Regularity of isoperimetric and time constants for a supercritical Bernoulli percolation

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Abstract: We consider an i.i.d. supercritical bond percolation on $\mathbb{Z}^d$, every edge is open with a probability $p > p_c(d)$, where $p_c(d)$ denotes the critical parameter for this percolation. We know that there exists almost surely a unique infinite open cluster $\mathcal{C}_p$ [18]. We are interested in the regularity properties of two distinct objects defined on this infinite cluster: the isoperimetric (or Cheeger) constant, and the chemical distance for supercritical Bernoulli percolation. The chemical distance between two points $x, y \in \mathcal{C}_p$ corresponds to the length of the shortest path in $\mathcal{C}_p$ joining the two points. The chemical distance between 0 and nx grows asymptotically like $n\mu_p(x)$. We aim to study the regularity properties of the map $p \to \mu_p$ on the supercritical regime. This may be seen as a special case of first passage percolation where the distribution of the passage time is $G_p = p\delta_1 + (1-p)\delta_\infty$, $p > p_c(d)$. It is already known that the map $p \to \mu_p$ is continuous (see [15]). We prove here that $p \to \mu_p$ satisfies stronger regularity properties, this map is almost Lipschitz continuous up to a logarithmic factor in $[p_0, 1]$ for any $p_0 > p_c(d)$. We prove an analog result for the Cheeger constant in dimension 2 for all intervals $[p_0, p_1] \subset (1/2, 1)$. For $d \geq 3$, we prove that the modified Cheeger constant defined in [16] is Lipschitz continuous on all intervals $[p_0, p_1] \subset (p_c(d), 1)$.

AMS 2010 subject classifications: primary 60K35, secondary 82B43.
Keywords: Regularity, percolation, time constant, isoperimetric constant.

1 Introduction

In this section, we give informal definitions in order to present the background and state our results. More rigorous definitions will be given in section 2.

1.1 Isoperimetric constant of the infinite cluster in dimension 2

In this section we gather definitions and notations presented in [15], see section 1.1. For a finite graph $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$, the isoperimetric constant is

*Research was partially supported by the ANR project PPPP (ANR-16-CE40-0016)
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defined as

$$\varphi_{\mathcal{G}} = \min \left\{ \frac{|\partial_{\mathcal{G}} A|}{|A|} \,:\, A \subset V(\mathcal{G}), 0 < |A| \leq \frac{|V(\mathcal{G})|}{2} \right\}$$

where $\partial_{\mathcal{G}} A$ is the edge boundary of A in $\mathcal{G}$, $\partial_{\mathcal{G}} A = \{e = (x,y) \in E(\mathcal{G}) : x \in A, y \notin A\}$, and $|B|$ denotes the cardinal of the finite set B.

We consider the isoperimetric constant $\varphi_n(p)$ of $\mathcal{C}_p \cap [-n,n]^d$, the intersection of the infinite component of i.i.d. supercritical percolation of parameter p with the box $[-n,n]^d$

$$\varphi_n(p) = \min \left\{ \frac{|\partial_{\mathcal{C}_p \cap [-n,n]^d} A|}{|A|} \,:\, A \subset \mathcal{C}_p \cap [-n,n]^d, 0 < |A| \leq \frac{|\mathcal{C}_p \cap [-n,n]^d)|}{2} \right\}.$$

In several papers (e.g. [2], [24], [25], [3]), it was shown that there exist constants $c, C > 0$ such that $c < n\varphi_n(p) < C$, with probability tending rapidly to 1. This led Benjamini to conjecture the existence of $\lim_{n\to\infty} n\varphi_n(p)$. In [27], Rosenthal and Procaccia proved that the variance of $n\varphi_n(p)$ is smaller than Cn^{2-d}, which implies $n\varphi_n(p)$ is concentrated around its mean for $d \geq 3$. In [4], Biskup, Louidor, Procaccia and Rosenthal proved the existence of $\lim_{n\to\infty} n\varphi_n(p)$ for $d = 2$. This constant is called Cheeger constant. In addition, a shape theorem was obtained: any set yielding the isoperimetric constant converges in the Hausdorff metric to the normalized Wulff shape $\widehat{W}_p$, with respect to a specific norm given in an implicit form (see (1) for precise definition of W_p and $\widehat{W}_p$). For additional background and a wider introduction on Wulff construction in this context, the reader is referred to [4]. In [15], Garet, Marchand, Procaccia and Théret proved the continuity of the Cheeger constant and of the Wulff shape with regard to the parameter p of the percolation in dimension $d = 2$. In this article, we obtain better regularity properties.

Theorem 1.1 (Regularity of the Cheeger constant in dimension 2)**.** *Let $p_c(2) < p_0 < p_1 < 1$. There exits a constant ν, depending on p_0 and p_1, such that for all $p \leq q$ in $[p_0, p_1]$*

$$\lim_{n\to\infty} n|\varphi_n(p) - \varphi_n(q)| \leq \nu(q-p)|\log(q-p)|\,.$$

Theorem 1.2 (Regularity of the associated Wulff crystal in dimension 2)**.** *Let $p_c(2) < p_0 < p_1 < 1$. There exits a constant ν' depending on p_0 and p_1, such that for all $p \leq q$ in $[p_0, p_1]$*

$$d_{\mathcal{H}}(W_p, W_q) \leq \nu'(q-p)|\log(q-p)|\,,$$

where W_p (resp. W_q) denotes the Wulff shape associated with parameter p (resp. q) and $d_{\mathcal{H}}$ is the Hausdorff distance between non empty compact sets of $\mathbb{R}^2$.

1.2 Isoperimetric constant in higher dimension

As introduced by Gold in [16], we define the modified Cheeger constant $\widehat{\varphi}_n(p)$ for $p > p_c(d)$ by

$$\widehat{\varphi}_n(p) = \min \left\{ \frac{|\partial_{\mathcal{C}_p} A|}{|A|} \,:\, A \subset \mathcal{C}_p \cap [-n,n]^d,\, 0 \leq |A| \leq \frac{|\mathcal{C}_p \cap [-n,n]^d|}{d!} \right\},$$

where $\partial_{\mathcal{C}_p} A$ denotes the open edges boundary of A within all of $\mathcal{C}_p$ as opposed to the definition of $\varphi_n(p)$ where we looked at the open edges boundary of A within $\mathcal{C}_p \cap [-n,n]^d$. The $d!$ on the volume upper bound ensures that there exists a set A that satisfies the volume upper bound and does not touch the boundary of the box $[-n,n]^d$. Therefore, the subgraphs A are treated as subgraphs of $\mathcal{C}_p$. In [16], Gold has shown that the limit $\lim_{n\to\infty} n\widehat{\varphi}_n(p)$ exists for all $p > p_c(d)$ for $d \geq 3$, this is what we call the modified Cheeger constant. As in dimension 2, a shape theorem is obtained in [16]. Any set yielding the infimum in the definition of the isoperimetric constant converges in a weaker sense to the normalized Wulff shape $\widehat{W_p}$, with respect to a specific norm β_p given in an implicit form (see (4) for precise definition of W_p and $\widehat{W_p}$).

The idea behind the modified Cheeger constant is the same than in dimension 2, we are trying to minimize the energy of the surface represented by the boundary of A. We obtain the following results.

Theorem 1.3 (Regularity of the modified Cheeger constant in higher dimension)**.** *Let $d \geq 3$. Let $p_c(d) < p_0 < p_1 < 1$. There exits a constant ν_d depending only on d, p_0 and p_1, such that for all $p, q \in [p_0, p_1]$,*

$$\lim_{n\to\infty} n|\widehat{\varphi}_n(q) - \widehat{\varphi}_n(p)| \leq \nu_d |q-p| \,.$$

Theorem 1.4 (Regularity of the associated Wulff crystal in higher dimension)**.** *Let $d \geq 3$. Let $p_c(d) < p_0 < p_1 < 1$. There exits a constant ν'_d depending only on d, p_0 and p_1, such that for all $p, q \in [p_0, p_1]$,*

$$d_{\mathcal{H}}(W_p, W_q) \leq \nu'_d |q-p| \,,$$

where W_p (resp. W_q) denotes the Wulff shape associated with parameter p (resp. q) and $d_{\mathcal{H}}$ is the Hausdorff distance between non empty compact sets of $\mathbb{R}^d$.

Remark 1.1. Actually, the Cheeger constant is also continuous at 1, this is not a consequence of Theorem 1.3 but it comes from the fact that the function $p \to \beta_p$ is continuous, see subsection 1.3 for more details.

1.3 Flow constant

The proof of Theorems 1.3 and 1.4 rely on the study of the norm β_p which is interesting in itself. In this section, we give a precise definition of this norm and state some results concerning the regularity of the map $p \to \beta_p$.

We consider a bond percolation on $\mathbb{Z}^d$ of parameter $p > p_c(d)$ with $d \geq 3$. We introduce now many notations used for instance in [28] concerning flows through cylinders, and in [16] concerning the modified Cheeger constant. Let A be a non-degenerate hyperrectangle, that is to say a rectangle of dimension $d-1$ in $\mathbb{R}^d$. Let $\overrightarrow{v}$ be one of the two unit vectors normal to A. Let $h > 0$, we denote by $\mathrm{cyl}(A,h)$ the cylinder of basis A and height $2h$ defined by

$$\mathrm{cyl}(A,h) = \{x + t\overrightarrow{v} \,:\, x \in A,\, t \in [-h,h]\} \,.$$

The set $\mathrm{cyl}(A,h) \setminus A$ has two connected components, denoted by $C_1(A,h)$ and $C_2(A,h)$. For $i = 1,2$, we denote by $C'_i(A,h)$ the discrete boundary of $C_i(A,h)$ defined by

$$C'_i(A,h) = \left\{x \in \mathbb{Z}^d \cap C_i(A,h) \,:\, \exists y \notin \mathrm{cyl}(A,h),\, \langle x,y\rangle \in \mathbb{E}^d\right\} \,.$$

We say that the set of edges E cuts $C_1'(A,h)$ from $C_2'(A,h)$ in $\mathrm{cyl}(A,h)$ if any path γ from $C_1'(A,h)$ to $C_2'(A,h)$ in $\mathrm{cyl}(A,h)$ contains at least one edge of E. We call such a set a cutset. For any cutset E, let $|E|_{o,p}$ denote the number of p-open edges in E. We shall call it the capacity of E. Define

$$\tau_p(A,h) = \min\left\{|E|_{o,p} : \ E \text{ cuts } C_1'(A,h) \text{ from } C_2'(A,h) \text{ in } \mathrm{cyl}(A,h)\right\} .$$

Note that it is a random quantity as $|E|_{o,p}$ is random, and that the cutsets in this definition are anchored at the border of A.

Proposition 1.1 (Definition of the norm β_p). *Let $d \geq 3$, $p > p_c(d)$, A be a non-degenerate hyperrectangle and $\overrightarrow{v}$ one of the two unit vectors normal to A. Let h an height function such that $\lim_{n\to\infty} h(n) = \infty$. The limit*

$$\beta_p(\overrightarrow{v}) = \lim_{n\to\infty} \frac{\mathbb{E}[\tau_p(nA, h(n))]}{\mathcal{H}^{d-1}(nA)}$$

exists and is finite. Moreover, the limit is independent of A and h and β_p is a norm.

This is a corollary of Proposition 3.5 in [28]. This was also shown in Proposition 3.4 in [16], where the author used the same arguments for a specific A and h: he took for A a square isometric to $[-1,1]^{d-1} \times \{0\}$ and $h(n) = n$ (*i.e.* $\mathrm{cyl}(nA, h(n)$ is a cube of size $2n$). As the limit does not depend on A and h, in the following for simplicity, we will take $h(n) = n$ and $A = S(\overrightarrow{v})$ where $S(\overrightarrow{v})$ is a square isometric to $[-1,1]^{d-1} \times \{0\}$ normal to $\overrightarrow{v}$. We will denote by $B(n, \overrightarrow{v})$ the cube $\mathrm{cyl}(nS(\overrightarrow{v}), n)$ and by $\tau_p(n, \overrightarrow{v})$ the quantity $\tau_p(nS(\overrightarrow{v}), n)$.

The norm β_p is called the flow constant. A straightforward application of Theorem 3.8 in [28] gives the existence of the following almost sure limit:

$$\lim_{n\to\infty} \frac{\tau_p(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \beta_p(\overrightarrow{v}) .$$

In [29], Rossignol and Théret studied the map $p \to \beta_p$ and proved its continuity on $(p_c(d), 1]$ (see Theorem 4 in [29]). We prove in this paper the following result.

Theorem 1.5 (Regularity of the flow constant). *Let $p_c(d) < p_0 < p_1 < 1$. There exists a constant κ_d depending only on d, p_0 and p_1, such that for all $p \leq q$ in $[p_0, p_1]$*

$$\sup_{x\in\mathbb{S}^{d-1}} |\beta_p(x) - \beta_q(x)| \leq \kappa_d |q-p| .$$

1.4 Chemical distance for supercritical Bernoulli percolation

We consider a supercritical independent and identically distributed Bernoulli bond percolation of parameter $p > p_c(d)$ on $\mathbb{Z}^d$. Let $\mathcal{C}_p'$ be the subgraph of $\mathbb{Z}^d$ whose edges are opened for the Bernoulli percolation of parameter p. For $x, y \in \mathbb{Z}^d$, we denote by $D^{\mathcal{C}_p'}(x,y)$ the length of the shortest p-open path joining x and y. This is called the chemical distance. If x and y are not in the same cluster of $\mathcal{C}_p'$, $D^{\mathcal{C}_p'}(x,y) = +\infty$. Actually, when x and y are in the same cluster,

$D^{\mathcal{C}'_p}(x,y)$ is of order $\|y-x\|_1$. In [1], Antal and Pisztora obtained the following large deviation upper bound:

$$\limsup_{\|y\|_1\to\infty}\frac{1}{\|y\|_1}\log\mathbb{P}[0\leftrightarrow y, D^{\mathcal{C}'_p}(0,y)>\rho]<0\,.$$

This result implies that there exists a constant ρ depending on the parameter p and the dimension d such that

$$\limsup_{\|y\|_1\to\infty}\frac{1}{\|y\|_1}D^{\mathcal{C}'_p}(0,y)\mathbb{1}_{0\leftrightarrow y}\leq\rho,\ \mathbb{P}_p\text{ a.s.}$$

These results were proved using renormalization arguments. It was improved later in [12] by Garet and Marchand, for the more general case of a stationary ergodic field. They proved that $D^{\mathcal{C}'_p}(0,x)$ grows linearly in $\|x\|_1$. More precisely, for each $y\in\mathbb{Z}^d\setminus\{0\}$, they proved the existence of a constant $\mu_p(y)$ such that

$$\lim_{\substack{n\to\infty\\0\leftrightarrow ny}}\frac{D^{\mathcal{C}'_p}(0,ny)}{n}=\mu_p(y),\ \mathbb{P}_p\text{ a.s.}.$$

The constant μ_p is a called the time constant. The map $p\to\mu_p$ can be extended to $\mathbb{Q}^d$ by homogeneity and to $\mathbb{R}^d$ by continuity. It is a norm on $\mathbb{R}^d$. This convergence holds uniformly in all directions, this is equivalent of saying that an asymptotic shape emerges. Indeed, the set of points that are at a chemical distance from 0 smaller than n asymptotically looks like $n\mathcal{B}_{\mu_p}$, where $\mathcal{B}_{\mu_p}$ denotes the unit ball associated with the norm μ_p.

In another paper [13], they studied the fluctuations of $D^{\mathcal{C}'_p}(0,y)/\mu_p(y)$ around its mean and obtained the following large deviations result:

$$\forall\varepsilon>0,\ \limsup_{\|x\|_1\to\infty}\frac{\ln\mathbb{P}_p\left(0\leftrightarrow x,\frac{D^{\mathcal{C}'_p}(0,y)}{\mu_p(y)}\notin(1-\varepsilon,1+\varepsilon)\right)}{\|x\|_1}<0\,.$$

In the same paper, they showed another large deviation result that, as a corollary, proves the continuity of the map $p\to\mu_p$ in $p=1$. In [14], Garet and Marchand obtained moderate deviations of the quantity $|D^{\mathcal{C}'_p}(0,y)-\mu_p(y)|$. In [15], Garet, Marchand, Procaccia and Théret work on the continuity of a more general time constant. A corollary of their work was the continuity of the map $p\to\mu_p$ in $(p_c(d),1]$. Our paper is in the continuity of [15], our aim is to obtain better regularity properties for the map $p\to\mu_p$ than just continuity. The following theorem is the heart of the paper.

Theorem 1.6 (Regularity of the time constant)**.** *Let $p_0>p_c(d)$. There exists a constant κ_d depending only on d and p_0, such that for all $p\leq q$ in $[p_0,1]$*

$$\sup_{x\in\mathbb{S}^{d-1}}|\mu_p(x)-\mu_q(x)|\leq\kappa_d(q-p)|\log(q-p)|\,.$$

We recall that $\mathcal{B}_{\mu_p}$ denotes the unit ball associated with the norm μ_p. From the previous theorem we can easily deduce the following regularity of the asymptotic shapes.

Corollary 1.1 (Regularity of the asymptotic shapes)**.** *Let $p_0 > p_c(d)$. There exists a constant κ'_d depending only on d and p_0, such that for all $p \leq q \in [p_0, 1]$,*

$$d_{\mathcal{H}}(\mathcal{B}_{\mu_q}, \mathcal{B}_{\mu_p}) \leq \kappa'_d (q-p)|\log(q-p)|$$

where $d_{\mathcal{H}}$ is the Hausdorff distance between non-empty compact sets of $\mathbb{R}^d$.

1.5 Background on first passage percolation

Most of the previous results concerning the chemical distance $D^{\mathcal{C}'_p}$ were actually obtained in the more general case of first passage percolation. First passage percolation was first introduced by Hammersley and Welsh [19] as a model for the spread of a fluid in a porous medium. Let us consider the lattice $\mathbb{Z}^d$, with each edge e we associate a non negative random variable $t(e)$ that models the time needed for the fluid to cross the edge e. If all the $t(e)$ are independent and identically distributed with a law G, then for all $x \in \mathbb{Z}^d \setminus \{0\}$, there exists a constant $\mu_G(x)$ that may be seen as the inverse of the speed of spread of the fluid in the direction of x. Under some assumptions on the moments of the variables $(t(e))$, the time needed for the fluid to go from 0 to nx is close to $n\mu_G(x)$ for large n. This result was proved by Cox and Durrett in [9] in dimension 2 under some integrability conditions on G, they also proved that μ_G is a semi-norm. Kesten extended this result to any dimension $d \geq 2$ in [21], and he proved that μ_G is a norm if and only if $G(\{0\}) < p_c(d)$. In the study of first passage percolation, μ_G is usually called the time constant.

It is possible to extend this model by doing first passage percolation on a random environment, for instance on the infinite cluster $\mathcal{C}_p$ of a supercritical Bernoulli percolation of parameter p. To do so, we attribute an infinite value to all p-closed edges, thus G is a probability measure on $[0, +\infty]$ and $\mathcal{C}_p$ can be seen as the infinite cluster of a supercritical Bernoulli percolation of parameter $G([0, \infty)) = p$. The existence of a time constant was first obtained in the context of stationary integrable ergodic field by Garet and Marchand in [12] and was later shown for an independent field without any integrability condition by Cerf and Théret in [6]. Note that in the case of $G_p = p\delta_1 + (1-p)\delta_\infty$, $p > p_c(d)$, the travel time coincides with the chemical distance. The results for the chemical distance may be seen as a corollary of the results for general laws G on $[0, +\infty]$.

The study of the continuity of the map $G \rightarrow \mu_G$ started in dimension 2 with the article of Cox [8]. He showed the continuity of this map under the hypothesis of uniform integrability: if G_n weakly converges toward G and if there exists an integrable law F such that for all $n \in \mathbb{N}$, F stochastically dominates G_n, then $\mu_{G_n} \rightarrow \mu_G$. In [10], Cox and Kesten prove the continuity of this map in dimension 2 without any integrability condition. Their idea was to consider a geodesic for truncated passage times $\min(t(e), M)$, and along it to avoid clusters of bad edges, that is to say edges with a passage time larger than some $M > 0$, by bypassing them with a short path in the boundary of this cluster. Note that by construction, the edges of the boundary have passage time smaller than M. Thanks to combinatorial considerations, they were able to obtain a precise control on the length of these bypasses. This idea was later extended to all the dimensions $d \geq 2$ by Kesten in [21], by taking a M large enough such that the percolation of the edges with a passage time larger than M is highly sub-critical: for such a M, the size of the clusters of bad edges can be controlled. This idea does not work anymore when we allow passage time to take infinite value. In [15], Garet, Marchand, Procaccia and Théret proved the continuity of the map $G \to \mu_G$ for general laws on $[0, +\infty]$ without any moment condition. More precisely, let $(G_n)_{n\in\mathbb{N}}$, and G probability measures on $[0, +\infty]$ such that G_n weakly converges toward G (we write $G_n \overset{d}{\to} G$), that is to say for all continuous bounded functions $f : [0, +\infty] \to [0, +\infty)$, we have

$$\lim_{n\to+\infty} \int_{[0,+\infty]} f dG_n = \int_{[0,+\infty]} f dG \,.$$

Equivalently, we say that $G_n \overset{d}{\to} G$ if and only if $\lim_{n\to+\infty} G_n([t, +\infty]) = G([t, +\infty])$ for all $t \in [0, +\infty]$ such that $x \to G([x, +\infty])$ is continuous at t. If moreover for all $n \in \mathbb{N}$, $G_n([0, +\infty)) > p_c(d)$ and $G([0, +\infty)) > p_c(d)$, then

$$\lim_{n\to\infty} \sup_{x\in\mathbb{S}^{d-1}} |\mu_{G_n}(x) - \mu_G(x)| = 0$$

where $\mathbb{S}^{d-1}$ is the unit sphere for the Euclidean norm.

There is another interpretation of first passage percolation: suppose that the variable $t(e)$ associated with an edge e represents the maximal amount of water that can cross the edge e per second. It follows a natural definition of the maximal flow through Ω, a finite subset of $\mathbb{Z}^d$, from a collection of sources to a collection of sinks. In this paper, we are not going to define properly what a maximal flow is in this more general context, for precise definitions and results we refer to [29]. However, we want to enlight the fact that the study of maximal flows, that has been initiated by Grimmett and Kesten [17] and Kesten [22], may be interpreted as a higher dimensional version of classical first passage percolation. Indeed, by the max-flow min-cut theorem, the study of maximal flows is equivalent to the study of the minimal weights of sets of edges, called cutsets, that cut the sources from the sinks in Ω. These cutsets can be seen as $(d-1)$-dimensional objects, whereas geodesics in the study of classical first passage percolation are one-dimensional objects.

We have explained that the time constant μ_p associated with the chemical distance $D^{\mathcal{C}'_p}$ can be seen as the time constant μ_G associated with first passage percolation where the distribution of the passage times is $G_p = p\delta_1 + (1 - p)\delta_\infty$. Similarly, the flow constant β_p can be seen as the flow constant ν_G associated with first passage percolation (see [29] for a precise definition) where the distribution of the $(t(e))_{e\in\mathbb{E}^d}$ is $G'_p = p\delta_1 + (1 - p)\delta_0$.

In [29], Rossignol and Théret proved in fact the continuity of the map $G \to \nu_G$ in this more general setting.

1.6 Idea of the proof

To study the regularity of the map $p \to \mu_p$, our aim is to control the difference between the chemical distance in the infinite cluster $\mathcal{C}_p$ of a Bernoulli percolation of parameter $p > p_c(d)$ with the chemical distance in $\mathcal{C}_q$ where $q \geq p$. The key part of the proof lies in the modification of a path. We couple the two percolations such that a p-open edge is also q-open but the converse does not necessarily hold. We consider a q-open path for some $q \geq p > p_c(d)$. Some of the edges of this path are p-closed, we want to build upon this path a p-open path by bypassing the p-closed edges. In order to bypass them, we use the idea of [15] and we build our bypasses at a macroscopic scale. This idea finds its inspiration in the works of Antal and Pisztora [26] and Cox and Kesten [10]. We have to consider an appropriate renormalization and we obtain a macroscopic lattice with good and bad sites. Good and bad sites correspond to boxes of size $2N$ in the microscopic lattice. We will do our bypasses using good sites at a macroscopic scale that will have good connectivity properties at a microscopic scale. The remaining of the proof consists in getting probabilistic estimates of the length of the bypass. In this article we improve the estimates obtained in [15]. We get an explicit expression of the appropriate size of a N-box. We use the idea of corridor that appeared in the work of Cox and Kesten [10] to have a better control on combinatorial terms and derive a more precise control of the length of the bypasses than the one obtained in [15]. The same construction of bypasses can be used to study both the chemical distance for $d \geq 2$ and the Cheeger constant in dimension 2.

For $d \geq 3$, thanks to the simpler definition of the norm β_p, we can avoid the renormalization procedure. We will control the difference between the maximal flow for percolations of parameter p and q with $p < q$ by considering an appropriate coupling of these two percolations and considering the number of q-open edges in a minimal cutset for the percolation of parameter p.

Here is the structure of the paper. In section 2, we introduce some definitions and preliminary results that are going to be useful in the following. The section 3 presents the renormalization process and how we modify a q-open path to turn it into a p-open path and how we can control the length of the bypasses. In section 4 and 5, we get probabilistic estimates on the length of the bypasses. Finally, in section 6 we prove the main theorem for the time constant Theorem 1.6 and Corollary 1.1. In section 7, we prove the two main theorems for the Cheeger constant in dimension 2, Theorem 1.1 and Theorem 1.2. In section 8 we prove Theorems 1.3 and 1.4, which are the two main theorems for the modified Cheeger constant for $d \geq 3$, and the corresponding result for the flow constant, Theorem 1.5.

The section 3 is a simplified version of the renormalization process that was already present in [15]. The simplification comes from the fact that we are not interested in general distributions but only on distributions G_p for $p > p_c(d)$ which have the advantage of taking only two values 1 or $+\infty$. The sections 4 and 5 are the most original part of this paper. In section 4, we rewrite an adaptation of several existing proofs for the paper to be self-contained. Except the use of a new coupling and the integration of the results obtained in sections 4 and 5, the sections 6 and 7 are very similar to sections 4 and 6 in [15]. Except the subsection 8.1, which is an adaptation of an existing proof, the section 8 is a generalization of arguments that were already present in section 7 for higher dimensions.

2 Definitions and preliminary results

In this section we give a formal definition of the time constant and of the Cheeger constant.

2.1 Regularized chemical distance and time constant

Let $d \geq 2$, we here consider the graph whose vertices are the points of $\mathbb{Z}^d$, and the set of edges $\mathbb{E}^d$: if two points of $\mathbb{Z}^d$ are neighbors (at Euclidean distance 1) we put an edge between them. We denote by 0 the origin of the graph. Let us recall the different distances in $\mathbb{R}^d$. Let $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we define $\|x\|_1 = \sum_{i=1}^d |x_i|$, $\|x\|_2 = \sqrt{\sum_{i=1}^d x_i^2}$ and $\|x\|_\infty = \max\{|x_i|, i = 1, \dots, d\}$. For $x, y \in \mathbb{Z}^d$, $p > p_c(d)$ and $\mathcal{C}$ a subgraph of $\mathbb{Z}^d$. A path $\gamma = (v_0, e_1, \dots, e_n, v_n)$ is a path from x to y if $v_0 = x$, $v_n = y$ and for all $i \in \{1, \dots, n\}$ the edge $e_i = \langle v_{i-1}, v_i \rangle \in \mathbb{E}^d$, we write by $|\gamma| = n$ the length of γ. We define

$$D^{\mathcal{C}}(x, y) = \inf\{|r| : r \text{ is a path from } x \text{ to } y \text{ in } \mathcal{C}\}$$

the chemical distance between x and y in $\mathcal{C}$. If x and y are not connected in $\mathcal{C}$, $D^{\mathcal{C}}(x, y) = \infty$. In the following, $\mathcal{C}$ will be $\mathcal{C}'_p$ the subgraph of $\mathbb{Z}^d$ whose edges are opened for the Bernoulli percolation of parameter $p > p_c(d)$.

To get round the fact that the chemical distance can take infinite value we introduce regularized chemical distance. Let $\mathcal{C} \subset \mathcal{C}'$ be a connected cluster, we define $\widetilde{x}^{\mathcal{C}}$ as the vertex of $\mathcal{C}$ which minimizes $\|x - \widetilde{x}^{\mathcal{C}}\|_1$ with a deterministic rule to break ties. Thus, $D^{\mathcal{C}'}(\widetilde{x}^{\mathcal{C}}, \widetilde{y}^{\mathcal{C}}) \leq D^{\mathcal{C}}(\widetilde{x}^{\mathcal{C}}, \widetilde{y}^{\mathcal{C}}) < \infty$. Typically, $\mathcal{C}$ is going to be the infinite cluster for Bernoulli percolation with a parameter small enough to get $\mathcal{C} \subset \mathcal{C}'$.

We can define the regularized time constant as in [14] or as a special case of [6].

Proposition 2.1. *Let $p > p_c(d)$. There exists a deterministic function $\mu_p : \mathbb{Z}^d \to [0, +\infty)$, such that for every $p' \in (p_c(d), p]$:*

$$\forall x \in \mathbb{Z}^d \lim_{n\to\infty} \frac{D^{\mathcal{C}_p}(\widetilde{0}^{\mathcal{C}_{p'}}, \widetilde{nx}^{\mathcal{C}_{p'}})}{n} = \mu_p(x) \text{ a.s. and in } L^1.$$

It is important to check that μ_p does not depend on $\mathcal{C}_{p'}$, the cluster we use to stabilize. This is done in Lemma 2.11 in [15]. As a corollary, we obtain the monotonicity of the map $p \to \mu_p$ which is non increasing, see Lemma 2.12 in [15].

Corollary 2.1. *For all $p_c(d) < p \leq q$ and for all $x \in \mathbb{Z}^d$,*

$$\mu_p(x) \geq \mu_q(x) .$$

2.2 Definition of the Cheeger constant in supercritical percolation on $\mathbb{Z}^2$

In this section we define properly the Cheeger constant and state some results obtained in [4].

Let us fix $p > p_c(2)$. For a path $r = (x_0, \dots, x_n)$ and for $2 \leq i \leq n-1$, an edge $e = (x_i, z)$ is a right-boundary edge if z is a neighbor of x_i between x_{i+1} and x_{i-1} in the clockwise direction. The right boundary $\partial^+ r$ is the set of right-boundary edges. A path is called right-most if each edge is used at most once in every orientation and it does not contain right-boundary edges. See Figure 1, the solid lines represent a right-most path and the dashed lines

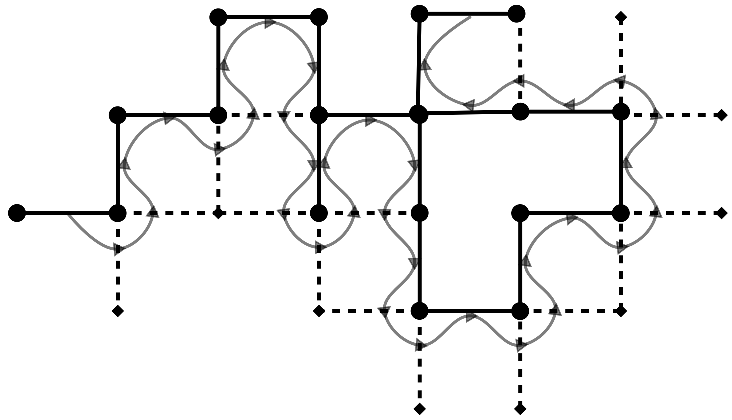


Figure 1 – A right-most path

correspond to the right boundary edges, the curly line shows the orientation of this path. For $x, y \in \mathbb{Z}^2$, we denote by $\mathcal{R}(x,y)$ the set of right-most paths from x to y. Note that this set is deterministic and does not depend on the state of the edges. For a path $r \in \mathcal{R}(x,y)$, define $\mathbf{b}_p(r) = |\{e \in \partial^+ r : \ e \text{ is } p\text{-open}\}|$, the number of p-open edges in the right-boundary $\partial^+ r$ of the path r. Let $\mathcal{C}_p$ be the infinite cluster for the Bernoulli percolation of parameter p. For $x, y \in \mathcal{C}_p$, we can find a p-open self-avoiding path which is also a right-most path, thus the right boundary distance $b_p(x,y) = \inf\{\mathbf{b}_p(r) : r \in \mathcal{R}(x,y),\ r \text{ is } p\text{-open}\}$ is finite. This distance converges uniformly to a norm on $\mathbb{R}^2$:

Proposition 2.2 (Lemma 2.13. in [15])**.** *For any $p > p_c(2)$, there exists a norm β_p on $\mathbb{R}^2$, such that for any $p_0 \in (p_c(2), p]$, $x \in \mathbb{R}^2$,*

$$\beta_p(x) := \lim_{n\to\infty} \frac{b_p(\widetilde{0}^{\mathcal{C}_{p_0}}, \widetilde{nx}^{\mathcal{C}_{p_0}})}{n} \ \mathbb{P}_p\text{-a.s. and in } L^1(\mathbb{P}_p).$$

Moreover, the convergence is uniform on $\mathbb{S}^1$.

We will need in what follows a control on the length of the right-most paths and a control on the amount of right boundary edges:

Lemma 2.1 (Proposition 2.9 in [4])**.** *There exist C, C', $\alpha > 0$ depending on p such that for all $n \in \mathbb{N}$,*

$$\mathbb{P}\left[\exists \gamma \in \bigcup_{x\in\mathbb{Z}^d} \mathcal{R}(0,x) : |\gamma| > n\,,\ \boldsymbol{b}_p(\gamma) \leq \alpha n\right] \leq Ce^{-C'n}\,.$$

Lemma 2.2 (Lemma 2.5 in [4])**.** *For every right-most path γ,*

$$\frac{|\gamma|}{3} - 2 \leq |\partial^+\gamma| \leq 3|\gamma|\,.$$

We can make a connection between the Cheeger constant and the norm β_p by considering a continuous isoperimetric problem. Let us first introduce some important notions. Let $\lambda : [0,1] \to \mathbb{R}^2$ be a continuous curve and ρ a norm, we define the ρ-length of λ to be

$$\mathrm{len}_\rho(\lambda) = \sup_{N\geq 1} \sup_{0\leq t_0 < \dots < t_N \leq 1} \sum_{i=1}^{N} \rho\left(\frac{\lambda(t_i)-\lambda(t_{i-1})}{\|\lambda(t_i)-\lambda(t_{i-1})\|_2}\right) \|\lambda(t_i)-\lambda(t_{i-1})\|_2 \,.$$

We say λ is a Jordan curve if λ is rectifiable (i.e. for any norm ρ, $\mathrm{len}_\rho(\lambda) < \infty$) $\lambda(0) = \lambda(1)$ (i.e. λ ends where it started) and λ is injective on $[0,1)$ (i.e. λ is self-avoiding). For any Jordan curve λ, we can define its interior $\mathrm{int}(\lambda)$ as the unique finite component of $\mathbb{R}^2 \setminus \lambda([0,1])$. We denote by Leb the Lebesgue measure on $\mathbb{R}^2$. The Cheeger constant can be represented as the solution of the following continuous isoperimetric problem:

Theorem 2.1 (Theorem 1.6 in [4])**.** *For every $p > p_c(2)$,*

$$\lim_{n\to\infty} n\varphi_n(p) = (\sqrt{2}\theta_p)^{-1} \inf\{\mathrm{len}_{\beta_p}(\lambda) : \lambda \textit{ is a Jordan curve, } \mathrm{Leb}(\mathrm{int}(\lambda)) = 1\}\,,$$

where $\theta_p = \mathbb{P}[0 \in \mathcal{C}_p]$.

Moreover one obtains a limiting shape for the sets that achieve the minimum in the definition of $\varphi_n(p)$. This limiting shape is given by the Wulff construction [31]. Define

$$W_p = \bigcap_{\widehat{n}:\|\widehat{n}\|_2=1} \{x \in \mathbb{R}^2 : \widehat{n}\cdot x \leq \beta_p(\widehat{n})\} \text{ and } \widehat{W_p} = \frac{W_p}{\sqrt{\mathrm{Leb}(W_p)}} \tag{1}$$

where $\cdot$ denotes the Euclidean inner product. The set $\widehat{W_p}$ is a minimizer for the isoperimetric problem associated with the norm β_p, and it gives the asymptotic shape of minimizer sets in the definition of $\varphi_n(p)$. We define β_p^* the dual norm associated with β_p, we have for $x \in \mathbb{R}^2$

$$\beta_p^*(x) = \sup\{x\cdot y : \beta_p(y) \leq 1\}\,. \tag{2}$$

The Wulff cristal W_p corresponds to the unit ball $\mathcal{B}_{\beta_p^*}$ for the norm β_p^*.

2.3 Definition of the modified Cheeger constant in supercritical percolation on $\mathbb{Z}^d$

The connection between the modified Cheeger constant and the norm β_p, defined in Proposition 1.1, goes as in dimension 2 through a continuous isoperimetric problem. Let $E \subset \mathbb{R}^d$ with Lipschitz boundary, let τ be a norm on $\mathbb{R}^d$. We define the τ-size of the boundary of E by

$$\mathcal{I}_\tau(E) = \int_{\partial E} \tau(\nu_E(x))\mathcal{H}^{d-1}(x)\,, \tag{3}$$

where ∂E denotes the boundary of E, $\nu_E(x)$ the unit exterior vector normal to E at the point $x \in \partial E$, which is defined for $\mathcal{H}^{d-1}$- almost every point of ∂E. We denote by $\mathcal{I}_p$ the function defined in (3) for $\tau = \beta_p$. The function $\mathcal{I}_p$ may be seen as a surface energy functional. This kind of functional was first widely studied in the context of the Ising model in dimension $d \geq 3$ (see for instance [5]).

As in dimension 2, we define the Wulff crystal associated with the norm β_p by

$$W_p = \bigcap_{\overrightarrow{v}\in\mathbb{S}^1} \left\{x \in \mathbb{R}^d \,:\, x\cdot\overrightarrow{v} \leq \beta_p(\overrightarrow{v})\right\} \text{ and } \widehat{W}_p = \frac{2^d W_p}{d!\mathcal{L}^d(W_p)}. \tag{4}$$

We define β_p^* the dual norm associated to β_p as in dimension 2 (see (2)), we have for all $x \in \mathbb{R}^d$:

$$\beta_p^*(x) = \sup\{x\cdot y : \beta_p(y) \leq 1\}.$$

The Wulff crystal W_p corresponds to the unit ball $\mathcal{B}_{\beta_p^*}$ for the norm β_p^*. As in dimension 2 the set $\widehat{W}_p$ is a minimizer for the isoperimetric problem associated with the norm β_p.

Theorem 2.2 (Theorem 1.3. in [16])**.** *Let $d \geq 3$, $p > p_c(d)$ and let β_p be the norm defined in Proposition 1.1 and $\widehat{W}_p$ defined in* (4)*. Then,*

$$\lim_{n\to\infty} n\widehat{\varphi}_n(p) = \frac{\mathcal{I}_p(W_p)}{\theta_p(d)\mathcal{L}^d(W_p)}$$

holds $\mathbb{P}_p$-almost surely. Moreover, the Wulff shape is a solution of the following isoperimetric problem:

$$\textit{minimize } \frac{\mathcal{I}_p(E)}{\mathcal{L}^d(E)} \textit{ subject to } \mathcal{L}^d(E) \leq 1.$$

Remark 2.1. Note that in Theorem 1.3, we obtain a better regularity in p for the modified Cheeger constant for $d \geq 3$ than for the Cheeger constant in dimension 2. The difference is due to the fact that the definition of the norm β_p is simpler than the definition of the norm β_p in dimension 2. In dimension $d \geq 3$, right-boundaries of right-most paths are replaced by cutsets. In dimension 2, we have to consider only p-open right-most paths whereas we do not have an equivalent restriction in dimension $d \geq 3$. Thus, we can avoid the renormalization step in dimensions greater than 3. However, the author of [16] pays the price of this simpler definition of the norm by a very technical proof that uses renormalization arguments.

3 Renormalization

In this section we present the renormalization process. We are here at a macroscopic scale, we define good boxes to be boxes with useful properties to build our modified paths.

3.1 Definition of the renormalization process

Let $p > p_c(d)$ be the parameter of an i.i.d Bernoulli percolation on the edges of $\mathbb{Z}^d$. For a large integer N, that will be chosen later, we set $B_N = [-N,N]^d \cap \mathbb{Z}^d$ and define the following family of N-boxes, for $\mathbf{i} \in \mathbb{Z}^d$,

$$B_N(\mathbf{i}) = \tau_{\mathbf{i}(2N+1)}(B_N)$$

where τ_b denotes the shift in $\mathbb{Z}^d$ with vector $b \in \mathbb{Z}^d$. $\mathbb{Z}^d$ is the disjoint union of this family: $\mathbb{Z}^d = \sqcup_{\mathbf{i}\in\mathbb{Z}^d} B_N(\mathbf{i})$. We need to introduce larger boxes that will help us to link N-boxes together. For $\mathbf{i} \in \mathbb{Z}^d$, we define

$$B'_N(\mathbf{i}) = \tau_{\mathbf{i}(2N+1)}(B_{3N}).$$

A connected cluster C is crossing for a box B, if for all d directions, there is an open path in $C \cap B$ connecting the two opposite faces of B. We define the diameter of a finite cluster $\mathcal{C}$ as

$$\mathrm{Diam}(\mathcal{C}) := \max_{\substack{i=1,\dots,d \\ x,y\in\mathcal{C}}} |x_i - y_i| \, .$$

Let $\mathcal{C}'_p$ be the subgraph of $\mathbb{Z}^d$ whose edges are opened for the Bernoulli percolation of parameter p. We recall that $\mathcal{C}_p$ is the infinite cluster of $\mathcal{C}'_p$, and we have $D^{\mathcal{C}_p}(x,y) = D^{\mathcal{C}'_p}(x,y)$ for every vertices x and y in $\mathcal{C}_p$, and $D^{\mathcal{C}_p}(x,y) = \infty$ if x or y are not in $\mathcal{C}_p$.

To define what a good box is, we have to list properties that a good box should have to ensure that we can build a modification of the path as we have announced in the introduction. We have to keep in mind that all the properties must occur with probability 1 when N goes to infinity.

Definition 3.1. *We say that the macroscopic site $\boldsymbol{i}$ is p-good if the following events occur:*

(i) There exists a unique p-cluster $\mathcal{C}$ in $B'_N(\boldsymbol{i})$ with diameter larger than N;

(ii) This p-cluster $\mathcal{C}$ is crossing for each of the 3^d N-boxes included in $B'_N(\boldsymbol{i})$;

(iii) For all $x,y \in B'_N(\boldsymbol{i})$, if x and y belong to $\mathcal{C}$ then $D^{\mathcal{C}'_p}(x,y) \leq 12\beta N$, for an appropriate β that will be defined later.

$\mathcal{C}$ is called the crossing p-cluster of the p-good box $B_N(\boldsymbol{i})$.

Let us define a percolation by site on the macroscopic grid given by the state of the boxes. Note that the state of the boxes are not independent, there is a short range dependence.

On the macroscopic grid $\mathbb{Z}^d$, we consider the standard definition of closest neighbor, that is to say x and y are neighbors if $\|x-y\|_1 = 1$. Let C be a connected set of macroscopic sites, we define its exterior vertex boundary

$$\partial_v C = \left\{ \begin{array}{c} \mathbf{i} \in \mathbb{Z}^d \setminus C : \mathbf{i} \text{ has a neighbour in C and is connected} \\ \text{to infinity by a } \mathbb{Z}^d\text{-path in } \mathbb{Z}^d \setminus C \end{array} \right\}.$$

For a bad macroscopic site $\mathbf{i}$, let us denote by $C(\mathbf{i})$ the connected cluster of bad macroscopic sites containing $\mathbf{i}$. If $C(\mathbf{i})$ is finite, the set $\partial_v C(\mathbf{i})$ is not connected in the standard definition but it is with a weaker definition of neighbors. We say that two macroscopic sites $\mathbf{i}$ and $\mathbf{j}$ are $*$-neighbors if and only if $\|\mathbf{i}-\mathbf{j}\|_\infty = 1$. Therefore, $\partial_v C(\mathbf{i})$ is an $*$-connected set of good macroscopic sites see for instance Lemma 2 in [30]. We adopt the convention that $\partial_v C(\mathbf{i}) = \{\mathbf{i}\}$ when $\mathbf{i}$ is a good site.

3.2 Modification of a path

Let us consider $p_c(d) < p \leq q$, we fix N in this section. Let us consider a q-open path γ. We consider a coupling of the two percolations of parameter p and q, that we will specify later, such that a p-open edge is q-open. Thus, some edges in γ are p-closed. We denote by γ_o the set of good edges in γ , *i.e.*, the edges that are p-open, and by γ_c the set of p-closed edges in γ. Our aim is to build a bypass for each edge in γ_c using only p-open edges. The proof will follow the proof of Lemma 3.2 in [15] up to some adaptations.

As the bypasses are going to be made at a macroscopic scale, we need to consider the N-boxes that γ crosses. We denote by $\Gamma \subset \mathbb{Z}^d$ the connected set of all the N-boxes visited by γ. Γ is a connected set in the standard definition.

We denote by Bad the random set of bad connected components on the macroscopic percolation given by the states of the N-boxes.

Lemma 3.1. *Let us consider $y, z \in \mathcal{C}_p$ such that the N-boxes of y and z belong to an infinite cluster of p-good boxes. Let us consider a q-open path γ joining y to z. Then there exists a p-open path γ' between y and z that has the following properties:*

(1) $\gamma' \setminus \gamma$ is a set of disjoint self avoiding p-open paths that intersect $\gamma' \cap \gamma$ at their endpoints;

(2) $|\gamma' \setminus \gamma| \leq \rho_d N \left(\sum_{C \in Bad : C \cap \Gamma \neq \emptyset} |C| + |\gamma_c| \right)$, where ρ_d is a constant depending only on the dimension d.

Remark 3.1. Note that here we don't need to introduce a parameter p_0 and require that the bypasses are p_0 open as in [15]. Indeed, this condition was required because finite passage time of edges were not bounded. This is the reason why it was needed in [15] to bypass bad edges with p_0-open edges. These p_0-open edges were precisely edges with passage time smaller than some constant M_0. In our context, we can get rid of this technical aspect because passage time when finite may only take the value 1.

Before proving Lemma 3.1, we need to prove the following lemma that gives a control on the length of a path between two points in a $*$-connected set of good boxes.

Lemma 3.2. *Let $\mathcal{I}$ be a set of $n \in \mathbb{N}^*$ macroscopic sites such that $(B_N(\boldsymbol{i}))_{i \in \mathcal{I}}$ is a $*$-connected set of p-good N-boxes. Let $x \in B_N(\boldsymbol{j})$ be in the p-crossing cluster of $B_N(\boldsymbol{j})$ with $\boldsymbol{j} \in \mathcal{I}$ and $y \in B_N(\boldsymbol{k})$ be in the p-crossing cluster of $B_N(\boldsymbol{k})$ with $\boldsymbol{k} \in \mathcal{I}$. Then, we can find a p-open path joining x and y of length at most $12\beta N n$.*

Proof of Lemma 3.2. Since $\mathcal{I}$ is a $*$-connected set of macroscopic sites, there exists a self-avoiding macroscopic $*$-connected path $(\varphi_i)_{1 \leq i \leq r} \subset \mathcal{I}$ such that $\varphi_1 = \mathbf{j}$, $\varphi_r = \mathbf{k}$. Thus, we get that $r \leq |\mathcal{I}| = n$. As all the sites in $\mathcal{I}$ are good, all the N-boxes of the path $(\varphi_i)_{1 \leq i \leq r}$ are good.

For each $2 \leq i \leq r-1$, we define x_i to be a point in the p-crossing cluster of the box $B_N(\varphi_i)$. We define $x_1 = x$ and $x_r = y$. For each $1 \leq i < r$, x_i and x_{i+1} both belong to $B'_N(\varphi_i)$. Using property (iii) of a p-good box, we can build a p-open path $\gamma(i)$ from x_i to x_{i+1} of length at most $12\beta N$.

By concatenating the paths $\gamma(1), \dots, \gamma(r-1)$ in this order, we obtain a p-open path joining x to y of length at most $12\beta N n$.

□

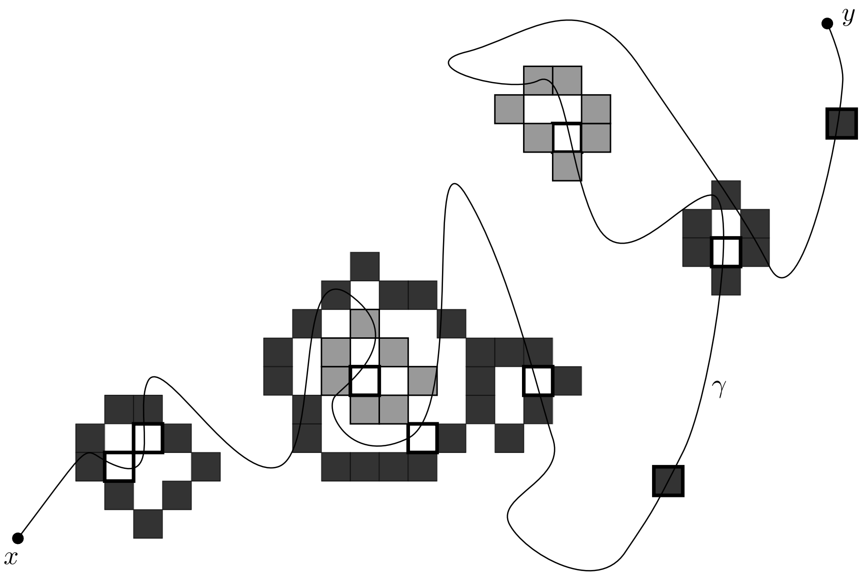
Proof of Lemma 3.1. Let $\varphi_0 = (\varphi_0(j))_{1\leq j\leq r_0}$ be the sequence of N-boxes γ visits. From the sequence φ_0, we can extract the sequence of N-boxes containing at least one p-closed edge of γ. We only keep the indices of the boxes containing the smallest extremity of a p-closed edge of γ for the lexicographic order. We obtain a sequence $\varphi_1 = (\varphi_1(j))_{1\leq j\leq r_1}$. Notice that $r_1 \leq r_0$ and $r_1 \leq |\gamma_c|$.

We want to bypass all the bad edges. Let us consider a bad edge e and $B_N(\mathbf{i})$ its associated N-box. There are two different cases:

— If $B_N(\mathbf{i})$ is a good box, we can build a bypass of e by staying in $B'_N(\mathbf{i})$. We will use the third property of good boxes to control the length of the bypass that will be at most $12\beta N$.

— If $B_N(\mathbf{i})$ is a bad box, we must build a bypass in the exterior vertex boundary $\partial_v C(\mathbf{i})$ that is an $*$-connected component of good boxes. We will use Lemma 3.2 to control the length of this bypass.

Before building our bypasses, we have to get rid of some pathological cases such as $\partial_v C(\varphi_1(j))$ coincide or is nested one in another or overlap. We are going to proceed to further extractions. From the sequence of $*$-connected component



□ : the boxes $(\varphi_1(j))_{1\leq j\leq r_1}$

■ : the sets of good boxes $(S_{\varphi_4(j)})_{1\leq j\leq r_4}$

■ : the sets of good boxes $(S_{\varphi_2(j)})_{1\leq j\leq r_2}$ that do not belong to $(S_{\varphi_4(j)})_{1\leq j\leq r_4}$

Figure 2 – Construction of the path γ' - First step

$(\partial_v C(\varphi_1(j)))_{1\leq j\leq r_1}$, we obtain a sequence of connected components such that each component appears only once. Note that two $*$-connected components of

$(\partial_v C(\varphi_1(j)))_{1\leq j\leq r_1}$ can be $*$-connected together, in that case they count as a unique connected component. We set $\varphi_2(1)=1$ and $S_{\varphi_2(1)}=\bigcup\{\partial_v C(\varphi_1(j)) : \partial_v C(\varphi_1(j)) \overset{*}{\sim} \partial_v C(\varphi_1(1))\}$ where $A\overset{*}{\sim}B$ means that the components A and B are $*$-connected. Assume $\varphi_2(1),\dots,\varphi_2(k)$ and $(S_{\varphi_2(j)})_{1\leq j\leq k}$ are constructed, we set $\varphi_2(k+1)=\inf\{\varphi_2(k)<j\leq r_1 : \partial_v C(\varphi_1(j))\not\subset \cup_{i=1}^k S_{\varphi_2(i)}\}$. If such a j exists we set $S_{\varphi_2(k+1)}=\bigcup\{\partial_v C(\varphi_1(i)) : \partial_v C(\varphi_1(i))\overset{*}{\sim}\partial_v C(\varphi_1(\varphi_2(k+1)))\}$, otherwise we stop the process. We obtain a sequence $(S_{\varphi_2(j)})_{1\leq j\leq r_2}$ where $r_2\leq r_1$. Next, we consider the case of nesting, that is to say when there exist $j\neq k$ such that $S_{\varphi_2(j)}$ is in the interior of $S_{\varphi_2(k)}$. In that case, we only keep the largest connected component: we obtain another subsequence $(S_{\varphi_3(j)})_{1\leq j\leq r_3}$ with $r_3\leq r_2$. Finally, we want to exclude a last case, when between the moment we enter for the first time in a given connected component and the last time we leave this connected component, we have explored other connected components of $(S_{\varphi_3(j)})_{1\leq j\leq r_3}$. That is to say we want to remove the macroscopic loops γ makes between different visits of the same $*$-connected components $S_{\varphi_3(j)}$ (see Figure 2). We iteratively extract from $(S_{\varphi_3(j)})_{1\leq j\leq r_3}$ a sequence $(S_{\varphi_4(j)})_{1\leq j\leq r_4}$ in the following way: $S_{\varphi_4(1)}=S_{\varphi_3(1)}$, assume $(S_{\varphi_4(j)})_{1\leq j\leq k}$ is constructed $\varphi_4(k+1)$ is the smallest indice $\varphi_3(j)$ such that γ visits $S_{\varphi_3(j)}$ after its last visit to $S_{\varphi_4(k)}$. We stop the process when we cannot find such j. Of course, $r_4\leq r_3$. $(S_{\varphi_4(j)})_{1\leq j\leq r_4}$ is a sequence of sets of good N-boxes that are all visited by γ.

Let us introduce some notations (see Figure 3), we denote by $\Psi_{in}(1)=\min\{j\geq 1,\gamma_j\in S_{\varphi_4(1)}\}$, $\Psi_{out}(1)=\max\{j\geq\Psi_{in}(1),\gamma_j\in S_{\varphi_4(1)}\}$. Assume $\Psi_{in}(1),\dots,\Psi_{in}(k)$ and $\Psi_{out}(1),\dots,\Psi_{out}(k)$ are constructed then $\Psi_{in}(k+1)=\min\{j\geq\Psi_{out}(k),\gamma_j\in S_{\varphi_4(k+1)}\}$, $\Psi_{out}(k+1)=\max\{j\geq\Psi_{in}(k+1),\gamma_j\in S_{\varphi_4(k+1)}\}$. Let $B_{in}(j)$ be the N-box in $S_{\varphi_4(j)}$ containing $\gamma_{\Psi_{in}(j)}$, $B_{out}(j)$ be the N-box in $S_{\varphi_4(j)}$ containing $\gamma_{\Psi_{out}(j)}$. Let $\gamma(j)$ be the section of γ from $\gamma_{\Psi_{out}(j)}$ to $\gamma_{\Psi_{in}(j+1)}$ for $1\leq j<r_4$, let $\gamma(0)$ (resp $\gamma(r_4)$) be the section of γ from y to $\gamma_{\Psi_{in}(1)}$ (resp. from $\gamma_{\Psi_{out}(r_4)}$ to z).

We have to study separately the beginning and the end of the path γ. Note that as the N-boxes of y and z both belong to an infinite cluster of good boxes, their box cannot be nested in a bigger $*$-connected components of good boxes of the collection $(S_{\varphi_4(j)})_{1\leq j\leq r_4}$. Thus, if $B_N(\mathbf{k})$, the N-box of y, contains a p-closed edge of γ, necessarily $S_{\varphi_4(1)}$ contains $B_N(\mathbf{k})$, $B_{in}(1)=B_N(\mathbf{k})$ and $\gamma_{\Psi_{in}(1)}=y$. Similarly, if $B_N(\mathbf{l})$, the N-box of z, contains a p-closed edge of γ, necessarily $S_{\varphi_4(r_4)}$ contains $B_N(\mathbf{l})$, $B_{out}(r_4)=B_N(\mathbf{l})$ and $\gamma_{\Psi_{out}(r_4)}=z$.

In order to apply Lemma 3.2, let us show that for every $j\in\{1,\dots,r_4\}$, $\gamma_{\Psi_{in}(j)}$ (resp. $\gamma_{\Psi_{out}(j)}$) belongs to the p-crossing cluster of $B_{in}(j)$ (resp. $B_{out}(j)$). Let us study separately the case of $\gamma_{\Psi_{in}(1)}$ and $\gamma_{\Psi_{out}(r_4)}$. If $\gamma_{\Psi_{in}(1)}=y$ then $\gamma_{\Psi_{in}(1)}$ belongs to the p-crossing cluster of $B_{in}(j)$. Suppose that $\gamma_{\Psi_{in}(1)}\neq y$. As $y\in\mathcal{C}_p$ and y is connected to $\gamma_{\Psi_{in}(1)}$ by a p-open path, $\gamma_{\Psi_{in}(1)}$ is also in $\mathcal{C}_p$. By the property (i) of a good box applied to $B_{in}(1)$, we get that $\gamma_{\Psi_{in}(1)}$ is in the p-crossing cluster of $B_{in}(1)$. We study the case of $\gamma_{\Psi_{out}(r_4)}$ similarly. To study $\gamma_{\Psi_{in}(j)}$ (resp. $\gamma_{\Psi_{out}(j)})$) for $j\in\{2,\dots,r_4-1\}$, we use the fact that by construction, thanks to the extraction φ_2, two different elements of $(S_{\varphi_4(j)})_{1\leq j\leq r_4}$ are not $*$-connected. Therefore, for $1\leq j<r_4$, $\|\gamma_{\Psi_{in}(j+1)}-\gamma_{\Psi_{out}(j)}\|_1\geq N$ and $\gamma(j)$, the section of γ from $\gamma_{\Psi_{out}(j)}$ to $\gamma_{\Psi_{in}(j+1)}$, has a diameter larger than N and contains only p-open edges. As $B_{out}(j)$ and $B_{in}(j+1)$ are good boxes, we obtain, using again property (i) of good boxes, that $\gamma_{\Psi_{out}(j)}$ and $\gamma_{\Psi_{in}(j+1)}$

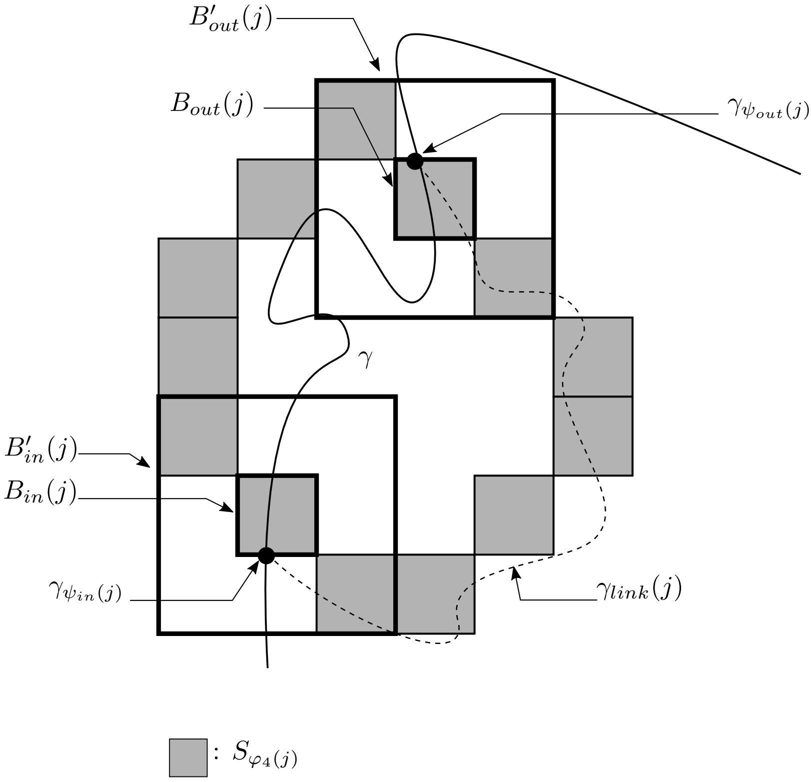


Figure 3 – Construction of the path γ' - Second step

belong to the p-crossing cluster of their respective boxes.

Finally, by Lemma 3.2, for every $j \in \{1, \dots, r_4\}$, there exists a p-open path $\gamma_{link}(j)$ joining $\gamma_{\Psi_{in}(j)}$ and $\gamma_{\Psi_{out}(j)}$ of length at most $12\beta N|S_{\varphi_4(j)}|$.

We obtain a p-open path γ' joining y and z by concatenating $\gamma(0), \gamma_{link}(1), \gamma(1), \dots, \gamma_{link}(r_4), \gamma(r_4)$ in this order. Up to cutting parts of these paths, we can suppose that each $\gamma_{link}(j)$ is a self-avoiding path, that all the $\gamma_{link}(j)$ are disjoint and that each $\gamma_{link}(j)$ intersects only $\gamma(j-1)$ and $\gamma(j)$ at their endpoints.

Let us estimate the quantity $|\gamma' \setminus \gamma|$, as $\gamma' \setminus \gamma \subset \cup_{i=1}^{r_4} \gamma_{link}(i)$, we obtain:

$$\begin{aligned}
|\gamma' \setminus \gamma| &\leq \sum_{j=1}^{r_4} |\gamma_{link}(j)| \\
&\leq \sum_{j=1}^{r_4} 12\beta N |S_{\varphi_4(j)}| \\
&\leq 12\beta N |\gamma_c| + 12\beta N \sum_{C \in Bad : C \cap \Gamma \neq \emptyset} |\partial_v C|
\end{aligned}$$

where the last inequality comes from the fact that each $S_{\varphi_4(j)}$ is the union of elements of $\{\partial_v C : C \in Bad; C \cap \Gamma \neq \emptyset\}$ and of good boxes that contain edges of γ_c. We conclude by noticing that $|\partial_v C| \leq 2d|C|$.

□

3.3 Deterministic estimate

When $q-p$ is small, we want to control the probability that the bypass of bad edges $\gamma' \setminus \gamma$ is big. We can notice in Lemma 3.1 that we need to control the bad connected components. This will be done in section 5. We will also need a deterministic control on $|\Gamma|$ which is the purpose of the following Lemma (this Lemma is an adaptation of Lemma 3.4 of [15]).

Lemma 3.3. *For every path γ of $\mathbb{Z}^d$, for every $N \in \mathbb{N}^*$, there exists a $*$-connected macroscopic path $\widetilde{\Gamma}$ such that*

$$\gamma \subset \bigcup_{\boldsymbol{i} \in \widetilde{\Gamma}} B'_N(\boldsymbol{i}) \text{ and } |\widetilde{\Gamma}| \leq 1 + \frac{|\gamma|+1}{N}\,.$$

Proof. Let $\gamma = (\gamma_i)_{1 \leq i \leq n}$ be a path of $\mathbb{Z}^d$ where γ_i is the i-th vertex of γ. Let Γ be the set of N-boxes that γ visits. We are going to define iteratively the macroscopic path $\widetilde{\Gamma}$.

Let $p(1) = 1$ and $\mathbf{i}_1$ be the macroscopic site such that $\gamma_1 \in B_N(\mathbf{i}_1)$. We suppose that $\mathbf{i}_1, \dots, \mathbf{i}_k$ and $p(1), \dots, p(k)$ are constructed. Let us define $p(k+1) = \min\{j > p(k) : \gamma_j \notin B'_N(\mathbf{i}_k)\}$. If this set is not empty, we set $\mathbf{i}_{k+1}$ to be the macroscopic site such that $\gamma_{p(k+1)} \in B_N(\mathbf{i}_{k+1})$. Otherwise, we stop the process, and we get that for every $j \in \{p(k), \dots, n\}$, $\gamma_j \in B'_N(\mathbf{i}_k)$. As n is finite, the process will eventually stop and the two sequences $(p(1), \dots, p(r))$ and $(\mathbf{i}_1, \dots, \mathbf{i}_r)$ are finite. We define $\widetilde{\Gamma} = (\mathbf{i}_1, \dots, \mathbf{i}_r)$. By construction,

$$\gamma \subset \bigcup_{\mathbf{i} \in \widetilde{\Gamma}} B'_N(\mathbf{i})\,.$$

Notice that for every $1 \leq k < r$, $\|\gamma_{p(k+1)} - \gamma_{p(k)}\|_1 \geq N$, thus $p(k+1) - p(k) \geq N$. This leads to $N(r-1) \leq p(r) - p(1) \leq n$, and finally,

$$|\widetilde{\Gamma}| \leq 1 + \frac{|\gamma|+1}{N}\,.$$

□

Remark 3.2. This Lemma implies that if Γ is the set of N-boxes that γ visits then

$$|\Gamma| \leq 3^d |\widetilde{\Gamma}| \leq 3^d \left(1 + \frac{|\gamma|+1}{N}\right)\,.$$

4 Control of the probability that a box is good

We need in what follows to control the quantity $\sum |C|$ where the sum is over all $C \in Bad$ such that $C \cap \Gamma \neq \emptyset$. In the macroscopic grid, all the sites are not independent from each other, there is a short range of dependence. Thus, such quantities are hard to compute. That's why we need to have a stochastic comparison with an i.i.d field of Bernoulli random variables where the independence make computations easier. For that purpose, we are going to use Liggett, Schonmann and Stacey's result in [23]. Let us assume that the probability that a box is good goes to 1 when N goes to infinity. Then, for any given parameter $\mathbf{p}$, we can always find a large enough N such that the field $(\mathbb{1}_{\{B_N(\mathbf{i}) \text{ is good}\}})_{\mathbf{i}\in\mathbb{Z}^d}$ stochastically dominates a family of independent Bernoulli random variables with parameter $\mathbf{p}$. However, we will need in what follows to have an explicit expression of N in terms of $\mathbf{p}$. This section was not present in [15] where they do not need an explicit control on N.

4.1 Explicit LSS

In this section, we want to find an explicit function $\psi := \psi(\mathbf{p})$ such that if N is such that $\mathbb{P}\,(B_N \text{ is a good box}) \geq \psi(\mathbf{p})$, then the field $(\mathbb{1}_{\{B_N(\mathbf{i}) \text{ is good}\}})_{\mathbf{i}\in\mathbb{Z}^d}$ stochastically dominates a family of independent Bernoulli random variables with parameter $\mathbf{p}$. We have to keep in mind that $\mathbf{p}$ is close to 1.

Proposition 4.1. *Let us define by $Y_{\mathbf{i}} = \mathbb{1}_{\{B_N(\mathbf{i}) \text{ is good}\}}$ and let $\mathbf{p} \geq \frac{1}{4}$. Let us assume there exists a constant k_d depending only on the dimension such that*

$$\mathbb{P}\,(B_N \text{ is a good box}) \geq 1 - (1-\sqrt{\mathbf{p}})^{k_d+1} .$$

Then the field $(\mathbb{1}_{\{B_N(\mathbf{i}) \text{ is good}\}})_{\mathbf{i}\in\mathbb{Z}^d}$ stochastically dominates a family of independent Bernoulli random variables with parameter $\mathbf{p}$.

Adaptation of the proof of Theorem 1.5 in [23]. Let $0 < \delta < 1$ such that we have $\mathbb{P}\,(B_N \text{ is a good box}) \geq \delta$. By construction, the family $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ is κ-dependent and identically distributed where κ is a constant depending on β, this dependence comes from property *(iii)* of a good box. Following the proof of Liggett, Schonmann and Stacey in [23], if we find two functions $\alpha := \alpha(\delta)$ and $\rho := \rho(\delta)$ satisfying:

$$0 < \alpha(\delta), \rho(\delta) < 1 \tag{5}$$

$$(1-\alpha(\delta))(1-\rho(\delta))^{k_d} \geq 1-\delta \tag{6}$$

$$(1-\alpha(\delta))\alpha(\delta)^{k_d} \geq 1-\delta \tag{7}$$

$$\lim_{\delta\to 1} \alpha(\delta)\rho(\delta) = 1 \tag{8}$$

where $k_d = |B(\kappa)| = (2\kappa+1)^d$ is the number of sites in a box of side 2κ, then $(Y_{\mathbf{i}})_{\mathbf{i}\in\mathbb{Z}^d}$ stochastically dominates a family of independent Bernoulli random variables with parameter $\pi(\delta) = \alpha(\delta)\rho(\delta)$. Here, we want $\pi(\delta) = \mathbf{p}$ and $\psi(\mathbf{p}) = \pi^{-1}(\mathbf{p})$. It is enough to assume that $(1-\alpha(\delta))(1-\rho(\delta))^{k_d} = 1-\delta$ to satisfy (6). Next, we choose α and ρ such that

$$1-\alpha = (1-\delta)^{\frac{1}{k_d+1}}$$

$$(1-\rho)^{k_d} = (1-\delta)^{\frac{k_d}{k_d+1}} .$$

Then, $\alpha(\delta) = \rho(\delta) = 1-(1-\delta)^{\frac{1}{k_d+1}}$ and the conditions (5), (6) and (8) are satisfied. The condition (7) is therefore equivalent to $\alpha(1-\pi/\alpha)^{-1} \geq 1$, it holds when $\pi = \mathbf{p} \geq 1/4$. As

$$\pi(\delta) = \left(1-(1-\delta)^{\frac{1}{k_d+1}}\right)^2 ,$$

we get $\pi^{-1}(\mathbf{p}) = 1-\left(1-\sqrt{\mathbf{p}}\right)^{k_d+1}$ when $\mathbf{p} \geq 1/4$. □

Remark 4.1. We did not try here to get an optimal condition on the probability a box is good. Even though this condition is not optimal and it is very likely that one can do better, this will be enough for our purpose.

4.2 Control of the probability of being a bad box

In this section, we want to prove that the probability of being a p-bad N-box decays exponentially fast with N. The main difficulty of this section is to get an exponential decay which is uniform in p. For that purpose, we are going to introduce a parameter $p_0 > p_c(d)$ and show that exponential decay is uniform for all $p \geq p_0$. Indeed, the speed will only depend on p_0.

Theorem 4.1. *Let $p_0 > p_c(d)$. There exist constants $C_0(p_0)$, $A(p_0)$ and $B(p_0)$ such that for all $p \geq p_0$ and for all $N > C_0$*

$$\mathbb{P}(B_N \text{ is } p\text{-bad}) \leq A(p_0)\exp(-B(p_0)N)\,.$$

The property (ii) of the definition of p-good box is a non-decreasing event in p. Thus, it will be easy to bound uniformly the probability that property (ii) is not satisfied by something depending only on p_0. However, for properties (i) and (iii) a uniform bound is more delicate to obtain. Before proving Theorem 4.1, we need the two following lemmas that deal with properties (i) and (iii).

Let $T_{m,N}(p)$ be the event that B_N has a p-crossing cluster and contains some other p-open cluster D having diameter at least m.

Lemma 4.1. *Let $p_0 > p_c(d)$, there exist $\mu = \mu(p_0, d) > 0$ and $\kappa = \kappa(p_0, d)$ such that for all $p \geq p_0$*

$$\mathbb{P}(T_{m,N}(p)) \leq \kappa N^{2d}\exp(-\mu m)\,. \tag{9}$$

Lemma 4.2. *Let $p_0 > p_c(d)$, there exist $\beta = \beta(p_0) > 0$, $\widehat{A} = \widehat{A}(p_0)$ and $\widehat{B} = \widehat{B}(p_0) > 0$ such that for all $p \geq p_0$*

$$\forall x \in \mathbb{Z}^d, \mathbb{P}(\beta\|x\|_1 \leq D^{\mathcal{C}'_p}(0,x) < +\infty) \leq \widehat{A}\exp(-\widehat{B}\|x\|_1)\,. \tag{10}$$

Remark 4.2. Lemma 4.2 is an improvement of the result obtained in [26], as the constants $\widehat{A}$ and $\widehat{B}$ are the same for all $p \geq p_0$. In the original result, the constants depend on p. To show this result, we slightly modify the proof of [26]. The proof is simplified by the use of the result of Liggett, Schonmann and Stacey [23] that was not published at that time.

Before proving these two lemmas, we are first going to prove Theorem 4.1 thanks to them.

Proof of Theorem 4.1. Let us fix $p_0 > p_c(d)$. Let us denote by $(iii)'$ the property that for all $x, y \in B'_N(\mathbf{i})$, if $\|x-y\|_\infty \geq N$ and if x and y belong to the p-crossing cluster $\mathcal{C}$ then $D^{\mathcal{C}'_p}(x,y) \leq 6\beta N$. Note that properties (ii) and $(iii)'$ imply property (iii). Indeed, thanks to the second property, we can find $z \in \mathcal{C} \cap B'_N(\mathbf{i})$ such that $\|x-z\|_\infty \geq N$ and $\|y-z\|_\infty \geq N$. Therefore, by applying the third property,

$$\begin{aligned} D^{\mathcal{C}'_p}(x,y) &\leq D^{\mathcal{C}'_p}(x,z) + D^{\mathcal{C}'_p}(z,y) \\ &\leq 12\beta N\,. \end{aligned}$$

Thus, we can bound the probability that a N-box is bad by the probability that it does not satisfy one of the properties (i), (ii) or $(iii)'$. Since we want to control the probability of B_N being a p-bad box uniformly in p, we will emphasize the dependence of (i), (ii) and $(iii)'$ in p by writing $(i)_p$, $(ii)_p$ and $(iii)'_p$. First, let us prove that the probability that a N-box does not satisfy property $(ii)_p$, i.e., the probability for a box not to have a p-crossing cluster, is decaying exponentially, see for instance Theorem 7.68 in [18] or [26]. There exist constants $\kappa_1(p_0)$ and $\kappa_2(p_0)$ such that for all $p \geq p_0$

$$\begin{aligned}\mathbb{P}(B_N \text{ does not satisfies } (ii)_p) &\leq \mathbb{P}(B_N \text{ does not satisfies } (ii)_{p_0}) \\ &\leq \kappa_1(p_0)\exp(-\kappa_2(p_0)N^{d-1})\,. \end{aligned} \tag{11}$$

Next, let us bound the probability that a N-box does not satisfy property $(iii)'_p$. Using Lemma 4.2, for $p \geq p_0$,

$$\begin{aligned}&\mathbb{P}[B_N \text{ does not satisfy } (iii)'_p] \\ &\quad\leq \sum_{x\in B'_N}\sum_{y\in B'_N}\mathbb{1}_{\|x-y\|_\infty\geq N}\mathbb{P}\left[6\beta N \leq D^{\mathcal{C}'_p}(x,y) < +\infty\right] \\ &\quad\leq \sum_{x\in B'_N}\sum_{y\in B'_N}\mathbb{1}_{\|x-y\|_\infty\geq N}\mathbb{P}\left[\beta\|x-y\|_\infty \leq D^{\mathcal{C}'_p}(x,y) < +\infty\right] \\ &\quad\leq \sum_{x\in B'_N}\sum_{y\in B'_N}\mathbb{1}_{\|x-y\|_\infty\geq N}\widehat{A}\exp(-\widehat{B}N) \\ &\quad\leq (6N+1)^{2d}\widehat{A}\exp(-\widehat{B}N)\,.\end{aligned}$$

Finally, by Lemma 4.1,

$$\begin{aligned}&\mathbb{P}(B_N \text{ is } p\text{-bad}) \\ &\quad\leq \mathbb{P}[B_N \text{ does not satisfies}(ii)_p] + \mathbb{P}[B_N \text{ satisfies } (ii)_p \text{ but not } (i)_p] \\ &\qquad\quad + \mathbb{P}[B_N \text{ does not satisfy } (iii)'_p] \\ &\quad\leq \kappa_1\exp(-\kappa_2N^{d-1}) + 3^d\kappa N^{2d}\exp(-\frac{\mu}{2}N) + (6N+1)^{2d}\widehat{A}\exp(-\widehat{B}N) \\ &\quad\leq A(p_0)e^{-B(p_0)N}\,.\end{aligned}$$

For the second inequality, we used inequality (11) and the fact that the event that the 3^d N-boxes of B'_N are crossing and there exist another p-open cluster of diameter larger than N in B'_N is included in the event there exists a N-box in B'_N that has a crossing property and contains another p-open cluster of diameter larger than $N/2$. The last inequality holds for $N \geq C_0(p_0)$, where $C_0(p_0)$, $A(p_0) > 0$ and $B(p_0) > 0$ depends only on p_0 and on the dimension d. □

Proof of Lemma 4.1. In dimension $d \geq 3$, we refer to the proof of Lemma 7.104 in [18]. The proof of Lemma 7.104 requires the proof of Lemma 7.78. The probability controlled in Lemma 7.78 is clearly non decreasing in the parameter p. Thus, if we choose $\delta(p_0)$ and $L(p_0)$ as in the proof of Lemma 7.78 for $p_0 > p_c(d)$, then these parameters can be kept unchanged for some $p \geq p_0$. Thanks to Lemma 7.104, we obtain

$$\forall p \geq p_0,\, \mathbb{P}(T_{m,N}(p)) \leq d(2N+1)^{2d} \exp\left(\left(\frac{m}{L(p_0)+1} - 1\right) \log(1-\delta(p_0))\right)$$
$$\leq \frac{d.3^d}{1-\delta(p_0)} N^{2d} \exp\left(-\frac{-\log(1-\delta(p_0))}{L(p_0)+1} m\right) .$$

We get the result with $\kappa = \frac{d.3^d}{1-\delta(p_0)}$ and $\mu = \frac{-\log(1-\delta(p_0))}{L(p_0)+1} > 0$.

In dimension 2, the result is obtained by Couronné and Messikh in the more general setting of FK-percolation in Theorem 9 in [7]. We proceed similarly as in dimension $d \geq 3$, the constant appearing in this theorem first appeared in Proposition 6. The probability of the event considered in this proposition is clearly increasing in the parameter of the underlying percolation, it is an event for the subcritical regime of the Bernoulli percolation. Let us fix a $p_0 > p_c(2) = 1/2$, then $1-p_0 < p_c(2)$ and we can choose the parameter $c(1-p_0)$ and keep it unchanged for some $1-p \leq 1-p_0$. In Theorem 9, we get the expected result with $c(1-p_0)$ for a $p \geq p_0$ and $g(n) = n$. □

Proof of Lemma 4.2. We follow the proof of Antal and Pisztora in [1]. We first have to define a renormalization, we keep the same definition for boxes B_N and B'_N. We say that the macroscopic site $\mathbf{i}$ is p-nice if the following events occur:

(i)$_p$ There exists a unique p-cluster $\mathcal{C}$ in $B'_N(\mathbf{i})$ with diameter larger than N;

(ii)$_p$ This p-cluster $\mathcal{C}$ is crossing for each of the 3^d N-boxes included in $B'_N(\mathbf{i})$.

$\mathcal{C}$ is called the crossing p-cluster of the p-nice box $B_N(\mathbf{i})$. Otherwise we say that the site is p-wrong. Note that this definition of a p-nice site is slightly different from the one used in [26]. They had to do additional work that required an additional property due to the fact that Liggett, Schonmann and Stacey's result [23] was not written yet.

Let us fix $p_0 > p_c(d)$. For all $p \geq p_0$, using Lemma 4.1, we have

$$\begin{aligned}
&\mathbb{P}(B_N \text{ is } p\text{-wrong})\\
&\quad\leq \mathbb{P}(B_N \text{ does not satisfies}(ii)_p) + \mathbb{P}(B_N \text{ satisfies } (ii)_p \text{ but not } (i)_p)\\
&\quad\leq \kappa_1(p_0)\exp(-\kappa_2(p_0)N^{d-1}) + 3^d \mathbb{P}(T_{\frac{N}{2},N}(p))\\
&\quad\leq \kappa_1(p_0)\exp(-\kappa_2(p_0)N^{d-1}) + 3^d \kappa N^{2d} \exp(-\frac{\mu}{2}N) .
\end{aligned}$$

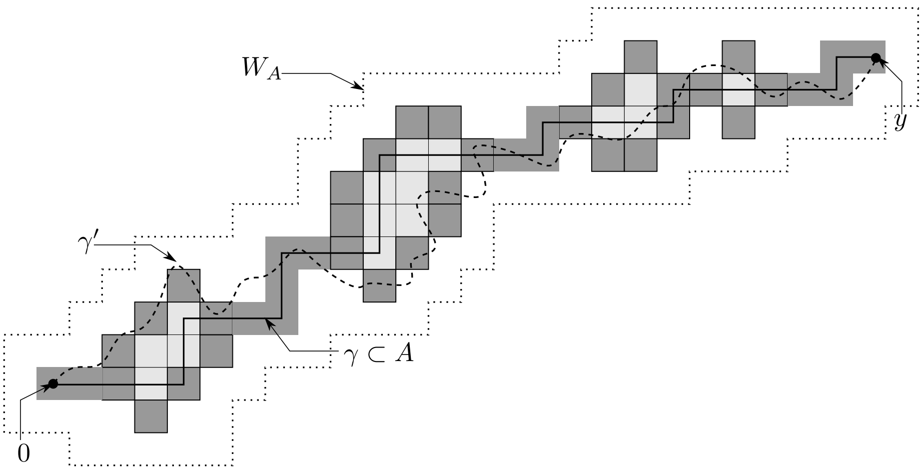
Using Liggett, Schonmann and Stacey's result, we know that there exists a function $\mathbf{p}(N)$ depending only on N and p_0, such that $\lim_{N\to\infty} \mathbf{p}(N) = 1$ and for all $p \geq p_0$, $(\mathbb{1}_{\{B_N(\mathbf{i}) \text{ is good}\}})_{\mathbf{i}\in\mathbb{Z}^d}$ stochastically dominates a family of independent Bernoulli random variables with parameter $\mathbf{p}(N)$.

Next, we have to build a short path with a length that we can easily control. We now consider 0 and $y \in \mathbb{Z}^d$. We assume that there exists a p-open path joining 0 and y. Let us denote by $\mathbf{a}(y)$ the indice of the N-box containing y. Let $n = \|\mathbf{a}(y)\|_1$ and $A = (\mathbf{a}_0, \dots, \mathbf{a}_n)$ be a macroscopic path joining $\mathbf{a}_0 = \mathbf{0}$ to $\mathbf{a}_n = \mathbf{a}(y)$. The different sites a_i may be nice or wrong. We denote by $Wrong$ the set of all wrong connected components on the macroscopic percolation given by the states of the N-boxes. For $\mathbf{i} \in \mathbb{Z}$, we denote by $C_{\mathbf{i}}$ the element of $Wrong$ containing $\mathbf{i}$ with the convention $C_{\mathbf{i}} = \emptyset$ if $\mathbf{i}$ is nice. We recall that $\partial_v C$ denotes the exterior vertex boundary and is $*$-connected.

Let us consider γ a path joining 0 and y that lies in A. Some of its edges may be p-closed. Using the modification of a path of Lemma 3.1, we know we can build a p-open path γ' joining 0 and y by bypassing p-closed edges of γ. Here the hypothesis that 0 and y belong to the infinite cluster $\mathcal{C}_p$ is replaced by the hypothesis that there exists a p-open path joining these two points. Of course, as nice boxes do not have the property (iii) of good boxes, we cannot hope to get a precise upper bound of the length of γ', this is the purpose of this section. However, we can include γ' in a bigger set of boxes whose size can be controlled, see Figure 4. By slightly modifying the proof of Lemma 3.1, it is easy to see that γ' is included in the following set:

$$W_A := \bigcup_{\mathbf{a}\in A}\left(\bigcup_{\mathbf{b}\in\partial_v C_{\mathbf{a}}} B'_N(\mathbf{b})\right) . \tag{12}$$

We recall the convention $\partial_v C_{\mathbf{a}} = \mathbf{a}$ if $\mathbf{a}$ is nice. Actually, the construction of such a path in the original article [26] is slightly different. Instead of taking connected wrong components they took $*$-connected wrong components. However the spirit of the proof remains the same. Here, we made the choice of being consistent with the previous section. As $\gamma' \subset W_A$ we obtain



: nice boxes

: the set of nice boxes $(\partial_v C)_{C\in Wrong: A\cap C\neq\emptyset}$

: the set of wrong connected components C such that $A \cap C \neq \emptyset$

Figure 4 – Construction of a path γ' included in W_A

$$\left\{\beta\|y\|_1 \leq D^{\mathcal{C}_p}(0,y) < +\infty\right\} \subset \{|W_A| \geq \beta\|y\|_1\} , \tag{13}$$

where $|W_A|$ denotes the number of edges inside the boxes of W_A (and not the number of N-boxes in W_A). As $|\partial_v C| \leq 2d|C|$,

$$\begin{aligned}
|W_A| &\leq 3^d(2N+1)^d \left(|A| + \sum_{C \in Wrong : A \cap C \neq \emptyset} |\partial_v C| \right) \\
&\leq 3^d(2N+1)^d \left(n + 1 + 2d \sum_{C \in Wrong : A \cap C \neq \emptyset} |C| \right) \\
&\leq c_d N^d \left(n + 1 + \sum_{C \in Wrong : A \cap C \neq \emptyset} |C| \right) \qquad (14)
\end{aligned}$$

where $c_d > 0$ only depends on the dimension d. Using the stochastic minoration, we get

$$\begin{aligned}
\mathbb{P}(\beta \|y\|_1 \leq D^{\mathcal{C}_p}(0, y) < +\infty) &\leq \mathbb{P}\left(n + 1 + \sum_{C \in Wrong : A \cap C \neq \emptyset} |C| \geq \frac{\beta \|y\|_1}{c N^d} \right) \\
&\leq \mathbb{P}_{\mathbf{p}(N)}\left(n + 1 + \sum_{C \in Wrong : A \cap C \neq \emptyset} |C| \geq \frac{\beta \|y\|_1}{c_d N^d} \right), \qquad (15)
\end{aligned}$$

where under $\mathbb{P}_{\mathbf{p}(N)}$ the field $(\mathbb{1}_{B_N(\mathbf{i}) \text{ is } p\text{-nice}})_{\mathbf{i} \in \mathbb{Z}^d}$ is i.i.d. with Bernoulli law of parameter $\mathbf{p}(N)$. In the spirit of the work of Fontes and Newman [11], we introduce $(\widetilde{C}(i))_{0 \leq i \leq n}$ an independent and identically distributed family of random sets of $\mathbb{Z}^d$ with the same law as C_0 under $\mathbb{P}_{\mathbf{p}(N)}$. We have the following stochastic domination:

$$\sum_{C \in Wrong : A \cap C \neq \emptyset} |C| \preceq \sum_{i=0}^{n} |\widetilde{C}(i)| \,. \qquad (16)$$

Therefore, the right hand side of (15) is smaller than

$$\mathbb{P}_{\mathbf{p}(N)}\left(\frac{1}{n+1} \sum_{i=0}^{n} (|\widetilde{C}(i)| + 1) \geq c_d^{-1} N^{-d} \frac{\beta \|y\|_1}{n+1} \right). \qquad (17)$$

For N large enough, and for some $h > 0$,

$$\mathbb{E}_{\mathbf{p}(N)}\left[\exp(h(|C_0| + 1)) \right] < \infty \,. \qquad (18)$$

We fix $N = N(p_0, d)$ such that (18) holds. For $\|y\|_1$ large enough, $\|y\|_1/(n+1) \geq N$. Finally, we choose $\beta = \beta(p_0, d)$ large enough such that:

$$\mathbb{E}_{\mathbf{p}(N)}\left[|C_0| + 1 \right] < \beta c^{-1} N^{-d+1} \,. \qquad (19)$$

By Cramer's theorem, the probability in (15) has an exponential decay in n with constants depending only on p_0 and d. And as $n = \|a(y)\|_1$, n is of the order of $\|y\|_1/N$, the probability in (15) also exponentially decays in $\|y\|_1$. $\square$

4.3 Conclusion

In this section, we want to deduce from the previous results an explicit form for N in order to have the expected stochastic comparison.

Proposition 4.2. *Let $p_c(d) < p_0$ and $p_c^{site}(d)$ be the critical parameter for independent Bernoulli site percolation on $\mathbb{Z}^d$. We consider $\boldsymbol{p} = 1-\alpha \in (p_c^{site}(d), 1)$ where $\alpha > 0$. There exist two positive constants $\widehat{C}_0$ and $\widehat{C}_1$ depending only on p_0 and the dimension d, such that for $N(p_0, d, \boldsymbol{p}) = N(\boldsymbol{p}) = \widehat{C}_0|\log\alpha| + \widehat{C}_1$, for any $p \geq p_0$ the field $(\mathbb{1}_{\{B_N(i) \text{ is } p\text{-good}\}})_{i\in\mathbb{Z}^d}$ stochastically dominates a family of independent Bernoulli random variables with parameter $\boldsymbol{p}$.*

Proof. By Proposition 4.1, in order to get the stochastic comparison with an i.i.d. field of Bernoulli random variable of parameter $\mathbf{p}$, we need

$$\mathbb{P}(B_N \text{ is } p\text{-bad}) \leq (1-\sqrt{\mathbf{p}})^{k_d+1} \,.$$

As the function $x \to \sqrt{x}$ is concave, its curve is below its tangent in 1, we get $\sqrt{x} \leq (x+1)/2$ and $\sqrt{1-x} \leq 1 - x/2$. Therefore,

$$(1-\sqrt{\mathbf{p}})^{k_d+1} = \left(1-\sqrt{1-\alpha}\right)^{k_d+1} \geq \left(\frac{\alpha}{2}\right)^{k_d+1} \,.$$

Thanks to Theorem 4.1, it is sufficient to choose $N > C_0$ and such that

$$A(p_0)e^{-B(p_0)N} = \left(\frac{\alpha}{2}\right)^{k_d+1}$$

that is to say

$$N = \frac{k_d+1}{B(p_0)}|\log\alpha| + \frac{(k_d+1)\log 2 + \log A(p_0)}{B(p_0)} + C_0 \,. \tag{20}$$

□

5 Probabilistic estimates

We can now use the stochastic minoration by a field of independent Bernoulli variables to control the probability that the quantity $\sum|C|$ is big, where the sum is over all $C \in Bad$ such that $C \cap \Gamma \neq \emptyset$. The proof of the following Lemma is in the spirit of the work of Cox and Kesten in [10] and relies on combinatorial considerations. These combinatorial considerations were not necessary in [15].

We consider a path γ and its associated lattice animal Γ. We need in the proof of the following Lemma to define Γ as a path of macroscopic sites, that is to say a path $(\mathbf{i}_k)_{k\leq r}$ in the macroscopic grid such that $\cup_{k\leq r} B_N(\mathbf{i}_k) = \Gamma$ (this path may not be self-avoiding). We can choose for instance the sequence of sites that γ visits. However, it is difficult to control the size of this sequence by the size of Γ. Therefore, we consider the path of the macroscopic grid $\widetilde{\Gamma}$ that was introduced in Lemma 3.3.

Lemma 5.1. *Let $p_c(d) < p_0$, let $0 < \varepsilon < 1 - p_c(d)$ and $p_c^{site}(d)$ be the critical parameter for independent Bernoulli site percolation on $\mathbb{Z}^d$. There exists $\boldsymbol{p}(\varepsilon) =$*

$1-\alpha(\varepsilon) \in (p_c^{site}(d), 1)$ and $C_\varepsilon \in (0,1)$ depending only on ε, such that if we choose $N(p_0, d, \boldsymbol{p}(\varepsilon))$ as in Proposition 4.2, then for all $p \geq p_0$, for every $n \in \mathbb{N}^*$

$$\mathbb{P}\left(\left\{|\widetilde{\Gamma}| \leq n\right\} \cap \left\{\sum_{C \in Bad: C\cap\Gamma\neq\emptyset} |C| \geq \varepsilon n\right\}\right) \leq C_\varepsilon^n$$

where Γ is the associated lattice animal of a path γ and $\widetilde{\Gamma}$ the macroscopic path given by Lemma 3.3.

Moreover, $\alpha(\varepsilon) = \alpha_d \varepsilon^r$ where α_d and r are constants depending only on d.

Proof. Let us consider a path γ, its associated lattice animal Γ and its associated path on the macroscopic grid $\widetilde{\Gamma} = (\widetilde{\Gamma}(k))_{0\leq k \leq r}$ such that $\gamma \subset \cup_{k=0}^r B'_N(\widetilde{\Gamma}(k))$. We first want to include $\widetilde{\Gamma}$ in a subset of the macroscopic grid. Of course, $\widetilde{\Gamma}$ is included in the hypercube of side $2r$ centered at $\widetilde{\Gamma}(0)$, but we need to have a more precise control. Let $K \geq 1$ be an integer that we will choose later. Let v be a site, we denote by $S(v) = \{w \in \mathbb{Z}^d : \|w-v\|_\infty \leq K\}$ the hypercube of side $2K$ centered at v and by $\partial S(v) = \{w \in \mathbb{Z}^d : \|w-v\|_\infty = K\}$ its inner boundary.

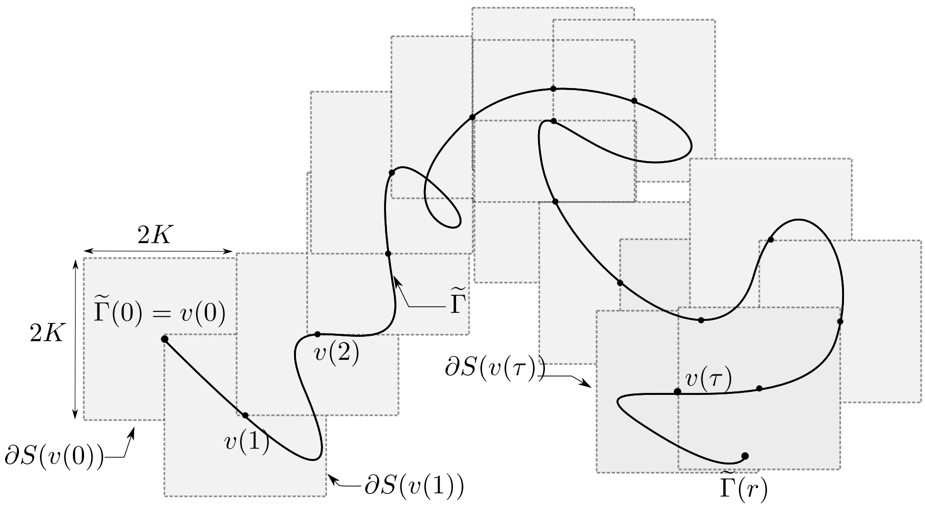


Figure 5 – Construction of $v(0), \dots, v(\tau)$

We define $v(0) = \widetilde{\Gamma}(0)$, $p_0 = 0$. If $p_0, \dots, p_k$ and $v(0), \dots, v(k)$ are constructed, we define if any

$$p_{k+1} = \min\left\{i \in \{p_k+1, \dots, r\} : \widetilde{\Gamma}(i) \in \partial S(v(k))\right\}$$

and $v(k+1) = \widetilde{\Gamma}(p_{k+1})$. If there is no such index we stop the process. Since $p_{k+1} - p_k \geq K$, there are at most $1+r/K$ such p_k. Notice that $1+r/K \leq 1+n/K$ on the event $\{|\widetilde{\Gamma}| \leq n\}$. We define $\tau = 1 + n/K$. Therefore, on the event $\{|\widetilde{\Gamma}| \leq n\}$, $\widetilde{\Gamma}$ is contained in the union of those hypercubes:

$$D(v(0), \dots, v(\tau)) = \bigcup_{i=0}^{\tau} S(v(i))\,.$$

If we stop the process for a $k < \tau$, we artificially complete the sequence until attaining τ by setting for $k < j \le \tau$, $v(j) = v(k)$. See figure 5, the corridor $D(v(0), \dots, v(\tau))$ is represented by the grey section.

Noticing that for all $1 \le k \le r$, there exists a $j \le \tau$ such that $\widetilde{\Gamma}(k)$ is in the strict interior of $S(v(j))$, we get that $\Gamma \subset \cup_{k=1}^{r} \{\mathbf{j}, \mathbf{j} \text{ is } *\text{-connected to } \widetilde{\Gamma}(k)\} \subset D(v(0), \dots, v(\tau))$. Thus,

$$\begin{aligned}
&\mathbb{P}\left(\left\{|\widetilde{\Gamma}| \le n\right\} \cap \left\{\sum_{C \in Bad: C \cap \Gamma \neq \emptyset} |C| \ge \varepsilon n\right\}\right) \\
&\quad \le \mathbb{P}\left(\bigcup_{v(0),\dots,v(\tau)} \left\{\sum_{C \in Bad: C \cap \Gamma \neq \emptyset} |C| \ge \varepsilon n \text{ and } \Gamma \subset D(v(0), \dots, v(\tau))\right\}\right) \\
&\quad \le \sum_{v(0),\dots,v(\tau)} \mathbb{P}\left(\sum_{C \in Bad: C \cap \Gamma \neq \emptyset} |C| \ge \varepsilon n \text{ and } \Gamma \subset D(v(0), \dots, v(\tau))\right) \\
&\quad \le \sum_{v(0),\dots,v(\tau)} \mathbb{P}\left(\sum_{\substack{C \in Bad: \\ C \cap D(v(0),\dots,v(\tau)) \neq \emptyset}} |C| \ge \varepsilon n\right) \\
&\quad \le \sum_{v(0),\dots,v(\tau)} \mathbb{P}_{\mathbf{p}}\left(\sum_{\substack{C \in Bad: \\ C \cap D(v(0),\dots,v(\tau)) \neq \emptyset}} |C| \ge \varepsilon n\right)
\end{aligned}$$

where the sum is over all sites $v(0), \dots, v(\tau)$ satisfying $v(0) = \Gamma(1)$, $v(k+1) \in \partial S(v(k)) \cup \{v(k)\}$ for all $0 \le k < \tau$. We use for the last inequality the stochastic comparison with an independent field of Bernoulli random variable of parameter $\mathbf{p} = \mathbf{p}(\varepsilon)$. Since $\partial S(v) \cup \{v\}$ contains at most $(c_d K)^{d-1}$ sites where $c_d \ge 1$ is a constant depending only on the dimension, the sum contains at most $(c_d K)^{(d-1)\tau} \le (c_d K)^{\frac{2n(d-1)}{K}} := C_2^n$ terms for n large enough.

Let us recall that for a bad macroscopic site $\mathbf{i}$, $C(\mathbf{i})$ denotes the connected cluster of bad macroscopic sites containing $\mathbf{i}$. Let us notice that the following event

$$\left\{\sum_{\substack{C \in Bad: \\ C \cap D(v(0),\dots,v(\tau)) \neq \emptyset}} |C| \ge \varepsilon n\right\}$$

is included in the event: there exist an integer ρ and distinct bad macroscopic sites $\mathbf{i}_1, \dots, \mathbf{i}_\rho \in D(v(0), \dots, v(\tau))$, disjoint connected components $\bar{C}_1, \dots, \bar{C}_\rho$ such that for all $1 \le k \le \rho$, $C(\mathbf{i}_k) = \bar{C}_k$ and $\sum_{k=1}^{\rho} |\bar{C}_k| \ge \varepsilon n$.

For any fixed $v(0), \dots, v(\tau)$, $D(v(0), \dots, v(\tau))$ contains at most $(\tau+1)(2K+1)^d \le (n/K + 2)(2K+1)^d \le 2n(3K)^d := C_3 n$ macroscopic sites, therefore

$$\mathbb{P}\left(\left\{|\widetilde{\Gamma}| \le n\right\} \cap \left\{\sum_{C \in Bad: C \cap \Gamma \neq \emptyset} |C| \ge \varepsilon n\right\}\right) \le$$

$$\sum_{v(0),\dots,v(\tau)}\sum_{j\geq \varepsilon n}\sum_{\rho=1}^{C_3 n}\mathbb{P}_{\mathbf{p}}\left(\begin{array}{c}\text{There exist distinct sites}\\ \mathbf{i}_1,\dots,\mathbf{i}_\rho\in D(v(0),\dots,v(\tau))\text{ and disjoint}\\ \text{connected components }\bar{C}_1,\dots,\bar{C}_\rho\text{ such that for}\\ \text{all }1\leq k\leq\rho, C(\mathbf{i}_k)=\bar{C}_k\text{ and }\sum_{k=1}^{\rho}|\bar{C}_k|=j\end{array}\right)$$

There are at most $\binom{C_3 n}{\rho}$ ways of choosing the sites $\mathbf{i}_1,\dots,\mathbf{i}_\rho$.

In order to count the connected components, we know that there are at most $(7^d)^l$ lattice animals containing a given site and of size l. Thus, if we fix the sites $\mathbf{i}_1,\dots,\mathbf{i}_\rho$ the number of possible choices of the connected components $\bar{C}_1,\dots,\bar{C}_\rho$ such that for all $1\leq k\leq\rho$, $C(\mathbf{i}_k)=\bar{C}_k$ and $\sum_{k=1}^{\rho}|\bar{C}_k|=j$ is at most:

$$\sum_{\substack{j_1,\dots,j_\rho\geq 1\\ j_1+\dots+j_\rho=j}}(7^d)^{j_1}\cdots(7^d)^{j_\rho}=(7^d)^j\sum_{\substack{j_1,\dots,j_\rho\geq 1\\ j_1+\dots+j_\rho=j}}1\,.$$

Next we need to estimate, for given sites $\mathbf{i}_1,\dots,\mathbf{i}_\rho$ and disjoint connected components $\bar{C}_1,\dots,\bar{C}_\rho$, the probability that for all $1\leq k\leq\rho$, $C(\mathbf{i}_k)=\bar{C}_k$. Let us recall that the probability for a site of being bad is $\alpha=\alpha(\varepsilon)=1-\mathbf{p}(\varepsilon)$. Therefore,

$$\begin{aligned}\mathbb{P}_{\mathbf{p}}\left(C(\mathbf{i}_k)=\bar{C}_k,1\leq k\leq\rho\right)&\leq\mathbb{P}\left(\forall 1\leq k\leq\rho,\forall\mathbf{j}\in\bar{C}_k,\mathbf{j}\text{ is bad}\right)\\&=\alpha^{\sum_{i=1}^{\rho}|\bar{C}_k|}\,.\end{aligned}$$

Finally,

$$\begin{aligned}&\mathbb{P}_{\mathbf{p}}\left(\begin{array}{c}\text{There exist distinct sites }\mathbf{i}_1,\dots,\mathbf{i}_\rho\in D(v(0),\dots,v(\tau))\\ \text{and disjoint connected components }\bar{C}_1,\dots,\bar{C}_\rho\text{ such that}\\ \text{for all }1\leq k\leq\rho, C(\mathbf{i}_k)=\bar{C}_k\text{ and }\sum_{k=1}^{\rho}|\bar{C}_k|=j\end{array}\right)\\&\qquad\leq\binom{C_3 n}{\rho}(7^d\alpha)^j\sum_{\substack{j_1,\dots,j_\rho\geq 1\\ j_1+\dots+j_\rho=j}}1\,.\end{aligned}$$

This implies

$$\begin{aligned}&\mathbb{P}\left(\left\{|\widetilde{\Gamma}|\leq n\right\}\cap\left\{\sum_{C\in Bad:C\cap\Gamma\neq\emptyset}|C|\geq\varepsilon n\right\}\right)\\&\qquad\leq C_2^n\sum_{j\geq\varepsilon n}(7^d\alpha)^j\sum_{\rho=1}^{C_3 n}\binom{C_3 n}{\rho}\sum_{\substack{j_1,\dots,j_\rho\geq 1\\ j_1+\dots+j_\rho=j}}1\,.\end{aligned}$$

Notice that

$$\sum_{\rho=1}^{C_3 n}\binom{C_3 n}{\rho}\sum_{\substack{j_1,\dots,j_\rho\geq 1\\ j_1+\dots+j_\rho=j}}1=\sum_{\substack{j_1,\dots,j_{C_3.n}\geq 0\\ j_1+\dots+j_{C_3.n}=j}}1=\binom{C_3 n+j-1}{j}\,.$$

To bound those terms we will need the following inequality, for $r\geq 3$, $N\in\mathbb{N}^*$ and a real z such that $0<ez(1+\frac{r}{N})<1$:

$$\sum_{j=N}^{\infty}z^j\binom{r+j-1}{j}\leq\nu\frac{(ez(1+\frac{r}{N}))^N}{1-ez(1+\frac{r}{N})}\tag{21}$$

where ν is an absolute constant. The proof of this inequality was not present in [10]. To show this inequality, we need a version of Stirling's formula with bounds: for all $n \in \mathbb{N}^*$, one has

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \leq n! \leq e\, n^{n+\frac{1}{2}} e^{-n} \,,$$

thus,

$$\begin{aligned}
\sum_{j=N}^{\infty} z^j \binom{r+j-1}{j} &= \sum_{j=N}^{\infty} z^j \frac{(r+j-1)!}{j!(r-1)!} \\
&\leq \sum_{j=N}^{\infty} z^j \frac{e\,(r+j-1)^{r+j-\frac{1}{2}} e^{-(r+j-1)}}{2\pi\, j^{j+\frac{1}{2}} (r-1)^{r-\frac{1}{2}} e^{-(r+j-1)}} \\
&= \sum_{j=N}^{\infty} \frac{e}{2\pi} z^j \left(\frac{r+j-1}{j}\right)^j \left(\frac{r+j-1}{r-1}\right)^{r-\frac{1}{2}} j^{-\frac{1}{2}} \\
&\leq \sum_{j=N}^{\infty} \frac{e}{2\pi} z^j \left(1+\frac{r}{N}\right)^j \left(1+\frac{j}{r-1}\right)^{r-1} \left(\frac{1}{j}+\frac{1}{r-1}\right)^{\frac{1}{2}} \\
&\leq \sum_{j=N}^{\infty} \frac{e}{2\pi} z^j \left(1+\frac{r}{N}\right)^j e^{(r-1)\log(1+j/(r-1))} \\
&\leq \sum_{j=N}^{\infty} \frac{e}{2\pi} (ez)^j \left(1+\frac{r}{N}\right)^j \\
&= \frac{e}{2\pi} \frac{(ez(1+\frac{r}{N}))^N}{1-ez(1+\frac{r}{N})}
\end{aligned}$$

where we use in the last inequality the fact that for all $x > 0$, $\log(1+x) \leq x$.

Using the inequality (21) and assuming $0 < e7^d\alpha(\varepsilon)(1+\frac{C_3}{\varepsilon}) < 1$, we get,

$$\begin{aligned}
\mathbb{P}\left(\left\{|\widetilde{\Gamma}| \leq n\right\} \cap \left\{\sum_{C \in Bad: C\cap\Gamma\neq\emptyset} |C| \geq \varepsilon n\right\}\right) &\leq C_2^n \sum_{j\geq \varepsilon n} (7^d\alpha)^j \binom{C_3 n + j - 1}{j} \\
&\leq \nu C_2^n \frac{\left[e7^d\alpha(\varepsilon)(1+\frac{C_3}{\varepsilon})\right]^{\varepsilon n}}{1-e7^d\alpha(\varepsilon)(1+\frac{C_3}{\varepsilon})}\,.
\end{aligned}$$

Let us recall that $C_2 = (c_d K)^{2(d-1)/K}$ and $C_3 = 2(3K)^d$. We have to choose $K(\varepsilon)$, $\alpha(\varepsilon)$ and a constant $0 < C_\varepsilon < 1$ such that $C_2\left[e7^d\alpha(\varepsilon)(1+\frac{C_3}{\varepsilon})\right]^\varepsilon < C_\varepsilon$ that is to say

$$(c_d K)^{\frac{2(d-1)}{K}} \left[e7^d\alpha(\varepsilon)(1+\frac{2(3K)^d}{\varepsilon})\right]^\varepsilon < C_\varepsilon \,. \tag{22}$$

Note that the condition (22) implies the condition $0 < e7^d\alpha(\varepsilon)(1+\frac{C_3}{\varepsilon}) < 1$.

We fix K the unique integer such that $\frac{1}{\varepsilon} \leq K < \frac{1}{\varepsilon}+1 \leq \frac{2}{\varepsilon}$. We recall that

$\varepsilon < 1$. Thus,

$$\begin{aligned}
(c_d K)^{\frac{2(d-1)}{K}} &\left[e7^d \alpha(\varepsilon)(1 + \frac{2(3K)^d}{\varepsilon}) \right]^{\varepsilon} \\
&\leq (c_d K)^{\frac{2d}{K}} \left[e7^d \alpha(\varepsilon) \frac{4(3K)^d}{\varepsilon} \right]^{\varepsilon} \\
&\leq \exp \left[\frac{2d}{K} \log(c_d K) + \varepsilon \log \left(e7^d \alpha(\varepsilon) \frac{4(3K)^d}{\varepsilon} \right) \right] \\
&\leq \exp \left[2d\varepsilon \log \left(\frac{2c_d}{\varepsilon} \right) + \varepsilon \log \left(e7^d \alpha(\varepsilon) \frac{4(3\frac{2}{\varepsilon})^d}{\varepsilon} \right) \right] \\
&\leq \exp \left[-2d\varepsilon \log \varepsilon + d\varepsilon \log(2c_d) + \varepsilon \log \left(4e(42)^d \alpha(\varepsilon) \frac{1}{\varepsilon^{d+1}} \right) \right].
\end{aligned}$$

We set

$$\alpha(\varepsilon) = (2c_d)^d \frac{\varepsilon^r}{4e(42)^d}$$

where r is the smallest integer such that $r \geq 3d + 2$. We obtain

$$\begin{aligned}
(c_d K)^{\frac{d}{K}} \left[e7^d \alpha(\varepsilon)(1 + \frac{2(3K)^d}{\varepsilon}) \right]^{\varepsilon} &\leq \exp((r - (3d+1))\varepsilon \log \varepsilon) \\
&\leq \exp(\varepsilon \log \varepsilon) < 1\,.
\end{aligned}$$

□

Remark 5.1. In order to have the control of Lemma 5.1, we only require that $\mathbf{p}(\varepsilon)$ goes polynomially fast to 1. This is an improvement of [15] that requires that $\mathbf{p}(\varepsilon)$ goes exponentially fast to 1. This is a key step to prove the announced regularity of the time constant.

6 Time constant

Lemma 6.1. *Let $p_c(d) < p \leq q$. Let us consider $y, z \in \mathbb{Z}^d$. We denote by $E_{y,z}$ the event that $y, z \in \mathcal{C}_p$ and the N-boxes containing y and z are good and belong to an infinite cluster of good boxes. Then, for $\delta > 0$*

$$\mathbb{P}\left[\begin{array}{c} E_{y,z} \cap \\ \left\{ D^{\mathcal{C}_p}(y,z) > D^{\mathcal{C}_q}(y,z)\left(1 + \rho_d N\left(\frac{q-p}{q} + \delta\right)\right) + \rho_d N \sum\limits_{C \in Bad : C \cap \Gamma \neq \emptyset} |C| \right\} \end{array} \right] \leq e^{-2\delta^2 \|z-y\|_1}.$$

where Γ is the lattice animal of N-boxes visited by an optimal path γ between y and z in $\mathcal{C}_q$.

Proof. As $y, z \in \mathcal{C}_p \subset \mathcal{C}_q$, there exists a q-open path joining y to z, let γ be an optimal one. Necessarily, we have $|\gamma| \geq \|y - z\|_1$. We consider the modification

γ' given by Lemma 3.1. As γ' is p-open,

$$\begin{aligned} D^{\mathcal{C}_p}(y,z) < |\gamma'| \le |\gamma \cap \gamma'| + |\gamma' \setminus \gamma| \\ \le |\gamma| + \rho_d \left(N|\gamma_c| + N \sum_{C \in Bad: C \cap \Gamma \neq \emptyset} |C| \right) \\ \le D^{\mathcal{C}_q}(y,z) + \rho_d \left(N|\gamma_c| + N \sum_{C \in Bad: C \cap \Gamma \neq \emptyset} |C| \right) . \end{aligned} \tag{23}$$

We want to control the size of γ_c. For that purpose, we want to introduce a coupling of the percolations q and p, such that if any edge is p is open then it is q is open, and we want the random path γ, which is an optimal q-open path between y and z, to be independent of the p-state of any edge, i.e., any edge is p-open or p-closed independently of γ. This is not the case when we use the classic coupling with a unique uniform random variable for each edge. Here we introduce two sources of randomness to ease the computations by making the choice of γ independent from the p-state of its edges. We proceed in the following way: with each edge we associate two independent Bernoulli random variables V and Z of parameters respectively q and p/q. Then $W = Z.V$ is also a Bernoulli random variable of parameter p. This implies

$$\begin{aligned} \mathbb{P}[W=0|V=1] &= \mathbb{P}[Z=0|V=1] \\ &= \mathbb{P}[Z=0] \\ &= 1 - \frac{p}{q} \\ &= \frac{q-p}{q} . \end{aligned}$$

Thus, we can now bound the following quantity by summing on all possible self-avoiding paths for γ. For short, we use the abbreviation s.a. for self-avoiding.

$$\begin{aligned} &\mathbb{P}\left[|\gamma_c| \ge |\gamma| \left(\frac{q-p}{q} + \delta \right) \right] \\ &\quad = \sum_{k=\|y-z\|_1}^{\infty} \sum_{\substack{|r|=k \\ \text{r s.a. path}}} \mathbb{P}\left[\gamma = r, |\gamma_c| \ge |\gamma| \left(\frac{q-p}{q} + \delta \right) \right] \\ &\quad = \sum_{k=\|y-z\|_1}^{\infty} \sum_{\substack{|r|=k \\ \text{r s.a. path}}} \mathbb{P}\left[\gamma = r, |r_c| \ge k \left(\frac{q-p}{q} + \delta \right) \right] \\ &\quad = \sum_{k=\|y-z\|_1}^{\infty} \sum_{\substack{|r|=k \\ \text{r s.a. path}}} \mathbb{P}\left[\gamma = r, |\{e \in r : Z(e) = 0\}| \ge k \left(\frac{q-p}{q} + \delta \right) \right] \\ &\quad = \sum_{k=\|y-z\|_1}^{\infty} \sum_{\substack{|r|=k \\ \text{r s.a. path}}} \mathbb{P}\left[\gamma = r \right] \mathbb{P}\left[|\{e \in r : Z(e) = 0\}| \ge k \left(\frac{q-p}{q} + \delta \right) \right] \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=\|y-z\|_1}^{\infty} \sum_{\substack{|r|=k \\ \text{r s.a. path}}} \mathbb{P}\left[\gamma = r\right] e^{-2\delta^2 k} \\
&\leq e^{-2\delta^2\|y-z\|_1} \qquad (24)
\end{aligned}
$$

where we use Chernoff bound in the second to last inequality (see Theorem 1 in [20]).

On the event $E_{y,z} \cap \left\{|\gamma_c| < |\gamma|\left(\frac{q-p}{q} + \delta\right)\right\}$, by (23), we get

$$
\begin{aligned}
D^{\mathcal{C}_p}(y,z) &\leq D^{\mathcal{C}_q}(y,z) + \rho_d \left(N|\gamma|\left(\frac{q-p}{q} + \delta\right) + N \sum_{C \in Bad : C \cap \Gamma \neq \emptyset} |C| \right) \\
&= D^{\mathcal{C}_q}(y,z)\left(1 + \rho_d N\left(\frac{q-p}{q} + \delta\right)\right) + \rho_d N \sum_{C \in Bad : C \cap \Gamma \neq \emptyset} |C|
\end{aligned}
$$

and the conclusion follows. □

Lemma 6.2. *Let $p_c(d) < p_0 \leq p \leq q$ and $\varepsilon > 0$, we choose $\boldsymbol{p}(\varepsilon)$ as in Lemma 5.1 and we set $N(\varepsilon) = N(p_0, d, \boldsymbol{p}(\varepsilon))$ as in Proposition 4.2. There exists $p_1(\varepsilon) > 0$ such that for all $x \in \mathbb{Z}^d$ with $\|x\|_1$ large enough,*

$$
\mathbb{P}\left(D^{\mathcal{C}_p}(\widetilde{0}^{\mathcal{C}_p}, \widetilde{x}^{\mathcal{C}_p}) \leq D^{\mathcal{C}_q}(\widetilde{0}^{\mathcal{C}_p}, \widetilde{x}^{\mathcal{C}_p})\left(1 + \rho_d \frac{q-p}{q} N(\varepsilon)\right) + \eta_d \varepsilon \|x\|_1 \right) \geq p_1(\varepsilon)
$$

where $\eta_d > 0$ is a constant depending only on d.

Proof. Let us fix $\varepsilon > 0$ and $N = N(\varepsilon)$ for short. Fix an $x \in \mathbb{Z}^d$ such that $\|x\|_1 \geq 18dN$. Let F_x be the following event: the N-boxes containing 0 and x and all the adjacent boxes belong to an infinite cluster of good boxes. Note that any point in the same $3N$-box as 0 or x is in a good box that belongs to an infinite cluster of good boxes.

For any y in the same $3N$-box as 0, for any z in the same $3N$-box as x, we recall that $E_{y,z}$ denote the event that $y, z \in \mathcal{C}_p$ and the N-boxes containing y and z belong to an infinite cluster of good boxes. Thus, $\|y - z\|_1 \leq \|y\|_1 + \|x\|_1 + \|x - z\|_1 \leq 9dN + \|x\|_1 \leq 2\|x\|_1$ and $\|y - z\|_1 \geq \|x\|_1 - 9dN \geq \frac{\|x\|_1}{2}$.

We obtain

$$
\begin{aligned}
&\mathbb{P}\left(D^{\mathcal{C}_p}(\widetilde{0}^{\mathcal{C}_p}, \widetilde{x}^{\mathcal{C}_p}) \geq D^{\mathcal{C}_q}(\widetilde{0}^{\mathcal{C}_p}, \widetilde{x}^{\mathcal{C}_p})\left(1 + \rho_d \frac{q-p}{q} N(\varepsilon)\right) + 6\varepsilon\beta\rho_d\|x\|_1 \right) \\
\leq& \mathbb{P}[F_x^c] \\
&+ \mathbb{P}\left(\begin{array}{c} F_x \cap \\ \left\{ D^{\mathcal{C}_p}(\widetilde{0}^{\mathcal{C}_p}, \widetilde{x}^{\mathcal{C}_p}) \geq D^{\mathcal{C}_q}(\widetilde{0}^{\mathcal{C}_p}, \widetilde{x}^{\mathcal{C}_p})\left(1 + \rho_d \frac{q-p}{q} N(\varepsilon)\right) + 6\varepsilon\beta\rho_d\|x\|_1 \right\} \end{array} \right) \\
=& \mathbb{P}[F_x^c] \\
&+ \sum_{y,z} \mathbb{P}\left(\begin{array}{c} F_x \cap \{\widetilde{0}^{\mathcal{C}_p} = y, \widetilde{x}^{\mathcal{C}_p} = z\} \cap \\ \left\{ D^{\mathcal{C}_p}(y,z) \geq D^{\mathcal{C}_q}(y,z)\left(1 + \rho_d \frac{q-p}{q} N(\varepsilon)\right) + 6\varepsilon\beta\rho_d\|x\|_1 \right\} \end{array} \right) \\
\leq& \mathbb{P}[F_x^c]
\end{aligned}
$$

$$+\sum_{y,z}\mathbb{P}\left(\begin{array}{c}E_{y,z}\cap\\ \left\{D^{\mathcal{C}_p}(y,z)\geq D^{\mathcal{C}_q}(y,z)\left(1+\rho_d\frac{q-p}{q}N(\varepsilon)\right)+3\varepsilon\beta\rho_d\|y-z\|_1\right\}\end{array}\right) \tag{25}$$

where the sum is over every y,z respectively belonging to the same $3N$-box than 0 and x.

Using the stochastic comparison and the FKG inequality, we get:

$$\mathbb{P}(F_x)\geq\theta_{site,\mathbf{p}(\varepsilon)}^{2.3^d}>0 \tag{26}$$

where $\theta_{site,\mathbf{p}(\varepsilon)}$ denotes the probability for a site to belong to the infinite cluster of i.i.d. Bernoulli site percolation of parameter $\mathbf{p}(\varepsilon)$.

On the event $E_{y,z}$, $y,z\in\mathcal{C}_p\subset\mathcal{C}_q$, we can consider $\gamma_{y,z}$ a geodesic in $\mathcal{C}_q$, and let $\Gamma_{y,z}$ be the set of boxes that $\gamma_{y,z}$ visits. By Lemma 6.1, we have for every $\delta>0$

$$\begin{aligned}
&\mathbb{P}\Big(E_{y,z}\cap\left\{D^{\mathcal{C}_p}(y,z)\geq D^{\mathcal{C}_q}(y,z)\left(1+\rho_d\frac{q-p}{q}N(\varepsilon)\right)+3\varepsilon\beta\rho_d\|y-z\|_1\right\}\Big)\\
&\leq\mathbb{P}\left(\begin{array}{c}E_{y,z}\cap\{|\gamma_{y,z}|\leq\beta\|y-z\|_1\}\cap\\ \left\{\rho_dN(\varepsilon)\left(D^{\mathcal{C}_q}(y,z)\delta+\sum_{C\in Bad:C\cap\Gamma_{y,z}\neq\emptyset}|C|\right)\geq3\varepsilon\beta\|y-z\|_1\right\}\end{array}\right)\\
&\quad+\mathbb{P}\left[\begin{array}{c}E_{y,z}\cap\\ \left\{D^{\mathcal{C}_p}(y,z)>D^{\mathcal{C}_q}(y,z)\left(1+\rho_dN\left(\frac{q-p}{q}+\delta\right)\right)+\rho_dN\sum|C|\right\}\end{array}\right]\\
&\quad+\mathbb{P}\left(E_{y,z}\cap\{|\gamma_{y,z}|>\beta\|y-z\|_1\}\right)\\
&\leq\mathbb{P}\left(\begin{array}{c}E_{y,z}\cap\{|\gamma_{y,z}|\leq\beta\|y-z\|_1\}\cap\\ \left\{\sum_{C\in Bad:C\cap\Gamma_{y,z}\neq\emptyset}|C|\geq\frac{3\varepsilon\beta\|y-z\|_1}{N(\varepsilon)}-\delta|\gamma_{y,z}|\right\}\end{array}\right)+e^{-2\delta^2\|y-z\|_1}\\
&\quad+\mathbb{P}\left(E_{y,z}\cap\{|\gamma_{y,z}|>\beta\|y-z\|_1\}\right)\\
&\leq\mathbb{P}\left(\begin{array}{c}E_{y,z}\cap\{|\gamma_{y,z}|\leq\beta\|y-z\|_1\}\cap\\ \left\{\sum_{C\in Bad:C\cap\Gamma_{y,z}\neq\emptyset}|C|\geq|\gamma_{y,z}|(\frac{3\varepsilon}{N(\varepsilon)}-\delta)\right\}\end{array}\right)+e^{-2\delta^2\|y-z\|_1}\\
&\quad+\mathbb{P}\left(E_{y,z}\cap\{|\gamma_{y,z}|>\beta\|y-z\|_1\}\right).
\end{aligned}$$

We set $\delta=\varepsilon/N(\varepsilon)$. We know by Lemma 3.3 that $|\widetilde{\Gamma}_{y,z}|\leq1+(|\gamma_{y,z}|+1)/N\leq1+2|\gamma_{y,z}|/N$. It implies

$$\begin{aligned}
&\mathbb{P}\left(E_{y,z}\cap\left\{D^{\mathcal{C}_p}(y,z)\geq D^{\mathcal{C}_q}(y,z)\left(1+\rho_d\frac{q-p}{q}N(\varepsilon)\right)+3\varepsilon\beta\rho_d\|y-z\|_1\right\}\right)\\
&\leq\mathbb{P}\left(E_{y,z}\cap\{|\gamma_{y,z}|>\beta\|y-z\|_1\}\right)+e^{-2(\varepsilon/N(\varepsilon))^2\|y-z\|_1}\\
&\quad+\mathbb{P}\left(|\widetilde{\Gamma}_{y,z}|\leq1+\frac{2\beta\|y-z\|_1}{N},\sum_{C\in Bad:C\cap\Gamma_{y,z}\neq\emptyset}|C|\geq\varepsilon\frac{2\beta\|y-z\|_1}{N}\right).
\end{aligned} \tag{27}$$

Moreover,

$$\begin{aligned}
\mathbb{P}\left(E_{y,z}\cap\{|\gamma_{y,z}|>\beta\|y-z\|_1\}\right)&\leq\mathbb{P}(\beta\|y-z\|_1\leq D^{\mathcal{C}_q}(y,z)<+\infty)\\
&\leq\widehat{A}\exp(-\widehat{B}\|y-z\|_1).
\end{aligned} \tag{28}$$

Finally, using Lemma 5.1,

$$\mathbb{P}\left(\begin{array}{c} |\widetilde{\Gamma}_{y,z}| \leq \left(1+\left\lfloor \frac{2\beta\|y-z\|_1}{N}\right\rfloor\right), \\ \sum_{C\in Bad: C\cap\Gamma_{y,z}\neq\emptyset} |C| \geq \varepsilon\left(1+\left\lfloor \frac{2\beta\|y-z\|_1}{N}\right\rfloor\right) \end{array}\right) \leq C_\varepsilon^{2\beta\|y-z\|_1/N(\varepsilon)} \tag{29}$$

where $C_\varepsilon < 1$.

Combining (25), (26), (27), (28) and (29), we obtain that

$$\begin{aligned}
&\mathbb{P}\left(D^{\mathcal{C}_p}(\widetilde{0}^{\mathcal{C}_p},\widetilde{x}^{\mathcal{C}_p}) \geq D^{\mathcal{C}_q}(\widetilde{0}^{\mathcal{C}_p},\widetilde{x}^{\mathcal{C}_p})\left(1+\rho_d\frac{q-p}{q}N(\varepsilon)\right)+6\varepsilon\beta\rho_d\|x\|_1\right)\\
&\leq 1-\theta_{site,\mathbf{p}(\varepsilon)}^{2.3^d}+\sum_{y,z}\left(C_\varepsilon^{2\beta\|y-z\|_1/N(\varepsilon)}+\widehat{A}e^{-\widehat{B}\|y-z\|_1}+e^{-2\varepsilon^2\|y-z\|_1/N(\varepsilon)^2}\right)\\
&\leq 1-\theta_{site,\mathbf{p}(\varepsilon)}^{2.3^d}+2(6N(\varepsilon)+1)^d\left(C_\varepsilon^{\beta\|x\|_1/N(\varepsilon)}+\widehat{A}e^{-\widehat{B}\|x\|_1/2}+e^{-\varepsilon^2\|x\|_1/N(\varepsilon)^2}\right)\\
&\leq 1-p_1(\varepsilon)
\end{aligned}$$

for an appropriate choice of $p_1(\varepsilon)>0$ and for every x large enough. $\square$

Proof of Theorem 1.6. Let $\varepsilon>0$, $\delta>0$, $p_0>p_c(d)$ and $x\in\mathbb{Z}^d$, consider $p_1(\varepsilon)$ as in Lemma 6.2 and $q\geq p\geq p_0$.

With the convergence of the regularized times given by Proposition 2.1, we can choose n large enough such that

$$\mathbb{P}\left(\mu_p(x)-\delta\leq\frac{D^{\mathcal{C}_p}(\widetilde{0}^{\mathcal{C}_p},\widetilde{nx}^{\mathcal{C}_p})}{n}\right)\geq 1-\frac{p_1(\varepsilon)}{3}$$

$$\mathbb{P}\left(\frac{D^{\mathcal{C}_q}(\widetilde{0}^{\mathcal{C}_p},\widetilde{nx}^{\mathcal{C}_p})}{n}\leq\mu_q(x)+\delta\right)\geq 1-\frac{p_1(\varepsilon)}{3}$$

$$\mathbb{P}\left(D^{\mathcal{C}_p}(\widetilde{0}^{\mathcal{C}_p},\widetilde{nx}^{\mathcal{C}_p})\leq D^{\mathcal{C}_q}(\widetilde{0}^{\mathcal{C}_p},\widetilde{nx}^{\mathcal{C}_p})\left(1+\rho_d\frac{q-p}{q}N(\varepsilon)\right)+\eta_d\varepsilon n\|x\|_1\right)\geq p_1(\varepsilon)\,.$$

The intersection of these three events has positive probability, we obtain on this intersection

$$\mu_p(x)-\delta\leq(\mu_q(x)+\delta)\left(1+\rho_d\frac{q-p}{q}N(\varepsilon)\right)+\eta_d\varepsilon\|x\|_1\,.$$

By taking the limit when δ goes to 0 we get

$$\mu_p(x)\leq\mu_q(x)\left(1+\rho_d\frac{q-p}{q}N(\varepsilon)\right)+\eta_d\varepsilon\|x\|_1\,.$$

By Corollary 2.1, we know that the map $p\to\mu_p$ is non increasing. We also know that $\mu_p(x)\leq\|x\|_1\mu_p(e_1)$ for $e_1=(1,0,\dots,0)$, for any $p>p_c(d)$ and any $x\in\mathbb{Z}^d$. Thus, for every $\varepsilon>0$,

$$\begin{aligned}
\mu_p(x)-\mu_q(x)&\leq\mu_q(x)\rho_d\frac{q-p}{q}N(\varepsilon)+\eta_d\varepsilon\|x\|_1\\
&\leq\mu_{p_0}(e_1)\|x\|_1\rho_d\frac{q-p}{p_c(d)}N(\varepsilon)+\eta_d\varepsilon\|x\|_1\\
&\leq\eta_d'(p_0)\|x\|_1(N(\varepsilon)(q-p)+\varepsilon)
\end{aligned}$$

where $\eta'_d(p_0)$ is a constant depending on d and p_0.

Using the result of Proposition 4.2,

$$\mu_p(x) - \mu_q(x) \leq \eta'_d \|x\|_1 \left(\left(\widehat{C}_0 |\log \alpha(\varepsilon)| + \widehat{C}_1 \right) (q-p) + \varepsilon \right) . \tag{30}$$

Following the result of Lemma 5.1, we set $\alpha(\varepsilon) := \alpha_d \varepsilon^r$ where $\alpha_d > 0$ and $r \geq 3d+2$ are constants depending only on d. By setting $\varepsilon = q - p$ in the inequality (30), we get

$$\mu_p(x) - \mu_q(x) \leq \eta'_d \|x\|_1 \left(\widehat{C'}_0 (q-p) |\log(q-p)| + \widehat{C'}_1 (q-p) \right) .$$

Then, there exists a constant $\eta''_d > 0$ (depending on d and p_0) such that

$$\mu_p(x) - \mu_q(x) \leq \eta''_d \|x\|_1 (q-p) |\log(q-p)| .$$

As $\mu_p(x) - \mu_q(x) \geq 0$ by Corollary 2.1, we obtain

$$|\mu_p(x) - \mu_q(x)| \leq \eta''_d \|x\|_1 (q-p) |\log(q-p)| . \tag{31}$$

By homogeneity, (31) also holds for all $x \in \mathbb{Q}^d$. Let us recall that for all $x, y \in \mathbb{R}^d$ and $p \geq p_c(d)$,

$$|\mu_p(x) - \mu_p(y)| \leq \mu_p(e_1) \|x - y\|_1 , \tag{32}$$

see for instance Theorem 1 in [6].

There exists a finite set $(y_1, \dots, y_m)$ of rational points of $\mathbb{S}^{d-1}$ such that

$$\mathbb{S}^{d-1} \subset \bigcup_{i=1}^{m} \{x \in \mathbb{S}^{d-1} : \|y_i - x\|_1 \leq (q-p)|\log(q-p)|\} .$$

Let $x \in \mathbb{S}^{d-1}$ and y_i such that $\|y_i - x\|_1 \leq (q-p)|\log(q-p)|$, we get

$$\begin{aligned}
&|\mu_p(x) - \mu_q(x)| \\
&\quad \leq |\mu_p(x) - \mu_p(y_i)| + |\mu_p(y_i) - \mu_q(y_i)| + |\mu_q(y_i) - \mu_q(x)| \\
&\quad \leq \mu_p(e_1) \|y_i - x\|_1 + \eta''_d \|y_i\|_1 (q-p) |\log(q-p)| + \mu_q(e_1) \|y_i - x\|_1 \\
&\quad \leq (2\mu_{p_0}(e_1) + \eta''_d)\, (q-p) |\log(q-p)|
\end{aligned}$$

where we use equation (32) in the second to last inequality and the monotonicity of the map $p \to \mu_p$ in the last inequality. □

Proof of Corollary 1.1. Let $p_0 > p_c(d)$. We consider the constant κ_d appearing in the Theorem 1.6. Let $p \leq q$ in $[p_0, 1]$. We recall the following definition of the Hausdorff distance:

$$d_{\mathcal{H}}(\mathcal{B}_{\mu_p}, \mathcal{B}_{\mu_q}) = \inf\{r \in \mathbb{R}^+ : \mathcal{B}_{\mu_p} \subset \mathcal{B}_{\mu_q}^r \text{ and } \mathcal{B}_{\mu_q} \subset \mathcal{B}_{\mu_p}^r\}$$

where $E^r = \{y : \exists x \in E, \|y - x\|_2 \leq r\}$. Thus, we have

$$d_{\mathcal{H}}(\mathcal{B}_{\mu_p}, \mathcal{B}_{\mu_q}) \leq \sup_{y \in \mathbb{S}^{d-1}} \left\| \frac{y}{\mu_p(y)} - \frac{y}{\mu_q(y)} \right\|_2 .$$

Note that $y/\mu_p(y)$ (resp. $y/\mu_q(y)$) is in the unit sphere for the norm μ_p (resp. μ_q) (see Figure 6). Let us define $\mu_p^{min} = \inf_{x\in\mathbb{S}^{d-1}} \mu_p(x)$. As the map $p \to \mu_p$ is uniformly continuous on the sphere $\mathbb{S}^{d-1}$ (see Theorem 1.2 in [15],) the map $p \to \mu_p^{min}$ is also continuous and $\mu^{min} = \inf_{p\in[p_0,1]} \mu_p^{min} > 0$. Finally

$$\begin{aligned} d_{\mathcal{H}}(\mathcal{B}_{\mu_p}, \mathcal{B}_{\mu_q}) &\leq \sup_{y\in\mathbb{S}^{d-1}} \left| \frac{1}{\mu_p(y)} - \frac{1}{\mu_q(y)} \right| \\ &\leq \sup_{y\in\mathbb{S}^{d-1}} \frac{1}{\mu_q(y)\mu_p(y)} |\mu_p(y) - \mu_q(y)| \\ &\leq \sup_{y\in\mathbb{S}^{d-1}} \frac{1}{(\mu^{min})^2} |\mu_p(y) - \mu_q(y)| \\ &\leq \frac{\kappa_d}{(\mu^{min})^2} (q-p)| \log(q-p)| \,. \end{aligned} \tag{33}$$

□

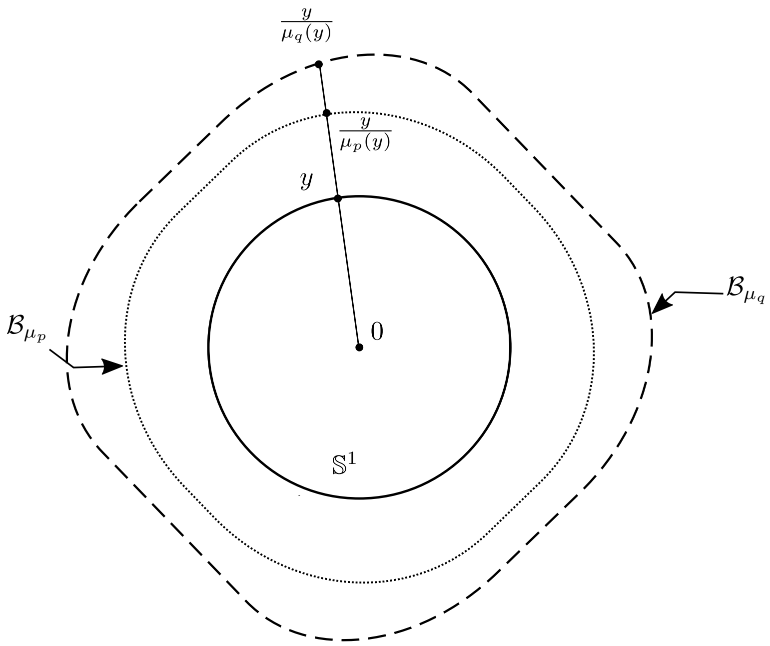


Figure 6 – Representation of $\mathbb{S}^1$, $\mathcal{B}_{\mu_p}$ and $\mathcal{B}_{\mu_q}$

Remark 6.1. At this stage, we were not able to obtain Lipschitz continuous property for $p \to \mu_p$. When p is very close to 1, we can avoid renormalization and bypass bad edges at a microscopic scale as in [9]. However, even in that case, we cannot obtain Lipschitz continuous regularity with the kind of combinatorial computations made in section 5 when we do our bypasses at a microscopic scale.

7 Cheeger constant in dimension 2

The main step in the proof of Theorem 1.1 is the following lemma:

Lemma 7.1. *For every $p_c(2) < p_0 < p_1 < 1$, there exists a constant κ depending on p_0 and p_1 such that for all $p \leq q$ in $[p_0, p_1]$*

$$\sup_{x\in\mathbb{S}^1} |\beta_p(x) - \beta_q(x)| \leq \kappa(q-p)|\log(q-p)|$$

where $\mathbb{S}^1$ denotes the Euclidean sphere.

Remark 7.1. Unlike the proof of Theorem 1.6, we don't have the monotonicity of the map $p \to \beta_p$. That is why we have an extra step in the proof of this Lemma that we did not have in the proof of Theorem 1.6.

Proof. Let $p_c(2) < p_0 \leq p \leq q$. Let $x \in \mathbb{S}^1$ and $n \in \mathbb{N}$, we denote by $\lfloor nx \rfloor$ the point y in $\mathbb{Z}^2$ which minimizes $\|nx - y\|_1$(with a deterministic rule to break ties). Let $x, y \in \mathbb{Z}^2$, we recall that $\mathcal{R}(x, y)$ is the set of right-most paths joining x and y, and for a path r, $\partial^+ r$ denotes the set of right boundary edges of r. For a path $r \in \mathcal{R}(x, y)$, let us denote by $\mathbf{b}_p(r) = |\{e \in \partial^+ r : e \text{ is } p\text{-open}\}|$ and if $x, y \in \mathcal{C}_p$, $b_p(x, y) = \inf\{\mathbf{b}_p(r) : r \in \mathcal{R}(x, y), r \text{ is } p\text{-open}\}$.

Step (i). Let $y \in \mathbb{Z}^2$. Let F_{p_0} be the event that 0 and y belong to $\mathcal{C}_{p_0}$. On the event F_{p_0}, we have $0, x \in \mathcal{C}_{p_0} \subset \mathcal{C}_p$. On the event F_{p_0}, we want to bound the quantity $\mathbf{b}_q(\widetilde{\gamma}_p) - \mathbf{b}_p(\widetilde{\gamma}_p)$ where $\widetilde{\gamma}_p$ is a random p-open path that achieves the infimum in $b_p(0, y)$. Thus, the idea is to introduce a coupling of the percolations of parameter p and q such that if an edge is p-open then it is q-open and $\widetilde{\gamma}_p$ is independent of the q-state of any edge. Unfortunately, we cannot find such a coupling but we can introduce a coupling that almost has this property. To do so, for each edge we consider two independent Bernoulli random variables U and V of parameters p and $(q-p)/(1-p)$. We say that an edge e is p-open if $U(e) = 1$ and that it is q-open if $U(e) = 1$ or $V(e) = 1$. Indeed,

$$\mathbb{P}[\{U = 1\} \cup \{V = 1\}] = p + (1-p)\frac{q-p}{1-p} = q\,.$$

The random path $\widetilde{\gamma}_p$ depends only on $(U(e))_{e\in\mathbb{E}}$. We can now bound the following quantity:

$$\begin{aligned}
&\mathbb{P}\left[F_{p_0}, \mathbf{b}_q(\widetilde{\gamma}_p) - \mathbf{b}_p(\widetilde{\gamma}_p) \geq |\partial^+\widetilde{\gamma}_p|\left(\frac{q-p}{1-p} + \delta\right)\right] \\
&\quad = \sum_{k=\|y\|_1}^{\infty} \sum_{\substack{|r|=k \\ \text{r s.a. path}}} \mathbb{P}\left[F_{p_0}, \widetilde{\gamma}_p = r, \mathbf{b}_q(r) - \mathbf{b}_p(r) \geq |\partial^+ r|\left(\frac{q-p}{1-p} + \delta\right)\right] \\
&\quad = \sum_k \sum_{|r|=k} \mathbb{P}\left[\begin{array}{c} F_{p_0}, \widetilde{\gamma}_p = r, \\ |\{e \in \partial^+ r : (U(e), V(e)) = (0,1)\}| \geq |\partial^+ r|\left(\frac{q-p}{1-p} + \delta\right) \end{array}\right] \\
&\quad \leq \sum_k \sum_{|r|=k} \mathbb{P}\left[F_{p_0}, \widetilde{\gamma}_p = r, |\{e \in \partial^+ r : V(e) = 1\}| \geq |\partial^+ r|\left(\frac{q-p}{1-p} + \delta\right)\right] \\
&\quad \leq \sum_k \sum_{|r|=k} \mathbb{P}\left[F_{p_0}, \widetilde{\gamma}_p = r\right] \mathbb{P}\left[|\{e \in \partial^+ r : V(e) = 1\}| \geq |\partial^+ r|\left(\frac{q-p}{1-p} + \delta\right)\right] \\
&\quad \leq \sum_{k=\|y\|_1}^{\infty} \sum_{\substack{|r|=k \\ \text{r s.a. path}}} \mathbb{P}\left[F_{p_0}, \widetilde{\gamma}_p = r\right] e^{-2\delta^2|\partial^+ r|}
\end{aligned}$$

$$\leq e^{-2\delta^2(\|y\|_1/3-2)} \tag{34}$$

where in the second to last inequality we use Chernoff bound, and in the last inequality we use the inequality of Lemma 2.2.

By Lemma 2.1, there exist $C, C', \alpha > 0$ (depending on p_0) such that $\forall p \geq p_0$, $\forall n$,

$$\mathbb{P}\left[\exists \gamma \in \bigcup_{x \in \mathbb{Z}^d} \mathcal{R}(0,x) : |\gamma| > n\,,\ \mathbf{b}_p(\gamma) \leq \alpha n\right] \leq C e^{-C'n}\,. \tag{35}$$

Let $n \in \mathbb{N}^*$. Let us now set $y = \lfloor nx \rfloor$. On the event F_{p_0}, by Lemma 2.2, we have $b_p(0, \lfloor nx \rfloor) \leq 3D^{\mathcal{C}_p}(0, \lfloor nx \rfloor) \leq 3D^{\mathcal{C}_{p_0}}(0, \lfloor nx \rfloor)$, thus using the result of Lemma 4.2, we know there exist positive constant $\beta, \widehat{A}, \widehat{B}$ depending only on p_0 such that for all $p \geq p_0$ and $x \in \mathbb{S}^1$,

$$\begin{aligned}\mathbb{P}[F_{p_0} \cap \{b_p(0, \lfloor nx \rfloor) \geq 3\beta n\}] &\leq \mathbb{P}[\beta n \leq D^{\mathcal{C}_{p_0}}(0, \lfloor nx \rfloor) < \infty]\\ &\leq \widehat{A} e^{-\widehat{B}\|\lfloor nx \rfloor\|_1}\\ &\leq \widehat{A} e^{-\widehat{B}(cn-2)}\end{aligned} \tag{36}$$

where in the last inequality, we use the fact that since norms in $\mathbb{R}^2$ are equivalent, there exists a $c > 0$ such that for all $x \in \mathbb{R}^2$, $c\|x\|_2 \leq \|x\|_1$. As $\|\lfloor nx \rfloor\|_1 \geq \|nx\|_1 - 2$, we get $\|\lfloor nx \rfloor\|_1 \geq cn - 2$. By the FKG inequality, we get

$$\mathbb{P}(F_{p_0}) \geq \theta_{p_0}^2 \tag{37}$$

where θ_{p_0} denotes the probability that 0 belongs to the infinite cluster of a Bernoulli bond percolation of parameter p_0 on $\mathbb{Z}^2$ ($\theta_{p_0} = \mathbb{P}[0 \in \mathcal{C}_{p_0}]$).

Finally, combining (34), (35), (36) and (37) and fixing $\alpha' = 3\beta/\alpha$ (that depends only on p_0), we obtain for all $p_c(2) < p_0 \leq p \leq q$, for all $x \in \mathbb{S}^1$,

$$\begin{aligned}&\mathbb{P}\Big[b_q(\widetilde{0}^{\mathcal{C}_q}, \widetilde{\lfloor nx \rfloor}^{\mathcal{C}_q}) > b_p(\widetilde{0}^{\mathcal{C}_p}, \widetilde{\lfloor nx \rfloor}^{\mathcal{C}_p}) + 3\alpha' n\left(\frac{q-p}{1-p} + \delta\right)\Big]\\
&\quad\leq \mathbb{P}[F_{p_0}^c] + \mathbb{P}_p[F_{p_0} \cap \{b_p(0, \lfloor nx \rfloor) > 3\beta n\}]\\
&\qquad + \mathbb{P}[F_{p_0} \cap \{\exists \gamma \in \mathcal{R}(0, \lfloor nx \rfloor) :\ |\gamma| > \alpha' n\,,\ \mathbf{b}_p(\gamma) \leq 3\beta n\}]\\
&\qquad + \mathbb{P}\left[F_{p_0} \cap \left\{|\widetilde{\gamma}_p| \leq \alpha' n\,,\ \mathbf{b}_p(\widetilde{\gamma}_p) - \mathbf{b}_q(\widetilde{\gamma}_p) \geq 3\alpha' n\left(\frac{q-p}{1-p} + \delta\right)\right\}\right]\\
&\quad\leq (1 - \theta_{p_0}^2) + \widehat{A} e^{-\widehat{B}(cn-2)} + C e^{-C'\alpha' n}\\
&\qquad + \mathbb{P}\left[F_{p_0} \cap \left\{\mathbf{b}_p(\widetilde{\gamma}_p) - \mathbf{b}_q(\widetilde{\gamma}_p) \geq |\partial^+ \widetilde{\gamma}_p|\left(\frac{q-p}{1-p} + \delta\right)\right\}\right]\\
&\quad\leq (1 - \theta_{p_0}^2) + \widehat{A} e^{-\widehat{B}(cn-2)} + C e^{-C'\alpha' n} + e^{-2\delta^2(\|\lfloor nx \rfloor\|_1/3-2)}\\
&\quad\leq (1 - \theta_{p_0}^2) + \widehat{A} e^{-\widehat{B}(cn-2)} + C e^{-C'\alpha' n} + e^{-2\delta^2((n-2)/3-2)}\end{aligned}$$

where we use Lemma 2.2 in the third to last inequality and in the last inequality we use the inequality $\|\lfloor nx \rfloor\|_1 \geq n - 2$.

For all $\delta > 0$, there exists $p_2(\delta) > 0$ such that for n large enough,

$$\mathbb{P}\Big[b_q(\widetilde{0}^{\mathcal{C}_q}, \widetilde{\lfloor nx \rfloor}^{\mathcal{C}_q}) > b_p(\widetilde{0}^{\mathcal{C}_p}, \widetilde{\lfloor nx \rfloor}^{\mathcal{C}_p}) + 3\alpha' n \left(\frac{q-p}{1-p} + \delta\right)\Big] \leq 1 - p_2(\delta)\,.$$

Thus, we have using Proposition 2.2

$$\beta_q(x) < \beta_p(x) + 3\alpha' \left(\frac{q-p}{1-p} + \delta\right)\,.$$

By letting δ go to 0,

$$\beta_q(x) \leq \beta_p(x) + 3\alpha' \frac{q-p}{1-p}\,. \tag{38}$$

Step (ii). We set $\alpha'' = 6\beta/\alpha = 2\alpha'$ (that still depends only on p_0) so that we will be able to apply Lemma 2.1 later. Let $\varepsilon > 0$ and let us consider a given q-open path. Some of its edges may not be p-open, we want to modify this path in order to obtain a p-open path that does not gain too many extra right-boundary edges. We choose $\mathbf{p}(\varepsilon)$ as in Lemma 5.1 and with this choice of $\mathbf{p}$, we set $N = N(\varepsilon) = N(p_0, d, \mathbf{p}(\varepsilon))$ as in Proposition 4.2. Let F' be the following event: the N-boxes containing 0 and $\lfloor nx \rfloor$ and all the adjacent boxes belong to an infinite cluster of good boxes. We denote by $E_{y,z}$ the event that $y, z \in \mathcal{C}_p$ and the N-boxes containing y and z belong to an infinite cluster of good boxes. As in equation (25), we have

$$\begin{aligned}\mathbb{P}&\left[b_p(\widetilde{0}^{\mathcal{C}_p}, \widetilde{\lfloor nx \rfloor}^{\mathcal{C}_p}) > b_q(\widetilde{0}^{\mathcal{C}_p}, \widetilde{\lfloor nx \rfloor}^{\mathcal{C}_p}) + 4\alpha'' n \rho_d N \left(\frac{q-p}{q} + \frac{3\varepsilon}{N}\right)\right]\\ &\leq \mathbb{P}[F'^c]\\ &\quad + \sum_{y,z} \mathbb{P}\left[E_{y,z} \cap \left\{b_p(y,z) > b_q(y,z) + 4\alpha'' n \rho_d N \left(\frac{q-p}{q} + \frac{3\varepsilon}{N}\right)\right\}\right]\end{aligned} \tag{39}$$

where the sum is over every y, z respectively belonging to the same $3N$-box than 0 and $\lfloor nx \rfloor$. Using the stochastic comparison and the FKG inequality, we get as in equation (26)

$$\mathbb{P}(F') \geq \theta^{18}_{site,\mathbf{p}(\varepsilon)}\,. \tag{40}$$

Let $y, z \in \mathbb{Z}^2$. On the event $E_{y,z}$, let $\gamma_{y,z} \in \mathcal{R}(y,z)$ be a q-open right-most path from y to z such that $b_q(y,z) = \mathbf{b}_q(\gamma_{y,z})$ and let $\Gamma_{y,z}$ be the set of N-boxes $\gamma_{y,z}$ visits. For short, we write γ for $\gamma_{y,z}$. We keep the same notations as in Lemma 3.1. On the event $E_{y,z}$, there exists a p-open path γ' given by Lemma 3.1, such that

$$|\gamma' \setminus \gamma| \leq \rho_d N \left(\sum_{C \in Bad : C \cap \Gamma_{y,z} \neq \emptyset} |C| + |\gamma_c|\right)\,.$$

By construction, γ' is a self-avoiding path. Since a self-avoiding path is also a right-most path, then $\gamma' \in \mathcal{R}(y,z)$. Note that $\mathbf{b}_p(\gamma') \leq \mathbf{b}_q(\gamma') \leq \mathbf{b}_q(\gamma) + 3|\gamma' \setminus \gamma|$.

We can bound γ_c as in the proof of Lemma 6.1 by using the same coupling of the percolations of parameter p and q. We obtain as in equation (24)

$$\mathbb{P}\left[E_{y,z}\cap\left\{|\gamma_c|\geq|\gamma|\left(\frac{q-p}{q}+\delta\right)\right\}\right]\leq e^{-2\delta^2\|y-z\|_1}\,. \tag{41}$$

For all $x\in\mathbb{S}^1$ and for all n large enough such that $\|y-z\|_1\leq\|\lfloor nx\rfloor\|_1+18N\leq 2n$ and $\|y-z\|_1\geq\|\lfloor nx\rfloor\|_1-18N\geq n/2$, we have for all $\delta>0$ and $\varepsilon>0$

$$\begin{aligned}
&\mathbb{P}\left[E_{y,z}\cap\left\{b_p(y,z)>b_q(y,z)+4\alpha''n\rho_d N\left(\frac{q-p}{q}+\frac{3\varepsilon}{N}\right)\right\}\right]\\
&\quad\leq\mathbb{P}[E_{y,z}\cap\{b_q(y,z)>6\beta n\}]\\
&\qquad+\mathbb{P}\left[E_{y,z}\cap\{\exists\gamma\in\mathcal{R}(y,z):\,|\gamma|>\alpha''n\,,\,\mathbf{b}_q(\gamma)\leq 6\beta n\}\right]\\
&\qquad+\mathbb{P}\left[E_{y,z}\cap\left\{|\gamma_c|\geq|\gamma|\left(\frac{q-p}{q}+\delta\right)\right\}\right]\\
&\qquad+\mathbb{P}\left[\begin{array}{c}E_{y,z}\cap\{|\gamma|\leq\alpha''n\}\\ \cap\left\{N\left(\alpha''n\delta+\sum_{C\in Bad:C\cap\Gamma_{y,z}\neq\emptyset}|C|\right)\geq 4\alpha''n\varepsilon\right\}\end{array}\right].
\end{aligned}\tag{42}$$

By Lemma 3.3, we know that $|\widetilde{\Gamma}_{y,z}|\leq 1+(|\gamma_{y,z}|+1)/N\leq 1+2|\gamma_{y,z}|/N$, thus by choosing $\delta=\varepsilon/N$, we obtain

$$\begin{aligned}
&\mathbb{P}\left[\begin{array}{c}E_{y,z}\cap\{|\gamma|\leq\alpha''n\}\\ \cap\left\{N\left(\alpha''n\delta+\sum_{C\in Bad:C\cap\Gamma_{y,z}\neq\emptyset}|C|\right)\geq 4\alpha''n\varepsilon\right\}\end{array}\right]\\
&\quad=\mathbb{P}\left[E_{y,z}\cap\{|\gamma|\leq\alpha''n\}\cap\left\{\alpha''n\frac{3\varepsilon}{N}\leq\sum_{C\in Bad:C\cap\Gamma_{y,z}\neq\emptyset}|C|\right\}\right]\\
&\quad\leq\mathbb{P}\left[|\widetilde{\Gamma}_{y,z}|\leq\left(1+\frac{2\alpha''n}{N}\right),\left(1+\frac{2\alpha''n}{N}\right)\varepsilon\leq\sum_{C\in Bad:C\cap\Gamma_{y,z}\neq\emptyset}|C|\right]\\
&\quad\leq C_\varepsilon^{2\alpha''n/N}
\end{aligned}\tag{43}$$

where $C_\varepsilon<1$ is defined in Lemma 5.1.

On the event $E_{y,z}$, $b_q(y,z)\leq 3D^{\mathcal{C}_q}(y,z)$ as $y,z\in\mathcal{C}_p\subset\mathcal{C}_q$ and $|\partial^+\gamma|\leq 3|\gamma|$. Thus,

$$\begin{aligned}
\mathbb{P}[E_{y,z}\cap\{b_q(y,z)>6\beta n\}]&\leq\mathbb{P}[\beta\|y-z\|_1\leq 2\beta n\leq D^{\mathcal{C}_q}(y,z)<\infty]\\
&\leq\widehat{A}e^{-\widehat{B}\|y-z\|_1}
\end{aligned}\tag{44}$$

where $\widehat{A}$ and $\widehat{B}$ are defined in Lemma 4.2 and depend only on p_0.

Combining (35),(41),(42),(43) and (44), we get

$$\begin{aligned}
&\mathbb{P}\left[E_{y,z}\cap\left\{b_p(y,z)>b_q(y,z)+4\alpha''n\rho_d N\left(\frac{q-p}{q}+\frac{3\varepsilon}{N}\right)\right\}\right]\\
&\quad\leq\widehat{A}e^{-\widehat{B}n/2}+Ce^{-C'\alpha''n}+e^{-\varepsilon^2n/N^2}+C_\varepsilon^{2\alpha''n/N}\,.
\end{aligned}\tag{45}$$

Finally, combining equations (39), (40) and (45):

$$
\begin{aligned}
&\mathbb{P}\left[b_p(\widetilde{0}^{\mathcal{C}_p}, \widetilde{\lfloor nx\rfloor}^{\mathcal{C}_p}) > b_q(\widetilde{0}^{\mathcal{C}_p}, \widetilde{\lfloor nx\rfloor}^{\mathcal{C}_p}) + 4\alpha'' n\rho_d N\left(\frac{q-p}{q}+\frac{3\varepsilon}{N}\right)\right] \\
&\qquad \leq 1-\theta^{18}_{site,\mathbf{p}(\varepsilon)} \\
&\qquad\quad + 2(6N+1)^2\left(\widehat{A}e^{-\widehat{B}n/2} + Ce^{-C'\alpha'' n} + e^{-\varepsilon^2 n/N^2} + C_\varepsilon^{2\alpha'' n/N}\right) \\
&\qquad \leq 1-p_3(\varepsilon)
\end{aligned}
$$

for an appropriate $p_3(\varepsilon) > 0$ and n large enough. Using Proposition 2.2, for every $x \in \mathbb{S}^1$, we obtain

$$
\beta_p(x) \leq \beta_q(x) + 3\alpha''\rho_d N(\varepsilon)\left(\frac{q-p}{q} + \frac{3\varepsilon}{N(\varepsilon)}\right). \tag{46}
$$

Step (iii) By Proposition 4.2, we obtain $N(\mathbf{p}) = \widehat{C}_0|\log\alpha| + \widehat{C}_1$. With the choice $\alpha(\varepsilon) = \alpha_2\varepsilon^8$ as in Lemma 5.1, we obtain $N(\varepsilon) = N(p_0, d, \mathbf{p}(\varepsilon)) = 6\widehat{C}_0|\log\varepsilon| + \widehat{C}_1'$ where $\widehat{C}_1'$ is a constant. We fix $p_c(d) < p_0 < p_1 < 1$. Let $p \leq q$ in $[p_0, p_1]$. Combining equations (38) and (46), and setting $\varepsilon = q-p$ as in (30), we similarly deduce that for all $x \in \mathbb{S}^1$

$$
|\beta_p(x) - \beta_q(x)| \leq \kappa(q-p)|\log(q-p)|
$$

where κ is a constant depending on p_0 and p_1. The conclusion follows. □

Remark 7.2. The choice of the coupling with V a Bernoulli random variable of parameter $(q-p)/(1-p)$ makes the term $1/(1-p)$ appear. Because of this term, we have to consider a parameter p away from 1 in the statement of Theorem 1.1. That is the technical reason of the upper bound p_1 that was not present for the study of the time constant. We do not know if it is just a technical issue or if the map $p \to \beta_p$ does not have nice regularity properties at 1.

Proof of Theorem 1.1. Let $p_c(2) < p_0 < p_1 < 1$ and κ be the constant defined in Lemma 7.1. Let $p < q \in [p_0, p_1]$ and λ be a rectifiable Jordan curve, with $\mathrm{Leb}(\mathrm{int}(\lambda)) = 1$. Recall that

$$
\mathrm{len}_{\beta_p}(\lambda) = \sup_{N\geq 1}\sup_{0\leq t_0<\cdots<t_N\leq 1}\sum_{i=1}^N \beta_p\left(\frac{\lambda(t_i)-\lambda(t_{i-1})}{\|\lambda(t_i)-\lambda(t_{i-1})\|_2}\right)\|\lambda(t_i)-\lambda(t_{i-1})\|_2 .
$$

Thus

$$
\begin{aligned}
&\mathrm{len}_{\beta_p}(\lambda) - \mathrm{len}_{\beta_q}(\lambda) \\
&\leq \sup_{N\geq 1}\sup_{0\leq t_0<\cdots<t_N\leq 1}\sum_{i=1}^N \left(\beta_p\left(\frac{\lambda(t_i)-\lambda(t_{i-1})}{\|\lambda(t_i)-\lambda(t_{i-1})\|_2}\right) - \beta_q\left(\frac{\lambda(t_i)-\lambda(t_{i-1})}{\|\lambda(t_i)-\lambda(t_{i-1})\|_2}\right)\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \|\lambda(t_i)-\lambda(t_{i-1})\|_2 \\
&\leq \sup_{x\in\mathbb{S}^1}|\beta_p(x)-\beta_q(x)|\,\mathrm{len}_{\|.\|_2}(\lambda) \\
&\leq \kappa(q-p)|\log(q-p)|\,\mathrm{len}_{\|.\|_2}(\lambda)
\end{aligned}
$$

where we use Lemma 7.1 in the last inequality. We proceed similarly for $\operatorname{len}_{\beta_q}(\lambda) - \operatorname{len}_{\beta_p}(\lambda)$, we obtain

$$|\operatorname{len}_{\beta_q}(\lambda) - \operatorname{len}_{\beta_p}(\lambda)| \leq \kappa(q-p)|\log(q-p)|\operatorname{len}_{\|.\|_2}(\lambda)\,. \tag{47}$$

The infimum in Theorem 2.1 is achieved (by compactness of the set of Lipschitz curves), so let us denote by λ_p (respectively λ_q) a rectifiable Jordan curve such that $\mathrm{Leb}(\mathrm{int}(\lambda_p)) = 1$ and $\operatorname{len}_{\beta_p}(\lambda_p) = \sqrt{2}\theta_p \lim_{n\to\infty} n\varphi_n(p)$ (respectively $\mathrm{Leb}(\mathrm{int}(\lambda_q)) = 1$ and $\operatorname{len}_{\beta_q}(\lambda_q) = \sqrt{2}\theta_q \lim_{n\to\infty} n\varphi_n(q)$). As β_p and β_q are norms and all the norms in $\mathbb{R}^2$ are equivalent, from $\operatorname{len}_{\beta_p}(\lambda_p) < \infty$ and $\operatorname{len}_{\beta_q}(\lambda_q) < \infty$, we can deduce that $\operatorname{len}_{\|.\|_2}(\lambda_p) < \infty$ and $\operatorname{len}_{\|.\|_2}(\lambda_q) < \infty$. We have

$$\begin{aligned}
\lim_{n\to\infty} n\varphi_n(p) &= \frac{\operatorname{len}_{\beta_p}(\lambda_p)}{\sqrt{2}\theta_p} \\
&\geq \frac{\operatorname{len}_{\beta_q}(\lambda_p) - \kappa(q-p)|\log(q-p)|\operatorname{len}_{\|.\|_2}(\lambda_p)}{\sqrt{2}\theta_p} \\
&\geq \frac{\operatorname{len}_{\beta_q}(\lambda_q)}{\sqrt{2}\theta_q} - \frac{\kappa(q-p)|\log(q-p)|\operatorname{len}_{\|.\|_2}(\lambda_p)}{\sqrt{2}\theta_{p_0}} \\
&\geq \lim_{n\to\infty} n\varphi_n(q) - \frac{\kappa(q-p)|\log(q-p)|\operatorname{len}_{\|.\|_2}(\lambda_p)}{\sqrt{2}\theta_{p_0}}
\end{aligned} \tag{48}$$

where we use equation (47), that λ_q is a minimizer for $\operatorname{len}_{\beta_q}$ and that the map $p \to \theta_p$ is non decreasing.

The map $p \to \theta_p$ is infinitely differentiable, see for instance Theorem 8.92 in [18]. As θ_p is positive for p in $[p_0, p_1]$, the map $p \to 1/\theta_p$ is differentiable on the compact $[p_0, p_1]$ and therefore is also Lipschitz on $[p_0, p_1]$. There exists a constant L depending on p_0, p_1 such that for all $p \leq q$ in $[p_0, p_1]$

$$\left|\frac{1}{\theta_p} - \frac{1}{\theta_q}\right| \leq L(q-p)\,. \tag{49}$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty} n\varphi_n(p) &\leq \frac{\operatorname{len}_{\beta_p}(\lambda_q)}{\sqrt{2}\theta_p} \\
&\leq \frac{\operatorname{len}_{\beta_q}(\lambda_q) + \kappa(q-p)|\log(q-p)|\operatorname{len}_{\|.\|_2}(\lambda_q)}{\sqrt{2}\theta_p} \\
&\leq \frac{\operatorname{len}_{\beta_q}(\lambda_q)}{\sqrt{2}\theta_p} + \frac{\kappa(q-p)|\log(q-p)|\operatorname{len}_{\|.\|_2}(\lambda_q)}{\sqrt{2}\theta_{p_0}} \\
&\leq \frac{\operatorname{len}_{\beta_q}(\lambda_q)}{\sqrt{2}\theta_q} + \frac{L}{\sqrt{2}}(q-p)\operatorname{len}_{\beta_q}(\lambda_q) + \frac{\kappa(q-p)|\log(q-p)|\operatorname{len}_{\|.\|_2}(\lambda_q)}{\sqrt{2}\theta_{p_0}} \\
&\leq \lim_{n\to\infty} n\varphi_n(q) + \left(\frac{L}{\sqrt{2}}\operatorname{len}_{\beta_q}(\lambda_q) + \frac{\kappa \operatorname{len}_{\|.\|_2}(\lambda_q)}{\sqrt{2}\theta_{p_0}}\right)(q-p)|\log(q-p)|\,.
\end{aligned} \tag{50}$$

Let $\beta_q^{min} = \inf_{x\in\mathbb{S}^1} \beta_q(x)$. By Lemma 7.1, the map $q \to \beta_q^{min}$ is continuous. We denote by $\beta^{min} > 0$ its infimum on the compact $[p_0, p_1]$. By definition of the length, $\mathrm{len}_{\|.\|_2}(\lambda_q) \leq \mathrm{len}_{\beta_q}(\lambda_q)/\beta^{min}$. Moreover, by Theorem 1.1 in [15], we know that the map $q \to \mathrm{len}_{\beta_q}(\lambda_q)$ is continuous. Thus, the quantity $\sup_{u\in[p_0,p_1]} \mathrm{len}_{\beta_u}(\lambda_u)$ is finite. Finally,

$$\begin{aligned} \lim_{n\to\infty} n\varphi_n(p) \leq &\lim_{n\to\infty} n\varphi_n(q) \\ &+ \left(\sup_{u\in[p_0,p_1]} \mathrm{len}_{\beta_u}(\lambda_u)\right)\left(L + \frac{\kappa}{\beta^{min}\sqrt{2}\theta_{p_0}}\right)(q-p)|\log(q-p)|\,. \end{aligned} \tag{51}$$

From equation (48), we obtain

$$\begin{aligned} \lim_{n\to\infty} n\varphi_n(p) &\geq \lim_{n\to\infty} n\varphi_n(q) - \frac{\kappa(q-p)|\log(q-p)|\,\mathrm{len}_{\|.\|_2}(\lambda_p)}{\sqrt{2}\theta_{p_0}} \\ &\geq \lim_{n\to\infty} n\varphi_n(q) - \frac{\kappa(q-p)|\log(q-p)|\,\mathrm{len}_{\beta_p}(\lambda_p)}{\beta_{min}\sqrt{2}\theta_{p_0}} \\ &\geq \lim_{n\to\infty} n\varphi_n(q) - \frac{\kappa\left(\sup_{u\in[p_0,p_1]} \mathrm{len}_{\beta_u}(\lambda_u)\right)}{\beta_{min}\sqrt{2}\theta_{p_0}}(q-p)|\log(q-p)|\,. \end{aligned} \tag{52}$$

Thus, combining (51) and (52), we deduce the existence of a constant ν depending only on p_0 and p_1, such that

$$\lim_{n\to\infty} n|\varphi_n(p) - \varphi_n(q)| \leq \nu(q-p)|\log(q-p)|\,.$$

$\square$

Proof of Theorem 1.2. Let $p_c(2) < p_0 < p_1 < 1$ and let $p \leq q$ in $[p_0, p_1]$. We consider β_p^* the dual norm of β_p, defined by

$$\forall x \in \mathbb{R}^2,\ \beta_p^*(x) = \sup\{x\cdot z \,:\, \beta_p(z) \leq 1\}\,.$$

Then β_p^* is a norm. The Wulff crystal associated with β_p is in fact the unit ball associated with β_p^*.

Note that the supremum is always achieved for a z such that $\beta_p(z) = 1$. Let $x \in \mathbb{R}^2$. Let $y \in \mathbb{S}^1$ be the direction that achieves the supremum for $\beta_p^*(x)$, thus we have

$$\beta_p^*(x) = x \cdot \frac{y}{\beta_p(y)}$$

and

$$\begin{aligned} \beta_p^*(x) - \beta_q^*(x) &\leq x\cdot\frac{y}{\beta_p(y)} - x\cdot\frac{y}{\beta_q(y)} \\ &= x\cdot\left(\frac{y}{\beta_p(y)} - \frac{y}{\beta_q(y)}\right) \\ &\leq \frac{\|x\|_2\|y\|_2}{\beta_p(y)\beta_q(y)}|\beta_p(y) - \beta_q(y)| \\ &\leq \frac{\|x\|_2}{(\beta^{min})^2}\sup_{z\in\mathbb{S}^1}|\beta_p(z) - \beta_q(z)| \\ &\leq \frac{\|x\|_2}{(\beta^{min})^2}\kappa(q-p)|\log(q-p)| \end{aligned}$$

where β^{min} was defined in the proof of Theorem 1.2 and where we use the Lemma 7.1. We proceed similarly for $\beta_q^*(x) - \beta_p^*(x)$, we obtain

$$|\beta_p^*(x) - \beta_q^*(x)| \leq \frac{\|x\|_2}{(\beta^{min})^2}\kappa(q-p)|\log(q-p)|$$

and finally

$$\sup_{x\in\mathbb{S}^1} |\beta_p^*(x) - \beta_q^*(x)| \leq \frac{\kappa}{(\beta^{min})^2}(q-p)|\log(q-p)|\,. \tag{53}$$

As the Wulff crystal is the unit ball associated with β_p^*, we can deduce the following result for the Wulff crystal W_p as in equation (33)

$$d_H(W_p, W_q) \leq \frac{\kappa}{(\beta^{min})^4}(q-p)|\log(q-p)|\,. \tag{54}$$

□

8 Modified Cheeger constant in dimension $d \geq 3$

In order to prove Theorems 1.3 and 1.4, we need to first adapt a proof of Zhang in [33] and to prove that the function $p \to \beta_p$ is uniformly Lipschitz continuous on any interval included in $(p_c(d), 1)$ (see Theorem 1.5). Except the subsection 8.1, this section is very similar to section 7, it uses the same kind of arguments but for higher dimension.

8.1 Adaptation of the Theorem 2 in [33]

In [33], Zhang obtained a control on the size of smallest minimal cutset corresponding to maximal flows in general first passage percolation, but his control depends on the distribution G of the variables $(t(e))_{e\in\mathbb{E}^d}$ associated with the edges. We only consider probability measures $G_p' = p\delta_1 + (1-p)\delta_0$ for $p > p_c(d)$, but we need to adapt Zhang's proof in this particular case to obtain a control that does not depend on p anymore. More precisely, let us denote by $\mathcal{N}_{n,p}$ the size of the smallest cutset that achieves the infimum in $\tau_p(n, \overrightarrow{v})$. We have the following control on $\mathcal{N}_{n,p}$.

Theorem 8.1 (Adaptation of Theorem 2 in [33])**.** *Let $p_0 > p_c(d)$. There exist constants C_1, C_2 and α that depend only on d and p_0 such that for all $p \in [p_0, 1]$, for all $n \in \mathbb{N}^*$,*

$$\mathbb{P}_p\left[\mathcal{N}_{n,p} > \alpha n^{d-1}\right] \leq C_1 \exp(-C_2 n^{d-1})\,.$$

Remark 8.1. The proof is going to be simpler than the proof of Theorem 2 in [33], because passage times in our context can take only values 0 or 1. We recall that we say that an edge is closed if its passage time is 0, otherwise we say it is open.

Let us first explain the idea behind that theorem. We can extend the notion of cutset defined in section 1.3 to sets that cut a given set from infinity. The capacity of a cutset E corresponds to $|E|_{o,p}$, *i.e.*, the number of p-open edges in E. We say that a cutset is a minimal cutset if it is a cutset of minimal capacity. We want to bound the size of the smallest minimal cutset that cuts a given set of vertices V from infinity. Let us consider $\mathcal{C}(V)$ the set that contains all the vertices that are connected to V by an open path. On the event that there exists a cutset of null capacity that cuts S to infinity, the set $\mathcal{C}(V)$ is finite and its edge boundary is a cutset of null capacity. However, this cutset may be very big and may contain too many extra edges. From this cutset, we want to build a "smoother" and so smaller one. We do renormalization at a scale t to be defined later, and we exhibit a set of boxes Γ_t that contain the exterior edge boundary of $\mathcal{C}(V)$ and contain a cutset of null capacity. By construction, each box of Γ_t has at least one $*$-neighbor in which an atypical event occurs (an event of probability that goes to 0 when t goes to infinity). As these events are atypical, we expect that Γ_t does not contain too many boxes.

As we are in a supercritical Bernoulli percolation, there is a set of measure close to 1 such that for any configuration ω in this set, a minimal cutset E has positive capacity, *i.e.*, contains open edges. To be able to do the construction of Γ_t even so, we slightly modify the configuration ω by closing all the open edges in E. This modification of ω allow us to build Γ_t and when we reopen the edges we have closed, some boxes of Γ_t remain unchanged, atypical events still occur. The number of boxes in Γ_t that change is upperbounded by the number of edges we have closed. As it is easy to bound this number of edges we have closed, we can obtain an upper bound on Γ_t by doing some combinatorial considerations on the number of boxes where an atypical event occurs. We can deduce an upperbound on the size of the minimal cutset Γ that belongs to Γ_t.

As the original proof is very technical, the adaptation of the proof is also technical. This proof is independent of the remaining of the paper, the reader may very well skip this proof.

Adaptation of the proof of Theorem 2 *in [33] to get Theorem 8.1.* We keep the same notations as in [33]. The following adaptation is not self-contained Let $p_0 > p_c(d)$ and $\overrightarrow{v} \in \mathbb{S}^{d-1}$. First notice that the construction of a linear cutset in section 2 of [33] is not specific to the set $B(k,m)$ and can be defined in the same way for any set of vertices. In particular we can replace $B(k,m)$ by $C_1'(nS(\overrightarrow{v}),n)$ and ∞ by $C_2'(nS(\overrightarrow{v}),n)$. We denote by $\mathcal{C}(n)$ the set that corresponds to $C(k,m)$ defined in Lemma 1 in [33]:

$$\mathcal{C}(n) = \{v \in \mathbb{Z}^d : \ v \text{ is connected by an open path to } C_1'(nS(\overrightarrow{v}),n)\}\,.$$

We denote by $\mathcal{G}(n)$ the event that $\mathcal{C}(n) \cap C_2'(nS(\overrightarrow{v}),n) = \emptyset$ (it corresponds to $\mathcal{G}(k,m)$ in [33]). On this event, there exists a closed cutset that cuts $C_1'(nS(\overrightarrow{v}),n)$ from $C_2'(nS(\overrightarrow{v}),n)$.

Thanks to the work we did in section 4.2 and in particular in Lemmas 4.1 and 4.2, the constants in Lemma 6 and 7 in [33] can be chosen such that they only depend on p_0 and d. The collection $(B_t(u))_{u\in\mathbb{Z}^d}$ is a partition of $\mathbb{Z}^d$ into boxes of size t, $\bar{B}_t(u) = \bigcup_{v\sim u} B_t(u)$ and the t-cubes in $\bar{B}_t(u)$ are precisely $B_t(u)$ and $B_t(v)$ for $v \sim u$. Let $p \geq p_0$, we say that $B_t(u)$ has a p-disjoint property if there exist two disconnected p-open clusters in $\bar{B}_t(u)$, both with vertices in $B_t(u)$ and in the boundary of $\bar{B}_t(u)$. We say that $B_t(u)$ has a p-blocked property if there is a p-open cluster in $\bar{B}_t(u)$ with vertices in $B_t(u)$ and in the boundary of $\bar{B}_t(u)$, but without vertices in a t-cube of $\bar{B}_t(u)$. Note that if $B_t(u)$ has a p-disjoint property and $\bar{B}_t(u)$ has a p-crossing cluster, then there is a p-open cluster of diameter greater than t different from the p-crossing cluster, so there is a t-cube in $\bar{B}_t(u)$ where the event $T_{t/2,t}(p)$ (as defined in [18] Lemma 7.104) occurs. Similarly, if $B_t(u)$ has a p-blocked property and $\bar{B}_t(u)$ has a p-crossing cluster, then there is a t-cube in $\bar{B}_t(u)$ where the event $T_{t/2,t}(p)$ occurs. Thus,

$$\begin{aligned}\mathbb{P}[B_t(u) \text{ has a } p\text{-disjoint property}] &\leq \mathbb{P}[\bar{B}_t(u) \text{ does not have a } p\text{-crossing cluster}]\\ &\quad + \mathbb{P}\left[\begin{array}{c} B_t(u) \text{ has a } p\text{-disjoint property} \\ \text{and } \bar{B}_t(u) \text{ has a } p\text{-crossing cluster}\end{array}\right]\\ &\leq \kappa_1(p_0)\exp(-\kappa_2(p_0)t^{d-1}) + 3^d\mathbb{P}[T_{t/2,t}(p)]\\ &\leq \kappa_1(p_0)\exp(-\kappa_2(p_0)t^{d-1})\\ &\quad + \kappa(p_0)3^d t^{2d}\exp(-\mu(p_0)t/2)\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}[B_t(u) \text{ has a } p\text{-blocked property}] &\leq \mathbb{P}[\bar{B}_t(u) \text{ does not have a } p\text{-crossing cluster}]\\ &\quad + \mathbb{P}\left[\begin{array}{c} B_t(u) \text{ has a } p\text{-blocked property} \\ \text{and } \bar{B}_t(u) \text{ has a } p\text{-crossing cluster}\end{array}\right]\\ &\leq \kappa_1(p_0)\exp(-\kappa_2(p_0)t^{d-1}) + 3^d\mathbb{P}[T_{t/2,t}(p)]\\ &\leq \kappa_1(p_0)\exp(-\kappa_2(p_0)t^{d-1})\\ &\quad + \kappa(p_0)3^d t^{2d}\exp(-\mu(p_0)t/2)\,.\end{aligned}$$

Let $p > p_0$, as we only focus here on the edges inside $B(n, \overrightarrow{v})$, we can use the probability measure $\mathbb{P}_{p,n}(\cdot)$ which is the product measure on the edges of $B(n, \overrightarrow{v})$. As a corollary of Lemma 4.1 in [28], there exists a deterministic cutset E that cuts $C_1'(nS(\overrightarrow{v}), n)$ from $C_2'(nS(\overrightarrow{v}), n)$ such that $|E| \leq c_d(2n)^{d-1}$ where c_d depends only on d but not on $\overrightarrow{v}$. Thus, we obtain that $\tau_p(n, \overrightarrow{v}) \leq |E| \leq c_d(2n)^{d-1}$. We denote by $E_{n,p}$ the cutset that achieves the infimum in $\tau_p(n, \overrightarrow{v})$ and such that $|E_{n,p}| = \mathcal{N}_{n,p}$ ($E_{n,p}$ corresponds to $W(k, m)$ in [33]).

For a configuration ω, we denote by $e_1, \dots, e_{J(\omega)}$ the p-open edges in $E_{n,p}$. We have $J(\omega) = \tau_p(n, \overrightarrow{v})(\omega) \leq c_d(2n)^{d-1}$. Assume that all the edges outside $B(n, \overrightarrow{v})$ are closed, it will not affect the probability measure $\mathbb{P}_{p,n}$. We denote by $\sigma(\omega)$ the configuration which coincides with ω except in edges $e_1, \dots, e_{J(\omega)}$ where the passage time is equal to 0. Thus, the set $E_{n,p}(\sigma(\omega))$ is a p-closed (for the configuration $\sigma(\omega)$) cutset that cuts $C_1'(nS(\overrightarrow{v}), n)$ from $C_2'(nS(\overrightarrow{v}), n)$. Note that the set of edges $E_{n,p}(\sigma(\omega))$ is determined by the configuration ω whereas we consider its capacity for $\sigma(\omega)$. The event $\mathcal{G}(n)$ occurs in the configuration $\sigma(\omega)$ and we can use the construction of section 2 in [33]: $\overline{\Gamma}_t$ contains a p-closed (for $\sigma(\omega)$) cutset Γ that cuts $C_1'(nS(\overrightarrow{v}), n)$ from $C_2'(nS(\overrightarrow{v}), n)$ and is contained in $B(n, \overrightarrow{v})$ (see Lemma 4 in [33]). We write $\Gamma(\omega)$ when we consider the edge set Γ with its edges capacities determined by the configuration ω.

We now change $\sigma(\omega)$ back to ω, the passage time of e_i changes from 0 to 1. $\Gamma(\omega)$ as a vertex set exists, it is still a cutset but it is no longer closed, all edges in $\Gamma(\omega)$ except the e_i are closed. Therefore, $|\Gamma(\omega)|_{o,p} \leq J(\omega)$, but by definition of $E_{n,p}$, we have $J(\omega) = |E_{n,p}(\omega)|_{o,p} \leq |\Gamma(\omega)|_{o,p} \leq J(\omega)$ and $|\Gamma(\omega)|_{o,p} = J(\omega)$. Moreover, for each ω, by definition of $\mathcal{N}_{n,p}(\omega)$, we get that $|\Gamma(\omega)| \geq \mathcal{N}_{n,p}(\omega)$.

Any cube $B_t(u)$ that intersects the boundary of nA belongs to Γ_t as it also intersects $\partial_e\mathcal{C}(n)$. Thanks to this remark, we avoid the part of Zhang's proof where he tries to find a vertex z in the intersection between the cutset $W(k, m)$ and a line L in order to find a cube that is in Γ_t. Thus, the term $\exp(\beta^{-1}n)$ in (6.19) is not necessary.

Note that for the t-cubes $B_t(u)$ in the boundary of $B(n, \overrightarrow{v})$, we cannot be sure that there exists a t-cube in $\bar{B}_t(u)$ with a blocked or disjoint property, but the number of boxes that intersect the boundary of $B(n, \overrightarrow{v})$ is bounded by $C_{d,t}n^{d-1}$ where $C_{d,t}$ depends only on d and t. Thus, if the number of t-cubes in Γ_t is greater than βn^{d-1}, then the number of t-cubes in Γ_t that does not intersect the boundary and that does not contain any edge among $e_1, \dots, e_J$ is greater than $(\beta - C_{d,t} - 2^{d-1}c_d)n^{d-1}$. All these t-cubes have at least one $*$-neighbor with a blocked or disjoint property. This leads to small modifications of constants in the proof of [32]. The remaining of the proof is the same. $\square$

8.2 Regularity of the norm β_p

In this section, we prove Theorem 1.5. We are going to use a proof quite similar to what we have done in section 7. Note that an important difference between β_p and the definition of the norm in dimension 2 is that we do not require the cutsets we consider to be open. In dimension 2, we require that the right-most paths in the definition of the norm are open, this is the reason why we need to perform a renormalization step. The proof is easier in dimension $d \geq 3$.

Our strategy is the following, we easily get that $\beta_p \leq \beta_q$ by properly coupling the percolations of parameters $p_c(d) < p < q$. The second inequality requires more work.

We denote by $E_{n,p}$ the random cutset of minimal size that achieves the minimum in the definition of $\tau_p(n, \overrightarrow{v})$. By definition, as $E_{n,p}$ is a cutset, we can upperbound $\tau_q(n, \overrightarrow{v})$ by the number of edges in $E_{n,p}$ that are q-open, which we expect to be at most $\tau_p(n, \overrightarrow{v}) + C(q-p)|E_{n,p}|$ where C is a constant. We next need to get a control of $|E_{n,p}|$ which is uniform in p of the kind $c_d(2n)^{d-1}$ where c_d does not depend on p. We can only hope a uniform control for all $p \in [p_0, 1]$ with $p_0 > p_c(d)$. This uniform control is given by Theorem 8.1.

Proof of Theorem 1.5. Let $p_c < p_0 < p_1 < 1, \overrightarrow{v} \in \mathbb{S}^{d-1}$, and p, q such that $p_0 < p < q < p_1$. First, we fix a cube $B(n, \overrightarrow{v})$ and we couple the percolations of parameters p and q in the standard way, *i.e.*, we consider the i.i.d. family $(U(e))_{e \in \mathbb{E}^d}$ distributed according to the uniform law on $[0,1]$ and we say that an edge e is p-open (resp. q-open) if $U(e) \geq p$ (resp. $U(e) \geq q$). Thanks to this coupling, we easily obtain that $\tau_p(\overrightarrow{v}, n) \leq \tau_q(\overrightarrow{v}, n)$ and by dividing by $(2n)^{d-1}$ and letting n go to infinity we conclude that

$$\beta_p(\overrightarrow{v}) \leq \beta_q(\overrightarrow{v}) \,. \tag{55}$$

Let $E_{n,p}$ be a random cutset of minimal size that achieves the minimum in the definition of $\tau_p(n, \overrightarrow{v})$. We take the same coupling as in Lemma 7.1 step (i). Let $\delta > 0$. We have,

$$\begin{aligned}
&\mathbb{P}\left[\tau_q(n, \overrightarrow{v}) > \tau_p(n, \overrightarrow{v}) + \left(\frac{q-p}{1-p} + \delta\right)\alpha n^{d-1}, \mathcal{N}_{n,p} < \alpha n^{d-1}\right] \\
&\qquad \leq \mathbb{P}\left[\tau_q(n, \overrightarrow{v}) - \tau_p(n, \overrightarrow{v}) > \left(\frac{q-p}{1-p} + \delta\right)|E_{n,p}|\right] \\
&\qquad \leq \sum_{\mathcal{E}} \mathbb{P}\left[E_{n,p} = \mathcal{E}, \#\{e \in \mathcal{E} : (U(e), V(e)) = (0,1)\} > \left(\frac{q-p}{1-p} + \delta\right)|\mathcal{E}|\right]
\end{aligned}$$

$$\leq \sum_{\mathcal{E}} \mathbb{P}[E_{n,p} = \mathcal{E}]\, \mathbb{P}\left[\#\{e \in \mathcal{E} : V(e) = 1\} > \left(\frac{q-p}{1-p} + \delta\right) |\mathcal{E}|\right]$$
$$\leq \exp(-2\delta^2 n^{d-1}) \tag{56}$$

where the sum is over all sets $\mathcal{E}$ that cut $C'_1(nS(\overrightarrow{v}), n)$ from $C'_2(nS(\overrightarrow{v}), n)$ in $B(n, \overrightarrow{v})$ and where we use in the last inequality Chernoff bound and the fact that $|E_{n,p}| \geq n^{d-1}$ (uniformly in $\overrightarrow{v}$).

Finally, using inequality (56) and Theorem 8.1, we get

$$\begin{aligned}
\mathbb{E}[\tau_q(n, \overrightarrow{v})] &\leq \mathbb{E}[\tau_q(n, \overrightarrow{v}) \mathbb{1}_{\mathcal{N}_{n,p} < \alpha n^{d-1}}] + \mathbb{E}[\tau_q(n, \overrightarrow{v}) \mathbb{1}_{\mathcal{N}_{n,p} \geq \alpha n^{d-1}}] \\
&\leq \mathbb{E}[\tau_p(n, \overrightarrow{v})] + \left(\frac{q-p}{1-p} + \delta\right) \alpha n^{d-1} \\
&\qquad + |B(n, \overrightarrow{v})| \left(\exp(-2\delta^2 n^{d-1}) + C_1 \exp(-C_2 n^{d-1})\right) \\
&\leq \mathbb{E}[\tau_p(n, \overrightarrow{v})] + \left(\frac{q-p}{1-p} + \delta\right) \alpha n^{d-1} \\
&\qquad + C_d (2n)^d \left(\exp(-2\delta^2 n^{d-1}) + C_1 \exp(-C_2 n^{d-1})\right) ,
\end{aligned}$$

where C_d is a constant depending only on d. Dividing by $(2n)^{d-1}$ and by letting n go to infinity, we obtain

$$\beta_q(\overrightarrow{v}) \leq \beta_p(\overrightarrow{v}) + \left(\frac{q-p}{1-p} + \delta\right) \frac{\alpha}{2^{d-1}} \tag{57}$$

and by letting δ goes to 0,

$$\beta_q(\overrightarrow{v}) \leq \beta_p(\overrightarrow{v}) + \kappa(q-p) \tag{58}$$

where $\kappa = \alpha/((1-p_1)2^{d-1})$.

Combining inequalities (55) and (58), we obtain that

$$\sup_{\overrightarrow{v} \in \mathbb{S}^{d-1}} |\beta_q(\overrightarrow{v}) - \beta_p(\overrightarrow{v})| \leq \kappa |q-p| \,. \tag{59}$$

□

8.3 Proof of Theorem 1.3

This proof uses the same arguments as in section 7. In the following W_p denotes the Wulff crystal for the norm β_p such that $\mathcal{L}^d(W_p) = 2^d/d!$. First, let us compute some useful inequalities. For any set $E \subset \mathbb{R}^d$ with Lipschitz boundary, by Theorem 1.5, we have

$$\begin{aligned}
|\mathcal{I}_p(E) - \mathcal{I}_q(E)| &= \left| \int_{\partial E} (\beta_p(\nu_E(x)) - \beta_q(\nu_E(x)))\, \mathcal{H}^{d-1}(dx) \right| \\
&\leq \int_{\partial E} |\beta_p(\nu_E(x)) - \beta_q(\nu_E(x))|\, \mathcal{H}^{d-1}(dx) \\
&\leq \kappa |q-p| \mathcal{H}^{d-1}(\partial E) \,.
\end{aligned} \tag{60}$$

Thanks to Theorem 1.5, we know that the function $p \to \beta_p$ is uniformly continuous on $[p_0, p_1]$. We denote by β^{min} and β^{max} its minimal and maximal value, i.e., for all $\overrightarrow{v} \in \mathbb{S}^{d-1}$ and $p \in [p_0, p_1]$, we have

$$\beta^{min} \leq \beta_p(\overrightarrow{v}) \leq \beta^{max} \,.$$

Using this inequality and the fact that the Wulff crystal is a minimizer for an isoperimetric problem, we get

$$\begin{aligned}
\mathcal{I}_p(W_p) &\leq \mathcal{I}_p(W_{p_0}) \\
&= \int_{\partial W_{p_0}} \beta_p(\nu_{W_{p_0}}(x))\mathcal{H}^{d-1}(dx) \\
&\leq \int_{\partial W_{p_0}} \beta^{max}\mathcal{H}^{d-1}(dx) \\
&\leq \mathcal{H}^{d-1}(\partial W_{p_0})\beta^{max}\,.
\end{aligned} \tag{61}$$

We also have

$$\begin{aligned}
\mathcal{H}^{d-1}(\partial W_p) &= \int_{\partial W_p} \mathcal{H}^{d-1}(dx) \\
&\leq \int_{\partial W_p} \frac{\beta_p(\nu_{W_p}(x))}{\beta^{min}}\mathcal{H}^{d-1}(dx) \\
&\leq \frac{\mathcal{I}_p(W_p)}{\beta^{min}} \\
&\leq \frac{\mathcal{I}_p(W_{p_0})}{\beta^{min}} \\
&\leq \frac{\mathcal{H}^{d-1}(\partial W_{p_0})\beta^{max}}{\beta^{min}}\,.
\end{aligned} \tag{62}$$

Finally, we obtain combining (60), (61), (62) and (49),

$$\begin{aligned}
\lim_{n\to\infty} n\widehat{\varphi}_n(p) &= \frac{\mathcal{I}_p(W_p)}{\theta_p(d)\mathcal{L}^d(W_p)} \\
&\geq \frac{\mathcal{I}_q(W_p) - \kappa|q-p|\mathcal{H}^{d-1}(\partial W_p)}{\theta_p(d)\mathcal{L}^d(W_p)} \\
&\geq \frac{\mathcal{I}_q(W_q)}{\theta_q(d)\mathcal{L}^d(W_q)} - \frac{\kappa|q-p|\mathcal{H}^{d-1}(\partial W_p)}{\theta_p(d)\mathcal{L}^d(W_q)} \\
&\geq \lim_{n\to\infty} n\widehat{\varphi}_n(q) - \frac{\kappa\, d!\,\beta^{max}\mathcal{H}^{d-1}(\partial W_{p_0})}{2^d\theta_{p_0}(d)\beta^{min}}|q-p|\,,
\end{aligned} \tag{63}$$

and,

$$\begin{aligned}
\lim_{n\to\infty} n\widehat{\varphi}_n(p) &= \frac{\mathcal{I}_p(W_p)}{\theta_p(d)\mathcal{L}^d(W_p)} \\
&\leq \frac{\mathcal{I}_p(W_q)}{\theta_p(d)\mathcal{L}^d(W_p)} \\
&\leq \frac{\mathcal{I}_q(W_q) + \kappa|q-p|\mathcal{H}^{d-1}(\partial W_q)}{\theta_p(d)\mathcal{L}^d(W_q)} \\
&\leq \frac{\mathcal{I}_q(W_q)}{\mathcal{L}^d(W_q)}\left(\frac{1}{\theta_q(d)} + L|q-p|\right) + \frac{\kappa\, d!\,\mathcal{H}^{d-1}(\partial W_q)}{2^d\theta_{p_0}(d)}|q-p| \\
&\leq \lim_{n\to\infty} n\widehat{\varphi}_n(q) + \left(\frac{L\mathcal{I}_q(W_q)d!}{2^d} + \frac{\kappa\, d!\,\mathcal{H}^{d-1}(\partial W_q)}{2^d\theta_{p_0}(d)}\right)|q-p|
\end{aligned}$$

$$\leq \lim_{n\to\infty} n\widehat{\varphi}_n(q) + \frac{d!\mathcal{H}^{d-1}(\partial W_{p_0})\beta^{max}}{2^d}\left(L + \frac{\kappa}{\theta_{p_0}(d)\beta^{min}}\right)|q-p|. \tag{64}$$

Thus combining (63) and (64), we get

$$\lim_{n\to\infty} n|\widehat{\varphi}_n(q) - \widehat{\varphi}_n(p)| \leq \nu_d |q-p| \tag{65}$$

where $\nu_d = d!\mathcal{H}^{d-1}(\partial W_{p_0})\beta^{max}\left(L + \kappa/(\theta_{p_0}(d)\beta^{min})\right)/2^d$.

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