Time Average in Micromagnetism

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Abstract - In this paper we study a model of ferromagnetic material governed by a nonlinear Laudau-Lifschitz equation coupled with Maxwell equations. We prove the existence of weak solutions. Then we prove that all points of the ω-limit set of any trajectories are solutions of the stationary model. Furthermore we derive rigourously the quasistatic model by an appropriate time average method.

1 Introduction.

In this paper we study the following system

$$\frac{\partial u}{\partial t} + u \wedge \frac{\partial u}{\partial t} = 2u \wedge H_e \ \text{ in } \mathbb{R}^+ \times \Omega, \tag{1.1}$$

where $H_e = \Delta u + H - \varphi(u)$,

$$\mu_0 \frac{\partial}{\partial t}(H + \bar{u}) + \text{curl } E = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3, \tag{1.2}$$

$$\varepsilon_0 \frac{\partial E}{\partial t} - \text{curl } H + \sigma \mathbf{1}_\Omega (E + f) = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3, \tag{1.3}$$

with the associated boundary conditions and initial data

$$\begin{cases} \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ E(0,x) = E_0(x) & \text{in } \mathbb{R}^3, \\ H(0,x) = H_0(x) & \text{in } \mathbb{R}^3. \end{cases} \tag{1.4}$$

We assume that

$$|u_0(x)| = 1 \text{ in } \Omega, \qquad \text{div } (H_0 + \bar{u}_0) = 0 \text{ in } \mathbb{R}^3. \tag{1.5}$$

In the above equations Ω is a smooth bounded open domain of $\mathbb{R}^3$, ν the unit normal on $\partial\Omega$, $\mathbf{1}_\Omega$ is the characteristic function of Ω, $\bar{u}$ is the extension of u by zero outside Ω.

This system of equations which couples the Landau-Lifschitz equation with Maxwell's equations describes electromagnetic waves propagation in a ferromagnetic medium confined to the domain Ω.

In the ferromagnetic model the magnetic moment denoted by u links the magnetic field H with the magnetic induction B through the relationship $B = \mu_0(H + \bar{u})$. Moreover u is a vector field which takes its values on S^2 the unit sphere of $\mathbb{R}^3$. The conductivity of the body Ω is denoted by $\sigma \in I\!R^{+\star}$, the anisotropic term is patterned by $\varphi(u)$ where $\varphi : I\!R^3 \to I\!R^3$ is the gradient of Φ a positively defined quadratic form of $I\!R^3$, f is a source term supported by $I\!R^+ \times \Omega$. Finaly ε_0 is the dielectric permittivity and μ_0 is the magnetic permeability.

This model is described in detail in [3], [11] and [15].

Remark 1.1 *When the solution of (1.1) is regular enough, this equation is equivalent to*

$$\frac{\partial u}{\partial t} = u \wedge H_e - u \wedge (u \wedge H_e) \ \textit{in}\ I\!R^+ \times \Omega. \tag{1.6}$$

In [14] A. Visintin establishes the existence of weak solutions of the system (1.6),(1.2)-(1.5). When H_e reduces to Δu, F. Alouges and A. Soyeur show in [2] the existence and the non uniqueness of the solutions of (1.1). F. Labbé establishes in [10] the non uniqueness of the solution for the quasistatic model. Numerical studies are carried on by P. Joly and O. Vacus in [9], and by F. Alouges in the steady state case in [1]. At least in the case when H_e reduces to H and $\Omega = I\!R^3$, J.L. Joly, G. Métivier and J. Rauch obtain existence and uniqueness results for the solutions of (1.6), (1.2), (1.3), (1.4).

Notations : in the sequel we denote $I\!H^1 = (H^1)^3$ and $I\!L^2 = (L^2)^3$.

2 Statement of the results.

Let us assume that

$$\left.\begin{array}{c} u_0 \in I\!H^1(\Omega)\ ,\ H_0 \in I\!L^2(I\!R^3)\ ,\ E_0 \in I\!L^2(I\!R^3)\ ,\ f \in I\!L^2(I\!R^+ \times \Omega), \\[2ex] |u_0| = 1\ a.e.\ ,\ \text{div}\,(H_0 + \bar{u}_0) = 0. \end{array}\right\} (\mathcal{H})$$

Definition 2.1 *We say that (u, E, H) is a weak solution of (1.1)-(1.5) if*

1. *(u, E, H) verifies*

$$\begin{array}{c} u \in L^\infty(I\!R^+; I\!H^1(\Omega))\ ,\ \dfrac{\partial u}{\partial t} \in I\!L^2(I\!R^+ \times \Omega)\ ,\ |u(t,x)| = 1\ \textit{a. e.}, \\[2ex] E \in L^\infty(I\!R^+; I\!L^2(I\!R^3)),\ \ H \in L^\infty(I\!R^+; I\!L^2(I\!R^3)). \end{array} \tag{2.1}$$

2. *For all $\Psi \in \mathcal{C}^\infty(I\!R^+; I\!H^1(\Omega))$,*

$$\begin{array}{l} \displaystyle\int_{I\!R^+\times\Omega} \left(\frac{\partial u}{\partial t}(t,x) + u(t,x) \wedge \frac{\partial u}{\partial t}(t,x)\right) \cdot \Psi(t,x)dx\,dt = \\[3ex] \displaystyle -2\int_{I\!R^+\times\Omega} \sum_{i=1}^{3} \left(u(t,x) \wedge \frac{\partial u}{\partial x_i}(t,x)\right) \cdot \frac{\partial \Psi}{\partial x_i}(t,x)dx\,dt \\[3ex] \displaystyle +2\int_{I\!R^+\times\Omega} u(t,x) \wedge \Big(\ H(t,x) - \varphi(u(t,x))\Big) \cdot \Psi(t,x)dx\,dt. \end{array} \tag{2.2}$$

3. *$u(0,x) = u_0(x)$ in the trace sense.*

4. *For all* $\Psi \in I\!H^1(I\!R^+ \times I\!R^3)$,

$$-\int_{I\!R^+\times I\!R^3} \Big(H(t,x) + \bar{u}(t,x) \Big) \cdot \frac{\partial \Psi}{\partial t}(t,x) dt\, dx + \int_{I\!R^+\times I\!R^3} E(t,x) \cdot \text{curl } \Psi(t,x) dx\, dt =$$

$$\int_{I\!R^3} H_0(x) \cdot \Psi(0,x) dx + \int_\Omega u_0(x) \cdot \Psi(0,x) dx. \tag{2.3}$$

5. *For all* $\Psi \in I\!H^1(I\!R^+ \times I\!R^3)$,

$$\begin{aligned} &-\int_{I\!R^+\times I\!R^3} E(t,x) \cdot \frac{\partial \Psi}{\partial t}(t,x) dx\, dt - \int_{I\!R^+\times I\!R^3} H(t,x) \cdot \text{curl } \Psi(t,x) dx\, dt \\ &+\sigma \int_{I\!R^+\times\Omega} \Big(E(t,x) + f(t,x) \Big) \cdot \Psi(t,x) dx\, dt = \int_{I\!R^3} E_0(x) \cdot \Psi(0,x) dx. \end{aligned} \tag{2.4}$$

6. *For all* $t > 0$, *we have the following energy estimate :*

$$\begin{aligned} \mathcal{E}(t) + \int_0^t \int_\Omega |\frac{\partial u}{\partial t}(t,x)|^2 dx\, dt + \frac{\sigma}{\mu_0} \int_0^t \int_\Omega |E(t,x)|^2 dx\, dt \leq \mathcal{E}(0) \\ + \frac{\sigma}{\mu_0} \int_0^t \int_\Omega |f(t,x)|^2 dx\, dt \end{aligned} \tag{2.5}$$

where

$$\mathcal{E}(t) = \int_\Omega \Big(|\nabla u(t,x)|^2 + 2\Phi(u(t,x)) \Big) dx + \int_{I\!R^3} \Big(|H(t,x)|^2 + \frac{\varepsilon_o}{\mu_0} |E(t,x)|^2 \Big) dx.$$

Theorem 2.1 *Under the assumption* ($\mathcal{H}$), *there exists at least one weak solution of (1.1)-(1.5).*

This theorem is established in section 3 using a Galerkine approximation for a relaxed problem.

Definition 2.2 *Let* u *be a weak solution of (1.1)-(1.5). We call* ω*-limit set of the trajectory* u *the following set*

$$\omega(u) = \Big\{ v \in I\!H^1(\Omega), \exists\, t_n, \lim t_n = +\infty, u(t_n,.) \rightharpoonup v \text{ in } I\!H^1(\Omega) \text{ weakly} \Big\}$$

From the energy estimate (2.5), for any u, $\omega(u)$ is non empty.

Theorem 2.2 *Under the assumption* ($\mathcal{H}$), *if* u *is a weak solution of (1.1)-(1.5), each point* v *in* $\omega(u)$ *is a weak solution of the steady state system*

$$v \in H^1(\Omega),\ |v| = 1 \text{ a.e.}, \tag{2.6}$$

$$\sum_{i=1}^3 \frac{\partial}{\partial x_i} \left(v \wedge \frac{\partial v}{\partial x_i} \right) + v \wedge (H - \varphi(v)) = 0 \text{ in } \Omega, \tag{2.7}$$

$$\begin{cases} H \in I\!L^2(I\!R^3), \\ \text{curl } H = 0 \text{ in } \mathcal{D}'(I\!R^3), \\ \text{div } (H + \overline{v}) = 0 \text{ in } \mathcal{D}'(I\!R^3). \end{cases} \tag{2.8}$$

Remark 2.1 *As v lies in $I\!H^1(\Omega)$, Δv lies in $I\!H^{-1}(\Omega)$ so the product $v \wedge \Delta v$ makes sense in $W^{-1,t}(\Omega)$ with $\frac{1}{t} = \frac{1}{2} + \frac{1}{6}$ (see J. Simon [13]). Moreover from the equation (2.7) this product belongs to $I\!L^2(\Omega)$.*

Theorem 2.2 is proved in section 4. The limit process for v is carried out thanks to the estimate

$$\int_{I\!R^+} \int_\Omega |\frac{\partial u}{\partial t}(t,x)|^2 dx\, dt < +\infty.$$

On the other hand an averaging technique is used to justify the limit for H.

The last part of this article is devoted to the validation when ε_0 and μ_0 go to zero of the quasi-stationary model. We suppose for this result that the source term f is zero.

Let us assume that

$$\left.\begin{array}{c} u_0 \in I\!H^1(\Omega)\ ,\ H_0 \in I\!L^2(I\!R^3)\ ,\ E_0 \in I\!L^2(I\!R^3), \\ \\ |u_0| = 1 \ a.e.\ ,\ \text{div}\ (H_0 + \bar{u}_0) = 0. \end{array}\right\} (\mathcal{H}_q)$$

Definition 2.3 *We say that u is a weak solution of the quasi-stationary model if*

1. *u satisfies*

$$u \in L^\infty(I\!R^+; I\!H^1(\Omega))\ ,\ \frac{\partial u}{\partial t} \in I\!L^2(I\!R^+ \times \Omega)\ ,\ |u| = 1\ a.e. \tag{2.9}$$

2. *For all $\Psi \in \mathcal{C}^\infty(I\!R^+; I\!H^1(\Omega))$,*

$$\int_{I\!R^+\times\Omega} \left(\frac{\partial u}{\partial t}(t,x) + u(t,x)\right) \wedge \frac{\partial u}{\partial t}(t,x) \cdot \Psi(t,x) dx\, dt =$$

$$-2 \int_{I\!R^+\times\Omega} \sum_{i=1}^{3} u(t,x) \wedge \frac{\partial u}{\partial x_i}(t,x) \cdot \frac{\partial \Psi}{\partial x_i}(t,x) dx\, dt \tag{2.10}$$

$$+2 \int_{I\!R^+\times\Omega} u(t,x) \wedge (H(t,x) - \varphi(u(t,x))) \cdot \Psi(t,x) dx\, dt,$$

3. *$u(0,x) = u_0(x)$ in the trace sense.*

4. *For all $t \in I\!R^+$, $H(t,x)$ is the unique solution of*

$$\left\{\begin{array}{l} \text{curl}\ H(t,\cdot) = 0, \\ \\ \text{div}\ (H(t,\cdot) + \bar{u}(t,\cdot)) = 0, \\ \\ H(t,.) \in I\!L^2(I\!R^3). \end{array}\right. \tag{2.11}$$

5. *For all t we have the following energy estimate*

$$\mathcal{E}_q(t) + \int_0^t \int_\Omega |\frac{\partial u}{\partial t}(t,x)|^2 dx\, dt \le \mathcal{E}_q(0), \tag{2.12}$$

where

$$\mathcal{E}_q(t) = \int_\Omega \Big(|\nabla u(t,x)|^2 + 2\Phi(u(t,x))\Big)\, dx + \int_{I\!R^3} |H(t,x)|^2 dx.$$

Theorem 2.3 *We consider two sequences* $(\varepsilon^n)_n$ *and* $(\mu^n)_n$ *which tend to zero as* $n \to +\infty$ *and such that* μ^n/ε^n *remains bounded.*

Under the assumption $(\mathcal{H}_q)$ *if* u^n *denote a weak solution of (1.1)-(1.5) with* $\varepsilon_0 = \varepsilon^n$ *and* $\mu_0 = \mu^n$*, there exists a subsequence still denoted* $(u^n)_n$ *such that* u^n *tends to a limit* u *in* $L^\infty(\mathbb{R}^+; \mathbb{H}^1(\Omega))$ *weak* $\star$ *where* u *is a solution of the quasi-stationary model (2.9)-(2.12).*

This result is obtained via a time average process on H which avoid the high frequency oscillations of H.

Proposition 2.1 *Every point of the* ω*-limit set of any trajectory of (2.9)-(2.11) is solution of the steady state model (2.7).*

This last result is straightforward from the estimate

$$\int_{\mathbb{R}^+}\int_\Omega |\frac{\partial u}{\partial t}(t,x)|^2 dx\, dt < +\infty$$

and from the continuity of the map $u \mapsto H$ given by (2.11).

3 Proof of the existence.

The main point is to establish that $|u| = 1$ almost everywhere. In order to construct a solution which satisfies this condition we first solve a relaxed problem $\mathcal{P}_\lambda$ where u^λ takes its values in $\mathbb{R}^3$. The penalization term takes the form $\frac{1}{\lambda}(|u|^2 - 1)u$, λ tends to 0.

In fact instead of (1.1) we solve the following equation

$$\frac{\partial u^\lambda}{\partial t} - u^\lambda \wedge \frac{\partial u^\lambda}{\partial t} - 2\Delta u^\lambda - 2\varphi(u^\lambda) + \frac{1}{\lambda}(|u^\lambda|^2 - 1)u^\lambda = 2H. \tag{3.1}$$

By a Galerkine process we construct a solution of (3.1) satisfying an energy estimate, that allows us to pass to the limit as λ goes to zero. This limit u takes its values on S^2 and by a suitable test function we show that u satisfies (1.1).

First step. Resolution of (3.1).

Let us recall that the eigenfunctions of the operator $A = -\Delta + I$ with domain

$$D(A) = \{u \in \mathbb{H}^2(\Omega), \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega\}$$

build an orthonormal basis $\{\varphi_k\}_k$ in $\mathbb{L}^2(\Omega)$ and an orthogonal basis in $\mathbb{H}^1(\Omega)$ and $\mathbb{H}^2(\Omega)$.

We denote V_N the N dimensional vector space spaned by $\{\varphi_k\}_{1\le k\le N}$.

Now we introduce the Hilbert space

$$\mathbb{H}_{\text{curl}}(\mathbb{R}^3) = \{\psi \in \mathbb{L}^2(\mathbb{R}^3), \text{curl}\,\psi \in \mathbb{L}^2(\mathbb{R}^3)\}$$

We denote $\{\psi_k\}_k$ an hilbertian basis of $\mathbb{H}_{\text{curl}}(\mathbb{R}^3)$ orthonormal in $\mathbb{L}^2(\mathbb{R}^3)$ and W_N the N dimensional vector space spaned by $\{\psi_k\}_{1\le k\le N}$.

In the approximate problem we seek (u_N, H_N, E_N) in $V_N \times W_N \times W_N$ such that

$$u_N(t,x) = \sum_{k=1}^{N} v_k(t)\varphi_k(x),$$

$$H_N(t,x) = \sum_{k=1}^{N} h_k(t)\psi_k(x),$$

$$E_N(t,x) = \sum_{k=1}^{N} e_k(t)\psi_k(x),$$

which satisfies

1. For any Φ_N in V_N,

$$\int_\Omega \left(\frac{\partial u_N}{\partial t}(t,x) - u_N(t,x) \wedge \frac{\partial u_N}{\partial t}(t,x)\right) \cdot \Phi_N(x)dx + 2\int_\Omega \nabla u_N(t,x) \cdot \nabla \Phi_N(x)dx$$
$$+\frac{4}{\lambda}\int_\Omega (|u_N(t,x)|^2 - 1)u_N(t,x) \cdot \Phi_N(x)dx \qquad (3.2)$$
$$-2\int_\Omega \Big(H_N(t,x) - \varphi(u_N(t,x))\Big) \cdot \Phi_N(x)dx = 0.$$

2. For any Ψ_N in W_N,

$$\mu_0 \int_{I\!R^3} \frac{\partial}{\partial t}\left(H_N(t,x) + \bar{u}_N(t,x)\right) \cdot \Psi_N(x)dx + \int_{I\!R^3} E_N(t,x) \cdot \text{curl } \Psi_N(x)dx = 0. \qquad (3.3)$$

3. For any Θ_N in W_N

$$\varepsilon_0 \int_{I\!R^3} \frac{\partial E_N}{\partial t}(t,x) \cdot \Theta_N(x)dx - \int_{I\!R^3} H_N(t,x) \cdot \text{curl } \Theta_N(x)dx$$
$$+\sigma \int_\Omega (E_N(t,x) + f(t,x)) \cdot \Theta_N(x)dx = 0. \qquad (3.4)$$

4. With the initial data

$$\begin{cases} u_N(0) = \Pi_{V_N}(u_0), \\ E_N(0) = \Pi_{W_N}(E_0), \\ H_N(0) = \Pi_{W_N}(H_0), \end{cases} \qquad (3.5)$$

where Π_{V_N} (resp. Π_{W_N}) denotes the orthogonal projection on V_N (resp. W_N).

Let us remark that $v \mapsto v - u \wedge v$ is one to one in $I\!R^3$ so the equation (3.2) can be solve for the derivative in time. Then by Cauchy Picard theorem there exists a local solution of (3.2)-(3.5).

The following *a priori* estimates show that, in fact, the approximate solution is global in time.

Taking $\Phi_N = \frac{\partial u_N}{\partial t}$ in (3.2) one has

$$\begin{aligned}&\int_\Omega |\frac{\partial u_N}{\partial t}(t,x)|^2 dx + \frac{d}{dt}\int_\Omega |\nabla u_N(t,x)|^2 dx + \frac{1}{\lambda}\frac{d}{dt}\int_\Omega (|u_N(t,x)|^2-1)^2 dx \\ &+2\frac{d}{dt}\int_\Omega \Phi(u_N(t,x)) = \int_\Omega \frac{\partial u_N}{\partial t}(t,x)\cdot H_N(t,x)\end{aligned} \tag{3.6}$$

Now we put $\Psi_n = H_N$ in (3.3)

$$\frac{\mu_0}{2}\frac{d}{dt}\int_{I\!R^3} |H_N(t,x)|^2 dx + \int_{I\!R^3} \text{curl } E_N(t,x)\cdot H_N(t,x)dx = -\mu_0 \int_\Omega \frac{\partial u_N}{\partial t}(t,x)\cdot H_N(t,x)dx \tag{3.7}$$

In the same way taking $\Theta_N = E_N$ in (3.4),

$$\begin{aligned}&\frac{1}{2}\varepsilon_0 \frac{d}{dt}\int_{I\!R^3} |E_N(t,x)|^2 dx - \int_{I\!R^3} H_N(t,x)\cdot \text{curl } E_N(t,x)dx \\ &+\sigma \int_\Omega \Big(|E_N(t,x)|^2 + f(t,x)\cdot E_N(t,x)\Big) dx = 0\end{aligned} \tag{3.8}$$

Combining (3.6), (3.7) and (3.8) we derive the following estimate through Young inequality

$$\begin{aligned}&\frac{d}{dt}\Big\{\int_\Omega |\nabla u_N(t,x)|^2 dx + \frac{1}{\lambda}\int_\Omega (|u_N(t,x)|^2-1)^2 dx + \int_\Omega \Phi(u_N(t,x))dx\Big\} \\ &+\frac{1}{2}\frac{d}{dt}\Big\{\int_{I\!R^3} (|H_N(t,x)|^2 + \frac{\varepsilon_0}{\mu_0}|E_N(t,x)|^2)dx\Big\} \\ &+\int_\Omega |\frac{\partial u_N}{\partial t}(t,x)|^2 dx + \frac{\sigma}{\mu_0}\int_\Omega |E_N(t,x)|^2 dx \le \frac{\sigma}{\mu_0}\int_\Omega |f(t,x)|^2 dx\end{aligned}$$

As $\Phi(u_N)$ is non negative we obtain the following bound for u_0 in $I\!H^1(\Omega)$, E_0 and H_0 in $I\!L^2(I\!R^3)$ and f in $L^2(I\!R^+ \times \Omega)$:
There exists constants k_i independant of N and λ such that

$$\|\nabla u_N\|_{L^\infty(I\!R^+;\, I\!L^2(\Omega))} \le k_1, \ \ \|\frac{\partial u_N}{\partial t}\|_{I\!L^2(I\!R^+\times\Omega)} \le k_2, \ \ \|u_N\|_{L^\infty(I\!R^+;I\!L^4(\Omega))} \le k_3,$$

$$\|E_N\|_{L^\infty(I\!R^+;\, I\!L^2(I\!R^3))} \le k_4, \ \ \|H_N\|_{L^\infty(I\!R^+;\, I\!L^2(I\!R^3))} \le k_5.$$

So we can suppose that there exists a subsequence still denoted (u_N, H_N, E_N) such that when N goes to $+\infty$,

$$\begin{aligned}&u_N \rightharpoonup u^\lambda && \text{in} && L^\infty(I\!R^+;\, I\!H^1(\Omega)) \text{ weak } \star, \\ &\frac{\partial u_N}{\partial t} \rightharpoonup \frac{\partial u^\lambda}{\partial t} && \text{in} && L^2(I\!R^+;\, I\!L^2(\Omega)) \text{ weak }, \\ &E_N \rightharpoonup E^\lambda && \text{in} && L^\infty(I\!R^+;\, I\!L^2(I\!R^3)) \text{ weak } \star, \\ &H_N \rightharpoonup H^\lambda && \text{in} && L^\infty(I\!R^+;\, I\!L^2(I\!R^3)) \text{ weak } \star.\end{aligned}$$

And according to Aubin's Lemma

$$u_N \to u^\lambda \text{ in } L^4(0,T;\mathbb{L}^4(\Omega)) \text{ strong } \text{ for all } T,$$

Taking the limit in the equation (3.2)-(3.5) we obtain

1. For any Φ in $\mathbb{H}^1(\Omega)$

$$\begin{aligned}
&\int_\Omega \frac{\partial u^\lambda}{\partial t}(t,x)\cdot\Phi(x)dx - \int_\Omega u^\lambda(t,x)\wedge\frac{\partial u^\lambda}{\partial t}(t,x)\cdot\Phi(x)dx \\
&+2\int_\Omega \nabla u^\lambda(t,x)\cdot\nabla\Phi(x)dx + \frac{4}{\lambda}\int_\Omega(|u^\lambda(t,x)|^2-1)u^\lambda(t,x)\cdot\Phi(x)dx \\
&-2\int_\Omega \Big(H^\lambda(t,x)-\varphi(u^\lambda(t,x))\Big)\cdot\Phi(x)dx = 0 \text{ in } L^2(\mathbb{R}^+_t).
\end{aligned} \tag{3.9}$$

2. For any Ψ in $\mathbb{H}_{\text{curl}}(\mathbb{R}^3)$,

$$\mu_0 < \frac{\partial H^\lambda}{\partial t} + \frac{\partial \bar{u}^\lambda}{\partial t}, \Psi > + \int_{\mathbb{R}^3} E^\lambda(t,x)\cdot \text{curl}\,\Psi(x)dx = 0 \text{ in } \mathcal{D}'(\mathbb{R}^+). \tag{3.10}$$

3. For any Θ in $\mathbb{H}_{\text{curl}}(\mathbb{R}^3)$

$$\begin{aligned}
&\varepsilon_0 < \frac{\partial E^\lambda}{\partial t}, \Theta > - \int_{\mathbb{R}^3} H^\lambda(t,x)\cdot\text{curl}\,\Theta(x)dx \\
&+\sigma\int_\Omega \Big(E^\lambda(t,x)+f(t,x)\Big)\cdot\Theta(x)dx = 0 \text{ in } \mathcal{D}'(\mathbb{R}^+)
\end{aligned} \tag{3.11}$$

4. With the initial data

$$\begin{aligned}
&u^\lambda(0) = u_0 \text{ in } \mathbb{L}^2(\Omega), \\
&E^\lambda(0) = E_0, \quad H^\lambda(0) = H_0 \text{ in } \Big(H_{\text{curl}}(\mathbb{R}^3)\Big)'.
\end{aligned} \tag{3.12}$$

As the L^2 (resp. L^∞) norm is lower semi continuous for the weak (resp. weak $\star$) topology we obtain the energy estimate

$$\begin{aligned}
\forall\, t>0,\ \ &\mathcal{E}_\lambda(t) + \int_0^t\int_\Omega |\frac{\partial u^\lambda}{\partial t}(t,x)|^2 dx\,dt + \frac{\sigma}{2\mu_0}\int_0^t\int_\Omega |E^\lambda(t,x)|^2 dx\,dt \\
&\le \frac{\sigma}{2\mu_0}\int_0^t\int_\Omega |f(t,x)|^2 dx\,dt + \mathcal{E}_\lambda(0),
\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}
\mathcal{E}_\lambda(t) = &\int_\Omega |\nabla u^\lambda(t,x)|^2 dx + \frac{1}{\lambda}\int_\Omega(|u^\lambda(t,x)|^2-1)^2dx + \int_\Omega \Phi(u^{\,l}(t,x))dx \\
&+\frac{1}{2}\int_{\mathbb{R}^3}\Big(|H^\lambda(t,x)|^2 + \frac{\varepsilon_0}{\mu_0}|E^\lambda(t,x)|^2\Big)dx.
\end{aligned}$$

Second step. Limit as λ tends to 0.

We first note that as $|u_0| = 1$, $\mathcal{E}_\lambda(0)$ does not depend on λ.

The estimate (3.13) allows us to suppose via the extraction of a subsequence that when λ goes to 0

$$\begin{aligned}
u^\lambda &\rightharpoonup u && \text{in } L^\infty(\mathbb{R}^+;\, \mathbb{H}^1(\Omega)) \text{ weak } \star,\\
\frac{\partial u^\lambda}{\partial t} &\rightharpoonup \frac{\partial u}{\partial t} && \text{in } \mathbb{L}^2(\mathbb{R}^+ \times \Omega) \text{ weakly},\\
u^\lambda &\to u && \text{in } L^2((0,T);\, \mathbb{L}^2(\Omega)) \text{ strongly for all } T > 0 \text{ and } a.e.,\\
E^\lambda &\rightharpoonup E && \text{in } L^\infty(\mathbb{R}^+;\, \mathbb{L}^2(\mathbb{R}^3)) \text{ weak } \star,\\
H^\lambda &\rightharpoonup H && \text{in } L^\infty(\mathbb{R}^+;\, \mathbb{L}^2(\mathbb{R}^3)) \text{ weak } \star.
\end{aligned}$$

• We remark, and it is the main point of the proof, that $|u| = 1$ $a.e.$ in $\mathbb{R}^+ \times \Omega$, as $u^\lambda \to u$ $a.e.$

• Now we derive the equation satisfied by u by taking in (3.9) $\Phi = u^\lambda(t,x) \wedge \xi(t,x)$ where ξ is any test function given in $\mathbb{L}^2_{loc}(\mathbb{R}^+; \mathbb{H}^2(\Omega))$.

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{\partial u^\lambda}{\partial t}(t,x) \cdot (u^\lambda(t,x) \wedge \xi(t,x)) dx\, dt\\
&- \int_0^T \int_\Omega u^\lambda(t,x) \wedge \frac{\partial u^\lambda}{\partial t}(t,x) \cdot (u^\lambda(t,x) \wedge \xi(x)) dx\, dt\\
&+2 \int_0^T \int_\Omega \sum_{i=1}^3 \frac{\partial u^\lambda}{\partial x_i}(t,x) \cdot \frac{\partial}{\partial x_i} \Big(u^\lambda(t,x) \wedge \xi(t,x)\Big)\, dx\, dt\\
&-2 \int_0^T \int_\Omega \Big(H^\lambda(t,x) - \varphi(u^\lambda(t,x))\Big) \cdot \Big(u^\lambda(t,x) \wedge \xi(t,x)\Big)\, dx\, dt\\
&+\frac{4}{\lambda} \int_0^T \int_\Omega (|u^\lambda(t,x)|^2 - 1) u^\lambda(t,x) \cdot \Big(u^\lambda(t,x) \wedge \xi(t,x)\Big)\, dx\, dt = 0
\end{aligned} \tag{3.14}$$

The last term of the left-hand side of (3.14) vanishes identically. Furthermore we remark that

$$\frac{\partial u^\lambda}{\partial x_i} \cdot \frac{\partial}{\partial x_i}(u^\lambda \wedge \xi) = -(u^\lambda \wedge \frac{\partial u^\lambda}{\partial x_i}) \cdot \frac{\partial \xi}{\partial x_i}.$$

Now we can take the limit when λ goes to 0 to obtain

$$\begin{aligned}
&\int_0^T \int_\Omega \left(\frac{\partial u}{\partial t}(t,x) - u(t,x) \wedge \frac{\partial u}{\partial t}(t,x)\right) \cdot (u(t,x) \wedge \xi(t,x))\, dx\, dt\\
&-2 \int_0^T \int_\Omega \sum_{i=1}^3 \frac{\partial \xi}{\partial x_i}(t,x) \cdot \left(u(t,x) \wedge \frac{\partial u}{\partial x_i}(t,x)\right) dx\, dt\\
&-2 \int_0^T \int_\Omega \Big(H(t,x) - \varphi(u(t,x))\Big) \cdot (u(t,x) \wedge \xi(t,x))\, dx\, dt = 0,
\end{aligned}$$

that is

$$\int_0^T \int_\Omega \left(\frac{\partial u}{\partial t}(t,x) + u(t,x) \wedge \frac{\partial u}{\partial t}(t,x) \right) \cdot \xi(t,x) dx\, dt$$

$$+2 \int_0^T \int_\Omega \sum_{i=1}^3 (u(t,x) \wedge \frac{\partial u}{\partial x_i}(t,x)) \cdot \frac{\partial \xi}{\partial x_i}(t,x) dx\, dt \tag{3.15}$$

$$-2 \int_0^T \int_\Omega u(t,x) \wedge (H(t,x) - \varphi(u(t,x))) \cdot \xi(t,x) dx\, dt = 0$$

as

$$\frac{\partial u}{\partial t} \cdot (u \wedge \xi) = -(u \wedge \frac{\partial u}{\partial t}) \cdot \xi, \text{ and } -(u \wedge \frac{\partial u}{\partial t}) \cdot (u \wedge \xi) = -\frac{\partial u}{\partial t} \cdot \xi$$

since $|u| = 1$ a.e. in $I\!R^+ \times \Omega$.

• Moreover as the L^2 (resp. L^∞) norm is lower semi continuous for the weak (resp. weak $\star$) the energy estimate (3.13) remains valid for $|u_0| = 1$.

• Next from (3.15) we derive that

$$\sum_{i=1}^3 \frac{\partial}{\partial x_i}(u \wedge \frac{\partial u}{\partial x_i}) \text{ belongs to } L^2_{loc}(I\!R^+; I\!L^2(\Omega))$$

so $u \wedge \dfrac{\partial u}{\partial \nu}$ makes sense in $L^2_{loc}(I\!R^+; I\!H^{-1/2}(\partial\Omega))$.

Moreover as $|u|^2 = 1$, one has $u \cdot \dfrac{\partial u}{\partial \nu} = 0$. So from the equality

$$\frac{\partial u}{\partial \nu} = \left(u \cdot \frac{\partial u}{\partial \nu} \right) u + u \wedge \left(u \wedge \frac{\partial u}{\partial \nu} \right) = \frac{\partial u}{\partial \nu}$$

which is valid in $H^{-1-\eta}(\partial\Omega)$ for any $\eta > 0$ according to the product of function in sobolev spaces (see L. Hörmander [6]) so in fact

$$\frac{\partial u}{\partial \nu} \text{ makes sense in } L^2_{loc}(I\!R^+;\ H^{-1-\eta}(\partial\Omega)) \text{ for any } \eta > 0.$$

• As the Maxwell equations are linear, it is straightforward to take the limit in (3.10) and (3.11) to obtain (2.3) and (2.4).

4 Description of the ω-limit set.

Consider a weak solution u of (1.1)-(1.5). From the energy estimate (2.5), the ω-limit set $\omega(u)$ is not empty. We denote u_∞ a point of this set.

Hence there exists a sequence $(t_n)_{n\geq 1}$, with $\lim_{n\to+\infty} t_n = +\infty$ such that $u(t_n, .)$ tends to u_∞ in $I\!H^1(\Omega)$ weak, in $I\!L^2(\Omega)$ strong, and almost everywhere in Ω. In particular one has $|u| = 1$ *a.e.* in Ω.

First step. Let be a a non negative real number. For s in $(-a, a)$ and x in Ω we define for n large enough

$$U_n(s,x) = u(t_n + s, x).$$

The sequence $(U_n)_{n\geq 1}$ tends to u_∞ in $\mathbb{L}^2((-a,a)\times\Omega)$ strongly and in $L^2((-a,a);\mathbb{H}^1(\Omega))$ weakly. In fact following [12], we have the estimate

$$\begin{aligned}\frac{1}{2a}\int_{-a}^{a}\int_\Omega |U_n(s,x)-u(t_n,x)|^2 dx\,ds &= \frac{1}{2a}\int_{-a}^{a}\int_\Omega \left|\int_0^s \frac{\partial u}{\partial t}(t_n+\tau,x)d\tau\right|^2 dx\,ds \\ &\leq \frac{1}{2a}\int_{-a}^{a}|s|\int_\Omega\int_{t_n-a}^{+\infty}\left|\frac{\partial u}{\partial t}(\tau,x)\right|^2 d\tau dx\,ds \\ &\leq a\int_{t_n-a}^{+\infty}\int_\Omega\left|\frac{\partial u}{\partial t}(\tau,x)\right|^2 dx\,d\tau.\end{aligned}$$

Now, as $\dfrac{\partial u}{\partial t}$ lies in $\mathbb{L}^2(\mathbb{R}^+\times\Omega)$, one gets

$$\lim_{n\to+\infty}\frac{1}{2a}\int_{-a}^{a}\int_\Omega |U_n(s,x)-u(t_n,x)|^2\,dx\,ds = 0.$$

Since $u(t_n,.)$ tends to u_∞ in $\mathbb{L}^2(\Omega)$ strongly, U_n tends to u_∞ in $L^2((-a,a);\mathbb{L}^2(\Omega))$ strongly.

Moreover we obviously see that the sequence $(\nabla U_n)_{n\geq 1}$ is bounded in $\mathbb{L}^2((-a,a)\times\Omega)$ so there exists a subsequence still noted $(U_n)_{n\geq 1}$ such that U_n tends to u_∞ in $L^2((-a,a);\mathbb{H}^1(\Omega))$ weakly, in $L^2((-a,a);\mathbb{L}^2(\Omega))$ strongly and almost everywhere in Ω.

Second step. We consider a $\mathcal{C}^\infty$ non negative function ρ_a supported by $(-a,a)$ satisfying

$$\rho_a(\tau) = 1 \text{ for } \tau\in(-a+1,a-1),$$

$$0\leq\rho_a(\tau)\leq 1,\ \ |\rho_a'(\tau)|\leq 2.$$

We set

$$H_a^n(x) = \frac{1}{2a}\int_{-a}^{a} H(t_n+s,x)\rho_a(s)ds$$

and

$$E_a^n(x) = \frac{1}{2a}\int_{-a}^{a} E(t_n+s,x)\rho_a(s)ds.$$

From the estimate (2.5), E and H are bounded in $L^\infty(\mathbb{R}^+;\,\mathbb{L}^2(\mathbb{R}^3))$. Then H_a^n and E_a^n are bounded in $\mathbb{L}^2(\mathbb{R}^3)$ independently of n and a. So by extracting a subsequence we may suppose that $(E_a^n\,,\,H_a^n)_{n\geq 1}$ converges in $\mathbb{L}^2(\mathbb{R}^3)$ weakly to (E_a,H_a) when n goes to $+\infty$.

Third step. In the weak formulation (2.2) we take as test function $\rho_a(t-t_n)\Psi(x)$ where Ψ is a function lying in $\mathcal{D}(\bar{\Omega})$. We obtain after the change of chart $s=t-t_n$

$$\begin{aligned}&\frac{1}{2a}\int_{-a}^{a}\int_\Omega\left(\frac{\partial U_n}{\partial t}(s,x)+U_n(s,x)\wedge\frac{\partial U_n}{\partial t}(s,x)\right)\cdot\Psi(x)\rho_a(s)dx\,ds \\ &+2\frac{1}{2a}\int_{-a}^{a}\int_\Omega\sum_{i=1}^{3}\left(U_n(s,x)\wedge\frac{\partial U_n}{\partial x_i}(s,x)\right)\cdot\frac{\partial\Psi}{\partial x_i}\rho_a(s)dx\,ds \\ &-2\frac{1}{2a}\int_{-a}^{a}\int_\Omega U_n\wedge\Big(H(t_n+s,x)-\varphi(U_n(s,x))\Big)\cdot\Psi(x)\rho_a(s)dx\,ds = 0.\end{aligned} \tag{4.1}$$

To pass through the limit in (4.1) we bound separately each term of (4.1).

• First term.

$$\left|\frac{1}{2a}\int_{-a}^{a}\int_{\Omega}\frac{\partial U_n}{\partial t}(s,x)\cdot\Psi(x)\rho_a(s)dx\,ds\right|$$

$$\le \frac{1}{2a}\int_{-a}^{a}\rho_a(s)\left(\int_{\Omega}\left|\frac{\partial U_n}{\partial t}(s,x)\right|^2 dx\right)^{1/2}\left(\int_{\Omega}|\Psi(x)|^2\,dx\right)^{1/2} ds$$

$$\le \frac{1}{\sqrt{2a}}\left(\int_{\Omega}|\Psi(x)|\,dx\right)^{1/2}\left(\int_{t_n-a}^{t_n+a}\int_{\Omega}\left|\frac{\partial u}{\partial t}(s,x)\right|^2 dx\,ds\right)^{1/2}$$

Since $\frac{\partial u}{\partial t}$ belongs to $\mathbb{L}^2(\mathbb{R}^+\times\Omega)$, this last term tends to zero as n goes to $+\infty$. In the same way, as U_n takes its values on S^2, one also has

$$\lim_{n\to+\infty}\frac{1}{2a}\int_{-a}^{a}\int_{\Omega}U_n(s,x)\wedge\frac{\partial U_n}{\partial t}(s,x)\rho_a(s)\cdot\Psi(x)dx\,ds=0$$

• Second term.

As $(U_n)_{n\geq 1}$ tends to u_∞ strongly in $\mathbb{L}^2((-a,a)\times\Omega)$, as $(\frac{\partial U_n}{\partial x_i})_{n\geq1}$ tends to $\frac{\partial u_\infty}{\partial x_i}$ weakly in $\mathbb{L}^2((-a,a)\times\Omega)$ and since $\frac{\partial \Psi}{\partial x_i}\rho_a$ belongs to $\mathbb{L}^\infty((-a,a)\times\Omega)$, the second term of (4.1) tends to

$$2\,\frac{1}{2a}\int_{-a}^{a}\rho_a(s)ds\int_{\Omega}\sum_{i=1}^{3}\left(u_\infty(x)\wedge\frac{\partial u_\infty}{\partial x_i}(x)\right)\cdot\frac{\partial\Psi}{\partial x_i}(x)dx.$$

• Third term.

$$\frac{1}{2a}\int_{-a}^{a}\int_{\Omega}U_n(s,x)\wedge H(t_n+s,x)\cdot\Psi(x)\rho_a(s)dx\,ds$$

$$=\frac{1}{2a}\int_{-a}^{a}\int_{\Omega}\left(U_n(s,x)-u_\infty(x)\right)\wedge H(t_n+s,x)\cdot\Psi(x)\rho_a(s)dx\,ds \tag{4.2}$$

$$+\frac{1}{2a}\int_{-a}^{a}\int_{\Omega}u_\infty(x)\wedge H(t_n+s,x)\cdot\Psi(x)\rho_a(s)dx\,ds.$$

The first term of (4.2) goes to zero as $(U_n-u_\infty)_n$ tends strongly to zero in $\mathbb{L}^2((-a,a)\times\Omega)$ and as H is bounded in $L^\infty(\mathbb{R}^+;\,\mathbb{L}^2(\mathbb{R}^3))$. The second term is equal to

$$\int_{\Omega}\left(u_\infty(x)\wedge H_a^n(x)\right)\cdot\Psi(x)dx,$$

and tends obviously to

$$\int_{\Omega}\left(u_\infty(x)\wedge H_a(x)\right)\cdot\Psi(x)dx.$$

As φ is linear, it is straightforward to take the limit in the last term.

So from equation (4.1) we derive that u_∞ solve the equation

$$\int_{\Omega}\sum_{i=1}^{3}\left(u_\infty(x)\wedge\frac{\partial u_\infty}{\partial x_i}(x)\right)\cdot\frac{\partial\Psi}{\partial x_i}(x)+\int_{\Omega}\left(u_\infty(x)\wedge\varphi(u_\infty(x))\right)\cdot\Psi(x)dx$$
$$\frac{2a}{\int_{-a}^{a}\rho(s)ds}\int_{\Omega}\left(u_\infty(x)\wedge H_a(x)\right)\cdot\Psi(x)dx=0. \tag{4.3}$$

Forth step. In order to obtain the desired result it remains to take the limit in (4.3) when a tends to $+\infty$.

We first remark that

$$\lim_{a\to+\infty} \frac{2a}{\int_{-a}^{a} \rho(s)ds} = 1.$$

Through estimate (2.5) and by definition of H_a, $(H_a)_{a\geq 1}$ is uniformly bounded in $\mathbb{L}^2(\mathbb{R}^3)$. Hence, by extraction we can suppose that H_a tends to H_∞ weakly in $\mathbb{L}^2(\mathbb{R}^3)$. So at the limit one has

$$-\int_\Omega \sum_{i=1}^{3} \left(u_\infty(x) \wedge \frac{\partial u_\infty}{\partial x_i}(x)\right) \cdot \frac{\partial \Psi}{\partial x_i}(x)dx + \int_\Omega u_\infty(x) \wedge \Big(H_\infty(x) - \varphi(u_\infty(x))\Big) \cdot \Psi(x)dx = 0$$

Fifth step. In order to derive the equation satisfied by H_∞ we first recall the equation verified by H_a^n and E_a^n.
In equation (2.3) we take $\Psi(t,x) = \theta_a(t-t_n)\nabla\xi(x)$ with ξ in $\mathcal{D}(\mathbb{R}^3)$ and θ_a is defined by

$$\theta_a(t) = \int_a^t \rho_a(s)ds.$$

We obtain that for every ξ in $\mathcal{D}(\mathbb{R}^3)$

$$-\int_{-a}^{a}\int_{\mathbb{R}^3} \Big(H(t_n+s,x) + \bar{u}(t_n+s,x)\Big) \cdot \nabla\xi(x)\rho_a(s)ds = \int_{\mathbb{R}^3} \Big(H_0(x) + \bar{u}_0(x)\Big) \cdot \nabla\xi(x)dx\theta_a(0).$$

As div $(H_0 + \bar{u}_0) = 0$ in $\mathcal{D}'(\mathbb{R}^3)$, we obtain after dividing by $2a$

$$\int_{\mathbb{R}^3} \left(H_a^n(x) + \frac{1}{2a}\int_{-a}^{a} \bar{u}(t_n+s,x)\rho_a(s)ds\right) \cdot \nabla\xi(x)dx = 0.$$

When n goes to $+\infty$ we obtain that

$$\int_{\mathbb{R}^3} \Big(H_a(x) + \bar{u}_\infty(x)\Big) \cdot \nabla\xi(x)dx = 0,$$

and so when a goes to infinity we get

$$\text{div } (H_\infty + \bar{u}_\infty) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3).$$

Now we take $\Psi(t,x) = \rho_a(t-t_n)\xi(x)$ in (2.4). We obtain that

$$\begin{aligned}
&\frac{1}{2a}\int_{-a}^{a}\int_{\mathbb{R}^3} E(t_n+s,x)\cdot\rho_a'(s)\xi(x)dx\,ds - \int_{\mathbb{R}^3} H_a^n(x)\cdot \text{curl } \xi(x)dx \\
&+\sigma\int_\Omega E_a^n(x)\cdot\xi(x)dx + \sigma\frac{1}{2a}\int_{-a}^{a}\int_\Omega f(t_n+s,x)\cdot\rho_a(s)\xi(x)dx\,ds \qquad (4.4)\\
&= \int_{\mathbb{R}^3} E_0(x)\cdot\xi(x)dx\ \rho_a(-t_n).
\end{aligned}$$

For n large enough, the righthand side of (4.4) vanishes identically.

Let us bound the first term of (4.4). As ρ_a' is identically zero on $(-a+1, a-1)$ and is bounded by 2, one has

$$\left|\frac{1}{2a}\int_{-a}^{a}\int_{\mathbb{R}^3} E(t_n+s,x)\cdot\rho_a'(s)\xi(x)dx\right| \leq \frac{1}{a}||\xi||_{\mathbb{L}^2(\mathbb{R}^3)}||E||_{L^\infty(\mathbb{R}^+;\,\mathbb{L}^2(\mathbb{R}^3))}. \qquad (4.5)$$

Moreover

$$\left|\frac{1}{2a}\int_{-a}^{a}\int_{I\!R^3} f(t_n+s,x)\cdot\rho_a(s)\xi(x)dx\,ds\right|$$

$$\leq \frac{1}{2a}\left(\int_{-a+t_n}^{a+t_n}||f(s)||^2_{I\!L^2(\Omega)}\right)^{1/2}\left(\int_{-a}^{a}\rho_a(s)^2ds\right)^{1/2}||\xi||_{I\!L^2(\Omega)},$$

that is

$$\left|\frac{1}{2a}\int_{-a}^{a}\int_{I\!R^3} f(t_n+s,x)\cdot\rho_a(s)\xi(x)dx\,ds\right| \leq \frac{1}{\sqrt{2a}}\left(\int_{-a+t_n}^{a+t_n}||f(s)||^2_{I\!L^2(\Omega)}\right)^{1/2}||\xi||_{I\!L^2(\Omega)} \qquad (4.6)$$

since $0 \leq \rho_a(s) \leq 1$.

When n goes to infinity, by extraction of a subsequence the first term of the left-hand side of (4.4) tends to a real α_a satisfying

$$|\alpha_a| \leq \frac{1}{2a}||E||_{L^\infty(I\!R^+;\, I\!L^2(I\!R^3))}||\xi||_{I\!L^2(\Omega)} \qquad (4.7)$$

Due to (4.6), the fourth term of the left-hand side of (4.4) goes to zero as

$$\int_{I\!R^+}\int_\Omega |f(t,x)|^2\,dx\,dt < +\infty.$$

Hence we obtain

$$\alpha_a - \int_{I\!R^3} H_a(x)\cdot \text{curl}\ \xi(x)dx + \sigma\int_\Omega E_a(x)\cdot\xi(x)dx = 0.$$

Then taking the limit as a goes to infinity, one has from (4.7)

$$\int_{I\!R^3} H_\infty(x)\cdot \text{curl}\ \xi(x)dx = \sigma\int_\Omega E_\infty(x)\cdot\xi(x)dx. \qquad (4.8)$$

In the same way, taking $\Psi(t,x) = \rho_a(t_n - t)\xi(x)$ in (2.3) we derive that

$$\int_{I\!R^3} E_\infty(x)\cdot \text{curl}\ \xi(x)dx = 0,$$

that is curl $E_\infty = 0$. So it is valid to take $\xi(x) = E_\infty(x)$ in (4.8) which leads to

$$\sigma\int_\Omega |E_\infty(x)|^2dx = 0.$$

This (4.8) gives curl $H_\infty = 0$. Finaly H_∞ is uniquely determined by

$$\begin{cases} \text{div}\ (H_\infty + \bar{u}_\infty) = 0 \text{ in } I\!R^3, \\ \text{curl}\ H_\infty = 0 \text{ in } I\!R^3, \\ H_\infty \in I\!L^2(I\!R^3). \end{cases}$$

Therefore u_∞ is a solution of the stationary model (2.6)-(2.8).

Remark 4.1 Following an idea of G. Métivier, it is possible to prove Theorem 2.2 without average Maxwell Equations. This is due to the fact that $H(t,.) - H(u(t))$ tends to zero in L^2_{loc} when t tends to $+\infty$ (see [8]).

5 Quasi-stationary model

The last part of this paper is devoted to the justification of the quasi-stationary model.

We recall that we suppose $f \equiv 0$.

We consider ε^n and μ^n such that ε^n, μ^n and ε^n/μ^n tend to zero. In the sequel we denote (u^n, H^n, E^n) a family of weak solutions of (1.1)-(1.5) with $\varepsilon_0 = \varepsilon^n$ and $\mu_0 = \mu^n$.

We recall the energy estimate satisfied by $(u^n,\ H^n,\ E^n)$.

$$\mathcal{E}^n(t) + \int_0^t \int_\Omega |\frac{\partial u^n}{\partial t}(t,x)|^2 dx\, dt + \frac{\sigma}{\mu^n} \int_0^t \int_\Omega |E^n(t,x)|^2 dx\, dt \le \mathcal{E}^n(0) \tag{5.1}$$

where

$$\mathcal{E}^n(t) = \int_\Omega \Big(|\nabla u^n(t,x)|^2 + 2\Phi(u^n(t,x)) \Big)\, dx + \int_{I\!R^3} \Big(|H^n(t,x)|^2 + \frac{\varepsilon^n}{\mu^n} |E^n(t,x)|^2 \Big)\, dx.$$

Since ε^n/μ^n remains bounded, the right hand-side term of (5.1) remains bounded uniformly in n. Therefore, by the energy estimate (5.1), u^n is bounded in $L^\infty(I\!R^+;\ I\!H^1(\Omega))$ and $\dfrac{\partial u^n}{\partial t}$ is bounded in $I\!L^2(I\!R^+ \times \Omega)$ uniformly in n. Furthermore H^n and $\sqrt{\varepsilon^n/\mu^n}\, E^n$ are uniformly bounded in $L^\infty(I\!R^+;\ I\!L^2(I\!R^3))$. Extracting a subsequence we can suppose that

$$u^n \rightharpoonup u \qquad \text{in } L^\infty(I\!R^+;\ I\!H^1(\Omega)) \text{ weak } \star,$$

$$u^n \to u \qquad \text{in } L^2((0,T);\ I\!L^2(\Omega)) \text{ strong for all } T > 0,$$

$$\frac{\partial u^n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \qquad \text{in } L^2((0,T);\ I\!L^2(\Omega)) \text{ weak for all } T > 0.$$

First step.

For any $a > 0$ we set

$$\begin{aligned} u_a^n(t,x) &:= \frac{1}{a}\int_0^a u^n(t+s,x)ds, \\ H_a^n(t,x) &:= \frac{1}{a}\int_0^a H^n(t+s,x)ds, \\ E_a^n(t,x) &:= \frac{1}{a}\int_0^a E^n(t+s,x)ds. \end{aligned} \tag{5.2}$$

Lemma 5.1 *For each $n \in I\!N$ and $a > 0$, $(u_a^n,\ H_a^n,\ E_a^n)$ satisfies the following estimates.*

$$\|u_a^n\|_{L^\infty(I\!R^+;\, I\!H^1(\Omega))} \le \|u^n\|_{L^\infty(I\!R^+;\, I\!H^1(\Omega))}, \tag{5.3}$$

$$\|\frac{\partial u_a^n}{\partial t}\|_{I\!L^2(I\!R^+\times\Omega)} \le \|\frac{\partial u^n}{\partial t}\|_{I\!L^2(I\!R^+\times\Omega)}, \tag{5.4}$$

$$\|H_a^n\|_{L^\infty(I\!R^+;\, I\!L^2(I\!R^3))} \le \|H^n\|_{L^\infty(I\!R^+;\, I\!L^2(I\!R^3))}, \tag{5.5}$$

$$\|E_a^n\|_{L^\infty(I\!R^+;\, I\!L^2(I\!R^3))} \le \|E^n\|_{L^\infty(I\!R^+;\, I\!L^2(I\!R^3))}. \tag{5.6}$$

Proof. The estimates (5.3), (5.5) and (5.6) follow directly from the definition (5.2).

For (5.4) we write

$$\frac{\partial u_a^n}{\partial t}(t,x) = \frac{1}{a}(u^n(t+a,x) - u^n(t,x)) = \int_0^1 \frac{\partial u^n}{\partial t}(t+\theta a, x)d\theta,$$

so

$$\int_{I\!R^+} |\frac{\partial u_a^n}{\partial t}(s,x)|^2 ds \leq \int_{I\!R^+} \left(\int_0^1 \frac{\partial u^n}{\partial t}(t+\theta a,x)d\theta\right)^2 dt \leq \int_{I\!R^+} |\frac{\partial u^n}{\partial t}(s,x)|^2 ds.$$

That is

$$\int_{I\!R^+\times\Omega} |\frac{\partial u_a^n}{\partial t}(s,x)|^2 ds\, dx \leq \int_{I\!R^+\times\Omega} |\frac{\partial u^n}{\partial t}(s,x)|^2 ds\, dx.$$

Lemma 5.2 *For every $a > 0$ we have the following estimate*

$$\|u_a^n - u^n\|_{L^\infty(I\!R^+;\, I\!L^2(\Omega))} \leq \sqrt{a}\,\|\frac{\partial u^n}{\partial t}\|_{I\!L^2(I\!R^+\times\Omega)}.$$

Proof. From the definition (5.2) one gets

$$\begin{aligned} u_a^n(t,x) - u^n(t,x) &= \frac{1}{a}\int_0^a (u^n(s+t,x) - u^n(t,x))ds \\ &= \frac{1}{a}\int_0^a \int_0^s \frac{\partial u^n}{\partial t}(t+\tau,x)d\tau\, ds, \end{aligned}$$

so

$$\begin{aligned} |u_a^n(t,x) - u^n(t,x)|^2 &\leq \left|\frac{1}{a}\int_0^a \int_0^s |\frac{\partial u^n}{\partial t}(t+\tau,x)|d\tau\, ds\right|^2 \\ &\leq \left|\int_0^a |\frac{\partial u^n}{\partial t}(t+\tau,x)|d\tau\right|^2 \\ &\leq a\int_t^{t+a} |\frac{\partial u^n}{\partial t}(s,x)|^2 ds, \end{aligned}$$

hence

$$\int_\Omega |u_a^n(t,x) - u^n(t,x)|^2 dx \leq a\int_{I\!R^+}\int_\Omega |\frac{\partial u^n}{\partial t}(s,x)|^2 ds\, dx.$$

Second step. We choose $a_n = (\varepsilon^n \mu^n)^{\frac{1}{4}}$, and we denote in the sequel

$$u_n := u_{a_n}^n, \quad H_n := H_{a_n}^n, \quad \text{and } E_n := E_{a_n}^n.$$

Thanks to the energy estimate (5.1) and Lemma 5.1, we can suppose after extraction of a subsequence that

$$\begin{aligned} &u_n \rightharpoonup u^\infty && \text{in } L^\infty(I\!R^+;\, I\!H^1(\Omega)) \text{ weak } \star, \\ &u_n \to u^\infty && \text{in } L^2((0,T);\, I\!L^2(\Omega)) \text{ strong for all } T > 0, \\ &H_n \rightharpoonup H^\infty && \text{in } L^\infty(I\!R^+;\, I\!L^2(I\!R^3)) \text{ weak } \star, \\ &\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u^\infty}{\partial t} && \text{in } I\!L^2(I\!R^+\times\Omega) \text{ weak}. \end{aligned}$$

Furthermore Lemma 5.2 ensures that $u^\infty = u$ and $u_n(0,\cdot) \to u_0(\cdot)$ in $I\!L^2(\Omega)$ strong.

Third step. For t given in $I\!R^+$ we take $\Psi(s,x) = \mathbf{1}_{[t,t+a[}(s)\xi(x)$ in (2.1). After dividing by a_n we obtain that

$$\int_\Omega \frac{\partial u_n}{\partial t}(t,x)\cdot\xi(x)dx + \int_\Omega \frac{1}{a_n}\int_0^{a_n} (u^n(t+s,x)\wedge \delta dt u^n(t+s,x))\cdot \xi(x)ds\, dx$$

$$+2\int_\Omega \frac{1}{a_n}\int_0^{a_n}\sum_{i=1}^3 \left(u^n(t+s,x)\wedge \frac{\partial u^n}{\partial x_i}(t+s,x)\right)\cdot \frac{\partial \xi}{\partial x_i}(x)ds\, dx$$

$$-2\int_\Omega \frac{1}{a_n}\int_0^{a_n} u^n(t+s,x)\wedge\Big(H^n(t+s,x)-\varphi(u^n(t+s,x))\Big)\cdot\xi(x)ds\, dx = 0.$$

Multiplying this last formula by a test function $\rho(t)$, we obtain after integration

$$\begin{aligned}
&\int_{I\!R^+\times\Omega} \frac{\partial u_n}{\partial t}(t,x)\cdot\xi(x)\rho(t)dx\, dt \\
&+\int_{I\!R^+\times\Omega}\frac{1}{a_n}\int_0^{a_n}\left(u^n(t+s,x)\wedge\frac{\partial u^n}{\partial t}(t+s,x)\right)\cdot\xi(x)\rho(t)ds\, dx\, dt \\
&+\int_{I\!R^+\times\Omega}\frac{1}{a_n}\int_0^{a_n}\sum_{i=1}^3\left(u^n(t+s,x)\wedge\frac{\partial u^n}{\partial x_i}(t+s,x)\right)\cdot\frac{\partial\xi}{\partial x_i}(x)\rho(t)ds\, dx\, dt \\
&-\frac{2}{a_n}\int_{I\!R^+\times\Omega}\int_t^{t+a_n} u^n(s,x)\wedge\Big(H^n(s,x)-\varphi(u^n(s,x))\Big)\cdot\xi(x)\rho(t)ds\, dx\, dt = 0.
\end{aligned} \tag{5.7}$$

As $\dfrac{\partial u_n}{\partial t} \rightharpoonup \dfrac{\partial u^\infty}{\partial t}$ in $I\!L^2(I\!R^+\times\Omega)$ weakly, the first term of (5.7) tends to

$$\int_{I\!R^+}\int_\Omega \frac{\partial u^\infty}{\partial t}(t,x)\cdot\xi(x)\rho(t)dx\, dt.$$

Let us now study the second term.

$$\begin{aligned}
&\int_{I\!R^+\times\Omega}\frac{1}{a_n}\int_0^{a_n} u^n(t+s,x)\wedge\frac{\partial u^n}{\partial t}(t+s,x)\cdot\xi(x)\rho(t)ds\, dx\, dt = \\
&\int_{I\!R^+\times\Omega}\rho(t)\xi(x)\cdot u^n(t,x)\wedge\left(\frac{1}{a_n}\int_0^{a_n}\frac{\partial u^n}{\partial t}(t+s,x)ds\right)dx\, dt \\
&+\int_{I\!R^+\times\Omega}\rho(t)\xi(x)\cdot\frac{1}{a_n}\int_0^{a_n}\Big(u^n(t+s,x)-u^n(t,x)\Big)\wedge\frac{\partial u^n}{\partial t}(s,x)ds\, dt\, dx.
\end{aligned}$$

The definition of u_n shows that this is equal to

$$\begin{aligned}
&\int_{I\!R^+\times\Omega}\frac{1}{a_n}\int_0^{a_n}\left(u^n(t+s,x)\wedge\frac{\partial u^n}{\partial t}(t+s,x)\right)\cdot\xi(x)\rho(t)ds\, dx\, dt = \\
&\int_{I\!R^+\times\Omega}\rho(t)\xi(x)\cdot\left(u^n(t,x)\wedge\frac{\partial u_n}{\partial t}(t,x)\right)dt\, dx \\
&+\int_{I\!R^+\times\Omega}\rho(t)\xi(x)\cdot\frac{1}{a_n}\int_0^{a_n}\Big(u^n(t+s,x)-u^n(t,x)\Big)\wedge\frac{\partial u^n}{\partial t}(s,x)ds\, dt\, dx.
\end{aligned} \tag{5.8}$$

The first term of (5.8) tends to

$$\int_{I\!R^+\times\Omega} \rho(t)\xi(x)\cdot\left(u^\infty(t,x)\wedge\frac{\partial u^\infty}{\partial t}(t,x)\right)dt\,dx$$

as

$$u^n \to u^\infty \text{ in } L^2_{loc}(I\!R^+; I\!L^2(\Omega)) \text{ strongly}$$

and

$$\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u^\infty}{\partial t} \text{ in } I\!L^2(I\!R^+\times\Omega) \text{ weakly.}$$

Now we prove that the second term goes to zero. We use the Cauchy-Schwarz inequality to obtain

$$A := \left|\int_{I\!R^+\times\Omega} \rho(t)\xi(x)\cdot\frac{1}{a_n}\int_0^{a_n}(u^n(t+s,x)-u^n(t,x))\wedge\frac{\partial u^n}{\partial t}(t+s,x)ds\,dx\,dt\right|$$

$$A \le \|\xi\|_{I\!L^\infty(\Omega)}\|\rho\|_{I\!L^\infty(I\!R^+)}\frac{1}{a_n}\left\{\int_{I\!R^+\times\Omega}\int_0^{a_n}\left(\int_0^s\frac{\partial u^n}{\partial t}(t+\tau,x)d\tau\right)^2 dx\,dt\,ds\right\}^{\frac{1}{2}}\times$$

$$\left\{\int_{I\!R^+\times\Omega}\int_0^{a_n}|\frac{\partial u^n}{\partial t}(t+s,x)|^2 ds\,dx\,dt\right\}^{\frac{1}{2}}.$$

Now by the Cauchy-Schwarz inequality and Fubini theorem we get

$$A \le \|\xi\|_{I\!L^\infty(\Omega)}\|\rho\|_{I\!L^\infty(I\!R^+)}\frac{1}{\sqrt{a_n}}\left\{\int_{I\!R^+\times\Omega}\int_0^{a_n}s\int_0^{a_n}|\frac{\partial u^n}{\partial t}(t+\tau,x)|^2 d\tau\,ds\,dt\,dx\right\}^{\frac{1}{2}}\|\frac{\partial u^n}{\partial t}\|_{I\!L^2(I\!R^+\times\Omega)}.$$

So after integration

$$A \le \frac{a_n}{\sqrt{2}}\|\xi\|_{I\!L^\infty(\Omega)}\|\rho\|_{I\!L^\infty(I\!R^+)}\|\frac{\partial u^n}{\partial t}\|^2_{I\!L^2(I\!R^+\times\Omega)}.$$

Hence by the energy estimate (5.1), A tends to zero as a_n.

In the same way as in the previous section we obtain finally

$$\int_{I\!R^+\times\Omega}\left(\frac{\partial u^\infty}{\partial t}(t,x)+u^\infty(t,x)\wedge\frac{\partial u^\infty}{\partial t}(t,x)\right)\cdot\xi(x)\rho(t)dx\,dt$$

$$+2\int_{I\!R^+\times\Omega}\sum_{i=1}^3\left(u^\infty(t,x)\wedge\frac{\partial u^\infty}{\partial x_i}(t,x)\right)\cdot\frac{\partial\xi}{\partial x_i}(x)\rho(t)dx\,dt \tag{5.9}$$

$$-2\int_{I\!R^+\times\Omega}u^\infty(t,x)\wedge\left(H^\infty(t,x)-\varphi(u^\infty(t,x))\right)\cdot\xi(x)\rho(t)dx\,dt=0.$$

Fourth step. As for the study of the ω-limit set we can prove that

$$\text{div }(H^\infty+\bar{u}^\infty)=0.$$

Now it remains to obtain

$$\text{curl } H^\infty = 0. \tag{5.10}$$

We recall that for all ξ in $\mathcal{D}(I\!R^3)$ and ρ in $\mathcal{D}([0,+\infty))$ we have according to (2.4) that

$$\begin{aligned} -\int_{I\!R^+\times I\!R^3} \varepsilon^n E^n(s,x)\cdot\frac{\partial\rho}{\partial t}(s)\xi(x)ds\,dx - \int_{I\!R^+\times I\!R^3} H^n(s,x)\cdot \text{curl}\,\xi(x)\rho(s)dx\,ds \\ +\sigma\int_{I\!R^+\times\Omega} E^n(s,x)\cdot\rho(s)\xi(x)ds\,dx = \int_{I\!R^3} E_0(x)\cdot\xi(x)\rho(0)dx. \end{aligned} \tag{5.11}$$

Formally, the identity (5.10) is obtained taking $\rho = \mathbf{1}_{(t,t+a_n)}$ in (5.11). Unfortunately this function is not regular enough, so we introduce a regularised function ρ_δ.

For each $\delta > 0$ given, $0 < \delta < a_n$, we denote

$$\rho_\delta(s) = \begin{cases} 1 & \delta \le s \le a_n - \delta \\ 0 & s \le 0 \text{ or } s \ge a_n \\ \text{linear} & 0 \le s \le \delta \text{ and } a_n - \delta \le s \le a_n \end{cases}$$

Now, for $\rho = \rho_\delta(s-t)$ equation (5.11) gives

$$\begin{aligned} &-\frac{\varepsilon^n}{a_n}\int_t^{t+\delta}\int_{I\!R^3} E^n(s,x)\frac{\partial\rho_\delta}{\partial t}(s-t)\cdot\xi(x)ds\,dx \\ &-\frac{\varepsilon^n}{a_n}\int_{t+a_n-\delta}^{t+a_n}\int_{I\!R^3} E^n(s,x)\frac{\partial\rho_\delta}{\partial t}(s-t)\cdot\xi(x)ds\,dx - \int_{I\!R^3} H^n_a(x)\cdot\text{curl}\,\xi(x)dx \\ &+\frac{\sigma}{a_n}\int_t^{t+a_n}\int_\Omega \rho_\delta(t-s)E^n(t,x)\cdot\xi(x)dx\,ds \\ &= -\frac{1}{a_n}\int_t^{t+a_n}\int_{I\!R^3} H^n(s,x)\cdot(1-\rho_\delta(s-t))\text{curl}\,\xi(x)dx\,ds. \end{aligned} \tag{5.12}$$

The two first terms of the left-hand side of (5.12) are bounded by

$$2\frac{\varepsilon^n}{a_n}\|E^n\|_{L^\infty(I\!R^+;\,I\!L^2(I\!R^3))}\|\xi\|_{I\!L^2(I\!R^3)}. \tag{5.13}$$

The last term of the left-hand side of (5.12) is bounded by

$$\sigma\|E^n\|_{L^\infty(I\!R^+;\,I\!L^2(\Omega))}\|\xi\|_{I\!L^2(\Omega)}. \tag{5.14}$$

The right-hand side of (5.12) is bounded by

$$2\frac{\delta}{a_n}\|H^n\|_{L^\infty(I\!R^+;\,I\!L^2(I\!R^3))}\|\text{curl}\,\xi\|_{I\!L^2(I\!R^3)}. \tag{5.15}$$

According to the energy estimate (5.1) we have

$$\begin{aligned} \|E^n\|_{L^\infty(I\!R^+;\,I\!L^2(I\!R^3))} &\le k\sqrt{\tfrac{\mu^n}{\varepsilon^n}} \\ \|E^n\|_{L^\infty(I\!R^+;\,I\!L^2(\Omega))} &\le k\sqrt{\mu^n} \end{aligned}$$

for some constant k
So by choosing $a_n = (\varepsilon^n\, \mu^n)^{\frac{1}{4}}$ and $\delta = a_n^2$ we get, for any test function φ

$$\int_{I\!R^3} H^\infty(t,x)\cdot \mathrm{curl}\, \xi(x)\varphi(t)dx\, dt = 0.$$

Fifth step. Energy estimate.
By convexity and thanks to the definition (5.2), one has

$$\int_\Omega |\nabla u_a^n(t,x)|^2 dx + 2\int_\Omega \Phi(u_a^n(t,x)) + \int_{I\!R^3} |H_a^n(t,x)|^2 dx$$
$$\leq \frac{1}{a}\int_0^a \left(\int_\Omega |\nabla u^n(t+s,x)|^2 dx + 2\int_\Omega \Phi(u^n(t+s,x)) + \int_{I\!R^3} |H^n(t+s,x)|^2 dx\right)$$
$$\leq \frac{1}{a}\int_0^a \mathcal{E}^n(t+s)ds.$$

On the other hand

$$\int_0^t \int_\Omega |\frac{\partial u_a^n}{\partial t}(s,x)|^2 dx\, ds = \int_0^t \int_\Omega \left|\frac{1}{a}\int_0^a \frac{\partial u^n}{\partial t}(\tau+s,x)d\tau\right|^2 dx\, ds$$
$$\leq \frac{1}{a}\int_0^a \int_0^{t+s} \int_\Omega |\frac{\partial u^n}{\partial t}(\tau,x)|^2 d\tau\, dx\, ds.$$

Hence

$$\int_\Omega |\nabla u_a^n(t,x)|^2 dx + 2\int_\Omega \Phi(u_a^n(t,x)) + \int_{I\!R^3} |H_a^n(t,x)|^2 dx + \int_0^t \int_\Omega |\frac{\partial u_a^n}{\partial t}(s,x)|^2 dx\, ds$$
$$\leq \frac{1}{a}\int_0^a \left(\mathcal{E}^n(t+s) + \int_0^{t+s}\int_\Omega |\frac{\partial u^n}{\partial t}(\tau,x)|^2 d\tau\, dx\right) ds \leq \mathcal{E}^n(0).$$

Since ε^n/μ^n tends to zero, $\mathcal{E}^n(0)$ tends to $\mathcal{E}_q(0)$. Therefore using the semi continuity of the norms for the weak topology, we derive the desired energy estimate (2.12).

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