Sequentialization and Procedural Complexity in Automata Networks

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Abstract. In this article we consider finite automata networks (ANs) with two kinds of update schedules: the parallel one (all automata are updated all together) and the sequential ones (the automata are updated periodically one at a time according to a total order w). The cost of sequentialization of a given AN h is the number of additional automata required to simulate h by a sequential AN with the same alphabet. We construct, for any n and q, an AN h of size n and alphabet size q whose cost of sequentialization is at least $n/3$. We also show that, if $q \geq 4$, we can find one whose cost is at least $n/2 - \log_q(n)$. We prove that $n/2 + \log_q(n/2+1)$ is an upper bound for the cost of sequentialization of any AN h of size n and alphabet size q. Finally, we exhibit the exact relation between the cost of sequentialization of h and its procedural complexity with unlimited memory and prove that its cost of sequentialization is less than or equal to the pathwidth of its interaction graph.

Keywords: Automata networks, intrinsic simulation, parallel update schedule, sequential update schedules, procedural complexity.

1 Introduction

In this article, we study finite automata networks (ANs). They are models classically used for representing and analyzing natural dynamical systems like genetic or neural networks [8,5]. Moreover, they are also computational models on which we study computability and complexity properties which is the purpose of this paper. An AN h can be seen as a transformation of A^n with A a finite alphabet. Here, n is the number of automata, and the i-th component of h is the update function of the i-th automaton. We consider them with two types of update schedules. With the parallel one, automata are updated all together, at each time step. In other words, we just apply h. With the sequential ones, automata are updated sequentialy, according to a total order w. They have been several works on the influence of the update schedules on the function computed by

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an AN [6,1]. Here, like in [7] we take the opposite approach. We have an AN h with a parallel update schedule and try to find an AN f with a sequential update schedule w which computes the same function. However, sometime it is impossible. For instance, the transformation of $\{0,1\}^2$ which exchanges the two values $h : (x_1, x_2) \mapsto (x_2, x_1)$ cannot be sequentialized. The famous XOR swap algorithm, $x_1 \leftarrow x_1 \oplus x_2$, $x_2 \leftarrow x_1 \oplus x_2$, $x_1 \leftarrow x_1 \oplus x_2$ does not apply here because we can only update one time each automaton beetween two time steps. However, what we can do is to consider the AN f with one additional automaton and the sequential update schedule $w := (3, 2, 1)$ which executes the three instructions $x_3 \leftarrow x_1$, $x_1 \leftarrow x_2$, $x_2 \leftarrow x_3$. We see that f with the update schedule w computes the transformation h if we only consider the 2 first automata. The goal of this paper is to determine the cost of sequentialization of an AN h, namely, the minimum number of additional automata that an AN f which sequentializes h will have. This paper is the direct sequel of [3] in which the same problem was studied for an alphabet of size 2 and with an imposed order of sequentialization. Definition 7, Theorem 1 and Lemma 3 are straightforward generalization of results published in [3]. All other results are new.

In Section 2, we define ANs, interaction graphs, the notion of a sequentialization and we present most of the notations that we use. In Section 3, we define the cost of sequentialization $\kappa(h, u)$ of an AN h respecting an order u. It is the minimum number of additional automata required for any AN f with a sequential update schedule w respecting the order u to compute h. We also define $\kappa^{\min}(h)$ which is like $\kappa(h, u)$ except that the sequential update schedules we consider are not constraint anymore. In Section 4, we give an upper and lower bounds for $\kappa(h, u)$ for the couple (h, u) which maximizes it. In Section 5, we prove different lower bounds depending on the alphabet size for $\kappa^{\min}(h)$ when h maximizes $\kappa^{\min}(h)$. In Section 6 we give the relation between $\kappa^{\min}$ and the procedural complexity as defined in [4]. Finally, In Section 7, we prove an upper bound for $\kappa^{\min}(h)$ depending on the pathwidth of the interaction graph of h.

2 Definitions and notations

For all $i \in \mathbb{N}$, the interval between 1 and i is denoted by $[i] := \{1, 2, \dots, i\}$. For all $i, j \in \mathbb{N}$, with $i \leq j$, the closed interval between i and j is denoted by $[i, j] := \{i, i+1, \dots, j\}$ and the open one by $]i, j[:= [i, j] \setminus \{i, j\}$. For any $q \geq 2$ and $n \in \mathbb{N}$, let $F(n, q)$ be the set of functions from $[0, q[^n$ to $[0, q[^n$ (also called transformations of $[0, q[^n$). For all $I = \{i_1, i_2, \dots, i_p\} \subseteq [n]$

with $i_1 < i_2 < \cdots < i_p$, the projection of x on I is denoted either by $\mathrm{pr}_I(x)$ or by x_I. In other words, $\mathrm{pr}_I(x) = x_I = (x_{i_1}, x_{i_2}, \ldots, x_{i_p})$. For all vectors $x := (x_1, \ldots, x_p)$ and $y := (y_1, \ldots, y_t)$, their concatenation is denoted by $xy := (x_1, \ldots, x_p, y_1, \ldots y_t)$.

Definition 1 (Coordinate functions). *Let $f \in F(n,q)$. For every $i \in [n]$, the i-th coordinate functions of f is the function $f_i := \mathrm{pr}_i \circ f$.*

This means that we have $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. In this paper, we make particular use of the superscript of a function f.

Definition 2 (Updates of a transformation). *For all $i \in [n]$, $f^i \in F(n,q)$ is the function which updates the i-th coordinate (*i.e.* executes f_i). For all $I \subseteq [n]$, f^I is the function which updates the coordinates of all elements of I synchronously. For any word $w := (w_1, w_2, \ldots, w_t)$ on the alphabet $[n]$, f^w is the function which updates sequentially the coordinates $w_1, \ldots, w_t$ in the order given by w.*

Formally, we have

$$\forall x \in A^n, \qquad f^i(x) := (x_1, \ldots, x_{i-1}, f_i(x), x_{i+1}, \ldots, x_n).$$
$$\forall x \in A^n, j \in [n], \qquad f^I(x)_j := \begin{cases} f_j(x) & \text{if } j \in I \\ x_j & \text{otherwise.} \end{cases}$$
$$\forall w = (w_1, w_2, \ldots, w_t) \in [n]^t, \; f^w := f^{w_t} \circ \cdots \circ f^{w_2} \circ f^{w_1}.$$

We say that f_i is a trivial coordinate function if for all $x \in A^n, f_i(x) = x_i$. The relation $y = f^i(x)$ can be expressed by $x \xrightarrow{f^i} y$. The set of permutations of $[n]$ is denoted by $\Pi([n])$. Let $w := (w_1, w_2, \ldots, w_t) \in \Pi([n])$. If $w_j = i$ then we say that i is updated at step $w(i) := j$.

Definition 3 (Sequentialization). *An* AN *$f \in F(m,q)$, with the sequential update schedule $w \in \Pi([m])$ sequentializes an* AN *$h \in F(n,q)$ with $m \geq n$ if $\mathrm{pr}_{[n]} \circ f^w = h \circ pr_{[n]}$.*

Remark 1. All the results of this paper remain true if we use the more general definition: $\exists I \subseteq [m]$, with $|I| = n$ such that $\mathrm{pr}_I \circ f^w = h \circ pr_I$.

Definition 4 (Interaction graph). *The interaction graph* $\mathrm{IG}(h)$ *of an* AN *$h \in F(n,q)$ is the directed graph $([n], E)$ with $(i,j) \in E$ if and only if i has an influence on j. More formally, $\forall i,j \in [n], (i,j) \in E$ if and only if $\exists x, y \in A^n$ such that $x_{[n]\setminus\{i\}} = y_{[n]\setminus\{i\}}$ and $h_j(x) \neq h_j(y)$.*

We denote by $\mathrm{IG}^*(h)$ be the undirected version of $\mathrm{IG}(h)$.

3 Cost of sequentialization

In this section, we define the main question tackled in this paper. For all $u \in \Pi([n])$, we say that $w \in \Pi([m])$ respects u, if all the coordinates of $[n]$ are updated in the same order in u and in w. In other words, $\forall i, j \in [n]$, if $u(i) < u(j)$ then $w(i) < w(j)$.

Definition 5 **($\kappa(h, u)$).** *Let $h \in F(n, q)$ and $u \in \Pi([n])$. The cost of sequentialization of h respecting u, denoted by $\kappa(h, u)$, is the smallest k such that there exists $f \in F(n+k, q)$ and $w \in \Pi([n+k])$, such that (f, w) sequentializes h and w respects u.*

Definition 6 **($\kappa^{\min}(h)$).** *Let $h \in F(n, q)$. The cost of sequentialization of h, denoted by $\kappa^{\min}(h)$, is the smallest k such that there is a $f \in F(n+k, q)$ and a $w \in \Pi([n+k])$, such that (f, w) sequentializes h.*

Clearly, $\kappa^{\min}(h) = \min(\{\kappa(h, u) \mid u \in \Pi([n])\})$. Given n and q, the maximal cost of sequentilization respectively with or without imposed order is denoted by $\kappa_{n,q} := max(\{\kappa(h, u) \mid h \in F(n, q) \text{ and } u \in \Pi([n])\})$ and $\kappa^{\min}_{n,q} := max(\{\kappa^{\min}(h) \mid h \in F(n, q)\})$, respectively. Example 1 shows that, for some (h, u), the difference between $\kappa^{min}(h)$ and $\kappa(h, u)$ is large.

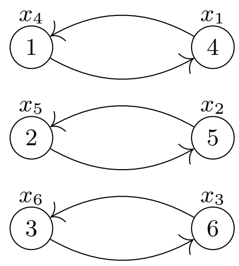

Fig. 1: Interaction graph of the AN *h of Example 1.*

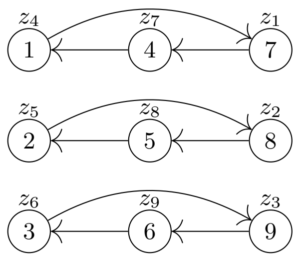

Fig. 2: Interaction graph of the AN *f of Example 1.*

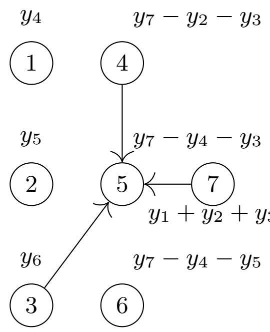

Fig. 3: Interaction graph of the AN *g of Example 1 with only inner edges of the automaton* 5 *displayed.*

Example 1. Let us consider the AN $h \in F(n, q)$ with $n = 6$ which computes the swaps of the values of 3 pairs of automata. In other words,

$$h : x \mapsto (x_4, x_5, x_6, x_1, x_2, x_3).$$

Figure 1 displays the interaction graph of h. Now, we consider the canonical sequential update schedule $u = (1, 2, \cdots, 6)$ and we want to find an AN f and a update schedule w which sequentializes h respecting u. To do so, let us consider an AN $f \in F(9, 2)$ and $w \in \Pi([9])$. First, we define the order $w := (7, 8, 9, 1, 2, 3, 4, 5, 6)$ which updates the $n/2$ additional automata of f before it updates the n first ones. Then, we take f which copies the values of the first set of automata in the third, the second in the first and the third in the second. Formally, $f : z \mapsto z_{[4,6]}z_{[7,9]}z_{[3]}$. Figure 2 shows the interaction graph of f. Now, a simple expansion of f^w gives us

$$z = z_{[3]}z_{[4,6]}z_{[7,9]} \xrightarrow{f^{7,8,9}} z_{[3]}z_{[4,6]}z_{[3]} \xrightarrow{f^{1,2,3}} z_{[4,6]}z_{[4,6]}z_{[3]} \xrightarrow{f^{4,5,6}} h(z_{[6]})z_{[3]}.$$

Thus, we have $\mathrm{pr}_{[n]} \circ f^w = h \circ \mathrm{pr}_{[n]}$. As a result, (f, w) sequentializes h respecting u and $\kappa(h, u) \leq 3$. Moreover, Lemma 3 (Section 4), shows that there are no smaller (f, w) which would suit. Thus, we have $\kappa_{6,q} \geq \kappa(h, u) = n/2 = 3$. Next, we define $g \in F(7, 2)$ and $v \in \Pi([7])$ such that (g, v) (with only one more automaton than h) sequentializes h (but without respecting u). First, we define the order $v := (7, 1, 4, 2, 5, 3, 6)$ which, instead of updating $[n]$ in the order u, updates the pairs of automata $(1, 4)$, $(2, 5)$ and $(3, 6)$ one by one. Then, we take g such that for all $y \in \{0, 1\}^7$,

$$g : y \mapsto (y_4, y_5, y_6, y_7 - y_2 - y_3, y_7 - y_4 - y_3, y_7 - y_4 - y_5, y_1 + y_2 + y_3).$$

Figure 3 depicts the interaction graph of g with only the inner edges of the automaton 5 displayed. As above, a simple expansion of g^v gives us $g^v : y \mapsto h(y_{[6]})(y_1 + y_2 + y_3)$. Thus, $\mathrm{pr}_{[n]} \circ g^v = h \circ \mathrm{pr}_{[n]}$ and g has 1 more automata than h. As a result, (f, w) sequentializes h and $\kappa^{\min}(h) \leq 1$. A generalization of this example shows that for all even n and $q \geq 2$, $\exists h \in F(n, q), u \in \Pi([n])$ such that $\kappa(h, u) \geq \kappa^{\min}(h) + n/2 - 1$.

4 Confusion graph and $\kappa_{n,q}$

In [3], the NECC graph was defined. This graph is very useful to compute $\kappa(h, u)$. We rather call it the confusion graph in this paper.

Definition 7 (Confusion graph). *Let us consider $h \in F(n, q)$ and the sequential update schedule $u \in \Pi([n])$. We call* confusion graph $G_{h,u}$ *the undirected graph whose vertices are all the configurations of $[0, q[^n$ and in which two configurations x and x' are neighbors if and only if $h(x) \neq h(x')$ and $\exists\ i \in [n]$, $h^{\{u_1,\ldots,u_i\}}(x) = h^{\{u_1,\ldots,u_i\}}(x')$.*

In the sequel, we denote by $\chi(G)$ the *chromatic number* of the graph G, namely the minimum number of colors of a proper coloring of its vertices. In [3], the exact relation between the chromatic number of the confusion graph $G_{h,u}$ and $\kappa(h,u)$ was proven in the case where $q = 2$. We propose in Theorem 1 a straightforward generalization for any alphabet size.

Theorem 1. *Let us consider $h \in F(n,q)$ and the sequential update schedule $u \in \Pi([n])$. Then we have $\kappa(h,u) = \lceil \log_q(\chi(G_{h,u})) \rceil$.*

In [3],the authors proved that for all n we can construct $h \in F(n,2)$ whose cost of sequentialization respecting the order $u \in \Pi([n])$ is $\lfloor n/2 \rfloor$. Lemma 3 bellow is a straightforward generalization for any alphabet size.

Lemma 3. *For all $n \in \mathbb{N}$ and $q \geq 2$ we have $\kappa_{n,q} \geq \lfloor n/2 \rfloor$.*

Moreover, in [3], the authors showed that $\forall n \in \mathbb{N}, \kappa_{n,2} \leq 2n/3+2$. Theorem 2 below shows that we have in fact, $\kappa_{n,q} \leq \lceil n/2 + \log_q(n/2+1) \rceil$ for any q. To prove it, we regroup all the configurations of the confusion graph $G_{h,u}$ which are equal in their second half ($x_{\{u_{n/2+1},\ldots,u_n\}} = x'_{\{u_{n/2+1},\ldots,u_n\}}$) and have the same image ($h(x) = h(x')$). We prove that a proper coloring of this graph is a proper coloring of the confusion graph. And then, we prove that the maximal degree of this factorized graph is at most $\lceil (n/2+1)q^{n/2} \rceil$. Since the chromatic number of a graph is at most its maximal degree (plus one), we deduce an upper bound for the chromatic number and then for $\kappa_{n,q}$.

Theorem 2. *For all $n \in \mathbb{N}$, $q \geq 2$ we have $\kappa_{n,q} \leq \lceil n/2 + log_q(n/2+1) \rceil$.*

5 Lower bounds for $\boldsymbol{\kappa}^{\mathbf{min}}_{\boldsymbol{n,q}}$

The goal of this section is to construct an AN with the biggest cost of sequentialization possible and thus deduce a lower bound for $\kappa^{\min}_{n,q}$. For any set I, the set of subsets of I of size k is denoted by $\binom{I}{k} := \{J \subseteq I \mid |J| = k\}$. For all $x \in A^n$ and $I \subseteq [n]$, let $x[I] := \{x' \in A^n \mid x'_{[n]\setminus I} = x_{[n]\setminus I}\}$ be the set of configurations of A^n which only differ from x in I. In Lemma 4, we prove that if we can find an encoding $b : \binom{[2k]}{k} \to A^n$ such that the sets $b(E)[E]$ with $E \in \binom{[2k]}{k}$ are disjoint, then there exists $h \in F(n,q)$ such that $\kappa^{min}(h) \geq k$. To do so, we define the function h such that for all $x \in b(E)[E]$, $h_E(x) = x_{[2k]\setminus E}$ and $h_{[2k]\setminus E}(x) = x_E$. For any $u \in \Pi([n])$, we can define E as the k first coordinates updated by u in $[2k]$ and consider $x = b(E)$. The set $x[E]$ is a clique in the confusion graph $G_{h,u}$. Indeed, any function which sequentializes h respecting u has, for any configuration in $x[E]$, to first erase the information in E and then to restore it in $[2k] \setminus E$. Since this clique is of size q^k, we have $\kappa_{h,u} \geq k$ for any u and $\kappa^{\min}(h) \geq k$.

Lemma 4. *Let $n, k \in \mathbb{N}$ and $q \geq 2$. If there is a function $b : \binom{[2k]}{k} \to [0, q[^n$ such that the sets $b(E)[E]$ with $E \in \binom{[2k]}{k}$ are disjoint then there exists a $h \in F(n, q)$ without trivial coordinate functions, with $\kappa^{min}(h) \geq k$.*

Using Lemma 4 we could easily show that for any $q \geq 2$ and $n \in \mathbb{N}$, we have $\kappa^{min}_{n,q} \geq \lfloor n/4 \rfloor$. Indeed, if we have $n = 4k$, we can use the second half of the configuration to encode the set E. In Theorem 3 we prove that for any alphabet, we can in fact encode any $E \in \binom{[2k]}{k}$ in a configuration x of size $3k$. To do so, we use the following technique: if $i \in \overline{E} := [2k] \setminus E$ then we have $x_i = 0$ if $i+1$ in E and 1 otherwise. Moreover, in $[2k+1, 3k]$, using the same technique, we indicate if each element of E is followed by another element of E or not. From this encoding and Lemma 4 we deduce a lower bound for $\kappa^{\min}$ for any alphabet.

Theorem 3. *For all $n \in \mathbb{N}$ and $q \geq 4$, we have $\kappa^{\min}_{n,q} \geq \lfloor n/2 - \log_q(n) \rfloor$.*

Theorem 4 below states that, if we have an alphabet of size at least 4, we can encode any $E \in \binom{[2k]}{k}$ in a configuration of size $2k + log_q(2k)$. To do so, we encode E in $[2k] \setminus E$ using the fact that in an alphabet of size 4 each coordinate can encode twice more information than with a bit. Then, we indicate in $[2k, 2k + log_q(2k)]$ where the reading for decoding starts. From this encoding and Lemma 4 we deduce a lower bound for $\kappa^{\min}$.

Theorem 4. *For all $q \geq 4, n \in \mathbb{N}$, $\kappa^{\min}_{n,q} \geq \lfloor n/2 - \log_q(n) \rfloor$.*

6 Procedural complexity

Now, we study the relation between $\kappa^{\min}$ and the procedural complexity as defined in [4]. The procedural complexity of h is the minimum number t of functions $g^{(1)}, \ldots, g^{(t)}$ (each of which update at most one coordinate) that are required for $g^{(t)} \circ \cdots \circ g^{(1)}$ to compute h. For all $q \geq 2$ and $n \geq 2$, let us denote by $F^*(n, q) \subseteq F(n, q)$ the set of functions which do not update more than one coordinate. In [4], the authors first studied the memoryless procedural complexity $\mathcal{L}(h)$. It is the necessary number of step to compute h with $g^{(1)}, \ldots, g^{(t)}$ of same size than h. Then, they studied $\mathcal{L}(h|m)$ which is the procedural complexity using functions $g^{(1)}, \ldots, g^{(t)}$ of a fixed size m. More formally , $\forall m \geq n$, $\mathcal{L}(h|m) :=$ smallest t such that

$\exists\, g^{(1)}, \ldots, g^{(t)} \in F^*(m,q)$ such that $\mathrm{pr}_{[n]} \circ g^{(t)} \circ \cdots \circ g^{(1)} = h \circ \mathrm{pr}_{[n]}$. Here, we also use $\mathcal{L}^*(h) := \min(\{\mathcal{L}(h|m) \mid n \leq m\})$ which is the procedural complexity with a size arbitrarily big. Let $\Omega(h)$ be the number of non-trivial coordinate functions of h. Theorem 5 shows that the procedural complexity of an AN h is equal to $\kappa^{min}(h) + \Omega(h)$. Furthermore, it shows that the minimum procedural complexity is reached when we use $\kappa^{\min}(h)$ additional automata. It is directly deduced from Lemma 5 and Lemma 6.

Theorem 5. *Let $h \in F(n,q)$ and $k := \kappa^{\min}(h)$. We have $\mathcal{L}^*(h) = \mathcal{L}(h|n+k) = \Omega(h) + k$.*

In Lemma 5, we prove that $\mathcal{L}^*(h) \leq \Omega(h) + \kappa^{min}(h)$. We use the fact that by definition of $k := \kappa^{min}(h)$ there is $f \in F(n+k,q)$ and $w \in \Pi([n+k])$ such that the $n+k$ instructions $f^{w_1}, \ldots, f^{w_{n+k}} \in F^*(n+k,q)$ compute h. With that, we already have $\mathcal{L}(h|n+k) \leq n+k$. Furthermore, for each i such that h_i is trivial, we can remove the function f^i of the list of instructions and still compute h. As a result, we have $\mathcal{L}(h|n+k) \leq n+k-(n-\Omega(h)) = \Omega(h)+k$, and by definition of $\mathcal{L}^*(h)$ we have $\mathcal{L}^*(h) \leq \mathcal{L}(h|n+k)$.

Lemma 5. *Let $h \in F(n,q)$ and $k := \kappa^{\min}(h)$. We have $\mathcal{L}^*(h) \leq \mathcal{L}(h|n+k) \leq \Omega(h) + k$.*

In Lemma 6, we prove that $\Omega(h) + k \leq \mathcal{L}^*(h)$ with $k := \kappa^{\min}(h)$. To do so, we take a set of functions $g^{(1)} \ldots, g^{(t)} \in F^*(m,q)$ which compute h. We consider an order $w \in \Pi([n])$ which updates all coordinate of $[n]$ in the same order that $g^{(1)} \ldots, g^{(t)}$ update them for the last time. Then we prove that h can be sequentialized respecting w with less than $\mathcal{L}^*(h) - \Omega(h)$ additional automata. Let $J = \{j_1, \ldots, j_\ell\}$ be the set of steps such that either $g^{(j_i)}$ updates a coordinate of $]n,m]$, either it updates a coordinate of $[n]$ that will be updated again later. We have $\ell = \mathcal{L}^*(h) - \Omega(h)$. Then, we define $c : A^n \to A^k$ such that $c_i(x)$ equals $(g^{(j_i)} \circ \cdots \circ g^{(1)}(x(0)^{m-n}))_a$ with a the coordinate updated by $g^{(j_i)}$. Then, we prove that c is a proper coloring of the confusion graph $G_{h,w}$ and that $\Omega(h) + k \leq \mathcal{L}^*(h)$.

Lemma 6. *Let $h \in F(n,q)$ and $k := \kappa^{\min}(h)$. We have $\Omega(h) + k \leq \mathcal{L}^*(h)$.*

In [4], Proposition 12 states that $\forall h \in F(n,q)$, we have $\mathcal{L}(h|n-1) \leq 2n-1$. In Corollary 1 bellow, we refine this bound using Theorem 2, Theorem 5 and the fact that $\forall h \in F(n,q), \Omega(h) \leq n$.

Corollary 1. *For all* $h \in F(n,q)$*,* $\mathcal{L}(h|m) \leq m$ *with* $m := n + \lceil n/2 + log_q(n/2+1)\rceil$*.*

In the following Corollary 2, we give a lower bound for the procedural complexity with unlimited memory. It is a direct corollary of Theorem 5, Lemma 4, Theorem 3, Theorem 4 in which we construct an AN h without trivial coordinate functions (and thus we have $\Omega(h) = n$).

Corollary 2. *For all* $n, q \geq 2$ *there is* $h \in F(n,q)$ *such that* $\mathcal{L}^*(h) \geq n + \lfloor n/3 \rfloor$*. Furthermore, if* $q \geq 4$ *there is* $h \in F(n,q)$ *such that* $\mathcal{L}^*(h) \geq n + \lfloor n/2 - log_q(n)\rfloor$*.*

7 Bound for $\kappa^{\min}(h)$ using interaction graph

Let us now present a way to upper bound $\kappa^{\min}(h)$ for an AN h using the pathwidth of the interaction graph of h [2].

Definition 8 (Pathwidth). *A path decomposition of an undirected graph* $G = (V,E)$ *is a sequence of subsets* $X_1, \ldots, X_p$ *of vertices such that*

- $\forall (v,v') \in E,\ \exists X_i$ *such that* $v, v' \in X_i$*.*
- *If* $v \in X_i$ *and* $v \in X_j$ *with* $i < j$ *then* $\forall k \in [i,j], v \in X_k$

The size of a path decomposition is the size of the largest X_ℓ *minus one. The* pathwidth $\mathrm{Pw}(G)$ *is the minimum size of a path decomposition of* G*.*

Theorem 6 shows that the pathwidth of the graph $IG^*(h)$ is an upper bound for $\kappa^{\min}(h)$. It can be deduced directly from Lemma 7 and Lemma 8.

Theorem 6. *For any* AN h*,* $\kappa^{\min}(h) \leq \mathrm{Pw}(\mathrm{IG}^*(h))$*.*

Lemma 7 shows that from a path decomposition of a graph G of size s, we can construct a partition c of its vertices in s sets, and an update schedule u with properties allowing an efficient sequentialization by Lemma 8. We define c (resp. u) using a greedy algorithm. We iterate the subsets $X_1, \ldots X_n$ of the path decomposition and choose the value $c(i)$ (resp. $u(i)$) the first (resp. last) time we see i.

Lemma 7. *Let* $G = ([n], E)$ *be an undirected graph and let* $s = \mathrm{Pw}(G)$*. Then there are functions* $c : [n] \to [s]$ *and* $u \in \Pi([n])$ *with the following property. For all* $i \in [n]$*, we have either* 1*) for all* k *neighbor of* i *in* G *we have* $u(i) \leq u(k)$ *or* 2*) for all* $j, k \in [n]$ *with* $c(i) = c(j)$*,* $u(i) < u(j)$ *and* k *neighbor of* j *in* G *we have* $u(i) \leq u(k)$*.*

Lemma 8 shows how to use c and u defined in Lemma 8 to sequentialize h respecting u. Each additional automaton j (denoted from 1 to s) computes the sum modulo q of the images $\{ h_i(x) \mid i \in [n] \text{ and } c(i) = j \}$. Then, each automaton of coordinate j can compute $h_j(x)$, either because all neighbors of j in G have not be updated yet, or because it can compute all $h_j(x)$ such that $i \neq j$ and $c(i) = c(j)$.

Lemma 8. *Let $h \in F(n,q)$. Let $G = \mathrm{IG}^*(h)$. If we have $c : [n] \to [s]$ and $u \in \Pi([n])$ such that G, c, u have the same properties as in Lemma 7, then we have $\kappa(h,u) \leq s$.*

8 Conclusion and future research

We have seen that $\lfloor n/2 - \log_q(n) \rfloor \leq \kappa_{n,q}^{\min} \leq \kappa_{n,q} \leq \lceil n/2 + log_q(n/2+1) \rceil$. Thus, for any fixed n, the limit of $\kappa_{n,q}^{min}$ and $\kappa_{n,q}$ when q tends to infinity is $n/2$. It is an argument in favor of the conjecture made in [3] which states that for any n and q, $\kappa_{n,q} = \lfloor n/2 \rfloor$ and which is still open. It would be interesting to investigate a variant of the problem presented in this paper, where additional automata are forbidden but several updates of the same automaton are allowed. The task is then to know, for given n and q, the minimum time $t(q,n)$ such that $\forall h \in F(n,q)$, $\exists f \in F(n,q)$, $w \in [n]^{t'}$ with $t' \leq t(q,n)$ such that $f^w = h$. The value of $t(2,2)$ is not defined because for the AN $h \in F(2,2)$ such that $(0,0) \xrightarrow{h} (0,1) \xrightarrow{h} (1,1) \xrightarrow{h} (1,0) \xrightarrow{h} (0,0)$ there are no such f. However, with computers, we established that $t(3,2) = 22$. We can easily see that $\mathcal{L}_{n,q} := \max(\{\mathcal{L}(h) \mid h \in F(n,q)\})$ is a lower bound for $t(n,q)$, and in [4], it is stated that $2n-1 \leq \mathcal{L}_{n,q} \leq 4n-3$.

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A Proof of Theorem 1

We can deduce Theorem 1 directly from Lemma 1 and Lemma 2.

Theorem 1 (Theorem 1). *Let us consider* $h \in F(n, q)$ *and the sequential update schedule* $u \in \Pi([n])$*. Then we have* $\kappa(h, u) = \lceil \log_q(\chi(G_{h,u})) \rceil$*.*

Lemma 1 shows that we can use any (f, w) which sequentializes h respecting u to construct a proper coloring of $G_{h,u}$. Indeed, we can color the vertices of the graph $G_{h,u}$ using the values of the additional automata of f after their update. Thus, this coloring does not use more than q^k colors with k the number of additional automata of f.

Lemma 1. *Let us consider* $h \in F(n, q)$ *and the sequential update schedule* $u \in \Pi([n])$*. Then we have* $\lceil \log_q(\chi(G_{h,u})) \rceil \leq \kappa(h, u)$*.*

Proof. Without loss of generality, let us say that u is the canonical sequential update schedule $(1, 2, \ldots, n)$. Let $k := \kappa(h, u)$, $m := n + k$, $f \in F(m, q)$ and $w \in \Pi([m])$ respecting u such that $\mathrm{pr}_{[n]} \circ f^w = h \circ \mathrm{pr}_{[n]}$. Let x, x' be two neighbors in the confusion graph $G_{h,u}$. Let $y := (0)^k$ (a word of size k containing only the letter 0). Let $z := xy$ and $z' := x'y$. Let us prove that $f^w(z)_{[n+1,m]} \neq f^w(z')_{[n+1,m]}$. For the sake of contradiction, let us say that $f^w(z)_{[n+1,m]} = f^w(z')_{[n+1,m]}$. Since x and x' are neighbors in $G_{h,u}$, we know that $h(x) \neq h(x')$ and $\exists\ i \in [n]$, $h^{[i]}(x) = h^{[i]}(x')$. Let us consider the biggest of these i. So we have $h^{[i+1]}(x) \neq h^{[i+1]}(x')$ and then $h_{i+1}(x) \neq h_{i+1}(x')$. Let $j = w(i+1)$. Let us prove that $f^{w_1,\ldots,w_{j-1}}(z) = f^{w_1,\ldots,w_{j-1}}(z')$. First, we have $f^{w_1,\ldots,w_{j-1}}(z)_{[n]} = h^{[i]}(x) = h^{[i]}(x') = f^{w_1,\ldots,w_{j-1}}(z')_{[n]}$. Furthermore, for all $a \in [n + 1, m]$ which is not updated before the step j in w we have $f^{w_1,\ldots,w_{j-1}}(z)_a = y_{a-n} = f^{w_1,\ldots,w_{j-1}}(z')_a$. Finally, for all $a \in [n + 1, m]$ updated before the step j in w we have $f^{w_1,\ldots,w_{j-1}}(z)_a = f^{w_1,\ldots,w_{j-1}}(z')_a$ because we assumed that $f^w(z)_{[n+1,m]} = f^w(z')_{[n+1,m]}$. As a result, $f^{w_1,\ldots,w_{j-1}}(z) = f^{w_1,\ldots,w_{j-1}}(z')$. However, $f_{w_j} \circ f^{w_1,\ldots,w_{j-1}}(z) = h_{i+1}(x) \neq h_{i+1}(x') = f_{w_j} \circ f^{w_1,\ldots,w_{j-1}}(z')$. This is a contradiction. Consequently, we have, $f^w(z)_{[n+1,m]} \neq f^w(z')_{[n+1,m]}$. More generally, if x and x' are neighbors in $G_{h,u}$ then $f^w(xy)_{[n+1,n+k]} \neq f^w(x'y)_{[n+1,n+k]}$. In other words, $c : x \mapsto f^w(xy)_{[n+1,n+k]}$ gives a proper coloring of the confusion graph $G_{h,u}$. As a result, the confusion graph needs at most q^k colors because $f^w(xy)_{[n+1,n+k]}$ is a word of size k on the alphabet q. Thus, $\chi(G_{h,u}) \leq q^k$ and $\lceil \log_q(\chi(G_{h,u})) \rceil \leq k \leq \kappa(h, u)$.

Conversely, Lemma 2 states that we can construct a couple (f, w) which sequentializes h respecting u from a proper coloring of $G_{h,u}$. If this coloring uses less than q^k colors then f is of size at most $n+k$ and then the cost of sequentialization is at most k.

Lemma 2. *Let us consider the* AN $h \in F(n,q)$ *and the sequential update schedule* $u \in \Pi([n])$. *Then we have* $\kappa(h,u) \leq \lceil \log_q(\chi(G_{h,u})) \rceil$.

Proof. Let $k := \lceil \log_q(\chi(G_{h,u})) \rceil$, $m := n+k$. Let $A := [0,q[$. Let $w \in \Pi([m])$ which first update the k last automata and then the n first automata in the same order than u. In other words, $w := (n+1,\ldots,n+k,u_1,\ldots,u_n)$. Let $c : A^n \to A^k$ be a proper coloring of $G_{h,u}$. For all $i \in [n]$, let us define $p^{(i)} : A^m \to P(A^n)$ with $P(A^n) := \{E \mid E \subseteq A^n\}$ the set of subsets of A^n. First, $p^{(1)} : z \mapsto \{z_{[n]}\}$ and then $\forall i \in [2,n]$, $p^{(i)} : z \mapsto \{x \in A^n \mid h^{\{u_1,\ldots,u_{i-1}\}}(x) = z_{[n]} \text{ and } c(x) = z_{[n+1,m]}\}$. Let $f \in F(n+k,q)$ such that:

- $\forall i \in [n+1,m],\ f_i = c_i \circ pr_{[n]}$.
- $\forall i \in [n],\ f_{u_i} : z \mapsto z_{u_i}$ if $p^{(i)}(z) = \emptyset$ and $h_{u_i}(x)$ with $x \in p^{(i)}(z)$ otherwise.

Let us prove that $\mathrm{pr}_{[n]} \circ f^w = h \circ \mathrm{pr}_{[n]}$. Let $x \in A^n$ and $z \in A^m$ with $z_{[n]} = x$. By, induction let us prove that,

$$\forall i \in [0,n], f^{w_1,\ldots,w_{k+i}}(z) = h^{\{u_1,\ldots,u_i\}}(x)c(x).$$

First, for $i = 0$, we have,

$$f^{w_1,\ldots,w_k}(z) = f^{n+1,\ldots,n+k}(z) = x(c_1(x), c_2(x), \ldots, c_k(x)) = xc(x).$$

Second, let $i \in [n]$ and let us suppose that,

$$f^{w_1,\ldots,w_{k+(i-1)}}(z) = h^{\{u_1,\ldots,u_{i-1}\}}(x)c(x).$$

We have $f_{w_{k+i}} \circ f^{w_1,\ldots,w_{k+(i-1)}}(z) = h_{u_i}(x')$ with $x' \in p^{(i)}(f^{w_1,\ldots,w_{k+(i-1)}}(z))$. We have $x \in p^{(i)}(f^{w_1,\ldots,w_{k+(i-1)}}(z))$ because $(f^{w_1,\ldots,w_{k+(i-1)}}(z))_{[n+1,m]} = c(x)$ and $(f^{w_1,\ldots,w_{k+(i-1)}}(z))_{[n]} = h^{\{u_1,\ldots,u_{i-1}\}}(x)$. Let us prove that $h_{u_i}(x') = h_{u_i}(x)$. For the sake of contradiction let us say that $h_{u_i}(x') \neq h_{u_i}(x)$. Thus, $h(x) \neq h(x')$. However, $x, x' \in p^{(i)}(f^{w_1,\ldots,w_{k+(i-1)}}(z))$ thus $h^{\{u_1,\ldots,u_{i-1}\}}(x) = h^{\{u_1,\ldots,u_{i-1}\}}(x')$ and $c(x) = c(x')$. Consequently, x and x' are neighbors in the confusion graph but they have the same color. This is a contradiction. As a result, $h_{u_i}(x') = h_{u_i}(x)$. Thus, $\forall i \in [0,n], f^{w_1,\ldots,w_{k+i}}(z) = h^{\{u_1,\ldots,u_i\}}(x)c(x)$. As a consequence, $f^w(z) = h(x)\ c(x)$ and $\mathrm{pr}_{[n]} \circ f^w = h \circ \mathrm{pr}_{[n]}$. And since f has k additional automata, we have $\kappa(h,u) \leq k = \lceil \log_q(\chi(G_{h,u})) \rceil$.

B Proof of Lemma 3

To prove Lemma 3, we can construct a couple (h, u) such that $G_{h,u}$ has a clique of size $q^{n/2}$. Since the chromatic number of a graph is at least the size of its biggest clique, we have $\chi(G_{h,u}) \geq q^{n/2}$. As a result, $\kappa_{h,u} = \log(\chi(G_{h,u})) \geq n/2$ and we get Lemma 3 from that.

Lemma 3 (Lemma 3). *For all $q \geq 2$ and $n \in \mathbb{N}$, we have $\kappa_{n,q} \geq \lfloor n/2 \rfloor$.*

Proof. Let $k := \lfloor n/2 \rfloor$. Let us consider $h \in F(n, q)$ such that:

- $\forall i \in [k],\ h_i : x \mapsto x_{i+k}$
- $\forall i \in [k+1, 2k],\ h_i : x \mapsto x_{i-k}$
- If n is odd let $h_n : x \mapsto x_n$.

We also consider the canonical sequential update schedule $u := (1, 2, \ldots, n)$. Let us consider the set of all configurations X which have only 0 in their second half. In other words, $X := \{x \in A^n \mid x_{[k+1,n]} = (0)^{n-k}\}$ ($(0)^{n-k}$ beeing a word of size $n-k$ containing only the letter 0). Let $x, x' \in X$ such that $x \neq x'$. We have $x_{[k+1,n]} = (0)^{n-k} = x'_{[k+1,n]}$. Thus, $x_{[k]} \neq x'_{[k]}$ and $\exists i \in [k]$ such that $x_i \neq x'_i$ and $h_{i+k}(x) = x_i \neq x'_i = h_{i+k}(x')$. Thus, $h(x) \neq h(x')$. However, when we update the first half of the automata, x and x' both become the configuration $(0)^n$. Indeed, $\forall i \in [k], f_i(x) = x_{i+k} = 0$. Then, we have $h^{[k]}(x) = (0)^n = h^{[k]}(x')$. As a result, (x, x') are neighbors in $G_{h,u}$. As a consequence, every two distinct vertices of X are neighbors. Thus, X is a clique. Moreover, X is a clique of size q^k. Thus, $\chi(G_{h,u}) \geq q^k$ and $\kappa(h, u) \geq \lceil log_q(\chi(G_{h,u}))\rceil \geq \lceil log_q(q^k)\rceil = k = \lfloor n/2 \rfloor$. Hence, $\forall q \geq 2, \forall n \in \mathbb{N},\ \kappa_{n,q} \geq \lfloor n/2 \rfloor$.

Remark 2. In [4], Theorem 5 shows that if $h \in F(n, q)$ is a permutation, then for any $u \in Pi([n])$ we have $\kappa(h, u) \leq n/2$ if n is even and $\lfloor n/2 \rfloor + 1$ otherwise. As a result, the problem is almost solved for the permutations.

C Proof of Theorem 2

Theorem 2 (Theorem 2). *For all $n \in \mathbb{N},\ q \geq 2$ we have $\kappa_{n,q} \leq \lceil n/2 + log_q(n/2 + 1)\rceil$.*

Proof. Let $h \in F(n, q)$ and $A := [0, q[$. Without loss of generality, let us say that u is the canonical sequential update schedule $(1, 2, \ldots, n)$. Let E be the set of edges of the confusion graph $G_{h,u}$. Let $X = \{X_1, \ldots, X_p\}$ be a partition of A^n, such that x, x' are in the same set X_i if and only if the two following conditions are respected:

- They are equal on the second half of the coordinates which will be updated in u. In other words, $x_{\{u(n/2+1),\dots,u(n)\}} = x'_{\{u(n/2+1),\dots,u(n)\}}$ or, more simply, $x_{]n/2,n]} = x'_{]n/2,n]}$ because we said that $u = (1, 2, \dots, n)$.
- They have the same image by h. In other words, $h(x) = h(x')$.

For all $x \in A^n$, let us denote by $X(x)$ the set $X_i \in X$ which contains x. Let $x^{(1)} \in X_1,\ x^{(2)} \in X_2,\ \dots,\ x^{(p)} \in X_p$. Let us consider the undirected graph $G' = (X, E')$ where two sets X_i and $X_{i'}$ are neighbors in G' if and only if there are two configurations $x \in X_i$ and $x' \in X_{i'}$ neighbors in the confusion graph $G_{h,u}$. Without loss of generality, let us consider the neighbors N of X_1 in G'. If $X_j \in N$ then $\exists x \in X_1, x' \in X_j$ such that $\exists i \in [n], h^{[i]}(x) = h^{[i]}(x')$ and $h(x) \neq h(x')$. Let us split N in $n/2+1$ sets:

- Let us denote by $N_{[n/2]}$ the set of sets X_j such that $\exists i \in [n/2], x \in X_1, x' \in X_j$ such that $h^{[i]}(x) = h^{[i]}(x')$ and $h(x) \neq h(x')$. Since $h^{[i]}(x') = h^{[i]}(x)$, we have $x'_{]i,n]} = x_{]i,n]}$ and $x'_{]n/2,n]} = x_{]n/2,n]} = x^{(1)}_{]n/2,n]}$ because $i \leq n/2$. In other words, $\forall X_j \in N_{[n/2]}$, we have $x' \in X_j$ such that $x'_{]n/2,n]} = x^{(1)}_{]n/2,n]}$. However, there is only $q^{n/2}$ such configurations x'. Thus, $|N_{[n/2]}| \leq q^{n/2}$.
- For all $i \in [n/2+1, n]$, let us denote by N_i, the set of sets X_j such that, $\exists x \in X_1, x' \in X_j$ such that $h^{[i]}(x) = h^{[i]}(x')$ and $h(x) \neq h(x')$. Let $X_j \in N_i$ and let $x \in X_1, x' \in X_j$ such that $h^{[i]}(x) = h^{[i]}(x')$. Thus, we have $x^{(j)}_{]i,n]} = x'_{]i,n]} = x_{]i,n]} = x^{(1)}_{]i,n]}$ because $i > n/2$. Thus, the value of $x^{(j)}_{[n/2+1,n]}$ is fixed on the interval $[i, n]$ and can vary only on the interval $[n/2+1, i]$. As a result,the second half of $x^{(j)}$ can take $q^{i-n/2}$ values. Furthermore, $h_{[i]}(x^{(j)}) = h_{[i]}(x) = h_{[i]}(x') = h_{[i]}(x^{(1)})$. Thus, the value of $h(x^{(j)})$ is fixed on the interval $[i]$ and can vary only on the interval $[i, n]$. As a result, $h(x^{(j)})$ can take q^{n-i} different values. Now if two configurations x' and x'' have the same image by h and are equal one their second half then they are in the same set X_j. Thus, $|N_i| \leq q^{i-n/2} * q^{n-i} = q^{n/2}$.

We have $N = N_{[n/2]} \cup N_{n/2+1} \cup \cdots \cup N_n$. Thus, $|N| \leq (n/2+1)q^{n/2}$. As a consequence, the degree of X^1 in G' is less than $(n/2+1)q^{n/2}$ (strictly less because X^1 is in N but is not neighbor of himself). As a result, $\chi(G') \leq d(G')+1 \leq (n/2+1)q^{n/2}$ with $d(G')$ the degree of G'. We can see that any coloring of this graph G' gives a proper coloring of the confusion graph. Indeed, we can color all the configurations of a set X_i in $G_{h,u}$ as we color X_i in G'. If two configurations x and x' are neighbors in the confusion graph $G_{h,u}$, then $X(x)$ and $X(x')$ are neighbors in G' and will not have the same color. Thus, $\chi(G_{h,u}) \leq \chi(G') \leq (n/2+1) * q^{n/2}$. As a consequence, according to Theorem 1, we have, $\kappa(h,u) \leq \lceil n/2 + \log_q(n/2+1) \rceil$. Hence, $\forall n \in \mathbb{N}, \kappa_{n,q} \leq \lceil n/2 + \log_q(n/2+1) \rceil$.

D Proof of Lemma 4

Lemma 4 (Lemma 4). *Let $n, k \in \mathbb{N}$ and $q \geq 2$. If there is a function $b : \binom{[2k]}{k} \to [0, q[^n$ such that the sets $b(E)[E]$ with $E \in \binom{[2k]}{k}$ are disjoint then there exists $h \in F(n,q)$ without trivial coordinate function, with $\kappa^{min}(h) \geq k$.*

Proof. Let $B := \bigcup_{E \in \binom{[2k]}{k}} b(E)[E]$. Let $a : B \to [0, q[^n$ such that $\forall E \in \binom{[2k]}{k}$, $\forall x \in b(E)[E]$, $a(x) = E$. Let $h \in F(n,q)$ such that: $\forall x \in B$,

- $h_{a(x)}(x) = x_{[2k] \setminus a(x)}$.
- $h_{[2k] \setminus a(x)}(x) = x_{a(x)}$.
- $\forall i \in [2k+1, n], h_i(x) = 0$.

and $\forall x \in [0, q[^n \setminus B$, $h(x) = (0)^n$. We can see that h does not have any trivial coordinate function. Indeed, for all $i \in [2k+1, n]$ we have $h_i : x \mapsto 0$ which is nontrivial. Furthermore, if we take $x, y \in b(E)[E]$ with $E = [k+1, 2k]$, and $x_E = (0)^k$ and $y_E = (1)^k$, we see that

$$\forall i \in [k],\ h_i(x) = x_{n/2+i} = 1 \neq 0 = y_{n/2+i} = h_i(y).$$

However, $\forall i \in [k], i \notin E$ and thus $x_i = y_i$ because $x, y \in E$. Thus, either $h_i(x) \neq x_i$ or $h_i(y) = y_i$. Either way, h_i is nontrivial. Thus, for all $i \in [k]$, h_i is nontrivial. The same way, we can prove that there are no trivial coordinate functions whose index is in $[k+1, 2k]$. As a result, h does not have any trivial coordinate function. Let us prove that $\forall u \in \Pi([n])$, $\kappa(h,u) \geq k$. Let us consider the sequential update schedule $u \in \Pi([n])$. Let $E \in \binom{[2k]}{k}$ be the set of the k first automata of $[2k]$ updated in u. Let $E' = [2k] \setminus E$. Furthermore, let i be the first step at which all automata of E are updated in u. In other words, we have $E \subseteq \{u_1, \ldots, u_i\}$ and $E' \cap \{u_1, \ldots, u_i\} = \emptyset$. Let $z = b(E)$. We will prove that $z[E]$ is a clique in the confusion graph $G_{h,u}$. Let $x, y \in z[E]$ with $x \neq y$. First let us prove that $h^{\{u_1,\ldots,u_i\}}(x) = h^{\{u_1,\ldots,u_i\}}(y)$. We have:

- $h^{\{u_1,\ldots,u_i\}}(x)_E = h^{\{u_1,\ldots,u_i\}}(x)_{a(x)} = x_{[2k]\setminus a(x)} = x_{E'} = z_{E'} = y_{E'} = y_{[2k]\setminus a(y)} = h^{\{u_1,\ldots,u_i\}}(y)_{a(y)} = h^{\{u_1,\ldots,u_i\}}(y)_E$.

- $h^{\{u_1,\dots,u_i\}}(x)_{E'} = (x)_{E'} = z_{E'} = y_{E'} = h^{\{u_1,\dots,u_i\}}(y)_{E'}$ because $E' \cap \{u_1,\dots,u_i\} = \emptyset$.
- $\forall j \in [2k+1,n]$, with $j \in \{u_1,\dots,u_i\}$ we have $h^{\{u_1,\dots,u_i\}}(x)_j = h_j(x) = 0 = h_j(y) = h^{\{u_1,\dots,u_i\}}(y)_j$.
- $\forall j \in [2k+1,n]$, with $j \notin \{u_1,\dots,u_i\}$ we have $h^{\{u_1,\dots,u_i\}}(x)_j = x_j = z_j = y_j = h^{\{u_1,\dots,u_i\}}(y)_j$.

As a result, $h^{\{u_1,\dots,u_i\}}(x) = h^{\{u_1,\dots,u_i\}}(y)$. Now, $x \neq y$ and $x, y \in z[E]$. Thus, $x_E \neq y_E$ and $h(x)_{E'} = x_E \neq y_E = h(y)_{E'}$. As a result, x and y are neighbors in $G_{h,u}$ and then $z[E]$ is a clique. Furthermore, $z[E]$ is of size q^k. Thus , $\chi(G_{h,u}) \geq q^k$. As a consequence, for any sequential update schedule u we have $\kappa(h,u) \geq k$ and then $\kappa^{min}(h) \geq k$.

E Proof of Theorem 3

Theorem 3 (Theorem 3). *For all $q \geq 2$ and $n \in \mathbb{N}$, $\kappa^{min}_{n,q} \geq \lfloor n/3 \rfloor$.*

Proof. Let $A := [0, q[$. In this proof, c^i refers to i times the composition of c. Let $n = 3k$. (if $n = 3k+1$ or $n = 3k+2$ we just add one or two useless automata and the demonstration is the same). Let $b : \binom{[2k]}{k} \to A^n$ such that $\forall E = \{e_1, e_2, \dots, e_k\} \in \binom{[2k]}{k}$, $\forall x \in b(E)[E]$ we have:

- $\forall e \in \overline{E} = [2k] \setminus E,\ x_e = 0$ if $e + 1 \in E$ and 1 otherwise.
- $x_{2k+1} = 0$ if $1 \in E$ and 1 otherwise.
- $\forall \ell \in [k-1],\ x_{2k+\ell+1} = 0$ if $j + 1 (\text{mod } 2k) \in E$ and 1 otherwise with $j = e_\ell$.

Let $a : B \to \binom{[2k]}{k}$ be the function which decodes the subset encoded in a configuration such that $a = g \circ c^{2k} \circ h$ with:

- $h : x \mapsto (x, \{1\}, \{\}, 1)$ if $x_{2k+1} = 0$ and $(x, \{\}, \{1\}, 1)$ otherwise.
- c such that for all $x \in B$, I, $\overline{I}$ subsets of $[n]$ and $e \in [2k]$:
  - If $e \in \overline{I}$:
    - If $x_e = 0$ then $c(x, I, \overline{I}, e) = (x, I \cup \{e+1\}, \overline{I}, e+1)$.
    - If $x_e = 1$ then $c(x, I, \overline{I}, e) = (x, I, \overline{I} \cup \{e+1\}, e+1)$.
  - $e \in I$:
    - If $|I| = k$ then $c(x, I, \overline{I}, e) = (x, I, \overline{I} \cup \{e+1\}, e+1)$.
    - Otherwise, let $\ell = |I|$ and $b = x_{2k+\ell+1}$.
      - If $b = 0$ then $c(x, I, \overline{I}, e) = (x, I \cup \{e+1\}, \overline{I}, e+1)$.
      - If $b = 1$ then $c(x, I, \overline{I}, e) = (x, I, \overline{I} \cup \{e+1\}, e+1)$.
- $g : (x, I, \overline{I}, q) \mapsto I$.

By induction, let us prove that:

$$\forall i \in [2k],\ \forall x \in b(E)[E],\ c^{i-1}(h(x)) = (x, E \cap [i], \overline{E} \cap [i], i).$$

First, for $i = 1$ we have $c^{i-1}(h(x)) = c^0(h(x)) = h(x)$. There are 2 cases:

- If $1 \in E$, then $x_{2k+1} = 0$ because $x_{2k+1} = 0$ if $1 \in E$ and 1 otherwise. Thus, $h(x) = (x, \{1\}, \{\}, 1)$. Furthermore, $E \cap [1] = \{1\}$ and $\overline{E} \cap [1] = \{\}$. As a result, we have $c^{i-1}(h(x)) = (x, E \cap [1], \overline{E} \cap [1], 1)$.
- If $1 \in \overline{E}$, then $x_{2k+1} = 1$ because $x_{2k+1} = 0$ if $1 \in E$ and 1 otherwise. Thus, $h(x) = (x, \{\}, \{1\}, 1)$. Furthermore, $E \cap [1] = \{\}$ and $\overline{E} \cap [1] = \{1\}$. As a result, we have $c^{i-1}(h(x)) = (x, E \cap [1], \overline{E} \cap [1], 1)$.

Next, let us suppose that for $i \in [2k[$, we have $c^{i-1}(h(x)) = (x, E \cap [i], \overline{E} \cap [i], i)$. Let $I = E \cap [i],\ \overline{I} = \overline{E} \cap [i],\ e = i$. Let $c^{i-1}(h(x)) = (x, I, \overline{I}, e)$.

- if $e \in \overline{E}$, then we have $e \in \overline{I}$. There are two cases:
  - If $e + 1 \in E$ then $x_e = 0$ because $\forall e \in \overline{E},\ x_e = 0$ if $e + 1 \in E$ and 1 otherwise. Then $c^i(h(x)) = (x, I \cup \{e + 1\}, \overline{I}, e + 1)$. As a result, $c^i(h(x)) = (x, E \cap [i + 1], \overline{E} \cap [i + 1], i + 1)$.
  - If $e + 1 \in \overline{E}$ then $x_e = 1$ because $\forall e \in \overline{E},\ x_e = 0$ if $e + 1 \in E$ and 1 otherwise. Then $c^i(h(x)) = (x, I, \overline{I} \cup \{e + 1\}, e + 1)$. As a result, $c^i(h(x)) = (x, E \cap [i + 1], \overline{E} \cap [i + 1], i + 1)$.
- if $e \in E$ then we have $e \in I$. There are two cases:
  - If $e + 1 \in E$. Then we have $|I| < k$ because $I = E \cap [i] \subseteq E$ and $|E| = k$ and $(e + 1) \in E \setminus I$. Let $\ell = |I|$. We have $e = e_\ell$. We have $x_{2k+\ell+1} = 0$ because $\forall j \in [k - 1],\ x_{2k+\ell+1} = 0$ if $j + 1 \in E$ and 1 otherwise with $j = e_\ell$. Then $c^i(h(x)) = (x, I \cup \{e + 1\}, \overline{I}, e + 1)$. As a result, $c^i(h(x)) = (x, E \cap [i + 1], \overline{E} \cap [i + 1], i + 1)$.
  - $e + 1 \in \overline{E}$. There are two cases:
    * If $e = e_k$. Then $|I| = k$, thus $c^i(h(x)) = (x, I, \overline{I} \cup \{e+1\}, e+1)$. As a result, $c^i(h(x)) = (x, E \cap [i + 1], \overline{E} \cap [i + 1], i + 1)$.
    * If $e = e_i$ with $i < k$. Then we have $|I| < k$. Let $\ell = |I|$. We have $e = e_\ell$. We have $x_{2k+\ell+1} = 1$ because $\forall j \in [k-1],\ x_{2k+\ell+1} = 0$ if $j + 1 \in E$ and 1 otherwise with $j = e_\ell$. Then $c^i(h(x)) = (x, I, \overline{I} \cup \{e + 1\}, e + 1)$. As a result, $c^i(h(x)) = (x, E \cap [i + 1], \overline{E} \cap [i + 1], i + 1)$.

By induction, we have $\forall x \in A^n,\ \forall i \in [2k]$,

$$c^{i-1}(h(x)) = (x, E^0 \cap V(i), E^1 \cap V(i), \overline{E}^0 \cap V(i), \overline{E}^1 \cap V(i), m+i+1, r(m+i+1)).$$

In particular, we have $c^{2k}(h(x)) = (x, E, \overline{E}, q)$. As a consequence, $a(x) = E$. Thus, all the sets $b(E)[E]$ with $E \in \binom{[2k]}{k}$ are disjoint. Using Lemma 4, we conclude that $\kappa_{n,q}^{min} \geq \lfloor n/3 \rfloor$.

F Proof of Theorem 4

Theorem 4 (Theorem 4). *For all* $q \geq 4, n \in \mathbb{N}$, $\kappa_{n,q}^{\min} \geq \lfloor n/2 - \log_q(n) \rfloor$.

Proof. Let $A := [0, q[$. Let $n = 2k + \lceil log_q(2k) \rceil$. Only in this proof, to simplify the use of modulo, we index the coordinates starting from 0 and not from 1. Furthermore, each addition or subtraction is done modulo $2k$, and we will consider that if $a < b$ then $[b, a] = [a, 2k[\cup [0, b]$. For all $I \subseteq [0, 2k[$, let $\triangle_I : E \mapsto |I \cap E| - |I \setminus E|$. Let us consider the two functions $M : \binom{[0,2k[}{k} \to [-k, k]$ and $m : \binom{[0,2k[}{k} \to [0, 2k[$ such that $\forall E \in \binom{[0,2k[}{k}$,

- $M(E) := \max(\{\triangle_{[i]}(E)) \mid i \in [0, 2k[\})$.
- $\triangle_{[m(E)]}(E) = M$.

For instance if we have $k := 4$, and $E := \{2, 4, 5, 6\}$ then,

i	0	1	2	3	4	5	6	7
$\in E$	No	No	Yes	No	Yes	Yes	Yes	No
$\triangle_{[0,i]}(E)$	-1	-2	-1	-2	-1	0	1	0

Furthermore, $m(E) = 6$ and $M(E) = \triangle_{[0,6]}(E) = 1$. For all $E \in \binom{[0;2k[}{k}$, let us denote by $E^0, E^1, \overline{E}^0, \overline{E}^1$ the subsets of $[0, 2k[$ such that

- $E^0 = \{e \in E \mid e - 1 \in \overline{E}\}$.
- $E^1 = E \setminus E^0 = \{e_1, e_2, \ldots, e_p\}$ with $e_1 - m(E) - 1 < e_2 - m(E) - 1 < \cdots < e_p - m(E) - 1$.
- $\overline{E}^0 = \{e' \in \overline{E} \mid e' + 1 \in E\}$.
- $\overline{E}^1 = \overline{E} \setminus \overline{E}^0 = \{e'_1, e'_2, \ldots, e'_p\}$ with $e'_1 - m(E) - 1 < e'_2 - m(E) - 1 < \cdots < e'_p - m(E) - 1$.

In other words, we sort the elements of E^1 and $\overline{E}^1$ in the order $m+1, m+2, \ldots, 2k-1, 0, 1, \ldots, m$. If we take again our example where $k := 4$, and $E := \{2, 4, 5, 6\}$ we have $E^0 = \{2, 4\}$, $E^1 = \{e_1 = 5, e_2 = 6\}$, $\overline{E}^0 = \{1, 3\}$ and $\overline{E}^1 = \{e'_1 = 7, e'_2 = 0\}$. Inded we have $e_1 - m(E) - 1 = 5 - 6 - 1 = 6 \leq e_2 - m(E) - 1 = 6 - 6 - 1 = 7$ and $e'_1 - m(E) - 1 = 7 - 6 - 1 = 0 \leq e'_2 - m(E) - 1 = 0 - 6 - 1 = 1$. Let $v : [0, 2k[\to A^{n-2k}$ be an injective function and let v^{-1} be the inverse function of v. Let $b : \binom{[0,2k[}{k} \to A^n$ such that if $x = b(E)$ then we have,

- $x_E = (0)^k$.
- $\forall e \in \overline{E}^0$, $x_e = 0$ if $e + 2 \in E$ and 1 otherwise.
- $\forall e'_j \in \overline{E}^1$, $x_{e'_j} = 2$ if $e_j + 1 \in E$ and 3 otherwise.

- $x_{[2k,n[} = v(m(E))$.

Again, with the same example, for all $y \in b(E)[E]$ we have:

i	0	1	2	3	4	5	6	7
$\in E$	No	No	Yes	No	Yes	Yes	Yes	No
y_i	3	1	y_2	0	y_4	y_5	y_6	2

Indeed,

- $y_3 = 0$ because $3 \in \overline{E}^0$ and $3 + 2 \in E$.
- $y_1 = 1$ because $1 \in \overline{E}^0$ and $1 + 2 \notin E$.
- $y_7 = 2$ because $e'_1 = 7 \in \overline{E}^1$ and $e_1 + 1 = 5 + 1 \in E$.
- $y_0 = 3$ because $e'_2 = 0 \in \overline{E}^1$ and $e_2 + 1 = 6 + 1 \notin E$.

Let $B =: \{x \in b(E)[E] \mid E \in \binom{[2k]}{k}\}$ be the set of configuration which encodes a set E. Let us consider the function $a : B \to \binom{[2k]}{k}$ which decodes the set encoded by any configuration of B such that $a = g \circ c^{2k} \circ h$ with:

- $h : x \mapsto (x, \emptyset, \emptyset, \emptyset, \emptyset, m + 1, 0)$ with $m = v^{-1}(x_{[2k,n[})$.
- c such that for all $I^0, I^1, \overline{I}^0, \overline{I}^1$ subsets of $[0, 2k[$, $e \in [0, 2k[$ and $q \in [0, 3[$,
  - if $q = 0$:
    * if $x_e = 0$ or 1 then $c(x, I^0, I^1, \overline{I}^0, \overline{I}^1, e, q) = (x, I^0, I^1, \overline{I}^0 \cup \{e\}, \overline{I}^1, e + 1, 1)$.
    * if $x_e = 2$ or 3 then $c(x, I^0, I^1, \overline{I}^0, \overline{I}^1, e, q) = (x, I^0, I^1, \overline{I}^0, \overline{I}^1 \cup \{e\}, e + 1, 0)$.
  - if $q = 1$:
    * if $x_{e-1} = 0$ then $c(x, I^0, I^1, \overline{I}^0, \overline{I}^1, e, q) = (x, I^0 \cup \{e\}, I^1, \overline{I}^0, \overline{I}^1, e + 1, 2)$.
    * if $x_{e-1} = 1$ then $c(x, I^0, I^1, \overline{I}^0, \overline{I}^1, e, q) = (x, I^0 \cup \{e\}, I^1, \overline{I}^0, \overline{I}^1, e + 1, 0)$.
  - if $q = 2$, let $j = |I^1|$, $e' = \overline{I}^1_j$ (the j-th element of $\overline{I}^1$ when we sort them it in the order $m + 1, m + 2, \ldots, 2k - 1, 0, 1, \ldots, m$).
    * if $x_{e'} = 2$ then $c(x, I^0, I^1, \overline{I}^0, \overline{I}^1, e, q) = (x, I^0, I^1 \cup \{e\}, \overline{I}^0, \overline{I}^1, e + 1, 2)$.
    * if $x_{e'} = 3$ then $c(x, I^0, I^1, \overline{I}^0, \overline{I}^1, e, q) = (x, I^0, I^1 \cup \{e\}, \overline{I}^0, \overline{I}^1, e + 1, 0)$.
- $g : (x, I^0, I^1, \overline{I}^0, \overline{I}^1, e, q) \mapsto I^0 \cup I^1$.

With the same example, let $y \in b(E)[E]$ and let us compute $a(y)$.

$$\begin{aligned}
y &= (3, 1, y_2, 0, y_4, y_5, y_6, 2)v(m(E)) \\
&\xrightarrow{h} (y, \emptyset, \emptyset, \emptyset, \emptyset, 7, 0) \\
&\xrightarrow{c} (y, \emptyset, \emptyset, \emptyset, \{7\}, 0, 0) \\
&\xrightarrow{c} (y, \emptyset, \emptyset, \emptyset, \{7, 0\}, 1, 0) \\
&\xrightarrow{c} (y, \emptyset, \emptyset, \{1\}, \{7, 0\}, 2, 1) \\
&\xrightarrow{c} (y, \{2\}, \emptyset, \{1\}, \{7, 0\}, 3, 0) \\
&\xrightarrow{c} (y, \{2\}, \emptyset, \{1, 3\}, \{7, 0\}, 4, 1) \\
&\xrightarrow{c} (y, \{2, 4\}, \emptyset, \{1, 3\}, \{7, 0\}, 5, 2) \\
&\xrightarrow{c} (y, \{2, 4\}, \{5\}\{1, 3\}, \{7, 0\}, 6, 2) \\
&\xrightarrow{c} (y, \{2, 4\}, \{5, 6\}\{1, 3\}, \{7, 0\}, 7, 0) \\
&\xrightarrow{g} \{2, 4, 5, 6\} = E.
\end{aligned}$$

Thus, we have $\forall y \in b(E)[E]$, $a(y) = E$. If we can prove that for all $E \in \binom{[0,2k[}{k}$ and for all $y \in b(E)[E]$, we have $a(y) = E$, then we prove that the sets $b(E)[E]$ are disjoint. Furthermore, with Lemma 4, we can conclude that $\kappa^{min}_{2k+log(k),q} \geq k$. In the remaining of the proof, we prove it formally for all k and E. Let $k \in \mathbb{N}$, $E \in \binom{[0,2k[}{k}$. Let $r_E : \ell \mapsto \begin{cases} 0 & \text{if } \ell \in \overline{E} \\ 1 & \text{if } \ell \in E^0 \\ 2 & \text{otherwise} \end{cases}$. By induction, let us prove that for all $i \in [0, 2k]$,

$$c^i(h(x)) = (x, E^0 \cap V(i), E^1 \cap V(i), \overline{E}^0 \cap V(i), \overline{E}^1 \cap V(i), m+i+1, r(m+i+1)).$$

with $V(0) = \emptyset$, and $\forall i \in [2k], V(i) = [m+1, m+i]$. First, let us prove that $m+1 \notin E$. For the sake of contradiction, let us say that $m+1 \in E$. Then we have $\triangle_{[m+1]}\ (E) = \triangle_{[m]}\ (E) + \triangle_{[m+1,m+1]}\ (E) = M(E) + 1$. This is absurd because $M(E) = \max(\{\triangle_{[i]}(E)) \mid i \in [2k]\})$. As a result $m+1 \notin E$. Thus, $r_E(m+0+1) = 0$. Furthermore,

$$\begin{gathered}
c^0(h(x)) = h(x) = (x, \{\}, \{\}, \{\}, \{\}, m+0+1, 0) \\
= (x, E^0 \cap V(0), E^1 \cap V(0), \overline{E}^0 \cap V(0), \overline{E}^1 \cap V(0), m+0+1, r_E(m+0+1)).
\end{gathered}$$

Next, let us suppose that the induction hypothesis hold for $i \in [0, 2k[$. Let $I = E \cap V(i)$, $\overline{I} = \overline{E} \cap V(i)$, $I^0 = E^0 \cap V(i)$, $I^1 = E^1 \cap V(i)$,

$\overline{I}^0 = \overline{E}^0 \cap V(i)$, $\overline{I}^1 = \overline{E}^1 \cap V(i)$, $e = m+i+1$ and $q = r_E(e)$. Thus, $c^i(h(x)) = (x, I^0, I^1, \overline{I}^0, \overline{I}^1, e, q)$. There are four cases:

- $e = m+i+1 \in \overline{E}^0$. As a consequence, we have $q = r_E(e) = 0$. Furthermore, we have $x_e = 0$ or 1 because $\forall e \in \overline{E}^0$, $x_e = 0$ if $e+2 \in E$ and 1 otherwise. Thus, $c^{i+1}(h(x)) = (x, I^0, I^1, \overline{I}^0 \cup \{e\}, \overline{I}^1, e+1, 1)$. By definition of $\overline{E}^0$, we have $e+1 \in E^0$ and then $r_E(e+1) = 1$. Thus, $c^{i+1}(h(x)) = (x, E^0 \cap V(i+1), E^1 \cap (V(i+1)), \overline{E}^0 \cap V(i+1), \overline{E}^1 \cap V(i+1), m+i+2, r_E(m+i+2))$.
- $e = m+i+1 \in \overline{E}^1$. As a consequence, we have $q = r_E(e) = 0$. Furthermore, we have $x_e = 2$ or 3 because $\forall e'_i \in \overline{E}^1$, $x_{e'_i} = 2$ if $e_i + 1 \in E$ and 3 otherwise. Thus, $c^{i+1}(h(x)) = (x, I^0, I^1, \overline{I}^0, \overline{I}^1 \cup \{e\}, e+1, 1)$. By definition of $\overline{E}^1$, we have $e+1 \in \overline{E}$ and then $r_E(e+1) = 0$. Thus, $c^{i+1}(h(x)) = (x, E^0 \cap V(i+1), E^1 \cap V(i+1), \overline{E}^0 \cap V(i+1), \overline{E}^1 \cap V(i+1), m+i+2, r_E(m+i+2))$.
- $e = m+i+1 \in E^0$. By induction hypothesis, we have $q = r_E(e) = 1$. Furthermore, by definition of E^0, $e-1 \in \overline{E}^0$. There are two subcases:
  - $e+1 \in E$. We have $x_{e-1} = 0$ because $\forall (e-1) \in \overline{E}^0$, $x_{e-1} = 0$ if $(e-1)+2 = e+1 \in E$ and 1 otherwise. Thus, $c^{i+1}(h(x)) = (x, I^0 \cup \{e\}, I^1, \overline{I}^0, \overline{I}^1, e+1, 2)$. And since $e+1 \in E$ and $e \notin \overline{E}$, then $e+1 \in E^1$ and $r_E(e+1) = 2$. As a result, $c^{i+1}(h(x)) = (x, E^0 \cap V(i+1), E^1 \cap V(i+1), \overline{E}^0 \cap V(i+1), \overline{E}^1 \cap V(i+1), m+i+2, r_E(m+i+2))$.
  - $e+1 \in \overline{E}$. We have $x_{e-1} = 1$ because $\forall (e-1) \in \overline{E}^0$, $x_{e-1} = 0$ if $(e-1)+2 = e+1 \in E$ and 1 otherwise. Thus, $c^{i+1}(h(x)) = (x, I^0 \cup \{e\}, I^1, \overline{I}^0, \overline{I}^1, e+1, 0)$. And since $e+1 \in \overline{E}$, $r_E(e) = 0$. As a result, $c^{i+1}(h(x)) = (x, E^0 \cap V(i+1), E^1 \cap V(i+1), \overline{E}^0 \cap V(i+1), \overline{E}^1 \cap V(i+1), m+i+2, r_E(m+i+2))$.
- $e = m+i+1 \in E^1$. As a consequence, we have $q = r_E(e) = 2$. Let $j = |I^1|$. Let us prove that $|\overline{I}^1| < |I^1|$. First, we have $|I^0| = |\overline{I}^0|$. Indeed, for all $u \in \overline{I}^0$, we have also $u \in \overline{E}^0$ and then $u+1 \in E^0$. Furthermore, $e \in E^1$ and thus $e-1 = m+i \notin \overline{E}^1$. Thus, $u \in [m+1, m+i+1[$, $u+1 \in [m+1, m+i+1]$. Consequently, $u+1 \in V(i)$ and $u+1 \in I^0$. As a result, for all $u \in \overline{I}^0$, we have $u+1 \in I^0$. As a consequence, $|\overline{I}^0| \leq |I^0|$. Reversely, for all $v \in I^0$, $v \in E^0$ and then $v-1 \in \overline{E}^0$. Furthermore, $m+1 \in \overline{E}$. Thus, $v \in]m+1, m+i+1]$ and $v-1 \in [m+1, m+i+1]$. As a result, $v-1 \in V(i)$ and $v-1 \in \overline{I}^0$. Consequently, for all $v \in I^0$, we have $v-1 \in \overline{I}^0$. As a consequence, $|I^0| \leq |\overline{I}^0|$ and

then $|I^0| = |\overline{I}^0|$. Now, $\triangle_{[m+i+1]}(E) = \triangle_{[m]}(E) + \triangle_{[m+1,m+i]}(E) + |\{e\}| = M(E) + |I| - |\overline{I}| + 1 = M(E) + |I^0| + |I^1| - |\overline{I}^0| - |\overline{I}^1| + 1 = M(E) + |I^1| - |\overline{I}^1| + 1$. If $|I^1| \geq |\overline{I^1}|$ then $\triangle_{[m+i+1]}(E) > M(E)$ which is absurd. Thus, $|I^1| < |\overline{I^1}|$. Let $e' = |\overline{I}^1_j|$. We have $e = e_j$ and $e' = e'_j$. There are two cases:

- $e + 1 \in E$. Then $x_{e'} = 2$ because $\forall e'_j \in \overline{E}^1$, $x_{e'_j} = 2$ if $e_j + 1 \in E$ and 3 otherwise. Thus, $c^{i+1}(h(x)) = (x, I^0, I^1 \cup \{e\}, \overline{I}^0, \overline{I}^1, e+1, 2)$. Furthermore, $e+1 \in E$ and $e \notin \overline{E}$ then $e+1 \in E^1$ and $r_E(e+1) = 2$. As a result, $c^{i+1}(h(x)) = (x, E^0 \cap V(i+1), E^1 \cap V(i+1), \overline{E}^0 \cap V(i+1), \overline{E}^1 \cap V(i+1), m+i+2, r_E(m+i+2))$.
- $e + 1 \in \overline{E}$. Then $x_{e'} = 3$ because $\forall e'_j \in \overline{E}^1$, $x_{e'_j} = 2$ if $e_j + 1 \in E$ and 3 otherwise. Thus, $c^{i+1}(h(x)) = (x, I^0, I^1 \cup \{e\}, \overline{I}^0, \overline{I}^1, e+1, 0)$. Furthermore, $e+1 \in E$ and $e \notin \overline{E}$ then $e+1 \in E^1$ and $r_E(e+1) = 0$. As a result, $c^{i+1}(h(x)) = (x, E^0 \cap V(i+1), E^1 \cap V(i+1), \overline{E}^0 \cap V(i+1), \overline{E}^1 \cap V(i+1), m+i+2, r_E(m+i+2))$.

By induction, we can see that $\forall i \in [2k[$, $c^{i+1}(h(x)) = (x, E^0 \cap V(i+1), E^1 \cap V(i+1), \overline{E}^0 \cap V(i+1), \overline{E}^1 \cap V(i+1), m+i+2, r_E(m+i+2))$. As a result, $a(x) = g(c^{2k}(h(x))) = g(x, E^0, E^1, \overline{E}^0, \overline{E}^1, m+2k, r_E(m+2k+1)) = E^0 \cup E^1 = E$. Since there is a function a such that $\forall E \in \binom{[0,2k[}{k}$, $\forall x \in b(E)[E]$, $a(x) = E$, we know that all the sets $b(E)[E]$ are disjoint. Using Lemma 4, we conclude that $\kappa^{min}_{2k+log(k),q} \geq k$.

G Proof of Lemma 5

Lemma 5 (Lemma 5). *Let $h \in F(n,q)$ and $k := \kappa^{\min}(h)$. We have $\Omega(h) + k \leq \mathcal{L}^*(h)$.*

Proof. Let $h \in F(n,q)$, $k := \kappa^{\min}(h)$, $A := [0,q[$ and $m := n + k$. By definition of $\kappa^{min}(h)$ there exists $f \in F(m,q)$ and $w \in \Pi([m])$ such that f^w simulates h. Thus, $\mathrm{pr}_{[n]} \circ f^{w_n} \circ \dots f^{w_1} = h \circ \mathrm{pr}_{[n]}$. By definition, $\forall i \in [m]$, f^i does not update more than one coordinate. Then, $f^{w_1}, \dots, f^{w_m} \in F^*(m,q)$. Let us consider the set T of the coordinates of the trivial functions of h and let $w' \in \Pi([m] \setminus T)$ be an order respecting w which does update the coordinates of T. In other words, $\forall i, j \in \Pi([m] \setminus T)$, if $w(i) < w(j)$ then $w'(i) < w'(j)$. Let us prove that $f^{w'} = f^w$. Let h_i be a trivial coordinate function. Thus, $\forall x \in A^n, h_i(x) = x_i$. And for all $y \in A^k$ and $z := xy$, we have $(f^w(z))_i = h_i(x) = x_i = z_i$. Furthermore, since $w \in \Pi([m])$, the coordinate i is updated only one time in w in step $j := w(i)$. Thus, $f_{w_j} \circ f^{w_1,\dots,w_{j-1}}(z) = (f^w(z))_i = z_i$. Furthermore, since i is not updated before the step j, we have $(f^{w_1,\dots,w_{j-1}}(z))_i = z_i$. As a result, $f^{w_j} \circ f^{w_1,\dots,w_{j-1}} = f^{w_1,\dots,w_{j-1}}$, $f^w = f^{w_1,\dots,w_{j-1},w_j,\dots,w_m}$. Using the same method for all j such that h_{w_j} is trivial we get $f^w = f^{w'}$. The order w' is of size $\Omega(h) + k$. As a result, we have $\mathcal{L}(h|n+k) \leq \Omega(h) + k$. And by definition of $\mathcal{L}^*(h)$ we have $\mathcal{L}^*(h) \leq \mathcal{L}(h|n+k)$.

H Proof of Label 6

Lemma 6 (Label 6). *Let $h \in F(n,q)$ and $k := \kappa^{\min}(h)$. We have $\Omega(h)+k \leq \mathcal{L}^*(h)$.*

Proof. Let $\ell := \mathcal{L}^*(h)$, $m \leq n$ and $g^{(1)}, \dots, g^{(\ell)} \in F^*(m,q)$ such that $\mathrm{pr}_{[n]} \circ g^{(\ell)} \circ \dots \circ g^{(1)} = h \circ \mathrm{pr}_{[n]}$. We can assume that for all $i \in [\ell]$, the function $g^{(i)}$ updates one coordinate. Otherwise, g^i would be the identity function and we could remove it and have $\ell > \mathcal{L}^*(h)$ which is absurd. Let $u \in [m]^t$ such that, for all $i \in [\ell]$, u_i is the coordinate updated by $g^{(i)}$. Let $I = \{i_1, i_2, \dots, i_p\}$ with $i_1 < i_2 < \dots < i_p$ the set of steps where a coordinate of $[n]$ is updated for the last time in u. In other words, $\forall j \in [p]$, $u_{i_j} \in [n]$ and $\forall i \in [i_j+1, \ell], u_i \neq u_{i_j}$. We know that $\Omega(h) \leq p$ because, to compute h, each coordinate of a nontrivial function of h needs to be updated at least once. Indeed, if h_i is nontrivial, then $\exists x \in A^n, h_i(x) \neq x_i$. If i is not updated in u, then $\forall y \in A^{m-n}$, $\mathrm{pr}_i \circ g^{(\ell)} \circ \dots \circ g^{(1)}(xy) = x_i \neq h_i(x)$ and h is not computed. Let $k := \ell - p \leq \mathcal{L}^*(h) - \Omega(h)$. Let v be an oder which updates all the coordinates of $[n]$ not updated by $g^{(1)}, \dots, g^{\ell}$. Let $u' := (u_{i_1}, u_{i_2}, \dots, u_{i_p})$ be an order which updates the coordinates of $[n]$ updated by $g^{(1)}, \dots, g^{\ell}$ in the same order that u updates them for the last time. Let $w := uv \in \Pi([n])$ be a permutation of $[n]$. Let $J = \{j_1, j_2, \dots, j_k\} = [\ell] \setminus I$ with $j_1 < j_2 < \dots < j_k$. For all $i \in [n]$, let $\tilde{g}^{(i)} : A^m \to A$ be the function which return the value of the coordinate updated by $g^{(i)}$. In other words, $\tilde{g}^{(i)} = pr_{u_i} \circ g^{(i)}$. Let $y := (0)^{m-n}$ (a word of size $m-n$ containing only the letter 0). Let $c : A^n \to A^k$, such that, $\forall x \in A^n$, $\forall i \in [k], c_i(x) = \tilde{g}^{(j_i)} \circ g^{(j_i - 1)} \circ \dots \circ g^{(1)}(xy)$. Let us prove that c give a proper coloring of the confusion graph $G_{h,w}$. Let $x, x' \in A^n$,be neighbors in the confusion graph $G_{h,w}$. In other words, $h(x) \neq h(x')$ but $\exists i \in [n], h^{\{w_1,w_2,\dots,w_i\}}(x) = h^{\{w_1,w_2,\dots,w_i\}}(x')$. For the sake of contradiction, let us say that $c(x) = c(x')$. Let $z := xy$ and $z' := x'y$. Let $e = \max(\{i \in [n] \mid h^{\{w_1,\dots,w_i\}}(x) = h^{\{w_1,\dots,w_i\}}(x')\})$. Let $b \in [p]$ be the last step of u in which the coordinate w_e is updated. Let

$r := g^{(b)} \circ \cdots \circ g^{(1)}(z)$ and $r' := g^{(b)} \circ \cdots \circ g^{(1)}(z')$. Let us prove that $r = r'$. For all $a \in [m]$ not yet updated in u at step b:

- if $a \in [n]$ then $r_a = x_a = x'_a = r'_a$ because $h^{\{w_1,w_2,\ldots,w_e\}}(x) = h^{\{w_1,w_2,\ldots,w_e\}}(x')$ and thus $x_{[n]\setminus\{w_1,\ldots,w_e\}} = x'_{[n]\setminus\{w_1,\ldots,w_e\}}$.
- if $a \in [n+1, m]$ then $r_a = z_a = y_{a-n} = z'_a = r'_a$.

For all $a \in [m]$ already updated in u at step b:

- if $a \in [n]$ and a is updated for the last time then $r_a = h_a(x) = h_a(x') = r'_a$ because $h^{\{w_1,w_2,\ldots,w_i\}}(x) = h^{\{w_1,w_2,\ldots,w_i\}}(x')$ and thus $h(x)_{[n]\setminus\{w_1,\ldots,w_e\}} = h(x')_{[n]\setminus\{w_1,\ldots,w_e\}}$.
- Otherwise, let $d < b$ be the last step in u before b such that a is updated. In other words, $u_d = a$ and $\forall i \in]d, b[, u_i \neq a$. We have $r_a = \tilde{g}^{(d)} \circ g^{(d-1)} \circ \cdots \circ g^{(1)}(z) = c_d(x) = c_{d'}(x) = \tilde{g}^{(d)} \circ g^{(d-1)} \circ \cdots \circ g^{(1)}(z') = r'_a$.

Thus, we have $g^{(b)} \circ \cdots \circ g^{(1)}(z) = g^{(b)} \circ \cdots \circ g^{(1)}(z')$ and thus $g^{(\ell)} \circ \cdots \circ g^{(1)}(z) = g^{(\ell)} \circ \cdots \circ g^{(1)}(z')$. However, $pr_{[n]} \circ g^{(\ell)} \circ \cdots \circ g^{(1)}(z) = h(x) \neq h(x') \neq g^{(\ell)} \circ \cdots \circ g^{(1)}(z')$. This is absurd, so if two configurations x, x' are neighbors in the confusion graph then $c(x) \neq c(x')$. Thus, c gives a proper coloring of the confusion graph $G_{h,w}$ and it uses at most $q^k = q^{\ell-p} \leq q^{\mathcal{L}^*(h)-\Omega(h)}$ colors. As a result, $\kappa(h, w) \leq \mathcal{L}^*(h) - \Omega(h)$.

I Proof of Lemma 7

Lemma 7 (Lemma 7). *Let $G = ([n], E)$ be an undirected graph and let $s = \mathrm{Pw}(G)$. Then there are functions $c : [n] \to [s]$ and $u \in \Pi([n])$ with the following property. For all $i \in [n]$, we have either 1) for all k neighbor of i in G we have $u(i) \leq u(k)$ or 2) for all $j, k \in [n]$ with $c(i) = c(j)$, $u(i) < u(j)$ and k neighbor of j in G we have $u(i) \leq u(k)$.*

Proof. Let $G = ([n], E)$, $s = \mathrm{Pw}(G)$ and $X_1, \ldots, X_p$ a minimal path decomposition of G. In other words:

- $\forall i \in [n], \forall a, b \in [p]$ with $a < b$, if $i \in X_a$ and $i \in X_b$ then $\forall \ell \in [a, b]$, $i \in X_\ell$.
- $\forall (i, j) \in E, \exists a \in [p]$ such that $i \in X_a$ and $j \in X_a$.
- $\forall i \in [n], \exists a \in [p]$ such that $i \in X_a$.
- $\forall i \in [p],\ |X_i| \leq s + 1$.

For all $i \in [n]$, let $X(i) = \{X \in \{X_1, \ldots, X_p\} \mid i \in X\}$. Let $b : i \mapsto \min(\{j \mid X_j \in X(i)\})$ and $e : i \mapsto \max(\{j \mid X_j \in X(i)\})$. We will assume that $\forall \{a, b\} \subseteq [p]$, we do not have $X_a \subseteq X_b$ since otherwise we could remove X_a and still have a valid path decomposition of same size. As a result, for all $a \in [p]$, there exists $j \in X_a$ such that $e(j) = a$. Indeed, if that was not the case, we would have $X_a \subseteq X_{a+1}$. Let $u \in \Pi([n])$ be an order respecting e and $v \in \Pi([n])$ be an order respecting b. In other words, for all $\{i, j\} \subseteq [n]$, if $e(i) < e(j)$ then $u(i) < u(j)$ and if $b(i) < b(j)$ then $v(i) < v(j)$. For all j in $[n]$ taken in the order v, let us define $c(j)$ as such:

- If, in the set of images by c of $X_{b(j)}$ already defined, there are value of $[s]$ not used then let $c(j)$ be the minimal of them. More formally, if $\{c(k) \mid k \in X_{b(j)} \text{ and } v(k) < v(j)\} \neq [s]$ then let $c(j) := \min([s] \setminus \{c(k) \mid k \in X_{b(j)} \text{ and } v(k) < v(j)\})$.
- Otherwise, if there is $k \in X_{b(j)}$ such that $v(k) < v(j)$ and $e(k) = b(j)$, then let us consider the i which minimize $u(i)$. In other words, let us consider i such that $u(i) = min(\{u(k) \mid k \in X_{b(j)}\})$ and let $c(j) := c(i)$. We remark that since $\forall k \in X_{b(j)}, b(j) \leq e(k)$, we have $e(i) = b(j)$ (and not $e(i) < b(j)$).
- Otherwise, let $c(j) := 0$. In this case, we have $b(j) = e(j)$ because $\forall k \in X_{b(j)} \setminus \{j\},\ b(j) < e(k)$ and by hypothesis $\forall a \in [p]$, there exists $j \in X_a$ such that $e(j) = a$.

We remark that with this construction of c, $\forall a \in [p]$ there is at most one $\{i, j\} \subseteq X_a$ such that $c(i) = c(j)$ because $|X_a| \leq s + 1$. Let $i \in [n]$. First, let us consider the case where $c(i)$ is defined using the third case. Then, we have $b(i) = e(i)$ and thus $X(i) = \{X_{b(i)}\}$. By definition of a path decomposition, for all neighbor k of i in G, we have $k \in X_{b(i)}$. Furthermore, $\forall k \in X_{b(i)},\ e(i) = b(i) < e(k)$. Thus, $\forall k \in X_{b(i)},\ u(i) < u(k)$. As a result, i respects the condition 1) for all k neighbor of i in G we have $u(i) \leq u(k)$. Next, let us assume that $c(i)$ is not defined using the third case. Let $j \in [n]$ such that $c(i) = c(j)$ and $u(i) < u(j)$. Let us prove that for all k neighbor of j in G, $u(i) \leq u(k)$. First let us prove that we have $e(i) \leq b(j)$. For the sake of contradiction, let us say that $b(j) < e(i)$. There are 2 cases:

- $v(i) < v(j)$. Thus, $b(i) \leq b(j) < e(i)$. However,
  - Since, $b(i) \leq b(j) < e(i)$, we have $b(j) \in [b(i), e(i)]$ and thus $i \in X_{b(j)}$. Thus, there exists $k \in X_{b(j)}$ (k := i) such that $c(k) = c(j)$ and $v(k) < v(j)$. As a result, $c(j)$ cannot have been defined using the first case of the definition.
  - We have $b(j) < e(i)$. Furthermore, by hypothesis, we know that there is $k \in X_{b(j)}$, such that $e(k) = b(j)$. Thus, $e(k) < e(i)$ and then $u(k) < u(i)$. As a consequence, we have $c(i) = c(j)$ but

$u(i) \neq min(\{u(k)|k \in X_{b(j)}\})$. As a result, $c(j)$ cannot have been defined using the second case of the definition.
  - We have $b(j) < e(i) \leq e(j)$ and thus $b(j) \neq e(j)$. As a result, $c(j)$ cannot have been defined using the third case of the definition.
- $v(j) < v(i)$ Thus, $b(j) \leq b(i) < e(i)$. However,
  - Since, $b(j) \leq b(i) < e(i) \leq e(j)$, we have $b(i) \in [b(j), e(j)]$ and thus $j \in X_{b(i)}$. Thus, there is $k \in X_{b(i)}$ such that $c(k) = c(i)$ and $v(k) < v(i)$. As a result, $c(i)$ cannot have been defined using the first case of the definition.
  - We have $b(i) \leq b(j) < e(i) \leq e(j)$ and thus $b(i) < e(j)$. Furthermore, we know that there is $k \in X_{b(i)}$, such that $e(k) = b(j)$. Thus, $e(k) < e(j)$ and then $u(k) < u(j)$. As a consequence, we have $c(j) = c(i)$ but $u(j) \neq min(\{u(k)|k \in X_{b(i)}\})$. As a result, $c(i)$ cannot have been defined using the second case of the definition.
  - By hypothesis, $c(i)$ is not defined using the third case of the definition.

All cases raise a contradiction. As a result, we have $e(i) \leq b(j)$. There are 2 cases:

- If $e(i) < b(j)$, then let us prove that for each k neighbor of j in G, $u(i) < u(k)$ (and then $u(i) \leq u(k)$). Let k be a neighbor of j in G. Thus, $\exists a \in [p], k \in X_a$ and $j \in X_a$ and thus $b(j) \leq b(k)$. As a consequence, $e(i) < b(j) \leq b(k) \leq e(k)$ and then $u(i) < u(k)$.
- If $e(i) = b(j)$, then let us prove that for each k neighbor of j in G, $u(i) \leq u(k)$. Let k be a neighbor of j in G. If $b(j) < b(k)$, then like in the previous case, we have $e(i) = b(j) < b(k) \leq e(k)$ and thus $u(i) < u(k)$. Otherwise, let us prove that $k \in X_{e(i)}$. We have $b(k) \leq b(j) = e(i)$ and since k is a neighbor of j then $e(i) = b(j) \leq e(k)$. Thus, $e(i) \in [b(k), e(k)]$ and thus $k \in X_{e(i)}$. We remark that i and j are in $X_{b(j)} = X_{e(i)}$ and $c(i) = c(j)$. Then, the value $c(j)$ corresponds to the second cases in the definition and we have $\forall k \in X_{b(j)}$, $u(i) \leq u(k)$ (the only case where $u(i) = u(k)$ being when $i = k$).

J Proof of Lemma 8

Lemma 8 (Lemma 8). *Let $h \in F(n, q)$. Let $G = \mathrm{IG}^*(h)$. If we have $c : [n] \to [s]$ and $u \in \Pi([n])$ such that G, c, u have the same properties as in Lemma 7, then we have $\kappa(h, u) \leq s$.*

Proof. For all $j \in [n]$, let $v(j) := \{k \in [n] \mid (k,j) \in E\}$. For all $i \in [n]$, let $g_i : A^{|v(i)|} \to A$ such that $g_i \circ pr_{v(i)} = h_i$. In other words, $\forall x \in A^n$, $g_i(x_{v(i)}) = h_i(x)$. We know that such a function exists by definition of the interaction graph. Let $I_1, I_2, \ldots, I_s$ a partition of $[n]$ such that $\forall \ell \in [s],\ I_\ell := \{i \in [n] \mid c(i) = \ell\}$. For all $j \in [n]$, let $I(j) = I_{c(j)}$. Let $w = (n+1, n+2, \ldots, n+s, u(1), \ldots, u(n))$. Without loss of generality, let us say that u is the canonical update schedule $(1, \ldots, n)$. Let $f \in F(n+s, q)$ such that $\forall z = xy \in A^{n+s}$,

- $\forall i \in [n], f_i(z) = \begin{cases} h_i(x) \text{ if } \forall k \in v(i),\ u(i) \le u(k) \\ y_{c(i)} - \sum\limits_{j \in I(i) \text{ with } u(j) < u(i)} x_j - \sum\limits_{j \in I(i) \text{ with } u(i) < u(j)} h_j(x) \text{ otherwise} \end{cases}$.
- $\forall \ell \in [s], f_{n+\ell}(z) = \sum\limits_{j \in I_\ell} h_j(x)$.

Let $y' = f^{w_1,\ldots,w_s}(z)_{[n+1,n+s]} = (\sum\limits_{j \in I_1} h_j(x), \ldots, \sum\limits_{j \in I_s} h_j(x))$. Let us prove by induction that, being assumed that $[0] = \emptyset$, we have $\forall i \in [0,n]$, $f^{w_1,\ldots,w_{s+i}}(z)_{[n]} = h^{[i]}(x)$. First $f^{w_1,\ldots,w_s}(z)_{[n]} = (xy')_{[n]} = x = h^{\emptyset}(x)$. Next, let $i \in [n]$, let us suppose that $f^{w_1,\ldots,w_{s+i-1}}(z)_{[n]} = h^{[i-1]}(x)$. Let $z' = x'y' = f^{w_1,\ldots,w_{s+i-1}}(z)$. There are two cases. If $\forall k \in v(i),\ u(i) \le u(k)$ then $f_i(z') = h_i(x')$. In this case we have, $x'_{v(i)} = x_{v(i)}$. Thus, $f_i(z') = h_i(x)$. Otherwise, we have $\forall j \in I(i)$ with $u(i) < u(j), \forall k \in v(j), u(i) < u(k)$. In other words, for each such k we have $x'_k = x_k$ and thus $x'_{v(j)} = x_{v(j)}$. Let $\ell = c(i)$. We have $f^i(z') = y'_\ell - \sum\limits_{j \in I_\ell \text{ with } u(j) < u(i)} x'_j - \sum\limits_{j \in I_\ell \text{ with } u(i) < u(j)} h_j(x')$

We know that:

- $y'_\ell = \sum\limits_{j \in I_\ell} h_j(x)$.
- $\forall j \in I_\ell$ with $u(j) < u(i),\ x'_j = h_j(x)$
- $\forall j \in I_\ell$ with $u(i) \le u(j),\ h_j(x') = g_j(x'_{v(j)}) = g_j(x_{v(j)}) = h_j(x)$ (because $\forall k \in v(j),\ (k,j) \in E$, and then, by hypothesis of this lemma, $u(i) \le u(k)$).

Thus,

$$\begin{aligned}
f_i(z') &= y'_\ell - \sum_{j\in I_\ell \text{ with } u(j)<u(i)} x'_j - \sum_{j\in I_\ell \text{ with } u(i)<u(j)} h_j(x') \\
&= \sum_{j\in I_\ell} h_j(x) - \sum_{j\in I_\ell \text{ with } u(j)<u(i)} h_j(x) - \sum_{j\in I_\ell \text{ with } u(i)<u(j)} h_j(x) \\
&= \sum_{j\in I_\ell \text{ with } u(j)=u(i+1)} h_j(x) \\
&= h_i(x)
\end{aligned}$$

.

As a result, in both case we have $f_i(z') = h_i(x)$. Moreover, $f^{w_1,\dots,w_{s+i}}(z)_{[n]} = f^{s+i}(z')_{[n]} = h^{[i+1]}(x)$ and by induction $\forall i \in [0,n]$, $f^{w_1,\dots,w_{s+i}}(z)_{[n]} = h^{[i]}(x)$. In particular, we have $f^w(z)_{[n]} = h(x)$ and then $\mathrm{pr}_{[n]} \circ f^w = h \circ \mathrm{pr}_{[n]}$. Thus, $\kappa(h,w) \leq s$. As a result, $\kappa^{min}(h) \leq s$.