Thin Layers in Micromagnetism

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Abstract - In this paper we study the solutions of micromagnetism equation in thin domain and we prove that the magnetic field induced by the magnetisation behaves like the projection of the magnetic moment on the normal to the domain.

1 Introduction

The aim of this work is to study the behavior of solutions of micromagnetism equations in thin domains. This paper concerns for example the magnetic microscopes composed by a thin layer of ferromagnetic material deposited on a dielectric point.

In the micromagnetism theory, a ferromagnetic material is characterized by a spontaneous magnetisation represented by a magnetic moment u defined on the domain Ω in which the material is confined. This moment satisfies $|u| \equiv 1$ on Ω and links the magnetic field H and the magnetic induction B by the relation $B = H + \bar{u}$, where $\bar{u}$ is the extension of u by zero outside Ω.

The magnetic field H satisfies curl $H = 0$ by static Maxwell Equations, and by the law of Faraday we have div $B =$ div $(H + \bar{u}) = 0$. Hence the magnetic moment u induces a magnetic field $H(u)$ given by :

$$\begin{cases} H(u) \in L^2(I\!R^3), \\ \text{curl } H(u) = 0 \text{ in } \mathcal{D}'(I\!R^3), \\ \text{div } (H(u) + \bar{u}) = 0 \text{ in } \mathcal{D}'(I\!R^3). \end{cases} \tag{1.1}$$

We will study two models of ferromagnetism.

Model I : steady state model
For $u \in H^1(\Omega; S^2) = \Big\{ v \in H^1(\Omega; I\!R^3), \text{ such that } |v| \equiv 1 \text{ almost everywhere} \Big\}$, we set

$$\mathcal{E}(u) = \int_\Omega |\nabla u|^2 + \int_{I\!R^3} |H(u)|^2.$$

The steady state configurations of u are the minimizers of $\mathcal{E}$ in the space $H^1(\Omega; S^2)$. They satisfy the following Euler equation :

$$-\Delta u - u|\nabla u|^2 - H(u) + (u, H(u))u = 0 \text{ in } \mathcal{D}'(\Omega).$$

Existence of the minimizers of $\mathcal{E}$ is proved in [6]. Regularity results about these minimizers are proved in [3] and [5].

Model II : Quasi-stationary Model
In this model u depends on the time t and satisfies the Landau Lifschitz equation :

$$\frac{\partial u}{\partial t} + u \wedge \frac{\partial u}{\partial t} = 2u \wedge (\Delta u + H(u)) \tag{1.2}$$

where $H(u)$ is defined by (1.1).
In [6] and [4], it is proved that if $u_0 \in H^1(\Omega; S^2)$, there exists at least one weak solution of (1.2) which satisfies :

$$\begin{cases} \bullet\ u \in L^\infty(I\!R^+; H^1(\Omega)), \\ \bullet\ |u| = 1 \ \ a.e. \\ \bullet\ \dfrac{\partial u}{\partial t} \in L^2(I\!R^+ \times \Omega), \\ \bullet\ \text{for all } \chi \in \mathcal{D}(I\!R^+ \times \overline{\Omega}), \\ \displaystyle\int_{R^+\times\Omega} \left(\frac{\partial u}{\partial t} + u \wedge \frac{\partial u}{\partial t}\right)\chi = -2\int_{R^+\times\Omega} \sum_{i=1}^{3} u \wedge \frac{\partial u}{\partial x_i} \cdot \frac{\partial \chi}{\partial x_i} + 2\int_{R^+\times\Omega} u \wedge H(u)\cdot\chi, \\ \bullet\ \text{for all } t \geq 0, \quad \mathcal{E}_{QS}(t) + \displaystyle\int_0^t \|\frac{\partial u}{\partial t}\|^2_{L^2(\Omega)} \leq \mathcal{E}_{QS}(0) \\ \text{where } \mathcal{E}_{QS}(t) = \displaystyle\int_\Omega |\nabla u(t,x)|^2 dx + \int_{I\!R^3} |H(u)(t,x)|^2 dx. \end{cases} \tag{1.3}$$

Our first result is the following

Theorem 1.1 *We set $\Omega_h = \omega \times [0, h]$, where ω is a regular bounded domain of $I\!R^2$.*
Let $u_h \in H^1(\Omega_h; S^2)$ be a minimizer of $\mathcal{E}$ in $H^1(\Omega_h; S^2)$.
For $(x, y, z) \in \omega\times]0, 1[$ we set $\widetilde{u}_h(x, y, z) = u_h(x, y, hz)$.
Then there exists a subsequence still denoted $\widetilde{u}_h$ such that $\widetilde{u}_h$ tends in $H^1(\omega \times [0, 1]; S^2)$ to a constant vector field e.
This constant e satisfies $|e| = 1$ and is contained in the plane of ω. Moreover it minimizes the following energy :

$$\tilde{\mathcal{E}}(\xi) = \int_{\partial\omega} |(\xi, \nu(y))|^2 d\sigma(y), \quad \text{for } \xi \in I\!R^2, \ \ |\xi| = 1, \tag{1.4}$$

where ν is the outward unitary normal to $\partial\omega$.

For the same kind of flat thin layer, we prove the following theorem which concerns weak solutions of Landau-Lifschitz equations (model II) :

Theorem 1.2 *Let $u_0 \in H^1(\omega; S^2)$. Let $v_h \in L^\infty(I\!R^+; H^1(\omega \times [0, h]; S^2))$ be a weak solution of (1.3) with initial data $v_h^0(x, y, z) = u_0(x, y)$. For $(t, x, y, z) \in I\!R^+\times\omega\times[0, 1]$, we set $\widetilde{v}_h(t, x, y, z) = v_h(t, x, y, hz)$.*
Then there exists a subsequence still denoted $\widetilde{v}_h$ such that $\widetilde{v}_h$ tends to a function $\widetilde{v}$ in $L^\infty(I\!R^+; H^1(\omega \times [0, 1]; S^2))$ $\star$ weak, in $L^2([0, T] \times \omega \times [0, 1])$ strong and almost everywhere.

Furthermore, $\widetilde{v}$ does not depend on z and satisfies :

$$
\left\{
\begin{array}{l}
\bullet\ \widetilde{v}(t=0) = u_0, \\[2mm]
\bullet\ \widetilde{v} \in L^\infty(I\!R^+; H^1(\omega)), \\[2mm]
\bullet\ |\widetilde{v}| = 1 \ \ a.e. \\[2mm]
\bullet\ \dfrac{\partial \widetilde{v}}{\partial t} \in L^2(I\!R^+ \times \omega), \\[2mm]
\bullet\ \ \text{for all } \chi \in \mathcal{D}(I\!R^+ \times \overline{\omega}), \\
\displaystyle\int_{I\!R^+\times\omega} \left(\frac{\partial \widetilde{v}}{\partial t} + \widetilde{v} \wedge \frac{\partial \widetilde{v}}{\partial t} \right) \chi = -2 \int_{I\!R^+\times\omega} \sum_{i=1}^{2} \widetilde{v} \wedge \frac{\partial \widetilde{v}}{\partial x_i} \cdot \frac{\partial \chi}{\partial x_i} - 2 \int_{R^+\times\omega} \widetilde{v} \wedge P(\widetilde{v}) \cdot \chi, \\
\text{with } P(\widetilde{v}) = -(\widetilde{v}, e_3)e_3 \text{ where } e_3 \text{ is the third vector of the canonical basis of } I\!R^3, \\
\bullet\ \text{for all } t \geq 0, \quad \mathcal{E}_2(t) + \displaystyle\int_0^t \| \frac{\partial \widetilde{v}}{\partial t} \|^2_{L^2(\omega)} \leq \mathcal{E}_2(0), \\
\ \text{where } \mathcal{E}_2(t) = \displaystyle\int_\omega |\nabla \widetilde{v}(t)|^2 + \int_\omega \widetilde{v}_3^2(t).
\end{array}
\right.
\tag{1.5}
$$

Remark 1.1 *When h goes to zero, the non local operator H behaves like the local operator P. We remark that $-P$ is the projection of u onto e_3, the normal to the domain.*

Remark 1.2 *Theorem 1.2 remains valid if we suppose that the initial data v_h^0 satisfies :*

1. $\widetilde{v_h^0} \longrightarrow u_0$ *in* $H^1(\omega \times [0,1])$ *strongly, where* $\widetilde{v_h^0}(x,y,z) = v_h^0(x,y,hz)$,
2. $\dfrac{1}{h} \dfrac{\partial \widetilde{v_h^0}}{\partial z}$ *tends to zero in* $L^2(\omega \times [0,1])$ *strongly.*

Afterward we will prove the same kind of theorems in a more complicated geometry.

Let us consider a surface $\mathcal{S} \subset I\!R^3$ such that $\overline{\mathcal{S}}$ is diffeomorphic to $\overline{B^2}$. We denote by $\vec{n}$ a regular unitary vector field defined on $\mathcal{S}$ and normal to $\mathcal{S}$.
We set

$$
O_h = \left\{ X \in I\!R^3,\ X = u + t\vec{n}(u),\ u \in \mathcal{S},\ t \in [0,h] \right\}.
$$

On the other hand, we endow $I\!R^3$ with a chart compatible with O_h : let ψ be a global diffeomorphism from $I\!R^3$ to $I\!R^3$ such that :

$$
\left\{
\begin{array}{l}
\text{there exists } R \text{ such that } \psi_{|^C B(0,R)} = Id_{|^C B(0,R)}, \\[2mm]
\psi(B^2 \times \{0\}) = \mathcal{S}, \\[2mm]
\forall\, x \in B^2, \ \ \forall\, z \in [0,h_0], \ \ \dfrac{\partial \psi}{\partial s}(x,s) = \vec{n}(\psi(x,0))
\end{array}
\right.
$$

We remark that $O_h = \psi(B^2 \times [0,h])$.

Remark 1.3 *This geometry describes the thin layer of ferromagnetic material in the magnetic microscopes.*

For this kind of thin layer, we first prove the following

Theorem 1.3 *Let $u_h \in H^1(O_h; S^2)$ be a minimizer of $\mathcal{E}$ in $H^1(O_h; S^2)$. We set $\underline{u_h} = u_h \circ \psi^{-1}$ ($\underline{u_h} \in H^1(B^2 \times [0,h]; S^2)$). We consider the rescaled $\widetilde{u_h}(x,y,z) = \underline{u_h}(x,y,hz)$.*
Then there exists a subsequence still denoted $\widetilde{u_h}$ such that $\widetilde{u_h}$ tends to $\widetilde{u}$ in $H^1(B^2 \times [0,1]; S^2)$ strong. The limit $\widetilde{u}$ does not depend on its third variable, and if we set $u(X) = \widetilde{u}(\psi^{-1}(X))$ for $X \in \mathcal{S}$, then u minimizes the following energy :

$$\tilde{\mathcal{E}}(v) = \int_{\mathcal{S}} |\nabla_{\mathcal{S}} v|^2 d\sigma + \int_{\mathcal{S}} |v \cdot \vec{n}|^2 d\sigma, \quad v \in H^1(\mathcal{S}; S^2).$$

Remark 1.4 *We can remark that the demagnetizing energy behaves in thin domains like an anisotropy energy forcing the magnetic moment to be tangential to the domain.*

For the solutions of Landau-Lifschitz equations in the domain O_h, we have the following

Theorem 1.4 *Let $u_0 \in H^1(\mathcal{S}; S^2)$. We consider $u_0^h \in H^1(O_h; S^2)$ satisfying :*

$$\begin{cases} u_0^h = u_0 \text{ in } \mathcal{S} \\ \\ u_0^h \text{ is constant along the lines } [x, x + h\vec{n}(x)] \text{ for } x \in \mathcal{S}. \end{cases}$$

let $v_h \in L^\infty(I\!R^+; H^1(O_h; S^2))$ be a weak solution of (1.3) in $I\!R^+ \times O_h$ with initial data u_0^h.
Like in Theorem 1.3 we set $\underline{v_h}(t,X) = v_h(t,\psi(X))$ for $t \in I\!R^+$ and $X \in B^2 \times [0,h]$ and we set $\widetilde{v_h}(t,x,y,z) = \underline{v_h}(t,x,y,hz)$.
Then extracting a subsequence, $\widetilde{v_h}$ tends to $\widetilde{v}$ in the $L^\infty(I\!R^+; H^1(B^2 \times [0,1]; S^2))$ $\star$ weak, $L^2([0,T] \times B^2 \times [0,1])$ strong and almost everywhere.
Furthermore $\widetilde{v}$ does not depend on his third variable and if we denote $v(X) = \widetilde{v}(\psi(X))$ for $X \in \mathcal{S}$, v satisfies

$$\begin{cases}
\bullet\ v(0,X) = u_0(X), \\
\bullet\ v \in L^\infty(R^+; H^1(\mathcal{S})), \\
\bullet\ |v| = 1 \ \text{a.e.} \\
\bullet\ \dfrac{\partial v}{\partial t} \in I\!L^2(I\!R^+ \times \mathcal{S}) \\
\bullet\ v \text{ is a weak solution of } \dfrac{\partial v}{\partial t} + v \wedge \dfrac{\partial v}{\partial t} = v \wedge \Big(\Delta_{\mathcal{S}} v - (v \cdot \vec{n})\vec{n}\Big), \\
\bullet\ \text{for all } t \geq 0,\ \mathcal{E}_3(t) + \displaystyle\int_0^t \|\frac{\partial v}{\partial t}\|^2_{L^2(\mathcal{S})} \leq \mathcal{E}_3(0) \\
\text{where } \mathcal{E}_3(t) = \displaystyle\int_{\mathcal{S}} |\nabla_{\mathcal{S}} v(t)|^2 d\sigma + \int_{\mathcal{S}} |v \cdot \vec{n}|^2 d\sigma.
\end{cases}$$

Remark 1.5 *In this more complicated geometry we remark that the same phenomenon than in flat domains occurs : the non local operator H behaves like the opposite of the projection onto the normal to the domain.*

Remark 1.6 *The hypothesis on the initial data u_0^h can be weakened in the following form : If we set $\widetilde{u_0^h}(x,y,z) = u_0^h(\psi(x,y,hz))$, we assume that*

1. *$\widetilde{u_0^h}$ tends to $\widetilde{u_0}$ in $H^1(B(0,1)\times[0,1])$ strongly,*

2. *$\dfrac{1}{h}\dfrac{\partial\widetilde{u_0^h}}{\partial z}$ tends to zero in $L^2(B(0,1)\times[0,1])$ strongly.*

This paper is organized as follows : Chapter two is devoted to the proof of Theorems 1.1 and 1.2. We prove Theorems 1.3 and 1.4 in Part 3.

2 Thin Domains of the form $\omega\times[0.h]$

2.1 Proof of Theorem 1.1

Let u_h be a minimizer of $\mathcal{E}$ on $H^1(\omega\times[0,h];S^2)$. We denote $\widetilde{u_h}(x,y,z) = u_h(x,y,hz)$. We consider K_h the rescaling of $H(u_h)$:

$$K_h(x,y,z) = H(u_h)(x,y,hz).$$

First step : energy estimate.
Let e be a constant vector field on Ω_h. We suppose that $|e| = 1$. We have $e \in H^1(\Omega_h;S^2)$, so $\mathcal{E}(u_h) \le \mathcal{E}(e)$.
We have,

$$\mathcal{E}(e) = \int_{I\!R^3} |H(e)|^2.$$

Now, there exists a constant C such that

$$\forall\, v\in L^2(I\!R^3),\ \ \|H(v)\|_{L^2(I\!R^3)} \le C\|v\|_{L^2(I\!R^3)}. \tag{2.6}$$

So,

$$\mathcal{E}(u_h) \le \mathcal{E}(e) \le C\|\bar{e}\|^2_{L^2(I\!R^3)} \le Ch$$

hence

$$\int_{\omega\times[0,h]} |\nabla u_h|^2 + \int_{I\!R^3} |H(u_h)|^2 \le Ch.$$

We re-scale this inequality and we obtain that :

$$\int_{\omega\times[0,1]} \left(|\frac{\partial\widetilde{u_h}}{\partial x}|^2 + |\frac{\partial\widetilde{u_h}}{\partial y}|^2 + \frac{1}{h^2}|\frac{\partial\widetilde{u_h}}{\partial z}|^2 \right) \ + \ \int_{I\!R^3} |K_h|^2 \le C. \tag{2.7}$$

Second step : limit when h goes to zero.

Extracting a subsequence we can suppose that there exists $\widetilde{u} \in H^1(\omega\times[0,1];S^2)$ and $K \in L^2(I\!R^3)$ such that $\widetilde{u_h}$ tends to $\widetilde{u}$ in H^1 weak, L^2 strong and almost everywhere and K_h tends to K in L^2 weak.

Furthermore we remark that $\dfrac{\partial\widetilde{u_h}}{\partial z}$ tends to zero in L^2 strong hence $\widetilde{u}$ does not depend on z.

We remark also that $|\widetilde{u}| = 1$ a.e.

Lemma 2.1 *Let $\widetilde{w_h} \in L^2(\omega\times]0,1[)$ such that $\widetilde{w_h} \longrightarrow \widetilde{w}$ in $L^2(\omega \times [0,1])$.*
For $(x,y,z) \in \omega\times]0,h[$, we set $w_h(x,y,z) = \widetilde{w_h}(x,y,\frac{z}{h})$. We consider the rescaling of $H(w_h)$ setting $W_h(x,y,z) = H(w_h)(x,y,hz)$.
Then, W_h tends to $P(\widetilde{w}) = -(\overline{\widetilde{w}}, e_3)e_3$ in $L^2(I\!R^3)$ strong as h goes to zero.

Remark 2.1 *We recall that $\overline{\widetilde{w}}$ is the extension of $\widetilde{w}$ by zero outside of $\omega \times [0,1]$.*

Proof of Lemma 2.1.
let us write the equations satisfied by $W_h = (W_h^1, W_h^2, W_h^3)$.
We know that $H(w_h)$ satisfies curl $H(w_h) = 0$ and div $(H(w_h) + \overline{w_h}) = 0$.
After rescaling we obtain that

$$\left\{\begin{array}{l} -\frac{1}{h}\frac{\partial W_h^2}{\partial z} + \frac{\partial W_h^3}{\partial y} = 0 \\ \\ \frac{1}{h}\frac{\partial W_h^1}{\partial z} - \frac{\partial W_h^3}{\partial x} = 0 \\ \\ \frac{\partial W_h^1}{\partial y} - \frac{\partial W_h^2}{\partial x} = 0 \\ \\ \frac{\partial}{\partial x}(W_h^1 + \overline{\widetilde{w_h}^1}) + \frac{\partial}{\partial y}(W_h^2 + \overline{\widetilde{w_h}^2}) + \frac{1}{h}\frac{\partial}{\partial z}(W_h^3 + \overline{\widetilde{w_h}^3}) = 0 \end{array}\right. \tag{2.8}$$

These equations are satisfied in $\mathcal{D}'(I\!R^3)$. Let us take the weak formulation of the first equation of (2.8) : we fix $\Psi \in \mathcal{D}(I\!R^3)$ and we have

$$\int_{I\!R^3} W_h^2 \frac{\partial \Psi}{\partial z} = h \int_{I\!R^3} W_h^3 \frac{\partial \Psi}{\partial y}.$$

Taking the limit when h tends to zero, since $W_h^2 \rightharpoonup K^2$ and $W_h^3 \rightharpoonup K^3$ in $L^2(I\!R^3)$ weak, we obtain that

$$\forall\, \Psi \in \mathcal{D}'(R^3), \quad \int_{I\!R^3} W^2 \frac{\partial \Psi}{\partial z} = 0,$$

hence W^2 does not depend on the variable z and since $W^2 \in L^2(I\!R^3)$, we deduce that $W^2 = 0$.
In the same way we prove that $W^1 = 0$.
Now, taking the limit in the weak formulation of the fourth equation in (2.8), we obtain that $W^3 + \overline{\widetilde{w}_3} = 0$. Therefore

$$W = -(\overline{\widetilde{w}}, e_3)e_3.$$

In order to prove that $W_h \longrightarrow W$ in $L^2(I\!R^3)$ strong, we remark that

$$\int_{I\!R^3} |W^h|^2 = h \int_{I\!R^3} |H(w_h)|^2$$

and by property of the operator H (since $-H$ is an orthogonal projection in $L^2(I\!R^3)$ onto the fields of gradients), we have :

$$\int_{I\!R^3} |W^h|^2 = -h \int_{I\!R^3} \overline{w_h} \cdot H(w_h) = -\int_{\omega\times[0,1]} \widetilde{w_h} \cdot W_h.$$

Since $\widetilde{w_h}$ tends to $\widetilde{w}$ in L^2 strong and since W_h tends to $-(\overline{\widetilde{w}}, e_3)e_3$ in L^2 weak, we obtain that $\|W_h\|_{L^2(I\!R^3)}$ tends to $\| - (\overline{\widetilde{w}}, e_3)e_3\|_{L^2(I\!R^3)}$.
Hence, W_h tends to $-(\overline{\widetilde{w}}, e_3)e_3$ in L^2 strong, which fulfills the proof of Lemma 2.1.

Applying Lemma 2.1 in our case, we obtain that K_h tends to $K = -(\overline{\widetilde{u}}, e_3)e_3$ in $L^2(I\!R^3)$ strong.

Third step : $\widetilde{u}$ is a constant.

Let e be a unitary vector contained in the plan of ω.
We set $e_h(x, y, z) \equiv e$ for $(x, y, z) \in \Omega_h$. We have :

$$\mathcal{E}(e_h) = \int_{I\!R^3} |H(e_h)|^2 = -\int_{\Omega_h} H(e_h) \cdot e_h.$$

We take the rescaling of the previous equality and we obtain :

$$\mathcal{E}(e_h) = -h \int_{\omega \times [0,1]} B_h \cdot e,$$

where $B_h(x, y, z) = H(e_h)(x, y, hz)$.

As in the second step we prove that $B_h \rightharpoonup \begin{pmatrix} 0 \\ 0 \\ e^3 \end{pmatrix} = 0$ in $L^2(I\!R^3)$ weak, so $\int_{\omega \times [0,1]} B_h \cdot e$ tends to zero when h tends to zero.
Now by minimality of u_h, we obtain after the rescaling that

$$\int_{\omega \times [0,1]} |\nabla_{xy} \widetilde{u_h}|^2 + \frac{1}{h^2} \int_{\omega \times [0,1]} |\frac{\partial \widetilde{u_h}}{\partial z}|^2 + \int_{R^3} |K_h|^2 \leq -\int_{I\!R^3} B_h \cdot e,$$

which implies first that $\nabla \widetilde{u} = 0$ i.e. $\widetilde{u}$ is a constant vector field, and that K_h tends strongly to zero in $L^2(I\!R^3)$, hence $K \equiv 0$, i.e. $\widetilde{u}_3 \equiv 0$.
So $\widetilde{u}$ is a constant unitary vector field contained in the plan of ω.

Fourth step : $\widetilde{u}$ minimizes $\widetilde{\mathcal{E}}$.
In order to prove that $\widetilde{u}$ minimizes $\widetilde{\mathcal{E}}$, we will compute an equivalent of $\mathcal{E}(e_h)$ when e_h is a constant unitary vector field defined on Ω_h and contained in the plane of ω. We denote ν the outward unitary normal to the domain ω.
We recall that of $H(e_h) = -\nabla \varphi_h$ and that :

$$\|H(e_h)\|^2_{L^2(I\!R^3)} = \int_{I\!R^3} |\nabla \varphi_h|^2 = -\int_{I\!R^3} \nabla \varphi_h \cdot \bar{e_h} = -\int_{\partial \Omega_h} \varphi_h (e_h, \nu) \; + \int_{\Omega_h} \varphi_h \mathrm{div}\, e_h.$$

Furthermore we have :

$$\varphi_h(x) = \int_{\Omega_h} \frac{1}{\|x - y\|} \mathrm{div}\, e_h(y)\, dy \; - \int_{\partial \Omega_h} \frac{1}{\|x - y\|} (e_h, \nu)(y) d\sigma(y).$$

Now, since e_h is a constant vector field,

$$\|H(e_h)\|^2_{L^2(I\!R^3)} = \int_{x \in \partial \Omega_h} \int_{y \in \partial \Omega_h} \frac{1}{\|x - y\|} (e_h, \nu)(x)(e_h, \nu)(y)\, d\sigma(x)\, d\sigma(y).$$

The boundary of Ω_h has three parts : $\partial\omega \times [0, h]$, $\omega \times \{0\}$, and $\omega \times \{h\}$.

Since e_h is a constant included in the plan of ω, the expression of $A_h = \|H(e_h)\|^2_{L^2(\mathbb{R}^3)}$ becomes :

$$A_h = h^2 \int_{(X,Y)\in\partial\omega^2,(z,z')\in[0,1]^2} \frac{1}{(\ \|X-Y\|^2 + h^2|z-z'|^2)^{\frac{1}{2}}}(e,\nu)(X)(e,\nu)(Y)d\sigma(X)\ d\sigma(Y)\ dz\ dz',$$

$$= 2h^2 \int_{X\in\partial\omega} (e,\nu)(X)G_h(X)d\sigma(X),$$

where

$$G_h(X) = \int_{Y\in\partial\omega}\int_{z\in[0,1]} \frac{1-z}{(\ \|X-Y\|^2+h^2z^2)^{\frac{1}{2}}}(e,\nu)(Y)\ d\sigma(Y)\ dz.$$

Hence

$$G_h(X) = G_h^1(X) + G_h^2(X),$$

with

$$G_h^1(X) = -\int_{Y\in\partial\omega} \left[(\|X-Y\|^2+h^2)^{\frac{1}{2}} - \|X-Y\|\right](e,\nu)(Y)d\sigma(Y)$$

and

$$G_h^2(X) = h\int_{Y\in\partial\omega} \text{Argsh}\left(\frac{h}{\|X-Y\|}\right)(e,\nu)(Y)d\sigma(Y).$$

Let us compute now an equivalent of G_h^1. We prove that

$$\frac{1}{-h^2\ln h}G_h^1(X) \text{ tends to } -\frac{1}{2}(e,\nu)(X)$$

as h tends to zero. Furthermore, this limit is uniform in $X\in\partial\omega$.
Let $u : \mathbb{R} \longrightarrow \mathbb{R}^2$, be a L-periodic normal parameterization of $\partial\omega$ (L is the length of $\partial\omega$). Let $X\in\partial\omega$.
Let $\varepsilon > 0$ be fixed. As the frontier $\partial\omega$ is regular and compact, there exists $\alpha_0 > 0$ such that :

$$\forall\ s_0\in\mathbb{R},\ \forall\ s\in]s_0-\alpha_0, s_0+\alpha_0[, \begin{cases} (1-\varepsilon)|s-s_0| \le \|u(s)-u(s_0)\| \le (1+\varepsilon)|s-s_0| \\ \text{and} \\ (e,\nu(s_0)) - \varepsilon \le (e,\nu(s)) \le (e,\nu(s_0)) + \varepsilon. \end{cases}$$

We remark that α_0 does not depend on X.
Even if it means translating the parameterization of $\partial\omega$, we can suppose that $X = u(0)$.
We have :

$$G_h^1(X) = -\int_{\frac{L}{2}}^{\frac{L}{2}} \left[(\|u(s)-u(0)\|^2+h^2)^{\frac{1}{2}} - \|u(s)-u(0)\|\right](e,\nu(s))ds.$$

We split the integral of G_h^1 in 2 parts using the periodicity of u :

$$\int_{\frac{L}{2}}^{\frac{L}{2}}\cdot = \int_{-\alpha_0}^{\alpha_0}\cdot + \int_{\alpha_0}^{L-\alpha_0}\cdot$$

Let us study the first part of this expression. If $s\in]-\alpha_0,\alpha_0[$,

$$\left[\left((1+\varepsilon)^2s^2 + h^2\right)^{\frac{1}{2}} - (1+\varepsilon)|s|\right]\left[(e,\nu(X)) - \varepsilon\right] \le$$

$$\left[(\|u(s)-u(0)\|^2+h^2)^{\frac{1}{2}}-\|u(s)-u(s_0)\|\right](e,\nu(u(s)))\leq$$

$$\left[\left((1-\varepsilon)^2s^2+\,h^2\right)^{\frac{1}{2}}-(1-\varepsilon)|s|\right]\left[(e,\nu(X))+\,\varepsilon\right]$$

We integrate this inequality between $-\alpha_0$ and α_0. We remark that

$$\lim_{h\longrightarrow 0}\frac{1}{-h^2\ln h}\int_{-\alpha_0}^{\alpha_0}\left[\left(\gamma^2s^2+\,h^2\right)^{\frac{1}{2}}-\gamma|s|\right]ds=-\frac{1}{\gamma}$$

Hence there exists $h_0>0$ such that for all $h<h_0$,

$$\frac{1-\varepsilon}{1+\varepsilon}\left[(e,\nu(X))-\,\varepsilon\right]\leq$$

$$\frac{1}{-h^2\ln h}\int_{-\alpha_0}^{\alpha_0}\left[(\|u(s)-u(0)\|^2+h^2)^{\frac{1}{2}}-\|u(s)-u(0)\|\right](e,\nu(s))ds\leq$$

$$\frac{1+\varepsilon}{1-\varepsilon}\left[(e,\nu(X))+\,\varepsilon\right].$$

We remark that h_0 is independent of X.

Let us study now the second part of the integral defining $G_1^h(X)$.
Since the parameterization of $\partial\omega$ is an embedding in $[0,L[$, there exists $\beta>0$ such that if $\alpha_0<|s_1-s_2|<L$ then $\|u(s_1)-u(s_2)\|\geq\beta$. Hence, for all $h>0$,

$$\left|\int_{\alpha_0}^{L-\alpha_0}\left[(\|u(s)-u(0)\|^2+h^2)^{\frac{1}{2}}-\|u(s)-u(0)\|\right](e,\nu(s))ds\right|\leq\int_{\alpha_0}^{L-\alpha_0}\left[(\beta^2+h^2)^{\frac{1}{2}}-\beta\right].$$

Now,

$$\frac{1}{-h^2\ln h}\int_{\alpha_0}^{L-\alpha_0}\left[(\beta^2+h^2)^{\frac{1}{2}}-\beta\right]$$

tends to zero when h tends to zero. So this part of $G_h^1(X)$ can be neglected, and we have proved that

$$\frac{1}{-h^2\ln h}G_h^1(X)\text{ tends uniformly in }\partial\omega\text{ to }-(e,\nu(X)).$$

In the same way we study now $G_h^2(X)$:

$$G_h^2(X)=h\int_0^L\text{Argsh}\left(\frac{h}{\|u(s)\|}\right)(e,\nu(u(s)))ds.$$

We remark now that

$$\lim_{h\longrightarrow 0}\frac{1}{-h^2\ln h}h\int_{-\alpha_0}^{\alpha_0}\text{Argsh}\left(\frac{h}{|s|}\right)=2,$$

hence, we prove that :

$$\frac{1}{-h^2\ln h}G_h^2(X)\text{ tends uniformly to }2(e,\nu(X)).$$

Therefore

$$\frac{1}{-h^2\ln h}G_h(X)\text{ tends uniformly on }\partial\omega\text{ to }(e,\nu(X))$$

Hence,

$$H(e_h) \text{ is equivalent to } -h^2 \ln h \int_{\partial\omega} (e,\nu(X))^2 d\sigma(X)$$

when h tends to zero.
Since u_h minimizes H on $\partial\omega \times [0,h]$, and since u_h tends to a constant which belongs to the plane of ω, this constant minimizes the energy :

$$\widetilde{\mathcal{E}}(e) = \int_{\partial\omega} (e,\nu(X))^2 d\sigma(X).$$

2.2 Proof of Theorem 1.2

Following [4] we build a weak solution of (1.2) which satisfies :

$$\mathcal{E}_h(t) + \int_0^t \int_{\Omega_h} \left|\frac{\partial v_h}{\partial t}\right|^2 \leq \mathcal{E}_h(0),$$

with

$$\mathcal{E}_h(t) = \int_{\Omega_h} |\nabla v_h(t,x)|^2 dx + \int_{I\!R^3} |H(v_h)|^2(t,x)dx.$$

As in the proof of Theorem 1.1 we consider the rescaling of v_h : we set $\widetilde{v_h}(t,x,y,z) = v_h(t,x,y,hz)$.
We set

$$\mathcal{F}_h(t) = \int_{\omega\times[0,1]} \left(|\nabla_{xy}\widetilde{v_h}|^2 + \frac{1}{h^2}|\frac{\partial\widetilde{v_h}}{\partial z}|^2\right)(t,x,y,z)dx\ dy\ dz + \int_{I\!R^3} |K_h(t,x,y,z)|^2 dx\ dy\ dz$$

where $K_h(t,x,y,z) = H(v_h)(t,x,y,hz)$.
The energy inequality writes :

$$\mathcal{F}_h(t) + \int_0^t \int_{\omega\times[0,1]} |\frac{\partial\widetilde{v_h}}{\partial t}|^2 \leq \mathcal{F}_h(0).$$

We remark that

$$\mathcal{F}_h(0) = \int_{\omega\times[0,1]} |\nabla_{xy}u_0|^2 + \int_{I\!R^3} |K_h(0)|^2$$

since u_0 does not depend on z.
Furthermore,

$$\int_{I\!R^3} |K_h(0)|^2 = \frac{1}{h}\int_{I\!R^3} |H(u_0^h)|^2$$

where $u_0^h(x,y,z) = u_0(x,y)\chi_{[0,h]}(z)$.
As the operator H is continuous on $L^2(I\!R^3)$ we obtain that $\int_{I\!R^3} |H(u_0^h)|^2 \leq Ch$.
Hence, there exists a constant C such that for all $h > 0$,

$$\left\{\begin{array}{l} \|\nabla_{xy}\widetilde{v_h}\|_{L^\infty(I\!R^+;L^2(\omega\times[0,1]))} \leq C \\ \\ \|\frac{\partial\widetilde{v_h}}{\partial z}\|_{L^\infty(I\!R^+;L^2(\omega\times[0,1]))} \leq Ch^2 \\ \\ \|\frac{\partial\widetilde{v_h}}{\partial t}\|_{L^2(I\!R^+\times\omega\times[0,1])} \leq C \\ \\ \|K_h\|_{L^\infty(I\!R^+;L^2(I\!R^3))} \leq C. \end{array}\right. \tag{2.9}$$

Extracting a subsequence, we can assume that :

- $\widetilde{v_h} \rightharpoonup v$ in $L^\infty(R^+; H^1(\omega \times [0,1]))$ $\star$ weak, $L^2([0,T] \times \omega \times [0,1])$ strong, and almost everywhere,
- $\dfrac{\partial \widetilde{v_h}}{\partial t} \rightharpoonup \dfrac{\partial v}{\partial t}$ in $L^2(I\!R^+ \times \omega \times [0,1])$ weakly,
- $K_h \rightharpoonup K$ in $L^\infty(I\!R^+; L^2(I\!R^3))$ $\star$ weak.

We know that $|v| \equiv 1$ since $\widetilde{v_h} \longrightarrow v$ almost everywhere.
Following the proof of Lemma 2.1, we obtain that

$$K = -(\overline{v}, e_3)e_3.$$

Now, in order to take the limit in Landau-Lifschitz equation, let us consider the weak formulation of (1.2). Let $\Phi \in \mathcal{D}(I\!R^+ \times \overline{\Omega_h})$. We have :

$$\int_{I\!R^+ \times \Omega_h} \left(\frac{\partial v_h}{\partial t} + v_h \wedge \frac{\partial v_h}{\partial t} \right) \Phi = -2 \sum_{i=1}^{3} \int_{I\!R^+ \times \Omega_h} v_h \wedge \frac{\partial v_h}{\partial x_i} \cdot \frac{\partial \Phi}{\partial x_i} \; + \; 2 \int_{I\!R^+ \times \Omega_h} v_h \wedge H(v_h) \cdot \Phi.$$

Let us take $\Phi(t,x,y,z) = \varphi(t,x,y)$ where $\varphi \in \mathcal{D}(I\!R^+ \times \overline{\omega})$ (i.e. φ does not depend on z).
Taking the same rescaling, we obtain that :

$$\int_{I\!R^+ \times \omega \times [0,1]} \left(\frac{\partial \widetilde{v_h}}{\partial t} + \widetilde{v_h} \wedge \frac{\partial \widetilde{v_h}}{\partial t} \right) \varphi = -2 \sum_{i=1}^{2} \int_{I\!R^+ \times \omega \times [0,1]} \widetilde{v_h} \wedge \frac{\partial \widetilde{v_h}}{\partial x_i} \cdot \frac{\partial \varphi}{\partial x_i} + 2 \int_{I\!R^+ \times \omega \times [0,1]} \widetilde{v_h} \wedge K_h \cdot \varphi.$$

It is now straightforward to take the limit of this expression when h tends to zero, since :

$$\left.\begin{array}{l} \dfrac{\partial \widetilde{v_h}}{\partial t} \rightharpoonup \dfrac{\partial v}{\partial t} \\[2ex] \dfrac{\partial \widetilde{v_h}}{\partial x_i} \rightharpoonup \dfrac{\partial v}{\partial x_i} \\[2ex] K_h \rightharpoonup -(\overline{v}, e_3)e_3 \end{array}\right\} \text{ in } L^2([0,T] \times \omega \times [0,1]) \text{ weakly.}$$

Now the integrand of the limit does not depend on z. We obtain that v is in $L^\infty(I\!R^+; H^1(\omega))$ and satisfies :

$$\int_{I\!R^+ \times \omega} \left(\frac{\partial v}{\partial t} + v \wedge \frac{\partial v}{\partial t} \right) \varphi = -2 \sum_{i=1}^{2} \int_{I\!R^+ \times \omega} v \wedge \frac{\partial v}{\partial x_i} \cdot \frac{\partial \varphi}{\partial x_i} + 2 \int_{I\!R^+ \times \omega} v \wedge (-(v, e_3)e_3) \cdot \varphi.$$

which is the weak formulation of

$$\frac{\partial v}{\partial t} + v \wedge \frac{\partial v}{\partial t} = 2v \wedge (\Delta v - (v, e_3)e_3)\,.$$

It remains to show that v satisfies the energy inequality. By lower semi-continuity of the norm for the weak topology, we obtain that

$$\mathcal{E}_2(t) + \int_0^t \int_\omega |\frac{\partial \widetilde{v}}{\partial t}|^2 \leq \; \liminf \frac{1}{h} \mathcal{E}_h(0),$$

with

$$\mathcal{E}_2(t) = \int_\omega |\nabla \widetilde{v}(t,x)|^2 dx + \int_\omega |v_3|^2(t,x)\, dx.$$

Let us compute $\liminf \frac{1}{h}\mathcal{E}_h(0)$. We have :

$$\mathcal{E}_h(0) = \int_{\omega\times[0,1]} |\nabla_{xy}u_0|^2 + \int_{I\!R^3} |H(u_0^h)|^2$$

where $u_0^h = u_0(x,y)\chi_{[0,h]}(z)$.
We set $K_0^h(x,y,z) = H(u_0^h)(x,y,hz)$.
So

$$\mathcal{E}_h(0) = h\int_\omega |\nabla_{xy}u_0|^2 + h\int_{I\!R^3} |K_0^h|^2.$$

By Lemma 2.1 we know that K_0^h tends to $(0,0,-\overline{u_0^3})$ in L^2 strong, hence

$$\frac{1}{h}\mathcal{E}_h(0) \longrightarrow \int_\omega |\nabla_{xy}u_0|^2 + \int_\omega |u_0^3|^2 = \mathcal{E}_2(0),$$

which concludes the proof of Theorem 1.2.

3 Non flat domain

We only detail the proof of Theorem 1.3. The proof of Theorem 1.4 follows with the same kind of arguments.

First step : geometrical preliminaries.
We recall that Ψ is a global diffeomorphism of $I\!R^3$ such that :

$$\begin{cases} \text{there exists } R \text{ such that } \psi_{|{}^C B(0,R)} = Id_{|{}^C B(0,R)}, \\ \psi(B^2\times\{0\}) = \mathcal{S}, \\ \forall\, x\in B^2,\ \ \forall\, z\in[0,h_0],\ \ \dfrac{\partial\psi}{\partial s}(x,s) = \vec{n}(\psi(x,0)) \end{cases}$$

We remark that $O_h = \psi(B^2\times[0,h])$.
We will use the following notations :

- $\varphi = \Psi^{-1}$,
- $g_{ij}(x) = (\dfrac{\partial\psi}{\partial x_i}(x), \dfrac{\partial\psi}{\partial x_j}(x))$,
- $g(x)$ is the matrix of the g_{ij}'s,
- $g_h(x_1,x_2,z) = g(x,y,hz)$,
- $\psi_h(x_1,x_2,z) = \psi(x_1,x_2,hz)$,
- g^{ij} are the coefficients of $g^{-1}(x)$,
- $\vec{\nu}(x_1,x_2) = \vec{n}(\Psi(x_1,x_2,0))$.

By property of the diffeomorphism ψ, we remark that $g(x)$ is on the form :

$$g(x) = \begin{pmatrix} \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and g^{-1} is on the same form.
In particular, $g^{13} \equiv g^{23} \equiv 0$.
On the other hand we remark that for all (x_1, x_2, z), we have $\varphi^3(\psi(x_1, x_2, z)) = z$ thus

$$\begin{cases} \nabla\varphi^3(\psi(x_1, x_2, z)) \cdot \dfrac{\partial \psi}{\partial x_1} = 0 \\ \\ \nabla\varphi^3(\psi(x_1, x_2, z)) \cdot \dfrac{\partial \psi}{\partial x_2} = 0 \\ \\ \nabla\varphi^3(\psi(x_1, x_2, z)) \cdot \dfrac{\partial \psi}{\partial z} = 1, \end{cases}$$

so $\nabla\varphi^3(\psi(x_1, x_2, z)) = \vec{n}(\psi(x_1, x_2, 0)) = \vec{\nu}(x_1, x_2)$.
We will now translate the energy in the new coordinates in order to perform the classical rescaling.

Second step : formulation of the energy in the new coordinates.

We have

$$\int_{\Omega_h} |\nabla u_h|^2 = \int_{B^2 \times [0,h]} \sum_{\alpha,\beta} g^{\alpha\beta}(x) \frac{\partial \underline{u_h}}{\partial x_\alpha} \frac{\partial \underline{u_h}}{\partial x_\beta} \sqrt{|g(x)|} dx_1 dx_2 dx_3.$$

and after rescaling we obtain that :

$$\int_{\Omega_h} |\nabla u_h|^2 = h \int_{B^2 \times [0,1]} \sum_{\alpha,\beta \in \{1,2\}} g^{\alpha\beta}(x, y, zh) \frac{\partial \widetilde{u_h}}{\partial x_\alpha} \frac{\partial \widetilde{u_h}}{\partial x_\beta} \sqrt{|\ g(x_1, x_2, hz)|} dx_1 dx_2 dz +$$

$$\frac{1}{h} \int_{B^2 \times [0,1]} |\frac{\partial \widetilde{u_h}}{\partial z}|^2 \sqrt{|\ g(x_1, x_2, hz)|} dx_1\ dx_2 dz.$$

In order to study the rescaling of H_h, we set $K_h(x, y, z) = H(u_h)(\psi(x, y, hz))$, and we have :

$$\int_{\mathbb{R}^3} |H(u_h)|^2 = \int_{\mathbb{R}^3} |K_h(x, y, z)|^2 |\text{Jac}\ \psi_h(x, y, z)|$$

and since $|\ \text{Jac}\ \psi_h(x, y, z)| = h\sqrt{|g(x, y, zh)|}$, we have

$$\int_{\mathbb{R}^3} |H(u_h)|^2 = h \int_{\mathbb{R}^3} |K_h(x, y, z)|^2 \sqrt{|g(x, y, zh)|} dx\ dy\ dz.$$

Third step : energy estimate.

Comparing the energy of u_h with the energy of a constant, using that there exists two constants μ and ν such that

$$\forall\, x \in \mathbb{R}^3,\ \forall\, \xi \in \mathbb{R}^3,\ \ \mu\|\xi\|^2 \le \sum_{i,j} g^{ij}(x)\xi_i\xi_j \le \nu\|\xi\|^2,$$

we obtain that there exists a constant C such that :

$$\left\{\begin{array}{l} \|\dfrac{\partial \widetilde{u_h}}{\partial x}\|_{L^2(B^2\times[0,1])} \leq C \\ \\ \|\dfrac{\partial \widetilde{u_h}}{\partial y}\|_{L^2(B^2\times[0,1])} \leq C \\ \\ \|\dfrac{\partial \widetilde{u_h}}{\partial z}\|_{L^2(B^2\times[0,1])} \leq Ch \\ \\ \|K_h\|_{L^2(I\!R^3)} \leq C \end{array}\right. \tag{3.10}$$

Extracting a subsequence, we can suppose that :

$$\left\{\begin{array}{l} \widetilde{u_h} \rightharpoonup \widetilde{u} \text{ in } H^1(B^2 \times [0,1]) \text{ weakly,} \\ \\ \widetilde{u_h} \longrightarrow \widetilde{u} \text{ in } L^2(B^2 \times [0,1]) \text{ strongly, and almost everywhere.} \\ \\ K_h \rightharpoonup K \text{ in } L^2(B^2 \times [0,1]) \text{ weakly.} \end{array}\right. \tag{3.11}$$

Fourth step : fundamental lemma.

Lemma 3.1 *Let $(\widetilde{w}_h)_h$ a sequence of $L^2(B^2 \times [0,1])$ such that $\widetilde{w}_h$ tends to $\widetilde{w}$ in L^2 strong. For $x \in \Omega_h$ we set $w_h(x) = \widetilde{w}_h(\varphi_1(x), \varphi_2(x), \frac{1}{h}\varphi_3(x))$ and we consider the rescaling of $H(w_h)$ setting :*

$$W_h(x_1, x_2, z) = H(w_h)(\psi(x_1, x_2, hz)).$$

Then W_h tends in $L^2(I\!R^3)$ strong to $-(\overline{\widetilde{w}} \cdot \vec{\nu})\vec{\nu}$.

Proof of the Lemma :
We have

$$H(w_h)(x) = W_h(\varphi_1(x), \varphi_2(x), \frac{1}{h}\varphi_3(x)).$$

By property of the operator H we know that $\|H(w_h)\|_{L^2(I\!R^3)} \leq \|w_h\|_{L^2(I\!R^3)}$. Hence there exists a constant C such that

$$\|W_h\|_{L^2(I\!R^3)} \leq C\|\widetilde{w_h}\|_{L^2(B^2\times[0,1])},$$

thus there exists a subsequence still denoted W_h such that W_h tends to W in $L^2(I\!R^3)$ weak.

Let us prove that $W = -(\overline{\widetilde{w}} \cdot \vec{\nu})\vec{\nu}$.
We write that div $(H(w_h) + \overline{w_h}) = 0$:

$$\forall\, \xi \in \mathcal{D}(I\!R^3), \quad \int_{I\!R^3} (H(w_k) + \overline{w_h}) \cdot \nabla\xi = 0$$

and taking $\xi(x) = \eta(\varphi_1(x), \varphi_2(x), \frac{1}{h}\varphi_3(x))$, we obtain

$$\int_{I\!R^3} \sum_{i=1}^{3}\sum_{j=1}^{2} (W_h^i + \overline{\widetilde{w_h^1}})(x) \cdot \frac{\partial \eta}{\partial x_j}(x)\frac{\partial \varphi_i}{\partial x_i}(\psi_h(x))\frac{1}{\sqrt{|\text{Jac } \varphi_h(\psi_h(x))|}}dx \;+$$

$$\frac{1}{h}\int_{I\!R^3}\sum_{i=1}^{3}(\frac{\partial\eta}{\partial x_3}(x)(W_h^i+\widetilde{\overline{w_h^i}})(x)\frac{\partial\varphi_3}{\partial x_i}(\psi_h(x))\frac{1}{\sqrt{|\text{Jac }\varphi_h(\psi_h(x))|}}dx=0$$

Multiplying by h and taking the limit when h goes to zero, one obtains that :

$$\int_{I\!R^3}\sum_{i=1}^{3}(\frac{\partial\eta}{\partial z}(x)(W^i+\widetilde{\overline{w^i}})(x)\frac{\partial\varphi_3}{\partial x_i}(\psi_0(x))\frac{1}{\sqrt{|\text{Jac }\varphi_0(\psi_0(x))|}}dX=0.$$

Now we know from the first step that $\nabla\varphi_3(\psi_0(X))=\vec{\nu}(X)$.
Hence we have obtain that :

$$\int_{I\!R^3}\frac{\partial\eta}{\partial z}\left(W+\overline{\widetilde{w}},\vec{\nu}\right)\sqrt{|g_0|}=0.$$

We deduce of this last assertion that $\left(W+\overline{\widetilde{w}},\vec{\nu}\right)\sqrt{|g_0|}$ does not depend on z and since this quantity is in L^2 we obtain that

$$\left(W+\overline{\widetilde{w}},\vec{\nu}\right)\equiv 0.$$

Let us write now that curl $H(w_h)=0$. Taking a test function of the same type and taking the limit when h goes to zero, we obtain that

$$\int_{I\!R^3}\frac{\partial\eta^2}{\partial z}\left(W^1\frac{\partial\varphi^3}{\partial z}(\psi_0(x))-W^3(x)\frac{\partial\varphi^3}{\partial x_1}(\psi(x))\right)\sqrt{|g_0(x)|}=0,$$

and using the same argument we get $W^1\vec{\nu}^3-W^3\vec{\nu}^1\equiv 0$.
Therefore we have shown that :

$$\begin{cases} W\wedge\vec{\nu}\equiv 0 \\ (W,\vec{n})\equiv-(\overline{\widetilde{w}},\vec{\nu}). \end{cases}$$

So

$$W\equiv-(\overline{\widetilde{w}},\vec{\nu})\vec{\nu}.$$

It remains to prove that W_h tends to W in $L^2(I\!R^3)$ strongly.

There exists a constant C independent of h such that :

$$\begin{aligned}\int_{I\!R^3}|W_h-W|^2 &\leq C\int_{I\!R^3}|W_h-W|^2\sqrt{|\,g_h(x)|}\\ &\leq C\int_{I\!R^3}|W_h|^2\sqrt{|\,g_h(x)|}\\ &\quad -2C\int_{I\!R^3}W_h\cdot W\sqrt{|\,g_h(x)|}+C\int_{I\!R^3}|W^2|\sqrt{|\,g_h(x)|}\end{aligned}$$

We remark now that

$$\begin{aligned}\int_{I\!R^3}|W_h|^2\sqrt{|\,g_h(x)|} &= \frac{1}{h}\int_{I\!R^3}|H(w_h)|^2\\ &= -\frac{1}{h}\int_{I\!R^3}\overline{w_h}H(w_h)\end{aligned}$$

by property of the operator H,

$$=-\int_{B(0,1)\times[0,1]}W_h\cdot\widetilde{w_h}\sqrt{|g_h(x)|}.$$

Now taking the limit when h tends to zero, using that

$$\begin{cases} W_h \rightharpoonup W \text{ in } L^2(I\!R^3) \text{ weakly}, \\ \\ \widetilde{w_h} \longrightarrow \widetilde{w} \text{ in } L^2(B(0,1)\times[0,1]) \text{ strong}, \\ \\ \sqrt{|g_h(x)|} \longrightarrow \sqrt{|g_0(x)|} \text{ in } L^\infty(B(0,1)\times[0,1]) \text{ strong}, \end{cases}$$

we deduce that $\|W_h - W\|_{L^2(I\!R^3)}$ tends to zero, thus W_h tends to W in $L^2(I\!R^3)$ strong.

Fifth step : end of the proof.

With (3.10) we obtain that $\dfrac{\partial \widetilde{u}}{\partial z} = 0$. Hence $\widetilde{u}$ does not depend on z and we can define u on $\mathcal{S}$ by $u(x) = \widetilde{u}(\psi^{-1}(x))$ for $X \in \mathcal{S}$.

Let $w \in H^1(\mathcal{S}; S^2)$. Let us prove that $\widetilde{E}(u) \leq \widetilde{E}(w)$.

We introduce $w_h \in H^1(\Omega_h; S^2)$ equal to w on $\mathcal{S}$ and constant along the radius $[x, x + h\vec{n}(x)]$ for $x \in \mathcal{S}$. We set $\underline{w_h} = w_h \circ \psi^{-1}$ and $\widetilde{w_h}(x_1, x_2, z) = \underline{w_h}(x_1, x_2, hz)$. We remark that $\widetilde{w_h}$ does not depend on z.
We set :

$$W_h(x_1, x_2, z) = H(w_h)(\psi(x_1, x_2, hz)).$$

Since $w_h \in H^1(\Omega_h; S^2)$, by minimality of u_h we have

$$\mathcal{E}(u_h) \leq \mathcal{E}(w_h).$$

We obtain that

$$\int_{B^2\times[0,1]} \sum_{\alpha,\beta\in\{1,2\}} g^{\alpha\beta}(x_1, x_2, zh) \frac{\partial \widetilde{u_h}}{\partial x_\alpha} \frac{\partial \widetilde{u_h}}{\partial x_\beta} \sqrt{|\ g(x_1, x_2, hz)|} dx_1 dx_2 dz$$

$$+ \int_{I\!R^3} |W_h(x_1, x_2, z)|^2 \sqrt{|g(x_1, x_2, zh)|} dx_1\ dx_2\ dz \leq$$

$$\int_{B^2\times[0,1]} \sum_{\alpha,\beta\in\{1,2\}} g^{\alpha\beta}(x_1, x_2, zh) \frac{\partial \widetilde{w_h}}{\partial x_\alpha} \frac{\partial \widetilde{w_h}}{\partial x_\beta} \sqrt{|\ g(x_1, x_2, hz)|} dx_1 dx_2 dz$$

$$+ \int_{I\!R^3} |W_h(x_1, x_2, z)|^2 \sqrt{|g(x_1, x_2, zh)|} dx_1 dx_2\ dz,$$

since $\dfrac{\partial \widetilde{w_h}}{\partial z} = 0$.
Now taking the limit when h goes to zero and using Lemma 3.1 we obtain that

$$\mathcal{E}(u) \leq \mathcal{E}(w).$$

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References

[1] F. Alouges et A. Soyeur, *On global weak solutions for Landau Lifschitz equations: existence and non uniqueness,* Nonlinear Anal., Theory Methods Appl. 18, No.11, 1071-1084 (1992).

[2] W.F. Brown, Micromagnetics, Interscience publisher, John Willey & Sons, New York, 1963.

[3] G. Carbou, *Regularity for Critical Points of a Non Local Energy,* Calc. Var. Partial Differential Equations **5** (1997), 409-433

[4] G. Carbou and F. Fabrie, *Time average in Micromagnetism,* Journal of Differential Equations **147**, 383-409 (1998) .

[5] R. Hardt, D. Kinderlehrer, *Soms Regularity Results in Ferromagnetism*, to appear.

[6] A. Visintin, *On Landau Lifschitz equation for ferromagnetism,* Japan Journal of Applied Mathematics, Vol. 2, n 1, p.p. 69-84, (1985).