip)**
The following resolvent condition (equivalent to the control of Schrödinger) holds

$$\|v\|^2 \leqslant m\|(\partial_x^2 + \partial_y^2 + \lambda)v\|^2 + \tilde{m}\|\Omega v\|^2, \quad v \in H^2(M) \cap H_0^1(M), \quad \lambda \in \mathbb{R}.$$

But it fails if m is replaced by a function such that $m(\lambda) \to 0$ as $\lambda \to \infty$.

The *unobserved "bouncing ball mode"* $v_n(x,y) = \chi(x)e_n(y) = \chi(x)\sqrt{2}\sin(n\pi y)$, with smooth amplitude $\chi \neq 0$, $\chi = 0$ on ω and out of $[0,a]$, disproves any better resolvent condition as $\lambda \to \infty$ since $(\partial_x^2 + \partial_y^2 + \lambda_n)v_n(x,y) = \partial_x^2 v_n(x,y) = \chi''(x)e_n(y)$. This quasimode "concentrates" on the (direction) of periodic geodesics with constant x (also responsible for the failure of the geodesics condition of Bardos-Lebeau-Rauch).

Due to the *additive tensor product structure* of $-\Delta_{x,y} = -\Delta_x \otimes \mathrm{id}_y - \mathrm{id}_x \otimes \Delta_y$, the resolvent condition for $A = -\Delta_{x,y}$ and $C = \omega \otimes \mathrm{id}_y$ is easily deduced (cf. [**1, 11**]) from the same resolvent condition for $A = -\Delta_x$ and $C = \omega$, implied by the better one proved in example 4.1. N.b. this exact controllability from a strip for the Schrödinger equation on a rectangle is due to Haraux'89 using lacunary trigonometric series.

N.b. if $\sqrt{-\Delta_{x,y}}$ had the same structure, this argument would prove the better resolvent condition (equivalent to waves control) disproved by the bouncing ball mode.

N.b. indeed, the better resolvent condition of example 4.1, that we used in the directions x, leads to a "gain of regularity" in this direction (cf. [**1, 2**]). Taking advantage of this in every directions, Burq-Zworski'12 [**3**] proved the resolvent condition for every observation Ω (this harder control result due to Jaffard'90 using trigonometric series, was extended with similar tools to any dimension by Komornik'92, to partially rectangular billiards [**1, 2**] and to potentials on two dimensional tori [**3**]).

N.b. in particular, the orthonormal basis of eigenfunctions obtained by a tensor product satisfies the resolvent condition. But, if $a^2 \notin \mathbb{Q}$, this fact is not sufficient to deduce the resolvent condition (through wavepackets mentioned about example 4.1), since there is no spectral gap, i.e. $\liminf(\lambda - \mu) = 0$ as $\lambda > \mu \to \infty$, λ and μ in the spectrum of A. *Resolvent conditions are more versatile than eigenfunctions ones.*

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