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# Equivalence of some subcritical properties in continuum percolation

Jean-Baptiste Gouéré\* and Marie Thérêt†

**Abstract:** We consider the Boolean model on  $\mathbb{R}^d$ . We prove some equivalences between subcritical percolation properties. Let us introduce some notations to state one of these equivalences. Let  $C$  denote the connected component of the origin in the Boolean model. Let  $|C|$  denotes its volume. Let  $\ell$  denote the maximal length of a chain of random balls from the origin. Under optimal integrability conditions on the radii, we prove that  $E(|C|)$  is finite if and only if there exists  $A, B > 0$  such that  $\mathbb{P}(\ell \geq n) \leq Ae^{-Bn}$  for all  $n \geq 1$ .

*Keywords :* Boolean model, continuum percolation, critical point.

## 1 Introduction

**The Boolean model.** The Boolean model is defined as follows. At each point of a homogeneous Poisson point process on the Euclidean space  $\mathbb{R}^d$ , we center a ball of random radius. We assume that the radii of the balls are independent, identically distributed and independent of the point process. The Boolean model is the union of the balls. There are three parameters:

- An integer  $d \geq 2$ . This is the dimension of the ambient space  $\mathbb{R}^d$ .
- A real number  $\lambda > 0$ . The intensity measure of the Poisson point process of centers is  $\lambda|\cdot|$  where  $|\cdot|$  denotes the Lebesgue measure on  $\mathbb{R}^d$ .
- A probability measure  $\nu$  on  $(0, +\infty)$ . This is the common distribution of the radii. We will also consider a random variable  $R$  whose distribution is  $\nu$ .

We will denote the Boolean model by  $\Sigma(\lambda, \nu, d)$  or  $\Sigma$ . We will also say that  $\Sigma$  is the Boolean model driven by the measure  $\lambda\nu$ .

More precisely, the Boolean model is defined as follows. Let  $\xi$  be a Poisson point process on  $\mathbb{R}^d \times (0, +\infty)$  with intensity measure  $\lambda|\cdot| \otimes \nu = |\cdot| \otimes \lambda\nu$ . Set

$$\Sigma(\lambda, \nu, d) = \bigcup_{(c,r) \in \xi} B(c, r)$$

where  $B(c, r)$  denotes the open Euclidean ball of  $\mathbb{R}^d$  with center  $c$  and radius  $r$ . We refer to the book by Meester and Roy [12] for background on the Boolean model, and to the book by Schneider and Weil [16] and the book by Last and Penrose [11] for background on Poisson processes. We also denote by  $S(c, r)$  the Euclidean sphere of  $\mathbb{R}^d$  with center  $c$  and radius  $r$ . We write  $S(r)$  when  $c = 0$ .

**Percolation in the Boolean model.** If  $A$  and  $B$  are two subsets of  $\mathbb{R}^d$ , we set

$$\{A \overset{\Sigma}{\longleftrightarrow} B\} = \{\text{There exists a path in } \Sigma \text{ from } A \text{ to } B\}$$

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and

$$\{0 \overset{\Sigma}{\longleftrightarrow} \infty\} = \{\text{The connected component of } \Sigma \text{ that contains the origin is unbounded}\}.$$

We denote by  $C$  the connected component of  $\Sigma$  that contains the origin. We denote by  $D$  the diameter of  $C$ . We denote by  $\#C$  the number of random balls contained in  $C$ . In other words,  $\#C$  is the following cardinality:

$$\#C = \text{card}(\{(c, r) \in \xi : c \in C\}).$$

A chain of length  $n \geq 1$  is a sequence  $((c_1, r_1), \dots, (c_n, r_n))$  of distinct points of  $\xi$  such that

$$\forall i \in \{2, \dots, n\}, B(c_{i-1}, r_{i-1}) \cap B(c_i, r_i) \neq \emptyset.$$

We say that the chain starts in  $A \subset \mathbb{R}^d$  if  $B(c_1, r_1)$  touches  $A$ . We say that the chain stops in  $A \subset \mathbb{R}^d$  if  $B(c_n, r_n)$  touches  $A$ . We denote by  $\ell$  the largest length of a chain starting in  $B(0, 1)$ . More precisely,

$$\ell = \sup\{n \geq 0 : \exists x_1, \dots, x_n \in \xi \text{ s.t. } (x_1, \dots, x_n) \text{ is a chain starting in } B(0, 1)\}. \quad (1)$$

Let us consider the sets

$$\begin{aligned} \Lambda &= \{\lambda > 0 : \mathbb{P}(0 \overset{\Sigma}{\longleftrightarrow} \infty) = 0\}, \\ \widehat{\Lambda} &= \{\lambda > 0 : \mathbb{P}(S(r) \overset{\Sigma}{\longleftrightarrow} S(2r)) \rightarrow 0 \text{ as } r \rightarrow \infty\} \end{aligned}$$

and the associated critical thresholds

$$\begin{aligned} \lambda_c &= \sup \Lambda_c, \\ \widehat{\lambda}_c &= \sup \widehat{\Lambda}_c. \end{aligned}$$

From

$$\mathbb{P}(0 \overset{\Sigma}{\longleftrightarrow} \infty) = \lim_{r \rightarrow \infty} \mathbb{P}(0 \overset{\Sigma}{\longleftrightarrow} S(2r))$$

we get

$$\widehat{\Lambda} \subset \Lambda$$

and thus

$$\widehat{\lambda}_c \leq \lambda_c.$$

When  $\lambda \in \Lambda$ , all the connected components of  $\Sigma$  are almost surely bounded. We say that  $\Sigma$  does not percolate. When  $\lambda \notin \Lambda$ , with probability one, one of the connected components of  $\Sigma$  is unbounded. We say that  $\Sigma$  percolates. The thresholds  $\lambda_c$  and  $\widehat{\lambda}_c$  are always finite. See for example the remark below Theorem 3.3 in [12]. If the radii are bounded, then  $\lambda_c = \widehat{\lambda}_c$ . This is a sharp threshold property. The sharpness of the transition in the discrete setting was proved independently by Menshikov [13] and by Aizenman-Barsky [3]. The first proof of the equality  $\lambda_c = \widehat{\lambda}_c$  relied on the analogous result in the discrete setting. We refer to [12] for the proof (see Theorem 3.5) and references. Ziesche gives in [18] a short proof of the equality  $\lambda_c = \widehat{\lambda}_c$  for bounded radii. It relies on a new and short proof of the analogous result in the discrete setting by Duminil-Copin and Tassion [8, 7]. Ahlbergh, Teixeira and Tassion gave in [2] a very complete picture of percolation in the two dimensional Boolean model. In particular, they established a sharp threshold property for the two dimensional Boolean model under a minimal integrability assumption ( $\mathbb{E}(R^2) < \infty$ ). See Theorem 1.1 in [2]. We also refer to [1] by Ahlbergh, Teixeira and Tassion and [15] by Penrose for further results about percolation in the complement of the Boolean model. Even more recently, Duminil-Copin, Raoufi and Tassion developed new methods to prove sharp threshold properties in a wide class of models via decision trees, see for instance [6, 5]. In a course given at the IHES [4], Duminil-Copin presented this new method and applied it to various models, including the Boolean model for which he announces sharp transition in any dimension under some moment conditions.

Assume in this paragraph that  $\mathbb{E}(R^d)$  is infinite. Then, for any positive  $\lambda$ , with probability one,  $\Sigma = \mathbb{R}^d$ . This can be shown easily by computing, for any  $r > 0$ , the probability that  $B(0, r)$  is covered by one random ball of  $\Sigma$ . See for example Theorem 16.4 in [11]. In this case, the model is therefore trivial from the percolation point of view:  $\Lambda = \widehat{\Lambda} = \emptyset$  and  $\lambda_c = \widehat{\lambda}_c = 0$ . As a consequence, in what follows, we will always assume that  $\mathbb{E}(R^d)$  is finite.

The following result is implicit in [9]. A proof is given in the Appendix of [10] (Theorem 11).

**Theorem 1.** Assume that  $\mathbb{E}(R^d)$  is finite. Then  $\widehat{\Lambda}$  is open and non-empty. In particular,  $\widehat{\lambda}_c$  and  $\lambda_c$  belong to  $(0, +\infty)$ .

In this paper, we want to investigate the connection between different percolation properties, such as the behavior of  $\mathbb{P}(S(r) \xleftrightarrow{\Sigma} S(2r))$  as  $r$  goes to  $\infty$ , the integrability properties of  $|C|$ ,  $\#C$  and  $D$ , and the tail of the distribution of  $\ell$ . We state first the following result.

**Theorem 2.** Let  $s > 0$ . The following statements are equivalent:

- For small enough  $\lambda$ ,  $\mathbb{E}(|C|^{s/d})$  is finite.
- For small enough  $\lambda$ ,  $\mathbb{E}(\#C^{s/d})$  is finite.
- For small enough  $\lambda$ ,  $\mathbb{E}(D^s)$  is finite.
- $\mathbb{E}(R^{d+s})$  is finite.

Moreover, if  $\mathbb{E}(R^{d+s})$  is finite, then  $\mathbb{E}(|C|^{s/d})$ ,  $\mathbb{E}(\#C^{s/d})$  and  $\mathbb{E}(D^s)$  are finite as soon as  $\lambda \in \widehat{\Lambda}$ .

In particular, if  $\mathbb{E}(R^{2d}) = \infty$  then  $\mathbb{E}(|C|) = \mathbb{E}(\#C) = \mathbb{E}(D^d) = \infty$  for any  $\lambda$ . Since we are interested in the finiteness of those expectations, we will naturally suppose that  $\mathbb{E}(R^{2d}) < \infty$  in the following theorem, which is the main result of this article.

**Theorem 3.** Assume that  $\mathbb{E}(R^{2d})$  is finite. The following statements are equivalent:

1.  $\mathbb{P}(S(r) \xleftrightarrow{\Sigma} S(2r)) \rightarrow 0$  as  $r \rightarrow \infty$ .
2. There exists  $A, B > 0$  such that, for all  $n \geq 1$ ,

$$\mathbb{P}(\ell \geq n) \leq A \exp(-Bn). \quad (2)$$

3.  $\mathbb{E}(D^d)$  is finite.
4.  $\mathbb{E}(|C|)$  is finite.
5.  $\mathbb{E}(\#C)$  is finite.

The main contribution of our work is the proof of  $4 \Rightarrow 2$ , *i.e.*, the fact that  $\mathbb{E}(|C|) < \infty$  implies the exponential decay of  $\ell$ , and this will be the core of the paper (see Section 3.2). Note that (2) does not imply that the decay of the tail of  $\#C$  is exponential. One can for example prove the following result, which is a simple consequence of Theorems 2 and 3.

**Corollary 4.** Let  $s > d$ . Assume  $\mathbb{E}(R^{2d}) < \infty$  and  $\mathbb{E}(R^{d+s}) = \infty$ . Let  $\lambda \in \widehat{\Lambda} = (0, \widehat{\lambda}_c)$ . Then there exists  $A, B$  such that

$$\mathbb{P}(\ell \geq n) \leq A \exp(-Bn).$$

However

$$\mathbb{E}(D^s) = \mathbb{E}(|C|^{s/d}) = \mathbb{E}(\#C^{s/d}) = \infty.$$

Combining Theorems 2 and 3 one also gets the following corollary.

**Corollary 5.** Let  $s \geq d$ . Assume that  $\mathbb{E}(R^{d+s})$  is finite. The following statements are equivalent:

1.  $\mathbb{P}(S(r) \xleftrightarrow{\Sigma} S(2r)) \rightarrow 0$  as  $r \rightarrow \infty$ .
2. There exists  $A, B > 0$  such that, for all  $n \geq 1$ ,  $\mathbb{P}(\ell \geq n) \leq A \exp(-Bn)$ .
3.  $\mathbb{E}(D^s)$  is finite.
4.  $\mathbb{E}(|C|^{s/d})$  is finite.
5.  $\mathbb{E}(\#C^{s/d})$  is finite.

The above results also yield equalities between some percolation thresholds. Such equalities were already proven in the case where  $R$  is bounded. We refer to Sections 3.4 and 3.5 of [12] and references therein for such results.

The proof of Theorem 2 is given in Section 2. The proof of Theorem 3 is given in Section 3. Corollaries 4 and 5 are straightforward consequences of Theorems 2 and 3, thus no additional proof is needed.

## 2 Proof of Theorem 2

The proof of Theorem 2 is divided into two parts. In Section 2.1, we prove the following result.

**Lemma 6.** *Let  $s > 0$ . If  $\mathbb{E}[R^{d+s}] < \infty$  and  $\lambda \in \widehat{\Lambda}$ , then  $\mathbb{E}(|C|^{s/d})$ ,  $\mathbb{E}(\#C^{s/d})$  and  $\mathbb{E}(D^s)$  are finite.*

Section 2.2 is devoted to the proof of the following result.

**Lemma 7.** *Let  $s > 0$ . If  $\mathbb{E}[R^{d+s}] = \infty$  then for every  $\lambda > 0$ ,  $\mathbb{E}(|C|^{s/d})$ ,  $\mathbb{E}(\#C^{s/d})$  and  $\mathbb{E}(D^s)$  are infinite.*

Theorem 2 is a straightforward consequence of Lemmas 6 and 7.

### 2.1 Proof of Lemma 6

We first establish the following result.

**Theorem 8.** *Let  $s > 0$ . Assume  $\mathbb{E}(R^{d+s}) < \infty$ . Let  $\lambda \in \widehat{\Lambda}$ . Then*

$$\int_0^\infty \alpha^{s-1} \mathbb{P}(S(\alpha) \xleftrightarrow{\Sigma} S(2\alpha)) d\alpha < \infty$$

and

$$\mathbb{E}(D^s) < \infty.$$

The result is implicit in [9]. We choose to give a detailed proof using intermediate results in Appendix A in [10] which themselves rely on results in [9].

Let us recall some notation from [9] or [10]. Let  $\alpha > 0$ .

- $\Sigma(B(0, \alpha))$  is the union of random balls of the Boolean model with centers in  $B(0, \alpha)$ .
- $G(0, \alpha)$  is the event "there exists a path from  $S(\alpha)$  to  $S(8\alpha)$  in  $\Sigma(B(0, 10\alpha))$ ".
- $\Pi(\alpha) = P(G(0, \alpha))$ .

Set

$$\varepsilon(\alpha) = \int_{[\alpha, +\infty)} r^d \nu(dr). \tag{3}$$

Note that, when  $\mathbb{E}(R^{d+s})$  is finite,

$$\int_0^\infty \alpha^{s-1} \varepsilon(\alpha) d\alpha < \infty. \tag{4}$$

The following proposition is stated in the same way in [10] as Proposition 12 in Appendix A.

**Proposition 9.** *There exists a constant  $K = K(d)$  such that, for any  $\alpha > 0$ ,*

$$\Pi(\alpha) \leq P(S(\alpha) \xleftrightarrow{\Sigma} S(2\alpha)) \leq K\Pi(\alpha/10) + \lambda K\varepsilon(\alpha/10), \tag{5}$$

$$\Pi(10\alpha) \leq K\Pi(\alpha)^2 + \lambda K\varepsilon(\alpha), \tag{6}$$

$$\Pi(\alpha) \leq \lambda K\alpha^d. \tag{7}$$

**Proof of Theorem 8.** This is a consequence of Proposition 9 above and Lemma 3.7 in [9]. Showing how to apply Lemma 3.7 would not be much shorter than adapting the proof in our context. Therefore we choose to give a full proof. Let  $s > 0$ . Assume  $\mathbb{E}(R^{d+s}) < \infty$  and let  $\lambda \in \widehat{\Lambda}$ . By (5),  $\Pi(\alpha)$  tends to 0 as  $\alpha$  tends to  $\infty$ . Therefore we can fix  $\alpha_0$  large enough such that, for all  $\alpha \geq \alpha_0/10$ ,

$$10^s K \Pi(\alpha) \leq \frac{1}{2}.$$

Then, for any  $\alpha \geq \alpha_0$ , using (6) and the definition of  $\alpha_0$ ,

$$\begin{aligned} \int_{\alpha_0}^{\alpha} r^{s-1} \Pi(r) dr &\leq \int_{\alpha_0}^{\alpha} r^{s-1} K \Pi(r/10)^2 dr + \int_{\alpha_0}^{\alpha} r^{s-1} \lambda K \varepsilon(r/10) dr \\ &\leq 10^s \int_{\alpha_0/10}^{\alpha/10} r^{s-1} K \Pi(r)^2 dr + \int_{\alpha_0}^{\infty} r^{s-1} \lambda K \varepsilon(r/10) dr \\ &\leq \frac{1}{2} \int_{\alpha_0/10}^{\alpha/10} r^{s-1} \Pi(r) dr + \int_{\alpha_0}^{\infty} r^{s-1} \lambda K \varepsilon(r/10) dr. \end{aligned}$$

Therefore, for any large enough  $\alpha$ ,

$$\int_{\alpha_0}^{\alpha} r^{s-1} \Pi(r) dr \leq \frac{1}{2} \int_{\alpha_0/10}^{\alpha_0} r^{s-1} \Pi(r) dr + \frac{1}{2} \int_{\alpha_0}^{\alpha} r^{s-1} \Pi(r) dr + \int_{\alpha_0}^{\infty} r^{s-1} \lambda K \varepsilon(r/10) dr.$$

Then, rearranging and using (4),

$$\int_{\alpha_0}^{\alpha} r^{s-1} \Pi(r) dr \leq \int_{\alpha_0/10}^{\alpha_0} r^{s-1} \Pi(r) dr + 2 \int_{\alpha_0}^{\infty} r^{s-1} \lambda K \varepsilon(r/10) dr < \infty.$$

Therefore,

$$\int_{\alpha_0}^{\infty} r^{s-1} \Pi(r) dr < \infty$$

and thus

$$\int_0^{\infty} r^{s-1} \Pi(r) dr < \infty.$$

By (4) and (5), this yields the first required result. The other result then follows from the fact that, for any  $\alpha > 0$ ,

$$\{D \geq 4\alpha\} \subset \{S(\alpha) \xrightarrow{\Sigma} S(2\alpha)\}.$$

□

**Proof of Lemma 6.** We suppose that  $\mathbb{E}(R^{d+s}) < \infty$  and  $\lambda \in \widehat{\Lambda}$ . By Theorem 8, we know that  $\mathbb{E}(D^s) < \infty$ . Since  $C \subset B(0, D)$ , this implies  $\mathbb{E}(|C|^{s/d}) < \infty$ . It remains to prove that  $\mathbb{E}(\#C^{s/d}) < \infty$ .

Let  $\kappa > 0$  be such that

$$\lambda v_d \kappa^d = \frac{1}{2},$$

where  $v_d$  denotes the volume of the unit ball in  $\mathbb{R}^d$ . For every  $u > 0$ , we have

$$\begin{aligned} \mathbb{P}(\#C \geq u) &\leq \mathbb{P}(C \subset B(0, \kappa u^{1/d}) \text{ and } \#C \geq u) + \mathbb{P}(C \not\subset B(0, \kappa u^{1/d})) \\ &\leq \mathbb{P}(\#\{(c, r) \in \xi : c \in B(0, \kappa u^{1/d})\} \geq u) + \mathbb{P}(D \geq \kappa u^{1/d}). \end{aligned} \quad (8)$$

Since  $\#\{(c, r) \in \xi : c \in B(0, \kappa u^{1/d})\}$  is a Poisson random variable with parameter  $u/2$ , we obtain

$$\mathbb{P}(\#\{(c, r) \in \xi : c \in B(0, \kappa u^{1/d})\} \geq u) \leq \exp\left(u \left(\frac{1}{2} - \ln(2)\right)\right)$$

and thus

$$\int_0^{\infty} du \frac{s}{d} u^{\frac{s}{d}-1} \mathbb{P}(\#\{(c, r) \in \xi : c \in B(0, \kappa u^{1/d})\} \geq u) < \infty.$$

Since  $\mathbb{E}(D^s) < \infty$ , we have

$$\int_0^{\infty} du \frac{s}{d} u^{\frac{s}{d}-1} \mathbb{P}(D \geq \kappa u^{1/d}) < \infty.$$

We conclude by (8) that  $\mathbb{E}(\#C^{s/d})$  is finite. □

## 2.2 Proof of Lemma 7

Set

$$A = \sup\{r > 0 : \exists c \in B(0, r/2) \text{ s.t. } (c, r) \in \xi\},$$

with the convention  $A = 0$  if the set is empty. Note that  $B(0, A/2)$  is covered by  $\Sigma$ . We first state the following preliminary result, which is essentially implicit in [9].

**Lemma 10.** *Let  $\lambda > 0$  and  $s > 0$ . Assume that  $\mathbb{E}(R^{d+s})$  is infinite. Then  $\mathbb{E}(A^s)$  is infinite.*

**Proof of Lemma 10.** For any  $a > 0$ ,

$$\mathbb{P}(A > a) = 1 - \exp\left(-\lambda v_d 2^{-d} \int_{(a, +\infty)} r^d \nu(dr)\right).$$

If  $\mathbb{E}(R^d)$  is infinite, then  $\mathbb{P}(A > a) = 1$  for all  $a > 0$  and thus  $A = +\infty$  almost surely and therefore  $\mathbb{E}(A^s) = \infty$ .

Assume henceforth that  $\mathbb{E}(R^d)$  is finite. Then

$$\mathbb{P}(A > a) \sim_{a \rightarrow \infty} \lambda v_d 2^{-d} \int_{(a, +\infty)} r^d \nu(dr).$$

Therefore, for some constant  $\gamma > 0$ , for all  $a > 0$ ,

$$\mathbb{P}(A > a) \geq \gamma \int_{(a, +\infty)} r^d \nu(dr)$$

and then

$$\begin{aligned} \mathbb{E}(A^s) &= \int_{(0, +\infty)} da \, s a^{s-1} \mathbb{P}(A > a) \\ &\geq \gamma \int_{(0, +\infty)} da \, s a^{s-1} \int_{(a, +\infty)} \nu(dr) r^d \\ &= \gamma \int_{(0, +\infty)} \nu(dr) r^d \int_{(0, r)} da \, s a^{s-1} \\ &= \gamma \int_{(0, +\infty)} \nu(dr) r^{d+s} \\ &= \gamma \mathbb{E}(R^{s+d}) \end{aligned}$$

which is infinite by assumption. □

**Proof of Lemma 7.** Let  $\lambda > 0$  and  $s > 0$ . We suppose that  $\mathbb{E}(R^{d+s})$  is infinite. By Lemma 10, we obtain that  $\mathbb{E}(A^s) = \infty$ . Since  $B(0, A/2)$  is covered by  $\Sigma$ , we know that  $D \geq A$  and  $|C| \geq v_d A^d / 2^d$ , thus  $\mathbb{E}(D^s) = \infty$  and  $\mathbb{E}(|C|^{s/d}) = \infty$ . It remains to prove that  $\mathbb{E}(\#C^{s/d}) = \infty$ .

Let  $r_0 > 0$  be such that  $\mathbb{P}(R \leq r_0) > 0$ . Set

$$A_{>r_0} = A \text{ if } A > r_0 \text{ and } A_{>r_0} = 0 \text{ otherwise.}$$

In other words,

$$A_{>r_0} = \sup\{r > r_0 : \exists c \in B(0, r/2) \text{ s.t. } (c, r) \in \xi\}$$

with the convention  $A_{>r_0} = 0$  if the set is empty. Note that  $B(0, A_{>r_0}/2)$  is covered by  $\Sigma$  and that  $A_{>r_0}$  is measurable with respect to  $\xi_{>r_0} = \xi \cap \mathbb{R}^d \times (r_0, +\infty)$ . Set

$$N = \text{card}(\{(c, r) \in \xi : c \in B(0, A_{>r_0}/2) \text{ and } r \leq r_0\}).$$

Conditionally on  $\xi_{>r_0}$ ,  $N$  is a Poisson random variable with parameter  $\alpha A_{>r_0}^d$  where

$$\alpha = \lambda \mathbb{P}(R \leq r_0) v_d / 2^d > 0.$$

But as  $B(0, A_{>r_0}/2)$  is contained in  $\Sigma$ , any random ball centered in  $B(0, A_{>r_0}/2)$  is contained in  $C$ . Therefore

$$\#C \geq N$$

and thus

$$\mathbb{E}(\#C^{s/d}) \geq \mathbb{E}(N^{s/d}) = \mathbb{E}(\mathbb{E}(N^{s/d} | \xi_{>r_0})).$$

Let  $\mu_0$  be such that, for any  $\mu \geq \mu_0$ , if  $X(\mu)$  is a Poisson random variable with parameter  $\mu$ , then  $\mathbb{P}(X(\mu) \geq \mu/2) \geq 1/2$ . Then,

$$\begin{aligned} \mathbb{E}(\#C^{s/d}) &\geq \mathbb{E}\left(\mathbb{E}(N^{s/d} | \xi_{>r_0}) \mathbf{1}_{\alpha A_{>r_0}^d \geq \mu_0}\right) \\ &\geq \mathbb{E}\left(\frac{1}{2} \left(\frac{\alpha A_{>r_0}^d}{2}\right)^{s/d} \mathbf{1}_{\alpha A_{>r_0}^d \geq \mu_0}\right) = \beta \mathbb{E}(A_{>r_0}^s \mathbf{1}_{\alpha A_{>r_0}^d \geq \mu_0}) \end{aligned}$$

where

$$\beta = \frac{\alpha^{s/d}}{2^{1+s/d}}.$$

To end the proof, it is therefore sufficient to check that  $\mathbb{E}(A_{>r_0}^s)$  is infinite. But this is a consequence of the infiniteness of  $\mathbb{E}(A^s)$  obtained by Lemma 10.  $\square$

### 3 Proof of Theorem 3

#### 3.1 Preliminary results.

For any  $r > 0$ , we denote by  $C(r)$  the connected component of  $\Sigma \cup B(0, r)$  which contains  $B(0, r)$ . If  $A$  and  $B$  are two subsets of  $\mathbb{R}^d$ , we denote by  $A + B$  the Minkowski sum of  $A$  and  $B$  defined by

$$A + B = \{a + b, a \in A, b \in B\}.$$

**Lemma 11.** *Let  $\lambda > 0$ . Assume that  $\mathbb{E}(|C|)$  is finite. Then*

$$\mathbb{E}(|C(1) + B(0, 1)|) < \infty.$$

**Proof.** This is a consequence of FKG inequality, see for instance Theorem 2.2 in [12]. Set

$$p = \mathbb{P}(B(0, 1) \subset \Sigma)$$

and note that  $p$  is positive. For any  $x \in \mathbb{R}^d$ , using FKG inequality in the third step and stationarity and definition of  $p$  in the fourth step, we get

$$\begin{aligned} \mathbb{P}(x \in C) &= \mathbb{P}(\{0\} \xleftrightarrow{\Sigma} \{x\}) \\ &\geq \mathbb{P}(\{B(0, 1) \xleftrightarrow{\Sigma} B(x, 1)\} \cap \{B(0, 1) \subset \Sigma\} \cap \{B(x, 1) \subset \Sigma\}) \\ &\geq \mathbb{P}(B(0, 1) \xleftrightarrow{\Sigma} B(x, 1)) \mathbb{P}(B(0, 1) \subset \Sigma) \mathbb{P}(B(x, 1) \subset \Sigma) \\ &= p^2 \mathbb{P}(B(0, 1) \xleftrightarrow{\Sigma} B(x, 1)) \\ &= p^2 \mathbb{P}(x \in C(1) + B(0, 1)). \end{aligned}$$

Therefore

$$\mathbb{E}(|C(1) + B(0, 1)|) = \int_{\mathbb{R}^d} dx \mathbb{P}(x \in C(1) + B(0, 1)) \leq p^{-2} \int_{\mathbb{R}^d} dx \mathbb{P}(x \in C) = p^{-2} \mathbb{E}(|C|).$$

As  $\mathbb{E}(|C|)$  is finite, the lemma is proven.  $\square$

**Lemma 12.** *Let  $K = K(d)$  be such that, for any  $r > 0$ , the ball  $B(0, r)$  can be covered by  $K(1 + r)^d$  balls of radius 1. Let  $\lambda > 0$ . Let  $r, s > 0$ . Then*

$$\int_{\mathbb{R}^d} dx \mathbb{P}(B(0, r) \xleftrightarrow{\Sigma} B(x, s)) = \mathbb{E}(|C(r) + B(0, s)|) \leq K^2(1 + r)^d(1 + s)^d \mathbb{E}(|C(1) + B(0, 1)|).$$



**Proof.** Write

$$\mathbb{E}(|C(r) + B(0, s)|) = \int_{\mathbb{R}^d} dx \mathbb{P}(x \in C(r) + B(0, s)) = \int_{\mathbb{R}^d} dx \mathbb{P}(B(0, r) \xleftrightarrow{\Sigma} B(x, s)).$$

Cover  $B(0, r)$  with at most  $K(1+r)^d$  balls of radius 1. Cover  $B(x, s)$  with at most  $K(1+s)^d$  balls of radius 1. If  $\{\mathbb{P}(B(0, r) \xleftrightarrow{\Sigma} B(x, s))\}$  holds, then there exists a path in  $\Sigma$  from one of the balls that cover  $B(0, r)$  to one of the balls that cover  $B(x, s)$ . Using union bound, stationarity and a change of variable, we thus get

$$\begin{aligned} \mathbb{E}(|C(r) + B(0, s)|) &\leq K^2(1+r)^2(1+s)^2 \int_{\mathbb{R}^d} dx \mathbb{P}(B(0, 1) \xleftrightarrow{\Sigma} B(x, 1)) \\ &= K^2(1+r)^d(1+s)^d \mathbb{E}(|C(1) + B(0, 1)|). \end{aligned}$$

The lemma is proven.  $\square$

### 3.2 Proof of 4 $\implies$ 2

This is the main part of the proof. For any  $n \geq 2$ , any  $r, s > 0$  and any  $x, y \in \mathbb{R}^d$  we consider the event

$$L_n(x, r, y, s) = \{\text{There exists a chain of length } n-1 \text{ starting in } B(x, r) \text{ and stopping in } B(y, s)\}$$

and we set

$$a_n(r, s) = \int_{\mathbb{R}^d} dy \mathbb{P}(L_n(0, r, y, s)). \quad (9)$$

Recall that  $R$  is a random variable with distribution  $\nu$ . Let  $S_i, i \geq 0$  be independent copies of  $R+1$ .

**Lemma 13.** For any  $\lambda > 0$ , any  $p \geq 2$  and any  $k \geq 1$ ,

$$\mathbb{P}(\ell \geq kp) \leq \left( \lambda [\mathbb{E}(a_p(S_1, S_2)^2)]^{1/2} \right)^k.$$

**Proof.** Set  $c_0 = 0$  and  $r_0 = 1$ . Let  $\lambda > 0$ ,  $p \geq 2$  and  $k \geq 1$ . We will use BK inequality, see the main theorem in [17] (and the remark (iii) above it concerning the choice of the definition of disjoint occurrence of increasing events). Before stating the inequality, let us recall informally some notations. The BK inequality apply directly to increasing events living on bounded region, thus define

$$L_p^n(x, r, y, s) = \left\{ \begin{array}{l} \text{There exists a chain } ((x_1, r_1), \dots, (x_{p-1}, r_{p-1})) \text{ of length } p-1 \\ \text{starting in } B(x, r) \text{ and stopping in } B(y, s) \\ \text{s.t. for all } i \in \{1, \dots, p-1\}, x_i \in [-n, n]^d \end{array} \right\}.$$

If  $((c_i, r_i))_{1 \leq i \leq k}$  are points of  $\mathbb{R}^d \times (0, +\infty)$ , we say that the increasing events  $L_p(c_{i-1}, r_{i-1}, c_i, r_i)$  (respectively  $L_p^n(c_{i-1}, r_{i-1}, c_i, r_i)$ ),  $i \in \{1, \dots, k\}$ , occur disjointly if there exists  $k$  chains, each of length  $p-1$ , using in total  $k(p-1)$  distinct random balls such that, for all  $i \in \{1, \dots, k\}$  the  $i$ -th chain starts in  $B(c_{i-1}, r_{i-1})$  and stops in  $B(c_i, r_i)$  (respectively and all the centers of the balls of these chains belong to  $[-n, n]^d$ ). We denote these events by

$$L_p(c_0, r_0, c_1, r_1) \circ \dots \circ L_p(c_{k-1}, r_{k-1}, c_k, r_k) \text{ and } L_p^n(c_0, r_0, c_1, r_1) \circ \dots \circ L_p^n(c_{k-1}, r_{k-1}, c_k, r_k)$$

or simply by

$$\circ_i L_p(c_{i-1}, r_{i-1}, c_i, r_i) \text{ and } \circ_i L_p^n(c_{i-1}, r_{i-1}, c_i, r_i).$$

By BK inequality, for all  $n \in \mathbb{N}$  we have

$$\mathbb{P}(L_p^n(c_0, r_0, c_1, r_1) \circ \dots \circ L_p^n(c_{k-1}, r_{k-1}, c_k, r_k)) \leq \prod_{i=1}^k \mathbb{P}(L_p^n(c_{i-1}, r_{i-1}, c_i, r_i)).$$

Taking the limit as  $n$  goes to infinity, we obtain

$$\mathbb{P}(L_p(c_0, r_0, c_1, r_1) \circ \dots \circ L_p(c_{k-1}, r_{k-1}, c_k, r_k)) \leq \prod_{i=1}^k \mathbb{P}(L_p(c_{i-1}, r_{i-1}, c_i, r_i)). \quad (10)$$

If  $\ell \geq kp$ , then there exists a chain of  $kp$  distinct balls starting in  $B(0, 1)$ . Taking one ball every  $p$ -th balls in this chain, we get a sequence  $(c_1, r_1), \dots, (c_k, r_k)$  of distinct points of  $\xi$  such that, with a slight abuse of notation, the event

$$\circ_i L_p(c_{i-1}, r_{i-1}, c_i, r_i)$$

holds for  $\xi \setminus \{(c_1, r_1), \dots, (c_k, r_k)\}$ . Therefore, again with a slight abuse of notation,

$$\mathbb{P}(\ell \geq kp) \leq \mathbb{E} \left( \sum_{(c_1, r_1), \dots, (c_k, r_k) \in \xi \text{ distinct}} \mathbb{1}_{\circ_i L_p(c_{i-1}, r_{i-1}, c_i, r_i)}(\xi \setminus \{(c_1, r_1), \dots, (c_k, r_k)\}) \right).$$

From Slivnyak's theorem, see Proposition 4.1.1 in [14], we get

$$\mathbb{P}(\ell \geq kp) \leq \lambda^k \int_{(\mathbb{R}^d)^k} dc_1 \dots dc_k \int_{(0, +\infty)^d} \nu(dr_1) \dots \nu(dr_k) \mathbb{P}(\circ_i L_p(c_{i-1}, r_{i-1}, c_i, r_i)).$$

By (10), this yields

$$\mathbb{P}(\ell \geq kp) \leq \lambda^k \int_{(\mathbb{R}^d)^k} dc_1 \dots dc_k \int_{(0, +\infty)^d} \nu(dr_1) \dots \nu(dr_k) \prod_{i=1}^k \mathbb{P}(L_p(c_{i-1}, r_{i-1}, c_i, r_i)).$$

Using stationarity and (9) we then get

$$\mathbb{P}(\ell \geq kp) \leq \lambda^k \int_{(0, +\infty)^d} \nu(dr_1) \dots \nu(dr_k) \prod_{i=1}^k a_p(r_{i-1}, r_i).$$

Distinguishing according to parity, we get

$$\mathbb{P}(\ell \geq kp) \leq \lambda^k \int_{(0, +\infty)^d} \nu(dr_1) \dots \nu(dr_d) \prod_{1 \leq i \leq k, i \text{ odd}} a_p(r_{i-1}, r_i) \prod_{1 \leq i \leq k, i \text{ even}} a_p(r_{i-1}, r_i).$$

Then, by Cauchy-Schwarz inequality,

$$\mathbb{P}(L_{kp}(1)) \leq \lambda^k A_p(k) B_p(k)$$

where

$$A_p(k) = \left[ \int_{(0, +\infty)^d} \nu(dr_1) \dots \nu(dr_d) \prod_{1 \leq i \leq k, i \text{ odd}} a_p(r_{i-1}, r_i)^2 \right]^{1/2}$$

and

$$B_p(k) = \left[ \int_{(0, +\infty)^d} \nu(dr_1) \dots \nu(dr_d) \prod_{1 \leq i \leq k, i \text{ even}} a_p(r_{i-1}, r_i)^2 \right]^{1/2}.$$

Let us get rid of the (easy but annoying) special case  $r_0 = 1$  as follows. Recall that  $R$  is a random variable with distribution  $\nu$  and that  $S_i, i \geq 0$  are independent copies of  $R + 1$ . As  $a_p(r, s)$  is non-decreasing in  $r$  and  $s$ , we have

$$A_p(k) \leq \left[ \mathbb{E} \left( \prod_{1 \leq i \leq k, i \text{ odd}} a_p(S_{i-1}, S_i)^2 \right) \right]^{1/2}$$

and

$$B_p(k) \leq \left[ \mathbb{E} \left( \prod_{1 \leq i \leq k, i \text{ even}} a_p(S_{i-1}, S_i)^2 \right) \right]^{1/2}.$$

As  $(S_i)_{i \geq 0}$  is an i.i.d. sequence, we get

$$A_p(k)B_p(k) \leq [\mathbb{E} (a_p(S_1, S_2)^2)]^{k/2}$$

and then

$$P(\ell \geq kp) \leq \left( \lambda [\mathbb{E} (a_p(S_1, S_2)^2)]^{1/2} \right)^k.$$

The lemma is proven.  $\square$

Recall that  $\ell$  is defined by (1).

**Lemma 14.** *Let  $\lambda > 0$ . Assume that  $\mathbb{E}(R^d)$  and  $\mathbb{E}(|C|)$  are finite. Then  $\ell$  is finite with probability one.*

**Remark.** We could remove the assumption  $\mathbb{E}(R^d)$  finite, as it is a consequence of the finiteness of  $\mathbb{E}(|C|)$ .

**Proof.** Let  $\lambda > 0$  be such that  $\ell$  is infinite with positive probability. We aim at proving that  $\mathbb{E}(|C|)$  is infinite. For any  $\eta > 0$ , set

$$\Sigma_{\leq \eta} = \bigcup_{(c,r) \in \xi: r \leq \eta} B(c,r).$$

If  $\eta$  is small enough, then  $\Sigma_{\leq \eta}$  does not percolate. Indeed,  $\Sigma_{\leq \eta}$  is a Boolean model driven by the measure

$$\lambda \nu(\cdot \cap (0, \eta]).$$

We can therefore couple  $\Sigma_{\leq \eta}$  with the Boolean model  $\Sigma_{\leq \eta}^+$  driven by the measure

$$\lambda \nu((0, \eta]) \delta_\eta$$

in such a way that

$$\Sigma_{\leq \eta} \subset \Sigma_{\leq \eta}^+.$$

But

$$\eta^{-1} \Sigma_{\leq \eta}^+$$

is a Boolean model driven by

$$\lambda \nu((0, \eta]) \eta^d \delta_1.$$

Therefore, as soon as

$$\lambda \nu((0, \eta]) \eta^d < \lambda_c(\delta_1, d),$$

$\eta^{-1} \Sigma_{\leq \eta}^+$  does not percolate, thus  $\Sigma_{\leq \eta}^+$  does not percolate and then  $\Sigma_{\leq \eta}$  does not percolate. In the remaining of the proof, we fix  $\eta > 0$  such that  $\Sigma_{\leq \eta}$  does not percolate.

Let  $C_{> \eta}$  denote the union of all the random balls of  $C$  whose radius is greater than  $\eta$ . We will repeatedly use the following property, that holds since  $\mathbb{E}(R^d)$  is finite: almost surely, whatever the bounded region  $B$  of  $\mathbb{R}^d$  we consider, the number of random balls of  $\Sigma$  that touches  $B$  is finite. If  $\ell$  is infinite, then for any  $n$  there exists a chain  $((c_1^n, r_1^n), \dots, (c_n^n, r_n^n))$  of balls starting in  $B(0, 1)$ . For all  $n$  the ball  $B(c_1^n, r_1^n)$  touches  $B(0, 1)$ , but by the previously stated property we know that a.s. there exists only a finite number of balls of  $\Sigma$  that touches  $B(0, 1)$ , thus an infinite number of those balls  $B(c_1^n, r_1^n)$  are equal to the same ball that we will denote by  $B_1$ . In other words, there exists an infinite number of chains of arbitrarily large length starting in  $B_1$ . But by the same property, we know that a.s. there exists only a finite number of balls of  $\Sigma$  that touches  $B(0, 1) \cup B_1$ , thus an infinite number of those chains (with arbitrarily large length) use a common first ball of  $\Sigma$  that we denote by  $B_2$ . By induction, we construct an infinite chain of distinct balls  $(B_i = B(x_i, s_i), i \geq 1)$  starting in  $B(0, 1)$ . Suppose that only a finite number of the radii  $s_i$  are bigger than  $\eta$ . Then there exists an infinite chain of balls of radii

smaller than or equal to  $\eta$ . By the previously stated property, we know that a.s. this infinite number of balls cannot stay in any bounded region, thus  $\Sigma_{\leq \eta}$  has to percolate, which is absurd by our choice of  $\eta$ . We conclude that an infinite number of these balls ( $B_i = B(x_i, s_i), i \geq 1$ ) satisfy  $s_i > \eta$ . A.s. these balls cannot stay in any bounded region, thus up to extraction we obtain a sequence  $(c_i, r_i)_{i \geq 0}$  of points of  $\xi$  such that, for all  $i \geq 1$ ,

$$B(c_i, r_i) \subset C, \quad \|c_i\| \geq \|c_{i-1}\| + 1, \quad r_i \geq \eta.$$

Therefore  $|C| = \infty$  almost surely on the event  $\{\ell = \infty\}$ . As this event occurs with positive probability we get  $\mathbb{E}(|C|) = \infty$ .  $\square$

**Lemma 15.** *Let  $\lambda > 0$ . Assume that  $\mathbb{E}(R^{2d})$  and  $\mathbb{E}(|C|)$  are finite. Then there exists  $p \geq 2$  such that*

$$\lambda [\mathbb{E}(a_p(S_1, S_2)^2)]^{1/2} < 1.$$

**Proof.** As  $\mathbb{E}(|C|)$  is finite,  $\ell$  is almost surely finite by Lemma 14. For any  $r > 0$ ,  $B(0, r)$  can be covered by a finite number of balls of radius 1. Therefore, by stationarity, the maximal number  $\ell(r)$  of balls in a chain starting in  $B(0, r)$  is almost surely finite. As a consequence, for all  $r > 0$ ,

$$\mathbb{P}(\ell(r) \geq p) \rightarrow 0 \text{ as } p \rightarrow \infty.$$

Thus, for all  $r, s > 0$  and  $y \in \mathbb{R}^d$ ,

$$\mathbb{P}(L_p(0, r, y, s)) \rightarrow 0 \text{ as } p \rightarrow \infty,$$

as  $L_p(0, r, y, s) \subset \{\ell(r) \geq p - 1\}$ . Moreover, for all  $r, s > 0, y \in \mathbb{R}^d$  and  $p \geq 2$ ,

$$\mathbb{P}(L_p(0, r, y, s)) \leq \mathbb{P}(B(0, r) \overset{\Sigma}{\leftarrow} B(y, s)). \quad (11)$$

But, by Lemma 12,

$$\int_{\mathbb{R}^d} dy \mathbb{P}(B(0, r) \overset{\Sigma}{\leftarrow} B(y, s)) \leq K^2(1+r)^d(1+s)^d \mathbb{E}(|C(1) + B(0, 1)|). \quad (12)$$

Moreover, as  $\mathbb{E}(|C|)$  is finite, we get

$$\mathbb{E}(|C(1) + B(0, 1)|) < \infty \quad (13)$$

by Lemma 11. Therefore, by dominated convergence theorem, for any  $r, s > 0$ ,

$$\int_{\mathbb{R}^d} dy \mathbb{P}(L_p(0, r, y, s)) \rightarrow 0 \text{ as } p \rightarrow \infty$$

that is, using Definition (9),

$$a_p(r, s) \rightarrow 0 \text{ as } p \rightarrow \infty.$$

By (11) and (12),

$$a_p(S_1, S_2) \leq K^2(1+S_1)^d(1+S_2)^d \mathbb{E}(|C(1) + B(0, 1)|).$$

Using (13) and the finiteness of  $\mathbb{E}(R^{2d})$  we get that the square of the right hand side of the above inequality is integrable. Using dominated convergence theorem again, we then get

$$\mathbb{E}(a_p^2(S_1, S_2)) \rightarrow 0 \text{ as } p \rightarrow \infty.$$

The lemma is proven.  $\square$

**Proof of 4  $\implies$  2.** By Lemma 15, there exists  $p \geq 2$  such that

$$\kappa := \lambda [\mathbb{E}(a_p(S_1, S_2)^2)]^{1/2} < 1.$$

By Lemma 13, for any  $k \geq 1$ ,

$$\mathbb{P}(\ell \geq kp) \leq \kappa^k.$$

Therefore there exists  $A, B > 0$  such that, for any  $n \geq 1$ ,  $\mathbb{P}(\ell \geq n) \leq A \exp(-Bn)$ .  $\square$

### 3.3 Proof of the others implications

**Proof of 2**  $\implies$  1. Let  $K = K(d)$  be such that, for any  $r \geq 1$ , the sphere  $S(r)$  can be covered by  $Kr^{d-1}$  balls of radius 1. Let  $r \geq 1$ . Let us first prove

$$\mathbb{P}(S(r) \overset{\Sigma}{\leftarrow} S(2r)) \leq \mathbb{P}(H(r)) + Kr^{d-1}\mathbb{P}(\ell \geq \sqrt{r}/2) \quad (14)$$

where

$$H(r) = \{\exists(c, s) \in \xi : s \geq \sqrt{r} \text{ and } B(c, s) \cap B(0, 2r) \neq \emptyset\}.$$

Indeed, if  $\{S(r) \overset{\Sigma}{\leftarrow} S(2r)\}$  holds, then there exists a chain of random balls from a point of  $S(r)$  to a point of  $S(2r)$ . If moreover  $H(r)$  does not hold, then the number of balls of this chain is at least  $\sqrt{r}/2$ . The sphere  $S(r)$  can be covered by  $Kr^{d-1}$  balls of radius 1 and the starting point of the chain is in one of these balls. Using the union bound and stationarity, we get (14).

As

$$\mathbb{P}(\ell \geq \sqrt{r}/2) \leq A \exp(-B\sqrt{r}/2),$$

we get

$$Kr^{d-1}\mathbb{P}(\ell \geq \sqrt{r}/2) \rightarrow 0 \text{ as } r \rightarrow \infty. \quad (15)$$

Furthermore,

$$\begin{aligned} \mathbb{P}(H(r)) &\leq \mathbb{E}(\text{card}[\{(c, s) \in \xi : s \geq \sqrt{r} \text{ and } B(c, s) \cap B(0, 2r) \neq \emptyset\}]) \\ &\leq \lambda \int_{[\sqrt{r}, \infty)} v_d(2r+s)^d \nu(ds) \\ &\leq \lambda \int_{[\sqrt{r}, \infty)} v_d(2s^2+s)^d \nu(ds) \end{aligned}$$

where  $v_d$  denotes the volume of the unit ball of  $\mathbb{R}^d$ . Therefore

$$\mathbb{P}(H(r)) \rightarrow 0 \text{ as } r \rightarrow \infty. \quad (16)$$

The result follows from (14), (15) and (16).  $\square$

**Remark.** Replacing  $\sqrt{r}$  by  $\alpha r / \log r$  for some big enough  $\alpha$  in the definition of the event  $H(r)$ , and replacing accordingly the event  $\{\ell \geq \sqrt{r}/2\}$  by  $\{\ell \geq \log r / (2\alpha)\}$ , we could weakened the hypothesis  $\mathbb{E}(R^{2d}) < \infty$  to  $\mathbb{E}(R^d (\log R)^\beta) < \infty$  for some  $\beta$ , but only for the implication  $2 \implies 1$ .

**Proof of 1**  $\implies$  3. This is a consequence of Theorem 2.  $\square$

**Proof of 3**  $\implies$  4. Note that  $C$  is a subset of  $B(0, D)$ . As a consequence,  $|C| \leq v_d D^d$ . The result follows.  $\square$

**Proof of 4**  $\iff$  5. There exists actually simple inequalities between  $\mathbb{E}(|C|)$  and  $\mathbb{E}(\#C)$ . Using Slivnyak's theorem, see Proposition 4.1.1 in [14], we get

$$\begin{aligned} \mathbb{E}(\#C) &= \mathbb{E} \left( \sum_{c \in \chi} 1_{c \in C} \right) \\ &= \lambda \int_{\mathbb{R}^d} dc \int_{(0, +\infty)} \nu(dr) \mathbb{P}(c \in C(\xi \cup (c, r))) \end{aligned}$$

where  $C(\xi \cup (c, r))$  denotes the connected component containing the origin in the Boolean model with the extra ball  $B(x, r)$ . Thus,

$$\begin{aligned}\mathbb{E}(\#C) &= \lambda \int_{\mathbb{R}^d} dc \int_{(0, +\infty)} \nu(dr) \mathbb{P}(B(c, r) \text{ touches } 0 \text{ or } B(c, r) \text{ touches } C) \\ &= \lambda \int_{\mathbb{R}^d} dc \int_{(0, +\infty)} \nu(dr) \mathbb{P}(c \in B(0, r) \cup (C + B(0, r))) \\ &= \lambda \mathbb{E}(|B(0, R) \cup (C + B(0, R))|)\end{aligned}$$

where  $R$  is independent of  $\xi$  and where the distribution of  $R$  is  $\nu$ . Thus,

$$\lambda \mathbb{E}(C) \leq \mathbb{E}(\#C). \quad (17)$$

Moreover, using Lemma 12,

$$\begin{aligned}\mathbb{E}(\#C) &\leq \lambda v_d \mathbb{E}(R^d) + \lambda \mathbb{E}(|C(1) + B(0, R)|) \\ &\leq \lambda v_d \mathbb{E}(R^d) + \lambda K^2 2^d \mathbb{E}([1 + R]^d) \mathbb{E}(|C(1) + B(0, 1)|).\end{aligned} \quad (18)$$

But, by Lemma 11,  $\mathbb{E}(|C(1) + B(0, 1)|)$  is finite. As  $\mathbb{E}(R^d)$  is also finite, the equivalence follows from (17) and (18).  $\square$

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