Teaching numeration units: why, how and limits?

Catherine Houdement, Frédérick Tempier

TEACHING NUMERATION UNITS: WHY, HOW AND LIMITS

Catherine Houdement[(1)]*, Frédérick Tempier*[(2)]
[(1)]*LDAR – Universités Paris Diderot et Rouen,*[(2)]*LDAR – Université Poitiers, France*

Abstract

In French teaching practices there is currently a lack of consideration for the decimal (base ten) principle of numeration system for whole numbers. This is the reason why we implemented two experiments to strengthen this principle, giving a key role to the use of numeration units (ones, tens, hundreds …). The first one was led in the context of designing a resource for teachers in grade 3, the second one involved training teachers in grades 1 and 2. We analyze how students and teachers take into account numeration units and put them in relation with standard representations of numbers. Both exemplify the complexity of the teaching-learning process of numeration.

Key words: Base-ten, Place value, Numeration units, Whole numbers, Teaching.

Introduction

As many scholars (among them Bednarz and Janvier, 1988; Kamii and Joseph 2004) we conceptualize the numeration as a network of skills about counting, grouping, representing the quantity in various ways and understanding the meaning of place-value. The place-value numeration system for whole numbers is based on two inseparable principles (Ross, 1989):

- The position of each digit in a written number corresponds to a unit (for example hundreds stand in the third place): this is the "positional principle";
- Each unit is equal to ten units of the immediately lower order (for example one hundred = ten tens): this is the "decimal principle".

Understanding this numeration system (and its relationship with oral numeration) is part of the school curriculum in all countries. Detailing what such understanding means and organizing it at primary school levels is probably less uniform: for instance Ma (1999) stresses the different ways used by US and Chinese teachers respectively to quote the decimal principle about the subtraction algorithm; in France (grades 1 to 5) the current curriculum (2008) does not refer to *place-value principles* before grade 3, and with no further details (Houdement and Chambris, 2013).

Supported with our study of the curriculum as well as some textbooks and some teachers' practices, we hypothesize an illusion of transparency of base-ten number concepts in the current French teaching (Tempier 2013). Of course it deals with position and associated value of a digit in the written number and the terms *ones, tens, hundreds* … - what we name *numeration units* (**NU**) as Chambris (2008) - are visible, principally used as names of position of each digit (positional principle). An indication of this illusion of transparency is the low percentage of success of 104 French 3-graders (8-9 year-old) in tasks involving relations between units (Tempier, 2013): "1 hundred = … tens" (48% success), "60 tens = … hundreds" (31% success) and "in 764 ones there are ... tens" (39% success). Yet, ones, tens, hundreds... are organized as a system of units: there are units of all orders and two units are always in a 10^n-to-1 ratio. The numeration units system is a named-values system (Fuson and Briars, 1990) which explains the positional base-ten system of whole numbers. This system has vector space properties (computation on numeration units) and allows different ways of writing: 5 ones 6 tens 3 hundreds or 2 hundreds 16 tens 5 units, with a number of units bigger than nine, which gives it great instrumental potential.

We are convinced of the interest of finely connecting three representation systems of numbers (Van de Walle, 2010): the two standard ways, *i.e.* the written numbers (**WN** as 56) and the spoken numbers (**SN** as fifty-six), and a third one, the in-numeration-units numbers (**NUN**) which are written as they are spoken. Teaching specifically and gradually the third system (NUN) in relation to the two others might facilitate understanding (1) of base-ten-place-value system (Tempier, 2013), (2) of computation algorithms (Ma, 1999), (3) of the decimal form of rational numbers, especially of decimal fractions. (4) It would resonate with the teaching of measurement units: length units, mass units ... (Chambris, 2008). And (5) this system provides an alternative to saying a WN (72 : 7 tens, 2 ones) without using the SN system which often doesn't reflect the way the numbers are written in Western languages (72 = soixante-douze in French, *i.e.* sixty-twelve). (6) It can also help to bridge WN and SN (65 = 6 tens 5 ones = sixty five).

The questions addressed in this paper are: how do teachers incorporate experiments supported with this proposal into their practice? What can be perceived of the acceptances or resistances of students and teachers?

Materials and methods

Two experiments supported with problems made us progress on these points: the first one in the context of a PhD thesis (Tempier, 2013), studying the use of a resource by four teachers in a design-based research (with a methodology of didactical engineering for the development of a resource) at grade 3 (8-9 year-old); the second one in a context of teacher education about six teachers of grades 1 and 2 (6-8 year-old) in the same school in a difficult area.

In both experiments we choose to introduce numeration units early in the mathematical organization of the year, so that they are available for working on computational techniques. We rely on two types of problems:

> Type A. Write in WN a quantity from a collection of objects (or from a NUN): we call these problems "counting problems";
>
> Type B. And the inverse problem that consists in producing a collection of objects (or a NUN) from a WN: we call these "ordering problems" (in reference to the ordering of a collection by a "shopkeeper").

In grades 1 & 2 the collection of objects is still present (either initially given or to build), whereas in grade 3, the work consists mainly in translating NUN in WN and *vice versa* (collection possibly used for validation). The collection consists of wooden sticks, a groupable ten base model (Van de Walle, 2010): this manipulative provides access to a first meaning of the NU, possibilities for organizing a collection that fits the various ways of writing with numeration-units. We use these possibilities of organization in the problems given to students.

Experiment 1

Here are three problems of the resource proposed to the teachers:

	Problem A1	**Problem A2**	**Problem B3**
Type of problem	Counting a collection totally organized (in groups of tens, hundreds...)	Counting a collection partially organized (NUN→WN) resulting of the union of two collections.	Ordering a collection (WN→NUN) taking into account constraints (*e.g.* "there are no more thousands of sticks")
Mathema-tical issues	Explaining the position principle	Converting units into higher order units	Converting units into lower order units
Examples	The 3 thousands are written in the fourth position of WN: 3024.	3 thousands 12 hundreds 1 ten 5 ones = 4 thousands 2 hundreds 1 ten 5 ones (because 10 h = 1 th)	2615 = 2 thousands 6 hundreds 1 ten 5 ones = 26 hundreds 1 ten 5 ones (because 1 th = 10 h).

Tab. 1: Three problems of the resource

We indicate thereafter the various uses of the numeration units in the implementations of the resource by the teachers.

Experiment 2

In experiment 2, the eleven teachers of the whole school reported difficulties in teaching and learning place-value system (for whole number in grade 1 and 2, for decimals in grades 3, 4 and 5 and for numbers beyond 1000) and the feeling of a lack of coherence in the practices. In the continuity of experiment 1, one of the authors developed a brief training in numeration units as an essential element for understanding written numbers and spoken numbers, supported by manipulative: wooden sticks but also students' fingers. Teachers agreed with the idea of types of problems from grade 1 to grade 5, with a game on the variables (size of numbers, organizing collections and various writings used). We will only talk about grades 1 and 2 (5 teachers) which have been more thoroughly surveyed.

Teachers implemented lessons where students were to produce a number of fingers or of a given collection of wooden sticks (less or more organized) in WN or NUN form (type A problem), or a collection of fingers or sticks corresponding to a WN or a NUN (type B problem).

Results

Experiment 1

In implementing Problem A1 (Tab. 1), all the teachers involved use the NU for highlighting the link between group and position in WN. The NU are mainly used to designate these positions; this is done in the "place value chart" with NU written on the top line. In implementing Problems A2 and B3 (Tab. 1), we could observe three types of resistances to an appropriate use of NU by teachers: avoidance of the use of units to describe the material groups, predominant reference to materials when conversions are involved and failure to use NU for writing conversions.

The first resistance was observed in only one class, Mrs. A.'s. Despite the proposed resources, the teacher only used the expressions "boxes", "bags", etc. to speak of the groups. Yet, she always used the NU to describe the positions in WN. See below how she came back to the addition carrying in the A2 problem.

> Mrs. A: eight plus four, what is… what is happening here in the hundreds?
> A student: twelve
> Mrs. A: we find our twelve bags but what is happening?
> A student: we carry 1
> Mrs. A *finishes writing the algorithm of addition on the blackboard:*
>
> | *Th* | *H* | *T* | *O* |
> |---|---|---|---|
> | *1* | | | |
> | *1* | *4* | *2* | *4* |
> | *+ 1* | *8* | *1* | *0* |
> | *3* | *2* | *3* | *4* |
>
> Mrs. A: well, that's why we carry 1, because we will keep only two bags for hundreds and here's ten that will make us a box of thousand more. [...] We now understand better the carrying: this is our small box of thousand more.

In Mrs. A.'s class, the NU are only used to label the positions of WN but not for material groups, which prevents the students from making sense of NU as quantities.

In the other classes, the NU are used to describe material groups, as proposed in the resource. However when there are more than ten units at an order, the teacher refers systematically to material groups (real or drawn). For example, in the implementation of Problem 2 (Mrs. B's class), a student (Marc) is asked to draw the union of two collections on the blackboard (a box for thousand, a bag for hundred…). Faced with Marc's difficulties with 12 bags, Mrs. B asks another student (Joris) to help him:

> Joris: in fact when you have twelve, you have more hundreds [...] If you got ten hundreds, what does it correspond to?
> Mrs. B: what can you do? If you got ten bags what can you do?
> Marc: ah yes, a thousand. *The student draws a new box on the blackboard.*
> Mrs. B: The ten [hundreds] you'll put them in a box. [...] When we have ten bags we can put them in a box. [...] So as soon as it exceeds ten we can put them in a thousand box.

While students use NU orally, the teacher reformulates by reducing to material issues (put ten bags in a box). This effective reference to the material is never questioned even if it becomes an obstacle. Mrs. C explains her confusion after a lesson during which students were asked to order 8004 sticks (problem B3) from a "shopkeeper" who didn't get thousands. She feels helpless with the difficulties encountered by students because she has no alternative but to use materials: " I had no way to help them because I have not eighty hundreds." The use of materials seems the only recourse available for teachers to explain and justify transformations of different stick organisations.

In the four classes is observed a third type of resistance, the failure to use the NU for writing conversions, even if NU are orally used. For example Mrs. C never writes a conversion (such as 12 hundreds = 1 thousand 2 hundreds), except basic conversions as 1 Th = 10 H, even if conversions are mentioned. At the blackboard Mrs. B reasons on the drawings of the groups without ever writing conversions. The previous excerpt shows it: she doesn't transpose the idea suggested by Joris into writing: 12 hundreds = 1 thousand 2 hundreds. When another student explains what he did to find the total number of sticks ("seven hundred and three hundred it's doing ten"), Mrs. B does not write the associated conversion (7H + 3H = 10H = 1Th) but continues to refer to materials ("I transformed my bags, I put it in a box because I have ten"). It prevents students from taking over the writing task of conversions between units and so from strengthening their constructive abstraction (Chandler and Kamii, 2009).

Experiment 2

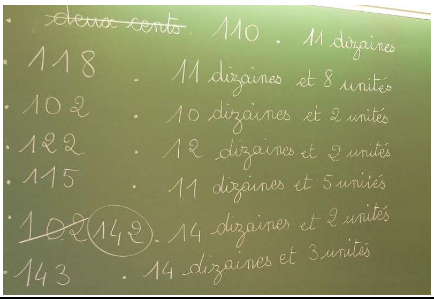
Consider two examples of students' and teacher' relation to numeration units.

In grade 2 (7-8 year-old), to count a collection (type A problem) the teacher gets her students to organizing collections in groups of ten and to using numeration-units to name bundles of sticks (tens, hundreds) and the ten fingers of a child (a ten). The teachers ask for a response without specifying its form, then the students give a standard spoken number resulting of a relatively easy counting by tens and ones (ten, twenty, thirty, thirty one, thirty two). In Tab. 2, first line, the two students of the group first counted ten, twenty…, ninety, a hundred, *two hundreds* and after the teacher noticed their error they announced to their peers *one hundred and ten*. Then the WN is collectively deduced from SN by touching the number strip and simultaneously counting one by one on the number strip, starting from a known number name:

95	96	97	98	99	100	101	102

Fig. 1: An extract of the usual number strip

When asked to count the tens (1, 2, 3 tens and 2 ones), it is more difficult, as was expected: students often waver for example between 110 tens and 11 tens on the second line. This is a sign of the difficulty of the twofold point of view on the NU: 1 ten must be understandable both as a multiplicity (ten ones) and a whole (one ten) (“composite unit of ten”, Steffe, 2004; Kamii and Joseph, 2004; Houdement and Chambris, 2013*)*.



~~Two hundreds~~ 110 11 tens
118 11 tens and 8 ones
102 10 tens and 2 ones
122.........12 tens and 2 ones

Fig. 2: The table of responses (grade 2, January)

With the help of the teacher the lines are progressively written on the blackboard. Sometimes the WN is obtained first, sometimes it is the NUN, as a description of a totally organized collection. For these students there are two ways to represent collections that are not yet articulated: the WN is deduced from the written translation of the SN by using the number strip, it is never deduced from the NUN by students, or even asked for by the teacher under this form.

Despite the training the teacher may think that the WN can only be deduced from the SN. Yet spoken numbers are rather part of an ordinal logic that consists in moving forward in the counting song, number-name by number-name, relying on specific numbers (ten, twenty, thirty…named *beacon numbers* by Mounier, 2010). This logic can be an obstacle in the cardinal logic in which a hundred is considered as ten tens, not only as a successor of ninety-nine, or ten more than ninety (“Numerical Composite of Ten” *versus* “Composite Unit of Ten”, Steffe, 2004).

In grade 1 (6-7-year-old), the teacher writes 47 and asks students to try to show 47 fingers, using a minimum of children with fingers up (type B problem).

The group is presenting their response. One student of the group remained in the back of the room.

The peers are protesting and saying: it does not work: *there are thirty four fingers* or *there are thirty seven fingers*! Probably the girl on the right tries to show the 4 of 47 (beside the 7). Some students suggest the girl raise ten fingers.

Fig. 3: Students producing a fingers collection for 47 (grade 1, January)

The students globally accept this suggestion (change four fingers up into ten fingers up): after an effective control by counting the problem is solved.

On the other hand when the teacher asks for the number of fingers up on the picture (Problem A2), nobody notices that there are 41 fingers up. Students seem to think that isolated ones (4 and 7) by different students cannot be put together to compose 11. Even if this work requires some flexibility −*i.e.* students being able to conceive simultaneously a ten and 10 units− this example shows how young students can rigidify their vision of quantity depending on the material: it is difficult for them to see a fourth ten in 4 fingers here and 7 fingers there (Chandler and Kamii, 2009).

Discussion and conclusion

Both experiments highlight the complexity of the teaching-learning process of the numeration. Grade 3 students seem quite ready to use numeration-units to refer to the collections; it is the teacher who checks this use by bringing them back to a description language of the materials in boxes, bundles.... Their use of the numeration units remains mostly oral or if written associated to the place-value chart. In this way they prevent students from conceptualising the relations between numeration units that are yet the goal of the session. To avoid the learning of two types of words, for materials and for numeration units (bundles and tens), we decided to introduce in grades 1-2 only the word ten (resp. hundred) to describe a bundle of ten (resp. of hundred). Teachers and a large part of students appropriate orally numeration units (tens, tens of fingers, tens of sticks) more easily than grade 3 students, and the same goes for the simple relations: 1 T=10 O, 1 H=10 T... But grade-2 students cannot yet deduce the relation between NUN and WN, despite juxtaposing the two writings on the blackboard (Fig. 2). Although informed, the teachers are surprised that students cannot deduce NUN from WN or *vice versa*, while the relation seems evident to them. But for these young students the only link between the two representations (NUN and WN) is the material and the different ways to organize it. Grade-1 students notice the presence of tens in a collection, first using the counting song ten-by-ten (ten, twenty, thirty: then 3 tens), then counting directly *the* tens. But they can also rigidify the representation of a number as full tens and isolated ones and no longer see 1 ten of fingers included in 7 and 4 fingers. This lack of cognitive flexibility is known by scholars (Chandler and Kamii, 2009), but is always difficult for teachers to understand.

Globally the in-service teachers are not fitted to manage this complexity:
- Due to a lack of institutional indicators: in French curriculum the decimal principle is only referred to as “grouping and exchanging tasks”“. The interest of numeration-units as a representation halfway between materials and WN remains hidden, the conversions seeming to be limited to manipulative.
- Due to a strong belief in some manipulative containing and directly displaying mathematical structures. Reference to the material often becomes the favourite way for teachers to help students not to forget the quantity associated with each numeration unit.
- Due to a lack of Mathematical Knowledge for Teaching (Ma, 1999; Ball et al., 2008) about the WN sense (the role of numeration units) and the differences

between WN and SN (particularly the obstacle that SN can represent for the understanding of WN).

This study might possibly indicate a need for further teacher education to enrich the teaching of WN, about the necessity (and the ways) of teaching various number representations (among them NUN) and their mutual links. We would be interested in knowing how other countries deal with this challenge.

References

Ball, D. L., Thames, M.H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal for Teacher Education*, 59, 389-407.

Bednarz, N., & Janvier, B. (1988). A constructivist approach to numeration in primary school: results of a 3-year intervention with the same group of children. *Educational Studies in Mathematics*, 19, 299-331.

Chambris, C. (2008). *Relations entre les grandeurs et les nombres dans les mathématiques de l'école primaire. Évolution de l'enseignement au 20e siècle. Connaissances des élèves actuels.* Thesis. Paris: Université Paris Diderot.

Chandler, C. C., & Kamii, C. (2009). Giving change when payment is made with a dime: The difficulty of tens and ones. *Journal for Research in Mathematics Education*, 40, 97-118.

Fuson, K.C., & Briars, D.J. (1990) Using a Base-Ten Blocks Learning/Teaching Approach for first and second grade Place-Value and Multidigit Addition and Subtraction. *Journal for Research in Mathematics Education,* 21, 180-206.

Houdement, C., & Chambris, C. (2013). Why and how to introduce numbers units in 1st and 2nd grades. *Proceedings of the 8th Conference of Europ. Research in Math.Education*. Turkey.

Kamii, C., & Joseph, L. L. (2004). *Young children continue to reinvent arithmetic-- 2nd grade: Implications of Piaget's theory*. Teachers College Press.

Ma, L. (1999). *Knowing and Teaching Elementary Mathematics*. Mahwah (NJ): Lawrence Erlbaum Associates.

Mounier, E. (2010). *Une analyse de la numération au CP. Vers de nouvelles pistes*. Thesis. Paris: Université Paris Diderot.

Ross, S. H. (1989). Parts, wholes and place value: A developmental view. *Arithmetic Teacher,* 36, 47-5l.

Steffe, L. P. (2004). PSSM from a constructivist perspective. In Clements, D.H., & Sarama, J. (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 221-251). Mahwah, NJ: Erlbaum.

Tempier, F. (2013). *La numération décimale de position à l'école primaire. Une ingénierie didactique pour le développement d'une ressource*. Thesis. Paris: Université Paris Diderot.

Van de Walle, J.A. (2010). *Elementary and middle school mathematics: Teaching developmentally* (Seventh edition). Boston: Pearson Education.