Partition games are pure breaking games

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Abstract

Taking-and-breaking games are combinatorial games played on heaps of tokens, where both players are allowed to remove tokens from a heap and/or split a heap into smaller heaps. Subtraction games, octal and hexadecimal games are well-known families of such games. We here consider the set of *pure breaking* games, that correspond to the family of taking-and-breaking games where splitting heaps only is allowed. The rules of such games are simply given by a list L of positive integers corresponding to the number of sub-heaps that a heap must be split into. Following the case of octal and hexadecimal games, we provide a computational testing condition to prove that the Grundy sequence of a given pure breaking game is arithmetic periodic. In addition, the behavior of the Grundy sequence is explicitly given for several particular values of L (e.g. when $1 \notin L$ or when L contains only odd values). However, despite the simplicity of its ruleset, the behavior of the Grundy function of the game having $L = \{1, 2\}$ is open.

1 Introduction and context

Integer partition theory, related to Ferrer diagrams and Young tableaus, is a classical subject in number theory and combinatorics, dating back to giants such as Lagrange, Goldbach and Euler; it concerns the number of ways you can write a given positive integer as a sum of specified parts. In most generality, to each positive integer n, there belongs a number $p(n)$, which counts the unrestricted number of ways this can be done. For example $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$, so $p(4) = 5$. We may index this partition number by saying exactly how many parts is required, and write $p_k(n)$ for the number of partitions of n in exactly k parts. Thus, in our example, $p_2(4) = 2$ and $p_3(4) = 1$. We could also define $p_{k,\ell} = p_k + p_\ell$ and so on. The number of partitions can be beautifully expressed via generating functions, where recurrence formulas, congruence relations, and several asymptotic estimates are known, proved more recently by famous number theorists such as Ramanujan, Hardy, Rademacher and Erdös in the early 1900s. About the same time, a theory of combinatorial games was emerging, via contributions by Bouton, Sprague and Grundy and others, seemingly unrelated to the full blossom of number theory.

An *integer partition game* can be defined by 2 players alternating turns and by specifying the legal partitions, say into exactly 2 or 3 parts, until the current player cannot find a legal partition of parts, and loses. Thus, from position 4, then $3 + 1, 2 + 2, 2 + 1 + 1$ are the legal move options—if you play to $2 + 2$ you win, and otherwise not. It turns out that the idea for how to win such games is coded in a 'game function', discovered independently by the mathematicians Sprague and Grundy, which, buy the way, has no apparent relation to the partitioning function. For example, the partition functions are nondecreasing, but if a Sprague-Grundy function is nondecreasing the game is usually rather trivial, such as the game of Nim on one heap. Let us begin by giving the relevant game theory background to our results, that most of the partition games, a.k.a. *pure breaking games*, are either periodic or arithmetic periodic.

1.1 Taking-and-breaking games: definitions and notations

Taking-and-breaking games [3] are 2-player impartial combinatorial games with alternating play. A game position is represented by a set of heaps of tokens. A move consists in choosing a single heap, removing

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some tokens from it, and possibly splitting the remaining heap into several heaps. If splitting is not allowed, we have (pure) *subtraction games*. In this case, the rules are given by a set S of positive integers which specifies the number of tokens that can be removed from the heap. When the heap may be split, the rulesets are often given by a code that specifies how many tokens can be removed and the number of heaps that one heap can be split into. For example, the family of games for which a heap can be split into at most two heaps is called *octal games*. This name is due to an explicit way to express any ruleset with an octal code $\mathbf{d_0.d_1 \dots d_k}$ with d_i an integer, $0 \leq d_i \leq 7$ for $1 \leq i \leq k$. More precisely, each value d_i with $i > 0$ can be encoded in binary with three digits $a_{i,2}a_{i,1}a_{i,0}$. The ruleset allows to remove i tokens from a heap and split it into j non-empty heaps if and only if $a_{i,j} = 1$. The value d_0 equals 0 or 4 according to whether it is allowed (value 4) or not (value 0) to split a heap without removing any token. The family of *hexadecimal games* is a natural extension of octal games, in which a heap can be split into at most three heaps. Variants of octal games where the ruleset also allows to split a heap without removing any token have also been considered in the literature, starting from Grundy's Game in 1939 [7].

The purpose of the current work is to extend such rulesets to allow a heap to be split into a selected number of heaps.

We first recall standard definitions in combinatorial game theory and use the notations introduced in [10] for taking-and-breaking games. In particular, a heap of size n will be denoted H_n. When the ruleset is clear, we associate the game played on the heap of size n with the positive integer n. A game with k heaps of respective sizes $a_1, \dots, a_k$ is considered as a *disjunctive sum* of heaps and will be denoted by a k-tuple $(a_1, \dots, a_k)$. An *option* of a game is a game that can be reached in one move.

The *Grundy value* of a game n, denoted by $\mathcal{G}(n)$, is a nonnegative integer given by

$$\mathcal{G}(n) = \text{mex}\{\mathcal{G}(O_i) \mid O_i \text{ is an option of } n\}$$

where mex(U) is the smallest nonnegative integer that does not belong to the set U. In the rest of the paper, an option of n over $\ell + 1$ non-empty heaps will be denoted $O_n = (i_0, \dots, i_\ell)$. The Grundy value of a game allows to determine the winner. Indeed, a game satisfies $\mathcal{G}(n) = 0$ if and only if it is a second player win.

From the Sprague-Grundy theory, one can compute the Grundy value of a k-heap game from the Grundy value of each 1-heap game. More precisely, we have

$$\mathcal{G}((a_1, \dots, a_k)) = \mathcal{G}(a_1) \oplus \dots \oplus \mathcal{G}(a_k)$$

where $\oplus$ is called the *Nim-sum* operator and corresponds to the XOR applied to integers written in binary. The Nim-sum of the same k terms i will be denoted $k \otimes i$. By definition of the XOR operator, it equals i or 0 according to the parity of k.

1.2 Regularities in taking-and-breaking games

Given a taking-and-breaking game, its $\mathcal{G}$-sequence is the sequence $\mathcal{G}(1), \mathcal{G}(2), \mathcal{G}(3), \dots$. Finding regularities in $\mathcal{G}$-sequences is a natural objective as it may lead to polynomial-time algorithms that compute the $\mathcal{G}$-values of the game. In particular, periodic behaviors are often observed. A game is said to be *ultimately periodic* with period p and preperiod n_0 if there exist n_0 and p such that $\mathcal{G}(n + p) = \mathcal{G}(n)$ for all $n \geq n_0$. *Periodic* games are those for which there is no preperiod.

For example, it is well known (see [10], Theorem 2.4) that all finite subtraction games are periodic. For octal games, the behavior of the $\mathcal{G}$-sequences is not fully understood. It has been conjectured by Guy that every octal game is ultimately periodic. Many games were proved to satisfy this conjecture, such as **0.106**, **0.165** or **0.454**. In some cases, the values of the period and the preperiod are huge (e.g. **0.454** has a period of 60620715 and a preperiod of 160949019). On his webpage [5], Flammenkamp maintains a list of octal games with known and unknown periodicities.

As explained in [9], some hexadecimal games also satisfy these properties of normal periodicity (e.g. **0.B3**, **0.33F**). In addition, another types of behavior have been exhibited for hexadecimal games, namely *arithmetic periodicity*. A taking-and-breaking game is said to be *arithmetic periodic* with period p, saltus s, and preperiod n_0 if there exist three integers n_0, p and s such that its $\mathcal{G}$-sequence satisfies $\mathcal{G}(n + p) = \mathcal{G}(n) + s$ for all $n \geq n_0$. This kind of behavior never occurs in octal games [2] but makes sense in the context of hexadecimal games where the Grundy values may not be bounded. For example, the games **0.13FF** or **0.9B** are proved to be arithmetic periodic with period 7 and saltus 4 in [9]. Note that normal and arithmetic periodicities are not the only kinds of regularities that have been detected in hexadecimal games. In [1], the game **0.205200C** is said to be *sapp-regular*, which means that the $\mathcal{G}$-sequence is an interlacing of two periodic subsequences with an arithmetic periodic one. This behavior also occurs in variants of octal games where pass moves are allowed [8]. In [9], the game **0.123456789** satisfies $\mathcal{G}(2m-1) = \mathcal{G}(2m) = m-1$, except $\mathcal{G}(2^k+6) = 2^k-1$. In [6], Grossman and Nowakowski introduce the notion of *ruler regularity* that arises in the games $\mathbf{0.20\ldots48}$ with an odd number of 0s in the hexadecimal code. Roughly speaking, it corresponds to a kind of arithmetic periodic sequence where new terms are regularly introduced that double the length of the apparent period.

For a better understanding of taking-and-breaking games, the question of how to detect a possible regularity using just a small number of computations is paramount. For example, the Subtraction Periodicity Theorem, found in the Chapter 4 of [10], ensures that for a given subtraction game on the set S, it suffices to find a repetition of $max(S)$ consecutive Grundy values, to establish ultimate periodicity. Concerning octal games, there is a similar result that has been extensively used to prove the ultimate periodicity of some $\mathcal{G}$-sequences.

Theorem 1. *[Octal periodicity test] Let G be an octal game $\mathbf{d_0.d_1\ldots d_k}$ of finite length k. If there exist $n_0 \geq 1$ and $p \geq 1$ such that*

$$\mathcal{G}(n+p) = \mathcal{G}(n) \ \forall n_0 \leq n < 2n_0 + p + k,$$

then G is ultimately periodic with period p and preperiod n_0.

Such kind of testing properties have also been considered for hexadecimal games. In [2], Austin yields a first set of conditions to guarantee the arithmetic periodicity of hexadecimal games with a saltus equal to a power of 2. A complementary result was later given by Howse and Nowakowski [9] for hexadecimal games having an arbitrary saltus. In both cases, several types of computations must be done. In particular, the arithmetic periodicity must be checked on a range of values much larger than in Theorem 1 (at least seven times the expected period).

1.3 Pure breaking Games

In view of the above results, there is a large gap between the understanding of octal and hexadecimal games. It turns out that allowing a heap to be split into three parts may significantly change the possible behaviors of the $\mathcal{G}$-sequences. In the current paper, we explore how the $\mathcal{G}$-sequences behave when increasing the number of possible splits of a heap. As one expects that the complexity of this generalization also increases accordingly, we have chosen to focus on breaking games only, i.e., games where it is not allowed to remove tokens from a heap. Grundy's game [7, 5] and Couples-are-Forever [4] are two well-known examples of such games. In the first one, a move consists in choosing a heap and splitting it into two heaps of different sizes. The latter one allows to split any heap of size at least three into two heaps. For both games, no regularity in the $\mathcal{G}$-sequence has been observed yet. In the current study, we will consider *pure* breaking games, i.e., games for which there is no additional constraint to the number of splits. The rulesets of such games will thus be given by a set of integers corresponding to the number of heaps that one heap can be split into.

Definition 2. *Let $L = \{\ell_1, \ldots, \ell_k\}$ be a set of positive integers, called the* cut numbers. *We define the* pure breaking game $\mathrm{PB}(L)$ *as the heap game such that n has the following options*

$$\{(i_0, \ldots, i_\ell) \mid \ell \in L, i_j > 0 \,\forall j \quad and \ i_0 + \ldots + i_\ell = n \ \}$$

In other words, in $\mathrm{PB}(L)$, a move consists in choosing a heap and splitting it into $k+1$ non-empty heaps with $k \in L$. Such a move will be called a *k-cut.* For example, for the game $\mathrm{PB}(L)$ with $L = \{1, 3\}$, the heap H_5 has the following set of options:

$$\{(1,4),(2,3),(1,1,1,2)\}$$

Without loss of generality, we will assume that each set L is ordered such that $\ell_1 < \ldots < \ell_k$. In this paper, we will consider instances of $\mathrm{PB}(L)$ for different sets L and examine their $\mathcal{G}$-sequence. A first result ensures that an equivalent of the octal game conjecture is not available for pure breaking games. Indeed, the following lemma establishes that if an even cut number belongs to L, the Grundy values are not bounded.

Lemma 3. *Let* $\mathrm{PB}(L)$ *be a pure breaking game, where* L *contains at least one even integer. Let* m *be the smallest even integer in* L.

For every pair $(x_1, x_2), x_1 \neq x_2$ *such that* $\mathcal{G}(x_1) = \mathcal{G}(x_2)$*, we have* $x_1 \not\equiv x_2 \bmod m$.

Proof. We reason by contradiction. Let x_1 and x_2 be two different integers such that $\mathcal{G}(x_1) = \mathcal{G}(x_2)$ and $x_1 \equiv x_2 \bmod m$. We have $x_1 = a_1 m + b$ and $x_2 = a_2 m + b$ for some $0 \leq b \leq m-1$. We assume without loss of generality that $x_1 > x_2$.

From a heap of x_1 counters, one can play to the option $O_{x_1} = (x_2, a_1 - a_2, \ldots, a_1 - a_2)$ obtained by an m-cut. Since m is even, $\mathcal{G}(O_{x_1}) = \mathcal{G}(x_2)$, and thus $\mathcal{G}(x_1) \neq \mathcal{G}(x_2)$, which contradicts our hypothesis. □

This result means that pure breaking games are somehow closer to hexadecimal games than octal games. One can then wonder whether the complexity of the $\mathcal{G}$-sequence increases with $\max(L)$. It does not seem to be the case, as we will show that in almost all cases, the $\mathcal{G}$-sequence is either periodic or arithmetic periodic. In Section 2, we consider several families of pure breaking games (e.g. those where $1 \notin L$, or those with only odd values in L) and prove their periodicity or arithmetic periodicity. For the remaining families, many games seem to have an arithmetic periodic behavior. To deal with them, we provide in Section 3 a set of testing conditions that are sufficient to show that a game is arithmetic periodic, and apply them to particular instances. Finally, in Section 4 we list the remaining sets L for which the regularity of the $\mathcal{G}$-sequence of $\mathrm{PB}(L)$ remains open.

2 Solving particular families of pure breaking games

In this section, we study specific families of pure breaking games. All the following results will be proved by contradiction. In each case, we will suppose that there exists an integer n for which the Grundy value is different from what was expected. By decomposing n into specific options, we will exhibit a contradiction. All the families will be proved to have arithmetic periodic sequences. We are going to use the following notation: $(m_1, \ldots, m_p)$ $(+s)$, which describes the arithmetic periodic sequence of period p and saltus s for which the first p values are $m_1, \ldots, m_p$. If a subsequence $(m_i, \ldots, m_j)$ is repeated q times, we will write $(m_i, \ldots, m_j)^q$. Thus, for example, the notation $(0,1,2)^2$ $(+3)$ denotes the arithmetic periodic sequence of period 6, saltus 3, and with first six values 0,1,2,0,1,2. We also use the notation $[\![a, b]\!]$ (with $a \leq b$) to describe the set of all the integers from a to b.

First, we study the games in which 1 is not an allowed cut number. In this case, optimal play is reduced to using only ℓ_1, and the Grundy sequence is arithmetic periodic with period ℓ_1 and saltus 1.

Proposition 4. *Let* $L = \{\ell_1, \ldots, \ell_k\}$ *be a set of cut numbers such that* $\ell_1 \geq 2$*. Then,* $\mathrm{PB}(L)$ *has a Grundy sequence of* $(0)^{\ell_1}$ $(+1)$.

Proof. We prove this result by contradiction. If n is a positive integer, then there exists a unique couple of nonnegative integers (a, b) such that : $0 \leq b \leq \ell_1 - 1$ and $n = a\ell_1 + b + 1$. We want to prove that for every positive integer n, $\mathcal{G}(n) = a$.
Assume that n is the smallest positive integer such that $\mathcal{G}(n) \neq a$.

Let $m \in L$. Suppose $\mathcal{G}(n) > a$. Then there exists $O_n = (a_0\ell_1 + b_0 + 1, \ldots, a_m\ell_1 + b_m + 1)$ an m-cut of n such that $\mathcal{G}(O_n) = a$. By minimality of n, $\mathcal{G}(O_n) = a_0 \oplus \ldots \oplus a_m = a$. Moreover, since O_n is an option of n, we have:

$$\sum_{i=0}^{m} (a_i\ell_1 + b_i + 1) = a\ell_1 + b + 1.$$

In particular, as $b < \ell_1$ we have $\sum_{i=0}^{m} a_i \leq a$. However, since $a = \bigoplus_{i=0}^{m} a_i \leq \sum_{i=0}^{m} a_i$ we have $\sum_{i=0}^{m} a_i = a$.

This implies that $1 + b = \sum_{i=0}^{m} (1 + b_i) = m + 1 + \sum_{i=0}^{m} b_i$.

This is a contradiction since $m \geq \ell_1$ which implies $b \geq \ell_1$.

Thus, there is no option of n with Grundy value a, hence $\mathcal{G}(n) < a$.

Now we prove that the heap of size n has options of Grundy values i for $i \in [\![0, a-1]\!]$. There are two cases:

1. If ℓ_1 is even, then for $i \in [\![0, a-1]\!]$, let $O_n = (i\ell_1 + b + 1, a-i, \ldots, a-i)$ be an ℓ_1-cut. This always exists since $\ell_1 \geq 2$. Moreover it is an option of n: $i\ell_1 + b + 1 + (a-i)\ell_1 = a\ell_1 + b + 1 = n$. Furthermore, we have $\mathcal{G}(O_n) = \mathcal{G}(i\ell_1+b+1) \oplus (\ell_1 \otimes \mathcal{G}(a-i))$. Since $\mathcal{G}(i\ell_1+b+1) = i$ by minimality of n, and ℓ_1 is even which implies $(\ell_1 \otimes \mathcal{G}(a-i)) = 0$, we have $\mathcal{G}(O_n) = i$.

2. Otherwise, for all $i \in [\![0, a-1]\!]$, we define an option O_n of n, obtained by an ℓ_1-cut, such that $\mathcal{G}(O_n) = i$. We have two subcases:

   2.1 If $a-i$ is odd, let

$$\begin{aligned} h_0 &= i\ell_1 + b + 1 \\ h_j &= \frac{1}{2}(a-i-1)\ell_1 + 1 \qquad \text{for } j = 1, 2 \\ h_j &= 1 \qquad \text{for } 3 \leq j \leq \ell_1 \end{aligned}$$

   This always exists since $\ell_1 \geq 3$ (if $\ell_1 = 3$ then there are only the first four heaps) and $(a-i-1)$ is even. Moreover, it is an option of n:

$$i\ell_1+b+1+2\left(\frac{1}{2}(a-i-1)\ell_1+1\right)+(\ell_1-2) = i\ell_1+b+1+(a-i-1)\ell_1+\ell_1 = a\ell_1+b+1 = n$$

   Furthermore, we have

$$\mathcal{G}(O_n) = \mathcal{G}(i\ell_1+b+1) \oplus \left(2 \otimes \mathcal{G}\left(\frac{1}{2}(a-i-1)\ell_1+1\right)\right) \oplus ((\ell_1-2) \otimes \mathcal{G}(1)) = i$$

   since $\mathcal{G}(i\ell_1+b+1) = i$ by minimality of n and $\mathcal{G}(1) = 0$.

   2.2 If $a-i$ is even, let

$$\begin{aligned} h_0 &= i\ell_1 + b + 1 \\ h_j &= \frac{1}{2}((a-i-1)\ell_1 + 1) \qquad \text{for } j = 1, 2 \\ h_j &= 2 \qquad \text{for } j = 3 \\ h_j &= 1 \qquad \text{for } 4 \leq j \leq \ell_1 \end{aligned}$$

   This always exists since $\ell_1 \geq 3$ (if $\ell_1 = 3$ then there are only the first four heaps) and $(a-i-1)$ and ℓ_1 are odd so $(a-i-1)\ell_1+1$ is even. Moreover, it is an option of n:

$$i\ell_1+b+1+2\cdot\frac{1}{2}((a-i-1)\ell_1+1)+2+(\ell_1-3) = i\ell_1+b+1+(a-i-1)\ell_1+\ell_1 = a\ell_1+b+1 = n$$

   Furthermore, we have

$$\mathcal{G}(O_n) = \mathcal{G}(i\ell_1+b+1) \oplus \left(2 \otimes \mathcal{G}\left(\frac{1}{2}((a-i-1)\ell_1+1)\right)\right) \oplus \mathcal{G}(2) \oplus ((\ell_1-3) \otimes \mathcal{G}(1)) = i$$

   since $\mathcal{G}(i\ell_1+b+1) = i$ by minimality of n and $\mathcal{G}(1) = \mathcal{G}(2) = 0$.

This proves that we have at least an option with Grundy value i for all $0 \leq i < a$, and thus that $\mathcal{G}(n) \geq a$, a contradiction.

Consequently, there is no counterexample to the sequence $(0)^{\ell_1}\ (+1)$. □

Now, we consider the pure breaking games in which the players are allowed to split a heap into two heaps. We first show that if L contains only odd cut numbers, then the Grundy sequence of $\mathrm{PB}(L)$ is periodic with period 2.

Proposition 5. *Let $L = \{1, \ell_2, \dots, \ell_k\}$ be a sequence of odd cut numbers. The game $\mathrm{PB}(L)$ has a Grundy sequence of $(0, 1)$ $(+0)$.*

Proof. We prove this result by contradiction. Let n be the smallest positive integer for which the Grundy value of a heap of size n does not match with the sequence $(0, 1)^1$ $(+0)$.

First assume that n is even. We will prove that all the options of n have Grundy value 0. Let O_n be an option of n. Note that O_n exists since $n \geq 2$ and $1 \in L$. Since all the values of L are odd, O_n contains an even number of non empty heaps whose sum is even. Hence O_n contains an even number of odd-sized heaps. Since all the heaps in O_n are strictly smaller than n, their Grundy values satisfy the sequence $(0, 1)^1$ $(+0)$, which implies that O_n contains an even number of heaps of Grundy value 1. Therefore, we have $\mathcal{G}(O_n) = 0$ and thus $\mathcal{G}(n) = 1$. Hence our counterexample n is necessarily odd.

We will show that n has no option of Grundy value 0. It is straightforward if n has no option. Otherwise, let O_n be an option of n. Since all the values of L are odd, O_n contains an even number of non empty heaps whose sum is odd. Hence O_n contains an odd number of odd-sized heaps and an odd number of even-sized heaps. Since all the heaps in O_n are strictly smaller than n, their Grundy values satisfy the sequence $(0, 1)^1$ $(+0)$, which implies that O_n contains an odd number of heaps of Grundy value 1. Hence $\mathcal{G}(O_n) = 1$ and thus $\mathcal{G}(n) = 0$.

Consequently, there is no counterexample to the sequence $(0, 1)^1$ $(+0)$. ☐

Next, we study the pure breaking games in which the players can split a heap into two, three or four heaps. In this case, even if the players are allowed to split a heap into more than four heaps, then the Grundy sequence is arithmetic periodic with period 1 and saltus 1.

Proposition 6. *Let $k \geq 3$ and $L = \{1, 2, 3, \ell_4, \dots, \ell_k\}$ be a sequence of cut numbers. The game $\mathrm{PB}(L)$ has a Grundy sequence of (0) $(+1)$.*

Proof. We prove this result by contradiction. Let n be the smallest positive integer such that $\mathcal{G}(n) \neq n-1$. Note that $n \geq 3$ since we have $\mathcal{G}(1) = 0$ and $\mathcal{G}(2) = 1$.

Suppose first that $\mathcal{G}(n) > n - 1$. Then n has an option $O_n = (h_0, \dots, h_\ell)$ such that:

$$\sum_{i=0}^{\ell} h_i = n \quad \text{and} \quad \bigoplus_{i=0}^{\ell} \mathcal{G}(h_i) = \bigoplus_{i=0}^{\ell} (h_i - 1) = n - 1.$$

However, $\sum_{i=0}^{\ell} (h_i - 1) = n - \ell - 1$, and since $\ell \geq 1$ we have

$$\mathcal{G}(O_n) = n - 1 > \sum_{i=0}^{\ell} (h_i - 1) \geq \bigoplus_{i=0}^{\ell} (h_i - 1) = \mathcal{G}(O_n),$$

a contradiction.

Thus, there is no option of n with Grundy value $n - 1$, which implies $\mathcal{G}(n) < n - 1$.

We now prove that, from a heap of n counters, we can play to an option of Grundy value m for all $m < n - 1$, which will lead to a contradiction.

If $m = n - 2$, then let $O_n = (1, n - 1)$ which is clearly an option of n with Grundy value $n - 2$ by minimality of n. Otherwise, let $m < n - 2$. There are two cases:

1. If n is even, then there are two subcases:

   1.1 If m is odd, $m \in [\![1, n - 3]\!]$, let

   $$O_n = (m + 1, \frac{n - 1 - m}{2}, \frac{n - 1 - m}{2})$$

   obtained by a 2-cut. It is an option of n and by minimality of n, $\mathcal{G}(O_n) = \mathcal{G}(m + 1) = m$.

1.2 If m is even, $m \in [\![0, n-4]\!]$, let:

$$O_n = (m+1, 1, \frac{n-m-2}{2}, \frac{n-m-2}{2})$$

obtained by a 3-cut. It is an option of n and by minimality of n, $\mathcal{G}(O_n) = \mathcal{G}(m+1) = m$.

2. If n is odd, then there are two subcases:

2.1 If m is odd, $m \in [\![1, n-4]\!]$, let:

$$O_n = (m+1, 1, \frac{n-m-2}{2}, \frac{n-m-2}{2})$$

obtained by a 3-cut. It is an option of n and by minimality of n, $\mathcal{G}(O_n) = \mathcal{G}(m+1) = m$.

2.2 If m is even, $m \in [\![0, n-3]\!]$, let:

$$O_n = (m+1, \frac{n-1-m}{2}, \frac{n-1-m}{2})$$

obtained by a 2-cut. It is an option of n and by minimality of n, $\mathcal{G}(O_n) = \mathcal{G}(m+1) = m$.

Thus, for both cases, $\mathcal{G}(n) \geq \text{mex}(\{0, ..., n-2\}) = n-1$, a contradiction.

Consequently, there is no counterexample to the sequence $(0)^1$ $(+1)$. □

Finally, we study the pure breaking games where the players can split a heap into 2, 4 or $2k+1$ heaps. In this case, the Grundy sequence is arithmetic periodic with period $2k$ and saltus 2. Note that this result includes the Grundy sequence of $\text{PB}(1, 2, 3)$.

Proposition 7. *Let $k \geq 1$ and $L = \{1, 3, 2k\}$ be a sequence of positive integers. Then,* $\text{PB}(L)$ *has a Grundy sequence of* $(0, 1)^k$ $(+2)$.

Proof. We want to prove that for all $n = 2ka + b + 1 \geq 1$, $\mathcal{G}(n) = 2a + (b \bmod 2)$. We are going to proceed by contradiction. Let $n = 2ka + b + 1$, $0 \leq b < 2k$, be the smallest positive integer such that $\mathcal{G}(n) \neq 2a + (b \bmod 2)$. Note that $n \geq 3$ since we have $\mathcal{G}(1) = 0$ and $\mathcal{G}(2) = 1$.

Assume first that $\mathcal{G}(n) > 2a + (b \bmod 2)$. Then n has an option $O_n = (2ka_0 + b_0 + 1, \ldots, 2ka_m + b_m + 1)$ with $m \in L$ such that $\mathcal{G}(O_n) = 2a + (b \bmod 2)$.

As O_n is an option of n with Grundy value $2a + (b \bmod 2)$ and n is minimal, we have, on one hand :

$$\mathcal{G}(O_n) = \bigoplus_{i=0}^{m} (2a_i + (b_i \bmod 2)) = 2 \bigoplus_{i=0}^{m} a_i + \bigoplus_{i=0}^{m} (b_i \bmod 2) = 2a + (b \bmod 2).$$

The second equality holds since 2 is a power of two and for all i, $(b_i \bmod 2) < 2$.
On the other hand we have:

$$n = \sum_{i=0}^{m} (2ka_i + b_i + 1) = 2k \sum_{i=0}^{m} a_i + \sum_{i=0}^{m} b_i + m + 1 = 2ka + b + 1.$$

Since a is the quotient of $n-1$ by $2k$, we have that $a_0 + \ldots + a_m \leq a$, and since $a_0 + \ldots + a_m \geq a_0 \oplus \ldots \oplus a_m$, we have $a = a_0 \oplus \ldots \oplus a_m = a_0 + \cdots + a_m$.

In particular $\sum_{i=0}^{m} b_i + m + 1 = b + 1$. Here we have two cases:

1. If $m = 2k$, then we have $b \geq m = 2k$, a contradiction.

2. If $m \in \{1, 3\}$, then we have:

$$b \bmod 2 = \bigoplus_{i=0}^{m} (b_i \bmod 2) = \left(\bigoplus_{i=0}^{m} b_i \right) \bmod 2 = \left(\sum_{i=0}^{m} b_i \right) \bmod 2 = \left(\sum_{i=0}^{m} b_i + m + 1 \right) \bmod 2 = (b+1) \bmod 2$$

(the third equality holds by Lemma 10, the fourth one since m is odd), a contradiction.

Thus, there are no options of n with Grundy value $2a+(b \bmod 2)$, which implies $\mathcal{G}(n) < 2a+(b \bmod 2)$.

We now prove that, from a heap of n counters, we can play to an option of Grundy value g for any g in $[\![0, 2a + (b \bmod 2) - 1]\!]$, which will lead to a contradiction. There are two cases:

1. If b is even, then $2a + (b \bmod 2) = 2a$ and from a heap of size n we can play to:

   1.1 for all $x \in [\![0, a-1]\!]$, the options:

   $$O_n = (2kx + b + 1, a - x, \ldots, a - x)$$

   obtained by a $2k$-cut. By minimality of n, $\mathcal{G}(O_n) = 2x$. By doing this, we obtain the even Grundy values in $[\![0, 2a-2]\!]$.

   1.2 if $b = 0$, for all $x \in [\![1, a-1]\!]$, the options:

   $$O_n = (2kx + b, 1, (a-x)k, (a-x)k)$$

   obtained by a 3-cut. By minimality of n, $\mathcal{G}(O_n) = 2(x-1) + (2k - 1 \bmod 2) = 2x - 1$ since $x \geq 1$. By doing this, we obtain the odd Grundy values in $[\![1, 2a-3]\!]$ and the value $2a-1$ is obtained by the option $O_n = (2ka, 1)$.

   1.3 if $b > 0$, for all $x \in [\![0, a-1]\!]$, the options:

   $$O_n = (2kx + b, 1, (a-x)k, (a-x)k)$$

   obtained by a 3-cut. By minimality of n, $\mathcal{G}(O_n) = 2x + (b-1 \bmod 2) = 2x+1$ since b is even. By doing this, we obtain the odd Grundy values in $[\![1, 2a-1]\!]$.

   Putting the three previous cases altogether, this implies $\mathcal{G}(n) \geq 2a$, being a contradiction.

2. If b is odd, then $2a + (b \bmod 2) = 2a + 1$, and from a heap of size n we can play to:

   2.1 for all $x \in [\![0, a-1]\!]$, the options:

   $$O_n = (2kx + b + 1, a - x, \ldots, a - x)$$

   obtained by a $2k$-cut. By minimality of n, $\mathcal{G}(O_n) = 2x + 1$. By doing this, we obtain the odd Grundy values in $[\![1, 2a-1]\!]$.

   2.2 for all $x \in [\![0, a-1]\!]$, the options:

   $$O_n = (2kx + b, 1, (a-x)k, (a-x)k)$$

   obtained by a 3-cut. By minimality of n, $\mathcal{G}(O_n) = 2x$. By doing this, we obtain the even Grundy values in $[\![0, 2a-2]\!]$ and the value $2a$ is obtained by the option $O_n = (2ka + b, 1)$.

   Altogether, this implies $\mathcal{G}(n) \geq 2a + 1$, a contradiction.

Consequently, there is no counterexample to the sequence $(0,1)^k$ $(+2)$. □

Note that when $k = 1$, the previous result gives the same result than Proposition 6 when $k = 3$ (and as such, $L = \{1, 2, 3\}$).

If the above results cover a large range of pure breaking games, there remain several families of games for which we were not able to have direct proofs. Yet, many of them seem to have an arithmetic periodic behavior. The next section is devoted to build a set of tests that would allow to prove (with a restricted number of computations) that a given game is arithmetic periodic. We then use this test to prove that some games have an arithmetic periodic sequence.

3 An arithmetic periodicity test for pure breaking games

The purpose of this section is to provide, for pure breaking games, a result similar to the octal and hexadecimal periodicity tests (see Theorem 1 for the first one, and see [9] for the latter one). We give an explicit way to prove that a pure breaking game is arithmetic periodic by computing as few values as we can. Recall that for octal games, the number of computations to prove the periodicity is in the range of twice the period, whilst it takes at least 7 times the period to prove the arithmetic periodicity of hexadecimal games (together with a couple of additional tests). In section 3.1, we prove that computing at most the first $4p$ values of the $\mathcal{G}$-sequence (where p is the expected period, which should be determined by a blind computation) is enough to prove arithmetic periodicity. We will also show that in some cases (depending on L), the first $3p$ values are even sufficient (section 3.2).

3.1 The AP-test

In this section, we describe the so-called *AP-test* that will be used to prove the arithmetic periodicity of a pure breaking game. First recall that if f is a function defined over an interval I, then f restricted to $J \subseteq I$ is noted $f|_J$; and the set of the images of f is $\mathrm{Im}(f) = \{f(x) \mid x \in I\}$. We now define the AP-test as follows:

Definition 8 (Arithmetic-Periodic Test (AP-test)). *Let* $\mathrm{PB}(L)$ *be a pure breaking game and denote by* $\mathcal{G}$ *its Grundy function. We say that* $\mathrm{PB}(L)$ *satisfies the* AP-test *if there exist a positive integer p and a power of two s such that:*

AP1. *for $n \leq 3p$, $\mathcal{G}(n+p) = \mathcal{G}(n) + s$,*

AP2. $\mathrm{Im}(\mathcal{G}|_{[\![1,p]\!]}) = [\![0, s-1]\!]$, *and*

AP3. *for all n in $[\![3p+1, 4p]\!]$ and for all g in $[\![0, s-1]\!]$, H_n admits an option O_n over $(m+1)$ non-empty heaps such that $m \geq 2, m \in L$ and $\mathcal{G}(O_n) = g$.*

The first two conditions are rather standard to prove the periodicity of taking-and-breaking games: similar conditions are required in the Subtraction Periodicity Theorem and in the Octal Games Periodicity Theorem. However, contrary to those, we need the saltus to be a power of two in order to prove the arithmeric periodicity. The third condition seems more unusual. We will see in the next subsection that for some values of L, the third condition $AP3$ can be directly deduced from $AP1$ and $AP2$ and does not need to be checked. We now state the main result of this section:

Theorem 9. *Let $L = \{\ell_1, \ldots, \ell_k\}$ be a set of positive integers, with $\ell_k \geq 2$ and such that* $\mathrm{PB}(L)$ *verifies the test AP. Then for all $n \geq 1$, $\mathcal{G}(n+p) = \mathcal{G}(n) + s$.*

In other words, if a pure breaking game verifies the AP-test, then it is arithmetic periodic. Note that in the AP-test, the saltus of the sequence is always a power of 2.

In order to prove this result, we need some technical lemmas. The first one is a well-known result that claims that the Nim-sum and the sum of the same set of positive integers have the same parity and that the Nim-sum cannot be greater than the sum.

Lemma 10. *Let $a_0, \ldots, a_m$ be $m+1$ positive integers. We have*

$$a_0 \oplus a_1 \oplus \cdots \oplus a_m \equiv (a_0 + \ldots + a_m) \bmod 2$$

and

$$a_0 + \cdots + a_m \geq a_0 \oplus \cdots \oplus a_m.$$

Proof. Let $a_0, \ldots, a_m$ be $m+1$ positive integers. Without loss of generality, we can assume that for some i, $a_0, a_1, \ldots, a_i$ are all odd and $a_{i+1}, a_{i+2}, \ldots, a_m$ are all even.
If i is odd, then there is an even number of odd integers, and their Nim-sum and their sum are even. If i is even, then there is an odd number of odd integers, their Nim-sum and their sum are then odd.

Now, let $N = a_0 \oplus \cdots \oplus a_m$, $S = a_0 + \cdots + a_m$ and $p = \lceil \log_2(S) \rceil$. There are, for $0 \leq i \leq p$, non-negative integers $b_{i,n}$ and $b_{i,j}$ such that $N = b_{0,n}2^0 + \cdots + b_{p,n}2^p$ and for all j, $a_j = b_{0,j}2^0 + \cdots + b_{p,j}2^p$. If $b_{i,n} = 1$ then there is at least one j such that $b_{i,j} = 1$, hence in the sum there is a term on 2^i. This being true for all $b_{i,n}$, the sum is such that $S \geq b_{0,n}2^0 + \cdots + b_{p,n}2^p = N$. □

In what follows, we will frequently make use of the fact that for every pair of positive integers p and n, there exists a unique couple (a,b) such that $n = 1+b+ap$ and $b < p$. The next result gives the closed formula corresponding to an arithmetic periodic behavior of the $\mathcal{G}$-sequence.

Lemma 11. *Let $L = \{\ell_1, \dots, \ell_k\}$ be a set of positive integers. In the game* $\mathrm{PB}(L)$*, if there exist two positive integers p and s, and $n_0 \geq p$ such that for all $n \leq n_0$, $\mathcal{G}(n+p) = \mathcal{G}(n) + s$ then for all $1 \leq n = ap+1+b \leq n_0 + p$ with $0 \leq b < p$, we have*

$$\mathcal{G}(n) = as + \mathcal{G}(1+b)$$

Proof. It is clear that for all $1 \leq n \leq p$, we have $n = ap+1+b$ with $a = 0$ and $0 \leq b < p$, and hence $\mathcal{G}(n) = as + \mathcal{G}(1+b)$.
Let $n = ap+b+1 \leq n_0+p$ be the smallest integer such that $\mathcal{G}(n) \neq as + \mathcal{G}(1+b)$. From the previous remark, we know that $n > p$. The Grundy value of n is:

$$\mathcal{G}(n) = \mathcal{G}(n-p) + s = \mathcal{G}((a-1)p+1+b) + s,$$

remark this equality holds since $n \leq n_0 + p$.
Since n is minimal and $(a-1)p+1+b < n$, we have $\mathcal{G}((a-1)p+1+b) = (a-1)s + \mathcal{G}(1+b)$, and thus

$$\mathcal{G}(n) = as + \mathcal{G}(1+b),$$

which contradicts our initial hypothesis. □

As a direct consequence, if Lemma 11 is satisfied with the two additional constraints:

- s is a power of 2
- $\mathcal{G}(n) < s$ for all $1 \leq n \leq p$,

then any disjunctive sum $G = (a_0p+1+b_0, \dots, a_mp+1+b_m)$ with $a_jp+1+b_j \leq n_0+p$ and $0 \leq b_j < p$ for all $0 \leq j \leq m$ satisfies

$$\mathcal{G}(G) = (a_0 \oplus \dots \oplus a_m)s + (\mathcal{G}(1+b_0) \oplus \dots \oplus \mathcal{G}(1+b_m)) \tag{1}$$

Theorem 9 will be proved by induction, with a rather technical base case. We consider a part of this base case in the following lemma to make the general proof more readable. Moreover, this lemma exposes why the condition $\ell_k \geq 2$ is necessary.

Lemma 12. *Let $L = \{\ell_1, \dots, \ell_k\}$ be a set of positive integers with $\ell_k \geq 2$ such that* $\mathrm{PB}(L)$ *verifies the test AP.*
Then for $i = 2, 3$, for all n in $[\![ip+1, (i+1)p]\!]$ and for all g in $[\![0, (i-1)s-1]\!]$, there is an option $O_n = (h_0, \dots, h_m), m \in L$ of n such that $m \geq 2$ and $\mathcal{G}(O_n) = g$.

Proof. Let $L = \{\ell_1, \dots, \ell_k\}$ be such a set.

- We first consider the case $i = 3$. Let $n = 3p+1+b \in [\![3p+1, 4p]\!]$ and $g \in [\![0, 2s-1]\!]$.
  If $g \in [\![0, s-1]\!]$ then condition $AP3$ ensures such an option exists.
  Now, for $g \in [\![s, 2s-1]\!]$, by the conditions $AP1$ and $AP2$, Lemma 11 can be applied, implying that $\mathcal{G}(n) = 3s + \mathcal{G}(1+b)$ and hence that there is an option O_n of n such that $\mathcal{G}(O_n) = g$. If $1 \notin L$, there is nothing to prove. Consequently, it suffices to prove that if $1 \in L$, and $O_n = (h_0, h_1)$ is an option of n obtained by a 1-cut, then $\mathcal{G}(O_n) \notin [\![s, 2s-1]\!]$. This result would indeed guarantee that all the options of n with Grundy value in $[\![s, 2s-1]\!]$ are obtained by m-cuts with $m \geq 2$.
  Assume $1 \in L$ and let $O_n = (h_0, h_1)$ be an option of n obtained by a 1-cut. There exist four unique nonnegative integers a_0, b_0, a_1, b_1 such that $0 \leq b_0, b_1 < p$ and $O_n = (a_0p+1+b_0, a_1p+1+b_1)$. As O_n is an option of n we have:

  $$(a_0+a_1)p + 1 + 1 + b_0 + b_1 = n = 3p + 1 + b$$

  which gives

  $$1 + b_0 + b_1 - b = (3 - a_0 - a_1)p.$$

  As $0 \leq a_0 + a_1 \leq 3$ and $b_0 + b_1 + 1 < 2p$, we have in one hand $0 \leq 1 + b_0 + b_1 - b < 2p$ and in the other hand that $1 + b_0 + b_1 - b \equiv 0 \pmod{p}$. Hence $1 + b_0 + b_1 - b \in \{0, p\}$. If it equals 0 then $a_0 + a_1 = 3$, otherwise $a_0 + a_1 = 2$. Without loss of generality the possible values for a_0, a_1 and $a_0 \oplus a_1$ are summarized in the following table:

a_0	a_1	$a_0 \oplus a_1$
0	2	2
	3	3
1	1	0
	2	3

In particular, we remark that $a_0 \oplus a_1 \neq 1$. And, by Property (1) we have : $\mathcal{G}(O_n) = (a_0 \oplus a_1)s + \mathcal{G}(1 + b_0) \oplus \mathcal{G}(1 + b_1) \notin [\![s, 2s - 1]\!]$ since s is a power of two and $\mathcal{G}(1 + b_0), \mathcal{G}(1 + b_1) < s$.

- We now consider the case $i = 2$. Let $n \in [\![2p + 1, 3p]\!]$ and $g \in [\![0, s - 1]\!]$.
  Let $n' = n + p \in [\![3p + 1, 4p]\!]$ and $g' = g + s \in [\![s, 2s - 1]\!]$.
  By the first part of the proof, we know that there is an option $O_{n'} = (a_{0,n'}p+1+b_{0,n'}, \dots, a_{m,n'}p+1+b_{m,n'})$ of n' such that $m \geq 2$ and $\mathcal{G}(O_{n'}) = g'$. Let $N = (a_{0,n'} \oplus \cdots \oplus a_{m,n'})$, $S = a_{0,n'} + \cdots + a_{m,n'}$ and $R = \mathcal{G}(1 + b_{0,n'}) \oplus \cdots \oplus \mathcal{G}(1 + b_{m,n'})$. Remark that $N = 1$ and $\mathcal{G}(O_{n'}) = Ns + R$ since we can apply Property (1) to $O_{n'}$ and $g' \in [\![s, 2s - 1]\!]$. We define the following m-cut option O_n of n by:

$$\begin{aligned} h_0 &= 1 + b_{0,n'} \\ h_j &= \frac{1}{2}(S-1)p + 1 + b_{j,n'} && \text{for } j = 1, 2 \\ h_j &= 1 + b_{j,n'} && \text{for } 3 \leq j \leq m \end{aligned}$$

  Remark that $S - 1 = S - N$ which is even and non-negative by Lemma 10.
  Note that O_n is indeed an option of n since we have that $h_0 + \cdots + h_m = (S - 1)p + (1 + b_{0,n'} + \cdots + 1 + b_{m,n'}) = n' - p = n$. By Property (1), we have $\mathcal{G}(O_n) = R = g' - s = g$. Hence, O_n is indeed an option of n with $m \geq 2$ and $\mathcal{G}(O_n) = g$.

□

We can now prove Theorem 9, meaning that if a pure breaking game verifies the AP-test, then its Grundy sequence is arithmetic periodic.

Proof of Theorem 9. Let us begin with some notations.
For all $1 \leq n \leq p$ we denote $r_n = \mathcal{G}(n)$; thus for $0 \leq a < 4$ and $n = ap + b + 1 \in [\![ap + 1, (a + 1)p]\!]$, and by Lemma 11 we have $\mathcal{G}(n) = \mathcal{G}(ap + b + 1) = as + r_{b+1}$. We recall that by Lemma 10, for a family of non-negative integers $a_0, \dots, a_m$, if $S = a_0 + \cdots + a_m$ and $N = a_0 \oplus \cdots \oplus a_m$ then $S \geq N$ and $S \equiv N \bmod 2$. In particular, $S - N$ is an even non-negative integer.

We will now prove by induction that for $n = ap + 1 + b \geq 1$, the following two properties hold:

(A) $\mathcal{G}(n) = as + r_{1+b}$ and

(B) for all $g \in [\![0, (a - 1)s - 1]\!]$, there is an option $O_n = (h_0, \dots, h_m)$ of n such that $m \geq 2$ and $\mathcal{G}(O_n) = g$.

Let $n = ap + 1 + b$ be the smallest positive integer such that either (A) or (B) is not verified. By Lemma 11, we know that (A) holds for all $n \leq 4p$. Moreover, by Lemma 12, we know that (B) holds for $a = 2, 3$, and it is trivially true for $a \leq 1$. Thus $n > 4p$.
Let $n = ap + 1 + b > 4p$. We consider two cases:

1. Assume (A) is not verified. Thus either $\mathcal{G}(n) < as + r_{1+b}$ or $\mathcal{G}(n) > as + r_{1+b}$.

   1.1 if $\mathcal{G}(n) < as + r_{1+b}$: by minimality of n, the heap of size $n' = n - 2p = a'p + 1 + b'$ verifies conditions (A) and (B). Let $O_{n'} = (a_{0,n'}p + 1 + b_{0,n'}, \dots, a_{m,n'}p + 1 + b_{m,n'})$ be an option of n' with Grundy value g, for some $g < (a' - 1)s$ and $m \geq 2$. Let $N = a_{0,n'} \oplus \cdots \oplus a_{m,n'}$,

$S = a_{0,n'} + \cdots + a_{m,n'}$ and $R = \mathcal{G}(1 + b_{0,n'}) \oplus \cdots \oplus \mathcal{G}(1 + b_{m,n'})$. Let O_n be the following option:

$$\begin{aligned} h_0 &= Np + 1 + b_{0,n'} \\ h_j &= \frac{1}{2}(S - N + 2)p + 1 + b_{j,n'} & \text{for } j = 1, 2 \\ h_j &= 1 + b_{m,n'} & \text{for } j > 2 \end{aligned}$$

This is an option of n since $h_0 + \cdots + h_m = (2 + S)p + 1 + b_{0,n'} + \cdots + 1 + b_{m,n'}$ and its Grundy value is $\mathcal{G}(O_n) = Ns + R = g$ by Property (1).
Hence, the heap of size n has options to all Grundy values in $[\![0, (a' - 1)s - 1]\!]$, *i.e.* $\mathcal{G}(n) \geq (a' - 1)s$.
We now change O_n into O'_n as follows:

$$\begin{aligned} h'_0 &= (N + 2)p + 1 + b_{0,n'} \\ h'_j &= \frac{1}{2}(S - N)p + 1 + b_{j,n'} & \text{for } j = 1, 2 \\ h'_j &= 1 + b_{j,n'} & \text{for } j > 2 \end{aligned}$$

This option is an option of n since $h'_0 + \cdots + h'_m = (2 + S)p + 1 + b_{0,n'} + \cdots + 1 + b_{m,n'}$ and its Grundy value is $\mathcal{G}(O'_n) = (N + 2)s + R = g + 2s$.
Hence, the heap of size n has options of Grundy values in $[\![2s, (a-1)s-1]\!]$. If $a > 4$ then with the previous remark, the heap of size n has options to all Grundy values in $[\![0, (a - 1)s - 1]\!]$. Otherwise, if $a = 4$, then we take an option $O_{n'} = (a_{0,n'}p+1+b_{0,n'}, \ldots, a_{m,n'}p+1+b_{m,n'})$ of $n' = 3p+1+b = n-p$ with Grundy value g in $[\![0, s-1]\!]$ and $m \geq 2$, which exists by Lemma 12. We note $S = a_{0,n'}+\cdots+a_{m,n'}$, $N = a_{0,n'} \oplus\cdots\oplus a_{m,n'}$ and $R = \mathcal{G}(1+b_{0,n'})\oplus\cdots\oplus\mathcal{G}(1+b_{m,n'})$. We transform it into an option $O_n = (h_0, \ldots, h_m)$ by:

$$\begin{aligned} h_0 &= (N + 1)p + 1 + b_{0,n'} \\ h_j &= \frac{1}{2}(S - N)p + 1 + b_{j,n'} & \text{for } j = 1, 2 \\ h_j &= 1 + b_{j,n'} & \text{for } 3 \leq j \leq m \end{aligned}$$

it is an option of n since $h_0 + \cdots + h_m = (S + 1)p + 1 + b_{0,n'} + \cdots + 1 + b_{m,n'} = n' + p = n$ and its Grundy value is $\mathcal{G}(O_n) = \mathcal{G}(O_{n'}) + s = g + s$.
Hence, even for $a = 4$, the heap of size n has options obtained by m-cuts, $m \geq 2$, to all Grundy values in $[\![0, (a - 1)s]\!]$, hence the heap of size n verifies (B).

Now, let $n'' = n - (a - 1)p = p + 1 + b$ and $g \in [\![0, s + r_{1+b} - 1]\!]$.
Let $O_{n''} = (a_{0,n''}p+1+b_{0,n''}, \ldots, a_{m,n''}p+1+b_{m,n''})$ be an option of n'' such that $\mathcal{G}(O_{n''}) = g$. It exists since the heap of size n'' verifies (B) by minimality of n. Please remark that as $n'' \leq 2p$, if there is a j such that $a_{j,n''} \neq 0$, then it is unique, without loss of generality, assume that $a_{0,n''} \in \{0, 1\}$ and for $j > 0$, $a_{j,n''} = 0$. Hence if $R = \mathcal{G}(1 + b_{0,n''}) \oplus \cdots \oplus \mathcal{G}(1 + b_{m,n''})$ then $\mathcal{G}(O_{n''}) = a_{0,n''}s + R$ by Property (1).
Let O_n be the following option:

$$\begin{aligned} h_0 &= (a_{0,n''} + a - 1)p + 1 + b_{0,n''} \\ h_j &= 1 + b_{j,n''} & \text{for } j > 0 \end{aligned}$$

This is an option of n since $h_0+\cdots+h_m = (a_{0,n''}+a-1)p+1+b_{0,n''}+1+b_{1,n''}+\cdots+1+b_{m,n''} = n''+(a-1)p = n$. Its Grundy value is $\mathcal{G}(O_n) = (a_{0,n''}+a-1)s+R = g+(a-1)s$. Hence, the heap of size n has options to all Grundy values in $[\![(a-1)s, as+r_{1+b}-1]\!]$. With the previous remarks, the heap of size n has options to all Grundy values in $[\![0, as + r_{1+b} - 1]\!]$.
Altogether, this means $\mathcal{G}(n) \geq as + r_{1+b}$, a contradiction.

1.2 Now, if $\mathcal{G}(n) > as + r_{1+b}$:
Let $O_n = (a_0p + 1 + b_0, \dots, a_mp + 1 + b_m)$ be an option of n with Grundy value $as + r_{1+b}$. Let $N = a_0 \oplus \dots \oplus a_m$, $S = a_0 + \dots + a_m$ and $R = \mathcal{G}(1 + b_0) \oplus \dots \oplus \mathcal{G}(1 + b_m)$. Remark that by Property (1) $a_0 \oplus \dots \oplus a_m = a$ and as $S \geq N$, $S = a$. Let $O_{n'}$ be the following option of $n' = n - 2p$:

$$\begin{aligned} h'_0 &= (a-2)p + 1 + b_0 \\ h'_j &= 1 + b_j \qquad \text{for } j > 1 \end{aligned}$$

This is an option of n' since $h'_0 + \dots + h'_m = (a-2)p + 1 + b = n - 2p$ and its Grundy value is $\mathcal{G}(O_{n'}) = (a-2)s + R = as + r_{1+b} - 2s = \mathcal{G}(n')$, a contradiction.

Hence, the heap of size n verifies (A).

2. Assume (B) is not verified:
By minimality of n, the heap of size $n' = n - 2p = a'p + 1 + b'$ verifies conditions (A) and (B). Let $O_{n'} = (h_{0,n'}, \dots, h_{m,n'}) = (a_{0,n'}p + 1 + b_{0,n'}, \dots, a_{m,n'}p + 1 + b_{m,n'})$ be an option of n' with Grundy value g, for some $g < (a'-1)s$ and with $m \geq 2$. Let $N = a_{0,n'} \oplus \dots \oplus a_{m,n'}$, $S = a_{0,n'} + \dots + a_{m,n'}$ and $R = \mathcal{G}(1 + b_{0,n'}) \oplus \dots \oplus \mathcal{G}(1 + b_{m,n'})$. Let O_n be the following option:

$$\begin{aligned} h_0 &= Np + 1 + b_{0,n'} \\ h_j &= \frac{1}{2}(S - N + 2) + 1 + b_{j,n'} \qquad \text{for } j = 1, 2 \\ h_j &= 1 + b_{m,n'} \qquad \text{for } j > 2 \end{aligned}$$

This is an option of n since $h_0 + \dots + h_m = 2p + h_{0,n'} + \dots + h_{m,n'}$ and its Grundy value is $\mathcal{G}(O_n) = Ns + R = g$.
Hence, the heap of size n has options obtained by m-cuts with $m \geq 2$ to all Grundy values in $[\![0, (a'-1)s - 1]\!]$.
We now change O_n into O'_n as follows:

$$\begin{aligned} h'_0 &= (N+2)p + 1 + b_{0,n'} \\ h'_j &= \frac{1}{2}(S - N) + 1 + b_{j,n'} \qquad \text{for } j = 1, 2 \\ h'_j &= 1 + b_{j,n'} \qquad \text{for } j > 2 \end{aligned}$$

This option is an option of n since $h'_0 + \dots + h'_m = 2p + h_{0,n'} + \dots + h_{m,n'}$ and its Grundy value is $\mathcal{G}(O'_n) = (N+2)s + R = g + 2s$.
Hence, the heap of size n has options obtained by m-cuts, $m \geq 2$ to all Grundy values in $[\![2s, (a-1)s - 1]\!]$. With the previous remark, this is true for all Grundy values in $[\![0, (a-1)s - 1]\!]$. Hence the heap of size n verifies (B), a contradiction.

□

3.2 Relaxed conditions on the AP-test

We now prove that for some families of games, the conditions $AP1$ and $AP2$ of the AP-test imply the condition $AP3$. We first prove that this is the case if the players are allowed to split a heap in at least one even and one odd number of heaps.

Proposition 13. *Let $L = \{\ell_1, \dots, \ell_k\}$ be a sequence of positive integers, $k > 1$. If* PB(L) *verifies the conditions $AP1$ and $AP2$ of the AP-test and there are $m_1, m_2 \in L$ of different parities such that $2 \leq m_1, m_2 \leq 2p + 1$; then* PB$(L)$ *verifies the AP-test.*

Proof. It suffices to prove that L verifies the condition $AP3$ of the AP-test. Without loss of generality, we can consider that m_1 is even and m_2 is odd. We prove that for all $n \in [\![3p+1, 4p]\!]$ and $g \in [\![0, s-1]\!]$ there is an option $O_n = (h_0, \ldots, h_m)$ of n such that $m \geq 2$ and $\mathcal{G}(O_n) = g$.
Let $n = 3p + 1 + b$ with $0 \leq b < p$ and $g \in [\![0, s-1]\!]$.
By $AP2$, there is $c \in [\![0, p-1]\!]$ such that $\mathcal{G}(1+c) = g$. Let $n' = n - 1 - c = 3p + b - c$. We consider two cases:

- if n' is even: let (q_1, r_1) be the unique couple such that $0 \leq r_1 < m_1$ and $n' = m_1 q_1 + r_1$. In particular, r_1 is even, since m_1 and n' are also even. Moreover $q_1 > 0$ since $m_1 \leq 2p + 1 \leq n'$. We define an option O_n of n by:

$$\begin{array}{rcll} h_0 & = & 1+c & \\ h_j & = & q_1 + \frac{1}{2} r_1 & \text{for } j = 1, 2 \\ h_j & = & q_1 & \text{for } 3 \leq j \leq m_1 \end{array}$$

  It is indeed an option of n since $h_0 + \cdots + h_{m_1} = 1 + c + m_1 q_1 + r_1 = 1 + c + n' = n$ and in the expression $\mathcal{G}(h_0) \oplus \cdots \oplus \mathcal{G}(h_m)$, the terms $\mathcal{G}(h_1)$ and $\mathcal{G}(h_3)$ appear an even number of times, which gives directly $\mathcal{G}(O_n) = \mathcal{G}(1+c) = g$.

- if n' is odd: let (q_2, r_2) be the unique couple such that $0 \leq r_2 < m_2$, $n' = m_2 q_2 + r_2$. Please remark that $q_2 > 0$ since $m_2 \leq 2p + 1 \leq n'$. As n' and m_2 are odd, either q_2 is even and r_2 is odd or vice versa.

  - if q_2 is even and r_2 is odd, we define the option O_n by:

$$\begin{array}{rcll} h_0 & = & 1+c & \\ h_j & = & \frac{3}{2} q_2 + \frac{1}{2}(r_2 - 1) & \text{for } j = 1, 2 \\ h_j & = & 1 & \text{for } j = 3 \\ h_j & = & q_2 & \text{for } 4 \leq j \leq m_2 \end{array}$$

    If $m_2 = 3$ then we only take the four first heaps. The option O_n is an option of n since $h_0 + \cdots + h_{m_2} = 1 + c + 3q_2 + r_2 - 1 + 1 + (m_2 - 1 - 2)q_2 = 1 + c + m_2 q_2 + r_2 = 1 + c + n' = n$. In the expression $\mathcal{G}(h_0) \oplus \cdots \oplus \mathcal{G}(h_{m_2})$ the terms $\mathcal{G}(h_1)$ and $\mathcal{G}(h_4)$ appear an even number of times and $\mathcal{G}(h_3) = 0$, hence $\mathcal{G}(O_n) = 1 + c = g$.

  - if q_2 is odd and r_2 is even, we define the option O_n by:

$$\begin{array}{rcll} h_0 & = & 1+c & \\ h_j & = & \frac{1}{2}(3q_2 - 1) + \frac{1}{2} r_2 & \text{for } j = 1, 2 \\ h_j & = & 1 & \text{for } j = 3 \\ h_j & = & q_2 & \text{for } 4 \leq j \leq m_2 \end{array}$$

    it is an option of n since $h_0 + \cdots + h_{m_2} = 1 + c + 3q_2 - 1 + r_2 + 1 + (m_2 - 3)q_2 = 1 + c + m_2 q_2 + r_2 = n$. In the expression $\mathcal{G}(h_0) \oplus \cdots \oplus \mathcal{G}(h_{m_2})$ the terms $\mathcal{G}(h_1)$ and $\mathcal{G}(h_4)$ appear an even number of times and $\mathcal{G}(h_3) = 0$, hence $\mathcal{G}(O_n) = g$.

In every case, there is an option O_n of n obtained by an m-cut, $m \geq 2$, such that $\mathcal{G}(O_n) = g$, *i.e.*, PB(L) verifies the condition $AP3$, which means PB(L) verifies the test AP. □

Now, we prove that if the players are allowed to split a heap in two or an odd number of heaps, and under some conditions, then the conditions $AP1$ and $AP2$ of the AP-test imply the condition $AP3$.

Proposition 14. *Let $L = \{1, \ell\}$ with $\ell > 2$ even. If* $\mathrm{PB}(L)$ *verifies the conditions $AP1$ and $AP2$ of the AP-test for some p with $\ell \le p$ and there are $x_1, x_2 \le p/2$ such that $\mathcal{G}(x_1) = \mathcal{G}(x_2) = 1$ and x_1 is odd and x_2 is even; then* $\mathrm{PB}(L)$ *verifies the AP-test.*

Proof. We are going to prove that the game $\mathrm{PB}(L)$ verifies the condition $AP3$, *i.e.*, that for $n \in [\![3p+1, 4p]\!]$ and for $g \in [\![0, s-1]\!]$, there exists an option O_n of n such that $\mathcal{G}(O_n) = g$. Since the condition $AP2$ is verified, this can be done by proving that for all $n \in [\![3p+1, 4p]\!]$ and for all $k \in [\![1, p]\!]$, there exists an option O_n of n such that $\mathcal{G}(O_n) = \mathcal{G}(k)$.
Let $n \in [\![3p+1, 4p]\!]$ and $k \in [\![1, p]\!]$. The proof is divided in four cases depending on the parities of k and n:

1. if $n = 2i$ is even:
   1. if $k = 2j$ is even, then let $O_n = (h_0, \ldots, h_\ell)$ be the following option, obtained by an ℓ-cut:

$$\begin{aligned} h_0 &= 2j \\ h_j &= i - j + 1 - \frac{1}{2}\ell \qquad \text{for } j = 1, 2 \\ h_j &= 1 \qquad \text{for } 3 \le j \le \ell \end{aligned}$$

      This option exists since $i \ge (3p+1)/2$, $j \le p/2$ and $\ell \le p$, hence $i - j + 1 - \ell/2 > p/2 > 0$. Moreover it is an option of n since $2j + 2i - 2j + 2 - \ell + 1 \times (\ell - 2) = n$ and its Grundy value is $\mathcal{G}(O_n) = \mathcal{G}(k)$ since except $2j$, all the other values in O_n appear an even number of times.
   2. if $k = 2j + 1$ is odd, then let O_n be the following option, obtained by an ℓ-cut:

$$\begin{aligned} h_0 &= 2j + 1 \\ h_j &= x_j \qquad \text{for } j = 1, 2 \\ h_j &= \frac{1}{2}(2i - 2j - \ell - x_1 - x_2 + 3) \qquad \text{for } j = 3, 4 \\ h_j &= 1 \qquad \text{for } 5 \le j \le \ell \end{aligned}$$

      This option exists since $i \ge (3p+1)/2$; $j, x_1, x_2 \le p/2$ and $\ell \le p$, hence $2i - 2j - \ell - x_1 - x_2 + 3 \ge 4$; and $2i - 2j - \ell - x_1 - x_2 + 3$ is even since $x_1 + x_2$ is odd. Moreover, it is an option of n since $2j + 1 + x_1 + x_2 + \ell - 4 + (2i - 2j - \ell - x_1 - x_2 + 3) = 2i = n$ and its Grundy value is $\mathcal{G}(O_n) = \mathcal{G}(k) \oplus \mathcal{G}(x_1) \oplus \mathcal{G}(x_2) = \mathcal{G}(k) \oplus 1 \oplus 1$ since the other values in O_n each appear an even number of times.

2. if $n = 2i + 1$ is odd:
   1. if $k = 2j$ is even, then let O_n be the following option, obtained by an ℓ-cut:

$$\begin{aligned} h_0 &= 2j \\ h_j &= x_j \qquad \text{for } j = 1, 2 \\ h_j &= \frac{1}{2}(2i - 2j - \ell - x_1 - x_2 + 5) \qquad \text{for } j = 3, 4 \\ h_j &= 1 \qquad \text{for } 5 \le j \le \ell \end{aligned}$$

      This option exists since $i \ge (3p+1)/2$; $j, x_1, x_2 \le p/2$ and $\ell \le p$, hence $2i - 2j - \ell - x_1 - x_2 + 5 \ge 6$; and $2i - 2j - \ell - x_1 - x_2 + 5$ is even since $x_1 + x_2$ is odd. Moreover, it is an option of n since $2j + x_1 + x_2 + \ell - 4 + (2i - 2j - \ell - x_1 - x_2 + 5) = 2i + 1 = n$ and its Grundy value is $\mathcal{G}(O_n) = \mathcal{G}(k) \oplus \mathcal{G}(x_1) \oplus \mathcal{G}(x_2) = \mathcal{G}(k)$ since the other values in O_n each appear an even number of times.

2. if $k = 2j+1$ is odd, then let O_n be the following option, obtained by an ℓ-cut

$$\begin{aligned} h_0 &= 2j+1 \\ h_j &= i-j+1-\frac{1}{2}\ell \qquad \text{for } j=1,2 \\ h_j &= 1 \qquad\qquad \text{for } 3 \le j \le \ell \end{aligned}$$

This option exists since $i \geq (3p+1)/2$ and $2j+1, \ell \leq p$, hence $i-j+1-\ell/2 > p/2 > 0$. Moreover it is an option of n since $2j+1+1\times(\ell-2)+2(i-j+1)-\ell = 2i+1$ and its Grundy value is $\mathcal{G}(O_n) = \mathcal{G}(2j+1) = \mathcal{G}(k)$ since all the other values in O_n appear an even number of times.

Hence, for all $k \in [\![1,p]\!]$, there exists an option of n with the same Grundy value. This implies that the condition $AP3$ is verified, and thus that the AP-test is verified for $\mathrm{PB}(L)$. □

We now prove that the conditions of the previous proposition are always verified for those games as long as $4\ell+3 \leq p$.

Corollary 15. *Let $L = \{1, \ell\}$ with $\ell > 2$ even. If $\mathrm{PB}(L)$ verifies the conditions $AP1$ and $AP2$ of the AP-test for some $p \geq 4\ell+3$, then $\mathrm{PB}(L)$ verifies the condition $AP3$ of the AP-test.*

Proof. By Proposition 14, we only need to prove that there exists $x_1, x_2 < p/2$ such that $\mathcal{G}(x_1) = \mathcal{G}(x_2) = 1$ and x_1 is odd and x_2 is even.
Remark that $\mathcal{G}(2) = 1$ since the only option is $(1,1)$ which has Grundy value 0. Hence we can assume $x_2 = 2$.
We claim that we can choose $x_1 = 2\ell+1$. In order to do that, we prove that the beginning of the Grundy sequence of the game $\mathrm{PB}(L)$ is $(0,1)^{\ell/2}$ and the following ℓ values are different from 1 and 0, and the $2\ell+1$-th value is 1. Note that we trivially have $\mathcal{G}(1) = 0$ and $\mathcal{G}(2) = 1$.
Let $k \leq \ell$ be the smallest integer such that $\mathcal{G}(k) \neq ((k \bmod 2) + 1 \bmod 2)$. The only possible options for k are obtained by 1-cuts. If k is odd, then all the options are of the form (i_0, i_1) with i_0 and i_1 of different parities, which have Grundy value 1 by minimality of k, a contradiction. If k is even, then all the options are of the form (i_0, i_1) with i_0 and i_1 of same parities, which have Grundy value 0 by minimality, a contradiction.
Now, let $k \in [\![\ell+1, 2\ell]\!]$. If k is odd, then k admits the 1-cut option $(k-\ell, \ell)$ of Grundy value 1 since ℓ is even, and the ℓ-cut option $(k-\ell, 1, \ldots, 1)$ of Grundy value 0. If k is even, it admits the ℓ-cut option $(k-\ell, 1, \ldots, 1)$ of Grundy value 1, and the 1-cut option $(k/2, k/2)$ of Grundy value 0. It thus implies that $\mathcal{G}(k) > 1$.

Finally, we prove $\mathcal{G}(2\ell+1) = 1$. We now set $k = 2\ell+1$.
From k, one can reach the value 0 by the option $(1, 2, \ldots, 2)$ obtained by an ℓ-cut. All the 1-cuts (i_0, i_1) are such that without loss of generality $i_0 > \ell$ and $i_1 \leq \ell$, so $\mathcal{G}((i_0, i_1)) \neq 1$ since $\mathcal{G}(i_1) < 2$ and $\mathcal{G}(i_2) \geq 2$. Assume there is an ℓ-cut $O_k = (i_0, \ldots, i_\ell)$ such that $\mathcal{G}(O_k) = 1$. If there is some j such that $i_j > \ell$, then it is unique and $\mathcal{G}(O_k) \geq 2$, hence, there is none: for all j, $i_j \leq \ell$. We necessarily have an odd number of i_j's, say $i_0, \ldots, i_e$ with e even, such that $\mathcal{G}(i_j) = 1$ for $j \in [\![0, e]\!]$. And for $j > e$, $\mathcal{G}(i_j) = 0$. Hence there is an even number of odd i_j's and an odd number of even ones, this gives directly that $2\ell+1$ is even, which is a contradiction. Therefore, $\mathcal{G}(2\ell+1) = 1$. Moreover, $2\ell+1 < p/2$ since $4\ell+3 \leq p$, hence it suffices to take $x_1 = 2\ell+1$ and $x_2 = 2$ to meet the conditions of Proposition 14 and thus the condition $AP3$ of the AP-test.

□

3.3 Applications of the AP-test

Table 1 summarizes the AP-test computations that have been made for some pure breaking games. Naturally, the games already solved in Section 2 are not in the table. All the games in this list satisfy the test and hence are proved to be arithmetic periodic. More specifically, Corollary 15 has been applied to the games $\{1,4\}$, $\{1,6\}$, $\{1,8\}$, and $\{1,10\}$. We note that for games of the form $\{1,\ell\}$, there seem rather long periods depending on ℓ, with always the same saltus. One can wonder whether this regularity holds for higher values of ℓ:

Conjecture 1. *Given $\ell \geq 2$, the game* $\mathrm{PB}(L)$ *with* $L = \{1, 2\ell\}$ *is arithmetic periodic of length* 12ℓ *and saltus* 8.

Surprisingly, when one adjoins new values to the games $\{1, 2\ell\}$ (with $\ell \geq 2$), the period simplifies significantly. These computations for small values lead to the following conjecture:

Conjecture 2. *Let K be a finite set of positive integers such that $2 \notin K$, $|K| \geq 2$ and K contains at least one even value. The game* $\mathrm{PB}(L)$ *with* $L = \{1\} \cup K$ *is arithmetic periodic with period* $(0, 1)^\ell$ *and saltus* 2*, where* 2ℓ *is the smallest even number in* K.

The case where $1, 2 \in L$ and $3 \notin L$ remains the hardest to understand. If Table 1 suggests an arithmetic periodic behavior when $|L| \geq 3$, we did not detect any regularity in the period. For example, when $|L| = 3$, the games $\{1, 2, 4\}$ and $\{1, 2, 6\}$ have identical Grundy sequences, whereas $\{1, 2, 5\}$ and $\{1, 2, 7\}$ are more singular. Even worse, the game $\{1, 2, 8\}$ is arithmetic periodic with a preperiod of positive length (which is not the case of the other sequences we computed). Note that for ultimately arithmetic periodic sequences, we use the notation $(i_1, \dots, i_e)$ $(m_1, \dots, m_p)$ $(+s)$ where $i_1, \dots, i_e$ are the e values of the preperiod, and the rest is as before the p first values of the arithmetic periodic sequence and s the saltus.

Sequence of integers	Sequence
$\{1, 4\}$	$((0,1)^2(2,3)^2, 1, 4, 5, 4, (3,2)^2(4,5)^2(6,7)^2)$ $(+8)$
$\{1, 6\}$	$((0,1)^3(2,3)^3, 1, 4, (5,4)^2(3,2)^3(4,5)^3(6,7)^3)$ $(+8)$
$\{1, 8\}$	$((0,1)^4(2,3)^4, 1, 4, (5,4)^3(3,2)^4(4,5)^4(6,7)^4)$ $(+8)$
$\{1, 10\}$	$((0,1)^5(2,3)^5, 1, 4, (5,4)^4(3,2)^5(4,5)^5(6,7)^5)$ $(+8)$
$\{1, 4\} \cup K$ with $K \subseteq \{3, 5, 6, 7, 8\}, K \neq \emptyset$	$(0,1)^2$ $(+2)$
$\{1, 6\} \cup K$ with $K \subseteq \{3, 5, 7, 8\}, K \neq \emptyset$	$(0,1)^3$ $(+2)$
$\{1, 8\} \cup K$ with $K \subseteq \{3, 5, 7\}, K \neq \emptyset$	$(0,1)^4$ $(+2)$
$\{1, 2, 4\} \cup K, \{1, 2, 6\} \cup K'$ with $K \subseteq \{6, 7, 8\}, K' \subseteq \{7, 8\}$	$(0, 1, 2, 3, 1, 4, 3, 2, 4, 5, 6, 7)$ $(+8)$
$\{1, 2, 5\} \cup K$ with $K \subseteq \{4, 6, 7, 8\}$	$(0, 1, 2, 3, 1, 4, 3, 6, 4, 5, 6, 7)$ $(+8)$
$\{1, 2, 7\}$	$(0, 1, 2, 3, 1, 4, 3, 2, 4, 5, 6, 7, 8, 9, 7, 6, 9, 8, 11, 10, 12, 13, 10, 11, 13, 12, 15, 14)$ $(+16)$
$\{1, 2, 8\}, \{1, 2, 7, 8\}$	$(0, 1, 2, 3, 1, 4)$ $(3, 2, 4, 5, 6, 7, 8, 9, 7, 11, 9, 8)$ $(+8)$

Table 1: Some pure breaking games for which the ultimate arithmetic periodicity is proved with the *AP*-test. All are purely arithmetic periodic, save for $\{1, 2, 8\}$ and $\{1, 2, 7, 8\}$ which are ultimately arithmetic periodic.

4 Conclusion and perspectives

We summarize in Table 2 the results obtained in Sections 2 and 3. The games are partitioned into three families: those for which the periodicity or arithmetic periodicity is proved, and those for which two or three conditions of the *AP*-test are required to prove that they are arithmetic periodic (if that is the case).

Among the families that are not solved, all of our computations on particular examples have shown ultimate arithmetic periodic behaviors, except for one: $\mathrm{PB}(\{1, 2\})$. This game has a Grundy sequence with a lot of regularity but some irregular values, as shown on Figure 1.

In view of our computations, we thus naturally propose the following conjecture.

Conjecture 3. *Every game* $\mathrm{PB}(L)$ *with* $L \neq \{1, 2\}$ *has a Grundy sequence either ultimately periodic or ultimately arithmetic periodic.*

	Sequence of integers	Sequence (if known)	Theorem
Solved	$\{\ell_1, \ldots, \ell_k\}$ $(\ell_1 > 1)$	$(0)^{\ell_1}$ $(+1)$	Proposition 4
	$\{1, \ell_2, \ldots, \ell_k\}$ $(\ell_i$ odd$)$	$(0,1)$ $(+0)$	Proposition 5
	$\{1, 2, 3, \ell_4, \ldots, \ell_k\}$	$(0)^1$ $(+1)$	Proposition 6
	$\{1, 3, 2k\}$ $(k \geq 1)$	$(0,1)^{\ell}$ $(+2)$	Proposition 7
Requires $AP1$ and $AP2$	$\{1, 2\ell, 2\ell' + 1, \ell_1, \ldots, \ell_k\}$		Proposition 13
	$\{1, 2\ell\}$ $(\ell \geq 2)$		Corollary 15
Requires $AP1$, $AP2$ and $AP3$	$\{1, \ell_1, \ldots, \ell_k\}$ $(\ell_i$ even, $k \geq 1)$		Theorem 9

Table 2: The pure breaking games.

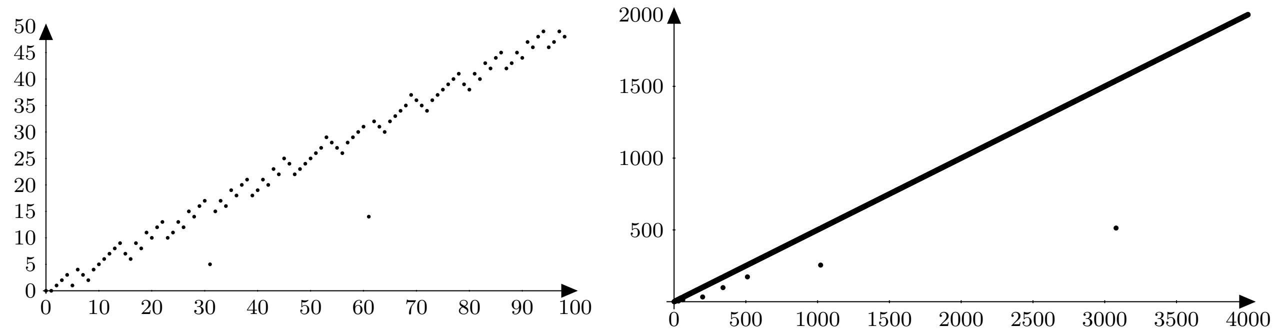

Figure 1: The Grundy sequence of $\mathrm{PB}(\{1,2\})$ for $n \leq 100$ and $n \leq 4000$.

For some games, the above conjecture is proved to be true but the expression of the period according to L is non trivial (e.g. $L = \{1, 2, 7\}$). This makes a general proof hard to obtain and motivates the testing conditions. If the AP-test is a rather short computation to prove the arithmetic-periodicity of a game, we are wondering whether the condition $AP3$ could be entirely removed from the test.

Open Problem 1. *Do the conditions $AP1$ and $AP2$ of the AP-test imply the conditions $AP3$ for any pure breaking game?*

In addition, the case of $\mathrm{PB}(\{1,2\})$ leaves a couple of open questions:

Open Problem 2. *What is the behavior of the Grundy sequence of* $\mathrm{PB}(\{1,2\})$*?*

Possibly other behaviors than periodicity and arithmetic periodicity could be expected for this game, as it is the case for hexadecimal games. Determining the number of occurrences of each Grundy value could be useful to help us understand this sequence. We already know from Lemma 3 that every Grundy value appears at most twice in the sequence of $\mathrm{PB}(\{1,2\})$ (apply Lemma 3 with $m = 2$).

Open Problem 3. *Does each Grundy value appear at least once in the sequence of* $\mathrm{PB}(\{1,2\})$*? More precisely, does each Grundy value appear exactly twice in the sequence of* $\mathrm{PB}(\{1,2\})$*?*

References

[1] M. Albert, R. J. Nowakowski & D. Wolfe, *Lessons in Play: An Introduction to Combinatorial Game Theory*, A.K. Peters (2007).

[2] R. B. Austin, *Impartial and Partizan Games*, M.Sc. Thesis, The University of Calgary (1976).

[3] E. R. Berlekamp, J. H. Conway, & R. K. Guy. *Winning ways for your mathematical plays (Vol. 1).* Natick, MA: AK Peters (2001).

[4] I. Caines, C. Gates, R. K. Guy & R. J. Nowakowski, Periods in taking and splitting games, *Amer. Math. Monthly* **106** (1999) 359–361.

[5] A. Flammenkamp, Sprague-Grundy values of some octal games, `wwwhomes.uni-bielefeld.de/achim/octal.html`.

[6] J. P. Grossman & R. J. Nowakowski, A ruler regularity in hexadecimal games, *Games of No Chance 4*, MSRI Publications, Cambridge University Press, Cambridge (2013).

[7] P. M. Grundy. Mathematics and games. *Eureka* **2** (1939) 6–9.

[8] D. G. Horrocks & R. J. Nowakowski, Regularity in the $\mathbb{G}$-sequences of octal games with a pass, *INTEGERS* **3** (2003), G1.

[9] S. Howse & R. J. Nowakowski, Periodicity and arithmetic-periodicity in hexadecimal games, *Theoretical Computer Science* **313** (2004) 463–472.

[10] A. N. Siegel, *Combinatorial Game Theory*, San Francisco, CA (2013).