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Using the buzzword of 'mathematical practice' researchers in history and philosophy of mathematics have in the last twenty years been working to adjust the classic image of mathematics that is essentially a heritage from the beginning of the 19th century filtered through the discussion on the foundations of mathematics in the first half of the twentieth century.[1] Depending on the interest of the researchers, 'mathematical practice' has been given different interpretations, most of them trying to get at the 'non-formal' or less visible parts of doing mathematics, taking into account contextual and cognitive factors and focussing on the process, not only on the product. The methodological question is how observations and data from mathematical practices can be made useable for the history and philosophy of mathematics. More, this usability for doing history and philosophy should be able to display in what ways practices may be constitutive and essential to mathematics, instead of reducing them to mere scaffolding that can be removed once results and theory are ready.

This project is closely intertwined with another ambition, viz. to extract mathematics from its arcane isolation and reinsert it into the realm of the mundane. Or, as Hilbert wrote, quoting an "old French mathematician"[2] in the famous lecture on mathematical problems before the International Congress of Mathematicians in Paris (1900), "A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street." [Hilbert 1900, p. 437] This ambition can be circumscribed more precisely, we should be "able to handle the details of actual events, handle them formally, and in the first instance be informative about them" [Sacks 1995, I, p. 622], viz. in a direct way so that you could tell the man in the street and that he could think about it and understand that it checks out.

The unraveling of mathematical practice's rich but complex texture is reciprocated by the communicability of mathematics itself.

[1] See e.g. the conferences and publications organised by the Association for the Philosophy of Mathematical Practice (APMP), http://institucional.us.es/apmp/.

[2] He was recently identified as Gergonne, see. June Barrow-Green and Reinhard Siegmund-Schultze, "The first man on the street tracing a famous Hilbert quote (1900) back to Gergonne (1825), Historia mathematica 2016.

1 The sociology of everyday life and its application to doing mathematics

In a certain way, the concept 'mathematical practice' has a function similar to 'social action' that was material in the formation of sociology as a science in the beginning of the 20th century. The question, then and now, is: What are the elementary, observable facts that constitute the data for sociology and how can they be processed in traceable and repeatable ways so that they may be integrated into a systematic body of knowledge?[3] For our take on mathematical practice, we will make use of the work of Alfred Schütz (1899–1959), who, in a nutshell, combined Max Weber's sociology with Husserl's phenomenological method [Schütz 1932]. The phenomenological substratum of Schütz's framework will enable us to take the nitty-gritty details of individual practice as a *time-based activity* into account. Since the history of mathematics traditionally attaches great importance to the details in the work of individual mathematicians, such a footing is necessary. The Weberian part of Schütz's approach then will introduce the crucial distinction between a social action *as performed* by someone and an action *as observed* by someone. This distinction, as important for a good understanding of what 'mathematical practice' might be as it was for sociology in the 1920s, is elemental for understanding how mathematics, just as any other tightly woven network of signs, is subject to new readings, transpositions and appropriations.[4] Finally, the ultimate objective of both Weber's and Schütz's sociology is to get at 'structures' of the mundane world, i.e. by assembling and processing the data of the social world to slowly knit together a spectrum of typical behaviour.[5]

1.1 An introduction to Schütz's sociology of everyday life

Schütz's sociology starts from the premiss that action and its meaning must always be considered in the *flow of time*. This being embedded in time imprints a temporal directionality onto each action and onto each meaning of this action. An action can be future directed and driven by what Schütz calls an *in-order-to motive*, or it can be directed to the past, explained as a *because motive*. Typically, the in-order-to motive as 'meaning-context' comes before completion of an action, so to say, *a priori*, whereas the because motive can only serve as 'meaning-context' after completion, *a posteriori*. This distinction is of paramount importance in Schütz's theory, since it is key to a grading of

[3] See the famous essays on the status of sociology by Max Weber [Weber].

[4] See [Goldstein 1995] for a programmatic stance.

[5] Schütz's ideas have so far not been applied to the study (and history) of mathematics. Ethnomethodology, in many ways heir to Schütz's thinking, has known a lot of work on scientific practices in general, and on mathematical practices too, see Eric Livingston's work, e.g. [Livingston 1986] on mathematical proof or more recently [Livingston 2008] on other formal activities close to mathematical proof. Livingston studies mathematics in action (mainly proof procedures), as being observed by the ethnomethodologist, he does not focus on past, historical practices.

kinds of meaning according to their temporal constitution, and it is key to the subtle analysis of intersubjective understanding.

Consider the grading of meaning. The meaning-context is the immediate meaning that accompanies the performance of an action, from just before till immediately after it. The projection of the action to perform for the actor himself is a plan of action in which the in-order-to motive figures.[6] Once the action is performed, the actor can attach meaning to his now completed action by assigning because motives to it. The sequencing and interlocking of in-order-to and because motives that underlie an action performed constitute the complete meaning-context of an action.

For the observer of the action, two points of perspective, two basic modes of meaning-interpretation are possible [Schütz 1932, pp. 132–136]. Either one "run[s] over the acts that constituted the experience of the producer" in quasi-simultaneity, this constitutes the *subjective meaning* of an action. Or one considers "the already constituted meaning-context of the thing produced" disregarding the actual production, this is the *objective meaning* of an action.

In a next step, Schütz tries to spell out a declination of the production of meaning in social relationships. This declination essentially runs over two dimensions, time and space. The basic social relationship is the *face-to-face relationship* where two actors during the same time share the same space and assume a 'Thou-orientation' towards each other [Schütz 1932, pp. 163–172]. According to Schütz, all other social relationships derive in some way from the face-to-face-relationship. The special feature of this relationship is that a particular interlocking of motives takes place that forms the basis for intersubjective understanding. In a face-to-face interaction between person A and person B, the in-order-to motive of an action by A becomes the because motive of the next action of B, which in its turn, as in-order-to motive, becomes the because motive of A etc. To give an everyday example, in a conversation, if I say something, I want my conversational partner to respond to it, normally, s/he takes the cue and answers, eliciting thereby my next turn in the conversation etc. This *reciprocity of perspectives*, as Schütz calls it, is the deep structure that can, at least structurally or formally, vouch for for mutual understanding. Indeed, due to the continual reciprocal mapping of perspectives, misunderstandings, noise, disregard etc. can be detected quickly and course of action can be taken to remedy the problem. Although the ultimate motive/meaning of the partner in a face-to-face relationship remains beyond reach, the reciprocity of perspectives provides a piecewise display sensitive to the temporal and spatial situatedness that allows for the best possible approximation of his/her motive/meaning. In all other social relationships, such a display is not given, but by analogy, one forms ideal types of such other relationships that have their roots in abstractions and reductions from experiences in face-to-face relationships.

If the togetherness in space is not given, but time is still a shared dimension, Schütz speaks of relationships within the *World of Contemporaries*. These contemporaries can be ordered according to their 'reachability', i.e. people whom

[6] More on plans of action in [Schütz 1951].

one has had face-to-face relationships with in the past (either frequently, occasionally or randomly), people whom face-to-face relationships are possible with (because they live in the neighbourhood, go around in the same social circles etc.), people whom 'indirect' face-to-face relationships are possible with (through mail, telephone, chat, etc.), and people whom face-to-face relationships are very unlikely with though one knows of their existence through indirect contact (print, television, hearsay etc.). Although the Contemporaries are as such, at a given time, not 'reachable' in a face-to-face relationship, I can orient my actions towards them as if they were there. By abstraction from previous experience in a face-to-face relationship I may construct an *ideal type* of the Contempary that will be the addressee of a 'They-orientation' of my actions. Though the simultaneity of time experience is still given, a common display of this simultaneity is lacking.[7] Therefore, through the construction of an ideal type, I 'transcend' the absence of a shared spatial situatedness.

If I want to orient my actions to someone in the past, I have to 'transcend' not only the spatial, but also the temporal limitations. In social relationships with the *World of Predecessors*, as Schütz calls it, I have to reconstitute in some way both the simultaneity of experienced time and the togetherness in space. Whereas in social relationships with Contemporaries I may, at least theoretically, think of a virtual reciprocity between me and the Other, such a reciprocity must be absent in a relationship with a Predecessor. Said differently, my actions can influence the actions of Contemporaries, although I may perhaps never enter into a face-to-face relationship with them, but I can never influence the action of a Predecessor, but my action or the Predecessor may affect me.

Signs, or symbols (the word preferred by Schütz), may function as a kind of 'virtual' display of social intersubjectivity when temporal and/or spatial togetherness is not given. In a face-to-face relationship, I may point to a thing, show something etc. to solve a disruption in mutual understanding. In the absence of the 'manipulatory sphere' typical of face-to-face, I need a device that 'transcends' temporal and spatial limitations, viz. symbols referring to an absent meaning-context. "The symbolic reference, however, is characterized by the fact that it transcends the finite province of meaning of everyday life so that only the appresenting member [symbol] of the related pair pertains to it, whereas the appresented member has its reality in another finite province of meaning." [Schütz 1955, p. 343]

[7] A spatial display in 'delay', however, is possible throughout, say e.g. letters, videos.

1.2 An initial application of Schütz's scheme to doing mathematics[8]

Mathematical discourse, just as, more generally, scientific discourse, has its own finite province of meaning, its own 'subuniverse'. Specifically, the theoretical thinker "puts his physical existence [...] in brackets", it "stand[s] outside social relationships" [Schütz 1945, p. 253].[9] This does not mean that theory is not in the flow of time, nor that it does not partake in the production of meaning as part of intersubjective understanding. It does mean however, that theory has no level of face-to-face relationships provided for, it deals, from the very start, with a 'typified', artificial world. The reciprocal interlocking of motives, so characteristic for the face-to-face relationships that we often substitute it for the sequencing of time itself, is absent in the process of theorising. Instead, the interlocking reduces to a near empty time scheme and the media of indirect communication, constitutive for how the imprint of symbols display, bear the dispositions of how the 'void' slots are ordered. The embedding in time of theorising unfolds not as the pattern of motives, but reveals the fabric at work in the succession of schemes.

Further, theorising suffers from the 'paradox of communication'. Although social relationships do not enter into the theorising itself, are in fact based upon idealisations of relationships in the real life world, in order to pass beyond the stage of mere individual imagination, theoretical discourse has to be reverted into in common-sense discourse so that it may be communicated to Others, be it for discussion, for teaching, etc. As a consequence, theorising always and irrevocably has to rely on the use of symbols. Since the flow of symbols in the process of communicating theoretical ideas cannot be checked against their equivalent interlocking of motives, mutual understanding has to be warranted through the introduction of supplementary devices. These supplementary devices are in essence arbitrarily chosen, mostly fashioned after an item out of the stock knowledge of social reality, known and subscribed to by persons communicating. The devices are added to or superimposed on the structure in the temporal and spatial configuration of symbols. For instance, Aristotle's writings together with a neatly developed style of comment and discussion (called Scholasticism), that was part of everyday university teaching, functioned as joining piece for mediating and codifying science before the so-called Scientific Revolution. The Galilean experiment, in its descriptiveness, repeatability and as an event that is publicly accessible equally mobilises everyday resources of accountability.

The unreachability of a We relationship reinforces and amplifies, as some kind of 'side effect', the indirect communication upheld with the World of Pre-

[8]What follows is my own development of some of Schütz's ideas, they do not necessarily coincide with Schütz's own views on science and mathematics. Schütz's writing on science is sparse, but it seems that he upheld a differentiation between the theoretical attitude (characteristic for science) and everyday experience with a divide that was rather difficult to cross [Lynch 1993, pp. 133–58; 299–308]. In the development here, more forms of mediation, not to say communication, are provided for.

[9]The "bracketing" here alluded to is Husserl's bracketing, needed for the phenomenological reduction.

decessors and (unreachable) Contemporaries. Whereas I would never engage a conversation on the topic of the weather ten years ago, the circuitous communication of science often opens up a dialogue with older results. This is not only true of science, of mathematics, but in general of every discourse that constitutes a more or less autonomous finite province of meaning and relies exclusively on symbols for tying (ideal) social relationships, such as e.g. literature, religion etc.

2 Illustrating the approach

A theory, as detailed as may be, remains sterile as long as it does not stand the test of actual application. Although the two following examples only uncover part of what the ideas presented in this paper may help to clarify, they give some intuition about how to apply the method and ideas to concrete cases. The theory does not allow for mechanical application, the temporal sequencing and spatial configuration of an action are too singular for each and every case. However, by way of some important concepts and distinctions, set out above, and by way of worked-out examples, set out below, a way of seeing and studying things may be apprehended.

Even if the examples are in some way randomly chosen, they were selected because they each illustrate some particularly interesting and important topics in the study of mathematical practice. The first example revolves around the multiple (historical) reconstructions of a mathematical practice that are possible for a given mathematical object, and how this impinges quite directly on the interpretation of this object in a philosophical discourse. The second example then, looks at how the introduction of the computer quite intrinsically changes mathematical practice, introducing new incongruities that cannot be easily lifted by classic strategies of intersubjective understanding, though they may offer new opportunities to devise such strategies.

2.1 Example I: Descartes' Chiliagon

In René Descartes' *Meditations* (1641), one finds a famous passage on the difference between imagination and intellect, nicely illustrated by the contemplation of a mathematical object, the chiliagon or 1000-sided polygon:

> "And to render this quite clear, I remark, in the first place, the difference that subsists between imagination and pure intellection [or conception]. For example, when I imagine a triangle I not only conceive (intelligo) that it is a figure comprehended by three lines, but at the same time also I look upon (intueor) these three lines as present by the power and internal application of my mind (acie mentis), and this is what I call imagining. But if I desire to think of a chiliagon, I indeed rightly conceive that it is a figure composed of a thousand sides, as easily as I conceive that a triangle is a figure composed of only three sides; but I cannot imagine the thousand sides

> of a chiliagon as I do the three sides of a triangle, nor, so to speak, view them as present with the eyes of my mind. And although, in accordance with the habit I have of always imagining something when I think of corporeal things, it may happen that, in conceiving a chiliagon, I confusedly represent some figure to myself, yet it is quite evident that this is not a chiliagon, since it in no wise differs from that which I would represent to myself, if I were to think of a myriogon, or any other figure of many sides; nor would this representation be of any use in discovering and unfolding the properties that constitute the difference between a chiliagon and other polygons. [...] Thus I observe that a special effort of mind is necessary to the act of imagination, which is not required to conceiving or understanding (ad intelligendum); and this special exertion of mind clearly shows the difference between imagination and pure intellection (imaginatio et intellectio pura). (Descartes, Sixth Meditation, 1641)

Indeed, as we have shown in the preceding section, mathematical objects 'transcend' easily aspects of the structuring of the everyday real world as we may see and experience it with our senses. The chiliagon, though distinctly graspable with mathematical theorising, is indistinguishable from the myriogon (10.000-gon) with the eye.

From the side of the Contemporaries This passage had an impressive posterity, quoted nearly *verbatim* in Port Royal's *La Logique ou l'art de penser* (1662) and resurfacing in Leibniz's theory of *cognitio symbolica* (1686). Although the chiliagon is nowadays still often recycled as illustration of Descartes' distinction, it has become a dummy example that needs not be accounted for. They satisfy themselves with the objective meaning of the chiliagon, with the "already constituted meaning-context of the thing produced", a 1000-sided polygon that needs no further exegesis. In the years immediately following Descartes' text, however, commentators felt the need to take the chiliagon seriously, viz. as a mathematical object that is the outcome of a mathematical practice.[10] Thus they try to get at the subjective meaning of the chiliagon, they tried to interlock the mathematical object with "the acts that constituted the experience of the producer". Interestingly, this interlocking did not target a reconstruction of the sequence of motives that might have led Descartes to include the chiliagon as an example, but proposed a sequence of motives that inserts the chiliagon in a practice of contemporaries. In other words, the reconstruction of motives that makes up the subjective meaning takes up Descartes' example (past-oriented) and tries to lace it up in a sequence of motives integrating it into the everyday mathematical life of the 17th century (future oriented).

[10]The most elaborate commentator in this respect is Christian Wolff in his *Psychologia empirica* (1732), where the formula for polygonal numbers serves as a paradigmatic example for *cognitio symbolica*. We will not discuss this case here because the cascade of reverberations with texts preceding Wolff would lead us too far astray.

The most obvious candidate for yielding a context was Euclid's *Elements*[11] where a regular polygon with three sides is constructed in Book I Proposition 1 (I.1), a square in I.46 and a pentagon, a hexagon and other regular polygons in Book IV. Unfortunately, a 1000-sided polygon cannot be constructed with rule and compass according to Euclidean standards.[12] Instead of going for the construction of the chiliagon as subjective meaning, the commentators go for the demonstrable properties of the chiliagon. In the *Logique* of Port-Royal one finds in apposition to the *verbatim* of Descartes the following [Arnauld and Nicole 1662, p. 28–29]:

> Je ne puis donc proprement m'imaginer une figure de mille angles; puisque l'image que j'en voudrois peindre dans mon imagination, me representeroit toute autre figure d'un grand nombre d'angles aussitost que celle de mille angles; & neanmoins je la puis concevoir tres-clairement & tres distinctement; *puisque j'en puis demontrer toutes les proprietez*; comme que tous ses angles ensemble sont égaux à 1996 angles droits: & par consequent c'est autre chose de s'imaginer, & autre chose de concevoir. (our italics)

This addition reinserts the chiliagon into a recognisable mathematical practice, that of a mathematical object that has properties accessible through Euclidean style demonstrations. In Port-Royal's *Logique* only the proposition appears, not the chain of derivations that constitutes the proof, only a result that serves as an index to its embedding in a practice. This omission is emended by French authors on geometry in the decades afterwards. Bernard Lamy's *Les elemens de geometrie* (1685) for instance, Livre II, théorème 7 reads: "[Le polygone] se reduit en autant de triangles qu'elle a de cotez, moins deux."[13] After the proof, since a triangle's angles add up to 2 right angles (180 degrees), follows a corollary (an application): "Ainsi tous les angles d'un Chiliogone, c'est à dire, d'une figure de mille cotez sont égaux à 1996 angles droits, ce qu'on concoit clairement, quoy qu'il soit impossible d'imaginer nettement un Chiliogon." [Lamy 1685, p. 84] This consigns the chiliagon and its property to a distinct slot within a network of proof.

Other sequencings of the chiliagon in a network of proof are conceivable. In Jacques Rohault's posthumous *Elemens d'Euclide* (1690), the chiliagon now finds itself neatly indexed in the traditional Euclidean framework. As a remark to Corollaries I to IV in Book I, Proposition XXXII we have: "Ainsi les Angles d'un Dodecagone valent vingt angles droits; & ceux d'un Chiliagone, ou d'une Figure de mille Cotez, valent 1996 Angles droits." [Rohault 1690, p. 67] This

[11] Euclid's *Elements* is here considered as a contemporary discourse of the 17th century. Of course, the situation is more entangled than that, since the *Elements* of the 17th century are part of a succession of 'readings' and appropriations.

[12] Only regular polygons with a number of sides equal to 3, 5, powers of 2 and combinations thereof can be constructed with rule and compass (according to Euclid and excepting the cases C.F. Gauss discovered in 1801, with a number of sides equal to a $2^n + 1$ being prime). Since $1000 = 2^3.5^3$ one could only get by Euclidean construction to a 40-gon, but since the division of an angle by 5 is impossible, the construction stops there.

[13] This corresponds to Proclus' Corollary 2 to Euclid Book I, prop. 32.

location of the chiliagon is an interesting one. In an influential discussion on the status of mathematics as a science (in the Aristotelian sense) that ran over two centuries (16th-17th), proposition I.32[14] functioned as one of the paradigmatic examples that were ruminated over and again [Mancosu 1992]. The main problem with the proof of I.32 was that Euclid constructed an exterior angle to prove the theorem. According to some 16th-17th century critics, this exterior angle was accessory and not essential to the main idea of the proof. As such, I.32 did not classify as a demonstration in compliance with the sprit of Aristotelian proof schemata where cause and effect are 'naturally' sequenced without the irruption of accessory or contingent events. Although as 21st century people we can only speculate on the exact bearing of Rohault's insertion of the chiliagon at this particular location in the chain of Euclidean proofs, we must at least contemplate the possibility that, by polythetically running over the acts that constitute the experience of a 17th century mathematician, the reconstruction of the subjective meaning might yield an additional layer of meaning to the chiliagon. This extra meaning would be that just as Euclid had to take refuge to some demonstrative artifices that did not readily or not all translate to strict Aristotelian thinking, the intellect, with the help of mathematical artifices, can sometimes conceive distinctly of objects that the imagination cannot easily grasp in a 'natural' way or "view them as present with the eyes of my mind".

From the side of the Predecessors The integration after the fact of the chiliagon in a Euclidean way of doing mathematics seems to be typical to French scholars divulging parts of Descartes' heritage.[15] However, in Descartes' own writing this meaning never appears. Marveling at the inclusion of this idiosyncratic 'chiliagon' in Descartes' text, a modern day historian may want to look for other embeddings. More specifically, as a historian being oriented towards the past, we may want to assemble a sequence of because motives that may have underlain acts guiding Descartes to his example. Although a reconstruction of the subjective meaning of the chiliagon in Descartes' passage has to remain speculative (we cannot prove beyond reprieve that Descartes thought along the lines we will set out), as a minimum, we can make sure that the elements of our reconstruction all qualify as historically sound building blocks for our argument.

Many-sided polygons around 1600 appear in just one context, that of fitting polygons into or onto a circle to approximate a value for the ratio of the circumference of the circle to its radius (an approximation for π in modern terms). This method goes back to the Greek mathematician Archimedes. In the year 1593 two treatises appeared that tackled the determination of the ratio of the circumference of the circle to its radius. The first one, *Variorum de Rebus Mathematics Reponsorum Liber VII* was written by François Viète, the second one, *Ideae mathematicae pars prima*, by Adriaan van Roomen. Their approaches, though both essentially go back to Archimedes' inscription of polygons in a cir-

[14] To construct a rectilinear angle equal to a given rectilinear angle on a given straight line and at a point on it.

[15] As noted in footnote 10, in the line of reception by Leibniz, Wolff and other German thinkers, the chiliagon gets transposed rather into an algebraic discourse.

cle, differed widely in spirit and philosophy and some polemics between the two ensued. Descartes surely was aware of this learned dispute. As is well known, he had studied Viète's works closely and had borrowed the locution *mathesis universalis* from van Roomen's work. The accessibility of these sources to Descartes is thus historically secured.

François Viète, well-known as one of the fathers of modern algebra, explains Archimedes' method of approximation using polygons, and another method, also due to Archimedes, using a logarithmic spiral [Vieta, pp. 391–400]. Viète's own contribution is a transposition of the polygon method into proto-algebraic language. Using a geometric progression of successively inscribed polygons with the number of sides equal to the ascending powers of 2, Viète derives an algebraic expression figuring successive square roots[16] that may be used as an iterative computational algorithm.

The contemporary book by van Roomen, a medecine professor from Louvain (later Würzburg) well-versed in mathematics, is less succinct. Over some 100 pages van Roomen spreads out a mixture of theorems, list-like calculations and numerical tables. As Viète, van Roomen constructs geometric progressions, not only the progression of polygons with a number of sides equal to powers of two, but also progressions starting with the triangle, pentagon, and 15-gon and proceeding with their doubles. The theorems provide relationships and formulae, the lists spell out the calculations[17], and the tables gather the information on the length of the side and area of the polygons. All in all, the book constitutes some kind of preliminary stage of a table of chords (of sines, of trigonometric functions in modern terms) [Bockstaele 1992]. The best-known result of van Roomen's exploit was a calculation of π to 16 decimal digits, thereby surpassing Viète's calculation of 1579 that gave 10 digits.

The preface to van Roomen's book also contained a mathematical problem (an equation of the 42nd degree) that was presented as a challenge to the public of mathematics amateurs. In a booklet, *Ad problema quod omnibus mathematicis totius orbis construendum proposuit Adrianus Romanus* (1595), Viète took up the challenge and solved it, much to the surprise of van Roomen and the mathematical community. In passing, Viète scorns van Roomen's tedious computations in the postface and claims that "one has to exercise not torture the ingenuity of the studious".[18] A similar, but more elaborate castigation occurs in yet another booklet by Viète, the *Appolonius Gallus* (1600) [Vieta, p. 338]:

> I remind you of one thing, bright Belgian. Whoever can solve a square on command, will as easily expose the root of $\frac{141{,}421{,}356}{100{,}000{,}000}$ as he who will expose the root of $\frac{141{,}421{,}356{,}237{,}309{,}505}{100{,}000{,}000{,}000{,}000{,}000}$. This requires more labour, but not more art. [...] More even, I say that he misspent work and time, and know by experience that no use can arise from

[16] Viz., $\sqrt{2+\sqrt{2}}$ for an octogon, $\sqrt{2+\sqrt{2+\sqrt{2}}}$ for a hexadecogon etc.

[17] The calculation of the square roots is not given in detail, this is caught up with in [van Roomen 1602]. The method of calculating square and cube roots is explained in a manuscript published posthumously [Bosmans 1904].

[18] Original text: "ad [exercendum non] cruciandum studiosorum ingenia" [Vieta, p. 324]

it. This curse of ingenuity is to be abhorred and this superfluous art to be shamed.[19]

Van Roomen went as far as a polygon with 644.245.509.440 ($= 3.5.2^{32}$) sides, Viète had stopped at 3.932.160 ($= 3.5.2^{18}$). Viète's criticism mainly targets the extravagant augmentation of significant digits and polygon sides, not so much the idea of putting values into tables.[20] The increase in digits adds nothing to the mathematical essence of the problem, the theorems and the formulae, though it boosts the amount of computational labour as well as the probability of errors. Van Roomen 'solves' the problem of errors by including the lengthy calculations in the book and apparently considers the amount of digits a contribution in itself to the discourse of mathematics.

In the argument between Viète and van Roomen, different attitudes towards the transposition of ancient Greek knowledge, still clad in geometric figures, into an emerging numerical and symbolic language can be seen at work. For Viète the pairing of syllogistic or Euclidean style proofs with symbolic expressions and their rules of formations, where the proofs guarantee that the adept symbolic operations and manipulations really work, exhaust what can be done theoretically. The application of the symbolic machinery in the execution of computations is therewith assured and the machinery can e.g. be used for the production of a table of chords. However, the computations themselves need not be recorded or communicated. In van Roomen's work theorems and formulae also appear, but apparently constitute a preamble to well-ordered detail of computation and its results. Using a simile from classical rhetorics, Viète wants mathematics to restrict itself to the exposition of the building blocks of important topics, the application of the topics, viz. the composition of a particular discourse, is left to the individual practitioner. For van Roomen, the application is as much part of mathematics as its theory, spelling out the topics and their relation and proportions is at the heart of mathematics itself.[21]

If the chiliagon in Descartes' quote in some way echoes parts of the Viète-van Roomen polemics, than clearly, Descartes favours Viète's view on mathematics. Rather than loosing one's self in the undergrowth of exorbitant computational detail that only the hardened intellect can still follow and distinguish, an immediate sensual accessibility of the results should prevail, just as symbolic algebra 'naturally' and transparently translates geometric reasoning – at least in some cases – as Descartes would show in *La géométrie* (1637).

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[19] Original text: "Unum est de quo te moneam, candide Belga. Qui jussus quadratum resolvere, exhibet radicem $\frac{141{,}421{,}356}{100{,}000{,}000}$ tam perite facit, quam qui radicem exhibet $\frac{141{,}421{,}356{,}237{,}309{,}505}{100{,}000{,}000{,}000{,}000{,}000}$. Hic plus opera confert, sed non plus artificii. [...] Immo vero dicam eum opera & ocio abuti, gnarus nullam inde nasci utilitatem. Abhorrenda autem est ingeniorum crux, & vitanda $\mu\alpha\tau\alpha\iota o\tau\epsilon\xi\nu\iota\alpha$."

[20] Viète himself had published a *Canon Mathematicus* with tables in 1579.

[21] This difference in attitude is somewhat, though not exactly, like the divide between the mathematicians preferring to extend the theory by generalisation, and mathematicians preferring to accumulate numerical results for the exploration of mathematics. See [Bullynck 2009] and [Bullynck 2011].

Intermediary conclusions This chiliagon example has served to illustrate the simple fact that a mathematical object can be subject to different 'readings', can be endowed with a multiplicity of meanings. The subtle differentiation of motives and meanings may help to clear things up and avoid misunderstandings and confusion. Although different 'readings' are possible, it is important to stress that the objective meaning of the chiliagon remains 'transcendent' to our everyday sensual perception of the life world. In other words, the 'readings' do not impinge upon the philosophical stance one chooses to take on mathematics, be it platonic, formalist, realist, or whatever. The 'readings' are rather effects from the sequencing of motives that underlie the acts making up the experience, in short, of the constitution of subjective meaning. The mode and way of attribution of subjective meaning to the chiliagon, as we have seen, depend on the situatedness in time (predecessors/contemporaries/successors) and on the orientation and interlocking of motives of those who effect the 'reading' (future/past). The successors of Descartes quite literally had to make 'sense' of the exotic chiliagon and therefore sequenced classic parts of Euclid. Descartes then might have motivated by the tensions surrounding the numerical and symbolic transpositions of mathematics that were negotiated in the debate between Viète and van Roomen. It may be clear from these examples that mathematical understanding is not a historical constant, not even between contemporaries. The incorporation of a mathematical object into a temporal sequence and spatial pattern of actions that make the mathematical practice can yield a multitude of 'readings'. Since the objective meaning is necessarily 'transcendental', the best possible approximation to it might well be the superposition of multiple understandings as parts of open ended process.

2.2 Example II: The CIRCLE algorithm

The computer has had and is having an impact on mathematics, but it seems particularly difficult to get a qualitative feel for this impact. Some particular phenomena have been celebrated as typifying achievements of a computer powered mathematics, such as non-deterministic algorithms (Monte Carlo), cellular automata, automated theorem proving, highly detailed visualisations, mathematical experiments etc. All these phenomena, however, have origins that can be traced back to pre-computer times. Rather, they appear to be effects, symptoms, highly amplified reverberations within mathematics due the influence of the computer on mathematics, not essentially typifying phenomena. The characteristic imprint of the computer on mathematics is this amplification, this resonance of the computer with mathematics. The study of this subtle resonance, though often requiring some painstaking detail, will be key to our modern day understanding of mathematics. The following example considers some junctures of mathematics and computer that are vital to such an analysis.

In 1972 a notorious internal technical memo from MIT, HAKMEM, was issued. Though only written up in 1972 it was actually a roundup of programming 'hacks' from the late 1950s onwards. Item 149, due to Marvin Minsky, reads [HAKMEM 1972, p. 73]:

> Here is an elegant way to draw almost circles on a point-plotting display. CIRCLE ALGORITHM:
> NEW X= OLD X - ϵ * OLD Y
> NEW Y= OLD Y + ϵ * NEW(!) X
> This makes a very round ellipse centered at the origin with its size determined by the initial point. ϵ determines the angular velocity of the circulating point, and slightly affects the eccentricity. If ϵ is a power of 2, then we don't even need multiplication, let alone square roots, sines and cosines! The "circle" will be perfectly stable because the points soon become periodic.

The computer on which this algorithm was discovered was the PDP-1 (1959–1963), a minicomputer that for a generation of hackers was their entry ticket to the digital revolution. The PDP-1 was revolutionary in many respects. It allowed a relatively great number of people (especially at universities) a personal access to computing, instead of only indirect access to a mainframe where one had to pass through specially trained operators. It was also designed with the philosophy, partially dictated by practical and financial constraints, that the users should be able to configure and program the computer. Finally, it came with a cathode ray tube (CRT, an early 'screen') that could display sets of points: 'the PDP-1 cathode ray display is convenient means for the computer to talk to the operator" [DEC 1963, p. 36].

From the side of the user The average computer user of today (anno 2011) would typically have nothing to do with either the programming or the hardware of the computer. Such a user was non existent in 1959 when the PDP-1 was introduced at MIT. The PDP-1 was a forerunner of the personal computer in the sense that this was the first mass produced computer that an arbitrary individual, if he take the time and invest the effort, could program and configure for himself. Being a computer user anno 1960 was thus defined over access to and engagement with hardware and software. Nevertheless, for the purpose of the paper, it is interesting to look at the CIRCLE algorithm from a user's perspective anno 2011.

For the unknowing, naive user only the end result of the algorithm matters, the objective meaning as product of the process. The end result is an image on the CRT screen that looks close enough like a circle. Indeed, although the program strictly speaking does not generate a circular pattern but rather a very circle-like ellipse (as can be proven, see p. 16), the eye cannot discern the difference. The imagination is tricked into the illusion of a circle by the intellect's artfulness. Next to the spatial stability of the displayed dots needed for the circle illusion, temporal logic has to fit too. That is, the circle must display fast enough on the screen so that the eye cannot perceive that it is actually plotted point by point, neither should it (or points of it) disappear. This was a tricky point for this early display of 1024×1024 points. On the PDP-1, "the plotting of a single point require[d] 50 microseconds" [DEC 1960,

p. 8], the processor's speed was 5 microseconds.[22] Further, the critical flicker frequency (CFF) of the human eye is the frequency at which a flickering light ceases to flicker and appears as a continuous light. Depending on the luminosity of the source the CFF is between 10 and 50 Hertz [Webvision 2011, Part VIII]. Putting these facts together, the plotting of the circle on a PDP-1 may not take longer than 100.000 microseconds, else the illusion breaks down altogether. Ben Gurley, who was the main engineer of the PDP-1, had devised an 8-line-program for plotting lines on the CRT display that took about some 25.000 microseconds [DEC 1960, p. 9]. Visual display on a PDP-1 that could convince the eye thus operated quite near the critical zone around the threshold of 100.000 microseconds. Minsky's circle algorithm, though some commands longer than Gurley's line algorithm still remains below that threshold, though by a slight difference only. These technical details hidden behind the interface that lets the human user interact with the computer are constitutive for the objective meaning realised by the user, for his impression that s/he sees a circle on the screen. But the details do not matter to him/her as long as everything works out fine. The scaffolding may be removed.

From the side of the programmer A PDP-1 user was inevitably a programmer. It meant learning to use the machine's code, studying some subroutines, all duly explained in the famed PDP Handbooks, but also engaging with the technical details, the input/output problematics of the peripheral devices. The CIRCLE algorithm is the outcome of this dynamic process that engaged the user with the PDP-1. Through luck, we even have the testimony of how Minsky arrived upon the small program [HAKMEM 1972, p. 73]:

> The circle algorithm was invented by mistake when I tried to save one register in a display hack! Ben Gurley had an amazing display hack using only about six or seven instructions, and it was a great wonder. But it was basically line-oriented. It occurred to me that it would be exciting to have curves, and I was trying to get a curve display hack with minimal instructions.

Indeed, a program of more than 50 lines of code that had to be iterated some 500 times already came frightfully close to an execution time surpassing the threshold of 0.1 seconds. A short program for displaying a circle was therefore a desideratum. Moreover, since memory was at a premium and it was time-consuming to load a value from the core memory, Minsky used only one fast storage register instead of one fast and one slow storage register. Economising this memory register, Minsky used only the accumulator (AC), input/output (IO) and memory registers in the CIRCLE loop to save cycle time. The algorithm is not bad, displaying one point of Minsky's pseudo-circle takes 215

[22] Remark also that "the program must always operate faster than a device" [DEC 1960, p. 7], else tricky synchronisation issues would pop up. *A fortiori*, the program must also operate faster than its user, else what would be the point of using a computer?

microseconds, or with bit shifts, 175 microseconds.[23] This is about three to four times as long as it took to display a point in Gurley's line algorithm (55 microseconds per point), but still below the threshold.

The three registers (AC, IO and memory) determine quite a lot of the art of programming the CIRCLE algorithm. Of course, in the community of hackers, the search for minimal programs for executing something was slowly developed into a sport, even into an art, an aesthetic in its own right, but one should not forget, this art was born out of necessity. The PDP-1 was a serial computer, therefore, all computational work was done in the accumulator. All machine orders for doing arithmetic are thus exclusively assigned to the AC register. The memory register then, only allows for transfer, to or from the AC register or IO register. The IO register is a very peculiar one, on the one hand, it is the register that communicates with input and output devices, on the other hand it is some kind of extension of the AC register, to which it is contiguous. Because of the contiguity, the AC and IO register can, for some actions, be treated as one sole register, viz. for binary shift and rotations, for 'bit-fiddling' as it as often called. Finally, the output on the cathode ray tube is triggered by values in the AC and IO register (functioning as X and Y values of a point). The writing of the CIRCLE program happens under this conditioning: Three registers, each with their own 'vocabulary' of orders where, starting from OLD X and Y, one needs to compute NEW X and Y. Programming with the limited resources of the PDP-1, aggravated by the temporal limitation due to the visual flicker threshold, the immediate consequence is a lot of bit-fiddling transfers between AC and IO registers to save both memory and cycle time. This economy, together with the specific operational vocabulary of commands associated with each of the 3 registers, turns programming the display on a PDP-1 partially into a combinatorial exercise.

In this combinatorics the OLD X gets overwritten by a NEW X, a bit like one forgets to transcribe or incorrectly transcribes a term of an equation after having transformed the equation in some way. In this particular case, it is a 'creative' error that discovers a new unthought of possibility. In contrast to the sequence of acts that make up the handling of a symbolic equation and where each step can be directly tracked following the trail of the preceding and succeeding steps, working on a program has a different temporal sequentiality. Manipulating an algebraic equation is goal-directed, it is gouverned by a sequence of in-order-to motives that would lead to a product, a solution or a simplest form. Essentially, this process is linear, even if one is sidetracked in some trial-and-error attempts, or if some calculations or graphs are introduced for heuristic purposes, or if it is the outcome of a collaboration, etc. The stages can always be merged into a linear proceeding.[24] The result of handling an equation is a last or final stage

[23]When working with numbers in binary notation, one can replace multiplication by a power of two with binary shifts ('shifting' the digits n places to the left or right). The binary shifts take only about a fourth of the time of a multiplication.

[24] In fact, the jotter of intermediary scribbling and calculations as amenable resources for doing mathematics that are always incorporable into practice may be typical of paper mathematics. On other media such as clay, sand etc. the erasability spells out other conditionings,

in the succession of symbol manipulations. In a computer program, the result is not on that same level of symbols, not in the same medium, it lies outside the codification of the program itself. The result is the process regimented by the program that is running on the computer. It cannot be linearly integrated into the process of programming, although compiling and executing the program is an important part of programming since it provides feedback, insight on what the program does or does not do when running.[25] Thus, may the first attempt at writing a program bear some kind of family resemblance to the mathematician's symbol manipulation, its writing-debugging feedback pattern with the program running on a computer distinguishes it from classic mathematics. A recognisable interlocking of in-order-to motives, programming to achieve some result, with because motives, picked up from the procedural output of the program, characterises programming a computer.

Nevertheless, programming can also be considered as a proper part of more classical mathematics. One can study a program and ask whether it stops, whether it provably produces an output etc. The hackers at MIT did exactly that. Knowing that Minsky's CIRCLE algorithm does not really create a real circle, but a close approximation, they set out to study the properties of the algorithm and its mathematics. Item 151 in HAKMEM shows that the program relies on a Chebyshev recurrence, item 152 derives the eccentricity of the ellipse the algorithm produces [HAKMEM 1972, p. 73]. The translation of the sequential algorithm into formulae allow for the CIRCLE algorithm to be reincorporated in a more traditional mathematical discourse, it allows for a renewed perspective and meaning of the algorithm. Thus, parallel to the verification the running of the program allows, a verification by a mathematician is often, though not always possible. These verifications have, however, a very different character and do not necessarily coincide.

From the side of the computer The previous reflections bring us to the side of the computer. Although the computer is not a person in the normal sense, both the user and the programmer entertain a face-to-face relationship with the computer. Time and space are shared and at least the user or programmer is oriented towards the computer. Given that the computer responds to buttons pushed, points on screens clicked etc., one may say that the computer is oriented towards the input of its users as well. Also, as the current metaphors indicate, the computer 'reads' the code, 'interprets' and 'executes' it, in other words, the computer runs over the lines of the program and performs a synthesis of the hardware acts incumbent to the orders, it constitutes its own experience, viz. 'the program running'. The subjective meaning of 'the program running', if we may call it thus, remains in principle unreachable, hidden for both user and programmer, with the possible exception of some intermittent signals. Only the end result of the program (its objective meaning) is actually readily accessible

e.g. once erased, you cannot go back.

[25] In the earliest days of computer programming (1946–1960), the feedback ran over halt orders and visual or auditive signalling.

to us.

This brings us to an important point: there is no reciprocity of perspectives between the user and the computer. More precisely, may user and computer share space and time, their experiences are only simultaneous in objective time (Bergson's *temps*), they are not simultaneous in experienced, subjective time (Bergon's *durée*). On the part of the programmer, s/he works on the level of minimizing the number of symbols (basic operations) taking into consideration hard- and software restraints. The result is a small program, i.e. a sequence of symbols. This can be collaboratively obtained in a We relationship with other programmers and face-to-face with the PDP-1, all within a time-frame of some hours. The programmer has an understanding of the program (a subjective meaning) based upon this social experience and sequencing of motives. The execution of the program by the computer, however, actuates a quite different process that shapes yet another meaning.

As we explained in section 2, the endowment of meaning by contemporaries is different according to the synthesis of their polythetic acts. This synthesis though, is in principle approximately reconstructible because of the reproducibility of the interlocking of motives after the model experienced in the reciprocal interlocking of motives in face-to-face relationships. These face-to-face relationships between human beings rely on the continual perception of the Other by the senses, particularly the eye and the ear, seeing and hearing. As we mentioned earlier (p. 14) the threshold for the illusion of continuity in vision lies around 10 to 50 Hz. The "boundary between intersyllabic gaps and listener-detected pauses" that guarantees the impression of a flow of speech or its interruption and that is material for conversational turn taking lies at 200 milliseconds [Brady 1965], the duration of meaningful speech units (phonemes) lies between 300 milliseconds (vowels) and 50 milliseconds (stops) [Fletcher 1953, pp. 58–67]. These thresholds determine directly the floor level of human interaction, of reciprocal social relationships. Visual or auditory interaction below this level lies beyond human interpretation. This is why we cannot get at what is going on inside somebody else's mind, except when it is tried to convey what is going on in a mind through social interaction, trough a reciprocal mapping of perspectives (at, say a frame rate of 10 per second and within a flow of sounds and interruptions that is amenable to conversational organisation).

In a face-to-face relationship with a computer, such reciprocal mapping is not possible, because the 'central processing cycles' do not match. Already the ENIAC (1946) ran on a program pulse of 5000 Hz, the PDP-1 at 200.000 Hz and ever since processor's clock rates have only increased. Although interaction between man and computer is possible via an interface (this is what the revolution of the personal computer is all about), a human being cannot "run over the acts that constituted the experience" of the computer. Only under special circonstances is a synchronisation of 'clock cycles' sometimes effected, e.g. for debugging, the execution of a program is slowed down.[26] In general,

[26] The LINC computer (1962) had even a special knob to diminish the processor's speed so the user could follow the processing of the data in 'real-time'.

there is an incongruity between the computer's actions and the human actions that cannot be remediated. An effect of this e.g., is the so-called unsurveyability of computer-assisted mathematical proofs.

3 Discussion

work-in-progress

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