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ON DISTRIBUTIONS OF EXPONENTIAL FUNCTIONALS OF THE PROCESSES WITH INDEPENDENT INCREMENTS

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Abstract. The aim of this paper is to study the laws of the exponential functionals of the processes X with independent increments, namely

$$I_t = \int_0^t \exp(-X_s)ds,\ t \geq 0,$$

and also

$$I_\infty = \int_0^\infty \exp(-X_s)ds.$$

Under suitable conditions we derive the integro-differential equations for the density of I_t and I_∞. We give sufficient conditions for the existence of a smooth density of the laws of these functionals with respect to the Lebesgue measure. In the particular case of Levy processes these equations can be simplified and, in a number of cases, solved explicitly.

KEY WORDS : process with independent increments, exponential functional, density, Kolmogorov-type equation.

MSC 2010 subject classifications: 60G51, 91G80

1. Introduction

This study was inspired by the questions arising in the mathematical finance, namely by the questions related with the perpetuities containing the liabilities, perpetuities subjected to the influence of the economical factors (see, for example, [20]), and also with the price of Asian options and similar questions (see, for instance, [18], [34], and references there). The study of the exponential functionals are also important in the insurance, since the insurance companies invest the money on risky assets. Then the distributions of these functionals appear very

naturally in the ruin problem (see for example [2], [25], [19], [33] and the references there).

In the mathematical finance the exponential functionals of the processes with independent increments (PII in short) arise very often. This fact is related with the observation that the log price process is usually not a homogeneous process on the relatively long time interval. For this reason several authors used for the modelling of the log price the process process X such that

$$X_t = \int_0^t g_{s-} dL_s$$

where L is Levy process and g is independent from L cádlàg random process for which the integral is well defined. In this case, the conditional distribution of the process X given σ -algebra generated by g, is a PII. Another important example of X is a Levy process L time changed by an independent increasing process $(\tau_t)_{t\geq 0}$ (cf.[13]), i.e.

$$X_t = L_{\tau_t}.$$

Again, conditionally to the process τ, the process X is PII.

In [27], [28] we proved the recurrent formulas for Mellin transform and we used these formulas to calculate the moments of exponential functionals of the processes with independent increments. In this paper we obtain the equations for the densities, when they exist, of the laws of exponential functionals I_t and I_∞ of PII processes.

Exponential functionals for Levy processes was studied in a big number of articles, the main part of them was related to the study of I_∞. The asymptotic behaviour of exponential functionals I_∞ was studied in [12], in particular for α-stable Levy processes. The authors also give an integro-differential equation for the density of the law of exponential functionals, when this density w.r.t. the Lebesgue measure exists. The questions related with the characterisation of the law of exponential functionals by the moments was studied in [8].

In more general setting, related to the Lévy case, the following functional

$$\int_0^\infty \exp(-X_{s-}) d\eta_s \tag{1}$$

where $X = (X_t)_{t\geq 0}$ and $\eta = (\eta_t)_{t\geq 0}$ are independent Lévy processes, was intensively studied. The conditions for finiteness of the integral (1) was obtained in [15]. The continuity properties of the law of this integral was studied in [7], where the authors give the condition for

absence of the atoms and also the conditions for absolute continuity of the laws of integral functionals w.r.t. the Lebesgue measure. Under the assumptions about the existence of the smooth density of these functionals, the equations for the density are given in [4], [5], [21].

In the papers [24] and [23], again for Levy process, the properties of the exponential functionals I_{τ_q} killed at the independent exponential time τ_q of the parameter $q > 0$, was investigated. In the article [23] the authors studied the existence of the density of the law of I_{τ_q}, they give an integral equation for the density and the asymptotics of the law of I_∞ at zero and at infinity, when X is a positive subordinator. The results given in [24] involve analytic Wiener-Hopf factorisation, Bernstein functions and contain the conditions for regularity, semi-explicite expression and asymptotics for the distribution function of I_{τ_q}. Despite numerous studies, the distribution properties of I_t and I_∞ are known only in a limited number of cases. When X is Brownian motion with drift, the distributions of I_t and I_∞ was studied in [14] and for a big number of specific processes X and η, like Brownian motion with drift and compound Poisson process, the distributions of I_∞ was given in [16].

The exponential functionals for diffusions stoped at first hitting time was studied in [26], where authors derive Laplace transform of the functionals and then, to find their laws, perform numerical inversion of Laplace transform. The relations between the hitting times and the occupation times for the exponential functionals was considered in [29], where the versions of identities in law such as Dufresne's identity, Ciesielski-Taylor's identity, Biane's identity, LeGall's identity was given.

In this article we consider a real valued process $X = (X_t)_{t\geq 0}$ with independent increments and $X_0 = 0$, which is a semi-martingale with respect to its natural filtration. We denote by (B, C, ν) a semi-martingale triplet of this process, which can be chosen deterministic (see [17], Ch. II, p.106). We suppose that $B = (B_t)_{t\geq 0}$, $C = (C_t)_{t\geq 0}$ and ν are absolutely continuous with respect to the Lebesgue measure, i.e.

$$B_t = \int_0^t b_s\, ds,\ \ C_t = \int_0^t c_s\, ds,\ \ \nu(dt, dx) = dtK_t(dx) \tag{2}$$

with càdlàg functions $b = (b_s)_{s\geq 0}, c = (c_s)_{s\geq 0}, K = (K_s(A))_{s\geq 0, A\in\mathcal{B}(\mathbb{R})}$. We assume that the compensator of the measure of jumps ν verify the

usual relation: for each $t \in \mathbb{R}^+$

$$\int_0^t \int_{\mathbb{R}} (x^2 \wedge 1) K_s(dx)\, ds < \infty. \tag{3}$$

For the the main result we will suppose an additional technical condition that

$$\int_0^t \int_{|x|>1} e^{|x|}\, K_s(dx)\, ds < \infty. \tag{4}$$

The last condition implies that $\mathbf{E}(|X_t|) < \infty$ for $t > 0$ (cf [30], Th. 25.3, p.159) so that the truncation of the jumps is no more necessary.

We recall that the characteristic function of X_t

$$\phi_t(\lambda) = \mathbf{E} \exp(i\lambda X_t)$$

is defined by the following expression: for $\lambda \in \mathbb{R}$

$$\phi_t(\lambda) = \exp\{i\lambda B_t - \frac{1}{2}\lambda^2 C_t + \int_0^t \int_{\mathbb{R}} (e^{i\lambda x} - 1 - i\lambda x)\, K_s(dx)\, ds\}.$$

We recall also that X is a semi-martingale if and only if for all $\lambda \in \mathbb{R}$ the characteristic function of X_t is of finite variation in t on finite intervals (cf. [17], Ch.2, Th. 4.14, p.106). Moreover, the process X always can be written as a sum of a semi-martingale and a deterministic function which is not necessarily of finite variation on finite intervals.

The article is organized as follows. The part 2. is devoted to the Kolmogorov type equation for the law of I_t. As known, the exponential functional $(I_t)_{t>0}$ is not a Markov process with respect to the filtration generated by the process X. It is continuous increasing process, which prevent the use of the stochastic calculus in an efficient way. For these reasons we fix t and we introduce a family of stochastic processes $V^{(t)} = (V_s^{(t)})_{0\le s\le t}$ indexed by t and such that $I_t = V_t^{(t)}$ (P-a.s.) (see Lemma 1). The construction of such processes is made via the time reversion of the process X at a fixed time t, and this gives us the process $Y^{(t)} = (Y_s^{(t)})_{0\le s\le t}$. We show that if X is PII then $Y^{(t)}$ is also PII, and if X is Levy process, then $Y^{(t)}$ is so (see Lemma 2). We prove then that $V^{(t)}$ is a Markov process with respect to the natural filtration of the process $Y^{(t)}$ (see Lemma 3), we find the generator and we give the Kolmogorov-type equations for $V^{(t)}$(see Theorem 1). Supposing the existence of the smooth density of the law of the process $V^{(t)}$ we derive the integro-differential equation for the density of the laws of $V_s^{(t)}, 0 < s < t$. The density of the law of $V_t^{(t)}$ can be obtained just by integration of the right-hand side of the equation for the density of the law of $V_s^{(t)}$ in s

on the interval $]0,t[$ since the laws of $V_t^{(t)}$ and $V_{t-}^{(t)}$ coincide. The last fact follows from the absence of the predictable jumps of the process X, and, hence, the ones of $Y^{(t)}$.

In the part 3. we consider the question of existence of the smooth density of the process $V^{(t)}$. The question of the existence of the density of the law of $V_s^{(t)}$, $0 \leq s \leq t$, of the class $C^{1,2}(]0,t[,\mathbb{R}^{+,*})$ is rather difficult question, which was open question in all cited papers on exponential functionals. We remark that in the case of Levy processes the laws of $V_t^{(t)}$ and I_t coincide. We will give in Proposition 2 sufficient conditions for the existence of the density of the class $C^{\infty}(]0,t[,\mathbb{R}^{+,*})$ when X is Levy process. For non-homogeneous diffusions we give a partial answer on this question in Corollary 1.

The part 4 is devoted to Levy processes. When X is Levy process, the equations for the density of I_t can be simplified due to the homogeneity (see Proposition 1). We present also the equations for the distribution functions of the laws of I_t and I_∞, since for these equations we have the explicit boundary conditions (cf. Corollary 2). In Corollary 3 we consider very known Brownian case. In Corollary 4 we give the equations for the case of Levy processes with integrable jumps, and in Corollary 5, we consider the case of exponential jumps.

2. Kolmogorov type equation for the density of the law of I_t

We introduce, for fixed $t > 0$, a time reversal process $Y = (Y_s)_{0\leq s\leq t}$ with

$$Y_s = X_t - X_{(t-s)-}.$$

Of course, this process depends on the parameter t, but we will omit this parameter for the simplicity of the notations.

For convenience of the readers we present here Lemma 1 and Lemma 2 proved in [27]. First result establish the relation between I_t and the process $Y = (Y_s)_{0\leq s\leq t}$.

Lemma 1. *For $t > 0$ ($\mathbf{P}$-a.s.)*

$$I_t = e^{-Y_t} \int_0^t e^{Y_s} ds.$$

Proof We write, using the definition of the process Y and the assumption that $X_0 = 0$:

$$e^{-Y_t}\int_0^t e^{Y_s}ds = \int_0^t e^{-Y_t+Y_s}ds = \int_0^t e^{-X_{(t-s)_-}+X_0}ds = \int_0^t e^{-X_s}ds = I_t.$$

The last equality holds after time change noticing that the integration of left-hand version and right-hand version of some càdlàg process w.r.t. the Lebesgue measure gives the same result. □

In the following lemma we claim that Y is PII and we precise its semi-martingale triplet. For that we introduce the functions $\bar{b} = (\bar{b}_u)_{0\le u\le t}$, $\bar{c} = (\bar{c}_u)_{0\le u\le t}$ and $\bar{K} = (\bar{K}_u)_{0\le u\le t}$ putting

$$\bar{b}_u = \mathbf{1}_{\{t\}}(u)(b_t - b_0) + b_{t-u}, \tag{5}$$

$$\bar{c}_u = \mathbf{1}_{\{t\}}(u)(c_t - c_0) + c_{t-u}, \tag{6}$$

$$\bar{K}_u(x) = \mathbf{1}_{\{t\}}(u)(K_t(x) - K_0(x)) + K_{t-u}(x) \tag{7}$$

where $\mathbf{1}_{\{t\}}$ is indicator function. It means, for instance for $\bar{b} = (\bar{b}_u)_{0\le u\le t}$ that

$$\bar{b}_u = \begin{cases} b_{t-u} & \text{if} \quad 0 \le u < t, \\ b_t & \text{if} \quad u = t. \end{cases}$$

So, the function $\bar{b}$ can have a discontinuity at t, since in general $b_0 \neq b_t$.

Lemma 2. *(cf. [27]) The process Y is a process with independent increments, it is a semi-martingale with respect to its natural filtration, and its semi-martingale triplet $(\bar{B}, \bar{C}, \bar{\nu})$ is given by: for $0 \le s \le t$,*

$$\bar{B}_s = \int_0^s \bar{b}_u du,\ \bar{C}_s = \int_0^s \bar{c}_u du,\ \bar{\nu}(du, dx) = \bar{K}_u(dx)\, du \tag{8}$$

To obtain an integro-differential equation for the density, we introduce two important processes related with the process Y, namely the process $V = (V_s)_{0\le s\le t}$ and $J = (J_s)_{0\le s\le t}$ with

$$V_s = e^{-Y_s}J_s, \qquad J_s = \int_0^s e^{Y_u}du. \tag{9}$$

We underline that the both processes depend of the parameter t, since it is so for the process Y.

We notice that according to Lemma 1, $I_t = V_t$, and then they have the same laws. As we will see, the process $V = (V_s)_{0\le s\le t}$ is a Markov process with respect to the natural filtration $\mathbb{F}^Y = (\mathcal{F}^Y_s)_{0\le s\le t}$ of the process Y and this fact will help us very much to find the equation for the density of the law of I_t.

Lemma 3. *The process $V = (V_s)_{0\le s\le t}$ is a Markov process with respect to the natural filtration $\mathbb{F}^Y = (\mathcal{F}_s^Y)_{0\le s\le t}$ of the process Y.*

Proof. We write that for $h > 0$

$$V_{s+h} = e^{-Y_{s+h}} \int_0^{s+h} e^{Y_u} du = e^{-(Y_{s+h}-Y_s)} \left[V_s + \int_s^{s+h} e^{Y_u - Y_s} du\right]$$

Then for all measurable bounded functions f

$$\mathbf{E}(f(V_{s+h}) \,|\, \mathcal{F}_s^Y) = \mathbf{E}\left(f(e^{-(Y_{s+h}-Y_s)}[V_s + \int_s^{s+h} e^{Y_u - Y_s} du]) \,|\, \mathcal{F}_s^Y\right) =$$

$$\mathbf{E}\left(f(e^{-(Y_{s+h}-Y_s)}[x + \int_s^{s+h} e^{Y_u - Y_s} du])\right)_{|x=V_s}$$

since Y is a process with independent increments. Hence, $\mathbf{E}(f(V_{s+h}) \,|\, \mathcal{F}_s^Y)$ is a measurable function of V_s and we conclude that V is Markov process with respect to the filtration generated by Y. □

We define the set of functions

$$\mathcal{C} = \{f \in \mathcal{C}_b^2 \,|\, \sup_{y\in\mathbb{R}^+} |f'(y)y| < \infty,\ \sup_{y\in\mathbb{R}^+} |f''(y)y^2| < \infty\}$$

and such that $f(0) = f'(0) = 0$. For $0 \le s \le t$ we put

$$\overline{a}_s = -\overline{b}_s + \frac{1}{2}\overline{c}_s + \int_{\mathbb{R}} (e^{-x} - 1 + x)\overline{K}_s(dx) \tag{10}$$

We notice that the conditions (3) and (4) imply that (λ-a.s.)

$$\int_{\mathbb{R}} |e^{-x} - 1 + x|\, \overline{K}_s(dx) < \infty,$$

so that, $\overline{a}_s$ is (λ-a.s.) well-defined. We introduce also for $f \in \mathcal{C}$ the generator $(\mathcal{A}_s^V)_{0\le s<t}$ of the process V via

$$\mathcal{A}_s^V(f)(y) = \tag{11}$$

$$(1+y\,\overline{a}_s)\, f'(y) + \frac{1}{2}\overline{c}_s\, f''(y)\, y^2 + \int_{\mathbb{R}} \left[f(ye^{-x}) - f(y) - f'(y)y(e^{-x}-1)\right] \overline{K}_s(dx)$$

Theorem 1. *Let us suppose that the conditions* (2),(3) *and* (4) *are satisfied. Then the infinitesimal generator $(\mathcal{A}_s^V)_{0\le s<t}$ of the Markov process V is defined by* (11). *In addition, for $0 \le s \le t$ and $f \in \mathcal{C}$*

$$\mathbf{E}(f(V_s)) = \int_0^s \mathbf{E}(\,\mathcal{A}_u^V(f)(V_u)\,)du \tag{12}$$

where $\mathcal{A}_t^V = \lim_{s\to t-} \mathcal{A}_s^V$. *If for* $0 < s \leq t$ *the density* p_s *w.r.t. the Lebesgue measure* λ *of the law of* V_s *exists and belongs to the class* $\mathcal{C}^{1,2}(]0,t[\times\mathbb{R}^{+,*})$, *then* λ*-a.s.*

$$\frac{\partial}{\partial s}p_s(y) = \frac{1}{2}\bar{c}_s\frac{\partial^2}{\partial y^2}(y^2\, p_s(y)) - \frac{\partial}{\partial y}((\bar{a}_s\, y + 1)\, p_s(y)) + \int_{\mathbb{R}}\left[e^x p_s(ye^x) - p_s(y) + (e^{-x}-1)\frac{\partial}{\partial y}(yp_s(y))\right]\bar{K}_s(dx) \tag{13}$$

and the density p_t *of the law of* I_t *verify:*

$$p_t(y) = \int_0^t \left\{\frac{1}{2}\bar{c}_s\frac{\partial^2}{\partial y^2}(y^2\, p_s(y)) - \frac{\partial}{\partial y}((\bar{a}_s\, y + 1)\, p_s(y)) + \int_{\mathbb{R}}\left[e^x p_s(ye^x) - p_s(y) + (e^{-x}-1)\frac{\partial}{\partial y}(yp_s(y))\right]\bar{K}_s(dx)\right\} ds \tag{14}$$

Remark 1. The existence and the uniqueness of the solution of the integro-differential equation given in Theorem 1 follows from the possibility to identify the characteristics of the corresponding process from the equation. Since the law of PII process, which is a semi-martingale, is uniquely defined by its characteristics, the solution of such equation exists and is unique.

Proof of Theorem 1. The proof of Theorem 1 will be divided in three parts : in the first part we prove a decomposition (20), then using limit passage we prove (12), and finally, in the third part we obtain the equations (13), (14) for the densities.

1) Proof of (20). For $f \in \mathcal{C}$ and $0 \leq s \leq t$ we write the Ito formula:

$$f(V_s) = f(V_0) + \int_0^s f'(V_{u-})dV_u + \frac{1}{2}\int_0^s f''(V_{u-})d < V^c >_u + \int_0^s\int_{\mathbb{R}}(f(V_{u-}+x) - f(V_{u-}) - f'(V_{u-})x)\,\mu_V(du,dx), \tag{15}$$

where μ_V is the measure of jumps of the process V. From the definition of the process V we can easily find that

$$dV_s = ds + J_s\, d(\, e^{-Y_s}\,)$$

and that

$$dV_s^c = -e^{-Y_{s-}}J_s\, dY_s^c = -V_{s-}\, dY_s^c, \tag{16}$$

$$d < V^c >_s = (e^{-Y_{s-}}J_s)^2\, d < Y^c >_s = V_{s-}^2\, d < Y^c >_s,$$

$$\Delta V_s = V_s - V_{s-} = e^{-Y_{s-}}J_s(e^{-\Delta Y_s} - 1) = V_{s-}(e^{-\Delta Y_s} - 1).$$

At the same time, again by the Ito formula we get the following decomposition

$$e^{-Y_s} = e^{-Y_0} + A_s + N_s. \tag{17}$$

In this decomposition the process $(A_s)_{s\geq 0}$ is defined as

$$A_s = \int_0^s e^{-Y_{u-}}[-\bar{b}_u + \frac{1}{2}\bar{c}_u + \int_{\mathbb{R}}(e^{-x} - 1 + x)\bar{K}_u(dx)]\, du \tag{18}$$

and it is a process of locally bounded vatiation on bounded intervals. In fact, let us introduce a sequence of stopping times: for $n \geq 1$

$$\tau_n = \inf\{0 \leq s \leq t \,|\, e^{-Y_s} \geq n\}$$

with $\inf\{\emptyset\} = \infty$. We notice that this sequence of stopping times tends to $+\infty$ as $n \to \infty$. Then, since $e^{Y_{s-}} < n$ on the stochastic interval $[\![0, \tau_n[\![$, we get from (2), (3) and (4) that

$$\mathrm{Var}(A)_{s\wedge\tau_n} \leq n \int_0^t \left[|\bar{b}_u| + \frac{1}{2}\bar{c}_u + \int_{\mathbb{R}} |e^{-x} - 1 + x|\, \overline{K}_u(dx)\right] du < \infty.$$

In (17) the process $N = (N_s)_{s\geq 0}$ is defined by

$$N_s = -\int_0^s e^{-Y_{u-}} d\bar{M}_u + \int_0^s \int_{\mathbb{R}} e^{-Y_{u-}}(e^{-x} - 1 + x)(\mu_Y(du, dx) - \bar{K}_u(dx)du) \tag{19}$$

In the relation (19), the process $\bar{M}$ is the martingale component of the semi-martingale decomposition of Y: $Y_s = \bar{B}_s + \bar{M}_s$, and μ_Y is the measure of jumps of the process Y. It should be noticed that since Y is a process with independent increments and $\bar{B}$ is deterministic, $\bar{M}$ is a martingale (see [32], Th. 58, p. 45) as well as its pure discontinuous part $\bar{M}^d$. Then, the process $(N_{s\wedge\tau_n})_{0\leq s\leq t}$ is a local martingale as a stochastic integral of a bounded function w.r.t. a martingale.

We put (16),(17),(18) and (19) into (15) to obtain a final decomposition for $f(V_s)$. To present this final decomposition we put for $y \geq 0$ and $x \in \mathbb{R}$

$$F(y, x) = f(ye^{-x}) - f(y) - f'(y)y(e^{-x} - 1)$$

and also

$$B^V_s = \int_0^s \left[f'(V_{u-})(1 + \bar{a}_u\, V_{u-}) + \frac{1}{2}f''(V_{u-})V^2_{u-}\, \bar{c}_u + \int_{\mathbb{R}} F(V_{u-}, x)\bar{K}_u(dx)\right] du$$

and

$$N^V_s = \int_0^s f'(V_{u-})\, V_{u-}[-d\bar{M}_u + \int_{\mathbb{R}}(e^{-x} - 1 + x)(\mu_Y(du, dx) - \bar{K}_u(dx)\, du] +$$

$$\int_0^s \int_{\mathbb{R}} F(V_{u-}, x)(\mu_Y(du, dx) - \bar{K}_u(dx)\, du).$$

Finally, we get a decomposition

$$f(V_s) = f(V_0) + B_s^V + N_s^V \tag{20}$$

In this decomposition B^V is a process with locally bounded variation and N^V is a local martingale. In fact, let us use the same sequence of stopping times τ_n as previously and let

$$D = \sup_{y\in\mathbb{R}} \max(|f(y)|, |f'(y)|, |f'(y)y|, |f''(y)y^2|).$$

Then, for $0 \le s \le t$

$$\mathrm{Var}(B^V)_{s\wedge\tau_n} \le D \int_0^t \left(1 + |\bar{b}_u| + \bar{c}_u + \int_{\mathbb{R}} |e^{-x} - 1 + x|\, \overline{K}_u(x)dx\right) du$$
$$+ \int_0^t \int_{\mathbb{R}} |F(V_{u-}, x)| \bar{K}_u(dx)\, du$$

The first term of the r.h.s. is finite since the functions $(\bar{B}_s)_{0\le s\le t}$ and $(\bar{C}_s)_{0\le s\le t}$ have finite variation on finite intervals and since (4) holds. Now, using the Taylor-Lagrange formula of the second order, we find that for $y > 0$ and $|x| \le 1$

$$|F(y, x)| = \frac{1}{2}|f''(y(1 + \theta(e^{-x} - 1)))|y^2(e^{-x} - 1)^2 \le \frac{D(e^{-x} - 1)^2}{2[1 + \theta(e^{-x} - 1)]^2}$$

where $0 < \theta < 1$. Since for $|x| \le 1$, $1 + \theta(e^{-x} - 1) \ge \frac{1}{e}$ and $|e^{-x} - 1| \le e|x|$, we find that $|F(V_{u-}, x)| \le \frac{1}{2}De^4x^2$.

For $|x| > 1$ we use Taylor-Lagrange formula of the first order to get

$$|F(y, x)| = |(f'(y(1 + \theta(e^{-x} - 1))) - f'(y))y(e^{-x} - 1)| \le$$
$$D\, \left[|1 + \theta(e^{-x} - 1)|^{-1} + 1\right]\, |e^{-x} - 1|$$

Again, for $x > 1$, $1+\theta(e^{-x}-1) \ge e^{-x}$, and for $x < -1$, $1+\theta(e^{-x}-1) \ge 1$. Moreover, for $x > 1$, $|e^{-x} - 1| \le 1$ and for $x < -1$, $|e^{-x} - 1| \le e^{-x}$. Finally,

$$|F(y, x)| \le C\left(e^{|x|}\mathbf{1}_{\{|x|>1\}} + x^2\, \mathbf{1}_{\{|x|\le 1\}}\right)$$

with some positive constant C. Then, the conditions (3) and (4) implies that

$$\int_0^t \int_{\mathbb{R}} |F(V_{u-}, x)| \bar{K}_u(dx)\, du < \infty.$$

Using above results we see that $(N^V_{s\wedge\tau_n})_{0\le s\le t}$ is a local martingale as an integral of a bounded function w.r.t. a martingale.

2) Proof of (12). Let $(\tau'_n)_{n\in\mathbb{N}}$ be the localising sequence for N_V and $\bar{\tau_n} = \tau_n \wedge \tau'_n$. Let $s \in [0, t[$ and $\delta > 0$ such that $s + \delta \leq t$. Then, from previous decomposition using the localisation we get:

$$\mathbf{E}(f(V_{(s+\delta)\wedge\bar{\tau_n}}) - f(V_{s\wedge\bar{\tau_n}}) \,|\, \mathcal{F}_s^Y) = \mathbf{E}(B^V_{(s+\delta)\wedge\bar{\tau_n}} - B^V_{s\wedge\bar{\tau_n}} \,|\, \mathcal{F}_s^Y)$$

Since f is bounded function and $\lim_{n\to\infty} \bar{\tau_n} = +\infty$, we can pass to the limit in the l.h.s. by the Lebesgue convergence theorem. The same can be done on the r.h.s. since the process $B^V = (B^V_s)_{0\leq s\leq t}$ is a process of bounded variation on bounded intervals, uniformly in s and n. After taking a limit as $n \to +\infty$ we get that

$$\mathbf{E}(f(V_{s+\delta}) - f(V_s) \,|\, \mathcal{F}_s^Y) = \mathbf{E}(B^V_{s+\delta} - B^V_s \,|\, \mathcal{F}_s^Y)$$

Now, we write the expression for $B^V_{s+\delta} - B^V_s$:

$$B^V_{s+\delta} - B^V_s = \int_s^{s+\delta} [f'(V_{u-})(1 + \bar{a}_u\, V_{u-}) + \frac{1}{2} f''(V_{u-}) V^2_{u-}\, \bar{c}_u + \int_{\mathbb{R}} F(V_{u-}, x) \bar{K}_u(dx)\,] du$$

We remark that

$$\lim_{\delta\to 0} \frac{B^V_{s+\delta} - B^V_s}{\delta} = f'(V_{s-})(1+\bar{a}_s\, V_{s-}) + \frac{1}{2} f''(V_{s-}) V^2_{s-}\, \bar{c}_s + \int_{\mathbb{R}} F(V_{s-}, x) \bar{K}_s(dx)$$

We show that the quantities $\frac{B^V_{s+\delta} - B^V_s}{\delta}$ are uniformly bounded, for small $\delta > 0$, by a constant. In fact, we can write that

$$\frac{|\, B^V_{s+\delta} - B^V_s \,|}{\delta} \leq \frac{C}{\delta} \int_s^{s+\delta} \left[\left(1 + \bar{a}_u + \frac{1}{2} \bar{c}_u \right) + \int_{\mathbb{R}} |F(V_{u-}, x)| \bar{K}_u(dx) \right] du$$

Then, using the estimations for $|F(V_{s-}, x)|$ obtained previously, and the fact that the sequences

$$\frac{1}{\delta} \int_s^{s+\delta} \bar{a}_u du,\ \frac{1}{\delta} \int_s^{s+\delta} \bar{c}_u du,\ \frac{1}{\delta} \int_s^{s+\delta} \int_{\mathbb{R}} \left(x^2 I_{\{|x|\leq 1\}} + e^{|x|} I_{\{|x|>1\}} \right) \bar{K}_u(dx) du$$

are uniformly bounded, for small values of $\delta > 0$, by a constant, we deduce that the quantities $\frac{|B^V_{s+\delta} - B^V_s|}{\delta}$ are uniformly bounded for small δ by a constant, too. Under these conditions we can exchange the limit and the conditional expectation and it gives us the expression for the generator of V at $0 \leq s < t$. As a conclusion, we get that for $0 \leq s < t$

$$\frac{d}{ds} \mathbf{E}(f(V_s)) = \mathbf{E} \mathcal{A}^V_s(f)(V_{s-}) \tag{21}$$

Let us prove that we can replace V_{s-} by V_s in the above expression. In fact, for $\lambda \in \mathbb{R}$

$$\mathbf{E}(e^{i\lambda \ln(\frac{V_s}{V_{s-}})}) = \mathbf{E}(e^{-i\lambda \Delta Y_s}) = \lim_{h\to 0+} \mathbf{E}(e^{-i\lambda(Y_{s+h}-Y_s)}) = 1$$

since the characteristics of Y are continuous in time. Hence, $V_{s-} = V_s$ (P-a.s.) and they have the same laws.

Then after the replacement of V_{s-} by V_s in (21) and the integration w.r.t. s we obtain (12). For $s = t$ we take $\lim_{s\to t-}$ in (12).

3) Proof of (13) *and* (14). We denote by P_s the law of V_s. Then from (12) we get that for $0 \le s \le t$

$$\int_0^s \int_0^\infty \Big[f'(y)(1 + y\,\bar{a}_u) + \frac{1}{2} f''(y) y^2\, \bar{c}_u + \tag{22}$$

$$\int_{\mathbb{R}} \big(f(ye^{-x}) - f(y) - f'(y)y(e^{-x} - 1) \big)\, \bar{K}_u(x)dx \Big]\, P_u(dy)\, du \;=\; \int_0^\infty f(y) P_s(dy)$$

Moreover, since for $s > 0$, the law P_s of V_s has a density p_s w.r.t. Lebesque measure, it gives

$$\int_0^s \int_0^\infty \Big[f'(y)(1 + y\,\bar{a}_u) + \frac{1}{2} f''(y) y^2\, \bar{c}_u + \tag{23}$$

$$\int_{\mathbb{R}} \big(f(ye^{-x}) - f(y) - f'(y)y(e^{-x} - 1) \big)\, \bar{K}_u(x)dx \Big]\, p_u(y)dy\, du$$

$$= \int_0^\infty f(y) p_s(y) dy$$

To obtain the equation for the density, we consider the set of continuously differentiable functions on compact support $\mathcal{C}^2_K \subseteq \mathcal{C}$. We differentiate the above equation with respect to s to get

$$\int_0^\infty \Big[f'(y)(1 + y\,\bar{a}_s) + \frac{1}{2} f''(y) y^2\, \bar{c}_s + \tag{24}$$

$$\int_{\mathbb{R}} \big(f(ye^{-x}) - f(y) - f'(y)y(e^{-x} - 1) \big)\, \bar{K}_s(x)dx \Big]\, p_s(y)$$

$$= \int_0^\infty f(y)\, \frac{\partial}{\partial s} p_s(y)\, dy$$

Using the integration by part formula we deduce that

$$\int_0^\infty f'(y)\, p_s(y) dy = -\int_0^\infty \frac{\partial}{\partial y}(p_s(y))\, f(y) dy$$

$$\int_0^\infty f'(y)\, y\, p_s(y) dy = -\int_0^\infty \frac{\partial}{\partial y}(y\, p_s(y))\, f(y) dy$$

$$\int_0^\infty f''(y)\, y^2\, p_s(y)dy = \int_0^\infty \frac{\partial^2}{\partial y^2}(y^2\, p_s(y))\, f(y)dy$$

By the change of the variables and by the integration by parts we obtain

$$\int_0^\infty \int_{\mathbb{R}} p_s(y)\ \left[f(ye^{-x}) - f(y) - f'(y)y(e^{-x}-1)\right] \bar{K}_s(dx)dy =$$

$$\int_0^\infty \left(\int_{\mathbb{R}} [e^x p_s(ye^x) - p_s(y) + (e^{-x}-1)\frac{\partial}{\partial y}(yp_s(y)]\, \bar{K}_s(dx) \right)\, f(y)dy$$

The mentioned relations together with the equation (24) gives that for all $f \in \mathcal{C}_K^2$:

$$\int_0^\infty f(y) \left[-\frac{\partial}{\partial s}p_s(y) + \frac{1}{2}\bar{c}_s \frac{\partial}{\partial y}(y^2 p_s(y)) - \frac{\partial}{\partial y}((\bar{a}_s y + 1)p_s(y) \right.$$

$$\left. + \int_{\mathbb{R}} e^x p_s(ye^x) - p_s(y) + (e^{-x}-1)\frac{\partial}{\partial y}(yp_s(y))\bar{K}_s(dx) \right] = 0$$

and it proves our claim about the equation for p_s.

We integrate the equation for p_s on the interval $]0, t-\delta[$ for $\delta > 0$ and we pass to the limit as $\delta \to 0$. Since the laws of V_{t-} and V_t coincide, we get in this way the equation for p_t. □

3. Some results about the existence of the smooth density

The question of the existence of the density of the law of V_s, $0 \le s \le t$, of the class $C^{1,2}(]0,t[, \mathbb{R})$ is rather difficult question, which was open question in all cited works on exponential functionals. We will give here a partial answer on this question via the known result on Malliavin calculus given in [9]. For the convenience of the readers we present this result here in the one-dimensional setting.

We consider the following stochastic differential equation:

$$X_t^x = x + \int_0^t a(X_{s-}^x)\, ds + \int_0^t b(X_{s-}^x)dW_s + \int_0^t \int_{\mathbb{R}} c(X_{s-}^x, z)(\mu - \nu)(ds, dz)$$

where $x \in \mathbb{R}$, a, b, c are measurable functions on $\mathbb{R}$ and $\mathbb{R}^2$ respectively, W is standard Brownian motion, and μ and ν are jump measure and its compensator of X^x. It is assumed that the solution of this equation exists and is unique, and also that the following assumptions hold.

Assumption (A-r):

(i) a and b are r-times differentiable with bounded derivatives of all order from 1 to r,

(ii) $c(\cdot, z)$ is r-times differentiable and there exists a sigma-finite measure G on $\mathbb{R}$ such that

(a) $c(0, \cdot) \in \bigcap_{2 \leq p < \infty} L^p(\mathbb{R}^*, G)$

(b) for $1 \leq n \leq r$, $\sup_y(\frac{\partial^n}{\partial y^n}(c(y, \cdot)) \in \bigcap_{2 \leq p < \infty} L^p(\mathbb{R}^*, G)$

Assumption (SB-(ζ, θ)): there exist $\epsilon > 0$ and $\delta > 0$ such that

$$b^2(y) \geq \frac{\epsilon}{1 + |y|^\delta}$$

Assumption (SC-bis) : for all $u \in [0, 1]$ there exists $\zeta > 0$ such that

$$|1 + u\frac{\partial}{\partial y}c(y, z)| > \zeta$$

Theorem 2.29 (cf. [9], p. 15) Suppose that the assumptions (A-(2r+10)), (SB-(ζ, θ)) and (SC-bis) are satisfied. Then for $t > 0$ the law of X_t^x has a density $p_t(x, y)$ w.r.t. the Lebesgue measure and the map $(t, x, y) \to p_t(x, y)$ is of class $C^r(]0, t] \times \mathbb{R} \times \mathbb{R})$.

To apply this theorem let us write stochastic differential equation for $(V_s)_{0 \leq s \leq t}$. For that we put for $0 \leq s \leq t$

$$\begin{cases} a_s(y) = y(-\bar{b}_s + \frac{1}{2}\bar{c}_s + \int_{\mathbb{R}}(e^{-z} - 1 + z)\bar{K}_s(dz) + 1, \\ b_s(y) = y\sqrt{\bar{c}_s}, \\ c_s(y, z) = y(e^{-z} - 1). \end{cases}$$

Proposition 1. *Suppose that*

$$\int_0^t \int_{\mathbb{R}} |e^{-z} - 1 + z|\bar{K}_s(dz) < \infty$$

and that $\bar{c}_s > 0$ for $0 < s \leq t$. Then the process $(V_s)_{0 \leq s \leq t}$ satisfy the following stochastic differential equation:

(25)

$$V_s = \int_0^s a_u(V_{u-})du - \int_0^s b_u(V_{u-})dW_u + \int_0^s \int_{\mathbb{R}} c_u(V_{u-}, z)(d\mu_Y - \bar{K}_u(dz)du)$$

where μ_Y is the jump measure of Y and W is DDS Brownian motion corresponding to the continuous martingale part Y^c of Y.

Proof. We recall that V_s is defined by (9). Let us introduce the process $\hat{Y}$ via the relation : for $0 \leq s \leq t$

$$e^{-Y_s} = \mathcal{E}(\hat{Y})_s \tag{26}$$

where $\mathcal{E}(\cdot)$ is Doléan-Dade exponential. Then,

$$V_s = \mathcal{E}(\hat{Y})_s \int_0^s \frac{du}{\mathcal{E}(\hat{Y})_u}$$

and we can see by the integration by part formula that $(V_s)_{0\leq s\leq t}$ is unique strong solution of the equation

$$dV_s = V_{s-} d\hat{Y}_s + ds \tag{27}$$

with the initial condition $V_0 = 0$. Using the definition of Doléan-Dade exponential we see that (26) is equivalent to

$$e^{-Y_s} = e^{\hat{Y}_s - \frac{1}{2}<\hat{Y}^c>} \prod_{0<u\leq s} (1 + \Delta\hat{Y}_u)\, e^{-\Delta\hat{Y}_u}$$

where $\hat{Y}^c$ is continuous martingale part of $\hat{Y}$. From this equality we find that $\hat{Y}^c_s = -Y^c_s$, $\ln(1+\Delta\hat{Y}_s) = -\Delta Y_s$ and that the semi-martingale characteristics $(\hat{B}, \hat{C}, \hat{\nu})$ of $\hat{Y}$ are:

$$\begin{cases} \hat{B}_s = -\bar{B}_s + \frac{1}{2}\bar{C}_s + \int_0^s \int_{\mathbb{R}} (e^{-z} - 1 + z)\bar{K}_u(dz)du \\ \hat{C}_s = \bar{C}_s \\ \hat{\nu}(ds, dz) = (e^{-z} - 1)\bar{K}_s(dz)ds \end{cases}$$

Since $(\bar{B}, \bar{C}, \bar{\nu})$ are absolutely continuous w.r.t. the Lebesgue measure with the derivatives $(\bar{b}, \bar{c}, \bar{K})$ we get that

$$\hat{Y}_s = \int_0^s (-\bar{b}_u + \frac{1}{2}\bar{c}_u + \int_{\mathbb{R}} (e^{-z} - 1 + z)\bar{K}_u(dz))du - \int_0^s \sqrt{\bar{c}_u}\, dW_u + \int_0^s \int_{\mathbb{R}} (e^{-z} - 1)(\mu_Y(du, dz) - \bar{K}_u(dz)du)$$

where W is DDS Brownian motion corresponding to the continuous martingale part of Y. Let us put this decomposition into (27) and we obtain (25). □

To apply the Theorem 2.29 from [9] we suppose that Y is a Levy process and we introduce the supplementary process

$$V^x_s = x + \mathcal{E}(\hat{Y})_s \int_0^s \frac{du}{\mathcal{E}(\hat{Y})_u}$$

with $x > 0$. We see that $V^x_s - x = V_s$, and $V^0_s = V_s$, and also that the density $p_s(x, y)$ of the law of V^x_s w.r.t. the Lebesgue measure and the density $p_s(y)$ of the law of V_s exist or not at the same time and are related: for all $x > 0$ and $y > 0$

$$p_s(x, y + x) = p_s(y).$$

So, the both densities are the same regularity w.r.t.(s, y). In addition we have : for all $s \geq 0$ and all $\omega \in \Omega$, $V_s^x(\omega) \geq x$.

Proposition 2. *Suppose that Y is Levy process with the triplet (b_0, c_0, K_0) and the following conditions are satisfied:*

(1) $c_0 > 0$,

(2) $\int_{z<-1} e^{-pz} K_0(dz) < \infty$ *for* $p \geq 2$,

(3) *there exists a constant* $A > 0$ *such that* $K_0(]A, +\infty[) = 0$.

Then, for $s > 0$, the law of V_s has a density p_s and the map $(s, y) \to p_s(y)$ is of class $C^\infty(]0, t], \mathbb{R}^{+,*})$.

Proof. When Y is Levy process, the functions a_s, b_s, c_s figured in (25) are independent on s and are equal to:

$$\begin{cases} a(y) = y(-b_0 + \frac{1}{2}c_0 - \int_{\mathbb{R}}(e^{-z} - 1 + z)K_0(dz)) + 1, \\ b(y) = y\sqrt{c_0}, \\ c(y, z) = y(e^{-z} - 1). \end{cases}$$

We consider now the process V^x with $x > 0$. This process satisfy the same equation as V with the replacement of V_{u-} and V_s by V_{u-}^x and V_s^x, and also with the replacement of the functions $a_s(y)$, $b_s(y)$ and $c_s(y, z)$ by $a(y), b(y), c(y, s)$ with $y \geq x$.

We see that the Assumption (A-r) is satisfied for all $r \geq 1$ with $G = K_0$, as well as the Assumption (SB-(ζ, θ)) putting $\epsilon = x^2 c_0$ and (SC-bis) taking $\zeta = \frac{1}{2}e^{-A}$. Then the map $(s, x, y) \to p_s(x, y)$ is of class $C^\infty(]0, t], [x, +\infty[\times[x, +\infty[)$, and the map $(s, y) \to p_s(x, y + x)$ is of class $C^\infty(]0, t], \mathbb{R}^{+,*} \times \mathbb{R}^{+,*})$. As a conclusion, the map $(s, y) \to p_s(y)$ is of class $C^\infty(]0, t], \mathbb{R}^{+,*})$. □

For non-homogeneous diffusion we obtain the following partial result.

Corollary 1. *Let $s > 0$ be fixed. Suppose that*

(1) $\int_0^s \bar{c}_u du > 0$,

(2) $\int_0^s \int_{z<-1} e^{-pz} K_s(dz) < \infty$ *for* $p \geq 2$,

(3) *there exist a constant* $A > 0$ *such that* $K_s(]A, +\infty[) = 0$ *for all* $0 < s \leq t$.

Then, the law of V_s has a density p_s such that the map $y \to p_s(y)$ is of class $C^\infty(\mathbb{R}^{+,*})$.

Proof. We notice that the law of Y_s coincide with the law of the Levy process with the triplet $(\frac{1}{s}\bar{B}_s, \frac{1}{s}\bar{C}_s, \frac{1}{s}\int_0^s \int_{\mathbb{R}} \bar{K}_s(dz)du)$ at the time s. Then the previous proposition can be applied and it gives the claim. □

4. When X is Levy process

In this section we consider a particular case of Levy processes. Namely, let X be Levy process with the parameters (b_0, c_0, K_0). As before we suppose that

$$\int_{\mathbb{R}} (x^2 \wedge 1) K_0(dx) < \infty \text{ and } \int_{|x|>1} e^{|x|} K_0(dx) < \infty \tag{28}$$

and we put

$$a_0 = -b_0 + \frac{1}{2}c_0 + \int_{\mathbb{R}} (e^{-x} - 1 + x) K_0(dx).$$

Due to the homogeneity of Levy process, the equation for the density can be simplified as we can see from the following proposition.

Proposition 3. *Suppose that* (28) *holds and the density of the law of* I_t *exists and belongs to the class* $\mathcal{C}^{1,2}(]0,t] \times \mathbb{R}^{+,*})$. *Then this density verify the following equation:*

$$\frac{\partial}{\partial t} p_t(y) = \frac{1}{2} c_0 \frac{\partial^2}{\partial y^2}(y^2\, p_t(y)) - \frac{\partial}{\partial y}((a_0 y + 1)\, p_t(y)) + \int_{\mathbb{R}} \left[e^x p_t(y e^x) - p_t(y) + (e^{-x} - 1) \frac{\partial}{\partial y}(y p_t(y)) \right] K_0(dx) \tag{29}$$

In particular case, when $I_\infty < \infty$ *(P-a.s.) and the density* p_∞ *of the law of* I_∞ *exists and belongs to the class* $\mathcal{C}^2(\mathbb{R}^{+,*})$, *we have*

$$\frac{1}{2} c_0 \frac{d^2}{dy^2}(y^2\, p_\infty(y)) - \frac{d}{dy}((a_0 y + 1)\, p_\infty(y)) + \int_{\mathbb{R}} \left[e^x p_\infty(y e^x) - p_\infty(y) + (e^{-x} - 1) \frac{d}{dy}(y p_\infty(y)) \right] K_0(dx) = 0 \tag{30}$$

Proof. In the case of Lévy processes we write that (P-a.s.)

$$V_s = e^{-Y_s}\, J_s = e^{-X_t + X_{(t-s)-}} \int_0^s e^{X_t - X_{(t-u)-}} du =$$

$$\int_0^s e^{X_{(t-s)-} - X_{(t-u)-}} du = \int_0^s e^{X_{(t-s)} - X_{(t-u)}} du.$$

Due to the homogeneity of the Lévy processes we have the following identity in law:

$$\mathcal{L}((X_{t-u} - X_{t-s})_{0\le s\le t}) = \mathcal{L}((X_{s-u})_{0\le s\le t})$$

Then,

$$\mathcal{L}(\int_0^s e^{X_{(t-s)}-X_{(t-u)}}du) = \mathcal{L}(\int_0^s e^{-X_{(s-u)}}du) = \mathcal{L}(\int_0^s e^{-X_u}du)$$

where the last equality is obtained by time change. As a conclusion, $\mathcal{L}(V_s) = \mathcal{L}(I_s)$ for $0 \le s \le t$, and, hence, $(p_s)_{0<s\le t}$ are the densities of the laws of $(I_s)_{0<s\le t}$. In addition, again due to the homogeneity, for all $0 \le s \le t$, $\bar{b}_s = b_{t-s} = b_0$, $\bar{c}_s = c_{t-s} = c_0$, $\bar{K}_s(dx) = K_{t-s}(dx) = K_0(dx)$. Then, from the Theorem 1 we obtain (29).

Again due to the homogeneity, for $0 < s \le t$, the generator $\mathcal{A}_s^V(f) = \mathcal{A}(f)$ where

$$\mathcal{A}(f)(y) = (1 + y\,a_0)\,f'(y) +$$
$$\frac{1}{2}c_0\,f''(y)\,y^2 + \int_{\mathbb{R}} \left[f(ye^{-x}) - f(y) - f'(y)y(e^{-x}-1)\right] K_0(dx)$$

and it does not depend on s. Moreover, $\mathcal{L}((V_s)_{0\le s\le t}) = \mathcal{L}((I_s)_{0\le s\le t})$ and the equality (12) become

$$\mathbf{E}f(I_s) = \int_0^s \mathbf{E}\mathcal{A}(f)(I_u)du.$$

We divide the both sides of the above equality by s and we let s go to infinity. Since f is bounded, we get zero as a limit on the l.h.s.. Since $I_t \overset{P-p.s.}{\longrightarrow} I_\infty$, we also get for $f \in \mathcal{C}$

$$\lim_{s\to\infty} \mathbf{E}\mathcal{A}(f)(I_s) = \mathbf{E}\mathcal{A}(f)(I_\infty).$$

Then, $\mathbf{E}\mathcal{A}(f)(I_\infty) = 0$ and we obtain (30) in the same way as in Theorem 1, by the integration by parts and the time change. □

Remark 2. Similar equation for the density of I_∞ in the case when $\int_{\mathbb{R}}(|x| \wedge 1)\,K_0(dx) < \infty$ was obtained in [12]. We recall that the condition on K_0 of [12] is stronger at zero then our condition. It should be mentioned that the authors of [12] did not obtained the equation for the density of I_t.

Corollary 2. *Under the conditions of Proposition 3, the distribution function F_t of I_t verify second order integro-differential equation*

$$\frac{\partial}{\partial t}F_t(y) = \frac{1}{2}c_0\frac{\partial}{\partial y}(y^2\,\frac{\partial}{\partial y}F_t(y)) - (a_0y + 1)\,\frac{\partial}{\partial y}F_t(y) + \tag{31}$$

$$\int_{\mathbb{R}} \left[F_t(ye^x) - F_t(y) + (e^{-x} - 1)\, y\, \frac{\partial}{\partial y} F_t(y)) \right] K_0(dx)$$

with following boundary conditions:

$$F_t(0) = 0, \quad \lim_{y \to +\infty} F_t(y) = 1.$$

When $I_\infty < \infty$, the similar equation is valid for the distribution function F_∞ of the law of I_∞:

$$\frac{1}{2} c_0 \frac{d}{dy}(y^2\, F'_\infty(y)) - (a_0 y + 1)\, F'_\infty(y) + \tag{32}$$

$$\int_{\mathbb{R}} \left[F_\infty(ye^x) - F_\infty(y) + (e^{-x} - 1)\, y\, F'_\infty(y)) \right] K_0(dx) = 0$$

with similar boundary conditions:

$$F_\infty(0) = 0, \quad \lim_{y \to +\infty} F_\infty(y) = 1.$$

Proof. We integrate each term of the equation of Theorem 1 on $[0, y]$ and we use the fact that

$$\int_0^y p_t(u) du = F_t(y)$$

since $F_t(0) = 0$. We take in account the fact that that the map $(t, u) \to p_t(u)$ is of class $C^{1.2}(\mathbb{R}^{+,*} \times \mathbb{R}^{+,*})$ to exchange the integration and the derivation. The same we do for $F_\infty(y)$. □

Corollary 3. *(cf.* [14], [11]*) Let us consider Brownian motion with drift, i.e.*

$$dX_t = b_0 dt + \sqrt{c_0} dW_t$$

where $c_0 \neq 0$ and $b_0 > 0$. Then the law of exponential functional I_t associated with X has a density which verify :

$$\frac{\partial}{\partial t} p_t(y) = \frac{1}{2} c_0 \frac{\partial^2}{\partial y^2}(y^2\, p_t(y)) - \frac{\partial}{\partial y}((a_0 y + 1)\, p_t(y))$$

In particular, for I_∞ we get that

$$p_\infty(x) = \frac{1}{\Gamma(\frac{2\, b_0}{c_0})\, x} \left(\frac{2}{c_0 x} \right)^{\frac{2 b_0}{c_0}} \exp\left(-\frac{2}{c_0 x} \right)$$

Proof. From the Proposition 3 we find the equation for p_t. From Corollary 1 we get the equation for F_∞ :

$$\frac{1}{2} c_0 \frac{d}{dy}(y^2\, F'_\infty(y)) - (a_0 y + 1)\, F'_\infty(y) = 0$$

This equation is equivalent to

$$\frac{1}{2}c_0 y^2\, F''_\infty(y) - ((a_0 - c_0)y + 1)\, F'_\infty(y) = 0$$

By the reduction of the order of the equation, we find that

$$F'_\infty(y) = C\, y^{2(\frac{a_0}{c_0}-1)}\, \exp\left(-\frac{2}{c_0 y}\right)$$

with some positive constant C. Using boundary conditions we calculate a constant C. We get that $C = \frac{1}{\Gamma\left(1-\frac{2a_0}{c_0}\right)}\left(\frac{c_0}{2}\right)^{2\frac{a_0}{c_0}-1}$ where $\Gamma(\cdot)$ is gamma function. Since $1 - \frac{2a_0}{c_0} = \frac{2b_0}{c_0}$, this gives us the final result. □

Remark 3. We recall that some rather complicated formulas related to the distribution of I_t for Brownian motion with drift was given in [14] and in [11], formula 1.10.4, p. 264. Recently in [10] we obtained the Laplace transform for the density of the exponential integral functional of the Brownian motion with drift

$$\hat{p}_\lambda(y) = \int_0^{+\infty} e^{-\lambda s} p_s(y) ds$$

where $\lambda > 0$. More precisely

$$\begin{aligned}\hat{p}_\lambda(y) &= \frac{1}{\lambda}\left(y\frac{c_0}{2}\right)^{-k} \frac{\Gamma\left(1 - \frac{2b_0}{c_0} + k\right)}{\Gamma\left(1 - \frac{2b_0}{c_0} + 2k\right)} \left\{ \frac{k}{y^{k+1}}\, M\left(k, 1 - \frac{2b_0}{c_0} + 2k, -\frac{2}{yc_0}\right)\right. \\ &\left. -\frac{2k}{c_0\, y^{k+2}(1 - \frac{2b_0}{c_0} + 2k)}\, M\left(k+1, 2 - \frac{2b_0}{c_0} + 2k, -\frac{2}{yc_0}\right)\right\},\end{aligned}$$

where $k = \frac{b_0 + \sqrt{b_0^2 + 2\lambda c_0}}{c_0}$ and M is the confluent hypergeometric function of the first kind known as Kummer's function. The Laplace transform in this case can be inverted in the usual way by the Bromwich-Mellin formula

$$p_s(y) = \int_{\lambda_0 - i\infty}^{\lambda_0 + \infty} e^{\lambda s}\, \hat{p}_\lambda(y) d\lambda$$

for any fixed $\lambda_0 > 0$ and $s > 0$.

Let us denote by ν^+ and ν^- the Levy measure of the positive and the negative jumps respectively, namely for $x > 0$

$$\nu^+([x, +\infty[) = \int_x^{+\infty} K_0(du), \;\; \nu^-(]-\infty, -x]) = \int_{-\infty}^{-x} K_0(du)$$

To simplify the notations we put also

$$\nu^+(x) = \nu^+([x, +\infty[),\ \nu^-(x) = \nu^-(]-\infty, -x])$$

Let us suppose in addition that

$$\int_{\mathbb{R}} |x|\, K_0(dx) < \infty.$$

Corollary 4. *Suppose that X is a Levy process with integrable jumps. Then, under the conditions of Proposition 3, the density p_t of I_t, verify :*

$$\frac{\partial}{\partial t}p_t(y) = \frac{1}{2}c_0\frac{\partial^2}{\partial y^2}(y^2\, p_t(y)) - \frac{\partial}{\partial y}((r_0 y + 1)\, p_t(y)) +$$

$$\int_y^{+\infty} p_t(z)\nu^+(\ln(\frac{z}{y}))\, dz + \int_0^y p_t(z)\nu^-(-\ln(\frac{z}{y}))\, dz$$

where $r_0 = a_0 - \int_{\mathbb{R}}(e^{-x} - 1)\, K_0(dx) = -b_0 + \frac{1}{2}c_0 + \int_{\mathbb{R}} x\, K_0(dx)$.

Proof. We take the equation (31) and we rewrite it in the following form:

$$\frac{\partial}{\partial t}F_t(y) = \frac{1}{2}c_0\frac{\partial}{\partial y}(y^2\, \frac{\partial}{\partial y}F_t(y)) - (r_0 y + 1)\, \frac{\partial}{\partial y}F_t(y) +$$

$$\int_{\mathbb{R}} [F_t(ye^x) - F_t(y)\,]\, K_0(dx)$$

Then we divide the integral over $\mathbb{R}$ in two parts integrating on $]0, +\infty[$ and $]-\infty, 0[$. We do the integration by parts :

$$\int_{\mathbb{R}} [F_t(ye^x) - F_t(y)] K_0(dx) =$$

$$\int_0^{+\infty} \frac{\partial}{\partial x}F_t(ye^x)\, ye^x\, \nu^+(x)dx + \int_{-\infty}^0 \frac{\partial}{\partial x}F_t\,(ye^x)\, ye^x\, \nu^-(x)dx$$

and we change the variables $z = ye^x$. We differentiate the result w.r.t. t and this gives us the claim. □

Corollary 5. *Suppose that for $x \in \mathbb{R}$*

$$K_0(x) = e^{-\mu x} I_{\{x>0\}}$$

Then, under the assumptions of Proposition 3, the density p_t of I_t, verify :

$$\frac{\partial}{\partial t}p_t(y) = \frac{1}{2}c_0\frac{\partial^2}{\partial y^2}(y^2\, p_t(y)) - \frac{\partial}{\partial y}((r_0 y + 1)\, p_t(y)) + \frac{y^\mu}{\mu}\int_y^\infty \frac{p_t(z)}{z^\mu}\, dz$$

In particular, for the density p_∞ of I_∞ we have :

$$\frac{1}{2}c_0\frac{\partial^2}{\partial y^2}(y^2\,p_t(y)) - \frac{\partial}{\partial y}((r_0 y+1)\,p_t(y)) + \frac{y^\mu}{\mu}\int_y^\infty \frac{p_t(z)}{z^\mu}\,dz = 0$$

Proof. We take in account that $\nu^+(x) = \frac{1}{\mu}e^{-\mu x}$ and $\nu^-(x) = 0$ for all $x > 0$ and this gives us the equation for p_t in this particular case. □

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