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Proper Generalized Decomposition and Variational Theory of Complex Rays: an alliance to to consider uncertainties over mid and high broad frequency bands

Andrea BARBARULO [1],[2], Pierre LADÈVEZE [3], Hervé RIOU [4], Matthieu GRAVELEAU [5]

[1] ENS-Cachan,LMT,barbarulo@lmt.ens-cachan.fr
[2] University of Cambridge,Engineering Department-Dynamics and Vibration Research Group,ab2105@cam.ac.uk
[3] ENS-Cachan,LMT,pierre.ladeveze@lmt.ens-cachan.fr
[4] ENS-Cachan,LMT,riou@lmt.ens-cachan.fr
[5] Ecole centrale de Nantes,Institut GeM,matthieu.graveleau@ec-nantes.fr

Résumé — In this paper an innovative model reduction technique is applied to the Variational Theory of Complex Rays (VTCR) in order to take account of uncertainties over a mid/high frequency band. Several methods have been developed to solve mid frequency problems, one of them is the VTCR, in order to solve noise and vibration problems in the medium-frequency range. The principal features of VTCR approach are the use of a weak formulation of the vibration problem, which allows to consider automatically boundary condition between sub-domains, without using any auxiliary equation such as Lagrange coefficients, and the use of Herglotz wave functions to represents the vibrational field ; those functions are an integral repartition of plane waves in all the direction, the unknowns of the problem are their amplitudes. In order to improve its performances over a frequencies range, Proper Generalized Decomposition (PGD) has been successfully applied in past works. The PGD is a model reduction technique that leads to construction of separated variables representations of the solution of models defined in tensor product spaces. Thanks to PGD, it is possible to write an approximation of wave amplitude on a reduced base, separating the polar contribution "θ" from the frequency dependent contribution "ω" leading to inexpensive broad band calculation. The main idea of this paper is to extend this model reduction to take into account parametric uncertainty. Being VTCR a pure deterministic method the easiest way to tackle stochastic problems would be to apply non-intrusive techniques, like Monte-Carlo simulation. The advantage of this technique is that it only requires to do several times the deterministic problem. Despite its simplicity, this method leads to important computational costs if considering complex cases and in particular broad bands. For that reason in this paper, the Proper Generalized Decomposition is proposed to consider uncertain parameters over a model order reduction, in addiction to the previous space/frequency decomposition on the reduced space. An iterative algorithm, based on residue minimization, is proposed in order to find the triplets (space, frequency and stochastic parameter) witch compose the solution over a reduced space. Finally an example is provided to show method efficacy on a stochastic mid-frequency broad bands and to show its potentialities to tackle high frequency.
Mots clés — Model Order Reduction, Proper Generalized Decomposition, Uncertainties, Monte Carlo

1 Introduction

Different frequency bands show a different sensibility to uncertainty parameters. This affirmation lies on some experimental evidence. An example of this phenomena in shown in Figure 1. These experiments, published in [1], are conducted on 98 nominally identical cars. The model and the production line is the same for every of the 98 cars. Experiments shows that low frequency dependency from the presence of little variation is little. This means that if a low frequency analysis is conducted deterministic models like FEM are effective and meaningful. The situation changes at higher frequencies. The effect of uncertainties are very strong in mid-frequency band and decisive in high-frequency. From this experimental evidence borns the consideration that a mid-frequency method should keep in account uncertainty. This is particularly true if an extension of these methods to large-frequency band is desired.

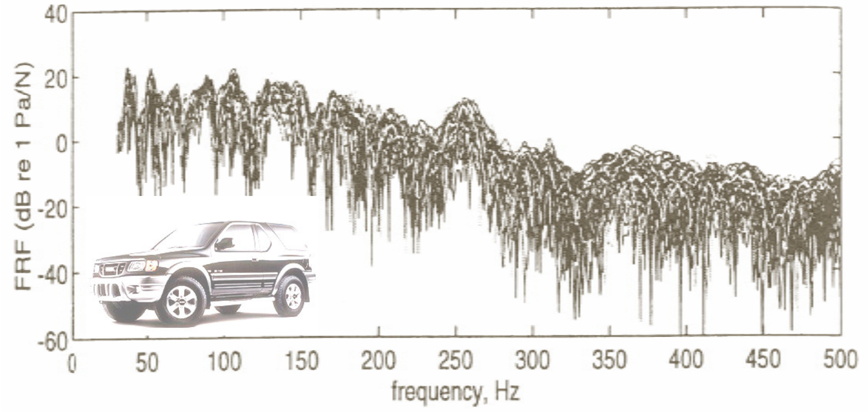


Fig. 1 – Frequency response of 98 “identical" cars from the same production line (courtesy of Toyota) [1]

Mid-frequency is a complex frequency range. Classical tools suited for the low or the high frequencies are generally not sufficiently accurate to give a reliable prevision in mid-frequency range. Indeed, on one hand the FEM techniques [2], established tools for low frequency, suffer from pollution errors and computational costs become prohibitive. On the other hand, the SEA technique [3], established tool for high frequency, remains too global and not optimal in this range of frequencies. It is a great challenge for modern computational dynamics, to account for all of these provisions while preserving a stochastic element .

The aim of this paper is to extend the Variational Theory of Complex Rays, a powerful mid-frequency technique, to broad frequency bands including the influence of a stochastic parameter.

The VTCR was introduced in [4] and belongs to the Trefftz family of methods which uses exact solutions of the governing differential equations for the expansion of the field variables. The decisive advantage of all Trefftz methods is that since they use exact solutions of the governing equations no refined discretization is necessary. Therefore, the model's size and the computational effort are considerably less than with element-based methods. The VTCR differs from other Trefftz methods in the choice of transmission conditions at the inter-element boundaries and in the types of shape functions utilized. It has already been shown to be capable of finding accurate solutions of vibration problems involving 3D plate assemblies [5], plates with heterogeneities [6] and shell structures [7] as well as solutions of acoustic problems [8].

Despite its efficacy VTCR presents two main disadvantage : the resulting formulation is totally frequency dependent and being a deterministic method extension to uncertainty is not straight forward. In other to overcome these limitations, a model order reduction through a separated representation, called Proper Generalized Decomposition, is applied over frequency and space .

Such a technique was proposed many years ago by Ladevèze for the resolution of complex nonlinear thermomechanical problems [9]

Today, the common name used for techniques involving a separated representation of the variables is Proper Generalized Decomposition (PGD). PGD belongs to the family of Reduced-Order Modeling (ROM) techniques, along with the ROM-POD method [10] and the reduced-basis element method [11], but in the case of PGD the construction of the representation takes into account the nature of the problem directly. The general form of a PGD separated representation of a function u of N variables is $u(x_1,...,x_N) \simeq u^M(x_1,...,x_N) = \sum_{m=1}^{M} u_m^1(x_1) \times ... \times u_m^N(x_N)$, M being the order of the approximation. Many applications of PGD, covering several domains, have already been presented : for example advanced nonlinear solvers using separated space-time representations ; multidimensional models ; the separation of physical spaces ; parametric models ; real-time simulations ; the quantification of uncertainties and stochastic parametric analysis. Reviews of recent works on PGD can be found in [12] and [13].

An application to acoustic mid frequency broad bands can be found in [14] and in this paper, using the same technique, an extension to a simple case with uncertain parameter is provided.

2 Reference problem and VTCR formulation

Let Ω_a be a 2-D dimensional acoustic bounded domain partitioned into n_{el} non-overlapping elements Ω_{aE} with intersection boundaries $\Gamma_{EE'}$. The acoustic reference problem to solve is : find p in $H^1(\Omega_a)$ such that :

$$\Delta p_E + k_a^2 p_E = 0 \quad \text{in } \Omega_{aE} \tag{1}$$

$$\left| \begin{array}{lll} p_E = p_{dE} & \text{on } \partial_p\Omega_{aE} & (a) \\ L_v[p_E] = v_{dE} & \text{on } \partial_v\Omega_{aE} & (b) \\ p_E - Z_E L_v[p_E] = h_{dE} & \text{on } \partial_Z\Omega_{aE} & (c) \end{array} \right. \tag{2}$$

$$\left| \begin{array}{ll} p_E - p_{E'} = 0 & \text{on } \Gamma_{EE'} \\ L_v[p_E] + L_v[p_{E'}] = 0 & \text{on } \Gamma_{EE'} \end{array} \right. \tag{3}$$

where p_E is the restriction of p to Ω_E, k_a is the acoustic wave vector, $L_v[p_E] = \dfrac{i}{\rho_0\omega}\dfrac{\partial p_E}{\partial n}$ is the velocity operator with the density ρ_0, the frequency ω and the outward normal n, Z_E an impedance coefficient and i the imaginary unit. Boundary conditions on the exterior boundaries of Ω_a are related to prescribed pressures p_{dE} (2-a), prescribed velocities v_{dE} (2-b), h_{dE} Robin equations (2-c) and interface continuity between the acoustic sub-cavities (3). VTCR formulation is obtained from the boundary value problem (1-2-3) by rewriting it in a weak form. After defining the following functional space :

$$A_{ad}^E = \left\{ p_E \quad | \quad \Delta p_E + k^2 p_E = 0, \forall x \in \Omega_{aE} \right\} \tag{4}$$

it can be shown (see [8]) that the boundary value problem (1-2-3) is equivalent to the following variational problem : find $(p_1, ...p_{n_{el}}) \in A_{ad}^1 \times ... \times A_{ad}^{n_{el}}$, such as :

$$\begin{aligned} &\sum_{\Omega_{aE}} \Re \Bigg\{ \int_{\partial_p\Omega_E} (p_E - p_{dE}) \overline{L_v[\delta p_E]} ds + \int_{\partial_v\Omega_E} \left(\overline{L_v[p_E] - v_{dE}} \right) \delta p_E ds \\ &\quad + \frac{1}{2} \int_{\partial_Z\Omega_E} \Big[\left((1 - Z_E L_v)[p_E] - h_{dE} \right) \overline{L_v[\delta p_E]} \\ &\qquad\qquad + \left(\overline{(L_v - 1/Z_E)[p_E] + h_{dE}/Z_E} \right) \delta p_E \Big] ds \\ &+ \sum_{\Gamma_{EE'}} \Re \Bigg\{ \frac{1}{2} \int_{\Gamma_{EE'}} \Big((p_E - p_{E'}) \overline{L_v[\delta p_E - \delta p_{E'}]} \\ &\qquad\qquad + \overline{L_v[p_E + p_{E'}]} (\delta p_E + \delta p_{E'}) \Big) ds \Bigg\} = 0 \\ &\qquad\qquad \forall (\delta p_1, ...\delta p_{n_{el}}) \in A_{ad}^1 \times ... \times A_{ad}^{n_{el}} \end{aligned} \tag{5}$$

The principle of the VTCR is to search an approximated solution $p_E^h \in A_{ad}^{E,h} \subset A_{ad}^E$ of the variational problem (5). It is possible to find different subspaces of A_{ad}^E, but the classic one used in the VTCR defines p_E^h as a sum of propagating waves :

$$p_E^h(x) = \int_{-\pi}^{\pi} X_E^h(\theta) e^{-ik(\theta)x} d\theta \tag{6}$$

where $X_E^h(\theta)$ describes the amplitudes of the plane waves propagating in the θ direction. The unknown is $X_E^h(\theta)$, which can be expressed as a Fourier series on the $2N+1$ first terms (see [15]) :

$$X_E^h(\theta) = \sum_{n=-N}^{N} X_{E,n}^h \int_{-\pi}^{\pi} e^{in\theta} d\theta \tag{7}$$

Once (6) and (7) are injected into the formulation (5), a resolution of a discretized finite dimension matrix is required in order to find the unknowns $X^h_{E,n}$:

$$\begin{bmatrix} \mathbb{K}_{1,1} & \dots & \mathbb{K}_{1,n_{el}} \\ \vdots & \ddots & \vdots \\ \mathbb{K}_{n_{el},1} & \dots & \mathbb{K}_{n_{el},n_{el}} \end{bmatrix} \begin{bmatrix} \mathbf{X}^h_1 \\ \vdots \\ \mathbf{X}^h_{n_{el}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_{n_{el}} \end{bmatrix} \tag{8}$$

or in a synthetic form

$$\mathbb{K}\mathbf{X} = \mathbf{F} \tag{9}$$

where $\underline{\underline{K}}_{i,j}$ and $\underline{F}_i$ are respectively the matrices of the bilinear forms and the vectors of the linear form of the variational formulation (5) and $\underline{X}^h_j$ is the vector of the amplitudes of the shape functions used in Ω_{aj}.

3 Proper Generalized Decomposition applied to broad band calculation with uncertainty

As it is clear from sections above, VTCR differs from other mid-frequency techniques (like [16], [17] and [18]) for being a deterministic formulation. Introducing uncertainty, without passing trough expensive techniques like Monte Carlo simulation [19], is hence a complex problem. The proposed solution is to extend the results found in [14] to uncertainty parameters as in [20]. The basic concept is to find a separate representation of the solution over frequency space and uncertainty with only few tests point arbitrarily chosen over the concerned spaces. If considering a system response over a wave band and a stochastic parameter $s \in S$ the final VTCR equation can be written as :

$$\mathbb{K}(\omega;s)\mathbf{X}(\omega;s) = \mathbf{F}(\omega;s) \tag{10}$$

the objective is to find an approximation of $\mathbf{X}$ such as :

$$\mathbf{X}(\omega;s) \approx \mathbf{X}^m(\omega;s) = \sum_{i=1}^{m} \hat{\mathbf{X}}_i \hat{\lambda}_i(\omega)\hat{\sigma}_i(s) \tag{11}$$

where $\hat{\mathbf{X}}_i$ is a space dependent vector, $\hat{\lambda}_i$ is a frequency dependent function and $\hat{\sigma}_i(s)$ is a function representing the uncertainty contribution. In order to simplify the formulation new operators are defined :

$$\int_I \bullet \, \mathrm{d}\omega = (\bullet)_\omega \quad ; \quad \int_S \bullet \, \mathrm{d}P(s) = (\bullet)_s \quad ; \quad \int_S \int_I \bullet \, \mathrm{d}\omega \mathrm{d}P(s) = (\bullet)_{\omega,s} \tag{12}$$

with $\| \bullet \| = (\bullet \cdot \mathbb{I} \cdot \bullet^*)$ where $\bullet^*$ is the complex conjugate operator.

In order to find the required approximation the norm of the scalar product of the residue, respect the m order of the approximation, is to minimize :

$$\left(\|\mathbf{R}_{m+1}\|^2\right)_{\omega,s} = \left(\|\mathbf{R}_m - \mathbb{K}\hat{\mathbf{X}}_{m+1}\hat{\lambda}_{m+1}(\omega)\hat{\sigma}_{m+1}(s)\|^2\right)_{\omega,s} \tag{13}$$

By minimizing equation 13 over the three variables, a triplet of equation is obtained :

$$\hat{\sigma}_{m+1} = \frac{\left(\Re(\hat{\lambda}_{m+1}\,\mathbf{R}^*_m \mathbb{K}\hat{\mathbf{X}}_{m+1})\right)_\omega}{\left(\hat{\lambda}_{m+1}\hat{\mathbf{X}}^*_{m+1}\,\mathbb{K}^*\mathbb{K}\hat{\mathbf{X}}_{m+1}\hat{\lambda}_{m+1}\right)_\omega} \tag{14a}$$

$$\hat{\lambda}_{m+1} = \frac{\left(\Re(\hat{\sigma}_{m+1}\,\mathbf{R}^*_m \mathbb{K}\hat{\mathbf{X}}_{m+1})\right)_s}{\left(\hat{\sigma}_{m+1}\hat{\mathbf{X}}^*_{m+1}\,\mathbb{K}^*\mathbb{K}\hat{\mathbf{X}}_{m+1}\hat{\sigma}_{m+1}\right)_s} \tag{14b}$$

$$\hat{\mathbf{X}}_{m+1} = \left(\hat{\lambda}_{m+1}\hat{\sigma}_{m+1}\,\mathbb{K}^*\mathbb{K}\hat{\sigma}_{m+1}\hat{\lambda}_{m+1}\right)^{-1}_{\omega,s}\left(\hat{\lambda}_{m+1}\hat{\sigma}_{m+1}\,\mathbb{K}^*\mathbf{R}_m\hat{\sigma}_{m+1}\hat{\lambda}_{m+1}\right)_{\omega,s} \tag{14c}$$

In order to perform the minimization a power type algorithm is required :

Algorithm 1: PGD for stochastic brad band problems

Initialization of $\mathbf{R}_0 = \mathbf{F}$;
for $i = 1$ *to* m **do**
  Initialization of $\hat{\mathbf{X}}_i^0$;
  Initialization of $\hat{\lambda}_i^0$;
  for $k = 1$ *à* k_{max} **do**
    Compute :
    $\hat{\sigma}_i^k = (\hat{\mathbf{X}}_i^{k-1}, \hat{\lambda}_i^{k-1}) \to (14a)$;
    $\hat{\lambda}_i^k = (\hat{\mathbf{X}}_i^{k-1}, \hat{\sigma}_i^k) \to (14b)$;
    $\hat{\mathbf{X}}_i^k = (\hat{\lambda}_i^k, \hat{\sigma}_i^k) \to (14c)$;
    Stationarity check $(\hat{\mathbf{X}}_i^k, \hat{\sigma}_i^k, \hat{\lambda}_i^k)$;
  Update of $\mathbf{R}_{i+1} = \mathbf{R}_i - \mathbb{K}\hat{\mathbf{X}}_i\hat{\sigma}_i\hat{\lambda}_i^k$;
  Convergency check;

It has to be noticed that any initialization would leads to convergency but an initialization choice somehow linked to the problem could leads to a sensible faster convergency.

4 Numerical example and validation

A simple 1D problem in Figure 2 is considered since is the first time VTCR is extended to uncertainty.

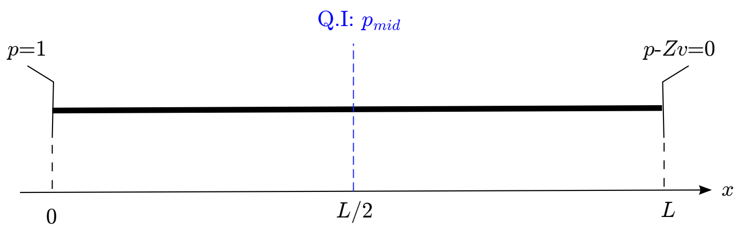


Fig. 2 – 1D problem

The example consists in a wave guide of $L = 2m$ where on $x = 0$ a unitary pressure is imposed and a variable Robin condition is applied on $x = L$. Considered fluid is air (density is $\rho = 1,125\text{kg.m}^{-3}$ and sound speed is $c = 320\text{m.s}^{-1}$). The problem is defined on a frequency band $f \in]1000, 4500[$ Hz and the real part of impedance is considered uncertain between 100 and 500 ($s = \Re(Z)$). This example is particularly useful in this explorative phase for two main reasons : there is a simple analytic reference to compare PGD approximation with and PDG results are of immediate and clear interpretation. For a 1D case the ensemble of admissible solution is found on the function family of $\{e^{ikx}, e^{-ikx}\}$ with $k = \omega/c$. For this reason the unknown pressure is expressed in the form :

$$p(x) = X^+ e^{ikx} + X^- e^{-ikx} \tag{15}$$

and v is :

$$v(x) = -\frac{k}{\rho_0\omega}(X^+ e^{ikx} - X^- e^{-ikx}) \tag{16}$$

The analytical solution is found solving :

$$\begin{cases} p(0) & & = 1 \\ p(L) & -Zv(L) & = h_{ed} \end{cases} \tag{17}$$

or in matrix form :

$$\begin{bmatrix} 1 & 1 \\ e^{ikL}\left(1 + \frac{Zk}{\rho_0\omega}\right) & e^{-ikL}\left(1 - \frac{Zk}{\rho_0\omega}\right) \end{bmatrix} \begin{pmatrix} X^+ \\ X^- \end{pmatrix} = \begin{pmatrix} 1 \\ h_{ed} \end{pmatrix} \tag{18}$$

In order to find a PGD-VTCR approximation a number of tests matrices K and vectors F is to find through a course discretization of the frequency and stochastic spaces. The variational formulation (5) can be easily particularized for 1D problems as :

$$\begin{aligned}&(X^+ + X^- - 1)\frac{k}{\rho_0\omega}(\delta X^+ - \delta X^-)+\\&\left(X^+ e^{ikL}\left(1+\frac{Zk}{\rho_0\omega}\right)+X^- e^{-ikL}\left(1-\frac{Zk}{\rho_0\omega}\right)-h_{ed}\right)\frac{k}{\rho_0\omega}(-\delta X^+ e^{-ikL}+\delta X^- e^{ikL})=0\end{aligned} \tag{19}$$

or in matrix form :

$$\begin{bmatrix}\dfrac{-Zk}{\rho_0} & \left(1-e^{-2ikL}\left(1-\frac{Zk}{\rho_0\omega}\right)\right)\\ \left(e^{2ikL}\left(1+\dfrac{Zk}{\rho_0\omega}\right)-1\right) & \dfrac{-Zk}{\rho_0}\end{bmatrix}\begin{pmatrix}X^+\\ X^-\end{pmatrix}=\begin{pmatrix}\left(1-h_{ed}e^{-ikL}\right)\\ \left(h_{ed}e^{ikL}-1\right)\end{pmatrix} \tag{20}$$

which is exactly the synthetic formulation expressed in 10. Once the test matrices are found a PGD iterative algorithm is applied to find the desired approximation :

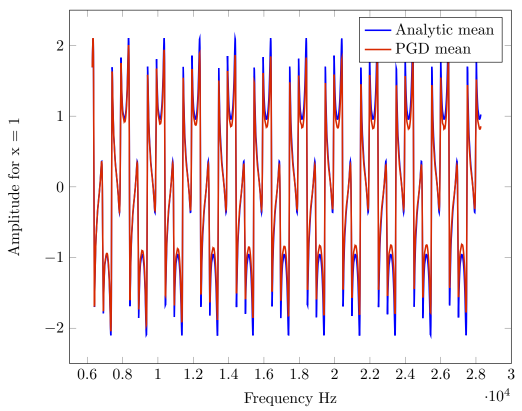
Algorithm 2: 1D algorithm

for $i = 1$ *to "stochastic test point number"* **do**
  for $j = 1$ *to "frequency test point number"* **do**
    Find $\mathbb{K}_{i,j} = \mathbb{K}(\omega, s)$;
    Find $\mathbf{F}_{i,j} = \mathbf{F}(\omega, s)$;

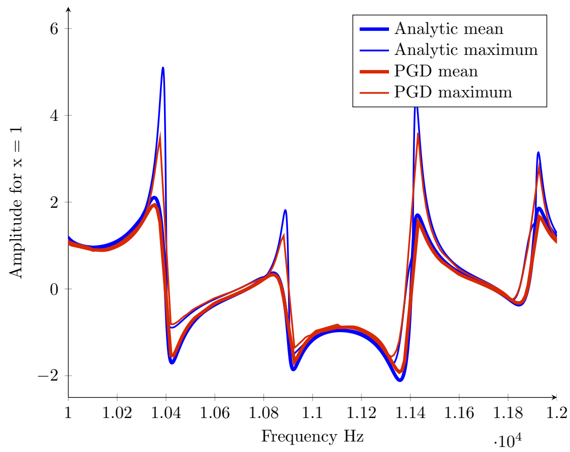
Apply algorithm 1 $\leftarrow (\{\mathbb{K}\}_{i,j}, \{\mathbf{F}\}_{i,j})$;
  $\hookrightarrow$ PGD triplets $\left\{\left(\hat{\mathbf{X}}(\theta), \hat{\lambda}(\omega)\hat{\sigma}(s)\right)\right\}_m$

Post-processing

Let's take an uniform distribution on Z, a PGD approximation can be built with 30 discretization points on stochastic and 1500 on frequency band. PGD solution is built with 150 triplets.



(a) FRF on frequency band totality



(b) Zoom on 1000 to 1200 Hz frequency band

Fig. 3 – PGD-VTCR mean pressure compared to mean reference

It can be noticed the very good agreement between the PGD approximation and the reference both for the general FRF shape and the peaks location. Looking at convergency issue in Figure 4 we can see that a fast regular and smooth convergency is provided with a modest numbers of algorithm sub-iterations (see Figure 5).

Finally an evaluation of triplets contribution on the solution has been conducted in Figure 6.

All results showed were found without making any choice on probability distribution. A simple post-processing is needed in order to include a chosen distribution.

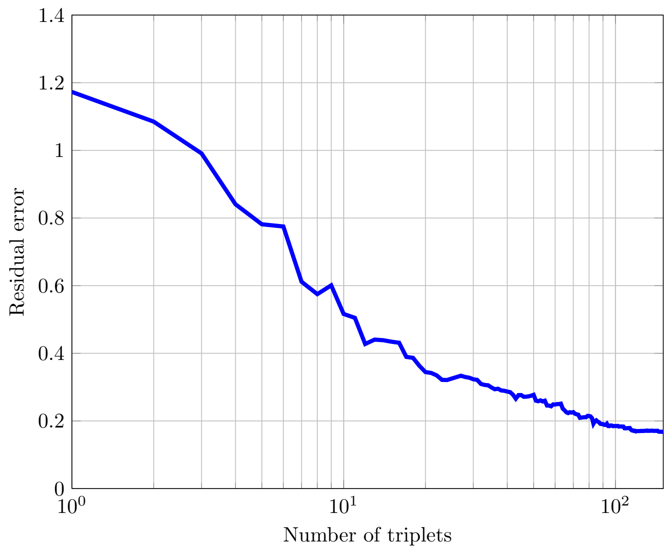


Fig. 4 – Residual error

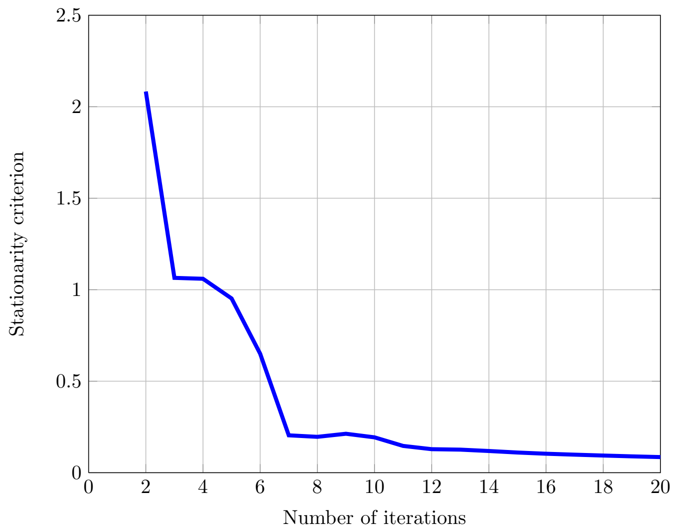
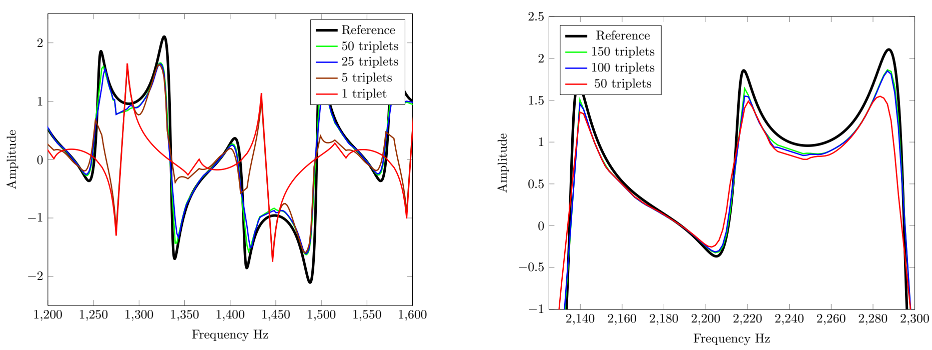


Fig. 5 – Evolution of sub-iteration number necessary to reach a stationarity criteria



(a) Solution reconstruction for different PGD approximation orders

(b) Zoom on the peaks to show high order contributions

Fig. 6 – Evolution of real pressure on middle point increasing decomposition's order

5 Références bibliographiques

Références

[1] M.S. Kompella and R.J. Bernhard. Measurement of the statistical variation of structural-acoustic caracteristics of automotive vehicles. In *SAE Noise and Vibration Conference*, volume 931272, 1993.

[2] O. C. Zienkiewicz. *The Finite Element Method*. McGraw-Hill, 1977.

[3] R. H. Lyon and G. Maidanik. Power flow between linearly coupled oscillators. *Journal of Acoustical Society of America*, 34 :640–647, 1962.

[4] P. Ladevèze. A new computational approach for structure vibrations in the medium frequency range. *Comptes Rendus Académie des Sciences Paris*, 332(2b) :849–856, 1996.

[5] P. Ladevèze, L. Arnaud, P. Rouch, and C. Blanzé. The variational theory of complex rays for the calculation of medium-frequency vibrations. *Engineering Computations*, (18) :193–214, 2001.

[6] P. Ladevèze, L. Blanc, P. Rouch, and C. Blanzé. A multiscale computational method for medium-frequency vibrations of assemblies of heterogeneous plates. *Computers and Structures*, 81 :1267–1276, 2003.

[7] H. Riou, P. Ladevèze, and P. Rouch. Extension of the variational theory of complex rays to shells for medium-frequency vibrations. *Journal of Sound and Vibration*, 272(1-2) :341–360, 2004.

[8] H. Riou, P. Ladevèze, and B. Sourcis. The multiscale vtcr approach applied to acoustics problems. *Journal of Computational Acoustics*, 16(4) :487–505, 2008.

[9] P. Ladevèze. *Nonlinear computationnal structural mechanics—new approaches and non-incremental methods of calculation*. Springer, Berlin, 1999.

[10] T. Lieu, C. Farhat, and M. Lesoinne. Reduced-order fluid/structure modeling of a complete aircraft configuration. *Computer Methods in Applied Mechanics and Engineering*, 195(41-43) :5730–5742, 2006.

[11] Y. Maday and E.M. Rønquist. A reduced-basis element method. *Journal of Scientific Computing*, 17(1) :447–459, 2002.

[12] F. Chinesta, A. Ammar, and E. Cueto. Recent advances and new challenges in the use of the proper generalized decomposition for solving multidimensional models. *Archives of Computational Methods in Engineering*, 4(17) :327–350, 2010.

[13] F. Chinesta, P. Ladeveze, and E. Cueto. A short review on model order reduction based on proper generalized decomposition. *Archives of Computational Methods in Engineering*, (18) :395–404, 2011.

[14] A Barbarulo, P. Ladevèze, H. Riou, and L. Kovalevsky. Proper generalized decomposition applied to linear acoustic : a new tool for broad band calculation. *Journal of Sound and Vibration*, 2012.

[15] L. Kovalevsky, P. Ladevèze, H. Riou, and M. Bonnet. The variational theory of complex rays for three-dimensional helmholtz problems. *Journal of Computational Acoustics*, 2012.

[16] V. Cotoni, R.S. Langley, and M.R.F. Kidner. Numerical and experimental validation of variance prediction in the statistical energy analysis of built-up systems. *Journal of Sound and Vibration*, 288(3) :701 – 728, 2005.

[17] C. Soize. Medium frequency linear vibrations of anisotropic elastic structures. *Recherche Aérospatiale*, pages 65–87, 1983.

[18] M.N. ICHCHOU, A. LE BOT, and L. JEZEQUEL. A transient local energy approach as an alternative to transient sea : Wave and telegraph equations. *Journal of Sound and Vibration*, 246(5) :829 – 840, 2001.

[19] N. Metropolis. The beginning of the monte carlo method. *Los Alamos Science*, 15(584) :125–130, 1987.

[20] A. Nouy. A generalized spectral decomposition technique to solve a class of linear stochastic partial differential equations. *Computer Methods in Applied Mechanics and Engineering*, 196(45-48) :4521 – 4537, 2007.