Mixture of consistent stochastic utilities, and a priori randomness * †

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Abstract

The purpose of this paper is to develop an explicit construction of consistent utilities, using the stochastic flows approach developed in [KM13] and [KM16]. Starting from a family of utility functions indexed by some parameter α (for example the risk aversion of different agents), the idea is to randomize α and construct a non standard stochastic utilities processes. Two approach are developed, the first one consists to built directly from the class $\{U^\alpha, \alpha \in \mathbb{R}\}$ a global one U as a sup-convolution. The second approach which is very different, consists to define from a class $(X^\alpha, Y^\alpha)_{\alpha\in\mathbb{R}}$ of monotonic processes a global pair (X^*, Y^*) as a mixture. The non standard stochastic utility is then obtained by composing stochastic flows and interpreted as the aggregate utility of all considered agents .

1 Introduction

In the next section, we introduce the market model, the consistent utility's definition and we recall some results that we will use extensively in this paper (established in [KM13] and [KM16]). In Section 3, we give ourselves a family of consistent utility processes indexed by a parameter α: $\{U^\alpha,\ \alpha \in \mathbb{R}\}$ and a finite positive Borel measure $m(d\alpha)$. Denoting by $\widetilde{U}^\alpha$ the dual convex conjugate of U^α, we define the convex process $\widetilde{U}^s(t, y) = \int \widetilde{U}^\alpha(t, y)m(d\alpha)$

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and we show, assuming that all $\widetilde{U}^\alpha$ generate the same optimal dual process denoted by Y^* (Pareto-Optimality principle), that $\widetilde{U}^s$ is the dual convex conjugate of a consistent utility U^s. We show also, Theorem 3.1, that $U^s(t,x)$ is the sup-convolution of the concave functions $U^\alpha(t,x)$. Moreover, the optimal wealth processes associated with U^s is a mixture of optimal wealths $X^{*,\alpha}$ (associated with consistent utilities U^α) starting from given initial conditions x^α fixed by the problem,

$$X^*(t,x) = \int X^{*,\alpha}(t, x^\alpha(x))m(d\alpha), \quad \text{where } x^\alpha(x) = (u_x^\alpha)^{-1}(u_x(x)), \ \int x^\alpha(x)m(d\alpha) = x.$$

Where we have denoted by u the inverse Fenchel-Transform of $\widetilde{U}^s(0,y)$. Note u is imposed by definition as sup-convolution of $U^\alpha(0,.)$, $\alpha \in \mathbb{R}$.

Inspired by this observation, in section 4, we give ourselves a finite positive Borel measure $m(d\alpha)$ and a family of deterministic utility functions $\{u^\alpha,\ \alpha \in \mathbb{R}\}$. In a first step, we generate, from $\{u^\alpha,\ \alpha \in \mathbb{R}\}$, using techniques of change of numeraire and probability a new family of consistent utilities processes $\{U^\alpha,\ \alpha \in \mathbb{R}\}$ whose optimal processes (strictly increasing with respect to their initial conditions) are denoted by $X^{*,\alpha}$ and $Y^{*,\alpha}$, $\alpha \in \mathbb{R}$. As processes $Y^{*,\alpha}$, $\alpha \in \mathbb{R}$ are not necessarily the same, we put aside these processes and we will consider another state density price that we denote $Y^*(y) := yY^*$ such that $Y^* X^{*,\alpha}$ is a martingale for any α. Thereafter, we will give ourselves a second family of strictly increasing positive functions $x^\alpha, \alpha \in \mathbb{R}$. This is any family that we, only, impose to satisfy $\int x^\alpha(x)m(d\alpha) = x,\ \forall x$.

The next step is then to build a new portfolio X^* that is strictly increasing with respect to its initial condition, as a mixture of $X^{*,\alpha}$, $\alpha \in \mathbb{R}$ as follows

$$X^*(t,x) = \int X^{*,\alpha}(t, x^\alpha(x))m(d\alpha),\ X_0^*(x) = \int x^\alpha(x)m(d\alpha) = x. \tag{1.1}$$

From this, after verifying that X^*Y^* is a martingale and denoting by $\mathcal{X}$ the inverse of X^* with respect to its initial condition, we generate a new consistent utility U from any initial condition u, using results of [KM13] and [KM16] as follows

$$U(t,x) := Y_t^* \int_0^x u_x(\mathcal{X}(t,z))dz.$$

Note: As it is defined this consistent utility, is not a mixture of processes U^α, $\alpha \in \mathbb{R}$. It's initial condition u is anyone, not necessarily equal to the sup-convolution of initial functions u^α as is the case of previous Section. In (1.2), $\{x^\alpha(.), \alpha \in \mathbb{R}\}$ are also arbitrary and not fixed as in Section 3.

These last two points give us additional degrees of freedom. Thus we can generate a large class of consistent utilities. Finally, in section 5, to provide a utility class even richer, we will build in the same way as for X^* a dual process Y^* as a mixture of optimal processes $Y^{*,\alpha}$ for a given, by analogy, a family of strictly increasing functions y^α, which play the role of x^α concerning Y^*, i.e.

$$Y^*(t,y) = \int Y^{*,\alpha}(t, y^\alpha(y))m(d\alpha),\ Y_0^*(y) = \int y^\alpha(y)m(d\alpha) \overset{assumption}{=} y. \tag{1.2}$$

The utility process proposed is, then, by results of [KM13] and [KM16], defined by

$$U(t,x) := \int_0^x Y_t^*(u_x(\mathcal{X}(t,z)))dz.$$

For any utility function u satisfying some integrability conditions.

2 Preliminaries

Utility function: Throughout the paper, we make the classical assumptions for an utility function V that is $V : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is increasing on $\mathbb{R}$, continuous on $\{V > -\infty\}$, differentiable and strictly concave on the interior of $\{V > -\infty\}$ and that V_x tends to zero when wealth tends to infinity,i.e.,

$$V_x(\infty) := \lim_{x\to\infty} V_x(x) = 0.$$

As regards the behavior of the (marginal) utility at the other end of the wealth scale we shall distinguish throughout the paper two cases.

Case 1 (negative wealth not allowed): in this setting we assume that V satisfies the conditions $V(x) = -\infty$, for $x < 0$, while $V(x) > -\infty$, for $x > 0$, and that

$$V_x(0) := \lim_{x\searrow 0} V_x(x) = \infty. \tag{2.1}$$

Case 2 (negative wealth allowed): in this case we assume that $V(x) = -\infty$, for all $x \in \mathbb{R}$, and that

$$V_x(-\infty) := \lim_{x\searrow -\infty} V_x(x) = \infty. \tag{2.2}$$

In the first part of this work, we restrict ourselves only to the first case before considering the second in the last section.

Prorper concave and convex functions: In mathematical convex analysis and optimization, a proper convex function is a convex function f taking values in the extended real number line such that $f(x) < +\infty$ for at least one x and $f(x) > -\infty$ for every x. That is, a convex function is proper if its effective domain is nonempty and it never attains $-\infty$. A proper concave function is any function g such that $f = -g$ is a proper convex function.

Some Properties: The infimal convolution (or epi-sum) of two functions f and g is defined as

$$(f \square g)(x) = \inf_{y\in\mathbb{R}} \{f(x-y) + g(y)\}.$$

(ii) The convex conjugate $\tilde{f}$ of a lower semi-continuous (lsc) proper concave function f is lsc proper convexe function.
(iii) Let $f_1, \dots, f_m$ be proper, convex and lsc functions on $\mathbb{R}$. Then the infimal convolution is convex and lsc (but not necessarily proper), and satisfies

$$F := (f_1 \Box \dots \Box f_m) \text{ is such that } \tilde{F} = \tilde{f}_1 + \dots + \tilde{f}_m$$

The Model: Let $W = (W_1, W_2, ..., W_n)^T$ be a n-standard Brownian motion ($n \geq d$), defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $(\mathcal{F}_t)_{t\geq 0}$ is the $\mathbb{P}$-augmented filtration generated by the Brownian motion W, i.e. $\mathcal{F}_t = \sigma(W_s, 0 \leq s \leq t)$. An admissible portfolio process $X^\kappa \in \mathbf{X}$ with strategy κ is an Itô martingale satisfying

$$dX_t^\kappa = X_t^\kappa \big[r_t dt + \kappa_t.(dW_t + \eta_t dt)\big],\ \ \kappa_t \in \mathcal{R}_t. \tag{2.3}$$

Where $\mathcal{R}_t$ denote the set (linear space) of admissible strategies at time t and by $\mathcal{R}_t^\perp$ its orthogonal. r and $\eta \in \mathcal{R}$ are a $\mathcal{F}$-progressively measurable processes playing the role of interest rate and the minimal risk premium.

In the following we will also need to define the family of admissible state prices density which is the set $\mathscr{Y}$ of positive semimartingales Y^ν whose the dynamics is

$$\frac{dY_t^\nu}{Y_t^\nu} = -r_t dt + (\nu_t - \eta_t).dW_t, \quad \nu_t \in \mathcal{R}^\perp \tag{2.4}$$

Then, for any wealth process $(X^\kappa,\ \ \kappa \in \mathcal{R})$, $Y^\nu X^\kappa$ is a local martingale. These processes are also called *adjoint processes*. Note that, for any $\nu \in \mathcal{R}^\perp$, $Y^\nu = Y^0 \mathcal{E}(\nu)$ where $\mathcal{E}(\delta)$ denote the exponential martingale given by $\mathcal{E}_t(\delta) = exp\Big(- \frac{1}{2}\int_0^t ||\delta_s||^2 ds + \int_0^t \delta_s.dW_s\Big)$.

Consistent Utilities Let $\mathscr{X}$ any set of wealth processes we now recall the definition of the $\mathscr{X}$-consistent stochastic utility.

Definition 2.1 (**$\mathscr{X}$-consistent Utility).** *A $\mathscr{X}$-consistent stochastic utility process $U(t,x)$ is a positive random field with the following properties:*

- $*$ **Concavity assumption :** *for $t \geq 0$, $x \mapsto U(t,x)$ is an increasing concave function, (in short utility function) .*
- **Consistency with the test-class:** *For any admissible wealth process $X \in \mathscr{X}$, $\mathbb{E}(U(t,X_t)) < +\infty$ and*
$$\mathbb{E}(U(t,X_t)/\mathcal{F}_s) \leq U(s,X_s),\ \forall s \leq t \ \ .a.s.$$
- **Existence of optimal wealth:** *For any initial wealth $x > 0$, there exists an optimal wealth process $X^* \in \mathscr{X}$, such that $X_0^* = x$, and $U(s,X_s^*) = \mathbb{E}(U(t,X_t^*)/\mathcal{F}_s)\ \forall s \leq t$.*

Convex conjugate of consistent stochastic utility The convex conjugate of a consistent stochastic utility is a random field defined by

$$\widetilde{U}(t,y) \stackrel{def}{=} \inf_{x>0, x\in Q^+} \big(U(t,x) - x\,y\big) \tag{2.5}$$

$\widetilde{U}(t,y)$ is a progressive decreasing convex dual utility, with first derivative $\widetilde{U}_y(t,.) = -U_x(t,.)^{-1}(y)$ is the inverse flow of the decreasing flow $U_x(t,x)$. Moreover, as it is shown in [KM13] and [KM16], $\widetilde{U}(t,y)$ is consistent with the family $\mathscr{Y}$ of state price densities, that is $\widetilde{U}(t,Y_t)$ is a submartingale and $\widetilde{U}(t,Y_t^*)$ is a martingale if $Y^*(t,y) = U_x\big(t, X_t^*(-\widetilde{U}_y(0,y))\big)$. $\widetilde{U}(t,y)$ is said to be *a $\mathscr{Y}$-consistent stochastic dual utility.*

Example: The Power Consistent Stochastic Utilities Denote $\mathscr{X}^+$ the set of all positive wealth processes. We are locking for $\mathscr{X}^+$-consistent power utilities $U^{(a)}(t,x) = Z_t^{(a)} \frac{x^{1-a}}{1-a}$ where a is the risk aversion coefficient and $Z^{(a)}$ a semimartingale that allows to satisfy the consistency property. As in the deterministic framework, the conjugate function $\widetilde{U}^{(a)}(t,y)$ is given by $\widetilde{U}^{(a)}(t,y) = -\tilde{Z}_t^{(a)} \frac{y^{1-\frac{1}{a}}}{1-\frac{1}{a}}$ with $\tilde{Z}_t^{(a)} = \big(Z_t^{(a)}\big)^{\frac{1}{a}}$.
Thanks to the consistency property, there exists an optimal portfolio $X_t^{(a),*}(x)$ s.t. $U^{(a)}(t, X_t^{(a),*}(x)) = \frac{1}{1-a} Z_t^{(a)} \big(X_t^{(a),*}(x)\big)^{1-a}$ is a martingale, and s.t. $U_x(t, X_t^{(a),*}(x)) = Y_t^{(a),*}(x^{-a})$ is a state price density process with initial condition x^{-a}. In particular, using the intuitive factorization $Z_t^{(a)} = Z_t^{(a,\sigma)}.Z_t^{(a,\perp)}$ where $Z_t^{(a,\perp)}$ is a exponential martingale $\mathcal{E}_t(\delta^\perp.W)$, whose the martingale part belongs to $\mathcal{R}^\perp$, we see that $Z_t^{(a,\sigma)}(X_t^{(a),*}(x))^{-a} = x^{-a} Y_t^0$, where Y_t^0 is the minimal state price density. The optimal wealth $X_t^{(a),*}(x)$ is linear with respect to the initial condition $X_t^{(a),*}(x) = x X_t^{(a),*}(1)$ where $X_t^{(a),*}(1)$ also denoted $X_t^{(a),*}$ is the optimal portfolio starting from $x=1$, hence

$$(I)\begin{cases} Z_t^{(a)} &= Z_t^{(a,\perp)}\, Y_t^0\, (X_t^{(a),*})^a, & \tilde{Z}_t^{(a)} = X_t^{(a),*}\big(Y_t^{(a),*}\big)^{\frac{1}{a}} \\ X_t^{(a),*}(x) &= x X_t^{(a),*}, & Y_t^{(a),*}(y) = y Z_t^{(a,\perp)}\, Y_t^0 \\ U^{(a)}(t,x) &= \frac{1}{1-a} Y_t^{(a),*} X_t^{(a),*} \big(\frac{x}{X_t^{(a),*}}\big)^{1-a}, & \widetilde{U}^{(a)}(t,y) = -\frac{Y_t^{(a),*} X_t^{(a),*}}{1-\frac{1}{a}} \big(\frac{y}{Y_t^{(a),*}}\big)^{1-\frac{1}{a}} \end{cases}$$

where we have used the fact that $Y^{(a),*}(y) = y Y^{(a),*}$ is linear with respect to y. Observe also that the optimal portfolio and the optimal price density depend on a only by the choice of the orthogonal volatility vector of $Z^{(a)}$.

Remark 2.1. *Identities $X_t^{(a),*}(x) = x X_t^{(a),*}$ and $Y_t^{(a),*}(y) = y Y_t^{(a),*}$ show that these processes are increasing in x and y with inverse flows $\mathcal{X}_t^{(a)}(x) = x/X_t^{(a),*}$ and $\mathcal{Y}_t^{(a)}(y) = y/Y_t^{(a),*}$. From this point, it is straightforward to check that $U^{(a)}(t,x)$ and $\widetilde{U}^{(a)}(t,y)$, taking $u = \frac{x^{1-a}}{1-a}$, have the following representations,*

$$U^{(a)}(t,x) = \int_0^x Y_t^{(a),*}\big(u_x(\mathcal{X}_t^{(a)}(z))\big)dz, \quad \widetilde{U}^{(a)}(t,y) = \int_y^{+\infty} X_t^{(a),*}\big(-\tilde{u}_y(\mathcal{Y}_t^{(a)}(z))\big)dz.$$

note, in particular, that $Y_t^{(a),*}(u_x(x)) = U_x^{(a)}(t, X_t^{(a),*}(x))$.
We will return, in the next section, to this interesting representations of stochastic utility and its dual convex and we recall, as established in [KM13] and [KM16], that this is not specific to the utilities of power type. Also, a second interpretation of these utilities and their stochastic optimal processes is given through a stochastic partial differential equations.

Main recent results The class of consistent stochastic utilities was studied in [KM13] and [KM16] under different aspects and different techniques. In [KM13] the approach proposed is an approach by stochastic PDE's, developed using tools of stochastic calculus such as the generalized Itô formula also called Itô-Ventzel formula and calculating inverse flows dynamics. While in [KM16] there is a direct approach which make minimal regularity assumptions of processes and therefore the general results are obtained in abstract form by methods of analysis and composition of monotonic flows. As it has established in [KM13] a large class of this stochastic utilities are solution of the following HJB-SPDE,

$$dU(t,x) = \Big[- x r_t U_x(t,x) + \frac{1}{2U_{xx}(t,x)} \|U_x(t,x)\eta_t + \gamma_x^\sigma(t,x)\|^2 \Big] dt + \gamma(t,x).dW_t \quad (2.6)$$

when the optimal policy κ^* is given by

$$x\kappa_t^*(x) = -\frac{1}{U_{xx}(t,x)}(U_x(t,x)\eta_t + \gamma_x^\sigma(t,x))$$

In this paper we are interested in stochastic utilities of this class. For this, some results that we used extensively in this paper and which are established in [KM13] and [KM16] are recalled. To get started, during this paper the optimal wealth process X^{κ^*} is denoted for simplicity by X^*. As in the classical theory of portfolio optimization in expected utility framework the process $U_x(t, X_t^*)$ has nice properties and a central place in the study of the dual problem. Indeed, it is showed in [KM13] that if U is a regular consistent progressive utility, its dual convex conjugate $\widetilde{U}(t,y)$ (convex decreasing stochastic flows, null if $y = 0$ satisfies) is consistent with the family of state density processes $\mathscr{Y}$. That is for any $Y^\nu \in \mathscr{Y}$, $\widetilde{U}(t, Y_t^\nu)$ is a submartingale and a martingale for an optimal dual choice given by $y\nu^*(t,y) = \gamma_x^\perp\big(t, -\widetilde{U}_y(t,y)\big)$.
Furthermore, for any $y > 0$ the equation $dY_t^{\nu^*} = Y_t^{\nu^*}\big(- r_t dt + \Big(\nu^*(t, Y_t^{\nu^*}) - \eta_t\Big).dW_t\big)$. admits at least one solution which is $Y_t^*(y) := U_x(t, X_t^*((U_x)^{-1}(0,y))$. In other words $U_x(t, X_t^*)$ is a price density process which is the optimum of the dual problem.
The purpose of recalling this results, see [KM13] and [KM16] for more details, is to highlight the role played by the processes $U_x(t, X_t^*)$ and especially emphasize the perfect symmetry between the primal problem whose optimum is X^* and the dual problem whose optimum is $Y^*(U_x(0,x)) := U_x\big(t, X_t^*(x)\big)$. This enables us in particular to prepare the next result which is the general theorem of stochastic construction of utilities which is

based on optimality conditions and the properties of optimal processes X^* and Y^*. Finally, in what follows it is sometimes easier to consider the dual convex $\widetilde{U}$ than the utility U as is the case of the decreasing stochastic utilities introduced in the next paragraph , which explains the interest that we bring to the duality in this work.
Next result, Theorem 4.3 in [KM13], show how by stochastic change of variable the consistent stochastic utilities are constructed. This new approach called "stochastic flow method" start from a wealth process X^* and a state density price Y^* and requires that the processes $X_t^*(x)$ and $Y_t^*(y)$ to be continuous and strictly increasing in x and y from 0 to $+\infty$.

Theorem 2.1. *Let (X^*, Y^*) a pair of wealth process and a state density price assumed to be continuous and increasing in x and y from 0 to $+\infty$ s.t. $X_t^*(0) = Y_t^*(0) = 0$, $X_t^*(+\infty) = Y_t^*(+\infty) = +\infty$ a.s. for any t. Denote by $\mathcal{X}$ and $\mathcal{Y}$ the inverse flows of X^* and Y^*, if $X_x^* Y^*$ is a martingale and u is an utility function s.t. $x \mapsto Y_t^*(u_x(\mathcal{X}(t,z)))$ is integrable near to zero. Then the process U defined by*

$$U(t,x) = \int_0^x Y_t^*\big(u_x(\mathcal{X}(t,z))\big)dz \tag{2.7}$$

is a $\mathscr{X}^+$-Consistent stochastic utility s.t. $Y_t^*(u_x(x)) = U_x(t, X_t^*(x))$. The associated optimal portfolio and the optimal state density price are X^* and Y^* and the convex conjugate is given by

$$\widetilde{U}(t,y) = \int_y^{+\infty} X_t^*\big(-\tilde{u}_y(\mathcal{Y}(t,z))\big)dz. \tag{2.8}$$

3 Sup Convolution of $\mathscr{X}$-consistent stochastic utilities

The results of this section are obtained in a general way by considering any set $\mathscr{X}$ of wealth processes (we do not need a model). Consider a family of $\mathscr{X}$-consistent stochastic utilities (with the same family of test portfolios $\mathscr{X}$). We are interested in mixtures of these utilities that meet the property of consistency with respect to the given class of wealth processes $\mathscr{X}$. For instance, let us assume that we want to test the gain of diversification over the different K business units of financial firm, equipped with different stochastic utilities $U^i(t,x)$ assumed to be **proper** concave functions, for example with different risk aversion coefficients. It should be noted that these utilities are not necessarily of the same type. The case where the utilities are all of the same type, for example a power or or exponential type, is a particular case.

The problem is to find the fair allocation of the wealth x between the different units, in the following sense: find the wealth $(x_1^*, x_2^*..., x_K^*)$ with $\sum_1^K x_i^* = x$ such that

$$U^s(t,x) = \sup\{\sum_1^K U^i(t,x_i) | \forall i\, x^i \geq 0, \text{and} \sum_1^K x_i = x\} \tag{3.1}$$

achieves its maximum on $(x_1^*, x_2^*..., x_K^*)$ and then study the new progressive utility. In particular, we are looking for sufficient conditions under which the new utility U^s is $\mathscr{X}$-consistent utility. In convex analysis, the utility $U^s(t,x)$ is known as the sup-convolution of the concave functions $U^i(t,x)$. Such utilities are easier to study from the dual point of view since $\widetilde{U}^s(y) = \sum_1^K \widetilde{U}^i(t,y)$. The same problem may be extended to a continuous set of units U^θ with a positive finite measure $m(d\theta)$.

Remark 3.1. *Observe, by assuming that the functions $U^i(t,.)$ are proper guarantee that $U^s(t,.)$ is well defined by* (3.1).

Mixture of convex dual utilities and sup-convolution Let us start with a family $\widetilde{U}^\theta(t,y)$ of convex consistent dual utilities with a common set of state prices densities $\mathscr{Y}$, and define $\widetilde{U}^s(t,y) = \int \widetilde{U}^\theta(t,y)m(d\theta)$. Assume that at any time t, the convex functions $\widetilde{U}(t,.)$ are proper continuously differentiable with continuously differentiable primal functions U^θ. Then, for any admissible state price density Y^ν, $\widetilde{U}^s(t,Y_t^\nu) = \int \widetilde{U}^\theta(t,Y_t^\nu)m(d\theta)$ is a submartingale, as sum of positive submartingales. The martingale property can be obtained only for a process Y_t^* such that for a.s θ, $\widetilde{U}^\theta(t,Y_t^*)$ is a martingale.
Since $X_t^*(x) = -\widetilde{U}_y^s(t,Y_t^*(u_x(x)))$, $X_t^*(x)$ is a mixture of optimal wealths,

$$X_t^*(x) = -\int \widetilde{U}_y^\theta\big(t,Y_t^*(u_x(x))\big)m(d\theta)$$

on the other hand, as Y^* is the optimal dual process for all the consistent utilities U^θ it follows, denoting by X^θ the associated optimal wealth, that $X_t^\theta(x) = -\widetilde{U}_y^\theta(t,Y_t^*(u_x^\theta(x))$. From this point, it is easy to check that $-\widetilde{U}_y^\theta(t,Y_t^*(u_x(x)) = X_t^\theta\big((u_x^\theta)^{-1}(u_x(x))\big)$ which implies

$$X^*(t,x) = \int X^{*,\theta}(t,x^\theta(x))m(d\theta), \quad \text{where } x^\theta(x) = (u_x^\theta)^{-1}(u_x(x)).$$

Let us now, show the following result,

Theorem 3.1. *Assume the existence of a state density process Y^* such that for a.s θ, $\widetilde{U}^\theta(t,Y_t^*)$ is a martingale (Y^* is the optimal dual process for $\widetilde{U}^s$), then*
(i) *The utility process U^s is given as the Sup-Convolution:*

$$U^s(t,x) = \sup\{\int U^\theta(t,x^\theta(x))m(d\theta); \int x^\theta(x)m(d\theta) = x\}$$

(ii) *The supremum is achieved at the family $\{\hat{x}^\theta(t,x) := (U_x^\theta)^{-1}(t,-(\widetilde{U}_y^s)^{-1}(t,x)),\ \theta\}$ satisfying the condition $\int \hat{x}^\theta(t,x)m(d\theta) = x$.*
(iii) *U^s is a consistent stochastic utility with optimal portfolio $X^*(t,x) = \int X^{*,\theta}(t,x^\theta(x))m(d\theta)$. Moreover, for any θ, the optimal wealth is $\hat{x}^\theta(t,X_t^*(x)) = X^{*,\theta}(t,x^\theta(x))$.*

Proof. (i) and (ii): As it is defined the random field $U^s(t,x)$ is given by

$$U^s(t,x) := \inf_{y>0}\{\widetilde{U}^s(t,y) + xy\} = \inf_{y>0}\{\int \widetilde{U}^\theta(t,y)m(d\theta) + xy\}$$

In particular for any family of functions $x^\theta(x) : \int x^\theta(x)m(d\theta) = x$, $U^s(t,x)$ can be written as

$$U^s(t,x) = \inf_{y>0}\{\int \big(\widetilde{U}^\theta(t,y) + yx^\theta(x)\big)m(d\theta)\} \tag{3.2}$$

On the other hand, as $\widetilde{U}^\theta$ is the convex conjugate of U^θ, it is rather obvious that

$$\widetilde{U}^\theta(t,y) + yx^\theta(x) \geq U^\theta(t, x^\theta(x))$$

Which leads, in (3.2), to

$$U^s(t,x) \geq \int U^\theta(t, x^\theta(x))m(d\theta) \text{ with } \int x^\theta(x)m(d\theta) = x.$$

Therefore

$$U^s(t,x) \geq \sup_{x^\theta:\ \int x^\theta(x)m(d\theta)=x} \int U^\theta(t, x^\theta(x))m(d\theta). \tag{3.3}$$

To conclude it remains to establish the reverse inequality. For this, observe, from the definition $\widetilde{U}^s(t,y) = \int \widetilde{U}^\theta(t,y)m(d\theta)$ and by integrability and regularity assumptions, that $\widetilde{U}^s_y(t,y) = \int \widetilde{U}^\theta_y(t,y)m(d\theta)$. Which gives, using $(U^s_x)^{-1}(t,y) = -\widetilde{U}^s_y(t,y),\ t \geq 0$ and $(U^\theta_x)^{-1}(t,y) = -\widetilde{U}^\theta_y(t,y),\ t \geq 0$ that $\int (U^\theta_x)^{-1}(U^s_x(x))m(d\theta) = x$ so that the family $\{\hat{x}^\theta(t,x) := (U^\theta_x)^{-1}(t, U^s_x(t,x)),\ \theta\}$ satisfies the condition

$$\int \hat{x}^\theta(t,x)m(d\theta) = x. \tag{3.4}$$

Second, using the dual identity $U^\theta(t,x) = \widetilde{U}^\theta(t, U^\theta_x(t,x)) + xU^\theta_x(t,x)$ it follows that

$$\begin{aligned} U^\theta(t, \hat{x}^\theta(t,x)) &= \widetilde{U}^\theta\Big(t, U^\theta_x\big(t, (U^\theta_x)^{-1}(t, U^s_x(t,x))\big)\Big) + (U^\theta_x)^{-1}\Big(t, U^s_x(t,x)\Big)U^\theta_x\Big(t, (U^\theta_x)^{-1}(t, U^s_x(t,x))\Big) \\ &= \widetilde{U}^\theta\Big(t, U^s_x(t,x)\Big) + U^s_x(t,x)(U^\theta_x)^{-1}(t, U^s_x(t,x)) \\ &= \widetilde{U}^\theta\Big(t, U^s_x(t,x)\Big) + U^s_x(t,x)\hat{x}^\theta(t,x) \end{aligned} \tag{3.5}$$

Let now point out that it follows from (3.2), for any $x^\theta(x) : \int x^\theta(x)m(d\theta) = x$, that

$$\begin{aligned} U^s(t,x) &= \inf_{y>0}\Big\{\int \big(\widetilde{U}^\theta(t,y) + yx^\theta(x)\big)m(d\theta)\Big\} \\ &\leq_{y=U^s_x(t,x)} \int \Big(\widetilde{U}^\theta(t, U^s_x(t,x)) + U^s_x(t,x)x^\theta(x)\Big)m(d\theta) \end{aligned}$$

Taking $x^\theta(x) = \hat{x}^\theta(t,x)$ and substituting the expression (3.5) into the last inequality yields

$$U^s(t,x) \leq \int U^\theta(t, \hat{x}^\theta(x))m(d\theta)$$

which gives the desired inequality:

$$U^s(t,x) \leq \sup_{x^\theta:\ \int x^\theta(x)m(d\theta)=x} \int U^\theta(t,x^\theta(x))m(d\theta).$$

Combining this with (3.3) yield (i) and (ii).
Let now focus on assertion (iii): by assumption $\widetilde{U}$ is the convex conjugate of U^s, which is consistent with the family of state density processes $\mathcal{Y}$, and achieves its maximum on Y^* also optimal for all utilities $\widetilde{U}^\theta$, this leads by analogy between the dual and primal problem, see [KM13], to the consistency of U^s. We have now to show at first the optimality of the process $X^*_t := \int X^{*,\theta}(t,x^\theta(x))m(d\theta)$ and second the martingale property of $U^s(t,X^*_t)$. For this, once again, from results of [KM13] and [KM16] and the correspondence primal-dual problem, the optimal primal process is $X^*_t(x) = -\widetilde{U}^s_y(t,Y^*_t(U^s(0,x)))$. This on the one hand but on the other one, as $Y^* = Y^{*,\theta}$, $\forall\theta$ it follows that $X^{*,\theta}_t(x) = -\widetilde{U}^\theta_y(t,Y^*_t(U^\theta_x(0,x)))$. Consequently,

$$\begin{aligned}
X^*_t(x) &= -\widetilde{U}^s_y(t,Y^*_t(U^s(0,x))) := -\int \widetilde{U}^\theta_y(t,Y^*_t(U^s(0,x)))m(d\theta) \\
&= \int X^{*,\theta}_t(-\widetilde{U}^\theta_y(0,U^s_x(0,x)))m(d\theta) = \int X^{*,\theta}_t(x^\theta(x))m(d\theta)
\end{aligned}$$

Hence the optimality of $\int X^{*,\theta}_t(x^\theta(x))m(d\theta)$. Now, the dual identity and the optimality of X^* and Y^* leed to

$$\begin{aligned}
U^s(t,X^*_t(x)) &= \widetilde{U}^s(t,-\widetilde{U}^s_y(t,X^*_t(x))) + \widetilde{U}^s_y(t,X^*_t(x))X^*_t(x)) \\
&= \widetilde{U}^s(t,Y^*_t(-\widetilde{U}^s_y(0,x))) + Y^*_t(-\widetilde{U}^s_y(0,x))X^*_t(x))
\end{aligned}$$

which implies, as it is the sum of two martingales, that $(U^s(t,X^*_t(x)))_t$ is a martingale. To achieve the proof, let us show the identity $\hat{x}^\theta(t,X^*_t(x)) = X^{*,\theta}(t,x^\theta(x))$, for this it suffices to use the definition of $x^\theta(t,.)$ above and the fact that $U^s_x(t,X^*_t(x)) = Y^*_t(U^s_x(0,x)) = U^\theta_x(t,X^{*,\theta}(-\widetilde{U}^\theta_y(0,U^s_x(0,x))$ and conclude using $x^\theta(x) := -\widetilde{U}^\theta_y(0,U^s_x(0,x)$, i.e.,

$$\begin{aligned}
x^\theta(t,X^*_t(x)) : &= -\widetilde{U}^\theta_y(t,U^s_x(t,X^*(t,x))) = -\widetilde{U}^\theta_y(t,U^\theta_x(t,X^{*,\theta}(-\widetilde{U}^\theta_y(0,U^s_x(0,x))))) \\
&= X^{*,\theta}_t(-\widetilde{U}^\theta_y(0,U^s_x(0,x)) = X^{*,\theta}_t(x^\theta(x))
\end{aligned}$$

□

Remark Let us comeback to the case of power utilities, developed above. The parameter θ plays now the role of risk aversion, that is $U^\theta(t,x) = Z^{(\theta)}_t \frac{x^{1-\theta}}{1-\theta}$, taking $Z^{(\theta),\perp}_t \equiv 1$ ($\Leftrightarrow$ the orthogonal part of the volatility vector of $Z^{(\theta)}$ is null), the state price density $Y^* = Y^0$ is optimal for all U^θ. According to (I), one easily gets that the convex conjugate $\widetilde{U}^s$ of the consistent utility U^s is given by

$$\widetilde{U}^s(t,y) = \int -\frac{1}{1-\frac{1}{\theta}} Y^0_t X^{(\theta)}_t \big(y/Y^0_t\big)^{1-\frac{1}{\theta}} m(d\theta)$$

with fixed initial condition

$$\widetilde{U}^s(0,y) = \tilde{u}^s(y) = \int -\frac{1}{1-\frac{1}{\theta}} y^{1-\frac{1}{\theta}} m(d\theta).$$

New interpretation and a Direct characterization of decreasing consistent utilities Herein, we are concerned with the class of increasing (in time) consistent utilities which was studied and fully characterized in the literature by Berrier & al. [FB09] and Musiela & al. [MZ08b]. Considered in a market model without interest rate ($r \equiv 0$), this utilities have a volatility vector γ identically zero. It is an example where the dual SPDE is easier to study than the primal one. Indeed, taking $\gamma = 0$ it follows, from equation (2.6), that U is a solution of the following PDE

$$dU(t,x) = \frac{1}{2} \frac{U_x(t,x)^2}{U_{xx}(t,x)} ||\eta_t||^2 dt \tag{3.6}$$

where the convex conjugate $\widetilde{U}$ satisfies

$$\widetilde{U}_t(t,y)(\omega) = -\frac{1}{2} y^2 \widetilde{U}_{yy}(t,y)(\omega) ||\eta_t(\omega)||^2 \tag{3.7}$$

which implies, by convexity, that $t \mapsto \widetilde{U}(t,y)$ is a decreasing function. Moreover, it is easy to recognize in this PDE that the right hand side of the equation is nothing other than the operator of diffusion $L^{GB}_{t,y}(\omega)$ of a geometrical Brownian motion with coefficients $\eta_t(\omega)$ and $r_t(\omega)$ applied to $\widetilde{U}$: $\widetilde{U}_t(t,y)(\omega) = -L^{GB}_{t,y}\widetilde{U}(t,y)(\omega)$. From this point, the idea is to look for positive solutions which are space-time harmonic functions of a geometric Brownian motion. Using the result of Widder, D.V [Wid63, Wid75], F. Berrier & al. [FB09] and Musiela & al. [MZ08b] show the following result which characterizes all regular dual convex conjugate of decreasing consistent utilities

Theorem 3.2. *Let $U(t,x)$ be a regular random field of class $\mathcal{C}^1 \times \mathcal{C}^3$ on (t,x). Assume U satisfies the PDE* (3.6)*. Then U is a consistent stochastic utility if and only if there exists a constant $C \in \mathbb{R}$ and a finite Borel measure m, supported on the interval* $(0,+\infty)$ *with everywhere finite Laplace transform, such that*

$$\begin{cases} \widetilde{U}(t,y) & = \int_{\mathbb{R}^*_+} \frac{1}{1-\frac{1}{\alpha}} \big(1 - y^{1-\frac{1}{\alpha}} e^{-\frac{1-\alpha}{2\alpha}\int_0^t ||\eta_s||^2 ds}\big) m(d\alpha) + C. \\ \widetilde{U}_y(0,y) & = -\int_{\mathbb{R}^*_+} y^{-\frac{1}{\alpha}} m(d\alpha) \end{cases}$$

Moreover the optimal wealth process is strictly increasing and regular with respect to its initial condition x.

These utilities were essentially studied by Zariphopoulou and al. [MZ08b] and by M. Tehranchi and al. [FB09]. In [MZ08b], the authors developed some examples with different measures m as well as properties of the associated optimal wealth.

Evidently, there is an interesting interpretation of these stochastic utilities: At date $t = 0$ the derivative $\widetilde{U}_y(0, y)$ can be easily interpreted as the integral of $-y^{-\frac{1}{\alpha}}$ weighted by the measure m, which is other than the derivative of the convex conjugate of power utility with risk aversion α. Hence, one can imagine that the investor starts from a power utility for which he pull at random the risk aversion, for any realization α he associate the power utility u^α weighted by m. The derivative of the convex conjugate of his utility at any date t is then the integral of the derivatives of the convex conjugates of power utility where the deterministic measure m becomes stochastic $m_t(d\alpha) := e^{-\frac{1-\alpha}{2\alpha}\int_0^t ||\eta_s||^2 ds} m(d\alpha)$. Moreover, the stochastic measure $m_t(d\alpha)$ is the unique one which ensure that the process $\widetilde{U}$ constructed is the derivative of a decreasing consistent utility.

This interpretation combined with the results of previous Pargraph, readily give us a direct charracterisation of the decreasing (in time) utility as a sup-convolution.

Theorem 3.3 (New Characterization)**.** *According to Theorem 3.1, any decreasing forward utility U is a Sup-Convolution:*

$$U(t, x) = \sup_{x^\alpha:\ \int x^\alpha(x) m(d\alpha) = x} \int \frac{(x^\alpha(x))^{1-\alpha}}{1-\alpha} e^{\frac{1-\alpha}{2}\int_0^t ||\eta_s||^2 ds} m(d\alpha)$$

4 Random risk aversion and decreasing consistent utilities

The interpretation of decreasing utilities in the last paragraph is the starting point for the rest of the paper where more general method to construct consistent utilities processes from a family of classical utilities functions is developed.
To illustrate this idea, the method proposed in this paper will be first developed in details in the context of power utilities that are present in this last result. This allows us to explain the different steps and different parameters that are involved in the construction in a simple case before describing the general method. On the other hand this allows us to recover from this procedure the decreasing consistent utilities of last Theorem, which will supports our approach.

4.1 Random Risk Aversion

To get started, we recall a result which gives sufficient conditions under which $\mathscr{X}^+$-Consistent stochastic utilities are obtained by combining a power utility function v with some positive process Z satisfying

$$\frac{dZ_t}{Z_t} = \mu_t^Z dt + \delta_t^Z . dW_t.$$

To alleviate the notations, for any random field $(M_t(z))_{t,z}$ the process $(M_t(1))_t$ (the process $(M_t)_t$ starting from 1 at $t=0$) is simply denoted by M_t. We also denote by $\mathcal{E}(\delta)$ the exponential local martingale defined by $\mathcal{E}_t(\delta) = exp\Big(-\frac{1}{2}\int_0^t ||\delta_s||^2 ds + \int_0^t \delta_s.dW_s\Big)$.
The following result is obtained from Proposition 5.1 recalled below stated for any utility function v.

Proposition 4.1. *Let u^α be a power utility with risk aversion α that is $u^\alpha(x) = \frac{x^{1-\alpha}}{1-\alpha}$. Assume that parameters of diffusion μ^Z, δ^Z of Z satisfy the following equation*

$$\mu_t^Z = -(1-\alpha)r_t - \frac{1-\alpha}{2\alpha}\|\eta_t + \delta_t^{Z,\sigma}\|^2. \tag{4.1}$$

Then the stochastic process U^α defined by $U^\alpha(t,x) = Z_t u^\alpha(x)$ is a $\mathscr{X}^+$-consistent utility with optimal policy

$$\kappa_t^{*,\alpha}(x) = \frac{1}{\alpha}\big(\eta_t + \delta_t^{Z,\sigma}\big)$$

In turn, the optimal wealth process $X^{\alpha,*}$ and the optimal dual process $Y^{\alpha,*}$ are given by,

$$\begin{cases} X_t^{*,\alpha}(x): & = xX_t^{*,\alpha} = xe^{\int_0^t \big(r_s + \frac{1}{\alpha}\big(\eta_s + \delta_s^Z\big).\eta_s\big)ds}\mathcal{E}_t\big(\frac{\eta+\delta^\sigma}{\alpha}\big) \\ Y_t^{\alpha,*}(y): & = U_x^\alpha(t, X_t^*(y^{-\frac{1}{\alpha}})) = yY_t^{\delta^\perp} = ye^{-\int_0^t r_s ds}\mathcal{E}_t(\delta^\perp - \eta) \end{cases} \tag{4.2}$$

Notice that the optimal wealth $X^{*,\alpha}$ and the optimal dual process $Y^{*,\alpha}$ are linear with respect to their initial conditions. This property will play an important role in the sequel. In particular, it provides an explicit formula for the corresponding inverse flows. Note also, that $Y^{*,\alpha}$ is independent on the risk aversion α.

Random risk aversion: At this stage the coefficient α, which is the relative risk aversion, was supposed constant, it is about the simplest case of the power $\mathscr{X}^+$-Consistent utilities. But it is completely conceivable that this risk aversion is in general random. Indeed we can imagine at date $t=0$ that the investor pulls at random the value of this coefficient. For every value α he associates:
(i) a weight $m(\alpha)$ (m is a finite positive measure s.t $\int_{\mathbb{R}_+^*} m(d\alpha) = 1$),
(ii) a proportion $x_\alpha(x)$ of its initial wealth (strictly increasing on x, $x_\alpha(x) \to \infty$ if $x \to \infty$ and $x_\alpha(0) = 0$) that he is going to invest on the financial market by considering the utility process U^α. He will so realize a wealth $X^{*,\alpha}(x_\alpha(x))$ associated with this edition and achieved by the optimal strategy $\kappa^{\alpha,*}$.
His final wealth is consequently the sum of the processes $X^{*,\alpha}(x_\alpha(x))$ weighted by the measure m, i.e., using the notation $X_t^{*,\alpha} := X_t^{*,\alpha}(1)$

$$\begin{cases} X_t^*(x) & = \int_{\mathbb{R}_+^*} X_t^{*,\alpha}(x_\alpha(x))m(d\alpha) = \int_{\mathbb{R}_+^*} x_\alpha(x)X_t^{*,\alpha}m(d\alpha), \\ X_0^*(x) & = x = \int_{\mathbb{R}_+^*} x_\alpha(x)m(d\alpha). \end{cases}$$

By monotony assumption of x_α, X^* is strictly increasing on x and satisfies

$$\frac{dX_t^*(x)}{X_t^*(x)} = r_t dt + \kappa_t^*(X_t^*(x)).\big(dW_t + \eta_t dt\big).$$

Where the volatility vector $\kappa_t^*(X_t^*(x))$ is given, using $\kappa_t^{*,\alpha}(x) = \frac{1}{\alpha}\big(\eta_t + \delta_t^{Z,\sigma}\big)$, by

$$\kappa_t^*(X_t^*(x)) := \Big[\int_{\mathbb{R}_+^*} \frac{X_t^{*,\alpha}(x_\alpha(x))}{X_t^*(x)} \frac{1}{\alpha} m(d\alpha)\Big]\big(\eta_t + \delta_t^{Z,\sigma}\big). \tag{4.3}$$

Denote by $\mathcal{X}$ the inverse flow of X^* and let u be an utility function such that $u_x(\mathcal{X}(x))$ is integrable near to zero (see [KM13, KM16] for explicit conditions on X^* and u such that this assumption is satisfied) and assume for the rest of this paragraph that

Assumption 4.1. *The process η and δ^Z are uniformly bounded.*

At first, this assumption implies that the process $X^* Y_t^{\delta^\perp}$ [1] is a martingale. Take $Y_t^*(y) = y Y_t^{\delta^\perp}$ ($Y_t^{\delta^\perp} := Y_t^{\delta^\perp}(1)$), it follows, according to Theorem 2.1 (Theorem 4.3 in [KM13]) that the process $U(t,x)$ defined by

$$U(t,x) = Y_t^{\delta^\perp} \int_0^x u_x(\mathcal{X}(t,z))dz, \tag{4.4}$$

is a consistent stochastic utility. The associated optimal portfolio and the optimal state density price are X^* and Y^*. Furthermore, the convex conjugate of U denoted by $\widetilde{U}$, is given by

$$\widetilde{U}(t,y) = \int_y^{+\infty} X^*(t, -\tilde{u}_y(\frac{z}{Y_t^{\delta^\perp}}))dz = \int_y^{+\infty} \int_{\mathbb{R}_+^*} x_\alpha(-\tilde{u}_y(\frac{z}{Y_t^{\delta^\perp}}))X_t^{*,\alpha} m(d\alpha)dz \tag{4.5}$$

with $\tilde{u}$ denote the convex conjugate of the given utility function u.
Thus we generate a family of non-standard consistent utilities from the optimal wealth processes $X^{*,\alpha}$ associated with a family of power consistent utilities. Notice that, this stochastic utilities depend, first on the family of functions x_α, $\alpha \in \mathbb{R}_+^*$, second on the choice of the measure m, third on the process Z (δ^Z) and finally on the initial condition u. This gives us a large family of consistent utilities.

Remark 4.1. *The initial utility function u is not necessarily a power one, the only restriction is that u must satisfies: $u_x(\mathcal{X}(x)))$ is integrable near to $x = 0$. In other words, the asymptotic behavior of u_x near to zero must be recompensed by that of $\mathcal{X}$.*

It is interesting to note that this stochastic utilities are built from simple initial utilities indexed by a parameter α by reasoning only in terms of the optimal wealth and optimal dual process by considering random editions of the parameter α.
To close this section let us show how the increasing consistent utilities (Theorem 3.2) are deduced from the class of utilities given by equations (4.4) and (4.5). For this let u in (4.5) be an utility function such that its inverse is given by

$$(u_x)^{-1}(x) = \int_{\mathbb{R}_+^*} x^{-\frac{1}{\alpha}} m(d\alpha)$$

[1] where we recall that $Y^{\delta^\perp}$ is the process defined by $\frac{dY_t^{\delta^\perp}}{Y_t^{\delta^\perp}} = -r_t dt + (\delta_t^\perp - \eta_t).dW_t$

and take $x_\alpha = [(u_x)^{-1}]^{-\frac{1}{\alpha}}$ it follows that $x_\alpha(u_x)(x) = x^{-\frac{1}{\alpha}}$. Take $r \equiv 0$ and the volatility of Z; $\delta \equiv 0$, it follows that $Y^{\delta^\perp}$ is Y^0 (the inverse of the market numeraire portfolio: $\nu = 0$ in (2.4)).

$$\widetilde{U}(t,y) = \int_{\mathbb{R}_+^*} \frac{1}{1-\frac{1}{\alpha}}\big(1 - y^{1-\frac{1}{\alpha}} X_t^{*,\alpha}(Y_t^0)^{\frac{1}{\alpha}}\big) m(d\alpha)$$

To conclude, remark that $X_t^{*,\alpha}(Y_t^0)^{\frac{1}{\alpha}} = \exp(-\frac{1-\alpha}{2\alpha}\int_0^t ||\eta_s||^2 ds)$ and finally

$$\widetilde{U}(t,y) = \int_{\mathbb{R}_+^*} \frac{1}{1-\frac{1}{\alpha}}\big(1 - y^{1-\frac{1}{\alpha}} e^{-\frac{1-\alpha}{2\alpha}\int_0^t ||\eta_s||^2 ds}\big) m(d\alpha).$$

Which is the convex conjugate of a decreasing consistent (forward) utilities in time (see Theorem 3.2).

4.2 Direct Characterization of Decreasing Stochastic Utilities from the Optimal Wealth

An interesting class of consistent utilities is the class of increasing consistent utilities which was studied and fully characterized in the literature by Berrier & al. [FB09] and Musiela & al. [MZ08b]. The volatility characteristic of these utilities is the null random field $\gamma \equiv 0$.

In contrast to previous sections, the utility volatility characteristics is given in place of the volatility coefficient. Since the optimal dual policy $\nu^* \equiv 0$ is equal to 0, the optimal density price is linear with respect to its initial condition and is given by$Y_t^*(y) = yY_t^0$. On the other hand, the optimal policy κ^* is given in terms of the risk tolerance $\frac{U_x}{U_{xx}}$ (the inverse of the absolute risk aversion) by

$$x\kappa^*(t,x) = -\frac{U_x(t,x)}{U_{xx}(t,x)}\eta_t \tag{4.6}$$

From now, the problem is so to characterize the utility process. On the beginning, observe that, under assumption $\gamma \equiv 0$, we have that, the utility process U and its dual convex $\widetilde{U}$ satisfy the following ODE

$$U_t(t,x) = -\frac{(U_x)^2(t,x)}{2U_{xx}(t,x)}||\eta_t||^2, \quad \widetilde{U}_t(t,y) = -\frac{1}{2}y^2\widetilde{U}_{yy}(t,y)||\eta_t||^2 \tag{4.7}$$

Clearly the dual equation is simpler to study than the primal. So we will invest the problem in terms of the dual convex. In addition, we will not work directly in terms of $\widetilde{U}$ but in terms of $\widetilde{U}_y(t, yY_t^0)$ because it has better properties. Indeed, by martingale property of $\widetilde{U}(t, yY_t^0)$, one can easily observe, using Itô-Ventzel's formula, that

$$d\widetilde{U}(t, yY_t^0) = -yY_t^0\widetilde{U}_y(t, yY_t^0)\eta_t dW_t \tag{4.8}$$

which becomes, noting by $\widetilde{V}(t,y) := \widetilde{U}(t, yY_t^0)$,

$$d\widetilde{V}(t,y) = -y\widetilde{V}_y(t,y)\eta_t dW_t \tag{4.9}$$

Assumption 4.2. *Assume the initial condition u is such that there exists a positive finite Borel measure μ supported on $\mathbb{R}$ such that $y \mapsto \tilde{u}(y)y^p$ is integrable with respect to μ for any $p \in \mathbb{R}$ and t.*

A simple example of such functions $\tilde{u}$ satisfying this integrability condition, is the class of $\tilde{u}$ bounded by a power function, that is there exists $p_0 \in \mathbb{R}$ such that $|\tilde{u}(y)| \leq cy^{-p_0},\ c, y > 0$ (considered in the financial framework by Karatzas & ali in [IS01]). In this case, one can easily sees that $y \mapsto \tilde{u}(y)y^p \leq y^{p-p_0}$ is integrable with respect to any measure ν supported on $]0,\infty[$ for any $p < p_0 - 1$.
Let now introduce the Mellin-Transform of the process $Z(t,y)$ which is an integral transform that may be regarded as the multiplicative version of the bilateral Laplace transform of $Z(t,e^y)$ and is defined by

$$M_t(p) := \int_0^{+\infty} y^{p-1}\widetilde{V}(t,y)\mu(dy)$$

which, by the martingale property of the process $\widetilde{U}(t, yY_t^0)$, is well defined, finite almost surely. Indeed

$$\mathbb{E}(M_t(p)) \leq \int_0^{+\infty} y^{p-1}\tilde{u}(y)\mu(dy) < +\infty.$$

Let us, now, focus on the dynamic of the process M, for this remark that the Mellin-Transform $\hat{M}$ of the process $y\widetilde{V}_y(t,y)$ is given, using integration by part and integrability conditions, by

$$\begin{aligned}\hat{M}_t(p) : &= \int_0^{+\infty} y^p\widetilde{V}_y(t,y)\mu(dy) = -p\int_0^{+\infty} y^p\widetilde{V}(t,y)\mu(dy)\\ &= -pM_t(p) + C(p)\end{aligned}$$

With $C(p) := \lim_{y\to 0} y^p\widetilde{V}(t,y)$ and it is null except for at most a single point p_0. So without loss of generality, we assume $C(p) = 0$ everywhere. This implies that the process $M_t(p)$ satisfies the following dynamic

$$dM_t(p) = pM_t(p)\eta_t.dW_t$$

which have a unique solution given by

$$M_t(p) = M_0(p)\mathcal{E}(p\int_0^t \eta_s.dW_s)$$

Where we recall that $\mathcal{E}\Big(p\int_0^t \eta_s.dW_s\Big) = exp\Big(p\int_0^t \eta_s.dW_s - \frac{p^2}{2}\int_0^t \|\eta_s\|^2 ds\Big)$. In particular, introducing the process Y^0 in the formula of $M_t(p)$, leads by easy computations to

$$M_t(p) = M_0(p)(Y_t^0)^{-p}e^{-p\frac{p+1}{2}\int_0^t \|\eta_s\|^2 ds} \tag{4.10}$$

To comeback to $\widetilde{V}(t,y)=\widetilde{U}(t,yY_t^0)$, we apply the Mellin inverse Transform, there exists a constant C_1 such that

$$\widetilde{U}(t,yY_t^0)=\int_{-\infty}^{+\infty} y^{-p}M_t(p)\mu(dp)+C_1 \tag{4.11}$$

which becomes, using (4.10),

$$\widetilde{U}(t,yY_t^0)=\int_{-\infty}^{+\infty}(yY_t^0)^{-p}e^{-p\frac{p+1}{2}\int_0^t\|\eta_s\|^2ds}M_0(p)\mu(dp)+C_1 \tag{4.12}$$

and finally, by change of variable $yY_t^0\mapsto y$, we conclude that

$$\widetilde{U}(t,y)=\int_{-\infty}^{+\infty} y^{-p}e^{-\frac{p(1+p)}{2}\int_0^t\|\eta_s\|^2ds}M_0(p)\mu(dp)+C_1$$

On the other hand, one can easily find that the condition $\widetilde{U}_t<0$ is true if and only if p is chosen s.t. $p>-1$ which leads to

$$\widetilde{U}(t,y)=\int_{-1}^{+\infty} y^{-p}e^{-\frac{p(1+p)}{2}\int_0^t\|\eta_s\|^2ds}M_0(p)\mu(dp)+C_1$$

by change of variable $q=p+1$

$$\widetilde{U}(t,y)=-\int_{0}^{+\infty}\frac{1}{1-q}y^{1-q}e^{-\frac{q(1-q)}{2}\int_0^t\|\eta_s\|^2ds}(1-q)M_0(q-1)\mu(dq)+C_1$$

by denoting ν the finite Borel measure supported on $[0,\infty[$ defined by $\nu(dp):=M_0(p)\mu(dp)/p$, we get

$$\widetilde{U}(t,y)=\int_{0}^{+\infty}\frac{1}{1-q}\big(1-y^{1-q}e^{-\frac{q(1-q)}{2}\int_0^t\|\eta_s\|^2ds}\big)\nu(dq)+C$$

For some constant C. In particular, we have that the initial data $\tilde{u}$ is necessarily of the form

$$\tilde{u}(y)=\int_{0}^{+\infty}\frac{1}{1-q}(1-y^{1-q})\nu(dq)+C \tag{4.13}$$

Which is other than the characterization of decreasing consistent utility showed by Zariphopoulou & ali and Tehranchi & ali using the Widder's Theorem characterizing space time harmonic functions.
Note in passing that the optimal wealth process X^* is given by the closed formula

$$X^*(t,x)=\int_{0}^{\infty}(u_x(x)Y_t^0)^{-p}e^{\frac{-p(1-p)}{2}\int_0^t\|\eta_s\|^2ds}\nu(dp) \tag{4.14}$$

and it is a strictly increasing with respect to it's initial condition x. Moreover, as the optimal dual process $Y^*(y)=yY^0$ which implies $\mathcal{Y}(y):=(Y^*)^{-1}(y)=y/Y^0$, one can easily verifies that the random field $\widetilde{U}_y$ is obtained by the general form of Theorem 2.1, that is $\widetilde{U}_y(t,y):=-X^*\Big(t,-\tilde{u}_y\big(\mathcal{Y}(t,y)\big)\Big)$.

Of course, this result was already established in [FB09] and & al. [MZ08b] but the novelty here, that merits to be integrated into this work, is that we propose a very simple and direct proof of the result, without using the time change techniques and Widder's results. But we must note that in their paper [MZ08b], M. Musiela and T. Zariphopoulou developed several examples with different measures m as well as properties of the associated optimal wealth, something we do not develop here.

More Examples Another point that needs to be clarified once and for all is the following: In a first reading one might think that the necessary condition (4.13) on the initial function u and the dependence (4.14) of the optimal wealth on this function is in contradiction with the rest of the paper where we emphasize that the choice of u does not play a very important role in our main results. Contrary to what one might think, this result is entirely consistent with previous paragraphs and even a nice example that supports the efficiency of our results. Indeed, starting from the optimal portfolio X^* equation (4.14), it is easily to construct a new consistent utility having X^* as optimal starting from an initial data v and generating a monotone optimal dual process $Y^{\nu^*}(y)$ (not necessarily equal to yY^0) with inverse $\mathcal{Y}$ as follows:

$$\begin{aligned}\widetilde{V}_y(t,y) &= -X^*\big(t,-\tilde{v}_y(\mathcal{Y}(t,y))\big)\\ &= -\int_0^\infty \big(u_x\big(-\tilde{v}_y(\mathcal{Y}(t,y))\big)Y_t^0\big)^{-p} e^{\frac{-p(1-p)}{2}\int_0^t \|\eta_s\|^2 ds}\nu(dp)\end{aligned}$$

Clearly this process is strictly convex non-zero volatility. Moreover $\widetilde{V}$ is the Fenchel-Transform of a consistent utility V. To be convinced, just apply the results of Theorem 2.7, equation (2.9) of [KM13], to the compound process $X^*\big(t,-\tilde{v}_y(\mathcal{Y}(t,y))\big)$, using the fact that $\mathcal{Y}$ is a solution of a SPDE since it is the inverse process of Y^* solution of a SDE. Integrate the dynamics in y and compare it to the dynamics (3.6) Theorem 3.2 of the dual convex of a consistent utility in [KM13].

5 General Construction

In the construction proposed in the previous section, the volatility δ^Z of Z is independent of α, while the investor can decide to assign to each function u^α a process Z^α whose volatility δ^α depends on the risk aversion α. Moreover, for simplicity we have chose the process $Y^*(y)$ proportional to $Y^{\delta^\perp}(1)$, i.e. $Y^*(y) = yY^{\delta^\perp}(1)$, so it is possible to construct by the same reasoning for X^* a process Y^* using a family of functions $y_\alpha \;\; \alpha \in \mathbb{R}_+^*$. All these observations combined with the possibility of taking $X^{*,\alpha}$ not necessarily positive (with a little more integrability conditions), open the way for a natural generalization to the construction proposed above even in the context of utility functions u^α that are not of a power type. This will be the target of this section where a generalization of the

method is proposed and a more general class of consistent utilities is built. The power and exponential utilities in what follows are also a concrete examples to illustrate the proposed method.
In order to pursue our investigations, we begin by recalling how it is possible to generate stochastic utilities from any family of utility functions u^α indexed, for example, by the parameter of risk aversion α others those of power type. The idea is the same as that of Proposition 4.1. Let u^α an utility function not necessary of power type and let N_t^α and Z_t^α two positive processes satisfying

$$\frac{dN_t^\alpha}{N_t^\alpha} = \mu_t^{N,\alpha} dt + \delta_t^{\alpha,N}.dW_t, \quad \frac{dZ_t^\alpha}{Z_t^\alpha} = \mu_t^{\alpha,Z} dt + \delta_t^{\alpha,Z}.dW_t, \quad Z_0 = 1.$$

Note that Z^α and N^α depend on the choice of the parameter α. Next result, showed at first in the PhD Thesis of Mrad M. [Mra09], generalizes Proposition 4.1. In particular, it gives a sufficient conditions on the triplet $(u^\alpha, N^\alpha, Z^\alpha)$ under which the process $U^\alpha(t,x) := Z_t^\alpha u^\alpha(x/N_t^\alpha)$ is a consistent utility.

Proposition 5.1. *Let u^α be an utility function.*

(i) Assume that N is an admissible positive wealth process, i.e.$(\delta^{N,\alpha} \in \mathcal{R}, \mu^{N,\alpha} = r + \eta.\delta^{N,\alpha})$ and Z is a martingale such that $ZX^\kappa/N, \kappa \in \mathcal{R}$ are local martingales. Then the process U^α defined by $U^\alpha(t,x) = Z_t u^\alpha(x/N_t)$ is a consistent stochastic utility with optimal wealth process $X^{\alpha,*} = N$.

(ii) If u^α is a power or exponential utility, then condition : "Z is martingale, ZX^κ/N is a local martingale for any $\kappa \in \mathcal{R}$ " can be relaxed.

- ✓ *In the case where u^α is a power utility the result is given by Proposition 4.1.*
- ✓ *If u^α is an exponential utility that is $u^\alpha(x) = \frac{1}{\alpha}e^{-\alpha x}$ it suffices to take Z and N satisfying $\mu^{\alpha,N} = r + \sigma^{\alpha,N}.\eta$, $\mu^{\alpha,Z} = \frac{1}{2}\|\eta - \delta^{\alpha,N} + \delta^{\alpha,Z,\sigma}\|^2$, $\delta^{\alpha,N} \in \mathcal{R}$ where the optimal policy $\kappa^{\alpha,*}$ is given by $x\kappa_t^{\alpha,*}(x) = x\delta_t^{\alpha,N} + \frac{N_t^\alpha}{\alpha}\big(\eta_t - \delta_t^{\alpha,N} + \delta^{\alpha,Z,\sigma}\big)$. Moreover, $x \mapsto x\kappa_t^{\alpha,*}(x)$ is globally Lipischitz function.*

In all cases $X^{*,\alpha}(x)$ is strictly increasing in x.

This result, whose the proof can be found in the last chapter of [Mra09], gives sufficient conditions under which U, defined above, is an $\mathscr{X}$-consistent stochastic utility. Note also that this example generalizes the one in [MZ08a] in which case u is an exponential utility and provides a similar sufficient conditions.
As in previous Section, the optimal wealth process associated to the stochastic utility U^α is denoted by $X^{*,\alpha} := X^{\kappa^{\alpha,*},\alpha}$ and satisfies

$$\frac{dX_t^{*,\alpha}}{X_t^{*,\alpha}} = r_t dt + \kappa_t^{\alpha,*}(X_t^{*,\alpha}).\big(dW_t + \eta_t dt\big)$$

Also, $Y^{*,\alpha}$ denote the optimal state density process given by $Y_t^{*,\alpha}(y) := U_x^\alpha\Big(t, X_t^{*,\alpha}((u_x^\alpha(x))^{-1}(y))\Big)$ which is attained by $\nu^{*,\alpha}$, that is

$$\frac{dY_t^{*,\alpha}(y)}{Y_t^{*,\alpha}(y)} = -r_t dt + \big(\nu_t^{\alpha,*}(Y_t^{*,\alpha}(y)) - \eta_t\big)dW_t, \quad \nu_t^{\alpha,*}(Y_t^{*,\alpha}(y)) \in \mathcal{R}_t^\perp,\ Y_0^{*,\alpha}(y) = y.$$

Random risk aversion : The idea developed in the previous section is generalized as follows.

At date $t = 0$ the investor pulls at random the value of the risk aversion coefficient. For every value α he associates:

(i) a weight $m(\alpha)$ (m is a finite positive measure s.t $\int_{\mathbb{R}_+^*} m(d\alpha) = 1$),

(ii) a proportion $x_\alpha(x)$ of its initial wealth (positive strictly increasing on x) that he is going to invest on the financial market by considering u^α as utility, he will so realize $X^{*,\alpha}(x_\alpha(x))$ as wealth (associated with this edition) achieved by the optimal policy $\kappa^{\alpha,*}$.

(iii) a function $y_\alpha(.), \alpha \in \mathbb{R}_+^*$ (strictly increasing y, $y_\alpha(y) \to \infty$ if $y \to \infty$ and null for $y = 0$).

His final (global) wealth is consequently the sum of the processes $X^{*,\alpha}(x_\alpha(x))$ weighted by the measure m, i.e.

$$X_t^*(x) = \int_{\mathbb{R}_+^*} X_t^{*,\alpha}(x_\alpha(x))m(d\alpha), \quad X_0^*(x) = x = \int_{\mathbb{R}_+^*} x_\alpha(x)m(d\alpha)$$

By monotonicity of x_α and that of $X^{*,\alpha}$, X^* is strictly increasing on x and satisfies

$$dX_t^*(x) = r_t X_t^*(x)dt + X_t^*(x)\kappa_t^*(X_t^*(x)).\big(dW_t + \eta_t dt\big)$$

where the volatility vector $\kappa_t^*(X_t^*(x))$ is given by

$$X_t^*(x)\kappa_t^*(X_t^*(x)) := \int_{\mathbb{R}_+^*} X_t^{*,\alpha}(x_\alpha(x))\kappa_t^{\alpha,*}(X_t^{*,\alpha}(x_\alpha(x)))m(d\alpha) \tag{5.1}$$

By analogy, we consider the state density price Y^* defined as the sum of the processes $Y^{\alpha,*}$ weighted by the measure m, i.e.

$$Y_t^*(y) = \int_{\mathbb{R}_+^*} Y_t^{*,\alpha}(y_\alpha(y))m(d\alpha), \quad Y_0^*(y) = y \overset{def}{=} \int_{\mathbb{R}_+^*} u_x^\alpha(y_\alpha(y))m(d\alpha) \tag{5.2}$$

Consequently, the increasing process $Y_t^*(x)$ solves

$$\frac{dY_t^*(y)}{Y_t^*(y)} = -r_t dt + \Big(\nu_t^*(Y_t^*(y)) - \eta_t\Big)dW_t.$$

with

$$\nu_t^*(Y_t^*(y)) := \int_{\mathbb{R}_+^*} \frac{Y_t^{\alpha,*}(x_\alpha(y))}{\int_{\mathbb{R}_+^*} Y_t^{\alpha,*}(y_\alpha(y))m(d\alpha)}\nu_t^{\alpha,*}(Y_t^{\alpha,*})m(d\alpha) \tag{5.3}$$

By definition, the pair of processes $(X_t^*(x), Y_t^*(y))$ is increasing with respect to its initial condition (x, y) and such that $X^*(x)Y^*(y)$ is a local martingale. Assume κ^* and ν^* to be bounded then X^*Y^* is a martingale. Furthermore, let $\mathcal{X}(t,z) = (X^*(t,.))^{-1}$ the inverse flow of X^* and u an utility function such that $x \mapsto Y^*(t, u_x(\mathcal{X}(t,z)))$ is integrable near to zero (see [KM13] for explicit conditions on Y^*, X^* and u such that this assumption is satisfied see also [KM16] for a more general framework). Then, according to Theorem 2.1, the process U defined by

$$U(t,x) = \int_0^x Y^*(t, u_x(\mathcal{X}(t,z)))dz$$

is a $\mathscr{X}$-Consistent stochastic utility. The associated optimal portfolio is X^* and the optimal state density price is Y^*, in particular $Y_t^*(u_x(x)) = U_x(t, X_t^*(x))$.
In the following paragraphs an explicit illustration of this method, based on utilities functions of power and exponential type, is given. Main tools are the results of Proposition 5.1.

5.1 Example 1: Consistent Utilities From Optimal processes Associated with Power Utilities Functions

In this paragraph, we are interested by applying the previous construction to the case where utilities u^α are of power type. For this, we consider a family $\{Z^\alpha; \alpha > 0\}$ such that Z^α satisfies, for each α, the following dynamics

$$\frac{dZ_t^\alpha}{Z_t^\alpha} = -\big((1-\alpha)r_t + \frac{1-\alpha}{2\alpha}\|\eta_t + \delta_t^{\alpha,\sigma}\|^2\big)dt + \delta_t^\alpha.dW_t, \quad Z_0 = 1.$$

To ensure that X^*Y^* is a martingale we make the following assumption

Assumption 5.1. *The minimal risk prime η is bounded and the family of the volatility vector processes δ^α are uniformly bounded.*

According to Proposition 4.1 the process $U^\alpha(t,x) = Z_t^\alpha \frac{x^{1-\alpha}}{1-\alpha}$, is a $\mathscr{X}^+$-Consistent dynamic utility such that the optimal policy κ^* is given by $\kappa_t^*(x) = \frac{1}{\alpha}\big(\eta_t + \delta_t^{\alpha,\sigma}\big)$. By Proposition 4.1, the optimal wealth process $X^{\alpha,*}$ and the optimal dual process $Y^{\alpha,*}$ associated with the power utility U^α are given by,

$$\begin{cases} X_t^{*,\alpha}(x) &= xX_t^{*,\alpha} = xe^{\int_0^t \big(r_s + \frac{1}{\alpha}\big(\eta_s + \delta_s^Z\big).\eta_s\big)ds}\mathcal{E}_t(\frac{\eta+\delta^{\alpha,\sigma}}{\alpha}) \\ Y_t^{\alpha,*}(y) &= yY_t^{\alpha,*} = ye^{-\int_0^t r_s ds}\mathcal{E}_t(\delta^{\alpha,\perp} - \eta). \end{cases} \tag{5.4}$$

From this and the previous construction, we are concerned with the following processes X^* and Y^* given by

$$\begin{cases} X_t^*(x) &= \int_{\mathbb{R}_+^*} x_\alpha(x) X_t^{*,\alpha} m(d\alpha), \quad X_0^*(x) = x = \int_{\mathbb{R}_+^*} x_\alpha(x)m(d\alpha) \\ Y_t^*(y) &= \int_{\mathbb{R}_+^*} y_\alpha(y) Y_t^{\alpha,*} m(d\alpha), \quad Y_0^*(y) = y = \int_{\mathbb{R}_+^*} y_\alpha(x)m(d\alpha) \end{cases} \tag{5.5}$$

By assumptions, X^* is a wealth process and Y^* is a state density process which are strictly increasing from 0 to ∞, we denote respectively $\mathcal{X}$ and $\mathcal{Y}$ their inverse with respect to their initial conditions. Consequently, for any utility function u such that $x \mapsto Y^*(t, u_x(\mathcal{X}(t,z)))$ is integrable near to zero the process U defined by

$$U(t,x) = \int_0^x Y^*(t, u_x(\mathcal{X}(t,z)))dz$$

is a consistent utility.

Although this class of utilities processes is simply generated from optimal processes associated with a power utilities (the simplest utilities we can consider) nevertheless it is a richer class. To be convinced it suffices to fix one of the parameters: x_α, y_α or the initial condition u.

5.1.1 The Case $x_\alpha(x) = xg(\alpha)$:

This choice implies that the wealth process X^* is linear with respect to its initial value x and is given by

$$X_t^*(x) = x \int_{\mathbb{R}_+^*} g(\alpha) X_t^{\alpha,*} m(d\alpha)$$

and $X_t^* := X_t^*(1) = \int_{\mathbb{R}_+^*} g(\alpha) X_t^{\alpha,*} m(d\alpha)$. In particular, we have the explicit formula for the inverse flow $\mathcal{X}$ of X^*, i.e., $\mathcal{X}_t(x) = x/X_t^*$. Composing the stochastic flows Y^* and $\mathcal{X}$, the derivative U_x of the stochastic utility constructed above satisfies

$$U_x(t,x) = Y_t^*(u_x(\mathcal{X}_t(x))) = \int_{\mathbb{R}_+^*} y_\alpha \circ \bar{u}_x\Big(\frac{x}{X_t^*}\Big) Y_t^{\alpha,*} m(d\alpha)$$

Integrating yields

$$U(t,x) = \int_{\mathbb{R}_+^*} \Big(\int_0^x y_\alpha \circ u_x\Big(\frac{z}{X_t^*}\Big)dz\Big) Y_t^{\alpha,*} m(d\alpha) \tag{5.6}$$

If, moreover, we take $y_\alpha(y) = yf(\alpha)$, then the utility processes U rewrites, after integration with respect to z,

$$U(t,x) = X_t^* \int_{\mathbb{R}_+^*} f(\alpha)\bar{u}\Big(\frac{x}{X_t^*}\Big) Y_t^{\alpha,*} m(d\alpha) = X_t^* Y_t^* u(\frac{x}{X_t^*}) \tag{5.7}$$

thus the utility process U is simply the transformation of the utility function u to a consistent one using the techniques of change of numeraire and probability: the numeraire N is the optimal portfolio X^* and the change of probability Z is the martingale Y^*X^*, which is in a perfect concordance with results of Proposition 5.1.

Particular form of the initial utility function u: Let $\{v^\alpha, \alpha > 0\}$ be a family of utilities functions (not necessarily of power type) and define the utility function v by

$$v_x(x) := \int_{\mathbb{R}_+^*} v_x^\alpha(x) m(d\alpha).$$

By definition v_x is strictly decreasing with inverse $(v_x)^{-1}$, take $y_\alpha(y) := v_x^\alpha((v_x)^{-1}(y))$ and observe that

$$y = \int_{\mathbb{R}_+^*} v_x^\alpha((v_x)^{-1}(y))m(d\alpha) = \int_{\mathbb{R}_+^*} y_\alpha(y)m(d\alpha).$$

Requirements of our construction being respected, the utility U is given by

$$U(t,x) = \int_{\mathbb{R}_+^*}\int_0^x v_x^\alpha\Big(\frac{z}{X_t^*}\Big)dzY_t^{\alpha,*}m(d\alpha)$$

Integrating, yields

$$U(t,x) = X_t^*\int_{\mathbb{R}_+^*} v^\alpha\Big(\frac{x}{X_t^*}\Big)Y_t^{\alpha,*}m(d\alpha) \tag{5.8}$$

which can be interpreted as the sum of consistent utilities $X_t^*Y_t^{\alpha,*}v^\alpha\Big(\frac{x}{X_t^*}\Big)$ which are the transformation of the utilities v^α by the same change of numeraire X^* and a different probability processes $X_t^*Y_t^{\alpha,*}$.

5.1.2 The Case $y_\alpha(y) = f(\alpha)y$:

In this case it is more convenient to calculate the dual convex of the utility U because the inverse of the state price density process Y^* is simply given by $(Y_t^*)^{-1}(y) = \frac{y}{Y_t^*}$ where we recall, $Y_t^* := Y_t^*(1) = \int_{\mathbb{R}_+^*} f(\alpha)Y_t^{\alpha,*}m(d\alpha)$. The dual convex conjugate $\widetilde{U}$ of U becomes

$$\begin{aligned}\widetilde{U}(t,y) &= \int_y^{+\infty} X_t^*\big((Y_t^*)^{-1}((u_x)^{-1}(z))\big)dz\\ &= \int_{\mathbb{R}_+^*}\Big[\int_y^{+\infty} x_\alpha\circ(u_x)^{-1}(\frac{z}{Y_t^*})dz\Big]X_t^{*,\alpha}m(d\alpha)\end{aligned} \tag{5.9}$$

It is important to note the symmetry between this equation and equation (5.6). In particular taking $x_\alpha = g(\alpha)x$ one get the dual conjugate of U given by (5.7) and $f(\alpha) = 1$, the dual conjugate of U given by (5.8).

Particular form of the initial utility function u: Let $\{v^\alpha, \alpha > 0\}$ be a family of utilities functions (not necessarily of power type) and define the utility function v via its conjugate by

$$(v_x)^{-1}(y) := \int_{\mathbb{R}_+^*}(v_x^\alpha)^{-1}(y)m(d\alpha).$$

By definition $(v_x)^{-1}$ is strictly increasing with inverse v_x, take $x_\alpha(x) := (v_x^\alpha)^{-1}(v_x(x))$ and observe that

$$x = \int_{\mathbb{R}_+^*}(v_x^\alpha)^{-1}(v_x(y))m(d\alpha) = \int_{\mathbb{R}_+^*} x_\alpha(x)m(d\alpha).$$

Requirements of our construction being respected, the convex conjugate $\widetilde{U}$ of U is given, taking $u = v$ in (5.9), by

$$\widetilde{U}(t,y) = \int_{\mathbb{R}_+^*}\Big[\int_y^{+\infty}(v_x^\alpha)^{-1}(\frac{z}{Y_t^*})dz\Big]X_t^{*,\alpha}m(d\alpha)$$

Denoting by $\tilde v$ the convex conjugate of v, integrating yields

$$\widetilde{U}(t,y) \quad = \quad Y_t^* \int_{\mathbb{R}_+^*} \tilde v^\alpha(\frac{z}{Y_t^*}) X_t^{*,\alpha} m(d\alpha) \tag{5.10}$$

which is, by analogy to (5.8), interpreted as the sum of the convex conjugate $Y_t^* X_t^{*,\alpha} \tilde v^\alpha(\frac{z}{Y_t^*})$ of consistent utilities $X_t^* Y_t^{\alpha,*} v^\alpha\Big(\frac{x}{X_t^*}\Big)$ in (5.8), which are the transformation of the utilities v^α by the same change of numeraire X^* and a different probability processes $X_t^* Y_t^{\alpha,*}$.

We can content ourselves with this simple example of our construction, but there is a sub-case of (5.10) corresponding to the case of decreasing consistent utilities which has been studied in the literature by Berrier & al and Musiela &al. This utilities can be obtain by a particular choice of functions x_α and the initial data u.

Case of decreasing X-Consistent utilities To get started let $\bar u$ be the function defined by

$$(\bar u_x)^{-1}(x) = \int_{\mathbb{R}_+^*} x^{-\frac{1}{\alpha}} m(d\alpha)$$

and take $x_\alpha = [(\bar u_x)^{-1}]^{-\frac{1}{\alpha}}$ it follows that $x_\alpha(\bar u_x)(x) = x^{-\frac{1}{\alpha}}$. Hence, always in the case where $y_\alpha(y) = yf(\alpha)$

$$\widetilde{U}(t,y) = Y_t^* \int_{\mathbb{R}_+^*} \Big[(1 - \frac{1}{1-\frac{1}{\alpha}} y^{1-\frac{1}{\alpha}} (\frac{1}{Y_t^*})^{-\frac{1}{\alpha}} dz\Big] X_t^{*,\alpha} m(d\alpha)$$

where we recall that $Y_t^* = \int_{\mathbb{R}_+^*} f(\alpha) Y_t^{\alpha,*} m(d\alpha)$.
From now taking $r \equiv 0$, $\delta^\alpha \equiv 0$, it follows that $Y^*(1)$ is Y^0 (the inverse of the market numeraire portfolio: $\nu = 0$ in (2.4)).

$$\widetilde{U}(t,y) = \int_{\mathbb{R}_+^*} \frac{1}{1-\frac{1}{\alpha}} \big(1 - y^{1-\frac{1}{\alpha}} X_t^{*,\alpha}(1)(Y_t^0)^{\frac{1}{\alpha}}\big) m(d\alpha)$$

To conclude, remark that $X_t^{*,\alpha}(1)(Y_t^0)^{\frac{1}{\alpha}} = exp(-\frac{1-\alpha}{2\alpha}\int_0^t ||\eta_s||^2 ds)$ and finally, one easily obtain the convex conjugate of decreasing consistent utilities.

$$\widetilde{U}(t,y) = \int_{\mathbb{R}_+^*} \frac{1}{1-\frac{1}{\alpha}} \big(1 - y^{1-\frac{1}{\alpha}} e^{-\frac{1-\alpha}{2\alpha}\int_0^t ||\eta_s||^2 ds}\big) m(d\alpha).$$

5.2 Example 2: Consistent Utilities From Optimal processes Associated with Exponential Utilities Functions

In this section u^α is an exponential utility with risk aversion α that is $u^\alpha(x) = 1 - \frac{1}{\alpha} e^{-\alpha x}$ According to Proposition 5.1, the numeraire N^α and the process Z^α are solutions of the following dynamics

$$\begin{cases} \dfrac{dN_t^\alpha}{N_t^\alpha} = \big(r_t + \delta_t^{\alpha,N}.\eta_t\big)dt + \delta_t^{\alpha,N}.dW_t, \quad \delta_t^{\alpha,N} \in \mathcal{R}_t, \quad t \geq 0, \quad N_0 = 1 \\ \dfrac{dZ_t^\alpha}{Z_t^\alpha} = \dfrac{1}{2} \|\eta_t - \delta_t^{N,\sigma} + \delta_t^{\alpha,Z,\sigma}\|^2 dt + \delta_t^{\alpha,Z}.dW_t, \quad t \geq 0, \quad Z_0 = 1 \end{cases}$$

To ensure that $X^* Y^*$ is a martingale we make the following assumption

Assumption 5.2. *The minimal risk prime η is bounded and the volatility vectors $\delta^{\alpha,N}$, $\delta^{\alpha,Z}, \alpha \in \mathbb{R}_+^*$ are uniformly bounded.*

The optimal policy $x\kappa^*$, according to Proposition 5.1 is given by

$$x\kappa_t^{\alpha,*}(x) = x\delta_t^{\alpha,N} + \frac{N_t^\alpha}{\alpha}\big(\eta_t - \delta_t^{\alpha,N} + \delta^{\alpha,Z,\sigma}\big).$$

In turn, the optimal portfolio $X^{*,\alpha}$ satisfies

$$dX_t^{*,\alpha}(x) = r_t X_t^{*,\alpha}(x)dt + \Big(X_t^{*,\alpha}(x)\delta_t^{\alpha,N} + \frac{N_t^\alpha}{\alpha}\big(\eta_t - \delta_t^{\alpha,N} + \delta^{\alpha,Z,\sigma}\big)\Big).\Big(dW_t + \eta_t dt\Big)$$

Applying Itô formula to the process $\frac{X_t^*(x)}{N_t^\alpha}$, simple calculations lead to

$$d\frac{X_t^{*,\alpha}(x)}{N_t^\alpha} = \frac{1}{\alpha}\Big(\eta_t - \delta_t^{\alpha,N} + \delta_t^{\alpha,Z,\sigma}\Big).\Big(dW_t + \big(\eta_t - \delta_t^{\alpha,N}\big)dt\Big)$$

Which is equivalent to

$$X_t^{\alpha,*}(x_\alpha(x)) = N_t^\alpha\Big[x_\alpha(x) + \frac{1}{\alpha}\int_0^t \Big(\eta_s - \delta_s^{\alpha,N} + \delta_s^{\alpha,Z,\sigma}\Big).\Big(dW_s + \big(\eta_s - \delta_t^{\alpha,N}\big)ds\Big)\Big]$$

Consequently the global wealth process is written

$$\begin{aligned} X_t^*(x) &= \int_{\mathbb{R}_+^*} X_t^{*,\alpha}(x_\alpha(x))m(d\alpha) \\ &= \Big[\int_{\mathbb{R}_+^*} x_\alpha(x)N_t^\alpha m(d\alpha) + \int_{\mathbb{R}_+^*} N_t^\alpha\Big(\frac{1}{\alpha}\int_0^t \Big(\eta_s - \delta_s^{\alpha,N} + \delta_s^{\alpha,Z,\sigma}\Big).\Big(dW_s + \big(\eta_s - \delta_s^{\alpha,N}\big)ds\Big)m(d\alpha)\Big] \end{aligned}$$

To achieve the construction in this exponential framework, after X^* we shall give an explicit form to Y^*. For this, we begin by calculating $Y^{\alpha,*}$ before integrating with respect to α and the measure m. From previous equations, it follows

$$e^{-\alpha\frac{X_t^{*,\alpha}(x)}{N_t^\alpha}} = e^{-\alpha x - \int_0^t \left(\eta_s - \delta_s^{\alpha,N} + \delta_s^{\alpha,Z,\sigma}\right).\left(dW_s + \left(\eta_s - \delta_s^{\alpha,N}\right)ds\right)}$$

Multiplying by Z_t^α, one can easily obtain $Z_t^\alpha e^{-\alpha\frac{X_t^{*,\alpha}(x)}{N_t^\alpha}} = e^{-\alpha x}\mathcal{E}_t(\delta^{Z,\perp} - \eta)$. Which implies

$$Y_t^{\alpha,*}(y) := U_x^\alpha(t, X_t^{*,\alpha}((u_x^\alpha)^{-1}(y))) = \frac{Z_t^\alpha}{N_t^\alpha}e^{-\alpha\frac{X_t^{*,\alpha}((u_x^\alpha)^{-1}(y))}{N_t}} = yY_t^{\delta^{\alpha,Z,\perp}}$$

where $Y^{\delta^{\alpha,Z,\perp}}$ denote the state price density process given by (2.4) for $\nu = \delta^{\alpha,Z,\perp}$ and with initial value equal to 1. Integrating with respect to α, the process $Y^*(y)$ is given by

$$Y_t^*(x) = \int_{\mathbb{R}_+^*} y_\alpha(y)Y_t^{\delta^{\alpha,Z,\perp}} m(d\alpha)$$

Case $N^\alpha = N$: In this case, it is immediate, using $\delta^{\alpha,N} = \delta^N$, that the global wealth process X^* is given by

$$X^*_t(x) = N_t\Big[x + \int_{\mathbb{R}^*_+} \frac{1}{\alpha}\int_0^t \Big(\eta_s - \delta^N_s + \delta^{\alpha,Z,\sigma}_s\Big).\Big(dW_s + \big(\eta_s - \delta^N_s\big)ds\Big)m(d\alpha)\Big] \qquad (5.11)$$

Where in the last line we have used the identity $X^*_0(x) = x = \int_{\mathbb{R}^*_+} x_\alpha(x)m(d\alpha)$ (see equation (4.3)). Hence X^* is strictly increasing with respect to its initial capital with inverse flow $\mathcal{X}$ given by,

$$\mathcal{X}_t(x) = \frac{x}{N_t} - M^\alpha_t \qquad (5.12)$$

$$M^\alpha_t = \int_{\mathbb{R}^*_+} \frac{1}{\alpha}\int_0^t \Big(\eta_s - \delta^N_s + \delta^{\alpha,Z,\sigma}_s\Big).\Big(dW_s + \big(\eta_s - \delta^N_s\big)ds\Big)m(d\alpha).$$

Let u be an utility function $u : \mathbb{R} \mapsto \mathbb{R}$ with good integrability conditions. Composing the stochastic flows Y^*, u_x and $\mathcal{X}$, the derivative U_x of the stochastic utility constructed satisfies

$$U_x(t,x) = Y_t\big(u_x(\mathcal{X}_t(x))\big) = \int_{\mathbb{R}^*_+} y_\alpha\Big(u_x\big(\frac{x}{N_t} - M^\alpha_t\big)\Big)Y^{\delta^{\alpha,Z,\perp}}_t m(d\alpha)$$

Integrating yields

$$U(t,x) = \int_{\mathbb{R}^*_+}\Big[\int_0^x y_\alpha\Big(u_x\big(\frac{z}{N_t} - M^\alpha_t\big)\Big)dz\Big]Y^{\delta^{\alpha,Z,\perp}}_t m(d\alpha)$$

Note that in this case of exponential utilities u^α when $N^\alpha = N$ the functions x_α, contrary to the power case, do not play any role.

✓ Case where $y_\alpha(y) = y$: the last identity becomes

$$U(t,x) = N_t\int_{\mathbb{R}^*_+} u\Big(\frac{z}{N_t} - M^\alpha_t\Big)Y^{\delta^{\alpha,Z,\perp}}_t m(d\alpha)$$

✓ Case where $y_\alpha(y) = e^{-\alpha(u_x)^{-1}(y)}$, we get

$$U(t,x) = N_t\int_{\mathbb{R}^*_+}\Big(1 - \frac{1}{\alpha}e^{-\alpha\frac{x}{N_t} + \alpha M^\alpha_t}\Big)Y^{\delta^{\alpha,Z,\perp}}_t m(d\alpha)$$

5.3 Example 3: Consistent Utilities From Optimal processes Associated with Different Types of Utilities Functions

In the examples above consistent utilities processes are constructed from a family of utility functions u^α either power type or exponential type. The most natural question one might ask is: This construction, is it valid if the utilities functions are of different types, such as a mixture of power and exponential utilities? The answer to this question, under uniform integrability assumptions of the parameters of diffusion of Z^α and N^α, is in fact positive. An intuitive explanation that makes this construction valid is the fact that despite the initial functions u^α are of a different types the optimal processes $(X^{\alpha,*})_{\alpha\in\mathbb{R}^*_+}$ and $(Y^{\alpha,*})_{\alpha\in\mathbb{R}^*_+}$ are such that $(X^{\alpha,*}Y^{\alpha',*})_{\alpha\neq\alpha'}$ are a martingales (portfolios versus state price density processes). Assuming uniform integrability assumptions of the diffusion parameters of Z^α and N^α the strictly increasing processes X^* and Y^* are, then, such that X^*Y^* is a martingale, which is enough to apply the general construction result of consistent utilities, Theorem 2.1.

Example Let us give an example based on a mixture of power and exponential utilities. To simplify, let δ_α be the Dirac measure in α and let the measure m be $\lambda\delta_{\alpha_1}+(1-\lambda)\delta_{\alpha_1}$. Let also u^{α_1} of a power type and u^{α_2} of an exponential type. Denoting by $M^{\alpha_2}_t := \int_0^t \Big(\eta_s-\delta^{\alpha_2,N}_s+\delta^{\alpha,Z,\sigma}_s\Big).\Big(dW_s+\big(\eta_s-\delta^{\alpha_2,N}_s\big)ds\Big)\Big]$, by Proposition 4.1 and Proposition 5.1, we get that

$$\begin{cases} X^{*,\alpha_1}_t(x) &= xX^{*,\alpha_1}_t, \quad X^{\alpha_2,*}_t(x)=N^{\alpha_2}_t\Big[x+M^{\alpha_2}_t\Big] \\ Y^{\alpha_1,*}_t(y) &= yY^{\delta^{\alpha_1,Z,\perp}}_t, \quad Y^{\alpha_2,*}_t(y)=yY^{\delta^{\alpha_2,Z,\perp}}_t \end{cases}$$

with X^{*,α_1}_t given by (5.4), it follows that

$$\begin{cases} X^*_t(x) &= x\big(\lambda X^{*,\alpha_1}_t+(1-\lambda)N^{\alpha_2}_t\big)+(1-\lambda)M^{\alpha_2}_tN^{\alpha_2}_t \\ Y^*_t(y) &= y\big(\lambda Y^{\delta^{\alpha_1,Z,\perp}}_t+(1-\lambda)Y^{\delta^{\alpha_2,Z,\perp}}_t\big) \end{cases}$$

Denote by $X^\lambda := \big(\lambda X^{*,\alpha_1}_t+(1-\lambda)N^{\alpha_2}_t\big)$ and $Y^\lambda := \big(\lambda Y^{\delta^{\alpha_1,Z,\perp}}+(1-\lambda)Y^{\delta^{\alpha_2,Z,\perp}}\big)$ and assume that X^λ and Y^λ are a.s. non null processes. Then, $X^*(x)$ and $Y^*(y)$ are strictly monotonous with respect to their initial conditions, with inverses flows

$$\mathcal{X}_t(x) \;=\; \tfrac{x-(1-\lambda)M^{\alpha_2}_tN^{\alpha_2}_t}{X^\lambda_t}, \quad \mathcal{Y}_t(y) \;=\; \frac{y}{Y^\lambda_t}$$

Let u an utility function defined on $\mathbb{R}$, by Theorem 2.1, the progressive utility U defined by

$$U(t,x)=Y^\lambda_t\int_0^x u_x\Big(\frac{z-(1-\lambda)M^{\alpha_2}_tN^{\alpha_2}_t}{X^\lambda_t}\Big)dz,$$

is a $\mathscr{X}$-consistent utility.

Conclusion The key point of this paper is to argue directly in terms of the optimal wealth and dual process and not in terms of consistent utilities for the simple reason that the sum of two consistent utilities is not a consistent utility, except in the very particular case where both optimal wealths and optimal dual processes are identical. On the other hand the sum of two acceptable wealths is always an acceptable wealth. Note also that the fact that $X^{\alpha,*}Y^{\alpha',*}, \alpha\neq\alpha'$ is martingale plays a crucial role in the construction proposed. Indeed, else the global wealth process X^* and the global state density price Y^* do not satisfy the necessary martingale condition, i.e. X^*Y^* is not a martingale.

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