Control from an Interior Hypersurface
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Abstract

We consider a compact Riemannian manifold $M$ (possibly with boundary) and $\Sigma \subset M \setminus \partial M$ an interior hypersurface (possibly with boundary). We study observation and control from $\Sigma$ for both the wave and heat equations. For the wave equation, we prove controllability from $\Sigma$ in time $T$ under the assumption ($TGCC$) that all generalized bicharacteristics intersect $\Sigma$ transversally in the time interval $(0, T)$. For the heat equation we prove unconditional controllability from $\Sigma$. As a result, we obtain uniform lower bounds for the Cauchy data of Laplace eigenfunctions on $\Sigma$ under $TGCC$ and unconditional exponential lower bounds on such Cauchy data.

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1 Introduction

Let $(M, g)$ be a compact $n$ dimensional Riemannian manifold possibly with boundary $\partial M$ and denote $\Delta_g$ the (non-positive) Laplace-Beltrami operator on $M$. We study the observability and controllability questions from interior hypersurfaces in $M$.

To motivate the more involved developments in control theory, let us start by stating (slightly informally) the counterpart of our observability/controllability results for lower bounds for eigenfunctions, i.e. solutions to

$$(-\Delta_g - \lambda^2)\phi = 0, \quad \phi|_{\partial M} = 0. \quad (1.1)$$

For more precision, see Section 1.3.
**Theorem 1.1.** Assume $M$ is connected and let $\Sigma$ be a nonempty interior hypersurface. Then there exists $c > 0$ so that for all $\lambda \geq 0$ and $\phi \in L^2(M)$ solutions to (1.1), we have

$$
\|\phi\|_{L^2(\Sigma)} + \|\partial_\nu \phi\|_{L^2(\Sigma)} \geq ce^{-\lambda/c}\|\phi\|_{L^2(M)}.
$$

(1.2)

Furthermore, if we assume that all generalized geodesics of some finite length cross $\Sigma$ transversally, then there is $c > 0$ so that for all $\lambda \geq 0$ and $\phi \in L^2(M)$ solutions to (1.1), we have

$$
\|\phi\|_{L^2(\Sigma)} + \|\langle \lambda^{-1} \partial_\nu \phi\rangle\|_{L^2(\Sigma)} \geq c\|\phi\|_{L^2(M)}.
$$

(1.3)

Here, we write $\langle \lambda \rangle := (1+|\lambda|^2)^{1/2}$. Generalized geodesics are usual geodesics of $g$ inside $\text{Int}(M)$, and reflect on $\partial M$ according to laws of geometric optics (see below). As far as the authors are aware (1.2) is the first general lower bound to appear for restrictions of Laplace eigenfunctions to hypersurfaces and (1.3) is the first uniform lower bound for such restrictions without either taking a full density subsequence of eigenfunctions or imposing restrictive assumptions on $M$. We will prove Theorem 1.1 in the process of studying controllability for the heat and wave equations from interior hypersurfaces. Because of this, we postpone further discussion of Theorem 1.1 (including optimality of (1.2) and (1.3)) to Section 1.3.

We define interior hypersurfaces as follows:

**Definition 1.2.** We say that $\Sigma$ is a hypersurface of $M$ if there is $\Sigma_0$ a compact embedded submanifold of $M$ of dimension $n-1$, possibly with boundary, such that $\Sigma$ is the closure of an open subset of $\Sigma_0$. The manifold $\Sigma_0$ shall be referred to as an extension of $\Sigma$.

- We say that $\Sigma$ is an interior hypersurface if moreover $\Sigma \subset \text{Int}(\Sigma_0) \subset \text{Int}(M)$.

- We say that $\Sigma$ is a compact interior hypersurface if it is a compact embedded submanifold of $\text{Int}(M)$ of dimension $n-1$, without boundary.

- We say that $\Sigma$ is cooriented if $\Sigma_0$ is (i.e. the normal bundle $T_{\Sigma_0}M/T\Sigma_0$ is an orientable vector bundle).

If not mentioned, all hypersurfaces considered in this paper are assumed to be coorientable.

In the particular case $\Sigma$ is a compact interior hypersurface, then it is an interior hypersurface with $\Sigma_0 = \Sigma$. Since $M$ is endowed with a Riemannian structure, the coorientability assumption is equivalent to that of having a smooth global vector field $\partial_\nu$ normal to $\text{Int}(\Sigma_0)$. Note that the coorientability condition can be slightly relaxed, see the discussion in Section 1.5 below.

Given an interior hypersurface $\Sigma$, the main goal of this paper is to study the controllability of some evolution equations with a control force of the form

$$
f_0 \delta_\Sigma + f_1 \delta_\Sigma',
$$

(1.4)

where the distributions $f_0 \delta_\Sigma$ and $f_1 \delta_\Sigma'$ are defined by

$$
\langle f_0 \delta_\Sigma, \phi \rangle = \int_\Sigma f_0 \phi d\sigma, \quad \langle f_1 \delta_\Sigma', \phi \rangle = -\int_\Sigma f_1 \partial_\nu \phi d\sigma.
$$

(1.5)

In this expression $\sigma$ denotes the Riemannian surface measure on $\Sigma$ induced by the metric $g$ on $M$. This contrasts with usual control problems for PDEs, for which the control function appears in the equation:

- either as a localized right handside (distributed or internal control) $\mathbb{1}_\omega f$, where $\omega$ is an open subset of $M$, and typically, the control function $f$ is in $L^2((0, T) \times \omega)$;

- or, in case $\partial M \neq \emptyset$, as a localized boundary term, e.g. under the form $u|_{\partial M} = \mathbb{1}_\Gamma f$, where $\Gamma$ is an open subset of $\partial M$, and typically, the control function $f$ is in $L^2((0, T) \times \Gamma)$ (here, $u$ denotes the function to be controlled).

---

1In case $M$ is oriented, note that $\Sigma_0$ is cooriented if it is oriented. However, if $M$ is not orientable, $\Sigma_0$ might be orientable without being coorientable, and vice versa.
Concerning the wave equation, the main result is the Bardos-Lebeau-Rauch Theorem [BLR92, BG97] providing a necessary and sufficient condition for the exact controllability with such control forces (see also e.g. [DL09, LL16, LLTT16] for recent developments). Concerning the heat equation, the question of null-controllability with internal or boundary control was solved independently by Lebeau-Robbiano [LR95] and Fursikov-Imanuvilov [FI96]. The aim of the present paper is threefold:

- Formulating a well-posedness result as well as an analogue of the Bardos-Lebeau-Rauch Theorem, for the wave equation with control like (1.4) (see Section 1.1);
- Formulating an analogue of the Lebeau-Robbiano-Fursikov-Imanuvilov Theorem for the heat equation with control like (1.4) (see Section 1.2);
- Formulating general lower bounds for restrictions on $\Sigma$ of eigenfunctions on $M$ (see Theorem 1.1 above and Section 1.3). These are analogues of the observability inequalities used to prove the above controllability statements and are of their own interest.

### 1.1 Controllability for the Wave Equation

In this section, we state our main result concerning the wave equation controlled by an interior hypersurface $\Sigma$, namely

\[
\begin{align*}
\Box v &= f_0 \delta_\Sigma + f_1 \delta_\Sigma^c & \text{on } (0, T) \times \text{Int}(M), \\
v &= 0 & \text{on } (0, T) \times \partial M, \\
(v, \partial_t v)_{t=0} &= (v_0, v_1) & \text{in } \text{Int}(M).
\end{align*}
\]

(1.6)

where $\Box$ denotes the D’Alembert operator on $\mathbb{R} \times M$,

\[
\Box = \partial_t^2 - \Delta_g.
\]

Before considering the control problem, we need to investigate conditions on $f_0, f_1$ under which the Cauchy problem of (1.6) is well-posed. Both the well-posedness and the control statements require the introduction of some geometric/microlocal definitions.

For a pseudodifferential operator $P$ on $\mathbb{R} \times M$, we write

\[
\text{Char}(P) = \{ q \in T^*(\mathbb{R} \times M) \setminus 0 \mid \sigma(P)(q) = 0 \}
\]

and $\sigma(P)$ denotes the principal symbol of $P$. In particular, writing $|\xi|_g = \sqrt{g(\xi, \xi)}$ the Riemannian norm of a cotangent vector, we are interested in

\[
\sigma(\Box)(t, x, \tau, \xi) = -\tau^2 + |\xi|_g^2, \quad \text{Char}(\Box) = \{ (t, x, \tau, \xi) \in T^*(\mathbb{R} \times M) \setminus 0 \mid |\xi|_g^2 = \tau^2 \}.
\]

Next, we define the glancing and the elliptic sets for $\Box$ above $\Sigma$ as

\[
\mathcal{G} = \text{Char}(\Box) \cap \tau(T^*(\mathbb{R} \times \text{Int}(\Sigma))), \quad \mathcal{G}^c = \iota^{-1}(\mathcal{G}), \\
\mathcal{E} = \{ q \in T^*(\mathbb{R} \times M) \setminus 0 \mid \sigma(\Box)(q) > 0 \} \cap \tau(T^*(\mathbb{R} \times \text{Int}(\Sigma))), \quad \mathcal{E}^c = \iota^{-1}(\mathcal{E}),
\]

(1.7)

where

\[
\iota : T^*(\mathbb{R} \times \text{Int}(\Sigma)) \hookrightarrow T^*(\mathbb{R} \times M)
\]

is the inclusion map. A more explicit expression of these sets in normal coordinates is given in Section 2.3 below.

Roughly speaking, the elliptic set $\mathcal{E}$ (resp. $\mathcal{E}^c$) consists in points $(t, x, \tau, \xi)$ in the whole phase space (resp. in tangential phase space to $\Sigma$) such that $x \in \text{Int}(\Sigma)$ in which no “ray of optics” for $\Box$ lives. The glancing set $\mathcal{G}$ (resp. $\mathcal{G}^c$) consists in points $(t, x, \tau, \xi)$ in the whole phase space (resp. in tangential phase space to $\Sigma$) such that $x \in \text{Int}(\Sigma)$, through which “rays of optics” for $\Box$ may pass tangentially. The complement of $\mathcal{G} \cup \mathcal{E}$ in the characteristic set of $\Box$ above $\mathbb{R} \times \text{Int}(\Sigma)$ is the set of point through which “rays of optics” for $\Box$ may pass transversally.
Definition 1.3. We say that \((\Sigma, T)\) satisfies the transverse geometric control condition \((T\, \text{GCC})\) if every generalized bicharacteristic of \(\Box\) intersects \(T^*(0, T) \times \text{Int}(\Sigma)\). We say that \(\Sigma\) satisfies \(T\, \text{GCC}\) if \((\Sigma, T)\) does for some \(T > 0\).

Definition 1.3 roughly says that \(T\, \text{GCC}\) is satisfied if every ray of geometric optics intersects \(\text{Int}(\Sigma)\) in the time interval \((0, T)\) at a transversal point, i.e. a non-tangential point. In case \(\partial M = \emptyset\), "generalized bicharacteristics" are only bicharacteristics of \(\Box\) and project on geodesics on \(M\) (see e.g. [DLRL14, Section 2.2]). For a precise definition of generalized bicharacteristics in case \(\partial M = \emptyset\) (and the geodesics of \(M\) have no contact of infinite order with \(\partial M\)), we refer to [MS78, Section 3], [Hör85, Chapter 24], or [LLTT16, Section 1.3.1]. For a simple example of a compact manifold \(M\) and a compact interior hypersurface \(\Sigma\) satisfying \(T\, \text{GCC}\), see Figure 1.

With these definitions in hand, our well-posedness result may be stated as follows.

Theorem 1.4. For all \((v_0, v_1) \in L^2(M) \times H^{-1}(M)\) and for all \(f_0 \in H^1_{\text{comp}}(\mathbb{R}_+^* \times \text{Int}(\Sigma))\) and \(f_1 \in L^2_{\text{comp}}(\mathbb{R}_+^* \times \text{Int}(\Sigma))\) such that
\[
\text{WF}^{-1}(f_0), \text{WF}^1(f_1) \cap G^\Sigma = \emptyset, \quad (1.9)
\]
there exists a unique \(v \in L^2_{\text{loc}}(\mathbb{R}_+^*; L^2(M))\) solution of (1.6).

With this well-posedness result and the definition of \(T\, \text{GCC}\), we now give a sufficient condition for the null-controllability of (1.6) from \(\Sigma\).

Theorem 1.5. Assume that the geodesics of \(M\) have no contact of infinite order with \(\partial M\) and that \((\Sigma, T)\) satisfies \(T\, \text{GCC}\). Then for any \((v_0, v_1) \in L^2(M) \times H^{-1}(M)\) there exist \((f_0, f_1) \in H^1_{\text{comp}}((0, T) \times \text{Int}(\Sigma)) \times L^2_{\text{comp}}((0, T) \times \text{Int}(\Sigma))\) with
\[
\text{WF}(f_0), \text{WF}(f_1) \cap (G^\Sigma \cup E^\Sigma) = \emptyset, \quad (1.10)
\]
so that the solution to (1.6) has \( v \equiv 0 \) for \( t \geq T \).

Here, WF stands for the usual \( C^\infty \) wavefront set. Theorem 1.5 follows from an observability inequality given in Theorem 4.1 below.

Of course, it is classical to check that a necessary condition for controllability to hold is that all generalized bicharacteristics intersect \( T_{0,T} \times \Sigma (\mathbb{R} \times M) \). As for the well-posedness problem, the issue of rays touching \( \mathbb{R} \times \Sigma \) only at points of \( g^Z \) is very subtle, and will be addressed in future work. See the discussion in Section 1.4 below.

### 1.2 Controllability from a hypersurface for the heat equation

We next consider the controllability of the heat equation from a hypersurface, namely

\[
\begin{cases}
(\partial_t - \Delta) v = f_0 \delta_Z + f_1 \delta_Z' & \text{on } (0, T) \times \operatorname{Int}(M), \\
v = 0 & \text{on } (0, T) \times \partial M, \\
v|_{t=0} = v_0 & \text{in } \operatorname{Int}(M).
\end{cases}
\]  

(1.10)

Well-posedness in the sense of transposition follows from the standard parabolic estimates, and is proved in Section 5.1. We only state a null-controllability result for (1.10).

**Theorem 1.6.** Suppose \( M \) is connected and \( \Sigma \) is any nonempty interior hypersurface. Then there exist \( C, c > 0 \) such that for all \( T > 0 \) and all \( v_0 \in H^{-1}(M) \), there exist \( f_0, f_1 \in L^2((0, T) \times \Sigma) \) with

\[
\|f_0\|_{L^2((0,T) \times \Sigma)} + \|f_1\|_{L^2((0,T) \times \Sigma)} \leq Ce^{\frac{c}{2}} \|v_0\|_{H^{-1}(M)},
\]

such that the solution \( v \) of (1.10) satisfies \( v|_{t=T} = 0 \).

Note that we also provide an estimate of the cost of the control as \( T \to 0^+ \), similar to the one in case of internal/boundary control [F96, Mil10].

### 1.3 Eigenfunction Restriction Bounds

As usual, the above two control results (or rather, the equivalent observability estimates) have related implication concerning eigenfunctions, stated in Theorem 1.1 above. We now formulate these results under the (stronger) form of resolvent estimates. Below, we write \( \langle \lambda \rangle := (1 + |\lambda|^2)^{1/2} \).

**Theorem 1.7** (Universal lower bound for eigenfunctions). Assume \( M \) is connected and \( \Sigma \) is a nonempty interior hypersurface. Then there exist \( C, c > 0 \) so that for all \( \lambda \geq 0 \) and all \( u \in H^2(M) \cap H^1_0(M) \) we have

\[
\|u\|_{L^2(M)} \leq Ce^{-\frac{c}{2}}(\|u\|_{L^2(\Sigma)} + \|\langle \lambda \rangle^{-1} \partial_u u\|_{L^2(\Sigma)} + \|(-\Delta - \lambda^2) u\|_{L^2(M)}).
\]

(1.11)

As far as the authors are aware, estimates (1.2)-(1.11) are the first general lower bounds to appear for restrictions of eigenfunctions. Moreover, these estimates are sharp in the sense that simultaneously neither the growth rate \( e^{\lambda t} \) nor the presence of both \( u \) and \( \partial_u u \) can be improved in general. This is demonstrated by the following example.

**Proposition 1.8.** Consider the manifold

\[
M = [-\pi, \pi] \times \mathbb{T}^3,
\]

with variables \((z, \theta)\), endowed with the warped product metric

\[
g(z, \theta) = dz^2 + R(z)^2 d\theta^2.
\]

Assume that \( R \) is smooth, that

\[
R(z) = R(-z), \quad R(z) \geq 1 \text{ for all } z \in [0, \pi], \quad R(0) = 1, \quad R(\frac{\pi}{2}) = \sqrt{5}, \quad R(\pi) = \frac{1}{\sqrt{2}}.
\]
Let \( \Sigma = \{ z = 0 \} \times \mathbb{T}^1 \subset M \). Then, there exist \( C, c > 0 \) and sequences \( \lambda_j^0 \to +\infty \) and \( \phi_j^0 \in L^2(M) \) such that

\[
(-\Delta - (\lambda_j^0)^2)\phi_j^0 = 0, \quad ||\phi_j^0||_{L^2(M)} = 1, \quad \phi_j^0(\pm \pi) = 0,
\]

with

\[
\partial_r \phi_j^0|_z = 0, \quad ||\phi_j^0||_{L^2(\Sigma)} \leq Ce^{-c\lambda_j^0}, \quad \text{and} \quad \phi_j^0|_z = 0, \quad ||\partial_r \phi_j^0||_{L^2(\Sigma)} \leq Ce^{-c\lambda_j^0}.
\]

This result is proved in Appendix B.

We expect that the symmetry in this example is the obstruction for removing one of the traces in the right handside of (1.2), and formulate the following conjecture.

**Conjecture 1.** Let \((M, g)\) be a Riemannian manifold and \(\Sigma\) an interior hypersurface with positive definite second fundamental form. Then there exists \(C, c, \lambda_0 > 0\) so that for all \((\lambda, \phi) \in [\lambda_0, \infty) \times L^2(M)\) satisfying (1.1), we have

\[
||\phi||_{L^2(\Sigma)} \leq Ce^{c\lambda}||\phi||_{L^2(\Sigma)}, \quad \text{and} \quad ||\phi||_{L^2(M)} \leq Ce^{c\lambda}||\lambda^{-1}\partial_r \phi||_{L^2(\Sigma)}.
\]

Note that if \(\Sigma\) has positive definite second fundamental form, then it is geodesically curved and in particular, not fixed by a nontrivial involution. This prevents the construction of counterexamples via the methods used to prove Lemma 1.8.

Under the geometric control condition \(\mathcal{T}\) GCC the estimate (1.11) can be improved.

**Theorem 1.9** (Improved lower bound for eigenfunctions under \(\mathcal{T}\) GCC). Assume that the geodesics of \((M, g)\) have no contact of infinite order with \(\partial M\) and that \(\Sigma\) satisfies \(\mathcal{T}\) GCC. Then there exists \(C > 0\) so that for all \(\lambda \geq 0\) and \(u \in H^2(M) \cap H^1_0(M)\), we have

\[
||u||_{L^2(M)} \leq C(||u||_{L^2(\Sigma)} + ||(\lambda^{-1} \partial_r u)||_{L^2(\Sigma)} + \lambda^{-1}||(-\Delta - \lambda^2)u||_{L^2(M)}).
\]

(1.12)

**Conjecture 2.** Let \((M, g)\) be a Riemannian manifold and \(\Sigma\) an interior hypersurface with positive definite second fundamental form satisfying \(\mathcal{T}\) GCC. Then there exists \(C, c, \lambda_0 > 0\) so that for all \((\lambda, \phi) \in [\lambda_0, \infty) \times L^2(M)\) satisfying (1.1) we have

\[
||\phi||_{L^2(\Sigma)} \leq C||\phi||_{L^2(\Sigma)}, \quad \text{and} \quad ||\phi||_{L^2(M)} \leq C||\lambda^{-1}\partial_r \phi||_{L^2(\Sigma)}.
\]

Other known lower bounds come from the quantum ergodic restriction theorem and apply to a full density subsequence of eigenfunctions rather than to the whole sequence \([TZ12, TZ13, DZ13, TZ17]\). These hold under an ergodicity assumption on the geodesic (or the billiard) flow, together with a microlocal asymmetry condition for the surface \(\Sigma\). This assumption states roughly that the measure of the set of geodesics through \(\Sigma\) whose tangential momenta agree at adjacent intersections with \(\Sigma\) is zero. In another direction, the work of Bourgain–Rudnick \([BR12, BR11, BR09]\) shows that on the torus \(\mathbb{T}^d\), \(d = 2, 3\) for any hypersurface \(\Sigma\) with positive definite curvature, (1.3) holds with the normal derivative removed from the left hand side. While the results of Bourgain–Rudnick do not hold on a general Riemannian manifold, we expect that either of the terms in the left hand side of (1.2) can be removed whenever \(\Sigma\) is not totally geodesic (which is even weaker that \(\Sigma\) having positive definite second fundamental form).

### 1.4 Weakening Assumption \(\mathcal{T}\) GCC

One might hope that Theorem 1.9 and its analog for the wave equation (the control result of Theorem 1.5 above and the observability inequality of Theorem 4.1 below) when the assumption \(\mathcal{T}\) GCC is replaced by the (weaker) assumption that

\[
\text{every generalized bicharacteristics of } \Box \text{ intersects } T^\ast((0, T) \times \text{Int}(\Sigma))
\]

(1.13)

(rather than \(T^\ast((0, T) \times \text{Int}(\Sigma)) \setminus \mathcal{G}^2\)). The following example shows that this is more subtle (see Appendix C for the proof).

**Proposition 1.10.** Assume \(M = S^2\) and \(\Sigma\) is a great circle. Then there exists a sequence \((\lambda_j, \phi_j)\) satisfying

\[
(-\Delta - \lambda_j^2)\phi_j = 0 \text{ together with } \lambda_j \to +\infty \text{ and } \phi_j|_\Sigma = 0, \quad ||\lambda_j^{-1}\partial_r \phi_j||_{L^2(\Sigma)} \leq \lambda_j^{-1/4}||\phi_j||_{L^2(M)}.
\]
In particular, this shows that Theorem 1.9 and associated observability inequality for the wave equation cannot hold under only (1.13). Moreover, the proof shows that \( \phi_j \) is microlocalized \( \lambda_j^j \) close to the glancing set on \( \Sigma \), this calculation suggests that one must scale the normal derivative and restriction of an eigenfunction as in [Gal16] to obtain an analog of Theorem 1.9 under (1.13). More precisely,

**Conjecture 3.** Suppose that \( \Sigma \) is a compact interior hypersurface. Then there exists \( C > 0 \) so that if \( (\lambda, \phi) \) satisfies (1.1), then

\[
\| (1 + \lambda^2 \Delta \phi)_\lambda \|_{L^2} + \| (1 + \lambda^2 \Delta \phi)_\lambda \|_{L^2(\Sigma)}^2 \leq \| \phi \|_{L^2(M)}.
\]

Suppose moreover that \( \Sigma \) satisfies (1.13). Then there exists \( C > 0 \) so that if \( (\lambda_j, \phi_j) \) satisfies (1.1), then

\[
\| \phi \|_{L^2(M)} \leq C\| (1 + \lambda^2 \Delta \phi)_\lambda \|_{L^2} + \| (1 + \lambda^2 \Delta \phi)_\lambda \|_{L^2(\Sigma)}^2,
\]

where \( \Delta \phi \) is the Laplace-Beltrami operator on \( \Sigma \) induced from \( (M, g) \), and the operator \( (1 + \lambda^2 \Delta \phi)_\lambda \) is defined via the functional calculus, see also [Gal16, Section 1].

### 1.5 Finite unions of hypersurfaces

In all of our results, one may replace \( \Sigma \) by any finite union of cooriented interior hypersurfaces \( \bigcup_{i=1}^m \Sigma_i \) where we replace the distribution \( f_0\delta_{\Sigma} + f_1\delta_{\Sigma} \) by

\[
\sum_{i=1}^m \left( f_{ij}^j \delta_{\Sigma} + f_{ij}' \delta_{\Sigma} \right). \tag{1.14}
\]

Then, all above results generalize with the sole modification that generalized bicharacteristics need only intersect one of the \( \Sigma_i \)'s transversally. This furnishes several simple examples for which our controllability/observability results for waves holds. Take e.g. \( \mathbb{T}^2 \approx [-\pi, \pi]^2 \) with \( \Sigma_1 = \{0\} \times \mathbb{T}^1 \) and \( \Sigma_2 = \mathbb{T}^1 \times \{0\} \).

This remark can also be used to remove the coorientability assumption. If the interior hypersurface \( \Sigma \) is not coorientable, we can cover it by a union of overlapping cooriented hypersurfaces \( \Sigma = \bigcup_{i=1}^m \Sigma_i \) and control from \( \Sigma \) by a sum like (1.14). In this context, we still obtain controllability results with controls supported by the hypersurface \( \Sigma \), but the form of the control is changed slightly.

### 1.6 Sketch of the proofs and organization of the Paper

We start in Section 2 with the introduction of coordinates, some geometric definitions and Sobolev spaces on \( \Sigma \).

Section 3 is devoted to the proof of (a slightly more precise version of) the well-posedness result of Theorem 1.4. The definition of solutions in the sense of transposition follows [Lio88]. The well-posedness result relies on a priori estimates on an adjoint equation – the free wave equation. The well-posedness statement then reduces to the proof of regularity bounds for restrictions on \( \Sigma \). This is done in Section 3.1. Namely, we show that if \( u \) is an \( H^1 \) solution to \( \Box u = 0 \), then the restriction \( (u|_{\Sigma}, \partial_{\nu}|_{\Sigma}) \) belong to \( H^1 \times H^{1,2} \) overall \( \mathbb{R} \times \Sigma \), and have the additional (microlocal) regularity \( H^1 \times L^2 \) everywhere except near Glancing points \( (\mathcal{G}^\Sigma) \). This fact is already known (see e.g. [Tat98]) but we rewrite a short proof for the convenience of the reader. Then, Section 3.2 is aimed at defining the appropriate spaces for the statement of the precise version of the well-posedness result. These are needed in particular to state the stability result associated to well-posedness, as well as to formulate the duality between the control problem and the observation problem. They are (loc and comp) Sobolev spaces on \( \mathbb{R} \times \Sigma \) that have different regularities near and away from the Glancing set \( \mathcal{G}^\Sigma \). With these spaces in hand, we define properly solutions of (1.6) and prove well-posedness in Section 3.3.

Section 4 is devoted to the proof of the control result of Theorem 1.4. Before entering the proofs, we briefly explain how Theorem 1.9 is deduced from the observability inequality of Theorem 4.1. Firstly, we prove in Section 4.1 that the condition \( T \mathcal{G} \) implies a stronger geometric statement. Namely, using the openness of the condition and a compactness argument, we prove that all rays intersect in \( (\epsilon, T - \epsilon) \) an open set of \( \Sigma \) “transversally” (i.e. \( \epsilon \) far away from the glancing region) for some \( \epsilon > 0 \). Secondly, this condition
The test functions we use are rather forcing terms $F$ of $\Sigma H$ having $\iff$. Throughout the article we shall use Fermi normal coordinates in a $(\Sigma_2$ Preliminary definitions

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Theorem 1.7 is sharp. Finally, Appendix C gives a proof of Proposition 1.10.

and 4. for the wave equation. Appendix B proves Proposition 1.8, i.e. constructs an example showing that

local Carleman estimate near $\Sigma$ observability

latter are deduced from standard parabolic regularity combined with Sobolev trace estimates. Then, to prove

observability/controllability, we proceed with the Lebeau-Robbiano method [LR95]. The starting point is a

local Carleman estimate near $\Sigma$, borrowed from [LR97], from which we deduce in Section 5.2 a global inter-

polation inequality for the operator $-\partial^2_x - \Delta_x$. Theorem 1.7 directly follows from this interpolation inequality.

To deduce the observability of the heat equation, we revisit slightly (in an abstract semigroup setting) the original Lebeau-Robbiano method (as opposed to the simplified one [LZ98, Mil06, LRL12], relying on a stronger spectral inequality) in Section 5.3. The interpolation inequality yields as usual an observability result for a finite dimensional elliptic evolution equation (i.e. cutoff in frequency), from which we deduce observability for the finite dimensional parabolic equation, with precise dependence of the constant with respect to the cutoff frequency and observation time. The latter argument simplifies the original one by using an idea of Ervedoza-Zuazua [EZ11b, EZ11a]. The observability of the full parabolic equation is finally deduced using the iterative Lebeau-Robbiano argument combining high-frequency dissipation with low frequency control/observation. We in particular use the method as refined by Miller [Mil10]. We explain in Section 5.4 how the heat equation observed by/controlled from $\Sigma$ fits into the abstract setting.

Appendix A contains some background information on pseudodifferential operators used in Sections 3 and 4. for the wave equation. Appendix B proves Proposition 1.8, i.e. constructs an example showing that Theorem 1.7 is sharp. Finally, Appendix C gives a proof of Proposition 1.10.

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2 Preliminary definitions

2.1 Fermi normal coordinates in a neighborhood $\Sigma_0$

Throughout the article we shall use Fermi normal coordinates in a (sufficiently small) neighborhood, say $V_\varepsilon$, of $\Sigma_0$. Namely since $\Sigma_0$ is cooriented, for $\varepsilon$ sufficiently small, there exists a diffeomorphism (see [Hör85,
and the associated sphere bundle, which, endowed with the induced topology, are locally compact metric spaces.

The map has the form

\[-\partial_t^2 + r(x_1, x', D_{x'}) + c(x, D),\]

where \(c(x, D)\) is a first order differential operator and \(r(x_1, x', D_{x'})\) is an \(x_1\)-family of second-order elliptic differential operators on \(\text{Int}(\Sigma_0)\), i.e. a tangential operator, with principal symbol \(r(x_1, x', \xi')\), \(\xi' \in T^*_x \text{Int}(\Sigma_0)\), that satisfies

\[r(x_1, x', \xi') \in \mathbb{R}, \quad C_1|\xi'|^2 \leq r(x_1, x', \xi') \leq C_2|\xi'|^2,\]  

for some \(0 < C_1 \leq C_2 < \infty\).

In these coordinates, note that we have in particular \(|x_1| = d(p, \Sigma)\), \(\partial_{x_i} = \partial_{x_i}\) (up to changing \(x_1\) into \(-x_1\)), as well as

\[\sigma(-\Delta_x) = \xi_1^2 + r(x_1, x', \xi')\]

and

\[-\Delta_{\Sigma_0} = r(0, x', D_{x'}), \quad \text{with} \quad \sigma(-\Delta_{\Sigma_0})(x', \xi') = r(0, x', \xi') =: \sigma(0, x', \xi').\]

where \(-\Delta_{\Sigma_0}\) is the Laplacian on \(\text{Int}(\Sigma_0)\) given by the induced metric on \(\Sigma_0\). We also recall that

\[\sigma(\Box) = -\tau^2 + \sigma(-\Delta_x) = -\tau^2 + |\xi|^2 = -\tau^2 + \xi_1^2 + r(x_1, x', \xi').\]

With a slight abuse of notation, we shall also denote by \((x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}\) and \((\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{n-1}\) associated cotangent variables) local coordinates in a neighborhood of a point in \(\text{Int}(\Sigma_0)\).

In these coordinates, the Hamiltonian vector field of \(\Box\) is given by

\[H_{\sigma(\Box)} = -2\tau \partial_t + 2\xi_1 \partial_{x_1} - \partial_{x_1} r(x_1, x', \xi') \partial_{\xi_1} + \partial_{x'} r(x_1, x', \xi') \partial_{\xi'} - \partial_{\xi'} r(x_1, x', \xi') \partial_{\xi'}\]  

and generates the Hamiltonian flow of \(\Box\) (these coordinates being away from the boundary \(\mathbb{R} \times \partial M\)).

### 2.2 The compressed cotangent bundle over \(M\)

This section is independent of the hypersurface \(\Sigma\) and is only aimed at defining, in case \(\partial M \neq \emptyset\), the space \(Z\) on which the Melrose-Sjöstrand bicharacteristic flow is defined, as well as some properties of the flow. In case \(\partial M = \emptyset\), this set is \(\text{Char}(\Box) \subset T^*(\mathbb{R} \times M) \setminus \emptyset\), the flow is the usual bicharacteristic flow of \(\Box\), and this section not needed and may be skipped. We refer to [MS78], [Leb96, Appendix A2] for more complete treatments.

We first embed \(M \hookrightarrow \tilde{M}\) into a manifold, \(\tilde{M}\), without boundary and write

\[T^*(\mathbb{R} \times M) := T^*_{\mathbb{R} \times \tilde{M}}(\mathbb{R} \times \tilde{M}).\]

Let \(bT^*(\mathbb{R} \times M) = (T^*(\mathbb{R} \times \text{Int}(M)) \setminus 0) \cup (T^*(\mathbb{R} \times \partial M) \setminus 0)\) denote the compressed cotangent bundle of \(\mathbb{R} \times M\) and

\[j : T^*(\mathbb{R} \times M) \rightarrow bT^*(\mathbb{R} \times M)\]

be the natural “compression” map. In any coordinates \((x', x_n)\) on \(M\) where \(x_n\) defines \(\partial M\) and \(x_n > 0\) on \(M\), \(j\) has the form

\[j(t, x, \tau, \xi) = (t, x, \tau, \xi', x_n \xi_n).\]  

The map \(j\) endows \(bT^*(\mathbb{R} \times M)\) with a structure of homogeneous topological space. We then write

\[Z = j(\text{Char}(\Box)), \quad \hat{Z} = Z \cup j(T^*_{\mathbb{R} \times \partial M}(\mathbb{R} \times M)),\]

and

\[S\hat{Z} = (\hat{Z} \setminus (\mathbb{R} \times M))/\mathbb{R}^*_+,\]

the associated sphere bundle, which, endowed with the induced topology, are locally compact metric spaces.
Away from the boundary, $j$ is a bijection and we shall systematically identify $b^{*}T^{*}(\mathbb{R} \times \text{Int}(M))$ with $T^{*}(\mathbb{R} \times \text{Int}(M))$ and $Z$ with $(\mathbb{R} \times \text{Int}(M))$. This will be the case in particular near the hypersurface $\mathbb{R} \times \Sigma$. Under the assumption that the geodesics of $M$ have no contact of infinite order with $\partial M$, and with $Z$ as in (2.4), the (compressed) generalized bicharacteristic flow for the symbol $\frac{1}{2}(-\tau^{2} + |\xi_{b}|^{2})$ is a (global) map

$$\varphi : \mathbb{R} \times Z \rightarrow Z, \quad (s, p) \mapsto \varphi(s, p)$$

(2.6)

We refer to [MS78, Section 3], [Hör85, Chapter 24], [BL01, Section 3.1] or [LLTT16, Section 1.3.1] for a definition. In particular, it has the following properties

- $\varphi$ coincides with the usual bicharacteristic flow of $\Delta$ (i.e. the Hamiltonian flow of $\sigma(\square)$) in the interior $\text{Char}(\square) \cap T^{*}(\mathbb{R} \times \text{Int}(M))$;
- $\varphi$ satisfies the flow property

$$\varphi(t, \varphi(s, p)) = \varphi(t + s, p), \quad \text{for all } t, s \in \mathbb{R}, p \in Z;$$

(2.7)

- $\varphi$ is homogeneous in the fibers of $Z$, in the sense that

$$M_{\lambda} \circ \varphi(s, \lambda \cdot) = \varphi(s, M_{\lambda} \cdot),$$

(2.8)

where $M_{\lambda}$ denotes multiplication in the fiber by $\lambda > 0$; Hence, it induces a flow on $SZ$.
- $\varphi : \mathbb{R} \times Z \rightarrow Z$ is continuous, see [MS78, Theorem 3.34].

### 2.3 Glancing Sets over $\Sigma$

For the following definitions, we use the above identification $b^{*}T^{*}_{\mathbb{R} \times \Sigma}((\mathbb{R} \times M)) \cong T^{*}_{\mathbb{R} \times \Sigma}((\mathbb{R} \times M))$ for the cotangent bundle of $\mathbb{R} \times M$ with foot points at $\mathbb{R} \times \Sigma_{0}$, since in this case, we may assume in Definition 1.2 that $\Sigma_{0} \cap \partial M = \emptyset$. Using the coordinates of Section 2.1, the map $\iota$ defined in (1.8) reads

$$\iota(t, x', \tau, \xi') = (t, 0, x', \tau, 0, \xi').$$

Still in coordinates, we define for $\varepsilon \geq 0$, the sets

$$G_{\varepsilon} = \{(t, 0, x', \tau, \xi_{1}, \xi') \in T_{\mathbb{R} \times \Sigma}^{*}((\mathbb{R} \times M)) \setminus \emptyset \mid \xi_{1}^{2} + r(0, x', \xi') = \tau^{2}, \xi_{1}^{2} \leq \varepsilon \tau^{2}\},$$

$$T_{\varepsilon} := \{(t, 0, x', \tau, \xi_{1}, \xi') \in T_{\mathbb{R} \times \Sigma}^{*}((\mathbb{R} \times M)) \setminus \emptyset \mid \xi_{1}^{2} + r(0, x', \xi') = \tau^{2}, \xi_{1}^{2} > \varepsilon \tau^{2}\}.$$

Let also

$$G_{\varepsilon}^{\Sigma} := \left\{(t, x', \tau, \xi') \in T^{*}((\mathbb{R} \times \text{Int}(\Sigma_{0})) \setminus \emptyset \mid x' \in \text{Int}(\Sigma), -\varepsilon \tau^{2} \leq \tau^{2} - r_{0}(x', \xi') \leq \varepsilon \tau^{2}\right\},$$

$$T_{\varepsilon}^{\Sigma} := \left\{(t, x', \tau, \xi') \in T^{*}((\mathbb{R} \times \text{Int}(\Sigma_{0})) \setminus \emptyset \mid x' \in \text{Int}(\Sigma), \varepsilon \tau^{2} < \tau^{2} - r_{0}(x', \xi')\right\},$$

$$E_{\varepsilon}^{\Sigma} := \left\{(t, x', \tau, \xi') \in T^{*}((\mathbb{R} \times \text{Int}(\Sigma_{0})) \setminus \emptyset \mid x' \in \text{Int}(\Sigma), \tau^{2} - r_{0}(x', \xi') < -\varepsilon \tau^{2}\right\}.$$

(2.9)

Observe that $G_{\varepsilon}^{\Sigma} \neq \pi(G_{\varepsilon})$ (although $G_{0}^{\Sigma} = \pi(G_{0})$) where

$$\pi : T^{*}_{\text{Int}(\Sigma_{0})}(\mathbb{R} \times M) \rightarrow T^{*}(\mathbb{R} \times \text{Int}(\Sigma_{0}))$$

is projection along $N^{*}(\mathbb{R} \times \text{Int}(\Sigma_{0}))$. In the above coordinates, $\pi(t, 0, x', \tau, \xi_{1}, \xi') = (t, x', \tau, \xi_{1}, \xi')$. Observe also that $G_{0} = G$ and $G_{\varepsilon}^{\Sigma} = G^{\Sigma}$ where $G$ and $G^{\Sigma}$ are defined in (1.7).

**Remark 2.1.** In these coordinates, $\Sigma_{0} = \{x_{1} = 0\}$ and, according to (2.2), we have $(dx_{1}, H_{\sigma(\square)}) = (dx_{1}, 2\xi_{1}\partial_{x_{1}}) = 2\xi_{1}$. Since $\xi_{1} \neq 0$ on $T_{0}$ this implies in particular that the vector field $H_{\sigma(\square)}$ is transverse to the Hypersurface $\mathbb{R} \times \Sigma$ on this set (which explains its name $T_{0}$).

With these definitions, $\mathcal{T}^{*}G_{\text{GCC}}$ can be written as

**Assumption GC-(0,T).** For all $p \in Z$,

$$\bigcup_{s \in \mathbb{R}} |\varphi(s, p)| \cap T_{0} \cap T^{*}(0, T) \times M \neq \emptyset.$$
2.4 Spaces on interior hypersurfaces

In case $\Sigma$ is a compact internal hypersurface, then the Sobolev spaces $H^s(\Sigma)$ have a natural definition. Here, we give a definition adapted to the case $\partial\Sigma \neq \emptyset$.

**Definition 2.2.** Let $S$ be an interior hypersurface of a $d$ dimensional manifold $X$, and let $S_0$ be an extension of $S$ (see Definition 1.2). Given $s \in \mathbb{R}$, we say that $u \in H^s(S)$ (extendable Sobolev space) if there exists $u \in H^s_{\text{comp}}(S_0)$ such that $u|_S = u$.

To put a norm on $H^s(S)$, let $\chi \in C^\infty(\text{Int}(S_0))$ such that $\chi = 1$ in a neighborhood of $S$. We denote by $(U_j, \psi_j)_{j \in J}$ an atlas of $S_0$ such that for all $j \in J$

$$U_j \cap \text{supp } \chi = \emptyset \quad \text{or} \quad U_j \cap \partial S_0 = \emptyset,$$

and write $J_S = \{ j \in J, U_j \cap \text{supp } \chi \neq \emptyset \}$ and $J_\partial = \{ j \in J, U_j \cap (\text{supp } \chi \setminus \text{Int}(S)) \neq \emptyset \} \subset J_S$ (possibly empty). Let $(\chi_j)_{j \in J}$ be a partition of unity of $S_0$ subordinated to $(U_j)_{j \in J}$. Given $u$, we define

$$\|u\|_{H^s(S)} = \sum_{j \in J_S \setminus J_\partial} \|\chi_j u\|_{H^s(T^*\mathbb{R}^{d-1})} + \inf_{\psi \in F} \sum_{j \in J_\partial} \|\chi_j \psi_j u\|_{H^s(T^*\mathbb{R}^{d-1})},$$

$$E_u := \{ u \in H^s_{\text{comp}}(\text{Int}(S_0)), u|_S = u \}.$$

The definition of the norm $H^s(S)$ depends on $S_0, \chi$, the choice of charts $(U_j, \psi_j)$ and the partition of unity $(\chi_j)$. One can however prove that, once $S_0$ and $\chi$ are fixed, two such choices of charts $(U_j, \psi_j)$ and partition of unity $(\chi_j)$ lead to equivalent norms $H^s(S)$. In what follows, $(U_j, \psi_j, \chi_j)$ shall be traces on $S_0$ of charts and partition of unity on $X$. In case $S$ is a compact interior hypersurface, then the spaces $\bar{H}^s(S)$, $\|\cdot\|_{\bar{H}^s(S)}$ coincides with usual $H^s(S)$ space.

3 Regularity of traces and well-posedness for the wave equation

The ultimate goal of the present section is to prove the well-posedness result for (1.6), see Theorem 1.4. Defining solutions by transposition as in [Lio88], this amounts to proving regularity of traces on $\Sigma$ of solutions to the free wave equation.

3.1 Regularity of traces

We start by giving estimates on the restriction to $\Sigma$ of a solution to

$$\begin{cases}
\Box u = F & \text{on } \mathbb{R} \times \text{Int}(M), \\
u = 0 & \text{on } \mathbb{R} \times \partial M, \\
(u, \partial_t u)|_{t = 0} = (u_0, u_1) & (u_0, u_1) \in H^1(M) \times L^2(M). 
\end{cases}$$

(3.1)

These bounds, indeed stronger bounds, can be found in [Tat98], but we choose to give the proof of the simpler estimates here for the convenience of the reader. They are closely related to the semiclassical restriction bounds from [BGT07, Tac10, Tac14, CHT15, Gal16].

**Proposition 3.1.** Fix $T > 0$. Then for any $A \in \Psi^0_{\text{ph}}(\mathbb{R} \times \text{Int}(\Sigma))$, with principal symbol vanishing in a neighborhood of $\partial^\infty_0$ and all $\varphi \in C^\infty_c(\mathbb{R})$, there exists $C > 0$ such that for any $(u_0, u_1) \in H^1(M) \times L^2(M)$ and $F \in L^2(\mathbb{R} \times M)$ with supp $F \subset [0, T] \times M$ the solution $u$ to (3.1) satisfies

$$\|\varphi(t)u(t)\|_{L^2(X)} + \|\varphi(t)\partial_t u(t)\|_{L^2(X)} + \|A\varphi(t)(u(t))\|_{L^2(X)} + \|A\varphi(t)(u(t))\|_{L^2(X)}^2 \
\leq C(C(\|u_0, u_1\|^2_{H^1(M) \times L^2(M)} + |F|_{L^2}^2) \quad (3.2)$

To prove Proposition 3.1 we need the following elementary lemma.

**Lemma 3.2.** Suppose that $S$ is an interior hypersurface of the $d$ dimensional manifold $X$ (in the sense of Definition 1.2) and $P \in \Psi^m_{\text{ph}}(\text{Int}(X))$ is elliptic on the conormal bundle to Int$(S_0)$. Then for any $s \in \mathbb{R}, k \geq 0$ and $\epsilon > 0$, there exists $C = C(\epsilon, k, \sigma) > 0$ so that for all $u \in C^\infty(M)$,

$$\|\partial^k \sigma u\|_{H^s(S)} \leq C(\|u\|_{H^{1/2+\epsilon}(X)} + \|Pu\|_{H^{1/2+\epsilon}(X)}).$$
Proof. We start by proving the case $k = 0$. In case $s > 0$, the stronger inequality $\|u_s\|_{H^s(S)} \leq C\|u\|_{H^{s+1/2}(X)}$ holds as a consequence of standard trace estimates [Hör85, Theorem B.2.7] (that the $H^s(S)$ norm is the appropriate one in case $S$ is not compact is made clear below).

We now assume that $s \leq 0$, and estimate each term in the definition (2.11) of $\|u_s\|_H^S(S_1)$ in local charts. For this, we use charts $(\Omega, \kappa_j)$ of $\text{Int}(X)$ such that $S_0 \subset \bigcup \Omega_j$, and such that $(\Omega_j \cap S_0, \kappa_j|_{S_0})$ satisfy the assumptions of Definition 2.2. In a neighborhood of $S_0$, we have $u = \sum_j \tilde{\chi}_j u$ (where $(\tilde{\chi}_j)$ is now a partition of unity of $S_0$ associated to $\Omega_j$, and hence, $(\tilde{\chi}_j|_{S_0})$ satisfies the assumptions of Definition 2.2), and estimating $\|u_s\|_H^S(S_1)$ amounts to estimating each

$$\|(\tilde{\chi}_j u)|_{S_0} \circ \kappa_j^{-1}\|_{H^j(\mathbb{R}^{d-1})} = \|\tilde{\chi}_j w\|_{\mathbb{R}^{d-1}}$$

with $\tilde{\chi}_j = \chi_j \circ \kappa_j^{-1}$ and $w = u \circ \kappa_j^{-1}$. We may now work locally, where $S$ is a subset of $\{x_1 = 0\}$, and estimate the trace of $z = \tilde{w}$.

Let $\chi \in C_c^\infty(\mathbb{R})$ have $\chi \equiv 1$ on $[-1, 1]$ with supp $\chi \subset [-2, 2]$, $0 \leq \chi \leq 1$ and fix $\delta > 0$ small enough so that $P$ (which, by abuse of notation, we use for the operator in local coordinates) is elliptic on a neighborhood of

$$\{\xi_1 \geq \delta^{-1}|\xi|^2\} \supset \mathcal{N}^*(0) \times \mathbb{R}^{d-1} = \{\xi' = 0\}$$

and let $\chi_\delta(\xi, \xi') = \chi \left(\frac{2|\xi'|}{\delta}\right)$ for $\xi_1 \neq 0$ and $\chi_\delta(0, \xi') = 0$. Then, we have

$$\|z|_{x_1 = 0}\|_{H^j(\mathbb{R}^{d-1})}^2 = \int_{\mathbb{R}^{d-1}} \langle \xi' \rangle^2 \left|\int_R (1 - \chi_\delta)\hat{\xi}(\xi_1, \xi')d\xi_1\right|^2 d\xi' \leq 2(A + B),$$

with

$$A = \int_{\mathbb{R}^{d-1}} \langle \xi' \rangle^2 \left|\int_R (1 - \chi_\delta)\hat{\xi}(\xi_1, \xi')d\xi_1\right|^2 d\xi', \quad B = \int_{\mathbb{R}^{d-1}} \langle \xi' \rangle^2 \left|\int_R \chi_\delta \hat{\xi}(\xi_1, \xi')d\xi_1\right|^2 d\xi'.$$

We now estimate each term. With the Cauchy Schwarz inequality, the first term is estimated by

$$A = \int_{\mathbb{R}^{d-1}} \langle \xi' \rangle^2 \left|\int_R (1 - \chi_\delta)\hat{\xi}(\xi_1, \xi')d\xi_1\right|^2 d\xi' \leq \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}^{d-1}} \langle \xi' \rangle^2 (1 - \chi_\delta)^2 d\xi_1\right)^2 \left(\int_{\mathbb{R}^{d-1}} \langle \xi' \rangle^2 d\xi_1\right) d\xi' \leq C_{x, \delta} \|z\|_{H^{j+1/2}(\mathbb{R}^{d-1})}^2,$$

since

$$\int_{\mathbb{R}^{d-1}} \langle \xi' \rangle^2 (1 - \chi_\delta)^2 d\xi_1 = \int_{\mathbb{R}^{d-1}} \frac{1}{(1 + t^2)^{j+1/2}} (1 - \chi \left(\frac{2|\xi'|}{\delta}\right))^2 dt \leq \int_{|t| \leq 2/\delta} \frac{1}{(1 + t^2)^{j+1/2}} dt =: C_{x, \delta}$$

(which is large since $s \leq 0$).

Again with the Cauchy Schwarz inequality, the second term is estimated by

$$B = \int_{\mathbb{R}^{d-1}} \langle \xi' \rangle^2 \left|\int_R \langle \xi \rangle^{1+2e} \chi_\delta \hat{\xi}(\xi_1, \xi')d\xi_1\right|^2 d\xi' \leq \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}^{d-1}} \langle \xi' \rangle^2 d\xi_1\right)^2 \left(\int_{\mathbb{R}^{d-1}} \langle \xi \rangle^{1+2e} \chi_\delta^2 d\xi_1\right) d\xi' \leq C_{x} \|\chi_\delta(D)z\|_{H^{j+2e}(\mathbb{R}^{d-1})}^2,$$

since

$$\int_{\mathbb{R}^{d-1}} \langle \xi \rangle^2 d\xi_1 = \langle \xi' \rangle^2 \int_{\mathbb{R}^{d-1}} \frac{1}{(1 + t^2)^{j+2e}} dt = \langle \xi' \rangle^{2j-2e} C_x.$$
with $C_\epsilon$ finite as soon as $\epsilon > 0$, and $|\xi'|^{2-2s} \leq 1$ since $s \leq 0$. Combining the last three estimates and recalling that $\tilde{\chi} = \tilde{\chi}w$ yields

$$\|\tilde{\chi}w\|_{L^2}^2 \leq C_\varepsilon \|\tilde{\chi}w\|^2_{L^2(E\cap \Gamma)} + C_\varepsilon \|\chi(D)\tilde{\chi}w\|^2_{H^{1/2}_{\sigma}(E\cap \Gamma)},$$

(3.3)

Now, according to the definition of $\chi$, the operator $P$ is elliptic on a neighborhood of supp(\tilde{\chi}) x supp($\chi$), a classical parametrix construction (see for instance [Hor85, Theorem 18.1.9]) implies, for any $N \in \mathbb{N}$,

$$\|\chi(D)\tilde{\chi}w\|_{H^{1/2}_{\sigma}(E\cap \Gamma)} \leq C_N(\tilde{\chi}Pw\|_{H^{1/2}_{\sigma}(E\cap \Gamma)} + \|\tilde{\chi}w\|_{H^{-s}(E\cap \Gamma)}),$$

(3.4)

where $\tilde{\chi}$ is supported in the local chart and equal to one in a neighborhood of supp($\tilde{\chi}$). Recalling that $w$ is the localization of $u$, and summing up the estimates (3.3)-(3.4) in all charts yields the sought result for $k = 0$.

We now show that the $k = 0$ case implies the $k > 0$ case. Let $\tilde{P} \in \Psi^{m}_{\text{phg}}(\text{Int}(X))$ be elliptic on $N^\circ(\text{Int}(S_0))$ with $\text{WF}(\tilde{P}) \subset \{\sigma(P) \neq 0\}$ (see e.g. Appendix A for a definition of $\text{WF}(A)$ for a pseudodifferential operator $A$). Then, applying the case $k = 0$ to the operator $\tilde{P}$, we obtain

$$\|\partial^s_u|\|_{\tilde{H}\ell^2} \leq C \left(\|\partial^s_u|\|_{\tilde{H}\ell^2} + \|\tilde{P}\partial^s_u|\|_{\tilde{H}\ell^2}\right) \leq C \left(\|u|\|_{\tilde{H}\ell^2} + \|\tilde{P}\partial^s_u|\|_{\tilde{H}\ell^2}\right).$$

(3.5)

Now, we write

$$\tilde{P}\partial^s_u| = \partial^s_u|\tilde{P}| + [\tilde{P}, \partial^s_u]|u.$$ Since $P$ is elliptic on $\text{WF}(\tilde{P})$, by the elliptic parametrix construction, we can find $E_1 \in \Psi^{k-1}_{\text{phg}}(\text{Int}(X))$, and $E_2 \in \Psi^{k-1}_{\text{phg}}(\text{Int}(X))$ so that

$$\tilde{P} = E_1P + E_2, \quad [\tilde{P}, \partial^s_u] = E_2P + E_2$$

with $E_i \in \Psi^{k^\circ}_{\text{phg}}(\text{Int}(X))$. Hence, we obtain

$$\|\tilde{P}\partial^s_u|\|_{\tilde{H}\ell^2} \leq C \left(\|E_1Pu\|_{\tilde{H}\ell^2} + \|E_2Pu\|_{\tilde{H}\ell^2} + \|u|\|_{\tilde{H}\ell^2}\right).$$

which, combined with (3.5), yields the result.

We now proceed with the proof of Proposition 3.1.

Proof of Proposition 3.1. First, observe that standard estimates for the Cauchy problem imply that for any $\tilde{\varphi} \in C^\infty_c(\mathbb{R})$,

$$\|\tilde{\varphi}u\|_{\tilde{H}} \leq C_{T,\tilde{\varphi}}(\|(u_0, u_t)|\|_{\tilde{H}^1(\mathbb{R})} + \|F\|_{L^1(0,T;L^1(\mathbb{R}))}),$$

so we may estimate by terms of the form $\|\tilde{\varphi}u\|_{\tilde{H}}$.

Second, notice that $N^\circ(\mathbb{R} \times \text{Int}(S_0)) \subset \{\tau = 0\}$, so $\tilde{\varphi}$ is elliptic on $N^\circ(\mathbb{R} \times \text{Int}(S_0))$ and hence Lemma 3.2 implies

$$\|\varphi(t)u|\|_{\tilde{H}^1(\mathbb{R})} + \|\varphi(t)\partial^s_u|\|_{\tilde{H}^1(\mathbb{R})} \leq C(\|\tilde{\varphi}(t)u\|_{\tilde{H}} + \|\tilde{\varphi}F\|_{L^1})$$

where $\tilde{\varphi} \in C^\infty_c(\mathbb{R})$ with $\tilde{\varphi} \equiv 1$ on supp $\varphi$. Now, observe that since $\tilde{\Sigma}$ is an interior hypersurface, we may work in a fixed compact subset, $K$ of Int($M$). Note also that there exists $\tilde{\Sigma}$ an interior hypersurface with $\tilde{\Sigma} \subset \text{Int}(S_0)$ so that $A = 1_{\tilde{\Sigma}}A1_{\tilde{\Sigma}}$.

We proceed by making a microlocal partition of unity on a neighborhood $T^*(\mathbb{R} \times K)$. It suffices to obtain the estimate

$$\|A(\text{Op}(\varphi)\varphi(t)u)|\|_{\tilde{H}^1(\mathbb{R})}^2 + \|A(\partial^s_u, \text{Op}(\varphi)\varphi(t)u)|\|_{\tilde{H}^1(\mathbb{R})}^2 \leq C(\|(u_0, u_t)|\|_{\tilde{H}^1(\mathbb{R})}^2 + \|F\|_{L^1}^2),$$

(3.6)

for $\varphi$ supported in a conic neighborhood of an arbitrary point, $q_0 = (t_0, \tau_0, x_0, \xi_0)$ in $T^*(\mathbb{R} \times K)$. We will focus on four regions: $q_0 \notin \text{Char}(\partial) \text{ (an elliptic point)}$; $q_0 \in \text{Char}(\partial)$ but away from $\Sigma$; $q_0 \in T^*_\Sigma(\mathbb{R} \times M) \cap T_0$ (a transversal point); and $q_0 \in T^*\Sigma(\mathbb{R} \times M) \cap T_0$ (a glancing point). In all regions, we shall use that given $\varphi \in S_0^0$ a cutoff to a conic neighborhood, $U$ of $q_0$, we have

$$\|\partial^s_u, \text{Op}(\varphi)\varphi(t)u|\|_{L^2} \leq \|\partial^s_u, \text{Op}(\varphi)\varphi(t)u|\|_{L^2} + \|\text{Op}(\varphi)\varphi(t)u|\|_{L^2} \leq C(\|\tilde{\varphi}u\|_{\tilde{H}} + \|\tilde{\varphi}F\|_{L^1}).$$

(3.7)
First start with $q_0$ in the elliptic region: $q_0 \notin \text{Char}(\Box)$. Shrinking the neighborhood if necessary, the microlocal ellipticity of $\Box$ near $q_0$ with (3.7) yields
\[ \| \text{Op}(\chi)\varphi u \|_{H^r} \leq C (\| \hat{\varphi} u \|_{H^r} + \| \hat{\varphi} F \|_{L^2}). \]
Hence, rough trace estimates imply
\[ \| (\partial_1 \text{Op}(\chi)\varphi u) \|_{L^2(L^2)} + \| (\text{Op}(\chi)\varphi u) \|_{H^r(L^2)} \leq C (\| \hat{\varphi} u \|_{H^r} + \| \hat{\varphi} F \|_{L^2}), \]
and boundedness of $A$ proves (3.6) in this case.

Second, suppose that $q_0 \in \text{Char}(\Box)$ but $x_0 \notin \Sigma$, then clearly there is a neighborhood $U$ of $q_0$ and $\chi$ elliptic at $q_0$ with $\text{supp} \chi \subset U$ so that
\[ \| (\partial_1 \text{Op}(\chi)\varphi u) \|_{L^2(L^2)} + \| (\text{Op}(\chi)\varphi u) \|_{H^r(L^2)} \leq C \| \hat{\varphi} u \|_{L^2} \]
and again boundedness of $A$ proves (3.6).

Third, suppose $q_0 \in T^*_R \Sigma^0 \chi^0 (\mathbb{R} \times M)$ is a transversal point. In that case, we use local Fermi normal coordinates (see Section 2.1) near $x_0$ so that $x_0 \mapsto (0,0)$. Note that since $q_0 \in \text{Char}(\Box)$, we have $\sigma(\Box)(q_0) = -\tau^2_0 + (\xi_0)^2 + r(x_0, \xi_0) = 0$. Since $q_0 \in T_0$, we have moreover $r_0(0, \xi_0^0) < 0$ and hence $\partial_1 \sigma(\Box)(q_0) = 2(\xi_0^0) \neq 0$. Therefore, by the implicit function theorem, there exist a neighborhood $U$ of $q_0$ and real valued symbols $b(\tau, x, \xi') \in C^\omega((-\varepsilon, \varepsilon); S^1_{\text{phg}}(T^* \mathbb{R} \times \{x_1 = 0\}))$ and $e(\tau, x, \xi) \in S^1_{\text{phg}}(T^* \mathbb{R} \times \mathbb{R}^n)$ elliptic near $q_0$ so that in $U$ we have
\[ \sigma(\Box) = e(\tau, x, \xi)(\xi_1^0 - b(\tau, x, \xi')). \]
Thus, letting $\tilde{\chi} \in S^0_{\text{phg}}(\mathbb{R} \times \mathbb{R}^n)$ with $\tilde{\chi} \equiv 1$ on $\chi$ and $\text{supp} \tilde{\chi} \cap N'((x_1 = 0)) = \emptyset$ (this is possible since we have $\text{Char}(\Box) \cap N'((x_1 = 0)) = \emptyset$ and $q_0 \in \text{Char}(\Box)$, so that we may assume $\text{supp} \chi \cap N'((x_1 = 0)) = \emptyset$), we have $b(\tilde{\chi}) \in S^1_{\text{phg}}(T^* \mathbb{R} \times \mathbb{R}^n)$ (see [Hör85, Theorem 18.1.35]) and in particular $\text{Op}(\chi) \text{Op}(\tilde{\chi}) \in \Psi^1_{\text{phg}}(\mathbb{R} \times \mathbb{R}^n)$. Therefore,
\[ \big[ \Box \text{Op}(\chi) = \text{Op}(\varepsilon) A_{x_1} - \text{Op}(b) \text{Op}(\tilde{\chi}) \big] \text{Op}(\chi) + R, \]
where $R \in \Psi^1_{\text{phg}}(\mathbb{R} \times \mathbb{R}^n)$ and hence, using a microlocal parametrix for $\text{Op}(\varepsilon)$ on $\text{supp} \chi$, we have, using (3.7)
\[ \| (A_{x_1} - \text{Op}(b)) \hat{\varphi} u \|_{H^r} \leq C (\| \hat{\varphi} u \|_{H^r} + \| \hat{\varphi} F \|_{L^2}) \]
and also
\[ \| (A_{x_1} - \text{Op}(b)) (\partial_1 \text{Op}(\chi)\varphi u) \|_{L^2} \leq C (\| \hat{\varphi} u \|_{H^r} + \| \hat{\varphi} F \|_{L^2}). \]
So, by Lemma A.1, we obtain
\[ \| \text{Op}(\chi)\varphi u \|_{L^2} \leq C (\| \hat{\varphi} u \|_{H^r} + \| \hat{\varphi} F \|_{L^2}), \]
\[ \| (\partial_1 \text{Op}(\chi)\varphi u) \|_{L^2} \leq C (\| \hat{\varphi} u \|_{H^r} + \| \hat{\varphi} F \|_{L^2}). \] (3.8)

Boundedness of $A$ and (3.8) implies (3.6).

Finally, it remains to show that for $q_0 \in T^*_R \Sigma^0 \chi^0 (\mathbb{R} \times M)$ a glancing point (i.e. with $\tau^2_0 - r_0(0, \xi_0^0) = 0$) and $\chi$ supported sufficiently close to $q_0$, we have
\[ \| A(\text{Op}(\chi)\varphi u) \|_{L^2} + \| A(\partial_1 \text{Op}(\chi)\varphi u) \|_{L^2} \leq C (\| \hat{\varphi} u \|_{H^r(M)} + \| \hat{\varphi} F \|_{L^2}). \]

Let $\psi \in C^\infty_c (\mathbb{R})$ with $\psi \equiv 1$ near 0 and define $\psi_{\varepsilon}(x_1) = \psi(\varepsilon^{-1} x_1)$. Then define
\[ A_{\varepsilon} u(x_1, x') = [\psi_{\varepsilon}(x_1)] AU(x_1, \cdot)(x'), \]
so that $A_{\varepsilon} u_{|x_1 = 0} = A u_{|x_1 = 0}$. Then by [Hör85, Theorem 18.1.35], $A_{\varepsilon} \text{Op}(\chi)\varphi(t) \in \Psi^0_{\text{phg}}(\mathbb{R} \times M)$ and for $\varepsilon > 0$ small enough and $\chi$ supported sufficiently close to $q_0$, $\sigma(A_{\varepsilon} \text{Op}(\chi)) = 0$. In particular, $A_{\varepsilon} \text{Op}(\chi)\varphi(t) \in \Psi^1_{\text{phg}}(\mathbb{R} \times M)$. Similarly, $A_{\varepsilon} \partial_1 \text{Op}(\chi)\varphi(t) \in \Psi^0_{\text{phg}}(\mathbb{R} \times M)$. Rough Sobolev trace estimates thus yield
\[ \| A_{\varepsilon} \text{Op}(\chi)\varphi(t) u_{|x_1 = 0} \|_{H^r(M)} \leq C (\| \hat{\varphi} u \|_{H^r(M)} + \| \hat{\varphi} F \|_{L^2}), \]
\[ \| A_{\varepsilon} \partial_1 \text{Op}(\chi)\varphi(t) u_{|x_1 = 0} \|_{L^2} \leq C (\| \hat{\varphi} u \|_{H^r(M)} + \| \hat{\varphi} F \|_{L^2}), \]
and the proof is finished.

\[ \Box \]
3.2 Microlocal spaces on the hypersurface

This section is aimed at defining the appropriate spaces for the statement of the well-posedness and control results in the present context. All along the section, a sequence $S = (e_j)_{j \in \mathbb{N}}$, $e_j \to 0$ is fixed and $\varepsilon, \varepsilon' \in S$. This precision is sometimes omitted for concision. Fix a family of interior hypersurfaces $\Sigma_\varepsilon$ with

$$\Sigma_\varepsilon' \subset \text{Int}(\Sigma_\varepsilon) \subset \Sigma_\varepsilon \subset \text{Int}(\Sigma), \quad \varepsilon < \varepsilon', \quad \bigcup_{\varepsilon > 0} \Sigma_\varepsilon = \text{Int}(\Sigma). \quad (3.9)$$

Let

$$\Gamma \subset T^* (\mathbb{R} \times \text{Int}(\Sigma)) \setminus 0 \text{ be a closed and conic set.} \quad (3.10)$$

We define spaces adapted to $\Gamma$, i.e., measuring different regularities near and away from $\Gamma$. In the applications below, we shall take $\Gamma = G^X$ for the study of the Cauchy problem and $\Gamma = \mathcal{E}^X \cup \mathcal{G}^X = \mathcal{E}^X$ for the study of the control problem.

To this end, let $\varepsilon \mapsto \Gamma_\varepsilon, \varepsilon \in S$, be a family of closed conic subset of $T^* (\mathbb{R} \times \text{Int}(\Sigma)) \setminus 0$ such that

$$\Gamma_\varepsilon \text{ is closed and conic for any } \varepsilon, \quad \Gamma_\varepsilon \subset \text{Int}(\Gamma_\varepsilon), \quad \varepsilon < \varepsilon', \quad \Gamma = \bigcap_{\varepsilon > 0} \Gamma_\varepsilon. \quad (3.11)$$

Next, fix a family of cutoff functions

$$\varphi_\varepsilon \in C^\infty_c ((0, T) \times \text{Int}(\Sigma)), \quad \varphi_\varepsilon \equiv 1 \text{ on } [\varepsilon, T - \varepsilon] \times \Sigma_\varepsilon. \quad (3.12)$$

and a family of cutoff operators

$$B^\varepsilon_f \in \Psi^0_{\text{phg}} ((0, T) \times \text{Int}(\Sigma)), \quad B^\varepsilon_f \text{ selfadjoint on } L^2 (\mathbb{R} \times \Sigma),$$

$$\text{WF}(B^\varepsilon_f) \cap \Gamma_\varepsilon = \emptyset, \quad \text{WF}(\varphi_\varepsilon (1 - B^\varepsilon_f)) \cap T^*_\varepsilon (\mathbb{R} \times \text{Int}(\Sigma)) \setminus \Gamma_\varepsilon = \emptyset, \quad (3.13)$$

$$\text{WF}(B^\varepsilon_f) \subset \text{Ell}(B^\varepsilon_f), \quad \varepsilon < \varepsilon' \in S, \quad B^\varepsilon_f \varphi_\varepsilon = B^\varepsilon_f \varphi_\varepsilon.$$

Note that once $\Gamma$ will be fixed, (see Sections 3.3 and 4), a more explicit expression for the symbol of the operators $B^\varepsilon_f$ will be given.

Next, we define for $k \geq s$, the Banach space

$$H^k_{\text{comp}, \Gamma, \varepsilon}(\Sigma_\varepsilon) = \left\{ f \in H^k_{\text{comp}}((0, T) \times \text{Int}(\Sigma)), \text{supp}(f) \subset [\varepsilon, T - \varepsilon] \times \Sigma_\varepsilon, (1 - B^\varepsilon_f) f \in H^k_{\text{comp}}((0, T) \times \text{Int}(\Sigma)) \right\},$$

normed by

$$\| f \|_{H^k_{\text{comp}, \Gamma, \varepsilon}(\Sigma_\varepsilon)}^2 := \| f \|_{H^k((0, T) \times \Sigma)}^2 + \| (1 - B^\varepsilon_f) f \|_{H^k(\mathbb{R} \times \Sigma)}^2$$

Notice that $(1 - B^\varepsilon_f)$ measures regularity in $\Gamma_\varepsilon$ and therefore, for $f \in H^k_{\text{comp}, \Gamma, \varepsilon}$, we have $f = \varphi_\varepsilon f$ and $\text{WF}(f) \subset T^* (\mathbb{R} \times \text{Int}(\Sigma)) \setminus \Gamma_\varepsilon$. We define the Fréchet space

$$H^k_{\text{comp}, \Gamma}(\Sigma_\varepsilon) = \bigcup_{\varepsilon > 0} H^k_{\text{comp}, \Gamma, \varepsilon}(\Sigma_\varepsilon) = \left\{ f \in H^k_{\text{comp}}((0, T) \times \text{Int}(\Sigma)), \text{WF}(f) \cap \Gamma = \emptyset \right\}$$

with topology given by the seminorms $\| \cdot \|_{H^k_{\text{comp}, \Gamma, \varepsilon}(\Sigma_\varepsilon)}$ (taken for a sequence of $\varepsilon$ going to zero). Functions/distributions in the space $H^k_{\text{comp}, \Gamma}(\Sigma_\varepsilon)$ are $H^k$ overall and microlocally $H^k$ ($k \geq s$) on $\Gamma$. In case $k = s$, we simply have $H^s_{\text{comp}, \Gamma}(\Sigma_\varepsilon) = H^s_{\text{comp}}((0, T) \times \text{Int}(\Sigma))$.

Similarly, we define for $k \leq s$, the vector space

$$H^k_{\text{loc}, \Gamma, \varepsilon}(\Sigma_\varepsilon) = \left\{ u \in \mathcal{D}'((0, T) \times \text{Int}(\Sigma)), \varphi_\varepsilon u \in H^k_{\text{comp}}((0, T) \times \text{Int}(\Sigma)), B^\varepsilon_f u \in H^s_{\text{comp}}((0, T) \times \text{Int}(\Sigma)) \right\},$$

endowed with the seminorm

$$\| u \|_{H^k_{\text{loc}, \Gamma, \varepsilon}(\Sigma_\varepsilon)}^2 := \| \varphi_\varepsilon u \|_{H^k((0, T) \times \Sigma)}^2 + \| B^\varepsilon_f u \|_{H^s((0, T) \times \Sigma)}^2.$$
We define as well the Fréchet space

$$H_{\text{loc},r}(\Sigma_T) = \bigcap_{\epsilon>0} H_{\text{loc},r,\epsilon}(\Sigma_T)$$

$$= \left\{ f \in \mathcal{D}'(0, T) \times \text{Int}(\Sigma), f \in H_{\text{loc}}^s(0, T) \times \text{Int}(\Sigma), Bf \in H_{\text{comp}}^s(0, T) \times \text{Int}(\Sigma) \right\}$$

for all $B \in \Psi^0((0, T) \times \text{Int}(\Sigma))$, s.t. $\text{WF}(B) \cap \Gamma = \emptyset$,

with topology given by the seminorms $\| \cdot \|_{H_{\text{loc},r,\epsilon}(\Sigma_T)}$. Functions/distributions in the space $H_{\text{loc},r}(\Sigma_T)$ are locally $H^k$ overall and microlocally $H^k$ ($s \geq k$) outside of $\Gamma$. Remark again that in case $k = s$, we simply have $H_{\text{loc},r}(\Sigma_T) = H_{\text{loc}}^s((0, T) \times \text{Int}(\Sigma))$.

**Lemma 3.3.** For $s, k \in \mathbb{R}, s \geq k$ the sesquilinear map

$$C_c^\infty((0, T) \times \text{Int}(\Sigma)) \times C_c^\infty((0, T) \times \text{Int}(\Sigma)) \rightarrow \mathbb{C}, \quad (f, u) \mapsto \int_{(0,T)\times \Sigma} f(t, x)\overline{u(t, x)}\,dt\,d\sigma(x),$$

extends uniquely as a continuous sesquilinear map

$$H_{\text{comp},r}^{-s,k}(\Sigma_T) \times H_{\text{loc},r}^k(\Sigma_T) \rightarrow \mathbb{C},$$

which we shall denote $(f, u)_{H_{\text{comp},r}^{-s,k} \times H_{\text{loc},r}^k}$. Moreover, for $(f, u) \in H_{\text{comp},r}^{-s,k} \times H_{\text{loc},r}^k$, we have

$$|(f, u)_{H_{\text{comp},r}^{-s,k} \times H_{\text{loc},r}^k}| \leq \| f \|_{H_{\text{comp},r}^{-s,k}(\Sigma_T)} \| u \|_{H_{\text{loc},r}^k(\Sigma_T)}. \quad \quad \text{Proof.} \quad \text{Let} \quad (f, u) \in C_c^\infty((0, T) \times \text{Int}(\Sigma)) \times C_c^\infty((0, T) \times \text{Int}(\Sigma)). \text{ Fix} \; \epsilon > 0 \text{ so that} \; \varphi \epsilon f = f. \text{ We compute}

$$|\langle f, u \rangle_{L^2((0,T)\times \Sigma)}| = | \langle f, \varphi \epsilon u \rangle_{L^2((0,T)\times \Sigma)} | 

\leq \left| \left( f, B_{\epsilon}^2 \varphi \epsilon u \right)_{L^2((0,T)\times \Sigma)} \right| + \left| \left( f, (1 - B_{\epsilon}^2) \varphi \epsilon u \right)_{L^2((0,T)\times \Sigma)} \right| 

\leq \| f \|_{H^{-s,k}(\Sigma_T)} \| B_{\epsilon}^2 \varphi \epsilon u \|_{H^{-s,k}(\Sigma_T)} + \| (1 - B_{\epsilon}^2) f \|_{H^{-s,k}(\Sigma_T)} \| \varphi \epsilon u \|_{H^{s,k}(\Sigma_T)} 

\leq \| f \|_{H^{-s,k}(\Sigma_T)} \| u \|_{H^{s,k}(\Sigma_T)}.$$

Then, the density of $C_c^\infty((0, T) \times \text{Int}(\Sigma))$ in $H_{\text{comp},r}^{-s,k}(\Sigma_T)$ and that of $C_c^\infty((0, T) \times \text{Int}(\Sigma))$ in $H_{\text{loc},r}^k(\Sigma_T)$ prove the statement. \hfill \Box

**Lemma 3.4.** For all $s, k \in \mathbb{R}$, $k \geq s$, we have $(H_{\text{comp},r}^{-s,k}(\Sigma_T))' = H_{\text{loc},r}^{-s,k}(\Sigma_T)$.

**Proof.** Lemma 3.3 proves $H_{\text{comp},r}^{-s,k}(\Sigma_T) \subset (H_{\text{comp},r}^{-s,k}(\Sigma_T))'$. Suppose $\mu \in (H_{\text{comp},r}^{-s,k}(\Sigma_T))'$. Then, since $C_c^\infty((0, T) \times \text{Int}(\Sigma)) \subset H_{\text{comp},r}^{-s,k}(\Sigma_T)$, $\mu \in \mathcal{D}'(0, T) \times \text{Int}(\Sigma)$. Fix $\epsilon > 0$. Then for $\varphi \in C_c^\infty((0, T) \times \text{Int}(\Sigma))$

$$|\langle \varphi, \mu \varphi \epsilon \rangle| = \| \varphi \|_{H^{s,k}(\Sigma_T)} \| \varphi \epsilon \|_{H^{-s,k}(\Sigma_T)} \| \varphi \epsilon \|_{H^{s,k}(\Sigma_T)}.$$

So, since $k \geq s$, we obtain in particular

$$|\langle \varphi, \mu \varphi \epsilon \rangle| \leq C_{\epsilon} \| \varphi \|_{H^{s,k}(\Sigma_T)} \| \varphi \epsilon \|_{H^{-s,k}(\Sigma_T)} \leq C_{\varphi} \| \varphi \|_{H^{s,k}(\Sigma_T)},$$

and hence $\varphi \mu \in H_{\text{comp},r}^{-s,k}(0, T) \times \text{Int}(\Sigma)$ with

$$\| \varphi \mu \|_{H^{-s,k}(\Sigma_T)} \leq C_{\varphi}. \quad (3.14)$$

Fix any $\epsilon \in \mathcal{S}$ and $\chi \in C_c^\infty((0, T) \times \text{Int}(\Sigma))$. Then there exists $\epsilon_0 > 0$ depending only on $\epsilon$ such that for $\epsilon_0 > \epsilon' > 0$, $B_{\epsilon'} \chi \in H_{\text{comp},r}^{-s,k}(\Sigma_T)$. Choose $\epsilon' < \epsilon_0$ small enough so that $\text{WF}(1 - B_{\epsilon'}^2) = \emptyset$. Then, we have

$$|\langle B_{\epsilon'}^2 \mu, \chi \rangle| = |\langle \mu, B_{\epsilon'}^2 \chi \rangle| \leq C_\epsilon \| B_{\epsilon'}^2 \chi \|_{H^{s,k}(\Sigma_T)} + \| (1 - B_{\epsilon'}^2) B_{\epsilon'}^2 \chi \|_{H^{-s,k}(\Sigma_T)} \leq C_\epsilon \| B_{\epsilon'}^2 \chi \|_{H^{s,k}(\Sigma_T)} + \| \chi \|_{H^{-s,k}(\Sigma_T)} \leq C_\epsilon \| B_{\epsilon'}^2 \chi \|_{H^{s,k}(\Sigma_T)}.$$

Therefore, $B_{\epsilon'}^2 \mu \in H_{\text{comp},r}^{-s,k}(0, T) \times \text{Int}(\Sigma)$ with

$$\| B_{\epsilon'}^2 \mu \|_{H^{-s,k}(\Sigma_T)} \leq C_{\epsilon'}.$$ 

This, together with (3.14) proves that $\mu \in H_{\text{loc},r}^{-s,k}(\Sigma_T)$, and hence the lemma. \hfill \Box
3.3 Definition of solutions and well-posedness

Observe that \( G^k \) and \( \widehat{G}^k \) satisfy (3.10) and for \( k \geq s \), we therefore have Fréchet spaces \( H_{\text{comp},G^k}^s(\Sigma_T) \), \( H_{\text{comp},\widehat{G}^k}^s(\Sigma_T) \) with dual spaces \( H_{\text{loc},G^k}^{-s,k}(\Sigma_T) \), \( H_{\text{loc},\widehat{G}^k}^{-s,k}(\Sigma_T) \).

With these definitions in hand, we can reformulate Proposition 3.1 as follows: For any \( T > 0 \), the map

\[
H_0^1(M) \times L^2(M) \times L^2(0,T;L^2(M)) \to H_{\text{loc},G^k}^{-1,k}(\Sigma_T) \times H_{\text{loc},\widehat{G}^k}^{-1,k}(\Sigma_T)
\]

\[
(u_0, u_1, F) \mapsto (u|_{\Sigma}, \partial_t u|_{\Sigma})
\]

(where \( u \) is solves (3.1)) is continuous.

We can now study the well-posedness for the control problem (1.6). We first recall that, given \( f_0, f_1 \in C^\infty(\mathbb{R} \times \Sigma) \), \( f_0 \delta_\Sigma \) and \( f_1 \delta_\Sigma \) are usual distributions defined by (1.5).

**Lemma 3.5.** Given \( T > 0 \), assume that the functions \( v \in C^\infty([0,T] \times M \setminus \Sigma) \cap C^1([0,T);L^2(M)) \) \( u, F \in C^\infty([0,T] \times \Sigma) \) solve

\[
\square v = f_0 \delta_\Sigma + f_1 \delta_\Sigma \text{ in } \mathcal{D}'((0,T) \times \text{Int}(M)), \quad \text{and} \quad \square u = F.
\]

Then, we have the identity

\[
\left[ (\partial_t v, u)_{L^2(M)} - (v, \partial_t u)_{L^2(M)} \right]_0^T + (v, F)_{L^2((0,T) \times M)} = \int_{(0,T) \times \Sigma} \left( f_0 u|_{\Sigma} - f_1 \partial_t u|_{\Sigma} \right) dtd\sigma.
\]

The duality property of Lemma 3.3, together with the formula of Lemma 3.5, valid for smooth functions, and (3.15) suggest that taking \( f_0 \in H_{\text{comp},G^k}^{-1,k}(\Sigma_T) \) and \( f_1 \in H_{\text{comp},\widehat{G}^k}^{-1,k}(\Sigma_T) \) could be an appropriate set of spaces for control functions, as well as the following definition of transposition solutions for the control problem.

**Definition 3.6.** Given \( T > 0 \), \((v_0, v_1) \in L^2(M) \times H^{-1}(M), f_0 \in H_{\text{comp},G^k}^{-1,k}(\Sigma_T), f_1 \in H_{\text{comp},\widehat{G}^k}^{-1,k}(\Sigma_T) \), we say that \( v \) is a solution of (1.6) if \( v \in L^2((0,T);L^2(M)) \) and for any \( F \in L^2((0,T);L^2(M)) \), we have

\[
\int_0^T (v, F)_{L^2(M)} dt = \langle v_1, u(0) \rangle_{H^{-1}(M),H^1(M)} - \langle v_0, \partial_t u(0) \rangle_{L^2(M)} + \langle f_0, u|_{\Sigma} \rangle_{H_{\text{comp},G^k}^{-1,k}(\Sigma_T),H_{\text{comp},G^k}^{-1,k}(\Sigma_T)} - \langle f_1, \partial_t u|_{\Sigma} \rangle_{H_{\text{comp},\widehat{G}^k}^{-1,k}(\Sigma_T),H_{\text{comp},\widehat{G}^k}^{-1,k}(\Sigma_T)},
\]

where \( u \) is the unique solution to

\[
\begin{aligned}
\square u &= F \quad \text{on } (0,T) \times \text{Int}(M) \\
(u, \partial_t u)|_{t=T} &= (0,0) \quad \text{in } \text{Int}(M).
\end{aligned}
\]

Note in particular that taking \( F \in C^\infty_c((0,T) \times \text{Int}(M)) \) implies that such a solution is a solution of the first equation of (1.6) in the sense of distributions.

**Theorem 3.7.** Let \( T > 0 \). For all \((v_0, v_1) \in L^2(M) \times H^{-1}(M) \) and for all \( f_0 \in H_{\text{comp},G^k}^{-1,k}(\Sigma_T) \) and \( f_1 \in H_{\text{comp},\widehat{G}^k}^{-1,k}(\Sigma_T) \), there exists a unique \( v \in L^2((0,T);L^2(M)) \) solution of (1.6) in the sense of Definition 3.6. The linear map

\[
L^2(M) \times H^{-1}(M) \times H_{\text{comp},G^k}^{-1,k}(\Sigma_T) \times H_{\text{comp},\widehat{G}^k}^{-1,k}(\Sigma_T) \to L^2((0,T);L^2(M))
\]

\[
(v_0, v_1, f_0, f_1) \mapsto v
\]

is continuous.

**Remark 3.8.** Note that, given two different times \( T < T' \), an initial data \((v_0, v_1)\) and control functions \( f_0, f_1 \) compactly supported in \((0,T) \subset (0,T')\), the above definition theorem yield two different solutions: one defined on \( (0,T) \) and one defined on \((0,T')\). However, one can observe that these two solutions coincide by extending all test functions \( F \in L^2((0,T);L^2(M)) \) by zero on \((T,T')\) to obtain test functions in \( L^2((0,T');L^2(M)) \). With this in mind, Theorem 1.4 is a direct consequence (and a simplified version) of Theorem 3.7.
Proof of Theorem 3.7. First, we define
\[
\ell(F) := \langle v_1, u(0) \rangle_{H^{-1}(\Omega), H^{1}(\Omega)} - \langle v_0, \partial_t u(0) \rangle_{L^2(\Omega)} + \langle f_0, u \rangle_{H^{-\frac{1}{2}}(\Sigma_T), H^{\frac{1}{2}}(\Sigma_T)} - \langle f_1, \partial_t u \rangle_{H^{\frac{1}{2}}(\Sigma_T), H^{-\frac{1}{2}}(\Sigma_T)},
\]
and prove that it is a continuous linear form on \(L^2(0, T; L^2(\Omega))\), with appropriate norm. We have
\[
|\ell(F)| \leq \|v_1\|_{H^{-1}(\Omega)} + \|v_0\|_{L^2(\Omega)} + R
\]
and
\[
|\ell(F)| \leq \|(v_0, v_1)\|_{L^2(0, T; L^2(\Omega))} + R,
\]
with
\[
R = \left| \langle f_0, u \rangle_{H^{-\frac{1}{2}}(\Sigma_T), H^{\frac{1}{2}}(\Sigma_T)} - \langle f_1, \partial_t u \rangle_{H^{\frac{1}{2}}(\Sigma_T), H^{-\frac{1}{2}}(\Sigma_T)} \right|.
\]
From the definition of the spaces in Section 3.2, there exists \(\epsilon > 0\) such that \((f_0, f_1) \in H_{\text{comp}, g^2, \epsilon}(\Sigma_T) \times H_{\text{comp}, g^2, \epsilon}^{0, \frac{1}{2}}(\Sigma_T)\) and hence, we obtain from Lemma 3.3,
\[
R \leq \|f_0\|_{H^{-\frac{1}{2}}(\Sigma_T)} \|u\|_{H^{\frac{1}{2}}(\Sigma_T)} + \|f_1\|_{H^{\frac{1}{2}}(\Sigma_T)} \|\partial_t u\|_{H^{-\frac{1}{2}}(\Sigma_T)}.
\]
Proposition 3.1 with \(A = 1 - B_{\epsilon}^{\text{loc}}\) (satisfying the appropriate conditions) then yields
\[
R \leq C \left( \|f_0\|_{H^{-\frac{1}{2}}(\Sigma_T)} + \|f_1\|_{H^{\frac{1}{2}}(\Sigma_T)} \right) \|F\|_{L^2(0, T; L^2(\Omega))}.
\]
Coming back to \(\ell\), we have obtained the existence of \(\epsilon \in \mathcal{S}, C \epsilon > 0\) such that
\[
|\ell(F)| \leq C \left( \|(v_0, v_1)\|_{L^2(0, T; L^2(\Omega))} + C \epsilon \|(f_0, f_1)\|_{H^{-\frac{1}{2}}(\Sigma_T) \times H^{\frac{1}{2}}(\Sigma_T)} \right) \|F\|_{L^2(0, T; L^2(\Omega))}.
\]
Hence, \(\ell\) is a continuous linear form on \(L^2(0, T; L^2(\Omega))\). There is thus a unique \(v \in L^2(0, T; L^2(\Omega))\) such that
\[
\ell(F) = \int_0^T (v(t), F(t))_{L^2(\Omega)} dt \text{ for all } F \in L^2(0, T; L^2(\Omega)),
\]
that is precisely the definition of a solution of (1.6) in Definition 3.6. This solution moreover satisfies, for \((f_0, f_1) \in H_{\text{comp}, g^2, \epsilon}^{-\frac{1}{2}}(\Sigma_T) \times H_{\text{comp}, g^2, \epsilon}^{\frac{1}{2}}(\Sigma_T)\), the estimate
\[
\|v\|_{L^2(0, T; L^2(\Omega))} \leq \|(v_0, v_1)\|_{L^2(0, T; L^2(\Omega))} + C \epsilon \|(f_0, f_1)\|_{H^{-\frac{1}{2}}(\Sigma_T) \times H^{\frac{1}{2}}(\Sigma_T)},
\]
which is the continuity statement. This concludes the proof of the Theorem. \(\square\)

4 Observability and controllability for the wave equation

The aim of this section is to study the observability of (3.1) from \(\Sigma\). In particular, we prove

Theorem 4.1 (Observability). Let \(\chi \in C^\infty(\mathbb{R})\) have \(\chi \equiv 1\) on \((-\infty, -\frac{1}{2}]\) and \(\text{supp} \chi \subset (-\infty, -\frac{1}{2}]\). Under Assumption GC-(0, T), there exists \(\delta_0 > 0\), so that for all \(\delta \in (0, \delta_0)\), all \(A_\delta \in \Psi^0_{\text{phg}}((0, T) \times \text{Int}(\Sigma))\) with principal symbol \(\chi(\Sigma(x, \cdot) + \frac{\partial \phi}{\partial t})\phi_\delta\), where \(\phi_\delta \in C^\infty_c((0, T) \times \text{Int}(\Sigma))\) with \(\phi_\delta \equiv 1\) on \([\delta, T - \delta] \times \Sigma_\delta\) where \(\Sigma_\delta\) is as in (3.9), for all \(N > 0\), there exists \(c_N > 0\) so that for any solution \(u\) to (3.1), we have
\[
c_N \|(u_0, u_1)\|^2_{H^{-\frac{1}{2}}(\Omega) \times L^2} \leq \|\phi_\delta \partial_t u\|_{H^{-\frac{1}{2}}(\Sigma_\delta)}^2 + \|\phi_\delta u\|_{H^{-\frac{1}{2}}(\Sigma_\delta)}^2 + \|A_\delta(\partial_t u)\|_{L^2(\Sigma_\delta)}^2 + \|A_\delta(u)\|_{L^2(\Sigma_\delta)}^2 + \|F\|_{L^2(0, T; L^2(\Omega))}.
\]
Let us briefly explain why the observability inequality of Theorem 4.1 implies Theorem 1.9.
Proof of Theorem 1.9. We apply Theorem 4.1 to the function \( u(t, x) = e^{\partial_t} v(x) \) with \( v \in H^1_0(M) \cap H^2(M) \). First observe that \( A_b \) is bounded on \( L^2 \) and hence
\[
\| A_b \partial_t u \|_{L^2(\mathbb{R} \times \Sigma)} \leq C \| \partial_t u \|_{L^2([0,T] \times \Sigma)} \leq C \| \partial_t v \|_{L^2(\Sigma)}.
\]
Observe also that there exists \( \delta_0 > 0 \) so that \( \varphi_\delta D_1 \) is elliptic on \( \text{WF}(A_b) \) and therefore,
\[
\|A_b \partial_t u\|_{H^1(\mathbb{R} \times \Sigma)} \leq C(\|\varphi_\delta D_1 u\|_{L^2(\mathbb{R} \times \Sigma)} + \|u\|_{L^2([0,T] \times \Sigma)} \leq C(\lambda) \|v\|_{L^2(\Sigma)}).
\]
Note also that
\[
\Delta u = e^{\partial_t} (-\Delta_x - \lambda^2) v
\]
and hence the right hand side of (4.1) is bounded by
\[
C(\|\partial_t v\|_{L^2(\Sigma)} + (\lambda) \|v\|_{L^2(\Sigma)} + (\|\Delta_x - \lambda^2\| v\|_{L^2([0,T] \times M)}).
\]
Finally, noticing that
\[
(u_{t=0}, \partial_t u_{t=0}) = (v, \lambda v),
\]
gives
\[
(\lambda) \|u\|_{L^2(\Sigma)} \leq \|(u_{t=0}, \partial_t u_{t=0})\|_{H^1(\Sigma) \times L^2(\Sigma)},
\]
finishing the proof of Theorem 1.9. \(\square\)

4.1 The geometric assumption \( T \) GC-C

To prove Theorem 4.1 we start with a dynamical lemma where we show that the \textit{a priori} weaker assumption GC-(0,\( T \)) implies the stronger assumption

\textbf{Assumption GC-}(\( \varepsilon, T \)). For all \( p \in Z \), we have
\[
\bigcup_{s \in \mathbb{R}} \{\varphi(s, p)\} \cap T_\varepsilon \cap T^*_{(e^{-\varepsilon} \leq 0)} (\mathbb{R} \times M) \neq \emptyset.
\]
Recall that \( Z \) is as in (2.4).

\textbf{Lemma 4.2.} Suppose that Assumption GC-(0,T) holds. Then there exists \( \varepsilon > 0 \) so that Assumption GC-(\( \varepsilon, T \)) holds.

\textbf{Proof.} We define \( Z^+_\varepsilon := Z \cap \{\tau = \pm 1, t = 0\} \). We shall show that Assumption GC-(0,T) implies the existence of \( \varepsilon > 0 \) such that
\[
\bigcup_{s \in \mathbb{R}} \{\varphi(s, p)\} \cap T_\varepsilon \cap T^*_{(e^{-\varepsilon} \leq 0)} (\mathbb{R} \times M) \neq \emptyset, \quad \text{for all } p \in Z^+_\varepsilon. \tag{4.2}
\]
We first show that (4.2) implies the lemma. With the identification \( b^*T^* (\mathbb{R} \times M) \approx T^* \mathbb{R} \times b^*T^* M \), consider \( p = (t_0, \tau, q) \in (T^* \mathbb{R} \times b^*T^* M) \cap Z \). Let \( M_\lambda \) be multiplication in the fiber by \( \lambda > 0 \). Then,
\[
p' = \varphi(t_0, \text{sgn} \tau, M_{|\tau|}^{-1} (t_0, \tau, q)) \in Z^+_\varepsilon \cup Z^-_\varepsilon.
\]
According to the homogeneity of \( \varphi \), see (2.8), and the flow property (2.7), we have
\[
\bigcup_{s \in \mathbb{R}} \{M_{|\tau|} \varphi(s, p)\} = \bigcup_{s \in \mathbb{R}} \{\varphi(s|\tau|, M_{|\tau|}^{-1} p)\} = \bigcup_{s \in \mathbb{R}} \{\varphi(s|\tau|, \varphi(t_0, \text{sgn} \tau, M_{|\tau|}^{-1} p))\} = \bigcup_{s \in \mathbb{R}} \{\varphi(s, p')\}. \tag{4.3}
\]
But, by (4.2), since \( p' \in Z^+_\varepsilon \cup Z^-_\varepsilon \), we have
\[
\bigcup_{s \in \mathbb{R}} \{\varphi(s, p')\} \cap T_\varepsilon \cap T^*_{(e^{-\varepsilon} \leq 0)} (\mathbb{R} \times M) \neq \emptyset,
\]
and hence homogeneity of \( \varphi, T_\varepsilon \) and \( T^*_{(e^{-\varepsilon} \leq 0)} (\mathbb{R} \times M) \) together with (4.3) completes the proof of the lemma from (4.2).
Theorem 3.34], the Melrose–Sjöstrand generalized bicharacteristic flow

Therefore, for each \( p \in Z_1^- \), there exists \( s_p > 0 \) and \( s_p \in (e_p, T - e_p) \) such that

\[
\phi(s_p, p) \in T_{e_p} \cap T_{(e_p, T - e_p)} \times \Sigma_p (\mathbb{R} \times M).
\]

Let \( \beta \) be a defining function for \( \Sigma \) near \( \phi(s_p, p) \), and consider \( g(s, q) = \beta \circ \pi_0 \circ \phi(s, q) \) for \( (s, q) \) in a neighborhood \( N_p \) of \( (s_p, p) \), where \( \pi_0 : T^* (\mathbb{R} \times \text{Int}(M)) \to \mathbb{R} \times \text{Int}(M) \) is the canonical projection. By [MS78, Theorem 3.34], the Melrose–Sjöstrand generalized bicharacteristic flow \( \phi \) is continuous and so \( g \) is continuous on \( N_p \).

Moreover, since \( \Sigma \) is an interior hypersurface, there exists \( \delta_p \) so that

\[
g(\cdot, q) : (s_p - \delta_p, s_p + \delta_p) \to j(T^* (\mathbb{R} \times \text{Int}(M)) \cap \text{Char}(\square)) \subset Z
\]
is C\(^1\) for \( q \) in a neighborhood of \( p \) since \( \phi \) coincides with the usual bicharacteristic flow of \( \phi \) near \( \phi(s_p, p) \).

Notice that \( \phi(s_p, p) \in T_{e_p} \) implies

\[
\partial_s g(s_p, p) = (d\beta(\pi_0 \circ \phi(s_p, p))) d\pi_0(\phi(s_p, p)), H_{\pi(\phi(s_p, p))} \neq 0.
\]

according to Remark 2.1. Hence by the implicit function theorem [Kum80], the equation \( g(s, q) = 0 \) defines a continuous function \( s = s(q) \) near \( q = p \). In particular, set

\[
\delta_0 = \min \left( \frac{s_p - T}{2}, \frac{T - s_p}{2} \right).
\]

Then there is a neighborhood, \( U_p \) of \( p \) and a continuous function \( s : U_p \to \mathbb{R} \) with \( s(p) = s_p \), such that

\[
\phi(s(q)) \in T_{e_p/2} \cap T_{(e_p/2, T - e_p/2)} \times \Sigma_{e_p/2} (\mathbb{R} \times M) \text{ and } |s(q) - s_p| < \delta_0 \text{ for all } q \in U_p.
\]

Since

\[
Z_1^- = j(\text{Char}(\square) \cap \{ \tau = -1, \, t = 0 \})
\]
is compact, we may extract from the cover \( Z_1^- \subset \bigcup_{p \in Z_1^-} U_p \) a finite cover \( \{ U_{e_p} \}_{i=1}^n \). Then taking \( \epsilon = \min_{1 \leq i \leq n} \epsilon_{e_p}/2, \) we have that for all \( p \in Z_1^- \),

\[
\bigcup_{s \in (0, T)} \{ \phi(s, p) \} \cap T_E \cap T_{(e_p, T - e_p)} \times \Sigma_p (\mathbb{R} \times M) \neq 0.
\]

In particular, (4.2) holds, which concludes the proof of the lemma. \( \square \)

### 4.2 Observability at High Frequency

The aim of this section is to prove the following proposition, which is a high-frequency version of Theorem 4.1. The estimates in these results differ in two respects: here in Proposition 4.4 there is \( \| \langle u_0, u_1 \rangle \|_{L^2_x H^{-1}} \) in the right handside, so that this estimate does not care about low frequencies. The treatment of low frequencies in Theorem 4.1 (see Section 4.3 below) requires the addition of observation terms in an arbitrary weak norm \( \| \varphi_0 \partial_t u_0 \|_{H^{-\infty}(\mathbb{R} \times \Sigma)} \) which is not needed here.

**Proposition 4.3.** Let \( \chi \in C^\infty_c (\mathbb{R}) \) have \( \chi \equiv 1 \) on \((-\infty, -1]\) and \( \text{supp} \chi \subset (-\infty, -\frac{1}{2}] \). Under Assumption GC-(0, T), there exists \( \delta_0 > 0 \), so that for all \( \delta \in (0, \delta_0) \), all \( A_\delta \in Q^0_{\text{phg}} ((0, T) \times \text{Int}(\Sigma)) \) with principal symbol \( \chi \frac{\partial \langle \partial t - \chi \rangle}{\partial t} \varphi_\delta \), where \( \varphi_\delta \in C^\infty_c ((0, T) \times \text{Int}(\Sigma)) \) with \( \varphi_\delta \equiv 1 \) on \( [\delta, T - \delta] \times \Sigma_\delta \) where \( \Sigma_\delta \) is as in (3.9), there exists \( c > 0 \) so that for any solution \( u \) to (3.1), we have

\[
c \| \langle u_0, u_1 \rangle \|_{L^2_x L^2_x} \leq \| A_\delta (u_{\Sigma_\delta}) \|_{L^2_x (\mathbb{R} \times \Sigma)}^2 + \| A_\delta (\partial_t u_{\Sigma_\delta}) \|_{L^2_x (\mathbb{R} \times \Sigma)}^2 + \| F \|_{L^2_x ((0, T) \times M)}^2 + \| \langle u_0, u_1 \rangle \|_{L^2_x H^{-1}}^2. \tag{4.4}
\]
We begin with two preliminary lemmas. We again work in fermi normal coordinates near \( \Sigma \). A more general version of version of the following Lemma is given in [Hor85, Lemma 23.2.8], but we decided to include a short proof in this particular context for the sake of readability.

**Lemma 4.4.** Denote \( \Box = -D_x^2 + D_{x'}^2 + r(x, D_x') + c(x)D_{x'} \), where \( r \) is defined in Section 2.1. For any \( 0 < \nu < 1 \), there exist \( \epsilon > 0 \) and \( \Lambda_x, \Lambda_x \in \mathcal{C}^\infty((-\epsilon, \epsilon); \Psi_{\text{phg}}^1(\mathbb{R} \times \mathbb{R}^{n-1})) \) with

\[
\sigma(\Lambda_x) = \sigma(\Lambda_x) = \sqrt{t^2 - r(x, \xi')} \text{ on } \{t^2 - r(x, \xi') \geq \nu t^2 \}
\]

such that for all \( b \in \mathcal{C}^\infty((-\epsilon, \epsilon); S^0_{\text{phg}}(T^*(\mathbb{R} \times \mathbb{R}^{n-1}))) \) with \( \text{supp } b \subset \{t^2 - r(x, \xi') \geq \nu t^2 \}, \)

\[
\text{Op}(b) \Box = \text{Op}(b)[(D_{x'} - \Lambda_\nu)(D_{x'} + \Lambda_\nu) + R]
\]

where \( R, \tilde{R} \in \mathcal{C}^\infty((-\epsilon, \epsilon); \Psi^{-\infty}_{\text{phg}}(\mathbb{R} \times \mathbb{R}^{n-1})) \).

**Proof.** Throughout this proof, we will write \( S^k_{\text{tan}} \) for \( C^\infty((-\epsilon, \epsilon); S^k_{\text{phg}}(T^*(\mathbb{R} \times \mathbb{R}^{n-1}))) \) and \( \Psi^k_{\text{tan}} \) for the corresponding quantization \( C^\infty((-\epsilon, \epsilon); \Psi^k_{\text{phg}}(\mathbb{R} \times \mathbb{R}^{n-1})) \). For an operator \( A \in \Psi^\infty_{\text{tan}} \), we will write

\[
\text{WF}(A) = \bigcup_{x_1} \text{WF}(A_{x_1}),
\]

where \( A_{x_1} \) is the pseudodifferential operator acting on \( \mathbb{R}^{n-1} \) at \( x_1 = y_1 \).

Given \( 0 < \nu < 1 \), we let \( \tilde{\chi}(t, x, \tau, \xi') \in S^0_{\text{tan}} \) with

\[
\tilde{\chi} \equiv 1 \text{ on } \{t^2 - r(x, \xi') \geq \nu t^2 / 3\}, \quad \text{supp } \tilde{\chi} \subset \{t^2 - r(x, \xi') \geq \nu t^2 / 4\}.
\]

Then, for \( (t, x, \tau, \xi') \in \text{supp } \tilde{\chi} \), we have the following factorization

\[
\sigma(\Box) = -t^2 + \xi_1^2 + r(x, \xi') = [\xi_1 + \sqrt{t^2 - r(x, \xi')}] [\xi_1 - \sqrt{t^2 - r(x, \xi')}].
\]

We thus let \( \lambda_0(t, x, \tau, \xi') = \sqrt{t^2 - r(x, \xi')} \) and \( \lambda_0 = \text{Op}(\tilde{\chi} \lambda_0) \).

Now, write \( D_{x'} = D_{x'} - \Lambda_0 + \lambda_0 \) so that

\[
\Box = (D_{x'} - \lambda_0)D_{x'} + \lambda_0D_{x'} - D_{x'}^2 + r(x, D_{x'}) + (D_{x'} - \lambda_0)c(x) + [c(x), D_{x'}] + \lambda_0c(x)
\]

\[
= (D_{x'} - \lambda_0)(D_{x'} + \lambda_0 + c(x)) + \lambda_0^2 + [\Lambda_0 + c(x), D_{x'}] - D_{x'}^2 + r(x, D_{x'}) + \lambda_0c(x)
\]

\[
= (D_{x'} - \lambda_0)Q_0 + \tilde{R}_0,
\]

where

\[
Q_0 = D_{x'} + \lambda_0 + c(x) \in \Psi^0_{\text{tan}}, \quad \Psi^1_{\text{tan}},
\]

\[
\tilde{R}_0 = \lambda_0^2 + [\Lambda_0 + c(x), D_{x'}] - D_{x'}^2 + r(x, D_{x'}) + \lambda_0c(x) \in \Psi^2_{\text{tan}}.
\]

First, we remark that \( \sigma(Q_0) = \xi_1 + \tilde{\chi} \lambda_0 \). Second, noting that \( \sigma(\tilde{R}_0) \) is independent of \( \xi_1 \), we take \( \xi_1 = \lambda_0 \) on \( \tilde{\chi} \equiv 1 \) in

\[
\sigma(\Box) = \xi_1^2 - t^2 + r(x, \xi') = (\xi_1 - \lambda_0 \tilde{\chi})\sigma(Q_0) + \sigma(\tilde{R}_0)
\]

to obtain \( \sigma(\tilde{R}_0) = 0 \) on that set. This yields \( \tilde{R}_0 = R_0 + E_0 \) with \( R_0 \in \Psi^1_{\text{tan}} \) and \( E_0 \in \Psi^2_{\text{tan}} \) with \( \text{WF}(E_0) \cap \{r^2 - r(x, \xi') \geq \nu r^2 \} = \emptyset \). Indeed for \( \chi_1 \in S^0_{\text{tan}} \) with \( \text{supp } \chi_1 \subset \{\tilde{\chi} \equiv 1\} \) and \( \chi_1 \equiv 1 \) in a neighborhood of \( \{r^2 - r(x, \xi') \geq \nu r^2 \} \), \( \sigma(\text{Op}(\chi_1) \tilde{R}_0) = 0 \). Thus,

\[
\tilde{R}_0 = E_0 + R_0, \quad E_0 = \text{Op}(\chi_1) \tilde{R} \in \Psi^1_{\text{tan}}, \quad R_0 = \text{Op}(1 - \chi_1) \tilde{R}.
\]

This implies the first factorization formula with \( R \in \Psi^1_{\text{tan}} \). We now proceed with an induction to improve this remainder term.
Suppose we have for some $j \geq 0$

$$\square = (D_{x_1} - \Lambda_{-,j})(D_{x_1} + \Lambda_{+,j}) + R_j + E_j$$  \hspace{1cm} (4.5)$$
with $\Lambda_{-,j} \in \Psi^1_{\text{tan}}$, with principal symbol $\lambda_0\bar{\chi}$, $R_j \in \Psi^{-\infty}_{\text{tan}}$, and $E_j \in \Psi^2_{\text{tan}}$ with $\text{WF}(E_j) \cap \{r^2 - r(x, \xi') \geq \nu r^2\} = \emptyset$. Now, we want to adjust $\Lambda_{-,j}$ to improve the error $R_j$. Let $\lambda_{j+1} \in S^{-j}_{\text{tan}}$ have

$$\sigma(R_j) + \lambda_{j+1}\sigma(\Lambda_{+,j} + \Lambda_{-,j}) = 0 \quad \text{in a neighborhood of } \bar{\chi} \equiv 1. \hspace{1cm} (4.6)$$
This is possible since $\sigma(\Lambda_{-,j}) = \bar{\chi}\lambda_0$ is elliptic on a neighborhood of $\bar{\chi} \equiv 1$.

Now, observe that

$$\text{Op}(\lambda_{j+1})(D_{x_1} + \Lambda_{+,j}) + R_j = \text{Op}(\lambda_{j+1})(\Lambda_{-,j} + \Lambda_{+,j}) + R_j + \text{Op}(\lambda_{j+1})(D_{x_1} - \Lambda_{-,j})$$
$$= \text{Op}(\lambda_{j+1})(\Lambda_{-,j} + \Lambda_{+,j}) + R_j + (D_{x_1} - \Lambda_{-,j})\text{Op}(\lambda_{j+1})$$
$$+ [\text{Op}(\lambda_{j+1}), D_{x_1} - \Lambda_{-,j}]$$
$$= R_{j+1,1} + E_{j+1,1} + (D_{x_1} - \Lambda_{-,j})\text{Op}(\lambda_{j+1}),$$
where, according to (4.6), we have $R_{j+1,1} \in \Psi^{-j}_{\text{tan}}$ and $E_{j+1,1} \in \Psi^{-1-j}_{\text{tan}}$ with $\text{WF}(E_{j+1,1}) \cap \{\bar{\chi} = 1\} = \emptyset$. So, coming back to (4.5), we now obtain

$$\square = (D_{x_1} - \Lambda_{-,j})(D_{x_1} + \Lambda_{+,j}) + R_j + E_j$$
$$= (D_{x_1} - \Lambda_{-,j})(D_{x_1} + \Lambda_{+,j}) - \text{Op}(\lambda_{j+1})(D_{x_1} + \Lambda_{+,j}) + E_{j+1,1} + (D_{x_1} - \Lambda_{-,j})\text{Op}(\lambda_{j+1})$$
$$= (D_{x_1} - \Lambda_{-,j} - \text{Op}(\lambda_{j+1}))(D_{x_1} + \Lambda_{+,j}) + E_{j+1,1} + (D_{x_1} - \Lambda_{-,j})\text{Op}(\lambda_{j+1})$$
$$+ (D_{x_1} - \Lambda_{-,j} - \text{Op}(\lambda_{j+1}))(D_{x_1} + \Lambda_{+,j}) + E_{j+1,1} + (D_{x_1} - \Lambda_{-,j})\text{Op}(\lambda_{j+1})$$
$$= R_{j+1,1} + E_{j+1,1} + (D_{x_1} - \Lambda_{-,j} - \text{Op}(\lambda_{j+1}))(D_{x_1} + \Lambda_{+,j}) + E_{j+1,1} + (D_{x_1} - \Lambda_{-,j})\text{Op}(\lambda_{j+1})$$
where $R_{j+1} \in \Psi^{-j}_{\text{tan}}$ and $E_{j+1} \in \Psi^{-1-j}_{\text{tan}}$ with $\text{WF}(E_{j+1}) \cap \{\bar{\chi} = 1\} = \emptyset$. Putting $\Lambda_{-,j+1} = \Lambda_{-,j} + \text{Op}(\lambda_{j+1})$ and $\Lambda_{+,j+1} = \Lambda_{+,j} + \text{Op}(\lambda_{j+1})$, we have (4.5) with $j$ replaced by $j + 1$. Since we modified $\Lambda_{-,j}$ and $\Lambda_{+,j}$ by terms in $\Psi^{-j}$, summing asymptotically and composing on the left with $\text{Op}(b)$ gives the desired result. Repeating the argument but starting with $D_{x_1} + \lambda_0$ on the left, we obtain the second factorization. \hfill $\square$

**Lemma 4.5.** Let $q_0 \in T_0 \cap T^*((0, T) \times \mathbb{R}^n)$. Suppose $b_0 \in S^0_{\text{phg}}(T^*((0, T) \times \mathbb{R}^{n-1}))$ with $\text{supp} b_0 \subset T_0^\supset$ and $b_0(\pi(q_0)) = 1$ (with $\pi$ as in (2.10)). Then there exists a neighborhood $U$ of $q_0$ so that for all $\bar{\chi} \in S^0_{\text{phg}}(T^*((0, T) \times \mathbb{R}^n))$ with $\text{supp} \bar{\chi} \subset U$ there exists $\bar{\phi} \in C^\infty((0, T) \times \mathbb{R}^n)$ such that

$$||\text{Op}(\bar{\chi})u||_{H^2} \leq C(||\text{Op}(b_0)D_{x_1}u||_{L^2} + ||\text{Op}(b_0)u||_{L^2} + ||\bar{\phi}u||_{L^2} + ||\bar{\phi}u||_{L^2}).$$

**Proof.** Write $q_0 = (t_0, 0, x_0', \tau_0, (\xi_0, \epsilon_0'))$. We consider the case $(\xi_0, \epsilon_1) > 0$ (the case $(\xi_0, \epsilon_1) < 0$ is treated similarly) and denote by $\lambda \in C^\infty((-\epsilon, \epsilon); S^0_{\text{phg}}(T^*_{\mathbb{R} \times \mathbb{R}^{n-1})))$ a smooth symbol such that $\lambda(t, x, \tau, \xi') = \sqrt{\tau^2 - r(x, \xi')}$ on a neighborhood of $(-\epsilon, \epsilon) \times \text{supp}(b_0)$. Let $b \in C^\infty((-\epsilon, \epsilon); S^0_{\text{phg}}(T^*_{\mathbb{R} \times \mathbb{R}^{n-1}}))$ solve

$$\partial_{t_0} b - H_j b = 0, \quad b_{t_1 = 0} = b_0. \hspace{1cm} (4.7)$$

Denote by $\Lambda_\pm$ the two operators given by Lemma 4.4 associated to $\lambda$, so that

$$\text{Op}(b)(D_{x_1} - \Lambda_-(\lambda)) = \text{Op}(b)(\square + R),$$
with $R \in C^\infty((-\epsilon, \epsilon) \times \mathbb{R} \times \mathbb{R}^{n-1})$. Letting $\Omega_\epsilon = \{x_1 \in (-\epsilon/2, \epsilon/2)\}$, by Lemma A.1, we have

$$||\text{Op}(b)(D_{x_1} + \Lambda_+\lambda)u||_{L^2(\Omega_\epsilon)} \leq C(||\text{Op}(b_0)(D_{x_1} + \Lambda_+\lambda)u||_{L^2(\Omega_\epsilon)} + ||(D_{x_1} - \Lambda_+\lambda)\text{Op}(b)(D_{x_1} + \Lambda_+\lambda)u||_{L^2(\Omega_\epsilon)}$$
$$\leq C(||\text{Op}(b_0)(D_{x_1} + \Lambda_+\lambda)u||_{L^2(\Omega_\epsilon)} + ||\text{Op}(b)(D_{x_1} - \Lambda_+\lambda)(D_{x_1} + \Lambda_+\lambda)u||_{L^2(\Omega_\epsilon)}$$
$$+ C(||(D_{x_1} - \Lambda_+\lambda)\text{Op}(b_0)(D_{x_1} + \Lambda_+\lambda)u||_{L^2(\Omega_\epsilon)}). \hspace{1cm} (4.8)$$
Let us now estimate each term in the right hand-side. First, taking $\varphi = 1$ in a neighborhood of the support of the kernel of $\text{Op}(b)$ intersected with $\Omega_\ast$, and $\varphi \in C_0^\infty((0, T) \times \mathbb{R}^n)$ with $\varphi = 1$ in a neighborhood of $\text{supp} \varphi$, we have

$$
\| \text{Op}(b)(D_{s_1} - \Lambda_\ast)(D_{s_1} + \Lambda_\ast)u\|_{L^2(\Omega_\ast)} = \| \text{Op}(b)(\Box + R)u\|_{L^2(\Omega_\ast)} \\
\leq C\|\tilde{\varphi}\Box + R)u\|_{L^2(\Omega_\ast)} \leq C(\|\tilde{\varphi}\Box u\|_{L^2} + \|\tilde{\varphi}u\|_{L^2}).
$$

(4.9)

where we used the $L^2$ boundedness of $R$ and $\text{Op}(b)$ together with the fact that the quantization $\text{Op}$ gives operators whose kernels are compactly supported in $((0, T) \times \mathbb{R}^n)^2$. Second, with $\tilde{b}_0 \in S^0_{\text{phg}}(T^\ast((0, T) \times \mathbb{R}^n))$ with $\text{supp} \tilde{b}_0 \subset T^\ast_0$ and $\tilde{b}_0 = 1$ in a neighborhood of $\text{supp}(b_0)$, we obtain

$$
\| \text{Op}(b_0)(D_{s_1} + \Lambda_\ast)u\|_{L^2(\Omega_\ast)} \leq \| \text{Op}(b_0)D_{s_1}u|_{s_1 = 0}\|_{L^2} + \| \text{Op}(b_0)\Lambda_\ast u|_{s_1 = 0}\|_{L^2} \\
\leq \| \text{Op}(b_0)D_{s_1}u|_{s_1 = 0}\|_{L^2} + \| \Lambda_\ast \text{Op}(b_0)u|_{s_1 = 0}\|_{L^2} + \| \text{Op}(\tilde{b}_0)u|_{s_1 = 0}\|_{L^2}.
$$

(4.10)

Third, according to (4.7), the tangential operator $\{(D_{s_1} - \Lambda_\ast), \text{Op}(b)\} \in C^\infty((-\epsilon, \epsilon); \Psi^0_{\text{phg}}((0, T) \times \mathbb{R}^n))$ has principal symbol $\frac{1}{i}(\xi_1 - \lambda, b) = \frac{1}{i}(-\partial_s b - H_s b) = 0$ and is hence in $C^\infty((-\epsilon, \epsilon); \Psi^{-1}_{\text{phg}}((0, T) \times \mathbb{R}^n))$. This yields

$$
\||[(D_{s_1} - \Lambda_\ast), \text{Op}(b)]D_{s_1}u\|_{L^2(\Omega_\ast)} \leq \||[(D_{s_1} - \Lambda_\ast), \text{Op}(b)]D_{s_1}u\|_{L^2(\Omega_\ast)} + C\|\tilde{\varphi}u\|_{L^2}.
$$

(4.11)

To eliminate the first term in the right handside, we let $\varphi \in S^0_{\text{phg}}(\mathbb{R} \times \mathbb{R}^n)$ with $\varphi = 0$ in a conic neighborhood of $|\tau, \xi^\prime| \leq \epsilon\xi_1$ and $\varphi = 1$ in a conic neighborhood of $|\tau, \xi^\prime| \geq 2\epsilon\xi_1$, with $\epsilon > 0$ small enough so that $\varphi = 1$ on $\text{Char}(\Box)$. We write

$$
[(D_{s_1} - \Lambda_\ast), \text{Op}(b)]D_{s_1}u = [(D_{s_1} - \Lambda_\ast), \text{Op}(b)]D_{s_1} \text{Op}(\varphi)u + [(D_{s_1} - \Lambda_\ast), \text{Op}(b)]D_{s_1}(1 - \text{Op}(\varphi))u
$$

and remark first that $\{(D_{s_1} - \Lambda_\ast), \text{Op}(b)\}D_{s_1} \text{Op}(\varphi) \in \Psi^0_{\text{phg}}(\mathbb{R} \times \mathbb{R}^n)$ due to pseudodifferential calculus (see [Hör85, Theorem 18.1.35]), and hence

$$
\||[(D_{s_1} - \Lambda_\ast), \text{Op}(b)]D_{s_1} \text{Op}(\varphi)u\|_{L^2(\Omega_\ast)} \leq C\|\tilde{\varphi}u\|_{L^2}.
$$

(4.12)

Now, $1 - \varphi$ vanishes in a conic neighborhood of $|\tau, \xi^\prime| \geq 2\epsilon\xi_1$, and hence we have $\{(D_{s_1} - \Lambda_\ast), \text{Op}(b)\}D_{s_1}(1 - \text{Op}(\varphi)) \in \Psi^1_{\text{phg}}(\mathbb{R} \times \mathbb{R}^n)$ with principal symbol vanishing in a neighborhood of $\text{Char}(\Box)$. The ellipticity of $\Box$ there yields

$$
\||[(D_{s_1} - \Lambda_\ast), \text{Op}(b)]D_{s_1}(1 - \text{Op}(\varphi))\|_{L^2(\Omega_\ast)} \leq C\|\Box u\|_{L^2} + \|\tilde{\varphi}u\|_{L^2}.
$$

(4.13)

In particular,

$$
\||[(D_{s_1} - \Lambda_\ast), \text{Op}(b)]D_{s_1}(1 - \text{Op}(\varphi))u\|_{L^2(\Omega_\ast)} \leq C(\|\tilde{\varphi}\Box u\|_{L^2} + \|\tilde{\varphi}u\|_{L^2}),
$$

which, combined with (4.9)-(4.10)-(4.11)-(4.12) in (4.8) implies

$$
\| \text{Op}(b)(D_{s_1} + \Lambda_\ast)u\|_{L^2(\Omega_\ast)} \leq C(\| \text{Op}(b_0)D_{s_1}u|_{s_1 = 0}\|_{L^2} + \| \text{Op}(\tilde{b}_0)u|_{s_1 = 0}\|_{H^r} + \|\tilde{\varphi}\Box u\|_{L^2} + \|\tilde{\varphi}u\|_{L^2}).
$$

For $\psi \in S^0_{\text{phg}}(\mathbb{R} \times \mathbb{R}^n)$ vanishing near $|\tau, \xi^\prime| \leq \epsilon\xi_1$ and such that $\psi = 1$ at $q_0$, $\text{Op}(\psi) \text{Op}(b)(D_{s_1} + \Lambda) \in \Psi^1_{\text{phg}}(\mathbb{R} \times \mathbb{R}^n)$. Moreover, since $(\xi_0, \xi_0^\prime) > 0$ and $b(t_0, 0, x_0^\prime, \tau_0, \xi_0^\prime) = 1$, the operator $\text{Op}(\psi) \text{Op}(b)(D_{s_1} + \Lambda)$ is elliptic at $q_0$. Therefore, for $\tilde{\chi}$ supported near enough to $q_0$, adjusting $\tilde{\varphi}$ if necessary, we finally obtain

$$
\| \text{Op}(\tilde{\chi})u\|_{H^r} \leq C(\| \text{Op}(\psi) \text{Op}(b)(D_{s_1} + \Lambda)u\|_{H^r} + \|\tilde{\varphi}u\|_{L^2}) \\
\leq C(\| \text{Op}(\tilde{b}_0)D_{s_1}u|_{s_1 = 0}\|_{L^2} + \| \text{Op}(\tilde{b}_0)u|_{s_1 = 0}\|_{H^r} + \|\tilde{\varphi}\Box u\|_{L^2} + \|\tilde{\varphi}u\|_{L^2}),
$$

which concludes the proof of the lemma (up to changing $b_0$ into $\tilde{b}_0$ in the statement).

We now turn to the proof of Proposition 4.3. We follow the general structure of proof introduced by Lebeau in [Leb96], using the microlocal defect measures of Gérard [Gér91] and Tartar [Tar90]. Note that from the quantitative estimate of Lemma 4.5, and in case $\partial M = \emptyset$, “constructive proofs” (i.e. using no contradiction argument, and hence no defect measures) of Proposition 4.3 are possible, see [LL17] or [LL16].
Proof of Proposition 4.3. We prove estimate (4.4) by contradiction. Assuming that estimate (4.4) is false, there exist a sequence of data $F_k \in L^2((0, T) \times M)$ and $(u_{0,k}, u_{1,k}) \in H^1_0(M) \times L^2(M)$ with

$$\|(u_{0,k}, u_{1,k})\|_{H^1 \times L^2} = 1$$

such that the associated solution $(u_k)$ to (3.1) satisfies

$$\|A_\delta(u_k)\|_{H^1(\R^d \times \Sigma)}^2 + \|A_\delta(\partial_t u_k)\|_{L^2(\R^d \times \Sigma)}^2 + \|F_k\|_{L^2((0,T)\times M)}^2 + \|(u_{0,k}, u_{1,k})\|_{L^2}^2 \to 0.$$  

(4.14)

Classical energy estimates for then yield $\|u_k\|_{H^1((0,T)\times M)} \leq C$ together with $\|u_k\|_{L^2((0,T)\times M)} \to 0$. Hence $u_k \to 0$ in $H^1$ and, possibly after taking a subsequence, we may assume (see [Gér91, Tar90] in the case without boundary and [Leb96] or [BL01] in the general case) there exists a nonnegative measure $\mu$ on $S\tilde{Z}$ (see (2.5) for a definition) so that,

$$(Au_k, u_k)_{H^1 \times L^2} \to \int (j^{-1})^* \sigma(A) d\mu,$$

(4.15)

for all $A \in \Psi^0_\delta((0,T) \times \Int(M))$. Moreover, letting $(x_1, x')$ be Fermi normal coordinates near $\partial M$, then the convergence (4.15) also holds for $A \in C^\infty((0, \epsilon); \Psi^2(\R \times \partial M_{x'}))$, for $\epsilon > 0$. Note that in both cases $(j^{-1})^* \sigma(A)$ lies in $C^\infty(S\tilde{Z})$ since $\sigma(A)$ is independent of $\xi_1$ for $x_1$ small enough.

Let us first show that $\mu \equiv 0$. Notice that Lemma 4.2 implies there exists $\epsilon > 0$ so that Assumption GC-($\epsilon, T$) holds. We first prove that $\mu = 0$ on a neighborhood of $\overline{\mathcal{T}_\epsilon \cap T_{(\epsilon, T-\epsilon)^x}}(\R \times M)$. Then, since $\mu$ is invariant under the generalized bicharacteristic flow $\varphi(s, \cdot)$ defined in (2.6) (which passes to the quotient space $S\tilde{Z}$ according to homogeneity (2.8)), see [Leb96, BL01], Assumption GC-($\epsilon, T$) implies $\mu \equiv 0$ (note that it is sufficient that $\text{supp}(\mu)$ is invariant).

Suppose $q_0 \in \overline{\mathcal{T}_\epsilon \cap T_{(\epsilon, T-\epsilon)^x}}(\R \times M)$. Then for $\delta < \epsilon$, we have $\varphi(A_{\delta})(\pi(q_0)) = 1$. Therefore, Lemma 4.5 applies with $\text{Op}(b_0) = A_{\delta}$ and hence for $\tilde{\chi}$ supported close enough to $q_0$, we have

$$\|\text{Op}(\tilde{\chi})u_k\|_{H^1} \leq C(\|A_\delta D_{x_1}u_k|_{x_1=0}\|_{L^2} + \|A_\delta u_k|_{x_1=0}\|_{H^1} + \|\tilde{\varphi} \Box u_k\|_{L^2(\partial M)} + \|\tilde{\varphi} u_k\|_{L^2(\partial M)})).$$

Now, the right hand side tends to 0 by assumption. Thus, pseudodifferential calculus together with (4.15), imply the existence of a conic neighborhood $U$ of $q_0$ so that $\mu(U/\R^*_+) = 0$. Since this is true for any $q_0 \in \overline{\mathcal{T}_\epsilon \cap T_{(\epsilon, T-\epsilon)^x}}(\R \times M)$, there is a conic neighborhood $U_1$ of $\overline{\mathcal{T}_\epsilon \cap T_{(\epsilon, T-\epsilon)^x}}(\R \times M)$ so that $\mu(U_1/\R^*_+) = 0$. Invariance of $\mu$ and Assumption GC-($\epsilon, T$) imply that $\mu$ vanishes identically, which precisely means

$$u_k \to 0 \text{ in } H^1((0, T) \times M).$$

(4.16)

Now, we denote

$$E_k(t) := \|\nabla u_k(t, \cdot)\|_{L^2(M)}^2 + \|\partial_t u_k(t, \cdot)\|_{L^2(M)}^2,$$

and observe from (4.13)-(4.14) that $E_k(0) \to 1$. Moreover, for all $s_1, s_2 \in [0, T]$, we have

$$|E_k(s_2) - E_k(s_1)| \leq \frac{1}{2} \int_{s_1}^{s_2} \partial_t E_k(t) dt \leq \|F_k\|_{L^2} \|u_k\|_{H^1} \to 0.$$

In particular, since this convergence is uniform in $s_1, s_2$,

$$\int_0^T E_k(t) dt = \int_0^T E_k(t) - E_k(0) dt + T E_k(0) \to T.$$

Together with (4.16), this yields

$$0 < T \leftarrow \int_0^T E_k(t) dt \leq \|u_k\|_{H^1}^2 \to 0,$$

and hence the sought contradiction. \qed
4.3 Observability: the Low Frequencies. From Proposition 4.3 to Theorem 4.1

There are different ways of writing the compactness-uniqueness argument of [BLR92] (both reducing the problem to a unique continuation property for Laplace eigenfunctions). The first one is the precise argument of [BLR92]: it uses again the geometric condition together with the propagation of wavefront sets (see also [LLT16]). A second form seems to be due to [BG02]: it is a bit longer but uses only that the observation region is time invariant. We write this version of the proof in the present context.

We first need a weak unique continuation property from a hypersurface. This is a weak version of Theorem 1.7, but we chose to give a proof since it is much less involved. Note that no compactness is assumed and no boundary conditions are prescribed here.

Lemma 4.6 (Unique continuation). Let $\Sigma$ be a nonempty interior hypersurface of a connected manifold $M$ and assume

$$(-\Delta_g - \lambda^2)u = 0 \quad \text{in } M, \quad u|_{\Sigma} = \partial_\nu u|_{\Sigma} = 0,$$

then $u$ vanishes identically.

Proof. Let $\Omega$ be a nonempty connected open set of $M$ such that $\Omega \cap \Sigma \neq \emptyset$ and $\Omega = \Omega^+ \cup (\Omega \cap \Sigma) \cup \Omega^-$ where the union is disjoint. Then, setting

$$v(x) = u(x) \quad \text{for } x \in \Omega^+, \quad v(x) = 0 \quad \text{for } x \in \Omega^-,$$

we have $v \in L^2(\Omega)$, with, moreover ($\partial_\nu$ pointing towards $\Omega^+$)

$$(-\Delta_g - \lambda^2)v = 0 - [v|_{\Sigma}]_\nu + (c(x)[v|_{\Sigma} - [\partial_\nu v]|_{\Sigma})\nu|_{\Sigma} = -u|_{\Sigma}_\nu + (c(x)u|_{\Sigma} - \partial_\nu u|_{\Sigma})\nu|_{\Sigma} = 0.$$

This follows from the jump formula written in Fermi coordinate charts $(x_1, x')$ with $\Omega^+ = \{x_1 > 0\} \cap \Omega$ and $-\Delta = -\partial^2_{x_1} + R(x_1, x', D') + c(x)D_x$, with $R$ tangential (see Section 2.1).

A classical unique continuation result for elliptic operators (see e.g. [LRL12, Theorem 4.2]) then implies that $v = 0$ in all $\Omega$. From the definition of $v$, this yields $u|_{\Omega^-} = 0$, and, using again the elliptic unique continuation result and the connectedness of $M$, implies that $u$ vanishes identically in $M$. \hfill $\square$

We next define for any $T > 0$ and $\varepsilon > 0$ the set of invisible solutions from $[\varepsilon, T - \varepsilon] \times \Sigma_{\varepsilon}$ where $\Sigma_{\varepsilon}$ is as in (3.9):

$$N(\varepsilon, T) = \{(u_0, u_1) \in H_0^1(M) \times L^2(M) \mid \text{satisfies } \partial_\nu u|_{\Sigma} = u|_{\Sigma} = 0 \text{ in } \mathcal{D}'((\varepsilon, T - \varepsilon) \times \Sigma_{\varepsilon})\}.$$ 

We have the following lemma, which is a consequence of Proposition 4.3.

Lemma 4.7. Suppose GC-(0,T) holds. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, we have $N(\varepsilon, T) = \{0\}$.

We denote by $\mathcal{A}$ the generator of the wave group, namely

$$\mathcal{A} = \begin{pmatrix} 0 & -\Id \\ -\Delta_g & 0 \end{pmatrix}, \quad D(\mathcal{A}) = (H^1 \cap H^1_0(M)) \times H^1_0(M),$$

so that the wave equation (3.1) with $F = 0$ may be rewritten as

$$\partial_t U + \mathcal{A} U = 0, \quad U|_{t=0} = U_0 = (u_0, u_1).$$

Proof. Step 1: $N(\varepsilon, T)$ is finite dimensional. First, Proposition 3.1 implies that $N(\varepsilon, T)$ is a closed linear subspace of $H^1_0(M) \times L^2(M)$ for all $\varepsilon > 0$. Since Assumption GC-(0,T) holds, we may apply Proposition 4.3. The kernel of the operator $A_g$ in (4.4) is compactly supported in $(0, T) \times \operatorname{Int}(\Sigma)$, and hence in $(\varepsilon_0, T - \varepsilon_0) \times \Sigma_{\varepsilon_0}$ for some $\varepsilon_0 > 0$. Thus, for all $0 < \varepsilon < \varepsilon_0$, the relaxed observability inequality (4.4) applied to elements of $N(\varepsilon, T)$ gives

$$c\| (u_0, u_1) \|^2_{H^1_0 \times L^2} \leq \| (u_0, u_1) \|^2_{L^2 \times H^1}, \quad \text{for all } (u_0, u_1) \in N(\varepsilon, T),$$

for all $(u_0, u_1) \in N(\varepsilon, T)$. \hfill $\square$
since the kernel of the operator $A_\delta$ is compactly supported in $(\varepsilon_0, T - \varepsilon_0) \times \Sigma_{\varepsilon_0}$, and $u|_{\Sigma}, \partial_t u|_{\Sigma}$ vanish on this set.

Using the compact imbedding $H^1_0 \times L^2 \subset L^2 \times H^{-1}$, this implies that the unit ball of $N(\varepsilon, T)$ for the $H^1_0 \times L^2$-norm is compact, that is, $N(\varepsilon, T)$ has finite dimension. Note also that it is thus complete for any norm.

**Step 2: $N(\varepsilon, T) \subset C^\infty(\overline{M})$ and $AN(\varepsilon, T) \subset N(\varepsilon, T)$.

Taking $\eta > 0$, sufficiently small (namely $\eta < \varepsilon_0 - \varepsilon$), we remark that (4.19) is also satisfied by all $U_0 = (u_0, u_1) \in N(\varepsilon + \eta, T)$. Taking $U_0 \in N(\varepsilon, T)$ implies that, for all $\varepsilon \in (0, \eta)$, we have $e^{-e^\delta} U_0 \in N(\varepsilon + \eta, T)$. We also have, for $\lambda_0$ sufficiently large,

\[
(\lambda_0 + \mathcal{A})^{-1} \frac{1}{\varepsilon} (\mathbb{I} - e^{-e^\delta}) U_0 = \frac{1}{\varepsilon} (\mathbb{I} - e^{-e^\delta}) (\lambda_0 + \mathcal{A})^{-1} U_0 \quad \text{as} \quad \varepsilon \to 0
\]

in $H^1_0 \times L^2$, as $(\lambda_0 + C) \mathcal{A}^{-1} U_0 \in D(\mathcal{A})$. As a consequence, the sequence $\left( \frac{1}{\varepsilon} (\mathbb{I} - e^{-e^\delta}) U_0 \right)_{\varepsilon > 0}$ is a Cauchy sequence in $N(T' - \eta)$ endowed with the norm $\| (\lambda_0 + \mathcal{A})^{-1} \cdot \|_{H^1_0 \times L^2}$. As all norms are equivalent in $N(\varepsilon + \eta, T)$, the sequence $\left( \frac{1}{\varepsilon} (\mathbb{I} - e^{-e^\delta}) U_0 \right)_{\varepsilon > 0}$ is thus also a Cauchy sequence in this space, endowed with the norm $\| \cdot \|_{H^1_0 \times L^2}$, which yields $\mathcal{A} U_0 \in H^1_0 \times L^2$. Hence, we have $N(\varepsilon, T) \subset D(\mathcal{A})$. This argument may be inductively repeated to prove that $N(\varepsilon, T) \subset D(\mathcal{A}^k)$ for all $k \in \mathbb{N}$, and yields in particular, that functions in $N(\varepsilon, T)$ are $C^\infty(\overline{M})$.

Take now $U_0 \in N(\varepsilon, T)$, and denote by $U(t)$, the associated solution of (3.1), or equivalently (4.18). Then, $u \in C^\infty(\mathbb{R} \times \overline{M})$, and using the fact that $\partial_t$ is tangential to the manifold $\mathbb{R} \times \Sigma$ (thus commuting with $\partial_\nu$), we obtain that $\partial_t u|_{\Sigma}(t, x) = 0$ and $\partial_\nu(\partial_t u)|_{\Sigma}(t, x) = 0$ for all $(t, x) \in [\varepsilon, T - \varepsilon] \times \Sigma$ (since this $U_0 \in N(\varepsilon, T)$ implies that this is satisfied by $u$). This is $\partial_t U|_{t=0} \in N(\varepsilon, T)$. Remarking then that we have $\mathcal{A} U_0 = -\partial_\nu U|_{t=0} \in N(\varepsilon, T)$, this implies $\mathcal{A} N(\varepsilon, T) \subset N(\varepsilon, T)$.

**Step 3: reduction to unique continuation for Laplace eigenfunctions: end of the proof.** Since $N(\varepsilon, T)$ is a finite dimensional subspace of $D(\mathcal{A})$, stable by the action of the operator $\mathcal{A}$, it contains an eigenfunction of $\mathcal{A}$. There exist $\mu \in \mathbb{C}$ and $U = (u_0, u_1) \in N(\varepsilon, T)$ such that $\mathcal{A} U = \mu U$, that is, given the definition of $\mathcal{A}$ in (4.17), $-\Delta_\Sigma u_0 = -\mu^2 u_0$ and $U_1 = -\mu u_0$. Hence $u_0$ is an eigenfunction of the Laplace-Dirichlet operator on $\Sigma$, associated to $-\mu^2 \in \mathbb{R}^+$, i.e., $\mu = i \lambda$, $\lambda \in \mathbb{R}$. The associated solution to (3.1) is $u(t, x) = e^{i \lambda t} u_0$, and $U_0 \in N(\varepsilon, T)$ implies $\partial_t u_0|_{\Sigma} = u_0|_{\Sigma} = 0$. This, together with the fact that $u_0$ is a Laplace eigenfunction and Lemma 4.6 proves that $u_0 = 0$ and then $U = 0$. This proves that $N(\varepsilon, T) = \{0\}$.

From Lemma 4.7, we can now conclude the proof of Theorem 4.1.

**Proof of Theorem 4.1.** We proceed by contradiction and suppose that the observability inequality (4.1) does not hold for any $\delta > 0$. Thus, for any $\delta > 0$, there exists a sequence $(u^k_0, u^k_1, F^k)_{k \in \mathbb{N}}$ of $H^1_0(M) \times L^2(M) \times L^2((0, T) \times M)$ such that, with $u^k$ the associated solution to (3.1), we have

\[
\| (u^k_0, u^k_1) \|_{H^1_0 \times L^2} = 1, \quad (4.20)
\]

\[
\| \partial \varphi_\delta \partial_\nu u^k \|_{L^2(\Sigma_{\delta})}^2 + \| \partial \varphi_\delta u^k \|_{L^2(\Sigma_{\delta})}^2 + \| F^k \|_{L^2((0, T) \times M)} \to 0, \quad (4.21)
\]

\[
\| A_\delta (\partial_\nu u^k) \|_{L^2(\Sigma_{\delta})} + \| A_\delta (\partial_\nu u^k) \|_{L^2(\Sigma_{\delta})} \to 0. \quad (4.22)
\]

From (4.20), we may extract a subsequence of $(u^k_0, u^k_1)$ converging weakly in $H^1_0 \times L^2$ to some $(u_0, u_1)$. Denote by $u$ the associated solution to (3.1), with $F = 0$. Since $F^k \to 0$ in $L^2$ we may further extract from $u^k$ a subsequence converging to $u$ weakly in $H^1((0, T) \times M)$. According to Proposition 3.1, we have $\partial_t u^k|_\Sigma \to \partial_t u|_\Sigma$ and $u^k|_\Sigma \to \partial_\nu u|_\Sigma$ weakly in $H^{-1}((0, T') \times \Sigma)$. But according to (4.21), this yields

\[
\varphi_\delta \partial_\nu u|_\Sigma = \varphi_\delta u|_\Sigma = 0,
\]

and in particular, taking $\delta < \varepsilon$,

\[
\partial_\nu u|_\Sigma = u|_\Sigma = 0, \quad \text{on} \ [\varepsilon, T - \varepsilon] \times \Sigma.
\]

Thus,

\[
(u_0, u_1) = (u(0), \partial_\nu u(0)) \in N(\varepsilon, T).
\]
So, from Lemma 4.7, we obtain \((u_0, u_1) = 0\). The imbedding \(H^1_0 \times L^2 \hookrightarrow L^2 \times H^{-1}\) being compact, this implies
\[
\|(u^k_0, u^k_1)\|_{L^2 \times H^{-1}} \rightarrow \|(u_0, u_1)\|_{L^2 \times H^{-1}} = 0.
\] (4.23)

Finally, Proposition 4.3 implies that (4.4) holds for any \(\delta < \delta_0\). Therefore, taking \(\delta < \min(\varepsilon, \delta_0)\) and using (4.20), (4.21), (4.22), (4.23) in the relaxed observability inequality (4.4), we obtain at the limit \(0 < \varepsilon \leq 0\), which is a contradiction. \(\square\)

4.4 Controllability of the Wave Equation

Theorem 1.5 is a straightforward corollary of the following theorem. Recall that \(\bar{E}^\varepsilon = E^\varepsilon \cup G^\varepsilon\).

**Theorem 4.8.** Assume \((\Sigma, T)\) satisfies Assumption GC-(0,T). Then there exists a continuous map
\[
L^2(M) \times H^{-1}(M) \ni (v_0, v_1) \mapsto (f_0, f_1) \in \bigcap_{N \in \mathbb{N}} H^{-1, N}_{\text{comp}, E^\varepsilon}(\Sigma_T) \times H^{0, N}_{\text{comp}, E^\varepsilon}(\Sigma_T)
\]
(the latter space being a Fréchet space when endowed with the seminorms of all \(H^{-1, N}_{\text{comp}, E^\varepsilon}(\Sigma_T) \times H^{0, N}_{\text{comp}, E^\varepsilon}(\Sigma_T)\)) so that the associated solution to (1.6) has \(v \equiv 0\) for \(t \geq T\).

**Proof.** Fix \(0 < T < T_1\). Then define
\[
L^2([T, T_1] \times M) = \{F \in L^2([0, T_1] \times M), \supp F \subset [T, T_1]\}
\]
and for \(N \geq \frac{1}{2}\) the map
\[
K : L^2([T, T_1] \times M) \rightarrow H^{-1, N}_{\text{loc}, E^\varepsilon}(\Sigma_T) \times H^{0, N}_{\text{loc}, E^\varepsilon}(\Sigma_T)
\]
given by
\[
F \mapsto (u_{|0,T\times\Sigma}, -\partial_t u_{|0,T\times\Sigma})
\]
where \(u\) solves
\[
\begin{align*}
\Box u &= F & \text{on } (0, T_1) \times \text{Int}(M), \\
u &= 0 & \text{on } (0, T_1) \times \partial M, \\
(u_{|t=T_1}, \partial_t u_{|t=T_1}) &= (0, 0) & \text{in } \text{Int}(M).
\end{align*}
\]
This map is well defined by (3.15). Define also the operator \(S : L^2([T, T_1] \times M) \rightarrow H^0_0(M) \times L^2(M)\) by
\[
S(F) := (u_{|t=0}, \partial_t u_{|t=0}).
\] (4.24)

Now, suppose that Assumption GC-(0,T) holds and let \(A_\delta\) as in Theorem 4.1. For \(\varepsilon > 0\) small \(B^\varepsilon\) is elliptic on \(\text{WF}(A_\delta)\) and hence using the elliptic parametrix construction we write
\[
A_\delta = GB^\varepsilon + R
\]
with \(R \in \Psi^-\infty((0, T) \times \text{Int}(\Sigma))\), and \(G \in \Psi^0_{\text{phg}}((0, T) \times \text{Int}(\Sigma))\). Therefore Theorem 4.1 implies that there exists \(\varepsilon > 0\) small enough depending only on \((\Sigma, T)\) and for all \(N \in \mathbb{N}\), there exists \(C_N > 0\) so that
\[
||S(F)||_{H^0_0(M) \times L^2(M)} \leq C_N||K(F)||_{H^{-1, N}_{\text{loc}, E^\varepsilon}(\Sigma_T) \times H^{0, N}_{\text{loc}, E^\varepsilon}(\Sigma_T)}.
\] (4.25)

Let \((v_0, v_1) \in H^{-1}(M) \times L^2(M)\) and define the linear functional \(\ell_N : \text{ran}(K) \rightarrow \mathbb{C}\) by
\[
\ell_N(K(F)) = \langle S(F), (-v_1, v_0) \rangle_{H^0_0(M) \times L^2(M), H^{-1}(M) \times L^2(M)}
\]
where \(S\) is defined in (4.24). Then, \(\ell_N\) is well defined and continuous by (4.25). In particular,
\[
|\ell_N(K(F))| \leq C_N||(v_0, v_1)||_{H^{-1}(M) \times L^2(M)} ||K(F)||_{H^{-1, N}_{\text{loc}, E^\varepsilon}(\Sigma_T) \times H^{0, N}_{\text{loc}, E^\varepsilon}(\Sigma_T)}.
\]
Since $\ell_N$ is a continuous linear functional defined on a subspace of $H^{1-N}_{loc,\Sigma} \times H^0_{loc,\Sigma}$ by the Hahn-Banach theorem $\ell_N$ extends to a continuous linear functional on the whole space (still denoted $\ell_N$) with

$$\|\ell_N(w_1, w_2)\| \leq C_N \| (v_0, v_1) \|_{H^{-1}(M) \times L^2(M)} \| (w_1, w_2) \|_{H^{1-N}_{loc,\Sigma} \times H^0_{loc,\Sigma}}.$$ 

Thus, by Lemma 3.4, there exists $(f_{0,N}, f_{1,N}) \in H^{-1,N}_{comp,\Sigma} \times H^{0,N}_{comp,\Sigma}$ so that for all $(w_1, w_2) \in H^{1-N}_{loc,\Sigma} \times H^0_{loc,\Sigma}$, we have

$$\ell_N(w_1, w_2) = \langle (w_1, w_2), (f_{0,N}, f_{1,N}) \rangle_{H^{1-N}_{loc,\Sigma} \times H^0_{loc,\Sigma}}.$$ 

and hence for some $\varepsilon' > 0$, 

$$\|(f_{0,N}, f_{1,N})\|_{H^{-1,N}_{loc,\Sigma} \times H^{0,N}_{loc,\Sigma}} \leq C_N \varepsilon' \| (v_0, v_1) \|_{H^{-1}(M) \times L^2(M)}.$$ 

Let $v$ be the unique solution to

$$\begin{cases}
\square v = f_{0,N} \delta_\Sigma + f_{1,N} \delta_\Sigma' & \text{on } (0, T) \times \text{Int}(M), \\
v = 0 & \text{on } (0, T) \times \partial M, \\
(v, \partial_t v)|_{t=0} = (v_0, v_1) & \text{in } \text{Int}(M),
\end{cases}$$

given by Definition 3.6 and Theorem 3.7. Then for any $F \in L^2([T, T_1] \times M)$ we have

$$\langle (v, F)_{L^2([T, T_1] \times M)} = \langle (v_1, u_0)_{H^{-1}(M) \times H^0(M)} - (v_0, \partial_t u_0)_{L^2(M)} + \langle (f_{0,N}, f_{1,N})_{H^{-1,N}_{comp,\Sigma} \times H^{0,N}_{comp,\Sigma}} \rangle_{H^{-1}(M) \times L^2(M)}$$

$$= \langle (v_1, -v_0), S(F) \rangle_{H^{-1}(M) \times L^2(M)} + \langle (f_{0,N}, f_{1,N}), K(F) \rangle_{H^{-1,N}_{comp,\Sigma} \times H^{0,N}_{comp,\Sigma}}$$

$$= \langle (v_1, -v_0), S(F) \rangle_{H^{-1}(M) \times L^2(M)} + \ell_N(K(F))$$

$$= \langle (v_1, -v_0), S(F) \rangle_{H^{-1}(M) \times L^2(M)} + \langle (v_1, -v_0), S(F) \rangle_{H^{-1}(M) \times L^2(M)} + \langle (v_1, -v_0), S(F) \rangle_{H^{1-N}_{loc,\Sigma} \times H^0_{loc,\Sigma}} = 0.$$ 

Since this is true for all $F \in L^2([T, T_1] \times M)$, we obtain $v \equiv 0$ on $[T, T_1] \times M$.

Now, for $k > N$, the inclusion $H^{-1,k}_{comp,\Sigma} \subset H^{-1,N}_{comp,\Sigma}$ is dense and the inclusion $H^{1-k}_{loc,\Sigma} \subset H^{1-N}_{loc,\Sigma}$ is dense. So, in particular, $\ell_N$ extends to a linear functional on $H^{1-N}_{loc,\Sigma} \times H^{0,N}_{loc,\Sigma}$ by density. This yields

$$\langle (w_1, w_2), (f_{0,k}, f_{1,k}) \rangle_{H^{1-N}_{loc,\Sigma} \times H^{0,N}_{loc,\Sigma}} = \langle (w_1, w_2), (f_{0,N}, f_{1,N}) \rangle_{H^{1-N}_{loc,\Sigma} \times H^{0,N}_{loc,\Sigma}}$$

for all $(w_1, w_2) \in H^{1-N}_{loc,\Sigma} \times H^{0,N}_{loc,\Sigma}$. This implies that $f_{0,k} = f_{0,N}$ and $f_{1,k} = f_{1,N}$ and hence that

$$f_{0,N} \equiv f_0 \in \bigcup_N H^{-1,N}_{comp,\Sigma}, \quad f_{1,N} \equiv f_1 \in \bigcup_N H^{0,N}_{comp,\Sigma},$$

which concludes the proof of the theorem. 

\[ \square \]

5 Controllability of the heat Equation

5.1 Well-posedness for the heat equation controlled from a hypersurface

The well-posedness theory is easier than that of the wave equation since the regularity theory for the heat equation directly implies that the traces of the solution on $\Sigma$ are “admissible” observations, in the usual sense, see [Cor07, Chapter 2.3] and [TW09, Chapter 4.3].

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Lemma 5.1. Given $T > 0$, assume that the functions $v \in C^\infty([0, T] \times M \setminus \Sigma) \cap C^0((0, T); L^2(M))$ and $f_0, f_1 \in C^\infty((0, T) \times \text{Int}(\Sigma))$ solve

$$\begin{aligned}
(\partial_t - \Delta)v &= f_0 \delta \Sigma + f_1 \delta' \Sigma \text{ in } \mathcal{D}'((0, T) \times \text{Int}(M)), \\
and (-\partial_t - \Delta)u &= F.
\end{aligned}$$

Then, we have the identity

$$\int_{(0, T) \times \Sigma} \left( f_0 u|_{\Sigma} - f_1 \partial_u|_{\Sigma} \right) d\sigma dt.$$

Also, we have the following “admissibility result” (regularity of traces).

Lemma 5.2. Given $T > 0$, there is $C > 0$ such that for all $F \in L^2((0, T) \times M)$, $u \in H^1_0(M)$ and $u$ associated solution of

$$\begin{aligned}
(-\partial_t - \Delta)u &= F \text{ on } (0, T) \times \text{Int}(M), \\
u &= 0 \text{ on } (0, T) \times \partial M, \\
|u|_{t=T} &= \tilde{u} \text{ in } \text{Int}(M),
\end{aligned}$$

we have

$$||\partial_t u||^2_{L^2((0,T);L^2(M))} + ||u||^2_{L^2((0,T);H^1(M))} \leq C||F||^2_{L^2((0,T)\times M)} + C||\tilde{u}||^2_{H^1(M)}.$$

Proof. This is a direct consequence of the regularity theory for the heat equation (5.1), namely $u \in C^0([0, T]; H^1_0(M)) \cap L^2(0, T; H^2(M)) \cap H^1(0, T; H^1_0(M))$, with

$$||u||^2_{L^2(0,T;H^1(M))} + ||u||^2_{H^1(0,T;H^1(M))} \leq C||F||^2_{L^2((0,T)\times M)} + C||\tilde{u}||^2_{H^1(M)},$$

see e.g. [Eva98, Chapter 7.1.3, Theorem 5]. The standard trace estimates then yield

$$||\partial_t u||^2_{L^2((0,T);H^1(M))} + ||u||^2_{L^2(0,T;H^1(M))} \leq C||u||^2_{L^2((0,T);H^1(M))},$$

which concludes the proof of the lemma. \hfill \Box

This suggests the following definition (see [Cor07, Chapter 2.3]) of solutions of the controlled heat equation (1.10).

Definition 5.3. Given $T > 0$, $v_0 \in H^1(M)$, $f_0 \in L^2(0, T; H^{1/2}_{\text{comp}}(\Gamma(\Sigma)))$, $f_1 \in L^2(0, T; H^{1/2}_{\text{comp}}(\text{Int}(\Sigma)))$, we say that $v$ is a solution of (1.6) if $v \in C^0([0, T]; H^{-1}(M))$ and for any $t \in [0, T]$, for any $\tilde{u} \in H^1_0(M)$, we have

$$\langle v(t), \tilde{u} \rangle_{H^{-1}(M)} = \langle v_0, \tilde{u} \rangle_{H^1(M)} + \int_0^t \langle f_0(s), u_0|_{\Sigma} \rangle_{H^{1/2}_{\text{comp}}(\Sigma), H^{-1/2}_{\text{comp}}(\Sigma)} - \langle f_1(s), \partial_s u_0|_{\Sigma} \rangle_{H^{1/2}_{\text{comp}}(\Sigma), H^{-1/2}_{\text{comp}}(\Sigma)} ds.$$

where $u$ is the unique solution to

$$\begin{aligned}
(-\partial_s - \Delta)u &= 0 \text{ on } (0, t) \times \text{Int}(M), \\
u &= 0 \text{ on } (0, t) \times \partial M, \\
|u|_{t=t} &= \tilde{u} \text{ in } \text{Int}(M),
\end{aligned}$$

\text{i.e. } u(s) = e^{(t-s)\Delta} \tilde{u}.

The following result is a direct consequence of (a slight variation on) [Cor07, Theorem 2.37] and the admissibility estimate of Lemma 5.2.

Theorem 5.4 (Well-posedness of the controlled heat equation). Let $T > 0$. There exist $C > 0$ such that for all $v_0 \in H^{-1}(M)$, $f_0 \in L^2(0, T; H^{1/2}_{\text{comp}}(\Gamma(\Sigma)))$, $f_1 \in L^2(0, T; H^{1/2}_{\text{comp}}(\text{Int}(\Sigma)))$, there exists a unique solution $v$ of (1.10) in the sense of Definition 5.3 and we have:

$$||v||_{L^2((0,T);L^2(M))} \leq C \left( ||v_0||_{H^{-1}(M)} + ||f_0||_{L^2((0,T);H^{1/2}_{\text{comp}}(\Sigma))} + ||f_1||_{L^2((0,T);H^{1/2}_{\text{comp}}(\Sigma))} \right).$$
5.2 Global interpolation inequality and universal lower bound for traces of eigenfunctions

We follow the general Lebeau-Robbiano method [LR95] and use moreover a Carleman estimate of [LR97]. We refer to [LRL12] for an exposition of these works.

The global strategy [LR95] is the following:

1. Local Carleman estimates
2. \( \implies \) local interpolation estimates
3. \( \implies \) a global interpolation estimate
4. \( \implies \) finite dimensional observability/controllability for an elliptic evolution equation
5. \( \implies \) finite dimensional observability/controllability for the heat equation
6. \( \implies \) observability/controllability for the heat equation.

Also, the unique continuation estimate for eigenfunctions of Theorem 1.7 can be deduced from the global interpolation estimate. The present section proves steps 1, 2, 3. The next section is devoted to that of steps 4, 5, 6.

In the following, for \( \alpha > 0 \), we set \( Y_\alpha = (-\alpha, \alpha) \times M \), \( \Sigma_\alpha = (-\alpha, \alpha) \times \Sigma \), and denote \( Q = -\partial_x^2 - \Delta_x \).

**Theorem 5.5** (Global interpolation). Let \( S > \beta > 0 \). For all \( \psi \in C^\infty_c(\Sigma_\beta) \) not identically vanishing, there exist \( C, \delta > 0 \) such that

\[
\|v\|_{H^1(Y_\delta)} \leq C \left( \|Qv\|_{L^2(Y_\beta)} + \|\psi v\|_{L^2(\Sigma_\beta)} + \|\psi \partial_\nu v\|_{L^2(\Sigma_\beta)} \right) \delta \|v\|_{H^\delta(Y_\delta)}^{1-\delta} \tag{5.3}
\]

for all \( v \in H^2(Y_S) \) such that \( v|_{(S^c)_\delta \times M} = 0 \).

If we were considering a second order elliptic operator \( Q \) on a manifold \( Y_S \) with smooth boundary, and with Dirichlet condition on the whole \( \partial Y_S \), this estimate would simply read

\[
\|v\|_{H^1(Y_S)} \leq C \left( \|Qv\|_{L^2(Y_S)} + \|\psi v\|_{L^2(\Sigma)} + \|\psi \partial_\nu v\|_{L^2(\Sigma)} \right).
\]

However, here \( Y_S = (S, S) \times M \) is not smooth at \( (S^c, S) \times \partial M \) and it is crucial for the next arguments that no boundary condition is prescribed at the boundary \((S^c \cup \{S\}) \times M\).

The proof of Theorem 5.5 follows from arguments of Lebeau and Robbiano [LR95, LR97]. The idea is that such interpolation inequalities follow locally from Carleman estimates, and then propagate well. Hence, our task is only

(i) to deduce from a local Carleman estimate near \( \Sigma_\beta \) that the traces at the boundary “control” a small nonempty open set near \( \Sigma_\beta \) (i.e. that (5.3) holds with, in the l.h.s. the local \( H^1 \) norm in this set)

(ii) to use a global interpolation inequality implying that such a small set “controls” the \( H^1(Y_\beta) \) norm, and then put the two inequalities together.

For the second point (ii), we can start from the following result of [LR95, Section 3, Estimate (1)].

**Theorem 5.6**. Let \( U \subset Y_S \) be any nonempty open set, then there is \( C > 0 \) and \( \delta_0 \in (0, 1) \) such that we have

\[
\|v\|_{H^1(Y_\delta)} \leq C \left( \|Qv\|_{L^2(Y_\delta)} + \|v\|_{H^1(Y_\delta)} \right)^{\delta_0} \|v\|_{H^1(Y_\delta)}^{1-\delta_0} \tag{5.4}
\]

for all \( v \in H^2(Y_S) \) such that \( v|_{(S^c)_\delta \times M} = 0 \).
As a consequence, it suffices to prove the first point (i), namely, that there exists such an $U$ such that, for some $C, \delta_1 > 0$ we have

$$
\|v\|_{H^s(U)} \leq C \left( \|Qv\|_{L^2(\mathbb{R}_+^n)} + \|\psi v\|_{L^2(\mathbb{R}_+^n)} + \|\psi \partial_x v\|_{L^2(\mathbb{R}_+^n)} \right)^\frac{2}{\delta_1} \|v\|_{H^s(\mathbb{R}_+^n)},
$$

(5.5)

which is now a local estimate. Indeed, (5.4) together with (5.5) directly yield (5.3) for $\delta = \delta_0 \delta_1$ (see e.g. [LR95, Lemma 4]).

To prove (5.5), we shall take $m \in \Sigma$ a point for which $\psi(m) \neq 0$, and assume that the set $U$ is a small neighborhood of $m$ intersected with a single side of $\Sigma$. Also, we shall say that $\partial_x$ is pointing towards $U$. We now work in the local Fermi normal coordinates near $m \in \Sigma$, described in Section 2.1. The operator $Q = -\partial_1^2 - \Delta_x$, still denoted by $Q$ in these coordinates, is given, modulo conjugation by a harmless exponential factor, by

$$
Q = -\partial_x^2 - \partial_1^2 + r(x_1, x', \frac{\partial_1}{t}),
$$

with principal symbol

$$
q = \xi_1^2 + \xi_s^2 + r(x_1, x', \xi'),
$$

where

- $(s, x')$ are the variables in $(-S, S) \times \Sigma$, $\xi_s \in \mathbb{R}$ is the cotangent variable associated to $s$;
- variables are in a neighborhood of zero in the half space $\mathbb{R}_+^{n+1} = \mathbb{R}_s \times \mathbb{R}_+ x_1 \times \mathbb{R}_x^{n-1}$ (we only estimate things on $x_1 > 0$, where $U$ is);
- $\partial_x$ is given by $\partial_{x_1}$ in these coordinates.

Now, the proof of (5.5) relies on the following Proposition [LR97, Proposition 1]. Here, the variable $s$ does not play a particular role: hence, in what follows, we only write (with a slight abuse of notation) $\partial_x$ for the overall variable, and accordingly $q = q(x, \xi) = q(x, x_1, x', \xi_x, \xi_1, \xi')$. We also use the notation

$$
q_\varphi(x, \xi) = q(x, \xi + id\varphi(x)).
$$

Proposition 5.7. Let $R > 0$ and $\varphi \in C^\infty$ in a neighborhood of $K := \mathbb{R}_+^{n+1} \cap \overline{B}(0, R)$ and such that

- $\partial_i \varphi \neq 0$ on $K$,
- (Hörmander subellipticity condition) $\forall (x, \xi) \in K \times \mathbb{R}_+^{n+1}$, $q_\varphi(x, \xi) = 0 \implies \{\text{Re}(q_\varphi), \text{Im}(q_\varphi)\}(x, \xi) > 0$.

Then, we have

$$
\begin{align*}
\frac{h^n}{\text{vol}(\mathbb{R}_+^{n+1})} & \leq h^{n-1} L^2(\mathbb{R}_+^{n+1}) + h^n L^2(\mathbb{R}_+^{n+1}) \\
& \quad \quad \text{ for all } u \in C^\infty(\mathbb{R}_+^{n+1}) \text{ such that supp}(u) \subset B(0, R) \text{ and } h \in (0, h_0).
\end{align*}
$$

(5.6)

The end of proof of Theorem 5.6 is then similar to [LR95] or [LZ98, Appendix].

End of the proof of Theorem 5.6. We first fix $R > 0$ small enough such that $B(0, R)$ is contained in the coordinate chart and that the set $\overline{B}(0, R) \cap \{x_1 = 0\}$ (where the observation shall take place) is contained in the set $\{\psi > 0\}$ (where $\psi$ is the cutoff function appearing in (5.3)). Second, we define the weight function $\varphi(x) = e^{-\mu|x-x'|^2} - e^{-\mu|x|^2}$, where $\mu > 0$ (large, to be chosen) and, for $a \in (0, R)$, we have $x^a = (0, \ldots, 0, -a) \notin \mathbb{R}_+^{n+1}$. Hence, $\varphi$ is smooth and satisfies $\partial_\alpha \varphi \neq 0$ on $K = \mathbb{R}_+^{n+1} \cap \overline{B}(0, R)$.

According to classical computations (see e.g. [LRL12, Lemma A.1]), $\varphi$ satisfies the Hörmander subellipticity condition on $K$ for $\mu$ large enough (depending on $R$ and $a$, and fixed from now on).

Note that levelsets of $\varphi$ are balls. Moreover, we have $\varphi(0) = 0$ and $\varphi(x) < 0$ if $|x - x'| > |x^a|$, and in particular on $\{x_1 = 0\}$.

For $\varepsilon > 0$ sufficiently small (depending on $R, a$ and $\mu$), the set $\{\varphi \geq -4\varepsilon\} \cap \mathbb{R}_+^{n+1}$ is contained in $B(0, R) \cap \mathbb{R}_+^{n+1}$, where (5.6) holds. Also, the set $\{\varphi \geq -4\varepsilon\} \cap \{x_1 = 0\} \subset B(0, R) \cap \{x_1 = 0\}$ is contained in the set $\{\psi > 0\}$. 

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Finally, setting
\[ U := \{ \varphi > -\frac{\varepsilon}{2} \} \cap \{ x_1 > 0 \}, \]
we have \( U \neq \emptyset \) since \( \varphi(0) = 0 \) and \( \varphi < 0 \) on \( \{ x_1 > 0 \} \).

We let \( \chi \in C^\infty(\mathbb{R}^{n+1}) \) such that \( \chi = 1 \) on \( \{ \varphi \geq -2\varepsilon \} \) and \( \chi = 0 \) on \( \{ \varphi \leq -3\varepsilon \} \), and apply (5.6) to \( u = \chi v \). We have \( \varphi \leq 0 \) on the support of \( u \) so that
\[
|v_{x_1}||\chi|_{L^2_\infty} \leq C|\chi v|_{L^2_\infty} \leq C|\chi v|_{L^2_\infty} \leq C|\chi v|_{L^2_\infty} \leq C \rho \|v\|_{H^2_\infty}.
\]

Using that \( \chi = 1 \) on \( \{ \varphi \geq -\frac{\varepsilon}{2} \} \subset \{ \varphi \geq -\varepsilon \} \) and \( U = \{ \varphi \geq -\frac{\varepsilon}{2} \} \subset \mathbb{R}^{n+1} \), we have that
\[
h|v|_{L^2_\infty} \geq h^2 |v_{x_1}|_{L^2_\infty} + h^3 |v_{x_1}||\chi|_{L^2_\infty} \geq \left( h^2 + h^3 \right) |v|_{L^2_\infty} \geq h^2 |v|_{L^2_\infty}.
\]

Finally, we have \( Q \varphi v = \chi Q v + [Q, \chi] v \), where \([Q, \chi]\) is a first order differential operator with coefficients supported in \( \{ -\varepsilon \leq \varphi \leq -2\varepsilon \} \subset \mathbb{R}^{n+1} \). Thus, we have
\[
h^2 |v|_{L^2_\infty} \leq \left( h^2 + h^3 \right) |v|_{L^2_\infty} \geq h^2 |v|_{L^2_\infty}.
\]

Combining the last three estimates with (5.6), we find that there is \( C, h_0 > 0 \) such that for any \( v \in C^\infty(\mathbb{R}^{n+1}) \), for all \( h \in (0, h_0) \), we have
\[
e^{-\frac{\varepsilon}{h}} |v|_{H^1(U)}^2 \leq C |v|_{L^2(\mathbb{R}^{n+1})}^2 + C \|v_{x_1}\|_{L^2(\mathbb{R}^{n+1})}^2 + C \|Q \varphi v\|_{L^2(\mathbb{R}^{n+1})}^2 + C e^{-\frac{\varepsilon}{h}} |v|_{H^1(K)}^2.
\]
and hence, for all \( h \in (0, h_0) \),
\[
|v|_{H^1(U)}^2 \leq C e^{-\frac{\varepsilon}{h}} \left( |v|_{L^2(\mathbb{R}^{n+1})}^2 + \|v_{x_1}\|_{L^2(\mathbb{R}^{n+1})}^2 + \|Q \varphi v\|_{L^2(\mathbb{R}^{n+1})}^2 + C e^{-\frac{\varepsilon}{h}} |v|_{H^1(K)}^2 \right).
\]

After an optimization in the parameter \( h \) (see [Rob95]), this yields the existence of \( C > 0 \) and \( \delta_1 \in (0, 1) \) such that
\[
|v|_{H^1(U)}^2 \leq C \left( |v|_{L^2(\mathbb{R}^{n+1})}^2 + \|v_{x_1}\|_{L^2(\mathbb{R}^{n+1})}^2 + \|Q \varphi v\|_{L^2(K)}^2 \right)^{\delta_1} |v|_{H^1(K)}^{2(1-\delta_1)},
\]
which, coming back to the original variables, implies (5.5), and then according to Theorem 5.6 and [LR95, Lemme 4]), concludes the proof of Theorem 5.5 (see the above discussion). \( \square \)

From Theorem 5.5, we deduce a proof of Theorem 1.7.

\textbf{Proof of Theorem 1.7.} For a non identically vanishing function \( \varphi \) such that \( \text{supp}(\varphi) \subset \Sigma_\beta \), we apply Theorem 5.5 to \( v(x) = e^{\Delta x} u(x) \in C^\infty((-S, S); H^2(M) \cap H^1_0(M)), \) which satisfies
\[
Q v = e^{\Delta x}(-\Delta x - \lambda^2) u, \quad \text{in } \text{Int}(Y_S),
\]
as well as \( v|_{(-S, S) \times \partial M} = 0 \). Hence, Equation (5.3) gives
\[
|v|_{L^2(Y_S)}^2 \leq C \left( e^{2\lambda x} |(-\Delta x - \lambda^2) u|_{L^2(Y_S)}^2 + \|v\|_{L^2(M)}^2 + \|\varphi v\|_{L^2(M)}^2 \right)^{\delta} |v|_{H^1(Y_S)}^{2(1-\delta)} \tag{5.7}
\]
and we estimate each remaining term. First, we have
\[
|v|_{L^2(Y_S)}^2 \geq C e^{2\lambda x} |u|_{L^2(M)}^2.
\]
Second, we write
\[ \| v \|_{H^2(Y_t)}^2 = \| \partial_t v \|_{L^2(Y_t)}^2 + \| \nabla v \|_{L^2(Y_t)}^2 + \| v \|_{L^2(Y_t)}^2 \]
\[ = \| u \|_{L^2(M)}^2 \int_{-s}^s 2 \lambda^2 e^{2\lambda s} + e^{2\lambda s} ds + ((-\Delta - \lambda^2)u, u)_{L^2(M)} \int_{-s}^s e^{2\lambda s} ds \]
\[ \leq C e^{C\lambda} (\| u \|_{L^2(M)}^2 + \| u \|_{L^2(M)}^2). \]

We may assume that \( \|(-\Delta - \lambda^2)u\|_{L^2(M)} \leq \|u\|_{L^2(M)} \) since otherwise the inequality (1.11) holds trivially, and therefore obtain
\[ \| v \|_{H^2(Y_t)}^2 \leq C e^{C\lambda} |u|_{L^2(M)}^2. \]

Third, we have
\[ \| \psi v \|_{L^2(\Sigma)}^2 + \| \psi \partial_s v \|_{L^2(\Sigma)}^2 \leq \int_{-s}^s e^{2\lambda s} (\| u \|_{L^2(\Sigma)}^2 + \| \partial_s u \|_{L^2(\Sigma)}^2) ds \]
\[ \leq 2S e^{2\lambda s} (\| u \|_{L^2(\Sigma)}^2 + \| \partial_s u \|_{L^2(\Sigma)}^2). \]

Plugging the above three inequalities in (5.7) and dividing by \( |u|_{L^2(M)}^{2(1-\theta)} \) (if non zero) yields the sought result.

\[ \square \]

### 5.3 From interpolation inequality to observability in an abstract setting: the original Lebeau-Robbiano method revisited

In this section, we explain how to deduce the observability estimate for the heat equation from the interpolation inequality of Theorem 5.5. This follows the Lebeau-Robbiano method introduced in [LR95] in its original form (used also in [Léa10]), as opposed to the simplified version (see e.g. [LZ98, LRL12]) which uses the stronger spectral inequality [JL99, LZ98] (which we do not prove in the present context). We explain how this method can be simplified using [Mil10, EZ11b, EZ11a].

We consider an abstract setting containing the above particular situation of the heat equation. Most results presented here still hold in the much more general abstract setting of [Mil10]. In Section 5.4 below, we explain how the problem of the heat equation controlled by a hypersurface is put in this general framework.

We denote by \( H \) (with norm \( \| \cdot \| \)) and \( K \) (with norm \( \| \cdot \|_K \)) two Hilbert spaces, namely the state space and the observation space. We denote by \( A : D(A) \subset H \rightarrow H \) a non-positive selfadjoint operator on \( H \), with compact resolvent. We denote by \( (\phi_j) \) an orthonormal basis of eigenfunctions associated to the eigenvalues \( \lambda_j^2 \geq 0 \) of \(-A\) (we keep the notation used for the Laplace operator) and set
\[ E_\lambda := \text{span}[\phi_j, \lambda_j \leq \lambda], \quad \lambda > 0. \quad (5.8) \]

The operator \( A \) generates a contraction semigroup \( (e^{\lambda t}) \) on \( H \). We denote by \( B \in \mathcal{L}(D(A); K) \) the observation operator. We say that \( B \) is an admissible observation for \( (e^{\lambda t}) \) if there is \( T > 0 \) and \( C_{\text{adm}, T} > 0 \) such that
\[ \| Be^{\lambda t} y \|_{L^2((0,T), K)} \leq C_{\text{adm}, T} \| y \|, \quad \text{for all } y \in D(A). \quad (5.9) \]

On account of the semigroup property, (5.9) holds for all \( T > 0 \) if it holds for some \( T \) (see [Cor07, Section 2.3]). Hence, under the above admissibility assumption, for any \( T > 0 \), the map \( u_0 \mapsto (t \mapsto Be^{\lambda t} u_0) \) extends uniquely as a continuous linear map \( H \rightarrow L^2(0, T; K) \), which we shall still denote \( Be^{\lambda t} \).

In our next lemma, we use the notation, for \( s \in \mathbb{N} \) and \( \tau > 0 \),
\[ \mathcal{H}_s = \bigcap_{n=0}^s H^{-n}(\mathbb{R}, \mathcal{D}((-A)^{n/2})); \]

normed by
\[ \| v \|_{\mathcal{H}_s} = \left( \sum_{n+s \leq s} \| \partial_n^s (I - A)^{n/2} v \|_{L^2(\mathbb{R}, \mathcal{H})}^2 \right)^{1/2}. \]

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Lemma 5.8. Let $S > \beta > 0$ and $\varphi \in C_c^\infty(-S, S)$. Assume there is $C > 0$ and $\delta \in (0, 1)$ such that for all $v \in H_\delta^0$, we have
\[
\|v\|_{H_\delta^0} \leq C \left( \|(-\partial_x^2 - A)v\|_{H_\delta^0} + \|\varphi(s)Bv\|_{L^2(-S, S; \mathbb{R})} \right)^\delta \|v\|_{H_\delta^0}^{1-\delta}.
\] (5.10)

Then, there exists $S, C, \delta > 0$ such that
\[
\|v_0\|^2 + \|v_1\|^2 \leq C e^{\beta\lambda} \|\varphi(s)Bv(s)\|^2_{L^2(-S, S; \mathbb{R})},
\] for all $\lambda > 0$, $(v_0, v_1) \in E_\lambda \times E_\lambda$,

with
\[
v(s) = \cosh(s \sqrt{-A}) v_0 + \frac{\sinh(s \sqrt{-A})}{\sqrt{-A}} v_1.
\] (5.11)

Note that in the formula (5.11), we extend $\cosh(s \sqrt{-A})$ (resp. $\frac{\sinh(s \sqrt{-A})}{\sqrt{-A}}$) by continuity by $\text{Id}$ (resp. by $s \text{Id}$) on $\ker(A)$. Thus, denoting by $\Pi_0$ the orthogonal projector on $\ker(A)$ and $\Pi_+ = \text{Id} - \Pi_0$, (5.11) can be rewritten more explicitly by
\[
v(s) = \cosh(s \sqrt{-A}) \Pi_+ v_0 + \Pi_0 v_0 + \frac{\sinh(s \sqrt{-A})}{\sqrt{-A}} \Pi_+ v_1 + s \Pi_0 v_1.
\]

Hence $v(s)$ in (5.11) is the unique solution to
\[(-\partial_x^2 - A)v = 0, \quad (v, \partial_v)_{|v=0} = (v_0, v_1).
\]

Proof of Lemma 5.8. Note first that with $v$ in (5.11), we have $(-\partial_x^2 - A)v(s) = 0$ so that, in (5.10), it suffices to estimate $\|v\|_{H_\delta^0}$ from above and $\|\tilde{v}\|_{H_{\delta/2}^0}$ from below. For $(v_0, v_1) \in E_\lambda \times E_\lambda$, we denote by $w_k = \Pi_0 v_k$, $k = 0, 1$, and $w^\pm = \frac{1}{2}(\Pi_+ v_0 \pm (-A)^{-1/2} \Pi_+ v_1)$. This is
\[
\Pi_+ v_0 = w^+ + w^-, \quad \Pi_+ v_1 = \sqrt{-A} w^+ - \sqrt{-A} w^-,
\]
and the parallelogram law yields
\[
\|(-A)^{\frac{1}{2}} \Pi_+ v_0\|^2 + \|(-A)^{\frac{1}{2}} \Pi_+ v_1\|^2 = 2\|((-A)^{\frac{1}{2}} w^+\|^2 + \|((-A)^{\frac{1}{2}} w^-\|^2).
\]

We also have, with $w^\pm = \sum_{0 \leq i, j \leq 1} w^\pm \phi_j$,
\[
v(s) = \cosh(s \sqrt{-A}) v_0 + \frac{\sinh(s \sqrt{-A})}{\sqrt{-A}} v_1 = e^{s \sqrt{-A}} w^+ + e^{-s \sqrt{-A}} w^- + w_0 + s w_1
= \sum_{0 \leq i, j \leq 1} (e^{s \lambda_j} w^+_j + e^{-s \lambda_j} w^-_j) \phi_j + w_0 + s w_1.
\]

Now, we estimate $\|v\|_{H_\delta^0}$ and $\|\tilde{v}\|_{H_{\delta/2}^0}$ in terms of $\lambda$. Firstly, we have
\[
\|v\|_{H_\delta^0}^2 \geq \|v\|_{H_\delta^0}^2 \geq \sum_{0 \leq i, j \leq 1} \int_0^\beta \left| e^{s \lambda_i} w^+_j + e^{-s \lambda_i} w^-_j \right|^2 ds + \int_0^\beta \|w_0 + sw_1\|^2 ds
= \sum_{0 \leq i, j \leq 1} \frac{e^{2\beta \lambda_j} - e^{-2\beta \lambda_j}}{2 \lambda_j} \left( |w^+_j|^2 + |w^-_j|^2 \right) + 4 \beta \text{Re}(w^+_j \overline{w^-_j}) + 2 \beta^2 |w_0|^2 + \frac{2}{3} \beta^3 |w_1|^2
= 2 \beta \sum_{0 \leq i, j \leq 1} Q_j((w^+_j, w^-_j) + (\overline{w^+_j}, \overline{w^-_j})) + 2 \beta |w_0|^2 + \frac{2}{3} \beta^3 |w_1|^2,
\]
where $Q_j$ is the matrix
\[
Q_j = \begin{pmatrix} \frac{\sinh(X_j)}{X_j} & \frac{1}{\sinh(X_j)} X_j \end{pmatrix}, \quad X_j = 2 \beta \lambda_j.
\]
The eigenvalues of $Q_j$ are $\frac{\sinh(X_j)}{X_j} \pm 1 \geq e^{X_j/2}$ on the set $[2\beta \lambda_0, +\infty]$, where $\lambda_0$ is the first non-zero eigenvalue of $-A$, and $\varepsilon$ only depends on $2\beta \lambda_0$. As a consequence, we obtain

$$\|v\|^2_{\mathcal{H}_g} \geq C \left( \|v_0\|^2 + \|v_1\|^2 \right).$$

Secondly, we also have

$$\|v\|^2_{\mathcal{H}_g} = \int_0^S \|\partial_s v\|^2 + \|(J - A)^{\frac{1}{2}} v\|^2 + \|v\|^2 ds \leq \sum_{0 < j \leq \lambda} (|w_j|^2 + |w_j|^2) \int_0^S (2\lambda_j^2 + 4)e^{2\lambda_j} ds + 2\int_0^S \|w_0 + sw_1\|^2 ds + \int_0^S \|w_1\|^2 ds \leq Ce^{3\lambda} \left( \|v_0\|^2 + \|v_1\|^2 \right).$$

Combining the last two estimates together with (5.10) yields

$$\|v_0\|^2 + \|v_1\|^2 \leq C\|\varphi(s) Bv\|_{L^2([-S,S];K)}^2 \left( Ce^{3\lambda} \left( \|v_0\|^2 + \|v_1\|^2 \right) \right)^{1-\delta},$$

and hence the sought result when dividing by $\left( \|v_0\|^2 + \|v_1\|^2 \right)^{1-\delta}$. \qed

The next step of the Lebeau-Robbiano method relies on a so-called “transmutation argument” to deduce from the observability of the elliptic system on $E_\lambda$ the observability of the heat equation on $E_{\lambda_\delta}$, with a precise estimate on the cost in terms of $\lambda$ and $T$ (observation time). Here, we use an idea of Ervedoza and Zuazua [EZ11b, EZ11a] to simplify the original argument of Lebeau and Robbiano [LR95] (who used the moment method of Russell to pass from the elliptic system to the parabolic system, and was quite technically involved, see [Léa10] for a review of the method).

**Lemma 5.9.** Assume that there exists $S, C, c > 0$ such that

$$\|v_0\| \leq C e^{-\delta} \left\| B \frac{\sinh(s \sqrt{-A})}{\sqrt{-A}} v_0 \right\|_{L^2([-S,S];K)}, \text{ for all } \lambda > 0, v_0 \in E_{\lambda}.$$  

Then, there exist $C, c > 0$ such that

$$\|e^{TA} u_0\| \leq C e^{-\delta} \|Be^{TA} u_0\|_{L^2(0,T,K)}, \text{ for all } T > 0, \lambda > 0, u_0 \in E_{\lambda}.$$  

Note that in the assumption of Lemma 5.9, $\frac{\sinh(s \sqrt{-A})}{\sqrt{-A}}$ can equivalently be replaced by $\cosh(s \sqrt{-A})$.

We need the following lemma, which is a slight variant on [EZ11b, EZ11a].

**Lemma 5.10.** Given $S, T > 0$, $\delta \in (0, 1)$, and $\alpha > S^2 \left( 1 + \frac{1}{\delta} \right)$, there exists a function $k_T \in C^\infty([0,T] \times [-S,S])$ satisfying

$$\left( \partial_t - \partial_s^2 \right) k_T = 0, \text{ for } (t, s) \in (0, T) \times (-S,S), \quad (5.12)$$

$$\begin{cases}
  k_T|_{t=0} = 0, \quad k_T|_{t=T} = 0, & \text{for } s \in (-S,S), \\
  k_T|_{s=0} = 0, \quad \partial_s k_T|_{s=0} = e^{-\alpha(1+\frac{s}{\mu})}, & \text{for } t \in (0, T),
\end{cases} \quad (5.13)$$

$$\|k_T(t, s)\| \leq |s| e^{\frac{1}{2} \left( \frac{s^2}{\delta} - \frac{1}{\delta} \right)} \tau = \min(t, T- t), \text{ for } (t, s) \in (0, T) \times (-S,S). \quad (5.14)$$

For the proof of Lemma 5.10, we follow [EZ11b, Section 3.1], where the authors go from the wave equation to the heat equation. Here, we use the method to go from an elliptic equation to heat equation. The only difference is that we take $g_{2\alpha+1} = g_1^{(k)}$ where Ervedoza and Zuazua [EZ11b, EZ11a] take $g_{2\alpha+1} = (-1)^k g_1^{(k)}$ in the proof below.
Sketch of proof of Lemma 5.10. The starting point is that, if it converges, then the function
\[ k_T(t, s) = \sum_{n \in \mathbb{N}} \frac{e^{-n^2}}{n^3} g_n(t), \quad g_{2k} = g_0, \quad g_{2k+1} = g_1, \quad k \in \mathbb{N}, \]  
(5.15)
solves (5.12). Choose \( g_0 = 0 \) and, for \( \alpha > 0 \), choose \( g_1 \) to be the Gevrey function
\[ g_1(t) = \begin{cases} e^{-\alpha \log 2} t^{1+\frac{1}{2}} & \text{if } t \in (0, T), \\ 0 & \text{otherwise}. \end{cases} \]
Then, [EZ11a, Lemma 3.1] yields for all \( \delta \in (0, 1) \), \( |g_{2k+1}(t)| = |g_1(t)| \leq e^{-\frac{t^2}{\delta^2}} \) with \( \tau = \min(t, T-t) \). This implies (see [EZ11b, Equation (3.8)]) that for all \( \delta \in (0, 1) \), \( S > 0 \) and \( \alpha > S^2 \left( 1 + \frac{1}{\delta} \right) \), the series (5.15) converges towards \( k_T \in C^\infty(\mathbb{R}^2 \times [-S, S]) \) with (5.14)-(5.13).

With this lemma, the proof of Lemma 5.9 follows [EZ11b, Section 3.1].

Proof of Lemma 5.9. We first pick \( \delta \in (0, 1) \), and \( \alpha > S^2 \left( 1 + \frac{1}{\delta} \right) \), and denote by \( k_T \) the kernel then furnished by Lemma 5.15. Given \( u_0 \in E_\lambda \), we define
\[ v(s) := \int_0^T k_T(t, s) e^{\lambda t} u_0 dt. \]
From the above properties of \( k_T \), the function \( v(s) \) satisfies
\[ (v, \partial_s v)|_{s=0} = \left( 0, \int_0^T g_1(t) e^{\lambda t} u_0 dt \right) \in E_\lambda \times E_\lambda, \]
where \( g_1(t) = e^{-\alpha \log 2} t^{1+\frac{1}{2}} \), together with
\[ \partial_s^2 v(s) = \int_0^T \partial_s k_T(t, s) u(t) dt = \int_0^T \partial_s k_T(t, s) e^{\lambda t} u_0 dt = -\int_0^T k_T(t, s) \partial_t e^{\lambda t} u_0 dt \]
\[ = -\int_0^T k_T(t, s) A e^{\lambda t} u_0 dt = -A \left( \int_0^T k_T(t, s) e^{\lambda t} u_0 dt \right) = -Av(s). \]
Hence, \( v(s) = \frac{\sinh(\sqrt{\lambda} t)}{\sqrt{\lambda}} \left( \int_0^T g_1(t) e^{\lambda t} u_0 dt \right) \), so that Lemma 5.9 yields the estimate
\[ \left\| \int_0^T g_1(t) u(t) dt \right\| \leq C e^{\lambda^2} \| Bv(s) \|_{L^2(-S, S; K)}. \]
Now, writing \( u_0 = \sum_j \alpha_j \phi_j \), we have
\[ \left\| \int_0^T g_1(t) e^{\lambda t} u_0 dt \right\|^2 = \sum_j \left( \int_0^T g_1(t) e^{-\lambda t^2} \alpha_j dt \right)^2 \]
\[ \geq \sum_j \left( \int_0^T g_1(t) dt \right)^2 e^{-\lambda T} \int |x|^2 = \left( \int_0^T g_1(t) dt \right)^2 \| e^{\lambda t} u_0 \|^2 \geq \frac{T^2}{9} e^{-\frac{9}{2}} \| e^{\lambda t} u_0 \|^2. \]
Also, we have from (5.14) the estimate
\[ \| Bv(s) \|_{L^2(-S, S; K)} = \int_{-S}^S \left\| \int_0^T k_T(t, s) e^{\lambda t} u_0 dt \right\|^2 ds \]
\[ \leq \left( \int_{[0, T]} k_T(t, s)^2 dtds \right) \int_0^T \| e^{\lambda t} u_0 dt \|^2 dt \]
\[ \leq C_5 T \int_0^T \| e^{\lambda t} u_0 dt \|^2 dt. \]
Combining the last three estimates concludes the proof of Lemma 5.9. \( \Box \)
From the low frequency observability estimate with precise cost, we may now deduce the full observability estimate. The original Lebeau-Robbiano strategy [LR95] does not provide with an optimal dependence on the blow-up of the constant as $T \to 0^+$. The modified and simplified argument of [Mil10] does so, and we follow it here.

**Lemma 5.11.** Assume $B : D(A) \subset H \to K$ is an admissible observation for $(e^{tA})$. Assume for some $a_0, a, b > 0$ we have

$$
\|e^{tA}y\| \leq a_0 e^{a_1 t + \frac{b}{2}} \|Be^{tA}y\|_{L^2(0,T;K)}, \quad \text{for all } y \in E_\lambda, \lambda > 0, T > 0. \quad (5.16)
$$

Then there is $C, c > 0$ such that we have

$$
\|e^{tA}y\| \leq Ce^{t_0^*} \|Be^{tA}y\|_{L^2(0,T;K)}, \quad \text{for all } y \in H, T > 0.
$$

A proof of this lemma (in much more generality) is included in the proof of [Mil10, Theorem 2.1], but we give it for the sake of readability. The key feature of the semigroup $(e^{tA})$ we shall use is that

$$
\|e^{tA}y\| \leq e^{-\varepsilon t^2} \|y\|, \quad \text{for all } y \in E_\lambda^+, \lambda > 0, t > 0. \quad (5.17)
$$

We also make use of the following particular case of [Mil10, Lemma 2.1].

**Lemma 5.12.** Let $T_0 > 0$, $q \in (0, 1)$ and $f : (0, T_0) \to \mathbb{R}_+$ increasing, such that $\lim_{t \to 0^+} f(t) = 0$. Assume that $B$ is an admissible observation for $(e^{tA})$ and that

$$
f(T)\|e^{tA}y\|^2 - f(qT)\|y\|^2 \leq \|Be^{tA}y\|^2 \quad \text{for all } T \in (0, T_0) \text{ and } y \in H.
$$

Then we have

$$
f((1 - q)T)\|e^{tA}y\|^2 \leq \|Be^{tA}y\|^2 \quad \text{for all } T \in (0, T_0) \text{ and } y \in H.
$$

**Proof of Lemma 5.11.** For $y \in H$, we decompose $y = y_1 + r_1$ with $y_1 \in E_\lambda$ and $r_1 \in E_\lambda^+$. Then, we estimate

$$
\|e^{tA}y\| \leq \|e^{tA}y_1\| + \|e^{tA}r_1\|. \quad (5.18)
$$

Concerning the second term in (5.18), we only use (5.17) to write

$$
\|e^{tA}r_1\| \leq e^{-\varepsilon t^2} \|r_1\| \leq e^{-\varepsilon t^2} \|y\|.
$$

Concerning the first term in (5.18), we write $e^{tA} = e^{t^\frac{1}{2}A} e^{(1-e^{t^2})t^\frac{1}{2}A}$ and apply (5.16) to $e^{(1-e^{t^2})t^\frac{1}{2}A} y_1 \in E_\lambda$ to obtain

$$
\|e^{tA}y_1\| \leq a_0 e^{a_1 t + \frac{b}{2}} \|Be^{t^\frac{1}{2}A} e^{(1-e^{t^2})t^\frac{1}{2}A} y_1\|_{L^2(0,T;K)}

\leq a_0 e^{a_1 t + \frac{b}{2}} (\|Be^{t^\frac{1}{2}A} e^{(1-e^{t^2})t^\frac{1}{2}A} y_1\|_{L^2(0,T;K)} + \|Be^{t^\frac{1}{2}A} e^{(1-e^{t^2})t^\frac{1}{2}A} r_1\|_{L^2(0,T;K)}).
$$

We remark that $\|Be^{t^\frac{1}{2}A} e^{(1-e^{t^2})t^\frac{1}{2}A} y_1\|_{L^2(0,T;K)} = \|Be^{t^\frac{1}{2}A} y_1\|_{L^2(0,T;K)} \leq \|Be^{t^\frac{1}{2}A} y_1\|_{L^2(0,T;K)}$ and estimate the last term using (5.9), and then (5.17) as

$$
\|Be^{t^\frac{1}{2}A} e^{(1-e^{t^2})t^\frac{1}{2}A} r_1\|_{L^2(0,T;K)} \leq C_{adm,eT} \|e^{(1-e^{t^2})t^\frac{1}{2}A} r_1\| \leq C_{adm,eT} e^{-\varepsilon t^2 (1-e^{t^2})T} \|r_1\|

\leq C_{adm,eT} e^{-\varepsilon t^2 (1-e^{t^2})T} \|y\|.
$$

Combining the above three estimates in (5.18) implies for all $y \in H$, $T > 0$ and $\lambda > 0$,

$$
\|e^{tA}y\| \leq a_0 e^{a_1 t + \frac{b}{2}} \|Be^{tA}y\|_{L^2(0,T;K)} + e^{-\varepsilon t^2 (1-e^{t^2})T} (a_0 e^{a_1 t + \frac{b}{2}} C_{adm,eT} + e^{-\varepsilon t^2 T} \|y\|).
$$

We notice that $C_{adm,eT} \leq C_{adm,T_0}$ for $T \leq T_0$ and $e \in (0, 1)$, and denote $m_1 := C_{adm,T_0} + \frac{1}{a_0}$. We then rewrite this estimate for $\lambda = \frac{1}{T}$, with $r > 0$ to be chosen, as

$$
\frac{1}{d_0} e^{-\varepsilon (\frac{r^2}{2} + \frac{r}{2})} \|e^{tA}y\| \leq \|Be^{tA}y\|_{L^2(0,T;K)} + m_1 e^{-\varepsilon T} \|y\|, \quad T \leq T_0.
$$
Writing \( f(T) = \frac{1}{2m_0} e^{-\frac{T}{r} (\varepsilon + \frac{r}{\varepsilon})} \), and assuming the parameters \( \varepsilon \in (0, 1), r > 0, q \in (0, 1) \) are such that

\[
\frac{1}{q} \left( \frac{a}{r} + \frac{b}{\varepsilon} \right) \leq 1 - \frac{\varepsilon}{r^2},
\]

(which we may, taking e.g. \( \varepsilon = q = 1/2 \) and \( r \) sufficiently small) we have \( \left( m_1 e^{-\frac{T}{r} \varepsilon} \right)^2 \leq f(qT) \) for \( T \in (0, T') \) for some \( T' \in (0, T_\alpha) \), and we obtain

\[
f(T)\|e^{TA}y\|^2 \leq \|Be^{TA}y\|^2_{L^q(0,T;K)} + f(qT)\|y\|^2.
\]

Lemma 5.12 implies

\[
f((1-q)T)\|e^{TA}y\|^2 \leq \|Be^{TA}y\|^2_{L^q(0,T;K)}, \quad T \in (0, T'), y \in H,
\]

which is the sought result for \( t \in (0, T') \). The case \( T > T' \) follows from the boundedness of the semigroup and the case \( T < T' \). \( \square \)

### 5.4 From interpolation inequality to the observability estimate for the heat equation

Let us now put the above context of the heat equation in the present abstract framework, and state the consequences of the above abstract setting. We have \( H = H^1_0(M), A = \Delta_D \) (the Dirichlet Laplacian) with \( D(A) = \{ u \in H^1(M), u|_{\partial M} = 0, \Delta q u|_{\partial M} = 0 \}. \) We also have \( K = L^2(\Sigma) \times L^2(\Sigma) \) as well as

\[
B : D(A) \subset H^1(M) \rightarrow L^2(\Sigma) \times L^2(\Sigma)
\]

\[
u \mapsto (u|_{\Sigma}, \partial_\nu u|_{\Sigma}).
\]

Lemma 5.2 implies that \( B \) is an admissible observation for \((e^{tA})\) in the sense of (5.9).

The first lemma is a consequence of the interpolation inequality of Theorem 5.5 and Lemma 5.8. Here, \( E_1 \) is defined by (5.8) where \( \phi_j, \lambda_j \) are an orthonormal basis of solutions to

\[
(-\Delta - \lambda_j^2)\phi_j = 0.
\]

**Lemma 5.13** (observability of finite dimensional elliptic equation). Assume \( M \) is connected and \( \Sigma \) is nonempty. Then, for all \( S > 0 \), there exists \( C, c > 0 \) such that for all \( \lambda > 0 \), all \( (v_0, v_1) \in E_1 \times E_1 \) and associated solution \( v \) of

\[
\begin{cases}
(-\partial_t^2 - \Delta)v = 0 & \text{on } (-S,S) \times \text{Int}(M), \\
v = 0 & \text{on } (-S,S) \times \partial M, \\
(v, \partial_\nu v)|_{t=0} = (v_0, v_1) & \text{in } \text{Int}(M),
\end{cases}
\]

we have

\[
\| (v_0, v_1) \|_{H^1 \times H'} \leq Ce^{c\lambda} \left( \|v\|_{L^2((-S,0) \times \Sigma)} + \|\partial_\nu v\|_{L^2((-S,0) \times \Sigma)} \right).
\]

This together with Lemma 5.9 this implies the following result.

**Lemma 5.14** (observability of finite dimensional heat equation with precise cost). Assume \( M \) is connected and \( \Sigma \) is nonempty. Then, there exists \( C, c > 0 \) such that for all \( \lambda, T > 0 \), all \( u_0 \in E_1 \) and associated solution \( u \) of

\[
\begin{cases}
(\partial_t - \Delta)u = 0 & \text{on } (0,T) \times \text{Int}(M), \\
u = 0 & \text{on } (0,T) \times \partial M, \\
u|_{t=0} = u_0 & \text{in } \text{Int}(M),
\end{cases}
\]

we have

\[
\| u(T) \|_{H'} \leq Ce^{c\lambda + \frac{T}{2}} \left( \|u_0\|_{L^2(0,T \times \Sigma)} + \|\partial_\nu u_0\|_{L^2(0,T \times \Sigma)} \right).
\]

Lemma 5.11 finally yields the following observability result.
Theorem 5.15 (observability for heat equation). Assume $M$ is connected and $\Sigma$ is nonempty. Then, there exist $C, c > 0$ such that for all $T > 0$, all $u_0 \in H^1(M)$ and associated solution $u$ of (5.20), we have

$$\|u(T)\|_{H^1} \leq Ce^T \left( \|u_0\|_{L^2((0,T)\times \Sigma)} + \|\partial_n u\|_{L^2((0,T)\times \Sigma)} \right).$$

From this observability estimate and the duality with the control problem (1.10), given by Definition 5.3, we deduce the null-controllability of the heat equation Theorem 1.6. The proof is classical and we omit it (see e.g. [Cor07, Chapter 2.3]).

A Facts and notations of pseudodifferential calculus

Here, we follow [Bur97, Section 1.1] or [DLRL14, Section 2.1] for the notation. We denote by $X$ an open set of a $d$ dimensional manifold, which, in the main part of the article, is, with $d = n - 1, n, n + 1$, one of the following:

$$X = \mathbb{R}^d, \quad X = \text{Int}(M), \quad X = (0, T) \times \text{Int}(M), \quad X = \text{Int}(\Sigma), \quad X = (0, T) \times \text{Int}(\Sigma), \quad X = \text{Int}(\Sigma). \quad \text{(A.1)}$$

We also denote by $x$ the variable in $X$ (whereas, in case $X = (0, T) \times \text{Int}(M)$ the variable in denoted $(t, x)$ in the main part of the article). We denote by $\pi_0 : T^* X \to X$ the canonical projection.

We write $S^m_{\text{hom}}(T^*X)$ for the set of positively homogeneous degree $m$ functions on $T^* X$ with compact support in $X$. That is, $a \in S^m_{\text{hom}}(T^*X)$ if and only if $a \in C^\infty(T^*X)$, $\pi_0(\text{supp}(a))$ is a compact of $X$, and there is $R > 0$ (depending on $a$) such that for $(x, \xi) \in T^* X$, with $|\xi| \geq R \Lambda \geq 1$, we have $a(x, \lambda \xi) = \lambda^m a(x, \xi)$. For any $m$, the restriction to the sphere $S^* X = T^* M / \mathbb{R}^*_+$

$$S^m_{\text{phg}}(T^*X) \to C^\infty(S^*X), \quad a(x, \xi) \to \lim_{\lambda \to \infty} \lambda^{-m} a(x, \lambda \xi), \quad \text{(A.2)}$$

is onto, which identifies, for $m$ fixed, smooth functions on the sphere with homogeneous symbols of degree $m$.

We also write $S^m_{\text{phg}}(T^*X)$ for the set of polyhomogeneous symbols of order $m$ on $X$ with compact support in $X$. That is, $a \in S^m_{\text{phg}}(T^*X)$ if and only if $a \in C^\infty(T^*X)$, $\pi_0(\text{supp}(a))$ is a compact of $X$, and there exist $a_j \in S^m_{\text{hom}}(T^*X)$, such that for all $N \in \mathbb{N}$, $a - \sum_{j=0}^N a_j \in S^{m-N}(T^*X)$. We recall that symbols in the class $S^m_{\text{phg}}(T^*\mathbb{R}^d)$ behave well with respect to changes of variables, up to symbols in $S^{m-1}(T^*\mathbb{R}^d)$ (see [Hör85, Theorem 18.1.17 and Lemma 18.1.18]).

We denote by $\Psi^m_{\text{phg}}(X)$ the space of polyhomogeneous pseudodifferential operators of order $m$ on $X$, with a compactly supported kernel in $X \times X$: one says that $A \in \Psi^m_{\text{phg}}(X)$ if

1. its kernel $K(x, y) \in \mathcal{D}'(X \times X)$ is such that $\text{supp}(K)$ is compact in $X \times X$;
2. $K(x, y)$ is smooth away from the diagonal $\Delta_X = \{(x, x); \; x \in X\}$;
3. for every coordinate patch $X_c \subset X$ with coordinates $X_c \ni x \mapsto \kappa(x) \in \tilde{X}_c \subset \mathbb{R}^d$ and all $\phi_0, \phi_1 \in C^\infty_c(X_c)$ the map

$$u \mapsto \phi_1(\kappa^{-1})^* A \kappa^* (\phi_0 u)$$

is in $\text{Op}(S^m_{\text{phg}}(\mathbb{R}^d \times \mathbb{R}^d))$. Note that for $a \in S^m_{\text{phg}}(\mathbb{R}^d \times \mathbb{R}^d)$ we write $\text{Op}(a)$ for the standard quantization of $a$.

In case $X$ is not compact (which happens in most examples of (A.1)), we also define a non-canonical quantization procedure $\text{Op} : S^m_{\text{phg}}(T^*X) \to \Psi^m_{\text{phg}}(X)$. For this, fix $\chi_n \in C^\infty_c(X; [0, 1])$ so that $\chi_n \to 1$ for all $x \in X$ uniformly on compact sets. Then fix $(X_c, \kappa_c)$ a coordinate atlas for $X$. Let $\psi_i \in C^\infty_c(X_i)$ be a partition of unity subordinate to $X_i$ and $\tilde{\psi}_i \in C^\infty_c(X_i)$ with $\text{supp} \psi_i \subset \{ \tilde{\psi}_i \equiv 1 \}$. For $a \in S^m_{\text{phg}}(X)$, notice that $a_i(x, \xi) := \psi_i(\kappa_i^{-1}(x))a(\kappa_i(x), (\partial \kappa_i)^{-1}(\xi))$ has $a_i \in S^m_{\text{phg}}(\mathbb{R}^d \times \mathbb{R}^d)$. We then define

$$\text{Op}(a) = \sum_i \chi_i \kappa_i^*[((\kappa_i^{-1})^* \tilde{\psi}_i) \text{Op}(a_i)(\kappa_i^{-1})^*(\tilde{\psi}_i \chi_i)]^*, \quad N := \inf \{ n \mid \text{supp} a \cap \text{supp}(1 - \chi_n) = \emptyset \}.$$

Note that for all $A \in \Psi^m_{\text{phg}}(X)$, there exists $a \in S^m_{\text{phg}}(T^*X)$ so that

$$\text{Op}(a) - A = R \in \Psi^{m-N}_{\text{phg}}(X)$$
(see e.g. [DZ, Appendix E]).

For \( A \in \Psi_m^m(X) \), we denote by \( \sigma_m(A) \in S^m_{\text{hom}}(T^*X) \) the principal symbol of \( A \) (see [Hör85, Chapter 18.1]). Note that the principal symbol is uniquely defined in \( S^m_{\text{hom}}(T^*X) \) because of the polyhomogeneous structure (see the remark following Definition 18.1.20 in [Hör85]). When it will not lead to confusion, we abuse notation slightly and write \( \sigma(A) \) for the principal symbol of a pseudodifferential operator without reference to the order. The applications \( \sigma_m \) and \( \text{Op} \) enjoy the following properties:

- The sequence
  \[
  0 \to \Psi_{m-1}^m(T^*X) \to \Psi_m^m(X) \xrightarrow{\sigma_m} S^m_{\text{hom}}(T^*X) \to 0
  \]
  is exact.
- \( \sigma_m \circ \text{Op} : S^m_{\text{phg}}(T^*X) \to S^m_{\text{hom}}(T^*X) \) is the natural projection map.
- For all \( A \in \Psi_m^m(X) \), \( \sigma_m(A^*) = \overline{\sigma_m(A)} \).
- For all \( A_1 \in \Psi_{m_1}^m(X) \) and \( A_2 \in \Psi_{m_2}^m(X) \), we have \( A_1A_2 \in \Psi_{m_1+m_2}^m(X) \) with
  \[
  \sigma_{m_1+m_2}(A_1A_2) = \sigma_{m_1}(A_1)\sigma_{m_2}(A_2).
  \]
- For all \( A_1 \in \Psi_{m_1}^m(X) \) and \( A_2 \in \Psi_{m_2}^m(X) \), we have \( [A_1, A_2] = A_1A_2 - A_2A_1 \in \Psi_{m_1+m_2-1}^m(X) \) with
  \[
  \sigma_{m_1+m_2-1}([A_1, A_2]) = \frac{1}{i} [\sigma_{m_1}(A_1), \sigma_{m_2}(A_2)].
  \]

Here, \( \{a_1, a_2\} \) denotes the Poisson bracket, given in local charts by
\[
\{a_1, a_2\} = \sum_i (\partial_{\xi_i} a_1 \partial_{\xi_i} a_2 - \partial_{\xi_i} a_1 \partial_{\xi_i} a_2).
\]

- If \( A \in \Psi_m^m(X) \), then \( A \) maps continuously \( H^k_{\text{loc}}(X) \) into \( H^{k-m}_{\text{comp}}(X) \). In particular, for \( m < 0 \), \( A \) is compact on \( H^k(X) \).

Given an operator \( A \in \Psi_m^m(X) \), we define \( \text{Char}(A) = \{ \rho \in T^*X \mid 0, \sigma_m(A)(\rho) = 0 \} \) its characteristic set and
\[
\text{Ell}(A) = (T^*X \setminus 0) \setminus \text{Char}(A)
\]
its elliptic set.

We define the wavefront set of an operator \( A \in \Psi_m^m(X) \), denoted by \( \text{WF}(A) \) as follows (see [Hör85, Proposition 18.1.26 p88]). We say \( (x_0, \xi_0) \in T^*X \) is not in \( \text{WF}(A) \) if there exists \( B \in \Psi^0_{\text{phg}}(X) \) with \( \sigma_0(B) = 1 \) and
\[
BA : \mathcal{D}'(X) \to C^\infty_c(X).
\]
Note that in local coordinates, the wavefront set this is given by the support of the full symbol of \( A \) (seen as a subset of \( S^\infty_{\mathbb{R}^d} \)).

Also, in the main part of the article, we use so-called “tangential” symbols, pseudodifferential operators and pseudodifferential calculus. We write \( a \in C^\infty((-\varepsilon, \varepsilon); S^m_{\text{phg}}(T^*\mathbb{R}^d)) \) if \( a = a(x_1, x', \xi') \) is a smooth \( x_1 \)-dependent family of symbols in the \( (x', \xi') \) variables. We write \( A \in C^\infty((-\varepsilon, \varepsilon); \Psi_{m}^m(\mathbb{R}^d)) \) for the associated operators. The rules of pseudodifferential calculus are then as above.

Finally, in the main part of the article, we use estimates for the hyperbolic Cauchy problem. We state the following Lemma from Hörmander [Hör85, Lemma 23.1.1].

**Lemma A.1.** Let \( \varepsilon > 0 \), suppose that \( \Lambda = \Lambda(x_1, x', \xi') \in C^\infty((-\varepsilon, \varepsilon); S^1_{\text{phg}}(T^*\mathbb{R}^d)) \) is real valued and write \( \Lambda = \text{Op}(\Lambda) \). Then for all \( s \in \mathbb{R} \), there exists \( C > 0 \) so that for \( x_1, y_1 \in (-\varepsilon, \varepsilon) \) and all \( u, f \) solutions of
\[
(D_{x_1} - \Lambda)u = f,
\]
we have
\[
\|u(x_1, \cdot)\|_{H^s(\mathbb{R}^d)} \leq C\|u(y_1, \cdot)\|_{H^s(\mathbb{R}^d)} + \|f\|_{L^2((-\varepsilon, \varepsilon); H^s(\mathbb{R}^d))}
\]
and moreover
\[
\|u(x_1, \cdot)\|_{H^s(\mathbb{R}^d)} \leq C\|u(y_1, \cdot)\|_{H^s(\mathbb{R}^d)} + \|f\|_{L^2((-\varepsilon, \varepsilon); H^s(\mathbb{R}^d))}.
\]

Note that the second estimate is obtained from the first one by integrating in \( y_1 \).
B Sharpness of Theorem 1.7: Proof of Proposition 1.8

We start with an abstract simple lemma linking the symmetries of the manifold with that of solutions to related elliptic problems.

**Lemma B.1.** Let \((M, g)\) be a compact Riemannian manifold possibly with boundary and suppose that there is an isometric involution \(j : M \to M\) (i.e. a diffeomorphism such that \(j^*g = g\) and \(j^2 = 1d\)) and a compact hypersurface \(\Sigma \subset M\) such that
\[
M = M_+ \cup M_- \cup \Sigma, \quad \text{Fix}(j) =: \Sigma
\]
where
\[
\text{Fix}(j) := \{x \in M \mid j(x) = x\}
\]
and \(j(M_+) = M_-\). Let \(V \in C^\infty(M)\) such that \(V \circ j = V\). Suppose that \(u, v\) solve
\[
\begin{align*}
(-\Delta_g + V)u &= 0 \quad \text{in } \text{Int } M_+, & u|_\Sigma &= 0, & u|_{\partial M} &= 0, \\
(-\Delta_g + V)v &= 0 \quad \text{in } \text{Int } M_+, & \partial_\nu v|_\Sigma &= 0, & v|_{\partial M} &= 0.
\end{align*}
\]
Then, \(u_o := \begin{cases} u(x) & x \in M_+ \cup \Sigma, \\ -u(j(x)) & x \in M_-, \end{cases}\) and \(u_e := \begin{cases} v(x) & x \in M_+ \cup \Sigma, \\ v(j(x)) & x \in M_-, \end{cases}\)
satisfy \(u_o, u_e \in C^\infty(\overline{M})\) and solve
\[
\begin{align*}
(-\Delta_g + V)u_o &= 0 \quad \text{in } M, & u_o|_{\partial M} &= 0, & u_o|_\Sigma &= 0, \\
(-\Delta_g + V)u_e &= 0 \quad \text{in } M, & u_e|_{\partial M} &= 0, & \partial_\nu u_e|_\Sigma &= 0.
\end{align*}
\]

**Proof.** Notice first that \(\partial M_+ = \Sigma \cup (\partial M \cap M_+)\) and by elliptic regularity, we have \(u, v \in C^\infty(\overline{M}_+)\). Moreover, if \(w_+ \in C^2(\overline{M}_+)\), then, in the distribution sense (with \(\partial_\nu\) pointing towards \(M_-\))
\[
(-\Delta_g + V)w(x) = \mathbb{1}_M(-\Delta_g + V)w_+ + \mathbb{1}_M(-\Delta_g + V)w_- - (w_+|_\Sigma - w_-|_\Sigma)\delta_\Sigma - (\partial_\nu w_+|_\Sigma - \partial_\nu w_-|_\Sigma)\delta_\Sigma.
\]
Hence,
\[
(-\Delta_g + V)w(x) = (-\Delta_g + V)w_- = 0
\]
as distributions and by elliptic regularity, \(u_o, u_e \in C^\infty\) and hence have the desired properties. \(\square\)

We may now proceed to the proof of Proposition 1.8.

**Proof of Proposition 1.8.** The Riemannian volume element is \(R(z)dzd\theta\) and the Laplace Beltrami operator is given by
\[
\Delta_g = \frac{1}{R(z)}\partial_z R(z)\partial_z + \frac{1}{R(z)^2}\partial_\theta^2 = \partial_z^2 + \frac{R'}{R} \partial_z + \frac{1}{R^2} \partial_\theta^2.
\]
The map
\[
T : L^2(M, R(z)dzd\theta) \to L^2(M, dzd\theta) \quad \text{where } \quad T u, \quad (T u)(z, \theta) = R(z)^{-\frac{1}{2}} u(z, \theta)
\]
is an isometry and the conjugated operator of \(\Delta_g\) is
\[
\bar{\Delta} = T \Delta_g T^{-1} = R^{1/2}\Delta_g R^{-1/2} = \bar{\partial}_z^2 + \frac{1}{4} \left(\frac{R'}{R}\right)^2 - \frac{1}{2} \frac{R''}{R} + \frac{1}{2} \bar{\partial}_\theta^2
\]
where
\[
V_1(z) = -\frac{1}{4} \frac{R'(z)}{R(z)} + \frac{1}{2} \frac{R''(z)}{R(z)^2}
\]
is a smooth \(\theta\)-independent potential on \(M\).  

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We now construct eigenfunctions of $\tilde{\Delta}$ under the form $\tilde{\phi}_k(z, \theta) = e^{i\theta} \psi_k(z)$. Setting $h = k^{-1}$, this amounts to find eigenfunctions of the operator

$$P_h := -h^2 \partial_z^2 + \frac{1}{R(z)^2} + h^2 V(z)$$

with Dirichlet boundary conditions on $\pm \pi$. We shall rather consider this operator on $(0, \pi)$, and then complete the construction by symmetry with Lemma B.1. Recall that the potential $V(z) = \frac{1}{R(z)^2}$ satisfies $V(0) = 1$, $V(\pi/2) = 1/5$, $V(\pi) = 2$. Denoting by $E_{hk}^\nu$ the eigenvalues of $P_h$ on $(0, \pi)$ associated to Dirichlet on $\pi$ and Neumann on 0 for $E_{hk}^\nu$, resp. Dirichlet on 0 for $E_{hk}^\nu$. The Weyl law (see e.g. [DS99, Corollary 9.7], [Zwo12, Theorem 6.8] or [SV97, Theorem 1.2.1]) implies

$$\#\{E_{hk}^\nu \leq \frac{1}{2}\} \sim_{h\to0} (2\pi h)^{-1} \left| \left\{(z, \xi) \in (0, \pi) \times \mathbb{R}, \frac{1}{5} \leq \xi^2 + V(z) \leq \frac{1}{2} \right\} \right|,$$

so that, recalling the form of $V$, there is $h_0 > 0$ such that for $h \in (0, h_0)$, the set $\{E_{hk}^\nu \leq \frac{1}{2}\}$ is nonempty. We pick such an eigenvalue $E_{hk}^\nu \in [\frac{1}{5}, \frac{1}{2}]$, and denote $\psi_h^\nu$ an associated eigenfunction, i.e., which satisfies

$$(P_h - E_{hk}^\nu)\psi_h^\nu = 0, \quad \psi_h^\nu(0) \neq 0, \quad \psi_h^\nu(\pi) = 0, \quad \partial_z \psi_h^\nu(0) = 0, \quad \psi_h^\nu(0).$$

Applying now Lemma B.1 on $M = [0, \pi]$ with $j(z) = -z$, so that $\text{Fix}(j) = \{0\}$ allows to extend $\psi_h^\nu$ by parity/impairness as solutions of

$$(P_h - E_{hk}^\nu)\psi_h^\nu = 0 \text{ on } (-\pi, \pi), \quad \psi_h^\nu(\pm \pi) = 0, \quad \partial_z \psi_h^\nu(0) = 0, \quad \psi_h^\nu(0) = 0.$$

Now, since $E_{hk}^\nu \in [\frac{1}{5}, \frac{1}{2}]$ and $V(0) = \frac{1}{R(0)^2} = 1 > \frac{1}{5}$, 0 is in the classically forbidden region, so that classical Agmon estimates (see e.g. [DS99, Chapter 6] or [Zwo12, Chapter 6]), yield for $e, h_0 > 0$ small enough, for any $N > 0$, the existence of $C, c > 0$ such that for all $h \in (0, h_0)$

$$\|\psi_h^\nu\|_{H^2(-\pi, \pi)} \leq C e^{-ch} \|\psi_h^\nu\|_{L^2(-\pi, \pi)}.$$

Coming back to the variables $(z, \theta)$, setting $(\lambda_{hk}^\nu)^2 = k^2 E_{k^2} \in k^2[\frac{1}{5}, \frac{1}{2}]$ and $\tilde{\phi}_k(z, \theta) = e^{ik\theta} \psi_k(\xi(z), \zeta)$, we have obtained for $k \geq k_0$ solutions to

$$(-\tilde{\Delta} - (\lambda_{hk}^\nu)^2)\tilde{\phi}_k^\nu = 0, \quad \tilde{\phi}_k^\nu(\pm \pi) = 0, \quad \|\tilde{\phi}_k^\nu\|_{L^2} \neq 0,$$

together with

$$\tilde{\phi}_k^\nu|_\Sigma = 0, \quad \|\tilde{\phi}_k^\nu|_\Sigma\|_{L^2} \leq C e^{-c\lambda_{hk}^\nu} \|\tilde{\phi}_k^\nu\|_{L^2(M)}, \quad \|\tilde{\phi}_k^\nu|_\Sigma\|_{L^2} \leq C e^{-c\lambda_{hk}^\nu} \|\tilde{\phi}_k^\nu\|_{L^2(M)}.$$

Setting $\phi_k^\nu(z, \theta) = R(z)^{-1/2} \tilde{\phi}_k(z, \theta) ||\tilde{\phi}_k^\nu||_{L^2}^{-1}$ concludes the proof of the lemma. \hfill \Box

C About TGGC: Proof of Proposition 1.10

Proof of Proposition 1.10. Here, $M = S^2$ and $\Sigma$ is an equator. We take the following coordinates on $S^2$:

$$[0, 2\pi] \times [0, \pi] \ni (\theta, \phi) \mapsto (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \in S^2,$$

and let $\Sigma := \{\phi = \frac{\pi}{2}\}.$

Then an orthonormal basis of Laplace eigenfunctions is given by

$$Y_l^m(\theta, \phi) = \left(\frac{(l - m)! (2l + 1)}{4\pi (l + m)!}\right)^{1/2} e^{im\phi} P_l^m(\cos \phi), \quad -l \leq m \leq l,$$

where $P_l^m$ is an associated Legendre function (see for example [OLBC10, Section 14.30]). For the definition of $P_l^m$ see [OLBC10, Section 14.2]. Note that

$$(-\Delta_{\Sigma} - \lambda_l^2)Y_l^m = 0, \quad \lambda_l := \sqrt{l(l + 1)} \sim_{l \to \infty} l.$$
We take $\phi_l = Y_l^{l-1}$, and recall that $\Sigma = \{ \phi = \frac{\pi}{2} \}$. By [OLBC10, Section 14.30ii and Section 14.5(i)], we have

$$\phi_l|_{\Sigma} = Y_l^{l-1} \left( \theta, \frac{\pi}{2} \right) = 0,$$

since $P_l^{l-1}(0) = 0$. Moreover, using [OLBC10, Equation (14.5.2)] together with the definition of $Y_l^{l-1}$, we have

$$|\partial_{\nu} \phi_l|_{\Sigma} = \left| \partial_{\nu} Y_l^{l-1} \left( \theta, \frac{\pi}{2} \right) \right| = \left| \frac{2l^{1/2}}{\Gamma(-l + \frac{1}{2})} \right| ~ c^{1/4}. \quad (C.1)$$

Indeed, note that for $l \geq 1$,

$$\Gamma \left( \frac{1}{2} - l \right) = \frac{(-1)^l \pi}{\Gamma(l + \frac{1}{2})} = \frac{\sqrt{\pi} 2^{2l} l!}{(2l)!} = \frac{2^l(-1)^l \sqrt{\pi}}{\prod_{j=1}^{l}(2j - 1)}$$

and

$$\prod_{j=1}^{l}(2j - 1)^2 = \prod_{j=2}^{l} \frac{2j - 1}{2j - 2} = e^{\sum_{j=2}^{l} \log(1 + \frac{1}{j})}.$$

Then, note that

$$\sum_{j=2}^{l} \log \left( 1 + \frac{1}{2j - 2} \right) = \frac{1}{2} \log l + O(1)$$

and in particular,

$$\prod_{j=1}^{l}(2j - 1)^2 = c l^4.$$

The above four lines finally prove (C.1). Therefore, for $l$ large enough, we obtain

$$\lambda_l^{-1/4} \sim l^{-1/4} \| Y_l^{l-1} \|_{L^2(S^2)} \geq c \| l^{-1} \partial_{\theta} Y_l^{l-1} \|_{L^2(\Sigma)},$$

which concludes the proof of the lemma. \qed

References


