Strong convergence rates of semi-discrete splitting approximations for stochastic Allen–Cahn equation

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Abstract. This article analyzes an explicit temporal splitting numerical scheme for the stochastic Allen-Cahn equation driven by additive noise, in a bounded spatial domain with smooth boundary in dimension $d \leq 3$. The splitting strategy is combined with an exponential Euler scheme of an auxiliary problem.

When $d = 1$ and the driving noise is a space-time white noise, we first show some a priori estimates of this splitting scheme. Using the monotonicity of the drift nonlinearity, we then prove that under very mild assumptions on the initial data, this scheme achieves the optimal strong convergence rate $\mathcal{O}(\delta t^{\frac{1}{4}})$. When $d \leq 3$ and the driving noise possesses some regularity in space, we study exponential integrability properties of the exact and numerical solutions. Finally, in dimension $d = 1$, these properties are used to prove that the splitting scheme has a strong convergence rate $\mathcal{O}(\delta t)$.

1. Introduction

The stochastic Allen-Cahn equation driven by an additional noise term models the effect of thermal perturbations, and plays an important role in the phase theory and the simulations of rare events in infinite dimensional stochastic systems (see e.g. [**13, 19, 27**]).

In this article, we mainly focus on deriving the optimal strong convergence rates of temporal splitting schemes for the stochastic Allen-Cahn equation driven by Wiener processes, including the cylindrical Wiener process and some more regular Wiener processes, under homogenous Dirichlet boundary conditions:

$$dX(t) = AX(t) + F(X(t))dt + dW^Q(t),\ t \in (0, T],\ \ X(0) = X_0, \tag{1}$$

where $F(x) = x - x^3$, $(W^Q(t))_{t\in[0,T]}$ is a generalized Wiener process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}(t))_{t\in[0,T]}, \mathbb{P})$ and $\mathcal{O} \in \mathbb{R}^d, d \leq 3$ is a bounded spatial domain with smooth boundary $\partial\mathcal{O}$.

Strong convergence of numerical approximations for Stochastic Partial Differential Equations(SPDEs) with globally Lipschitz continuous coefficients has been extensively studied in the last twenty years (see e.g. [**1, 14, 17, 21, 25**]). For SPDEs with non-Lipschitz coefficients, there only exist a few results about the strong convergence rates of numerical schemes (see e.g. [**2, 3, 8, 9, 12**]). The

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strong convergence rates of numerical schemes, especially the temporal discretization, is far from being understood and it is still an open problem to derive general strong convergence rates of numerical schemes for SPDEs with non-globally Lipschitz coefficients.

For the discretization of equations such as the stochastic Allen-Cahn equation, the main difficulty is the polynomial growth of the non-globally Lipschitz continuous coefficient F. It is very delicate and necessary to design efficient numerical schemes for stochastic equations with this type of nonlinearities. The authors in [**20**] study a fully implicit split-step scheme combined with the backward Euler scheme, and show that the scheme converges strongly with a rate $\mathscr{O}(\delta t^{\frac{1}{2}})$ for Eq. (1) with $d \leq 3$, driven by some Q-Wiener processes. We refer to [**12**] for the analysis of finite element methods applied to stochastic Allen-Cahn equations with multiplicative noise. For Eq. (1), with $d = 1$ driven by a space-time white noise, first, the authors in [**3**] obtain the strong convergence rate results for a nonlinearity-truncated Euler-type scheme. Similar strong convergence results are then obtained in [**2**] for a nonlinearity-truncated fully discrete scheme. A backward Euler-Spectral Galerkin method has been considered in [**22**] by using stochastic calculus in martingale type 2 Banach spaces. Recently, the authors in [**4**] propose some splitting schemes and prove the proposed schemes are strongly convergent without strong convergence rates.

In this work, we give a systemic analysis of the properties of a splitting scheme and its strong convergence rates for approximating Eq. (1) with $d \leq 3$ driven by different kinds of noise. We first introduce the splitting scheme with a time-step size $\delta t > 0$, defined by:

$$
\begin{aligned}
Y_n &= \Phi_{\delta t}(X_n), \\
X_{n+1} &= S_{\delta t} Y_n + \int_{t_n}^{t_{n+1}} S(t_{n+1} - s) dW^Q(s),
\end{aligned} \tag{2}
$$

where $\Phi_{\delta t}(z) = \frac{z}{\sqrt{z^2 + (1-z^2)e^{-2\delta t}}}$ is the phase flow of $dX = F(X(t))dt, t \in [0, \delta t], X_0 = z$, and $S_{\delta t} = S(\delta t) = e^{A\delta t}$. This type of splitting scheme, in a stochastic context, has been first proposed in [**4**], and it is convenient for practical implementations since it is explicit and strongly convergent without a taming or truncation strategy. Note that an exponential Euler scheme is used in the second step of the splitting strategy.

In this article, we first derive the optimal strong convergence rate of the splitting scheme in the case of space-white time noise, using a variational approach. This gives a positive answer to the question asked in [**4**], concerning the strong convergence rate of splitting schemes for the stochastic Allen-Cahn equation. We would like to mention that these splitting-up based methods have many applications on approximating SPDEs with the Lipschitz nonlinearity, and are also used for approximating SPDEs with non-Lipschitz or non-monotone nonlinearities (see e.g. [**7, 9, 11, 15**]).

In order to analyze the strong convergence rate of this splitting method for different types of noise, different approaches are required. In the case of space-time white noise, there are three main steps to derive the strong convergence rate. Following [**4**], the first step is constructing an auxiliary problem, with a modified nonlinearity $\Psi_{\delta t}$ instead of F, such that the splitting scheme can be viewed as a standard exponential Euler scheme applied to the auxiliary problem. Even though the exponential Euler applied to the original equation may be divergent, the solutions of the numerical scheme and of the auxiliary problem are proved to be bounded in $L^p(\Omega; L^q)$, for all finite p, q. Thus no taming or truncation strategy is required to ensure the boundedness of numerical solutions. The second step is based on the monotonicity properties of the nonlinearities F and $\Psi_{\delta t}$, appearing in the exact and auxiliary problems respectively. In addition, since the noise is additive and an exponential Euler scheme is used with no discretization of the stochastic convolution, one is lead to study some PDEs with random coefficients. The last step consists in applying properties of the stochastic convolution and stochastic calculus results in martingale type 2 Banach spaces, to deduce the optimal strong convergence rate $\mathscr{O}(\delta t^{\frac{1}{4}})$ in $L^p(\Omega; C(0, T; L^q))$, $p \geq q = 2m$, $m \in \mathbb{N}^+$, i.e.,

$$\Big\| \sup_{t\in[0,T]} \|X^N(t) - X(t)\|_{L^q} \Big\|_{L^p(\Omega)} \leq C(X_0, T, p, q)\delta t^{\frac{1}{4}}.$$

This variational approach can also be used to obtain the strong convergence rate $\mathscr{O}(\delta t^{\frac{1}{2}})$, in the case of more regular Q-Wiener processes, in dimension $d \leq 3$.

In the case of $\mathbb{H}^1$-valued Q-Wiener processes, in dimension $d = 1$, we get higher strong convergence rates of this splitting method, thanks to exponential integrability properties of the exact and numerical solutions. To the best of our knowledge,this is the first result with strong convergence order 1 about the temporal numerical schemes approximating the stochastic Allen–Cahn equation. For similar approaches to derive the strong convergence rates of numerical schemes, we refer to [**7, 16, 18**] and the references therein. We first study stability and exponential integrability properties of the exact solution, in dimensions $d \leq 3$, and obtain results of their own interest beyond analysis of numerical schemes. Then, in dimension $d = 1$, a new auxiliary processes Z^N is constructed, and some a priori estimate and exponential integrability properties of Z^N are studied. We then prove that the scheme, in this context, has strong convergence order equal to 1:

$$\sup_{n\leq N} \Big\| \|X^N(t_n) - X(t_n)\|_{L^q} \Big\|_{L^p(\Omega)} \leq C(X_0, Q, T, p, q)\delta t.$$

This strong convergence result is restricted to dimension $d = 1$, due to a loss of exponential integrability of the auxiliary process Z^N in higher dimension. Further study is required to overcome this issue.

This article is organized as follows. Some preliminaries are given in Section 2. The variational approach to deal with the case of space-time white noise and Q-Wiener processes, as well as some properties of the auxiliary problem, and one main strong convergence rate result, are given in Section 3. In Section 4, stability and exponential integrability properties, of the exact solution and of a new auxiliary processes, are studied. Finally, we establish the optimal strong convergence rate of this proposed scheme in dimension 1.

We use C to denote a generic constant, independent of the time step size δt, which differs from one place to another.

2. Preliminaries

In this section, we first introduce some useful notations and further assumptions. Let $T > 0$, δt is the time step size, N is the positive integer such that $N\delta t = T$, and let $\{t_k\}_{k\leq N}$ be the grid points, defined by $t_k = k\delta t$. We denote by $\mathbb{H} = L^2(\mathcal{O})$, $L^q = L^q(\mathcal{O})$, $1 \leq q < \infty$ and $\mathcal{E} = \mathcal{C}(\mathcal{O})$. A is the Dirichlet Laplacian operator, which generates an analytic and contraction C_0-semigroup $S(t), t \geq 0$ in $\mathbb{H}$ and L^q. It is well-known that the assumptions on $\mathcal{O}$ implies that the existence of the eigensystem $\{\lambda_k, e_k\}_{k\in\mathbb{N}^+}$ of $\mathbb{H}$, such that $\lambda_k > 0$, $-Ae_k = \lambda_k e_k$ and $\lim\limits_{k\to\infty} \lambda_k = \infty$. Let $\mathbb{W}^{r,q}$ is the Banach space equipped with the norm $\|\cdot\|_{\mathbb{W}^{r,q}} := \|(-A)^{\frac{r}{2}} \cdot \|_{L^q}$ for the fractional power $(-A)^{\frac{r}{2}}, r \geq 0$. The identities $\mathbb{H}^1 = H_0^1$ and $\mathbb{H}^2 = H_0^1 \cap H^2$ are frequently used in Section 4.

Given two separable Hilbert spaces $\mathcal{H}$ and $\widetilde{H}$, we denote by $\mathcal{L}_2^0(\mathcal{H}, \widetilde{H})$ the space of Hilbert-Schmidt operators from $\mathcal{H}$ into $\widetilde{H}$, equipped with the usual norm given by $\|Q\|_{\mathcal{L}_2^0(\mathcal{H},\widetilde{H})} = (\sum_{k\in\mathbb{N}^+} \|Qe_k\|^2_{\widetilde{H}})^{\frac12}$, where $\mathbb{N}^+ = \{1, 2, \cdots\}$, and the result does not depend on the orthonormal basis $\{e_k\}_{k\in\mathbb{N}^+}$ of $\mathcal{H}$. We denote by $\mathcal{L}_2^{\mathbf{s}} := \mathcal{L}_2^0(\mathbb{H}, \mathbb{H}^{\mathbf{s}})$, for $\mathbf{s} \in \mathbb{N}$.

Given a Banach space E, we denote by $R(\widetilde{H}, E)$ the space of γ-radonifying operators endowed with the norm defined by $\|Q\|_{\gamma(\widetilde{H},E)} = (\widetilde{\mathbb{E}}\| \sum_{k\in\mathbb{N}^+} \gamma_k Q e_k\|_E^2)^{\frac12}$, where $(\gamma_k)_{k\in\mathbb{N}^+}$ is a sequence of independent $\mathcal{N}(0,1)$-random variables on a probability space $(\widetilde{\Omega}, \widetilde{\mathscr{F}}, \widetilde{\mathbb{P}})$. We also need the Burkerholder inequality in martingale-type 2 Banach spaces $E = L^q, q \in [2,\infty)$, (see e.g. [**5, 26**]): for some $C_{p,E} \in (0,\infty)$,

$$
\begin{aligned}
\Big\| \sup_{t\in[0,T]} \Big\| \int_0^t \phi(r) dW(r) \Big\|_E \Big\|_{L^p(\Omega)} &\leq C_{p,E} \|\phi\|_{\mathcal{L}^p(\Omega; L^2([0,T]; \gamma(\widetilde{H}; E)))} \\
&= C_{p,E} \Big(\mathbb{E} \Big(\int_0^T \|\phi(t)\|^2_{\gamma(\widetilde{H};E)} dt \Big)^{\frac{p}{2}} \Big)^{\frac1p}
\end{aligned}
\tag{3}
$$

and the following property (see [**26**]): for some $C_q \in (0,\infty)$,

$$
\|\phi\|^2_{\gamma(\widetilde{H},L^q)} \leq C_q \Big\| \sum_{k\in\mathbb{N}^+} (\phi e_k)^2 \Big\|_{L^{\frac{q}{2}}}, \quad \phi \in \gamma(\widetilde{H}, L^q). \tag{4}
$$

The process $W := \sum_{k\in\mathbb{N}^+} \beta_k e_k$ is the $\mathbb{H}$-valued cylindrical Wiener process, where $(\beta_k)_{k\in\mathbb{N}^+}$ are independent Brownian motions defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}(t))_{t\in[0,T]}, \mathbb{P})$. The driving noise is $W^Q := \sum_k \beta_k Q e_k$, where Q is a bounded operator from $\mathbb{H}$ to E. When $Q = I$, $E = \mathbb{H}$, W^Q is the standard cylindrical Wiener process, which corresponds to the case of space-time white noise. In Sections 3 and 4, we will also consider more regular cases, with assumptions $Q \in \mathcal{L}_2^{\mathbf{s}}$, $\mathbf{s} \in \mathbb{N}$.

The solution of the stochastic Allen-Cahn equation, Eq. (1), is interpreted in a mild sense,

$$
X(t) = S(t)X_0 + \int_0^t S(t-s)F(X(s))ds + \int_0^t S(t-s)dW^Q(s). \tag{5}
$$

Let $\omega(t) = \int_0^t S(t-s)dW^Q(s)$ be the so-called stochastic convolution. Then note that $Y(t) = X(t) - \omega(t)$ solves a random PDE (written in mild form):

$$
Y(t) = S(t)X_0 + \int_0^t S(t-s)F(Y(s) + \omega(s))ds. \tag{6}
$$

We now introduce an auxiliary problem, and several auxiliary processes. The auxiliary problem is coming from writing the solution of the splitting scheme Eq. (2)

as follows:

$$\begin{aligned}X_{n+1} &= S_{\delta t}\Phi_{\delta t}(X_n) + \int_{t_n}^{t_{n+1}} S(t_{n+1}-s)dW^Q(s)\\ &= S_{\delta t}X_n + \delta t S_{\delta t}\Psi_{\delta t}(X_n) + \int_{t_n}^{t_{n+1}} S(t_{n+1}-s)dW^Q(s),\end{aligned}$$

where $\Psi_{\delta t}(z) = \frac{\Phi_{\delta t}(z)-z}{\delta t}, \Psi_0(z) = F(z),\ \Phi_0(z) = z$. Thus we get for all $n \in \{0,\dots,N\}$

$$X_n = S(t_n)X_0 + \delta t\sum_{k=0}^{n-1} S(t_{n+1}-t_k)\Psi_{\delta t}(X_k) + \sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} S(t_{n+1}-s)dW^Q(s).$$

A continuous time interpolation, such that $X^N(t_n) = X_n$ for all $n \in \{0,\dots,N\}$, is defined by

(7)
$$X^N(t) = S(t)X_0 + \int_0^t S((t-\lfloor s\rfloor_{\delta t}))\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))ds + \int_0^t S(t-s)dW^Q(s),$$

where $\lfloor s\rfloor_{\delta t} = \max\{0,\delta t, 2\delta t,\cdots\}\cap[0,s]$.

As observed in [**4**], the proposed splitting scheme can be viewed as the exponential Euler method applied to the following auxiliary SPDE:

$$dX^{\delta t}(t) = AX^{\delta t}(t)dt + \Psi_{\delta t}(X^{\delta t}(t))dt + dW^Q(t), \quad X^{\delta t}(0) = X_0. \tag{8}$$

The associated mild formulation is given by

$$X^{\delta t}(t) = S(t)X_0 + \int_0^t S(t-s)\Psi_{\delta t}(X^{\delta t}(s))ds + \int_0^t S(t-s)dW^Q(s).$$

Let $Y^{\delta t}(t) = X^{\delta t}(t) - \omega(t)$, where ω is the stochastic convolution. Then $Y^{\delta t}$ is also solution of a random PDE:

$$Y^{\delta t}(t) = S(t)X_0 + \int_0^t S(t-s)\Psi_{\delta t}(Y^{\delta t}(s)+\omega(s))ds. \tag{9}$$

We quote the following results from [**4**]. The estimates may be derived with elementary calculations.

LEMMA 2.1. *For every $\delta t_0 \in (0,1)$ and $\delta t \in [0,\delta t_0)$, the mapping $\Phi_{\delta t}$ is globally Lipschitz continuous, and the mapping $\Psi_{\delta t}$ is locally Lipschitz continuous and satisfies a one-side Lipschitz condition. More precisely, for $q = 2m$, $m \in \mathbb{N}^+$,*

$$\begin{aligned}|\Phi_{\delta t}(z_1) - \Phi_{\delta t}(z_2)| &\le e^{C\delta t_0}|z_1 - z_2|,\\ (\Psi_{\delta t}(z_1) - \Psi_{\delta t}(z_2))(z_1-z_2)^{q-1} &\le e^{C\delta t_0}|z_1-z_2|^q,\\ |\Psi_{\delta t}(z_1) - \Psi_{\delta t}(z_2)| &\le C(\delta t_0)|z_1-z_2|(1+|z_1|^2+|z_2|^2),\\ |\Psi_{\delta t}(z_1) - \Psi_0(z_1)| &\le C(\delta t_0)\delta t(1+|z_1|^5).\end{aligned}$$

3. Strong convergence rate analysis of the splitting scheme approximating stochastic Allen-Cahn equation by a variational approach

This section is devoted to the application of a variational approach, to derive strong convergence rates for the splitting scheme defined by Eq. (2). The study includes the cases of the cylindrical Wiener process ($Q = I$, $q = 1$) and of L^q-valued Q-Wiener processes ($Q \in \gamma(\mathbb{H}, L^q)$).

We recall that in [**4**] it is proved that the scheme is convergent, when $d = 1$ and $Q = I$. Precisely, assume that $X_0 \in \mathbb{H}^{\beta_1} \cap \mathcal{E}$, for some $\beta_1 > 0$. Then

$$\lim_{\delta t \to 0} \mathbb{E}\Big[\sup_{n \le N} \|X^N(t_n) - X(t_n)\|^p\Big] = 0.$$

However, it is well-known that the standard approach used to derive strong rates of convergence using a Gronwall's inequality argument, cannot be applied, when the nonlinearity is not globally Lipschitz continuous. Additional properties, precisely giving exponential integrability for the exact and numerical solutions, are required.

Instead, in the present section, we overcome this issue using a variational approach, based on a different decomposition of the error introduced below.

For convenience, throughout this article, we assume that X_0 is a deterministic function and that $\sup_{k \in \mathbb{N}^+} \|e_k\|_{\mathcal{E}} \le C$. The typical example to ensures that $\sup_{k \in \mathbb{N}^+} \|e_k\|_{\mathcal{E}} \le C$ is the d dimensional cube $[0,1]^d$.

3.1. A priori estimates and spatial regularity properties. We first deal with the case $Q = I$, $d = 1$ and recall the following well-known result about the stochastic convolution (see e.g. [**10**]): for $2 \le q < \infty$,

$$\mathbb{E}\Big[\sup_{t \in [0,T]} \|\omega(t)\|_{L^q}^p\Big] \le C_p(T), \quad \mathbb{E}\Big[\sup_{t \in [0,T]} \|\omega(t)\|_{\mathcal{E}}^p\Big] \le C_p(T) < \infty.$$

The following lemma states standard a priori estimates for the processes X, X^N and $Y^{\delta t}$ defined by Eq. (1), (7) and (9) respectively. For convenience, throughout this paper, we omit the mollification procedure to get the evolution of $\|\cdot\|_{L^q}$.

Lemma 3.1. *Let $d = 1$, $Q = I$, $q = 2m$, $m \in \mathbb{N}^+$, $p \ge 1$ and $X_0 \in L^q$. Then X,$Y^{\delta t}$ and X^N satisfy*

$$\mathbb{E}\Big[\sup_{t \in [0,T]} \|X(t)\|_{L^q}^p\Big] < C(T,p,q)(1 + \|X_0\|_{L^q}^p),$$

and

$$\mathbb{E}\Big[\sup_{t \in [0,T]} \|Y^{\delta t}(t)\|_{L^q}^p\Big] + \mathbb{E}\Big[\sup_{t \in [0,T]} \|X^N(t)\|_{L^q}^p\Big] < C(T,p,q)(1 + \|X_0\|_{L^q}^p).$$

Proof. For the a priori estimate for the exact solution X, we refer to [**10**]. Thus we focus on the a priori estimate of $Y^{\delta t}$ and X^N. The definition, Eq. 9, of $Y^{\delta t}$ and the one-side Lipschitz condition on $\Psi_{\delta t}$ (see Lemma 2.1), combined with

Hölder and Young inequalities, imply that for $2 \le q < \infty$,

$$
\begin{aligned}
\|Y^{\delta t}(t)\|_{L^q}^q &\le \|X_0\|_{L^q}^q + q\int_0^t \langle AY^{\delta t}(s), (Y^{\delta t}(s))^{q-2}Y^{\delta t}(s)\rangle ds \\
&\quad + q\int_0^t \langle \Psi_{\delta t}(Y^{\delta t}(s)+\omega(s)), (Y^{\delta t}(s))^{q-2}Y^{\delta t}(s)\rangle ds \\
&\le \|X_0\|_{L^q}^q + q\int_0^t \langle \Psi_{\delta t}(Y^{\delta t}(s)+\omega(s)) - \Psi_{\delta t}(\omega(s)), (Y^{\delta t}(s))^{q-2}Y^{\delta t}(s)\rangle ds \\
&\quad + q\int_0^t \langle \Psi_{\delta t}(\omega(s)), (Y^{\delta t}(s))^{q-2}Y^{\delta t}(s)\rangle ds \\
&\le \|X_0\|_{L^q}^q + C(\delta t_0, q)\int_0^t \|Y^{\delta t}(s)\|_{L^q}^q ds \\
&\quad + C(\delta t_0, q)\int_0^t \|\Psi_{\delta t}(\omega(s))\|_{L^q}\|Y^{\delta t}(s))\|_{L^q}^{q-1} ds \\
&\le \|X_0\|_{L^q}^q + C(\delta t_0, q)\int_0^t \|Y^{\delta t}(s)\|_{L^q}^q ds + C(\delta t_0, q)\int_0^t (1+\|\omega(s)\|_{L^{3q}}^{3q}) ds.
\end{aligned}
$$

Using the moment estimate on the stochastic convolution above, applying the Gronwall's inequality concludes the proof for $Y^{\delta t}$.

The estimate for X^N is proved using similar arguments. First, note that it is sufficient to control the values of X^N at the grid points t_n, $n \le N$:

$$
\mathbb{E}\Big[\sup_{t\in[0,T]} \|X^N(t)\|_{L^q}^p\Big] \le C(p,q,T)\mathbb{E}\Big[\sup_{n\le N} \|X^N(t_n)\|_{L^q}^p\Big].
$$

By the definition of $X^N(t_n) = X_n$, $n \le N$ and the Lipschitz continuity of $\Phi_{\delta t}$ stated in Lemma 2.1, since $S(t)$ is a contraction semigroup, we obtain

$$
\begin{aligned}
\|X^N(t_n) - \omega(t_n)\|_{L^q} &\le \Big\|S(\delta t)\Phi_{\delta t}(X^N(t_{n-1})) - S(\delta t)\omega(t_{n-1})\Big\|_{L^q} \\
&\le \Big\|\Phi_{\delta t}(X^N(t_{n-1})) - \Phi_{\delta t}(\omega(t_{n-1}))\Big\|_{L^q} + \Big\|\Phi_{\delta t}(\omega(t_{n-1})) - \omega(t_{n-1})\Big\|_{L^q} \\
&\le e^{C\delta t}\Big\|X^N(t_{n-1}) - \omega(t_{n-1})\Big\|_{L^q} + C\delta t(1+\|\omega(t_{n-1})\|_{L^{3q}}^3).
\end{aligned}
$$

Then using the discrete Gronwall's inequality, and the estimate on the stochastic convolution, we get

$$
\mathbb{E}\Big[\sup_{n\le N} \|X^N(t_n)\|_{L^q}^p\Big] \le C(T,q,p),
$$

which concludes the proof of X^N. □

We now study spatial regularity properties of the processes X^N and X. We first state a Lemma (see [**10**]) concerning the factorization method.

Lemma 3.2. *Assume that $p > 1$, $r \ge 0$, $\gamma > \frac{1}{p} + r$ and that E_1 and E_2 are Banach spaces such that*

$$
\|S(t)x\|_{E_1} \le Mt^{-r}\|x\|_{E_2}, \quad t \in [0,T], x \in E_2.
$$

Set $G_\gamma f(t) := \int_0^t (t-s)^{\gamma-1}S(t-s)f(s)ds$, then, for $\gamma > \frac{1}{p} + r$, one has

$$
\|G_\gamma f\|_{C([0,T];E_1)} \le C(M)\|f\|_{L^p(0,T;E_2)},
$$

if $f \in L^p([0,T];E_2)$.

LEMMA 3.3. *Assume that* $d=1$, $Q=I$, $p\geq 2$ *and* $\|X_0\|_{\mathbb{H}^{\beta_1}}<\infty$, $\beta_1>0$. *The solution* u *satisfy the following estimate: if* $\beta<\min(\frac{1}{2},\beta_1)$, *then*

$$\mathbb{E}\Big[\sup_{0\leq s\leq T}\|X(t)\|^p_{\mathbb{H}^\beta}\Big]\leq C(p,T,\beta,X_0)<\infty.$$

PROOF. It is known (see e.g. [**10**]) that, for $\beta<\frac{1}{2}$,

$$\mathbb{E}\Big[\sup_{0\leq s\leq T}\|\omega(s)\|^p_{\mathbb{H}^\beta}\Big]\leq C(p,T).$$

Thus we only need to study the regularity of $S(t)X_0$ and of the deterministic convolution $\int_0^t S(t-s)F(X(s))ds$. First,

$$\|S(t)X_0\|_{\mathbb{H}^{\beta_1}}\leq \|X_0\|_{\mathbb{H}^{\beta_1}}.$$

For the deterministic convolution, by the Fubini theorem, we have

$$\int_0^t S(t-s)F(X(s))ds=\frac{\sin\gamma\pi}{\pi}\int_0^t (t-s)^{\gamma-1}S(t-s)Y_\gamma(s)ds,$$
$$Y_\gamma(t)=\int_0^t (t-s)^{-\gamma}S(t-s)F(X(s))ds$$

where we choose $\gamma<\frac{1}{4}$ such that the regularity result also holds for the stochastic convolution. Notice that $\|S(t)x\|_{\mathbb{H}^\beta}\leq Mt^{-\frac{\beta}{2}}\|x\|_{\mathbb{H}}$, for $\beta>0$. Taking $E_1=\mathbb{H}^\beta$ and $E_2=\mathbb{H}$, $r=\frac{\beta}{2}$. Lemma 3.2 yields that for large enough p and $\gamma>\frac{\beta}{2}+\frac{1}{p}$,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big\|\int_0^t S(t-s)F(X(s))ds\Big\|^p_{\mathbb{H}^\beta}\Big]\\
&\leq C\mathbb{E}\Big[\int_0^T\Big\|Y_\gamma(t)\Big\|^p_{\mathbb{H}}dt\Big]\\
&\leq C\mathbb{E}\Big[\Big(\int_0^T t^{-2\gamma}\|S(t)\|_{\mathcal{L}(\mathbb{H},\mathbb{H})}\Big(1+\sup_{r\in[0,T]}\|X(r)\|^3_{L^6}\Big)dt\Big)^p\Big]\\
&\leq C(T,p,X_0).
\end{aligned}$$

This concludes the proof. $\square$

Using standard arguments, including the use of a discrete Gronwall's lemma, and the two lemmas stated above, one may derive the following almost sure result (see [**4**] for similar arguments): assume $d=1$, $Q=I$, $\beta<\frac{1}{2}$, $X_0\in\mathbb{H}^{\beta_1}\cap\mathcal{E}$, $\beta_1>0$. Then almost surely, for some $C(\omega)\in(0,\infty)$, one has

$$\sup_{n\leq N}\|X^N(t_n)-X(t_n)\|\leq C(\omega)\delta t^{\min(\frac{\beta}{2},\frac{\beta_1}{2})}.$$

We omit the details. As explained above, the variational approach used below allows us to go beyond this result and get a strong rate of convergence.

3.2. Optimal strong convergence rate in space-time white noise case. We are now in position to apply the variational approach developed in [**3**] in order to obtain strong convergence rates for the splitting scheme (2).

We first state the main result of this section.

THEOREM 3.1. *Assume that $d = 1$, $Q = I$, $\|X_0\|_{L^{9q}} < \infty$, $p \geq q = 2m$, $m \in \mathbb{N}^+$ and $\eta < \frac{1}{q}$. Then X^N satisfies*

$$\left\| \sup_{t\in[0,T]} \|X^N(t) - X(t)\|_{L^q} \right\|_{L^p(\Omega)} \leq C(T, X_0, p, q)\delta t^{\min(\frac{1}{4},\eta)},$$

If in addition $\|X_0\|_{\mathbb{W}^{\beta,3q}} < \infty$, $\beta > 0$, then

$$\left\| \sup_{t\in[0,T]} \|X^N(t) - X(t)\|_{L^q} \right\|_{L^p(\Omega)} \leq C(T, X_0, p, q)\delta t^{\min(\frac{1}{4},\frac{\beta}{2}+\eta)},$$

Note that, for $q \in [2, 4)$, the first estimate of Theorem 3.1 gives order of convergence $\frac{1}{4}$. If $q \in [4,\infty)$,the order of convergence $\frac{1}{4}$ is obtained thanks to the second estimate, under the assumption $\beta > \frac{1}{2} - \frac{2}{q}$.

Observe that the error can be decomposed as follows:

$$\begin{aligned}\|X^N(t) - X(t)\|_{L^q} &\leq \|X^N(t) - \omega(t) - Y^{\delta t}(t)\|_{L^q} + \|Y^{\delta t}(t) + \omega(t) - X(t)\|_{L^q} \\ &\leq \|X^N(t) - X^{\delta t}(t)\|_{L^q} + \|Y^{\delta t}(t) - Y(t)\|_{L^q}.\end{aligned}$$

Then Theorem 3.1 is a straightforward consequence of the two auxiliary results stated below.

PROPOSITION 3.1. *Assume that $d = 1$, $Q = I$, $\|X_0\|_{L^{5q}} < \infty$. Then the proposed method X^N is strongly convergent to X and satisfies*

$$\left\| \sup_{t\in[0,T]} \|Y^{\delta t}(t) - Y(t)\|_{L^q} \right\|_{L^p(\Omega)} \leq C(T, X_0, p, q)\delta t,$$

where $p \geq q = 2m$, $m \in \mathbb{N}^+$.

Note that $Y^{\delta t}(t) - Y(t) = X^{\delta t}(t) - X(t)$, and in the case $q = 2$, Proposition 3.1 has already been proved in [**4**].

PROPOSITION 3.2. *Assume that $d = 1$, $Q = I$, $\|X_0\|_{L^{9q}} < \infty$, $p \geq q = 2m$, $m \in \mathbb{N}^+$. Then the proposed method X^N satisfies for $\eta < \frac{1}{q}$,*

$$\left\| \sup_{t\in[0,T]} \|X^N(t) - X^{\delta t}(t)\|_{L^q} \right\|_{L^p(\Omega)} \leq C(T, X_0, p, q)\delta t^{\min(\frac{1}{4},\eta)}. \tag{10}$$

If in addition assume that $\|X_0\|_{\mathbb{W}^{\beta,3q}} < \infty$, $\beta > 0$, then for $\eta < \frac{1}{q}$,

$$\left\| \sup_{t\in[0,T]} \|X^N(t) - X^{\delta t}(t)\|_{L^q} \right\|_{L^p(\Omega)} \leq C(T, X_0, p, q)\delta t^{\min(\frac{1}{4},\eta+\frac{\beta}{2})}. \tag{11}$$

It now remains to prove Propositions 3.1 and 3.2.

Proof of Proposition 3.1. Note that $Y^{\delta t}(0) = Y(0)$, and recall that $Y^{\delta t}(t) - Y(t) = X^{\delta t}(t) - X(t)$. Using the differential forms of the random PDEs (6) and (9),

$$\begin{aligned}
&\|Y^{\delta t}(t) - Y(t)\|_{L^q}^q \\
&= q\int_\epsilon^t \langle (Y^{\delta t}(s) - Y(s))^{q-2}(Y^{\delta t}(s) - Y(s)), AY^{\delta t}(s) - AY(s)\rangle ds \\
&\quad + q\int_\epsilon^t \langle (X^{\delta t}(s) - X(s))^{q-2}(X^{\delta t}(s) - X(s)), \Psi_{\delta t}(X(s)) - \Psi_{\delta t}(X(s)))\rangle ds \\
&\quad + q\int_\epsilon^t \langle (X^{\delta t}(s) - X(s))^{q-2}(X^{\delta t}(s) - X(s)), \Psi_{\delta t}(X(s))) - F(X(s)))\rangle ds.
\end{aligned}$$

Thanks to Lemma 2.1, combined with Young's inequality and Gronwall's lemma, we obtain

$$\begin{aligned}
&\|Y^{\delta t}(t) - Y(t)\|_{L^q}^q \\
&\le C(T)\delta t^q \int_0^T \Big(1 + \Big\|Y(s) + \omega(s)\Big\|_{L^{5q}}^{5q}\Big) ds,
\end{aligned}$$

It remains to use the a priori estimates of Lemma 3.1 to conclude the proof. □

To prove Proposition 3.2, we follow the approach from [**3**], and we introduce an additional auxiliary process,

$$\widehat{Y}(t) := S(t)X_0 + \int_0^t S(t-s)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))ds$$

for which the following auxiliary result is satisfied.

Lemma 3.4. *Assume that $d = 1$, $Q = I$, $\|X_0\|_{L^{3q}} < \infty$, $p \ge q \ge 2$. Then for $0 < \eta < 1$, $s \ge \delta t$,*

(12)
$$\mathbb{E}\Big[\Big\|\widehat{Y}(s) + \omega(s) - X^N(\lfloor s\rfloor_{\delta t})\Big\|_{L^q}^p\Big] \le C(T,p,\eta,X_0)(1 + (\lfloor s\rfloor_{\delta t})^{-\eta p})(\delta t)^{\min(\frac14,\eta)p}.$$

If assume in addition that $\|X_0\|_{\mathbb{W}^{\beta,q}} < \infty$, $\beta > 0$, then we have

(13)
$$\mathbb{E}\Big[\Big\|\widehat{Y}(s) + \omega(s) - X^N(\lfloor s\rfloor_{\delta t})\Big\|_{L^q}^p\Big] \le C(T,p,\eta,X_0)(1 + (\lfloor s\rfloor_{\delta t})^{-\eta p})(\delta t)^{\min(\frac14,\frac{\beta}{2}+\eta)p}.$$

Proof of Lemma 3.4. By the definition of $\widehat{Y}$ and X^N, we get, for $s \ge \delta t$,

$$\begin{aligned}
&\Big\|\widehat{Y}(s) + \omega(s) - X^N(\lfloor s\rfloor_{\delta t})\Big\|_{L^q} \\
&= \Big\|S(s)X_0 - S(\lfloor s\rfloor_{\delta t})X_0\Big\|_{L^q} \\
&\quad + \Big\|\int_0^s S(s-r)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))dr - \int_0^{\lfloor s\rfloor_{\delta t}} S(\lfloor s\rfloor_{\delta t} - \lfloor r\rfloor_{\delta t})\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))dr\Big\|_{L^q} \\
&\quad + \Big\|\int_0^s S(s-r)dW^Q(r) - \int_0^{\lfloor s\rfloor_{\delta t}} S(\lfloor s\rfloor_{\delta t} - r)dW^Q(r)\Big\|_{L^q} \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

Thanks to the smoothing properties of the semigroup $S(t)$, we have for arbitrary $\eta < 1$,

$$\begin{aligned} I_1 &\le \Big\| S(\lfloor s \rfloor_{\delta t})(S(s - \lfloor s \rfloor_{\delta t}) - I)X_0 \Big\|_{L^q} \\ &\le C \Big\| A^{\eta} S(\lfloor s \rfloor_{\delta t}) \Big\|_{\mathcal{L}(L^q, L^q)} \Big\| A^{-\eta}(S(s - \lfloor s \rfloor_{\delta t}) - I)X_0 \Big\|_{L^q} \\ &\le C(\lfloor s \rfloor_{\delta t})^{-\eta} \delta t^{\eta} \|X_0\|_{L^q}. \end{aligned}$$

Then we turn to estimate the term I_2, for $s \ge \epsilon$,

$$\begin{aligned} I_2 &\le \Big\| \int_0^{\lfloor s \rfloor_{\delta t}} \Big(S(s-r) - S(\lfloor s \rfloor_{\delta t} - \lfloor r \rfloor_{\delta t}) \Big) \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t})) dr \Big\|_{L^q} \\ &\quad + \Big\| \int_{\lfloor s \rfloor_{\delta t}}^{s} S(s-r) \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t})) dr \Big\|_{L^q} \\ &\le \int_0^{\lfloor s \rfloor_{\delta t}} \Big\| S(s-r)(S(r - \lfloor r \rfloor_{\delta t}) - I) \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t})) \Big\|_{L^q} dr \\ &\quad + \int_0^{\lfloor s \rfloor_{\delta t}} \Big\| S(\lfloor s \rfloor_{\delta t} - \lfloor r \rfloor_{\delta t})(S(s - \lfloor s \rfloor_{\delta t}) - I) \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t})) \Big\|_{L^q} dr \\ &\quad + C\delta t \sup_{s \in [0,T]} \Big\| \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t})) \Big\|_{L^q}. \end{aligned}$$

Similar to the estimate of I_1, combing with Lemma 2.1, we obtain for $0 < \eta < 1$,

$$I_2 \le C(T)(\delta t^{\eta} + \delta t)(1 + \sup_{n \le N} \|X^N(t_n)\|_{L^{3q}}^3).$$

Thanks to the Burkholder inequality, Eq. (3), from Section 2.1,

$$\begin{aligned} \mathbb{E}[I_3^p] &\le C(p) \mathbb{E}\Big[\Big\| \int_0^{\lfloor s \rfloor_{\delta t}} \big(S(s-r) - S(\lfloor s \rfloor_{\delta t} - r) \big) dW(r) \Big\|_{L^q}^p \Big] \\ &\quad + C(p) \mathbb{E}\Big[\Big\| \int_{\lfloor s \rfloor_{\delta t}}^{s} S(s-r) dW(r) \Big\|_{L^q}^p \Big] \\ &\le C(p) \Big(\int_0^{\lfloor s \rfloor_{\delta t}} \Big\| S(s-r) - S(\lfloor s \rfloor_{\delta t} - r) \Big\|_{\gamma(\mathbb{H}, L^q)}^2 dr \Big)^{\frac{p}{2}} \\ &\quad + C(p) \Big(\int_{\lfloor s \rfloor_{\delta t}}^{s} \Big\| S(s-r) \Big\|_{\gamma(\mathbb{H}, L^q)}^2 dr \Big)^{\frac{p}{2}}. \end{aligned}$$

Thanks to Eq. (4),

$$\|\phi\|_{\gamma(\mathbb{H}, L^q)}^2 \le C_q \Big\| \sum_{k \in \mathbb{N}^+} (\phi e_k)^2 \Big\|_{L^{\frac{q}{2}}}, \quad \phi \in \gamma(\mathbb{H}, L^q).$$

Recall that it is assumed that $\sup_{k \in \mathbb{N}^+} \|e_k\|_{\mathcal{E}} \le C < \infty$. Moreover, one has the following useful inequality: for any $\gamma > 0$ and any $\alpha \in [0, 1]$,

$$\sup_{r \in (0, \infty)} \sum_{k \in \mathbb{N}^+} r^{\frac{1}{2} + \alpha} \lambda_k^{\alpha} e^{-\gamma \lambda_k r} = C(\gamma, \alpha) < \infty.$$

Using these properties,

$$
\begin{aligned}
\mathbb{E}[I_3^p] &\le C\Big(\int_0^{\lfloor s\rfloor_{\delta t}} \sum_{k\in\mathbb{N}^+} \Big\|S(\lfloor s\rfloor_{\delta t}-r)\big(S(s-\lfloor s\rfloor_{\delta t})-I\big)e_k\Big\|_{L^q}^2 dr\Big)^{\frac{p}{2}} \\
&\quad + C\Big(\int_{\lfloor s\rfloor_{\delta t}}^s \sum_{k\in\mathbb{N}^+} \|S(s-r)e_k\|_{L^q}^2 dr\Big)^{\frac{p}{2}} \\
&\le C\Big(\int_0^{\lfloor s\rfloor_{\delta t}} \sum_{k\in\mathbb{N}^+} e^{-2\lambda_k(\lfloor s\rfloor_{\delta t}-r)}(e^{-\lambda_k(\lfloor s\rfloor_{\delta t}-s)}-1)^2 dr\Big)^{\frac{p}{2}} \\
&\quad + C\Big(\int_{\lfloor s\rfloor_{\delta t}}^s \sum_{k\in\mathbb{N}^+} e^{-2\lambda_k(s-r)} dr\Big)^{\frac{p}{2}} \\
&\le C\Big(\int_0^{\lfloor s\rfloor_{\delta t}} \sum_{k\in\mathbb{N}^+} e^{-2\lambda_k(\lfloor s\rfloor_{\delta t}-r)}\lambda_k^{\frac{1}{2}}\delta t^{\frac{1}{2}} dr\Big)^{\frac{p}{2}} + C\Big(\int_{\lfloor s\rfloor_{\delta t}}^s (s-r)^{-\frac{1}{2}} dr\Big)^{\frac{p}{2}} \\
&\le C\Big(\int_0^{\lfloor s\rfloor_{\delta t}} r^{-\frac{3}{4}} dr\,\delta t^{\frac{1}{2}}\Big)^{\frac{p}{2}} + C\delta t^{\frac{p}{4}} \le C\delta t^{\frac{p}{4}}.
\end{aligned}
$$

Combining the estimates of I_1, I_2 and I_3, we obtain for $s\ge\delta t$,

$$
\begin{aligned}
\mathbb{E}\Big[\Big\|\widehat{Y}(s)+\omega(s)-X^N(\lfloor s\rfloor_{\delta t})\Big\|_{L^q}^p\Big] &\le C(p)\Big(\mathbb{E}[I_1^p]+\mathbb{E}[I_2^p]+\mathbb{E}[I_3^p]\Big) \\
&\le C(T,p,q,X_0)\Big((\lfloor s\rfloor_{\delta t})^{-\eta p}\delta t^{\eta p}+\delta t^{\frac{p}{4}}\Big) \\
&\le C(T,p,q,X_0)(1+(\lfloor s\rfloor_{\delta t})^{-\eta p})\delta t^{\min(\frac{1}{4},\eta)p},
\end{aligned}
$$

which shows the first assertion.

If in addition we have $\|X_0\|_{\mathbb{W}^{\beta,q}}<\infty$, $\beta>0$, alternatively we have

$$
\begin{aligned}
I_1 &\le \Big\|A^\eta S(\lfloor s\rfloor_{\delta t})\|_{\mathcal{L}(L^q,L^q)}\|A^{-\eta}(S(s-\lfloor s\rfloor_{\delta t})-I)X_0\Big\|_{L^q} \\
&\le C(\lfloor s\rfloor_{\delta t})^{-\eta}\delta t^{\eta+\frac{\beta}{2}}\|X_0\|_{\mathbb{W}^{\beta,q}},
\end{aligned}
$$

where $\eta+\frac{\beta}{2}\le 1$, $0<\eta<1$. Combing the previous estimation on I_2 and I_3, this concludes the proof. □

It now remains to prove Proposition 3.2, using Lemma 3.4.

PROOF OF PROPOSITION 3.2. We first show the estimation (10) with the rough initial datum $X_0\in L^{9q}$. Due to the definition of X^N and $Y^{\delta t}$, we have

$$
\begin{aligned}
&\|X^N(t)-\omega(t)-Y^{\delta t}(t)\|_{L^q} \\
&= \Big\|\int_0^t S(t-\lfloor s\rfloor_{\delta t})\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))ds - \int_0^t S(t-s)\Psi_{\delta t}(Y^{\delta t}(s)+\omega(s))ds\Big\|_{L^q} \\
&\le \Big\|\int_0^t S(t-\lfloor s\rfloor_{\delta t})\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))ds - \int_0^t S(t-s)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))ds\Big\|_{L^q} \\
&\quad + \Big\|\int_0^t S(t-s)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))ds - \int_0^t S(t-s)\Psi_{\delta t}(Y^{\delta t}(s)+\omega(s))ds\Big\|_{L^q}.
\end{aligned}
$$

The first term is controlled by the smoothing properties of $S(t)$ and the uniformly boundedness of $\Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t}))$. For $0 < \eta_1 < 1$, we have

$$\begin{aligned}
&\Big\| \int_0^t S(t - \lfloor s \rfloor_{\delta t}) \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t})) ds - \int_0^t S(t-s) \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t})) ds \Big\|_{L^q} \\
&= \Big\| \int_0^t A^{\eta_1} S(t-s) A^{-\eta_1} (S(s - \lfloor s \rfloor_{\delta t}) - I) \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t})) ds \Big\|_{L^q} \\
&\le \int_0^t C(t-s)^{-\eta_1} (s - \lfloor s \rfloor_{\delta t})^{\eta_1} \Big\| \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t})) \Big\|_{L^q} ds \\
&\le C(\eta) \delta t^{\eta_1} \int_0^t (1 + \|X^N(\lfloor s \rfloor_{\delta t})\|^3_{L^{3q}}) ds.
\end{aligned}$$

We use the auxiliary process $\widehat{Y}$ and (12) in Lemma 3.4 to deal with the second term since

$$\Big\| \int_0^t S(t-s) \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t})) ds - \int_0^t S(t-s) \Psi_{\delta t}(Y^{\delta t}(s) + \omega(s)) ds \Big\|_{L^q} = \|Y^{\delta t}(t) - \widehat{Y}(t)\|_{L^q}.$$

By the one-sided Lipschitz continuity of $\Psi_{\delta t}$, Hölder and Young inequality, we have for $\delta t \le t$,

$$\begin{aligned}
&\|Y^{\delta t}(t) - \widehat{Y}(t)\|^q_{L^q} \\
&= \|Y^{\delta t}(\delta t) - \widehat{Y}(\delta t)\|^q_{L^q} + q \int_{\delta t}^t \langle (Y^{\delta t}(s) - \widehat{Y}(s))^{q-2} (Y^{\delta t}(s) - \widehat{Y}(s)), AY^{\delta t}(s) - A\widehat{Y}(s) \rangle ds \\
&\quad + q \int_{\delta t}^t \langle (Y^{\delta t}(s) - \widehat{Y}(s))^{q-2} (Y^{\delta t}(s) - \widehat{Y}(s)), \Psi_{\delta t}(Y^{\delta t}(s) + \omega(s)) - \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t})) \rangle ds \\
&\le \|Y^{\delta t}(\delta t) - \widehat{Y}(\delta t)\|^q_{L^q} + q \int_{\delta t}^t \Big\langle (Y^{\delta t}(s) - \widehat{Y}(s))^{q-2} (Y^{\delta t}(s) - \widehat{Y}(s)), \\
&\qquad \Psi_{\delta t}(Y^{\delta t}(s) + \omega(s)) - \Psi_{\delta t}(\widehat{Y}(s) + \omega(s)) \Big\rangle ds \\
&\quad + q \int_{\delta t}^t \langle (Y^{\delta t}(s) - \widehat{Y}(s))^{q-2} (Y^{\delta t}(s) - \widehat{Y}(s)), \Psi_{\delta t}(\widehat{Y}(s) + \omega(s)) - \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t})) \rangle ds \\
&\le \|Y^{\delta t}(\delta t) - \widehat{Y}(\delta t)\|^q + C(q) \int_{\delta t}^t \|Y^{\delta t}(s) - \widehat{Y}(s)\|^q_{L^q} ds \\
&\quad + \int_{\delta t}^t \Big\| \Psi_{\delta t}(\widehat{Y}(s) + \omega(s)) - \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t})) \Big\|^q_{L^q} ds.
\end{aligned}$$

Then Gronwall's inequality yields that for $t \ge \delta t$,

$$\begin{aligned}
\|Y^{\delta t}(t) - \widehat{Y}(t)\|^q_{L^q} &\le e^{CT} \|Y^{\delta t}(\delta t) - \widehat{Y}(\delta t)\|^q_{L^q} \\
&\quad + e^{CT} \int_{\delta t}^T \|\Psi_{\delta t}(\widehat{Y}(s) + \omega(s)) - \Psi_{\delta t}(X^N(\lfloor s \rfloor_{\delta t}))\|^q_{L^q} ds.
\end{aligned}$$

Using Lemma 2.1 and Hölder inequality,

$$\|\Psi_{\delta t}(z_1) - \Psi_{\delta t}(z_2)\|_{L^q} \le C\|z_1 - z_2\|_{L^{3q}} (1 + \|z_1\|^2_{L^{3q}} + \|z_2\|^2_{L^{3q}})$$

leads that

$$\|Y^{\delta t}(t)-\widehat{Y}(t)\|_{L^q}\le C(T,q)\Bigg(\|Y^{\delta t}(\delta t)-\widehat{Y}(\delta t)\|_{L^q}+\sup_{s\in[0,T]}\Big(1+\|\widehat{Y}(s)\|^2_{L^{3q}}+\|\omega(s)\|^2_{L^{3q}}$$

$$+\|X^N(\lfloor s\rfloor_{\delta t})\|^2_{L^{3q}}\Big)\Big(\int_{\delta t}^T\Big\|\widehat{Y}(s)+\omega(s)-X^N(\lfloor s\rfloor_{\delta t})\Big\|^q_{L^{3q}}\Big)^{\frac{1}{q}}ds\Bigg).$$

Since for $t\le\delta t$,

$$\begin{aligned}\sup_{t\in[0,\delta t]}\|Y^{\delta t}(t)-\widehat{Y}(t)\|_{L^q}&\le C\Big\|\int_0^t S(t-s)\Psi_{\delta t}(Y^{\delta t}(s)+\omega(s))ds\Big\|_{L^q}\\&\quad+C\Big\|\int_0^t S(t-s)\Psi_{\delta t}(X^N(\lfloor s\rfloor_{\delta t}))ds\Big\|_{L^q}\\&\le C\delta t\sup_{s\in[0,T]}\Big(1+\|Y^{\delta t}(s)\|^3_{L^{3q}}+\|\omega(s)\|^3_{L^{3q}}+\|X^N(s)\|^3_{L^{3q}}\Big).\end{aligned}$$

Taking expectation, together with the above results and a priori estimate in Lemma 3.1, Hölder inequality and Minkowski's inequality, leads that, for $p\ge q$, $\eta<\frac{1}{q}$,

$$\begin{aligned}&\mathbb{E}\Big[\sup_{t\in[\delta t,T]}\|Y^{\delta t}(t)-\widehat{Y}(t)\|^p_{L^q}\Big]\\&\le C(T,\eta,p,q)\Bigg(\mathbb{E}\Big[\|Y^{\delta t}(\delta t)-\widehat{Y}(\delta t)\|^p_{L^q}\Big]+\mathbb{E}\Bigg[\sup_{s\in[0,T]}\Big(1+\|\widehat{Y}(s)\|^{2p}_{L^{3q}}+\|\omega(s)\|^{2p}_{L^{3q}}\\&\quad+\|X^N(\lfloor s\rfloor_{\delta t})\|^{2p}_{L^{3q}}\Big)\Big(\int_{\delta t}^T\Big\|\widehat{Y}(s)+\omega(s)-X^N(\lfloor s\rfloor_{\delta t})\Big\|^q_{L^{3q}}ds\Big)^{\frac{p}{q}}\Bigg]\Bigg)\\&\le C\delta t\mathbb{E}\Big[\sup_{s\in[0,T]}\Big(1+\|Y^{\delta t}(s)\|^{3q}_{L^{3q}}+\|\omega(s)\|^{3q}_{L^{3q}}+\|X^N(s)\|^{3q}_{L^{3q}}\Big)\Big]\\&\quad+C\mathbb{E}\Bigg[\sup_{s\in[0,T]}\Big(1+\|\widehat{Y}(s)\|^{2p}_{L^{3q}}+\|\omega(s)\|^{2p}_{L^{3q}}+\|X^N(\lfloor s\rfloor_{\delta t})\|^{2p}_{L^{3q}}\Big)\\&\quad\Big(\int_{\delta t}^T\Big\|\widehat{Y}(s)+\omega(s)-X^N(\lfloor s\rfloor_{\delta t})\Big\|^q_{L^{3q}}ds\Big)^{\frac{p}{q}}\Bigg]\\&\le C\delta t\mathbb{E}\Big[\sup_{s\in[0,T]}\Big(1+\|Y^{\delta t}(s)\|^{3p}_{L^{3q}}+\|\omega(s)\|^{3p}_{L^{3q}}+\|X^N(s)\|^{3p}_{L^{3q}}\Big)\Big]\\&\quad+C\Big\|\sup_{s\in[0,T]}\Big(1+\|\widehat{Y}(s)\|^{2p}_{L^{3q}}+C\|\omega(s)\|^{2p}_{L^{3q}}+\|X^N(\lfloor s\rfloor_{\delta t})\|^{2p}_{L^{3q}}\Big\|_{L^2(\Omega)}\\&\quad\times\Big\|\Big(\int_{\delta t}^T\Big\|\widehat{Y}(s)+\omega(s)-X^N(\lfloor s\rfloor_{\delta t})\Big\|^q_{L^{3q}}ds\Big)^{\frac{1}{q}}\Big\|^p_{L^{2p}(\Omega)}.\end{aligned}$$

By Minkowski's inequality and (12) in Lemma 3.4, we get for $\eta < \frac{1}{q}$,

$$\begin{aligned}
&\Big\| \Big(\int_{\delta t}^{T} \Big\| \widehat{Y}(s) + \omega(s) - X^N(\lfloor s \rfloor_{\delta t}) \Big\|_{L^{3q}}^q ds \Big)^{\frac{1}{q}} \Big\|_{L^{2p}(\Omega)} \\
&\le \Big(\int_{\delta t}^{T} \Big(\mathbb{E}\Big[\Big\| \widehat{Y}(s) + \omega(s) - X^N(\lfloor s \rfloor_{\delta t}) \Big\|_{L^{3q}}^{2p} \Big] \Big)^{\frac{q}{2p}} ds \Big)^{\frac{1}{q}} \\
&\le C \Big(1 + \Big(\int_{\delta t}^{T} \lfloor s \rfloor_{\delta t}^{-\eta q} ds \Big)^{\frac{1}{q}} \Big) \delta t^{\min(\frac{1}{4}, \eta)} \le C(T, p, q, \|X_0\|_{L^{9q}}) \delta t^{\min(\frac{1}{4}, \eta)},
\end{aligned}$$

which combing with the above estimation, yields that

$$\mathbb{E}\Big[\sup_{t \in [\delta t, T]} \|Y^{\delta t}(t) - \widehat{Y}(t)\|_{L^q}^p \Big] \le C(T, p, q, \|X_0\|_{L^{9q}}) \delta t^{\min(\frac{1}{4}, \eta)}.$$

By the continuity of $Y^{\delta t}(t)$ and $\widehat{Y}(t)$, together the above estimations, we have for

$$\begin{aligned}
\mathbb{E}\Big[\sup_{t \in [0, T]} \|Y^{\delta t}(t) - \widehat{Y}(t)\|_{L^q}^p \Big] &\le \mathbb{E}\Big[\sup_{t \in [\delta t, T]} \|Y^{\delta t}(t) - \widehat{Y}(t)\|_{L^q}^p \Big] \\
&\quad + \mathbb{E}\Big[\sup_{t \in [0, \delta t]} \|Y^{\delta t}(t) - \widehat{Y}(t)\|_{L^q}^p \Big] \\
&\le C(T, p, q, \|X_0\|_{L^{9q}}) \delta t^{\min(\frac{1}{4}, \eta)},
\end{aligned}$$

which establishes the first assertion (10). For the estimation (11), we use (13) to estimate the term $\mathbb{E}\Big[\sup_{t \in [0,T]} \|Y^{\delta t}(t) - \widehat{Y}(t)\|_{L^q}^p \Big]$ and the arguments are similar. □

By this variational approach, we can deduce that if $d = 1$, $Q = I$, $p \ge q = 2m$, $m \in \mathbb{N}^+$, $\beta > 0$, $\eta < \frac{1}{q}$, $X_0 \in \mathbb{W}^{\beta, q} \cap \mathcal{E}$. Then the strong convergence rate result still holds, i.e.,

$$\Big\| \sup_{t \in [0, T]} \|X^N(t) - X(t)\|_{L^q} \Big\|_{L^p(\Omega)} \le C(T, p, q, X_0) \delta t^{\min(\frac{1}{4}, \frac{\beta}{2} + \eta)},$$

by using the estimation

$$\|\Psi_{\delta t}(z_1) - \Psi_{\delta t}(z_2)\|_{L^q} \le C \|z_1 - z_2\|_{L^q} (1 + \|z_1\|_{\mathcal{E}}^2 + \|z_2\|_{\mathcal{E}}^2)$$

and the procedures of Theorem 3.1. This above result gives the answer to the problem about the strong convergence rates of splitting schemes appeared in [**4**].

Remark 3.1. *This above variational approach, combining with some further analysis on the discrete stochastic convolution, may also be available for obtaining the optimal strong convergence rates of other splitting schemes, such as the splitting exponential Euler scheme and the splitting implicit Euler scheme in* [**4**]. *This extension will be studied in future works.*

To conclude this section, we give extensions of Theorem 3.1, when Eq. (1) is driven by a Q-Wiener process, in dimension $d \le 3$. We only sketch the proofs of the parts which require nontrivial modifications. Note that the order of convergence depends on the Hölder regularity exponents for the process X.

Corollary 3.1. *Let $d \le 3$, $p \ge q = 2m$, $m \in \mathbb{N}^+$, $\beta_1 > 0$, $\eta < \frac{1}{q}$. Assume that $X_0 \in \mathbb{W}^{\beta_1, q} \cap \mathcal{E}$ and that the operators A and Q satisfy: $Ae_k = -\lambda_k e_k$, $Qq_k =$*

$\sqrt{q_k}e_k$, $q_k > 0$, $k \in \mathbb{N}^+$, *with eigenfunctions such that* $\|e_k\|_{\mathcal{E}} \le C$, $\|\nabla e_k\| \le C\lambda_k^{\frac{1}{2}}$. *Suppose that* $\sum\limits_{k\in\mathbb{N}^+} q_k\lambda_k^{2\beta-1} < \infty$, *for some* $0 < \beta < \frac{1}{2}$. *Then we have*

$$\Big\| \sup_{t\in[0,T]} \|X^N(t) - X(t)\|_{L^q} \Big\|_{L^p(\Omega)} \le C(T,p,q,X_0)\delta t^{\min(\beta,\frac{\beta_1}{2}+\eta)}.$$

Proof. To prove that Lemma 3.1 holds true, it is sufficient to check the estimate $\mathbb{E}[\sup\limits_{t\in[0,T]} \|\omega(t)\|_{\mathcal{E}}^p] \le C(T,p,Q)$ for $p \ge 1$. This is a consequence of [**10**, Theorem 5.25].

It now remains to explain modifications concerning Lemma 3.4. More precisely, the control of the term I_3 is modified as follows:

$$\begin{aligned}
\mathbb{E}[I_3^p] &\le C(p)\mathbb{E}\Big[\Big\| \int_0^{\lfloor s\rfloor_{\delta t}} \big(S(s-r) - S(\lfloor s\rfloor_{\delta t} - r)\big)dW(r)\Big\|_{L^q}^p\Big] \\
&\quad + C(p)\mathbb{E}\Big[\Big\| \int_{\lfloor s\rfloor_{\delta t}}^s S(s-r)dW(r)\Big\|_{L^q}^p\Big] \\
&\le C\Big(\int_0^{\lfloor s\rfloor_{\delta t}} \sum_{k\in\mathbb{N}^+} (e^{-\lambda_k(s-r)} - e^{-\lambda_k(\lfloor s\rfloor_{\delta t}-r)})^2 q_k dr\Big)^{\frac{p}{2}} + C\Big(\int_{\lfloor s\rfloor_{\delta t}}^s \sum_{k\in\mathbb{N}^+} e^{-2\lambda_k(s-r)}q_k dr\Big)^{\frac{p}{2}} \\
&\le C\Big(\sum_{k\in\mathbb{N}^+} \lambda_k^{-1}(1-e^{-\lambda_k(s-\lfloor s\rfloor_{\delta t})})q_k\Big)^{\frac{p}{2}} + C\Big(\sum_{k\in\mathbb{N}^+} \frac{q_k}{\lambda_k}(1-e^{-2\lambda_k(s-\lfloor s\rfloor_{\delta t})})\Big)^{\frac{p}{2}} \\
&\le C(\sum_{k\in\mathbb{N}^+} q_k\lambda_k^{2\beta-1})^{\frac{p}{2}}\delta t^{\beta p}.
\end{aligned}$$

Applying the same techniques as above concludes the proof of Corollary 3.1. □

Corollary 3.2. *Let* $d = 1$, $\beta > 0$ *and* $p \ge 2$. *If* $Q \in \mathcal{L}_2^0$, $X_0 \in \mathbb{H}^\beta \cap \mathcal{E}$, *then there exists a constant* $C = C(X_0,Q,T,p)$ *such that*

$$\Big\| \sup_{t\in[0,T]} \|X^N(t) - X(t)\| \Big\|_{L^p(\Omega)} \le C\delta t^{\frac{1}{2}}.$$

If $d = 2,3$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathcal{L}_2^0} < \infty$, $X_0 \in \mathbb{H}^\beta \cap \mathcal{E}$, *then there exists a constant* $C' = C'(X_0,Q,T,p)$ *such that*

$$\Big\| \sup_{t\in[0,T]} \|X^N(t) - X(t)\| \Big\|_{L^p(\Omega)} \le C'\delta t^{\frac{1}{2}}.$$

Proof. We first show the first assertion. The assumptions ensures that the method to obtain the strong convergence rates of the splitting scheme in the case $Q = I$ is also available for the case $Q \in \mathcal{L}_2^0$. We only need to show that the a priori estimate of ω, and I_3 possess higher convergence speed than the case $Q = I$. The Sobolev embedding theorem, the regularity result of stochastic convolution in [**10**, Theorem 5.15] and Burkerholder inequality yield that for $p \ge 2$, there exsits $\frac{1}{4} < \beta < \frac{1}{2}$ such that

$$\mathbb{E}[\sup_{t\in[0,T]} \|\omega(t)\|_{\mathcal{E}}^p] \le \mathbb{E}[\sup_{t\in[0,T]} \|(-A)^\beta\omega(t)\|^p] \le C(Q,T,p),$$

and

$$\begin{aligned}
\mathbb{E}[I_3^p] &\leq C(p)\mathbb{E}\Big[\Big\|\int_0^{\lfloor s\rfloor_{\delta t}}\big(S(s-r)-S(\lfloor s\rfloor_{\delta t}-r)\big)dW^Q(r)\Big\|^p\Big]\\
&\quad+C(p)\mathbb{E}\Big[\Big\|\int_{\lfloor s\rfloor_{\delta t}}^s S(s-r)dW^Q(r)\Big\|^p\Big]\\
&\leq C(p)\mathbb{E}\Big[\Big(\int_0^{\lfloor s\rfloor_{\delta t}}\Big\|(-A)^{-\frac12}(S(s-\lfloor s\rfloor_{\delta t})-I)\Big\|^2\Big\|(-A)^{\frac12}S(\lfloor s\rfloor_{\delta t}-r)Q\Big\|^2_{\mathcal{L}_2^0}dr\Big)^{\frac p2}\Big]\\
&\quad+C(p)\mathbb{E}\Big[\Big(\int_{\lfloor s\rfloor_{\delta t}}^s\Big\|S(s-r)Q\Big\|^2dr\Big)^{\frac p2}\Big]\leq C(Q)\delta t^{\frac p2}.
\end{aligned}$$

The above properties, combined with the procedures in the proof of Theorem 3.1 shows the first assertion.

Denote $W_\gamma=\int_0^t(t-s)^{-\gamma}S(t-s)(-A)^{\frac12}dW^Q(s)$. When $d=2,3$, $\|(-A)^{\frac12}Q\|_{\mathcal{L}_2^0}<\infty$, Sobolev embedding theorem $\mathbb{H}^{1+2\beta}\hookrightarrow\mathcal{E}$, $\frac14<\beta<\frac12$, together with the the fractional method and Lemma 3.2, yields that for $p>2$, $\frac14<\beta<\frac12$, $\frac12>\gamma>\beta+\frac1p$,

$$\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[0,T]}\|\omega(s)\|_{\mathbb{E}}^p\Big]&\leq\mathbb{E}\Big[\sup_{s\in[0,T]}\|\omega(s)\|^p_{\mathbb{H}^{1+2\beta}}\Big]\leq C\mathbb{E}\Big[\sup_{s\in[0,T]}\|G_\gamma W_\gamma(s)\|^p_{\mathbb{H}^{2\beta}}\Big]\\
&\leq C\int_0^T\mathbb{E}\Big[\|W_\gamma(s)\|^p_{\mathbb{H}^{2\beta}}\Big]ds\\
&\leq C\Big(\int_0^T s^{-2\gamma}\|S(s)(-A)^{\frac12}Q\|^2_{\mathcal{L}_2^0}ds\Big)^{\frac p2}\leq C(T,Q,p).
\end{aligned}$$

Combining with the continuity of stochastic convolution

$$\begin{aligned}
\mathbb{E}[I_3^p]&\leq C(p)\mathbb{E}\Big[\Big(\int_0^{\lfloor s\rfloor_{\delta t}}\Big\|(-A)^{-\frac12}(S(s-\lfloor s\rfloor_{\delta t})-I)\Big\|^2\Big\|(-A)^{\frac12}S(\lfloor s\rfloor_{\delta t}-r)Q\Big\|^2_{\mathcal{L}_2^0}dr\Big)^{\frac p2}\Big]\\
&\quad+C(p)\mathbb{E}\Big[\Big(\int_{\lfloor s\rfloor_{\delta t}}^s\Big\|S(s-r)Q\Big\|^2dr\Big)^{\frac p2}\Big]\leq C(Q)\delta t^{\frac p2}.
\end{aligned}$$

The a priori estimate of $\mathbb{E}[\sup_{t\in[0,T]}\|\omega(t)\|^p_{\mathcal{E}}]$ and some procedures in the proof of Theorem 3.1, we get the second assertion. □

4. Higher strong convergence rate using exponential integrability properties (regular noise, dimension 1)

This section is devoted to two contributions. First, we investigate exponential integrability properties of the exact and numerical solutions X and X^N, in dimension $d=1,2,3$. We also derive useful a priori estimates in the $\mathbb{H}^2$ norm. This requires additional regularity conditions on the operator Q, and the initial condition X_0: it is assumed that $\|(-A)^{\frac12}Q\|_{\mathcal{L}_2^0}<\infty$ and $X_0\in\mathbb{H}^2$. Second, we prove that the splitting scheme, Eq. (2), in the one-dimensional case $d=1$, has a strong order of convergence equal to 1. Note that this higher order of convergence may be obtained since the stochastic convolution is not discretized. Up to our knowledge, this is the first proof that a temporal discretization scheme has a strong order of convergence equal to 1 for the stochastic Allen-Cahn equation.

Like in Section 3, it is assumed for simplicity that the initial condition X_0 is deterministic. The extension of the results below to random X_0 is straightforward under appropriate assumptions: for instance, conditions of the type $\mathbb{E}[e^{c\|X_0\|^2_{\mathbb{H}^1}}] < \infty$ for some $c < \infty$ are required when studying exponential integrability properties.

4.1. A priori estimates and exponential integrability properties. In order to show an improved strong error estimate, with order 1 in some cases, we need to prove additional a priori estimates, and to study the exponential integrability properties for $d = 1, 2, 3$, in some well-chosen Banach spaces.

We first state the following result. The proof is standard, using Itô's formula and the one-sided Lipschitz condition on F, and it is thus left to the interested readers.

LEMMA 4.1. *Assume that $d \le 3$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathcal{L}_2^0} < \infty$ and $X_0 \in \mathbb{H}^1$. Let $p \ge 1$. Then the solution X of Eq.* (1) *satisfies the a priori estimates*

$$\mathbb{E}\Big[\Big(\sup_{t\in[0,T]} \|X(t)\|^2 + \int_0^T \|X(t)\|^2_{\mathbb{H}^1} dt + \int_0^T \|X(t)\|^4_{L^4} dt\Big)^p\Big] \le C(X_0, T, Q, p)$$

and

$$\mathbb{E}\Big[\Big(\sup_{t\in[0,T]} \|X(t)\|^2_{\mathbb{H}^1} + \int_0^T \|X(t)\|^2_{\mathbb{H}^2} ds\Big)^p\Big] \le C(X_0, T, Q, p).$$

To show the exponential integrability of X, we quote an exponential integrability lemma, see [**8**, Lemma 3.1], see also [**6**] for similar results.

LEMMA 4.2. *Let H be a Hilbert space, and let X be an H-valued adapted stochastic process with continuous sample paths, satisfying $X_t = X_0 + \int_0^t \mu(X_r)dr + \int_0^t \sigma(X_r)dW_r$ for all $t \in [0,T]$, where almost surely $\int_0^T (\|\mu(X_t)\| + \|\sigma(X_t)\|^2)dt < \infty$.*

Assume that there exist two functionals $\overline{U}$ and $U \in \mathcal{C}^2(H;R)$, and $\alpha \ge 0$, such that for all $t \in [0,T]$

$$DU(x)\mu(x) + \frac{\mathrm{tr}\big[D^2U(x)\sigma(x)\sigma^*(x)\big]}{2} + \frac{\|\sigma^*(x)DU(x)\|^2}{2e^{\alpha t}} + \overline{U}(x) \le \alpha U(x).$$

Then

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\exp\Big(\frac{U(X_t)}{e^{\alpha t}} + \int_0^t \frac{\overline{U}(X_r)}{e^{\alpha r}} dr\Big)\Big] \le e^{U(X_0)}.$$

We are now in position to state a first exponential integrability result, which will be improved below. For the reader's convenience, we omit standard truncations and regularization procedures.

PROPOSITION 4.1. *Let $d \le 3$, and assume that $\|(-A)^{\frac{1}{2}}Q\|_{\mathcal{L}_2^0} < \infty$ and $X_0 \in \mathbb{H}^1$. Then for any $\rho, \rho_1 > 0$, there exist $\alpha = \lambda(\rho, Q) \in (0,\infty)$ and $\alpha_1 = \lambda(\rho_1, Q) \in (0,\infty)$, such that*

$$\mathbb{E}\Big[\exp\Big(e^{-\alpha t}\rho\|X(t)\|^2 + 2\rho\int_0^t e^{-\alpha s}\|\nabla X(s)\|^2 ds + 2\rho\int_0^t e^{-\alpha s}\|X(s)\|^4_{L^4} ds\Big)\Big] \le e^{\rho\|X_0\|^2}.$$

and

$$\mathbb{E}\Big[\exp\Big(\Big(e^{-\alpha_1 t}\rho_1\|\nabla X(t)\|^2 + 2\rho_1\int_0^t e^{-\alpha_1 s}\|AX(s)\|^2 ds\Big)\Big)\Big] \le e^{\rho_1\|\nabla X_0\|^2}.$$

PROOF. Define $\mu(x) = Ax - x^3 + x, \sigma(x) = Q$, $U(x) = \rho\|x\|^2$ and $U_1(x) = \rho_1\|\nabla x\|^2$. Then note that for $\rho > 0$,

$$\begin{aligned}
&\langle DU(x), \mu(x)\rangle + \frac{1}{2}\mathrm{tr}[D^2U(x)\sigma(x)\sigma^*(x)] + \frac{1}{2}\|\sigma(x)^*DU(x)\|^2 \\
&= 2\rho\langle x, Ax - x^3 + x\rangle + \rho\|Q\|^2_{L_2^0} + 2\rho^2\|Q^*x\|^2 \\
&\le -2\rho\|\nabla x\|^2 + 2\rho\|x\|^2 - 2\rho\|x\|^4_{L^4} + \rho\|Q\|^2_{L_2^0} + 2\rho^2\|x\|^2\|Q\|^2_{L_2^0} \\
&\le -2\rho\|\nabla x\|^2 - 2\rho\|x\|^4_{L^4} + \rho\|Q\|^2_{L_2^0} + (2\rho + 2\rho^2\|Q\|^2_{L_2^0})\|x\|^2.
\end{aligned}$$

Let $\alpha \ge 2\rho + 2\rho^2\|Q\|^2_{L_2^0}$, and define

$$\overline{U}(x) = 2\rho\|\nabla x\|^2 + 2\rho\|x\|^4_{L^4} - \rho\|Q\|^2_{\mathcal{L}_2^0}.$$

Then one may apply Lemma 4.2, which yields

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(e^{-\alpha t}\rho\|X(t)\|^2 + 2\rho\int_0^t e^{-\alpha s}\|\nabla X(s)\|^2 ds + 2\rho\int_0^t e^{-\alpha s}\|X(s)\|^4_{L^4} ds\Big)\Big] \\
&\le \mathbb{E}\Big[e^{\frac{\rho\|Q\|^2_{\mathcal{L}_2^0}}{\lambda}} e^{\rho\|X_0\|^2}\Big] \le e^{\rho\|X_0\|^2}.
\end{aligned}$$

The second inequality is obtained with similar arguments and the fact that $\mathbb{H}^1 = H_0^1$:

$$\begin{aligned}
&\langle DU_1(x), \mu(x)\rangle + \frac{1}{2}\mathrm{tr}[D^2U_1(x)\sigma(x)\sigma^*(x)] + \frac{1}{2}\|\sigma(x)^*DU_1(x)\|^2 \\
&\le -2\rho_1\langle Ax, Ax\rangle - 6\rho_1\langle \nabla x, \nabla x x^2\rangle + \rho_1\|\nabla Q\|^2_{\mathcal{L}_2^0} + (2\rho_1 + 2\rho_1^2\|\nabla Q\|^2_{\mathcal{L}_2^0})\|\nabla x\|^2.
\end{aligned}$$

It remains to apply Lemma 4.2, to get for $\alpha_1 \ge 2\rho_1 + 2\rho_1^2\|\nabla Q\|^2_{\mathcal{L}_2^0}$,

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\Big(e^{-\alpha_1 t}\rho_1\|X(t)\|^2 + 2\rho_1\int_0^t e^{-\alpha_1 s}\|AX(s)\|^2 ds\Big)\Big)\Big] \\
&\le \mathbb{E}\Big[e^{\frac{\rho_1\|\nabla Q\|^2_{\mathcal{L}_2^0}}{\alpha_1}} e^{\rho_1\|\nabla X_0\|^2}\Big] \le e^{\rho_1\|\nabla X_0\|^2}.
\end{aligned}$$

This concludes the proof of Proposition 4.1. □

The use of Gagliardo–Nirenberg–Sobolev inequalities (see e.g. [**24**]) then allows us to improve the result of Proposition 4.1 as follows: we control exponential moments of the type $\mathbb{E}\Big[\exp\Big(\int_0^T c\|X(s)\|^2_{\mathcal{E}} ds\Big)\Big]$ with arbitrarily large parameter $c \in (0, \infty)$. This result is crucial in the approach used below to obtain higher rates of convergence for the splitting scheme.

PROPOSITION 4.2. *Let $d \le 3$, and assume that $\|(-A)^{\frac{1}{2}}Q\|_{\mathcal{L}_2^0} < \infty$ and $X_0 \in \mathbb{H}^1$. Then the solution X of* (1) *satisfies, for any $c > 0$,*

$$\mathbb{E}\Big[\exp\Big(\int_0^T c\|X(s)\|^2_{\mathcal{E}} ds\Big)\Big] \le C(c, d, T, X_0, Q) < \infty.$$

PROOF. Assume first that $d = 1$. Then we use the Gagliardo–Nirenberg–Sobolev inequality $\|X\|_{\mathcal{E}} \le C_1\|\nabla X\|^{\frac{1}{3}}\|X\|^{\frac{2}{3}}_{L^4}$.

Thanks to Young's inequality, for all $\epsilon \in (0,1)$, there exists $C_1(\epsilon) \in (0,\infty)$ such that

$$\|X\|_{\mathcal{E}}^2 \le C_1 \|\nabla X\|^{\frac{2}{3}} \|X\|_{L^4}^{\frac{4}{3}} \le \Big(\epsilon \|\nabla X\|^2 + \epsilon \|X\|_{L^4}^4 + C_1(\epsilon)\Big).$$

Choose $\epsilon = \epsilon(c) \le \frac{\rho}{ce^{\alpha T}} \le 1$. Then, using Cauchy-Schwarz inequality, one gets

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\int_0^T c\|X(s)\|_{\mathcal{E}}^2 ds\Big)\Big] \\
&\le \mathbb{E}\Big[\exp\Big(\int_0^T \epsilon c\|\nabla X\|^2 + \epsilon c \|X\|_{L^4}^4 + C_1(\epsilon,c)\Big) ds\Big)\Big] \\
&\le e^{C_1(\epsilon,c)T}\sqrt{\mathbb{E}\Big[\exp\Big(\int_0^T 2\epsilon c\|\nabla X\|^2 ds\Big)\Big]}\sqrt{\mathbb{E}\Big[\exp\Big(\int_0^T 2\epsilon c\|X\|_{L^4}^4 ds\Big)\Big]} \\
&\le C(c,1,T,X_0,Q),
\end{aligned}$$

thanks to Proposition 4.1, since $2\epsilon c \le \frac{\rho}{ce^{\alpha T}}$. This concludes the treament of the case $d = 1$.

When $d = 2$, resp. $d = 3$, we apply the Gagliardo-Nirenberg-Sobolev inequality, $\|X\|_{\mathcal{E}} \le C_2 \|AX\|^{\frac{1}{3}} \|X\|_{L^4}^{\frac{2}{3}}$, resp. $\|X\|_{\mathcal{E}} \le C_3 \|AX\|^{\frac{3}{5}} \|X\|_{L^4}^{\frac{2}{5}}$. In both cases, applying Young's inequality, for any $\epsilon \in (0,1)$, there exists $C_d(\epsilon) \in (0,\infty)$ such that

$$\|X\|_{\mathcal{E}}^2 \le \Big(\epsilon\|AX\|^2 + \epsilon\|X\|_{L^4}^4 + C_d(\epsilon)\Big).$$

Choose $\epsilon = \epsilon(c) \le \min(\frac{\rho}{ce^{\alpha T}}, \frac{\rho_1}{e^{\alpha_1 T}}) \le 1$. Then

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\int_0^t c\|X(s)\|_{\mathcal{E}}^2 ds\Big)\Big] \\
&\le \mathbb{E}\Big[\exp\Big(\int_0^T \epsilon c\|AX\|^2 + \epsilon c\|X\|_{L^4}^4 + C_d(\epsilon,c)\Big) ds\Big)\Big] \\
&\le C(\epsilon,c,C_d,T)\sqrt{\mathbb{E}\Big[\exp\Big(\int_0^T 2\epsilon c\|AX\|^2 ds\Big)\Big]}\sqrt{\mathbb{E}\Big[\exp\Big(\int_0^T 2\epsilon c\|X\|_{L^4}^4 ds\Big)\Big]} \\
&\le C(c,C_d,T,X_0,Q),
\end{aligned}$$

using Proposition 4.1, and the condition on ϵ.

This concludes the proof of Proposition 4.2. □

To conclude this section, we give an additional a priori estimate, with higher order spatial regularity for the solution X of Eq. (1).

PROPOSITION 4.3. *Let $d \le 3$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathcal{L}_2^0} < \infty$ and $X_0 \in \mathbb{H}^2$. Then the solution $X \in \mathbb{H}^2$, a.s. Moreover for any $p \ge 2$,*

$$\sup_{s\in[0,T]} \mathbb{E}\Big[\|X(s)\|_{\mathbb{H}^2}^p\Big] \le C(T,Q,X_0,p).$$

PROOF. By the mild form of Y, we get

$$\|Y(t)\|_{\mathbb{H}^2} \le \|S(t)X_0\|_{\mathbb{H}^2} + \Big\|\int_0^t S(t-s)F(Y+\omega(s))ds\Big\|_{\mathbb{H}^2}.$$

The boundedness of $S(\cdot)$ and the calculus inequality in the Sobolev spaces (see e.g. [**23**]) leads that

$$\begin{aligned}
&\|S(t-s)F(Y(s)+\omega(s)))\|_{\mathbb{H}^2}\\
&\le C\Big(\|Y(s)+\omega(s)\|_{\mathbb{H}^2}+\|X(s)\|_{\mathbb{H}^2}\|X(s)\|_{\mathcal{E}}^2+\|X(s)^2\|_{\mathbb{H}^2}\|X(s)\|_{\mathcal{E}}\Big)\\
&\le C\Big(\|Y(s)+\omega(s)\|_{\mathbb{H}^2}+\|X(s)\|_{\mathbb{H}^2}\|X(s)\|_{\mathcal{E}}^2+\|X(s)\|_{\mathbb{H}^2}^2\|X(s)\|_{\mathcal{E}}\Big).
\end{aligned}$$

Gronwall inequality, together with Sobolev embedding theorem, implies that

$$\begin{aligned}
\|Y\|_{\mathbb{H}^2}\le C\exp(C\int_0^T\|X(s)\|_{\mathcal{E}}\|X(s)\|_{\mathbb{H}^2}ds)&\\
\Big(\sup_{t\in[0,T]}\|S(t)X_0\|+\int_0^T(1+\|X(s)\|_{\mathcal{E}}\|X(s)\|_{\mathbb{H}^2})\|\omega(t)\|_{\mathbb{H}^2}dt\Big)&.
\end{aligned}$$

Taking expectation, the exponential integrability in Proposition 4.2, and the regularity of the stochastic convolution,

$$\begin{aligned}
\sup_{t\in[0,T]}\mathbb{E}\Big[\|A\omega(s)\|^p\Big]&\le\sup_{t\in[0,T]}\mathbb{E}\Big[\|A\int_0^tS(t-s)dW^Q(s)\|^p\Big]\\
&\le C\sup_{t\in[0,T]}\mathbb{E}\Big[\Big(\int_0^t\|(-A)^{\frac12}S(t-s)(-A)^{\frac12}Q\|_{\mathcal{L}_2^0}^2ds\Big)^{\frac p2}\Big]\le C(T,Q,p),
\end{aligned}$$

yields that

$$\begin{aligned}
\mathbb{E}[\|X(s)\|_{\mathbb{H}^2}^p]&\le C\mathbb{E}[\|\omega(s)\|_{\mathbb{H}^2}^p]+C\Big(\mathbb{E}\Big[\exp(2pC\int_0^T\|X(s)\|_{\mathcal{E}}\|X(s)\|_{\mathbb{H}^2}ds)\Big]\Big)^{\frac12}\\
&\quad\times\Big(\sup_{t\in[0,T]}\|S(t)X_0\|_{\mathbb{H}^2}^{2p}+\mathbb{E}\Big[\Big(\int_0^T\|X(s)\|_{\mathcal{E}}\|X(s)\|_{\mathbb{H}^2}\|\omega(s)\|_{\mathbb{H}^2}ds\Big)^{2p}\Big]\Big)^{\frac12}.
\end{aligned}$$

By Gagliardo–Nirenberg inequality in $d=1,2,3$, and Young inequality, we get

$$\begin{aligned}
\|X(s)\|_{\mathcal{E}}\|X(s)\|_{\mathbb{H}^2}&\le C\|X(s)\|_{\mathbb{H}^2}\|\nabla X(s)\|^{\frac12}\|X(s)\|_{L^4}^{\frac12}\\
&\le\epsilon\|X(s)\|_{\mathbb{H}^2}^2+\epsilon\|\nabla X(s)\|^2+\epsilon\|X(s)\|_{L^4}^4+C(\epsilon),\quad d=1\\
\|X(s)\|_{\mathcal{E}}\|X(s)\|_{\mathbb{H}^2}&\le C\|X(s)\|_{\mathbb{H}^2}^{\frac43}\|X(s)\|_{L^4}^{\frac23}\\
&\le\epsilon\|X(s)\|_{\mathbb{H}^2}^2+\epsilon\|X(s)\|_{L^4}^4+C(\epsilon),\quad d=2\\
\|X(s)\|_{\mathcal{E}}\|X(s)\|_{\mathbb{H}^2}&\le C\|X(s)\|_{\mathbb{H}^2}^{\frac85}\|X(s)\|_{L^4}^{\frac25}\\
&\le\epsilon\|X(s)\|_{\mathbb{H}^2}^2+\epsilon\|X(s)\|_{L^4}^4+C(\epsilon),\quad d=3.
\end{aligned}$$

Combining with Proposition 4.1, we get the boundedness of this exponential moment $\exp\Big(C\int_0^T\|X(s)\|_{\mathcal{E}}\|X(s)\|_{\mathbb{H}^2}ds\Big)$. The estimation of $\mathbb{E}\Big[\Big(\int_0^T\|X(s)\|_{\mathcal{E}}\|X(s)\|_{\mathbb{H}^2}\|\omega(s)\|_{\mathbb{H}^2}ds\Big)^{2p}\Big]$

is similar. Gagliardo–Nirenberg–Sobolev inequality, together with Sobolev embedding $L^4 \hookrightarrow \mathbb{H}^1$, yields that for $d = 1$,

$$\mathbb{E}\Big[\Big(\int_0^T \|X(s)\|_{\mathcal{E}}\|X(s)\|_{\mathbb{H}^2}\|\omega(s)\|_{\mathbb{H}^2}ds\Big)^{2p}\Big] \le C\mathbb{E}\Big[(\int_0^T \|X(s)\|_{\mathbb{H}^2}^2 ds)^{2p}\Big] + C\mathbb{E}\Big[\int_0^T \|X(s)\|_{\mathbb{H}^1}^{4p}\|\omega(s)\|_{\mathbb{H}^2}^{4p}ds\Big],$$

for $\quad d = 2$,

$$\mathbb{E}\Big[\Big(\int_0^T \|X(s)\|_{\mathcal{E}}\|X(s)\|_{\mathbb{H}^2}\|\omega(s)\|_{\mathbb{H}^2}ds\Big)^{2p}\Big] \le C\mathbb{E}\Big[(\int_0^T \|X(s)\|_{\mathbb{H}^2}^2 ds)^{2p}\Big] + C\mathbb{E}\Big[\int_0^T \|X(s)\|_{\mathbb{H}^1}^{4p}\|\omega(s)\|_{\mathbb{H}^2}^{6p}ds\Big],$$

for $\quad d = 3$,

$$\mathbb{E}\Big[\Big(\int_0^T \|X(s)\|_{\mathcal{E}}\|X(s)\|_{\mathbb{H}^2}\|\omega(s)\|_{\mathbb{H}^2}ds\Big)^{2p}\Big] \le C\mathbb{E}\Big[(\int_0^T \|X(s)\|_{\mathbb{H}^2}^2 ds)^{2p}\Big] + C\mathbb{E}\Big[\int_0^T \|X(s)\|_{\mathbb{H}^1}^{4p}\|\omega(s)\|_{\mathbb{H}^2}^{10p}ds\Big].$$

Combining the a priori estimate in Lemma 4.1 and the above inequalities, we finish the proof. □

4.2. Strong convergence order 1 of the splitting scheme. In this part, we focus on the sharp strong convergence rate of X^N in $d = 1$. The main reason why we could not obtain higher strong convergence rate in $d = 2, 3$ is that this splitting up strategy will destroy the exponential integrability in $L^4([0, T]; L^4)$ and $L^2([0, T]; \mathbb{H}^2)$ of the original equation and that the a priori estimate of the auxiliary process Z^N in $\mathbb{H}^2$ can not be obtained, since the Sobolev embedding $\mathcal{E} \hookrightarrow \mathbb{H}^1$ does not hold. We also remark the a priori estimate in Lemma 4.3 holds for $d = 2, 3$. The study of higher strong convergence order for numerical schemes in high dimensional case will be studied in future works.

We first state the main result of this section.

THEOREM 4.1. *Assume that $d = 1$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathcal{L}_2^0} < \infty$ and $X_0 \in \mathbb{H}^2$. The proposed method possesses strong convergence order 1, i.e, for any $p \ge 1$,*

$$\sup_{n\le N} \mathbb{E}\Big[\Big\|X(t_n) - X^N(t_n)\Big\|^p\Big] \le C\delta t^p.$$

To obtain the higher strong order of the splitting scheme, we consider the following auxiliary predictable right continuous process Z^N such that $Z^N(t_n) = X^N(t_n)$, $n \le N$. The process Z^N is defined by recursion. Let $Z^N(0) := X_0$ and on each subinterval $[t_{n-1}, t_n]$, $1 \le n \le N$,

$$Z^N(t) = \Phi_{t-t_{n-1}}(Z^N(t_{n-1})), \quad t \in [t_{n-1}, t_n),$$
$$Z^N(t_n) = S(\delta t)\Phi_{\delta t}(Z^N(t_{n-1})) + \int_{t_{n-1}}^{t_n} S(t_n - s)dW^Q(s).$$

Since when $t \in [t_{n-1}, t_n)$,

$$dZ^N = F(Z^N(t))dt,$$

we rewrite the definition of Z^N into an integration form,

(14)
$$Z^N(t) = Z^N(t_{n-1}) + \int_{t_{n-1}}^{t} F(Z^N(s))ds, \quad t \in [t_{n-1}, t_n),$$

(15)
$$Z^N(t_n) = S(\delta t)Z^N(t_{n-1}) + \int_{t_{n-1}}^{t_n} S(\delta t)F(Z^N(s))ds + \int_{t_{n-1}}^{t_n} S(t_n - s)dW^Q(s).$$

Letting n be $n-1$ in the above equation and then plugging it into Eq. (14) yields that

$$\begin{aligned}Z^N(t) = S(\delta t)Z^N(t_{n-2}) &+ \int_{t_{n-2}}^{t_{n-1}} S(\delta t)F(Z^N(s))ds + \int_{t_{n-1}}^{t} F(Z^N(s))ds \\ &+ \int_{t_{n-2}}^{t_{n-1}} S(t_{n-1} - s)dW^Q(s), \quad t \in [t_{n-1}, t_n).\end{aligned}$$

Repeating this process, we get, for $t \in [t_{n-1}, t_n)$,

$$\begin{aligned}Z^N(t) = S(t_{n-1})X_0 &+ \int_0^{t_{n-1}} S(t_{n-1} - \lfloor s \rfloor_{\delta t})F(Z^N(s))ds \\ &+ \int_{t_{n-1}}^{t} F(Z^N(s))ds + \int_0^{t_{n-1}} S(t_{n-1} - s)dW^Q(s),\end{aligned}$$

and

$$Z^N(t_n) = S(t_n)X(0) + \int_0^{t_n} S(t_n - \lfloor s \rfloor_{\delta t})F(Z^N(s))ds + \int_0^{t_n} S(t_n - s)dW^Q(s).$$

4.2.1. *A priori estimate for the auxiliary process.* In order to get the strong convergence order, we also need the following a priori estimations of Z^N.

Lemma 4.3. *Assume that $d = 1$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathcal{L}_2^0} < \infty$, $\|X_0\|_{\mathbb{H}^1} < \infty$. Then for $p \geq 2$, the auxiliary process Z^N satisfies*

$$\mathbb{E}\Big[\sup_{s\in[0,T]} \|Z^N(s)\|_{\mathbb{H}^1}^p\Big] \leq C(X_0, p, T, Q).$$

Proof. We first show the estimation of $\sup_{s\in[0,T]} \mathbb{E}[\|Z^N(s)\|_{\mathbb{H}^1}^p] \leq C(T, p, Q, X_0)$. Since similar arguments in Lemma 3.1 implies that $\sup_{s\in[0,T]} \mathbb{E}[\|Z^N(s)\|^p] \leq C(T, p, Q, X_0)$, it sufficient to show $\sup_{s\in[0,T]} \mathbb{E}[\|\nabla Z^N(s)\|^p] \leq C(T, p, Q, X_0)$. For simplify the presentation, we only present the case $p = 2$. Consider the linear SPDE $d\widehat{Z} = A\widehat{Z}dt + dW^Q(t)$ in local interval $[t_{n-1}, t_n]$ with $\widehat{Z}(t_{n-1}) = \Phi_{\delta t}(Z^N(t_{n-1}))$, we have $\widehat{Z}(t_n) = Z^N(t_n)$. By Itô formula, we have

$$\begin{aligned}\|\nabla Z^N(t_n)\|^2 = \|\nabla\Phi_{\delta t}(Z^N(t_{n-1}))\|^2 &- 2\int_{t_{n-1}}^{t_n} \langle A\widehat{Z}, A\widehat{Z}\rangle ds \\ &+ 2\int_{t_{n-1}}^{t_n} \langle \nabla\widehat{Z}, \nabla dW(s)\rangle + \int_{t_{n-1}}^{t_n} \|\nabla Q\|_{\mathcal{L}_2^0}^2 ds.\end{aligned}$$

Then taking expectation yields that

$$\mathbb{E}[\|\nabla Z^N(t_n)\|^2] \le \mathbb{E}[\|\nabla \Phi_{\delta t}(Z^N(t_{n-1}))\|^2] + \int_{t_{n-1}}^{t_n} \|\nabla Q\|^2_{\mathcal{L}_2^0} ds.$$

Since $\Phi_{t-t_{n-1}} Z^N(t_{n-1})$ is the solution of $d\widetilde{Z} = F(\widetilde{Z})dt$ with $\widetilde{Z}(t_{n-1}) = Z^N(t_{n-1})$, the similar arguments yields that

$$\|\nabla \Phi_{t-t_{n-1}}(Z^N(t_{n-1}))\|^2 \le e^{C\delta t}\|\nabla Z^N(t_{n-1})\|^2.$$

Combing the above estimations, we have for $t \in [t_{n-1}, t_n)$,

$$\begin{aligned}\mathbb{E}\Big[\|\nabla Z^N(t)\|^2\Big] &\le e^{C\delta t}\mathbb{E}\Big[\|\nabla Z^N(t_{n-1})\|^2\Big] \\ &\le e^{C\delta t}\Big(e^{C\delta t}\mathbb{E}\Big[\|\nabla Z^N(t_{n-2})\|^2\Big] + C\delta t\Big) \\ &\le e^{CT}\|X_0\|^2 + C(Q,T),\end{aligned}$$

which implies that $\sup_{s\in[0,T]} \mathbb{E}[\|\nabla Z^N(s)\|^2] \le C(T,2,Q,X_0)$. Similarly, we obtain the uniformly boundedness of $\sup_{s\in[0,T]} \mathbb{E}\Big[\|Z^N(s)\|^p_{\mathbb{H}^1}\Big]$, $p \ge 2$.

Now we are in position to show the desired result. By the argument in Lemma 3.1, we have $\mathbb{E}[\sup_{n\in N} \|X(t_n)\|^p_{L^q}] \le C$, $q = 2m$. Then we aim to prove that $\mathbb{E}[\sup_{n\in N} \|\nabla X(t_n)\|^p] \le C$. By the similar procedure of the previous proof of Lemma 3.1, we get

$$\begin{aligned}\|\nabla\Big(X^N(t_n) - \omega(t_n)\Big)\|^2 \le (1+\delta t)\Big\|\nabla\Big(\Phi_{\delta t}(X^N(t_{n-1})) - \Phi_{\delta t}(\omega(t_{n-1}))\Big)\Big\|^2 \\ + C\delta t(1 + \|\omega(t_{n-1})\|^6_{\mathbb{H}^1}).\end{aligned}$$

Now, consider the SDEs $d\widetilde{Z}_i = F(\widetilde{Z}_i)dt$ with different inputs $\widetilde{Z}_1(t_{n-1}) = X^N(t_{n-1}))$ and $\widetilde{Z}_2(t_{n-1}) = \omega(t_{n-1})$, we get $d(\widetilde{Z}_1 - \widetilde{Z}_2) = (F(\widetilde{Z}_1) - F(\widetilde{Z}_2))dt$ for $t \in [t_{n-1}, t_n]$. Further calculations, together with Gagliardo–Nirenberg, Holder and Young inequalities, yield that

$$\begin{aligned}&\|\nabla\Big(\Phi_{t-t_{n-1}}(X^N(t_{n-1})) - \Phi_{t-t_{n-1}}(\omega(t_{n-1}))\Big)\|^2 \\ &\le \|\nabla(X^N(t_{n-1}) - \omega(t_{n-1}))\|^2 - \int_{t_{n-1}}^t \langle(\widetilde{Z}_1 - \widetilde{Z}_2)\nabla(\widetilde{Z}_1^2 + \widetilde{Z}_1\widetilde{Z}_2 + \widetilde{Z}_2^2), \nabla\widetilde{Z}_1 - \nabla\widetilde{Z}_2\rangle ds \\ &\le \|\nabla(X^N(t_{n-1}) - \omega(t_{n-1}))\|^2 + C\int_{t_{n-1}}^t \|\nabla\widetilde{Z}_1 - \nabla\widetilde{Z}_2\|^2 ds \\ &\quad + C\int_{t_{n-1}}^t \|\nabla(\widetilde{Z}_1^2 + \widetilde{Z}_1\widetilde{Z}_2 + \widetilde{Z}_2^2)\|^2 \|(\widetilde{Z}_1 - \widetilde{Z}_2)\|^2_{\mathcal{E}} ds \\ &\le \|\nabla(X^N(t_{n-1}) - \omega(t_{n-1}))\|^2 + C\int_{t_{n-1}}^t \|\nabla\widetilde{Z}_1 - \nabla\widetilde{Z}_2\|^2 ds \\ &\quad + C\int_{t_{n-1}}^t (\|\widetilde{Z}_1\|^4_{\mathbb{H}^1} + \|\widetilde{Z}_2\|^4_{\mathbb{H}^1})\|\widetilde{Z}_1 - \widetilde{Z}_2\|^2 ds.\end{aligned}$$

On the other hand, the monotonicity of F yields that the solution of $d\widetilde{Z} = F(\widetilde{Z})dt$ satisfies for $t \in [t_{n-1}, t_n]$,

$$\sup_{t\in[t_{n-1},t_n]} \|\widetilde{Z}(t)\|^2_{\mathbb{H}^1} \le e^{C\delta t}(1 + \|\widetilde{Z}(t_{n-1})\|^2_{\mathbb{H}^1}).$$

The above inequality yields that

$$\begin{aligned}
&\|\nabla\Big(\Phi_{t-t_{n-1}}(X^N(t_{n-1})) - \Phi_{t-t_{n-1}}(\omega(t_{n-1}))\Big)\|^2 \\
&\le e^{C\delta t}(\|\nabla(X^N(t_{n-1}) - \omega(t_{n-1}))\|^2 + e^{C\delta t}\delta t(1 + \|X^N(t_{n-1})\|^6_{\mathbb{H}^1} + \|\omega(t_{n-1})\|^6_{\mathbb{H}^1}).
\end{aligned}$$

Then discrete Gronwall's inequality leads that

$$\|X^N(t_n)\|^2_{\mathbb{H}^1} \le C\|X_0\|^2_{\mathbb{H}^1} + C\|\omega(t_n)\|^2_{\mathbb{H}^1} + C\sum_{j=0}^{n-1}\delta t(1 + \|X^N(t_j)\|^6_{\mathbb{H}^1} + \|\omega(t_j)\|^6_{\mathbb{H}^1}).$$

Taking expectation, we obtain for any $p \ge 2$,

$$\mathbb{E}\Big[\sup_{n\le N}\|X^N(t_n)\|^p_{\mathbb{H}^1}\Big] \le C\Big(1 + \|X_0\|^p_{\mathbb{H}^1} + \sup_{n\le N}\mathbb{E}\Big[\|X^N(t_n)\|^{3p}_{\mathbb{H}^1}\Big] + \mathbb{E}\Big[\sup_{n\le N}\|\omega(t_n)\|^{3p}_{\mathbb{H}^1}\Big]\Big).$$

Denote $W_\gamma = \int_0^t (t-s)^{-\gamma}S(t-s)(-A)^{\frac{1}{2}}dW^Q(s)$. By the fractional method and Lemma 3.2, we have for $\beta < \frac{1}{2}$, $\frac{1}{2} > \gamma > \beta + \frac{1}{3p}$,

$$\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[0,T]}\|\omega(s)\|^{3p}_{\mathbb{H}^1}\Big] &\le \mathbb{E}\Big[\sup_{s\in[0,T]}\|\omega(s)\|^{3p}_{\mathbb{H}^{1+2\beta}}\Big] \le C\mathbb{E}\Big[\sup_{s\in[0,T]}\|G_\gamma W_\gamma(s)\|^{3p}_{\mathbb{H}^{2\beta}}\Big] \\
&\le C\int_0^T \mathbb{E}\Big[\|W_\gamma(s)\|^{3p}_{\mathbb{H}^{2\beta}}\Big]ds \\
&\le C\Big(\int_0^T s^{-2\gamma}\|S(s)(-A)^{\frac{1}{2}}Q\|^2_{\mathcal{L}_2^0}ds\Big)^{\frac{3p}{2}} \le C(T, Q, p),
\end{aligned}$$

which implies that $\mathbb{E}\Big[\sup_{n\le N}\|X^N(t_n)\|^p_{\mathbb{H}^1}\Big] \le C(T, X_0, p, Q)$. Then the definition of Z^N yields that

$$\mathbb{E}\Big[\sup_{s\in[0,T]}\|Z^N(s)\|^p_{\mathbb{H}^1}\Big] \le C\Big(1 + \mathbb{E}\Big[\sup_{n\le N}\|X^N(t_n)\|^{3p}_{\mathbb{H}^1}\Big]\Big) \le C(T, X_0, Q, p),$$

which completes the proof. $\square$

Similar to the procedures in the proof of Proposition 4.2, we show the following exponential integrability of Z^N which is the key to get the higher strong convergence rate. The rigorous proof is that in each local interval, we first apply the truncated argument and the spectral Galerkin method, then use the Itô formula and Fatou lemma to get the evolution of Lyapunouv functions. For convenience, we omit these procedures.

Proposition 4.4. *Assume that $d = 1$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathcal{L}_2^0} < \infty$, $\|X_0\|_{\mathbb{H}^1} < \infty$. Then we have for any $c > 0$,*

$$\mathbb{E}\Big[\exp\Big(\int_0^T c\|Z^N(s)\|^2_{\mathcal{E}}ds\Big)\Big] \le C(X_0, T, Q, c). \tag{16}$$

Proof. In each subinterval $[t_{n-1},t_n]$, we define the process $\widehat{Z}$ as the solution of $d\widehat{Z}=A\widehat{Z}dt+dW^Q(t)$, with $\widehat{Z}(t_{n-1})=\Phi_{\delta t}Z^N(t_{n-1})$. Denote $\mu(x)=Ax,\sigma(x)=Q$, $U(x)=\rho\|x\|^2$ and $U_1(x)=\rho_1\|\nabla x\|^2$. we get for $\rho,\rho_1>0$,

$$\begin{aligned}
&\langle DU(x),\mu(x)\rangle+\frac{1}{2}\mathrm{tr}[D^2U(x)\sigma(x)\sigma^*(x)]+\frac{1}{2}\|\sigma(x)^*DU\|^2\\
&=2\rho\langle x,Ax\rangle+\rho\|Q\|^2_{L_2^0}+2\rho^2\|Qx\|^2\\
&\le -2\rho\|\nabla x\|^2+\rho\|Q\|^2_{L_2^0}+2\rho^2\|Q\|^2_{L_2^0}\|x\|^2.
\end{aligned}$$

Lemma 4.2 yields that for $\alpha\ge 2\rho^2\|Q\|^2_{L_2^0}$,

$$\mathbb{E}\Big[\exp\Big(e^{-\alpha t_n}\rho\|Z^N(t_n)\|^2\Big)\Big]\le e^{C\delta t}\mathbb{E}\Big[\exp\Big(e^{-\alpha t_{n-1}}\rho\|\Phi_{\delta t}Z^N(t_{n-1})\|^2\Big)\Big].$$

Since $\Phi_{t-t_{n-1}}Z^N(t_{n-1})$ is the solution of $d\widetilde{Z}=F(\widetilde{Z})dt$ with $\widetilde{Z}(t_{n-1})=Z^N(t_{n-1})$ in $[t_{n-1},t_n]$, similar calculation, together with Hölder and Young inequality, yields

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(e^{-\alpha t_{n-1}}\rho\|\Phi_{\delta t}Z^N(t_{n-1})\|^2\Big)\Big]\\
&=\mathbb{E}\Big[\exp\Big(e^{-\alpha t_{n-1}}\rho\|Z^N(t_{n-1})\|^2-e^{-\alpha t_{n-1}}2\rho\int_{t_{n-1}}^{t_n}\|Z^N(s)\|^4_{L^4}ds\\
&\quad+e^{-\alpha t_{n-1}}2\rho\int_{t_{n-1}}^{t_n}\|Z^N(s)\|^2ds\Big)\Big]\\
&\le e^{C\delta t}\mathbb{E}\Big[\exp\Big(e^{\alpha t_{n-1}}\rho\|Z^N(t_{n-1})\|^2-e^{-\alpha t_{n-1}}\rho\int_{t_{n-1}}^{t_n}\|Z^N(s)\|^4_{L^4}ds\Big)\Big]\\
&\le e^{C\delta t}\mathbb{E}\Big[\exp\Big(e^{-\alpha t_{n-1}}\rho\|Z^N(t_{n-1})\|^2\Big)\Big].
\end{aligned}$$

Then repeating the above procedures,

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(e^{-\alpha t_n}\rho\|Z^N(t_n)\|^2\Big)\Big]&\le e^{C\delta t}\mathbb{E}\Big[\exp\Big(e^{-\alpha t_{n-1}}\rho\|Z^N(t_{n-1})\|^2\Big)\Big]\\
&\le e^{Ct_n}e^{\rho\|X_0\|^2}.
\end{aligned}$$

For $t\in[t_{n-1},t_n)$, we similarly have

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(e^{-\alpha t}\rho\|Z^N(t)\|^2+\int_0^t e^{-\alpha s}\rho\|Z^N(s)\|^4_{L^4}ds\Big)\Big]\\
&\le\mathbb{E}\Bigg[\mathbb{E}\Big[\exp\Big(e^{-\alpha t}\rho\|Z^N(t)\|^2+\int_{t_{n-1}}^t e^{-\alpha s}\rho\|Z^N(s)\|^4_{L^4}ds\Big)\Big|\mathscr{F}_{t_{n-1}}\Big]\\
&\quad\times\exp\Big(\int_0^{t_{n-1}}e^{-\alpha s}\rho\|Z^N(s)\|^4_{L^4}ds\Big)\Bigg]\\
&\le e^{C\delta t}\mathbb{E}\Big[\exp\Big(e^{-\alpha t_{n-1}}\rho\|Z^N(t_{n-1})\|^2+\int_0^{t_{n-1}}e^{-\alpha s}\rho\|Z^N(s)\|^4_{L^4}ds\Big)\Big]\\
&\le e^{Ct_n}e^{\rho\|X_0\|^2}.
\end{aligned}$$

Next, we focus on the exponential integrability in $\mathbb{H}^1$. Since $d\widehat{Z} = A\widehat{Z}dt + dW^Q(t)$ in $[t_{n-1}, t_n]$, with $\widehat{Z}(t_{n-1}) = \Phi_{\delta t} Z^N(t_{n-1})$, for $\rho_1 > 0$, we have

$$\begin{aligned}
&\langle DU_1(x), \mu(x)\rangle + \frac{1}{2}\mathrm{tr}[D^2U_1(x)\sigma(x)\sigma^*(x)] + \frac{1}{2}\|\sigma(x)^* DU_1(x)\|^2 \\
&= -2\rho_1\langle Ax, Ax\rangle + \rho_1\|\nabla Q\|^2_{L_2^0} + 2\rho_1^2\|\nabla Q\|^2_{L_2^0}\|\nabla x\|^2,
\end{aligned}$$

which yields that for $\alpha_1 \geq 2\rho_1^2\|\nabla Q\|^2_{L_2^0}$,

$$\mathbb{E}\Big[\exp\Big(e^{-\alpha_1 t_n}\rho_1\|\nabla Z^N(t_n)\|^2\Big)\Big] \leq e^{C\delta t}\mathbb{E}\Big[\exp\Big(e^{-\alpha_1 t_{n-1}}\rho_1\|\nabla \Phi_{\delta t} Z^N(t_{n-1})\|^2\Big)\Big].$$

Then the fact that $\Phi_{t-t_{n-1}} Z^N(t_{n-1})$ is the solution of $d\widetilde{Z} = F(\widetilde{Z})dt$ in $[t_{n-1}, t_n]$, with $\widetilde{Z}(t_{n-1}) = Z^N(t_{n-1})$, yields that for $\alpha_1 \geq 2\widetilde{\rho}_1$, $\widetilde{\rho}_1 = e^{2\rho_1^2\|\nabla Q\|^2_{L_2^0}T}\rho_1$,

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(e^{-\alpha_1 t_{n-1}}\rho_1\|\nabla \Phi_{\delta t} Z^N(t_{n-1})\|^2 + \int_{t_{n-1}}^{t_n} e^{-\alpha_1 s} \\
&\quad 2\rho_1\langle \nabla Z^N(s), (Z^N(s))^2\nabla Z^N(s)\rangle ds\Big)\Big] \\
&\leq e^{C\delta t}\mathbb{E}\Big[\exp\Big(e^{-\alpha_1 t_{n-1}}\rho_1\|\nabla Z^N(t_{n-1})\|^2\Big)\Big]
\end{aligned}$$

Repeating the above procedures and taking $\alpha_1 \geq \max(2\rho_1^2\|\nabla Q\|^2_{L_2^0}, 2e^{2\rho_1^2\|\nabla Q\|^2_{L_2^0}T}\rho_1)$, we obtain

$$\sup_{t\in[0,T]}\mathbb{E}\Big[\exp\Big(e^{-\alpha_1 t}\rho_1\|\nabla Z^N(t)\|^2\Big)\Big] \leq Ce^{\rho_1\|\nabla X_0\|^2}.$$

Now, we are in position to show the desired result (16). Gagliardo–Nirenberg–Sobolev inequality $\|Z^N\|_{\mathcal{E}} \leq C_1\|\nabla Z^N\|^{\frac{1}{3}}\|Z^N\|^{\frac{2}{3}}_{L^4}$, together with Hölder and Young inequalities, implies that

$$\begin{aligned}
&\mathbb{E}\Big[\exp\Big(\int_0^T c\|Z^N(s)\|^2_{\mathcal{E}}ds\Big)\Big] \\
&\leq \mathbb{E}\Big[\exp\Big(\int_0^T \frac{1}{2}\epsilon_1\|\nabla Z^N(s)\|^2 + \frac{1}{2}\epsilon_2\|Z^N(s)\|^4_{L^4} + C(\epsilon_1,\epsilon_2,c))ds\Big)\Big] \\
&\leq C(T,\epsilon_1,\epsilon_2,c)\sqrt{\mathbb{E}\Big[\exp\Big(\int_0^T \epsilon_1\|\nabla Z^N(s)\|^2 ds\Big)\Big]}\sqrt{\mathbb{E}\Big[\exp\Big(\int_0^T \epsilon_2\|Z^N(s)\|^4_{L^4}ds\Big)\Big]}.
\end{aligned}$$

Choosing $\epsilon_2 \leq e^{-\alpha T}\rho$, we have

$$\sqrt{\mathbb{E}\Big[\exp\Big(\int_0^T \epsilon_2\|Z^N(s)\|^4_{L^4}ds\Big)\Big]} \leq e^{CT}e^{\frac{\rho}{2}\|X_0\|^2}.$$

Taking $\epsilon_1 \leq \frac{e^{-\alpha_1 T}\rho_1}{T}$, together with Jensen inequality, yields that

$$\begin{aligned}
\sqrt{\mathbb{E}\Big[\exp\Big(\int_0^T \epsilon_1\|\nabla Z^N(s)\|^2 ds\Big)\Big]} &\leq \sup_{s\in[0,T]}\sqrt{\mathbb{E}\Big[\exp\Big(T\epsilon_1\|\nabla Z^N(s)\|^2\Big)\Big]} \\
&\leq e^{CT}e^{\frac{\rho}{2}\|\nabla X_0\|^2}.
\end{aligned}$$

The above two estimations leads the desired result.

$\square$

4.2.2. *Strong convergence order 1 of the splitting scheme.* After establish the a priori estimates and the exponential integrability of both the exact and numerical solutions, we are in position to give the other main result on the strong convergence rate of the splitting scheme.

Proof of Theorem 4.1. The mild representation of X (5) and X^N (8) yields that

$$\begin{aligned}\|X(t_n)-X^N(t_n)\| &\le \Big\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} S(t_n-s)(F(X(s))-F(Z^N(s))ds\Big\|\\ &\quad+\Big\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}(S(t_n-s)-S(t_n-t_j))F(Z^N(s))ds\Big\| \quad := II_1+II_2.\end{aligned}$$

By the smoothing properties of $S(t)$, $\mathbb{H}^1$ is an algebra and $|F(z)|\le C(1+|z|^3)$, for $0<\eta<1$, II_2 is treat as follows:

$$\begin{aligned}II_2 &= \Big\|\int_0^{t_n}(-A)^{\eta}S(t_n-s)(-A)^{-\eta}(I-S(s-\lfloor s\rfloor_{\delta t}))F(Z^N(s))ds\Big\|\\ &\le C\delta t^{\frac12+\eta}\Big(1+\sup_{s\in[0,T]}\Big\|Z^N(s)\Big\|_{\mathbb{H}^1}^3\Big)\int_0^{t_n}\Big\|(-A)^{\eta}S(t_n-s)\Big\|ds\\ &\le C\delta t^{\frac12+\eta}\Big(1+\sup_{s\in[0,T]}\Big\|Z^N(s)\Big\|_{\mathbb{H}^1}^3\Big).\end{aligned}$$

For convenience, we introduce the mapping G such that $F(z_1)-F(z_2)=G(z_1,z_2)(z_1-z_2)$, $z_1,z_2\in R$, where $G(z_1,z_2)=-(z_1^2+z_2^2+z_1z_2)+1$. II_1 is decomposed as

$$\begin{aligned}II_1 &\le \Big\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} S(t_n-s)G(X(s),Z^N(s))(X(t_j)-Z^N(t_j))ds\Big\|\\ &\quad+\Big\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} S(t_n-s)G(X(s),Z^N(s))(X(s)-X(t_j))ds\Big\|\\ &\quad+\Big\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} S(t_n-s)G(X(s),Z^N(s))(Z^N(s)-Z^N(t_j))ds\Big\|\\ &:= II_{11}+II_{12}+II_{13}.\end{aligned}$$

Direct calculations, together with Sobolev embedding and Gagliardo–Nirenberg inequality, yields that

$$II_{11}\le C\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\Big(\|X(s)\|_{\mathcal{E}}^2+\|Z^N(s)\|_{\mathcal{E}}^2+1\Big)ds\|X(t_j)-Z^N(t_j)\|.$$

and

$$
\begin{aligned}
II_{13} &\le 2C\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \Big(1+\|X(s)\|_{L^6}^2+\|Z^N(s)\|_{L^6}^2\Big)\Big\|Z^N(s)-Z^N(t_j)\Big\|_{L^6}ds\\
&\le 2C\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \sup_{s\in[0,t_n]}\Big(1+\|X(s)\|_{L^6}^2+\|Z^N(s)\|_{L^6}^2\Big)\Big\|\int_{t_j}^{s}F(Z^N(r))dr\Big\|_{L^6}ds\\
&\le 2C\delta t\sup_{s\in[0,t_n]}\Big(1+\|X(s)\|_{L^6}^4+\|Z^N(s)\|_{L^6}^4+\|Z^N(s)\|_{L^{18}}^6\Big)\\
&\le 2C\delta t\sup_{s\in[0,t_n]}\Big(1+\|X(s)\|_{\mathbb{H}^1}^4+\|Z^N(s)\|_{\mathbb{H}^1}^6\Big).
\end{aligned}
$$

For II_{12}, we have

$$
\begin{aligned}
II_{12} \le& \Big\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} S(t_n-s)G(X(t_j),Z^N(t_j))(X(s)-X(t_j))ds\Big\|\\
&+\Big\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} S(t_n-s)\Big(G(X(s),Z^N(s))-G(X(t_j),Z^N(s))\Big)(X(s)-X(t_j))ds\Big\|\\
&+\Big\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} S(t_n-s)\Big(G(X(t_j),Z^N(s))-G(X(t_j),Z^N(t_j))\Big)(X(s)-X(t_j))ds\Big\|\\
:=& II_{121}+II_{122}+II_{123}.
\end{aligned}
$$

Using the mild form of $X(s)$ (5) and Sobolev embedding $\mathbb{H}^1\hookrightarrow\mathcal{E}$, we have

$$
\begin{aligned}
III_{121} \le& \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} \Big\|S(t_n-s)G(X(t_j),Z^N(t_j))\int_{t_j}^{s}S(s-r)F(X(r))dr\Big\|ds\\
&+\Big\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} S(t_n-s)G(X(t_j),Z^N(t_j))\int_{t_j}^{s}S(s-r)dW^Q(r)ds\Big\|\\
&+C\sum_{j=0}^{n-1}\delta t^2(\|X(t_j)\|_{\mathcal{E}}^2+\|Z^N(t_j)\|_{\mathcal{E}}^2)\|(-A)X(t_j)\|\\
\le& C\delta t\sup_{s\in[0,t_n]}\Big(1+\|X(s)\|_{\mathbb{H}^1}^5+\|Z^N(s)\|_{\mathbb{H}^1}^5\Big)\\
&+C\sum_{j=0}^{n-1}\delta t^2(\|X(t_j)\|_{\mathbb{H}^1}^2+\|Z^N(t_j)\|_{\mathbb{H}^1}^2)\|(-A)X(t_j)\|\\
&+\Big\|\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} S(t_n-s)G(X(t_j),Z^N(t_j))\int_{t_j}^{s}S(s-r)dW^Q(r)ds\Big\|.
\end{aligned}
$$

For the last term, taking expectation, together with the independence of increments of Wiener process, the adaptivity of X, Fubini theorem and Burkholder-Davis-Gundy inequality, yields that for $p \geq 2$,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big\| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} S(t_n - s) G(X(t_j), Z^N(t_j)) \int_{t_j}^{s} S(s-r) dW^Q(r) ds \Big\|^p\Big] \\
&= \mathbb{E}\Big[\Big\| \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \int_{r}^{t_{j+1}} S(t_n - s) G(X(t_j), Z^N(t_j)) S(s-r) ds dW^Q(r) \Big\|^p\Big] \\
&\leq C(p) \mathbb{E}\Big[\Big(\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Big\| \int_{r}^{t_{j+1}} S(t_n - s) G(X(t_j), Z^N(t_j)) S(s-r) ds Q \Big\|_{\mathcal{L}_2^0}^2 dr \Big)^{\frac{p}{2}}\Big] \\
&\leq C(p) \delta t^p \Big(\sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Big(\|X(t_j)\|_{L^p(\Omega;L^6)}^2 + \|Z^N(t_j)\|_{L^p(\Omega;L^6)}^2 + 1 \Big) \sum_{k \in \mathbb{N}^+} \Big\| Q e_k \Big\|_{\mathbb{H}^1}^2 ds \Big)^{\frac{p}{2}} \\
&\leq C(T, Q, X_0, p) \delta t^p.
\end{aligned}
$$

The definition of G implies that G is symmetric and $|G(z_1, z_2) - G(z_1, z_3)| \leq |z_1||z_2 - z_3| + |z_2 - z_3||z_2 + z_3|$. Based on this property, we estimate III_{122} and III_{123} as

$$
\begin{aligned}
&III_{122} + III_{123} \\
&\leq 2C \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|X(s) - X(t_j)\|_{L^6}^2 (\|X(s)\|_{L^6} + \|X(t_j)\|_{L^6} + \|Z^N(s)\|_{L^6}) ds \\
&\quad + 2C \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|X(s) - X(t_j)\|_{L^6} \|Z^N(s) - Z^N(t_j)\|_{L^6} (\|Z^N(s)\|_{L^6} + \|Z^N(t_j)\|_{L^6} + \|X(t_j)\|_{L^6}) ds.
\end{aligned}
$$

The continuity of X, the right continuity of Z^N and Sobolev embedding theorem lead that for $s \in [t_j, t_{j+1})$, $\eta < 1$,

$$
\begin{aligned}
&\|X(s) - X(t_j)\|_{L^6} \\
&\leq \|(S(s) - S(t_j)) X(0)\|_{L^6} + \|\int_0^s S(s-r) F(X(r)) dr - \int_0^{t_j} S(t_j - r) F(X(r)) dr\|_{L^6} \\
&\quad + \|\int_0^s S(s-r) dW^Q(r) - \int_0^{t_j} S(t_j - r) dW^Q(r)\|_{L^6} \\
&\leq C \delta t^{\frac{1}{2}} \|X_0\|_{\mathbb{H}^2} + \|\int_0^{t_j} (S(s-r) - S(t_j - r)) F(X(r)) dr\|_{L^6} + \|\int_{t_j}^{s} S(s-r) F(X(r)) dr\|_{L^6} \\
&\quad + \|\int_0^{t_j} (S(s-r) - S(t_j - r)) dW^Q(r)\|_{L^6} + \|\int_{t_j}^{s} S(s-r) dW^Q(r)\|_{L^6} \\
&\leq C \delta t^{\min(\frac{1}{2}, \eta)} \sup_{r \in [0,T]} \Big(\|X_0\|_{\mathbb{H}^2} + \|X(r)\|_{\mathbb{H}^1} + \|X(r)\|_{\mathbb{H}^1}^3 \Big) + \Big\| \int_{t_j}^{s} S(s-r) dW^Q(r) \Big\|_{L^6} \\
&\quad + \|\int_0^{t_j} (S(s-r) - S(t_j - r)) dW^Q(r)\|_{L^6},
\end{aligned}
$$

where the two stochastic convolution terms can be bounded by Sobolev embedding $L^6 \hookrightarrow \mathbb{H}^1$ and similar estimations for I_3 in Corollary 3.2, and

$$\begin{aligned}
&\|Z^N(s) - Z^N(t_j)\|_{L^6} \\
&\le \| \int_{t_{n-1}}^{s} F(Z^N(r))dr\|_{L^6} \le C\delta t \sup_{r\in[0,T]} \Big(1 + \|Z^N(r)\|_{\mathbb{H}^1} + \|Z^N(r)\|_{\mathbb{H}^1}^3\Big).
\end{aligned}$$

The above estimations, together with Young and Hölder inequality, implies that

$$\begin{aligned}
&III_{122} + III_{123} \\
&\le 2C\delta t^{\min(1,2\eta)} \sup_{s\in[0,t_n]} \Big(1 + \|X_0\|_{\mathbb{H}^2}^4 + \|X(s)\|_{\mathbb{H}^1}^{12} + \|Z^N(s)\|_{\mathbb{H}^1}^{12}\Big) \\
&\quad + 2C \sup_{s\in[0,t_n]} \Big(\|Z^N(s)\|_{\mathbb{H}^1} + \|X(s)\|_{\mathbb{H}^1}\Big) \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \Bigg(\Big\| \int_{t_j}^{s} S(s-r)dW^Q(r)\Big\|_{\mathbb{H}^1}^2 \\
&\quad + \| \int_0^{t_j} (S(s-r) - S(t_j - r))dW^Q(r)\|_{\mathbb{H}^1}^2\Bigg) ds.
\end{aligned}$$

Since $\|X(t_n) - Z^N(t_n)\| \le II_{11} + II_2 + II_{13} + II_{121} + II_{122} + II_{123}$, the discrete Gronwall's inequality in [**7**, Lemma 2.6] yields that

$$\begin{aligned}
\|X(t_n) - Z^N(t_n)\| \le C \exp\Big(2 \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|X(s)\|_{\mathcal{E}}^2 + \|Z^N(s)\|_{\mathcal{E}}^2 ds\Big) \\
\times \Big(II_2 + II_{13} + II_{121} + II_{122} + II_{123}\Big).
\end{aligned}$$

Then taking expectation, together with Hölder inequality, the a priori estimates in Lemma 4.1, Propositions 4.3 and 4.4, the continuity of stochastic convolution in the proof of Corollary 3.2 and exponential integrability of X and Z^N in Propositions

4.2 and 4.4, we obtain for $p \geq 1$, $\frac{1}{2} < \eta < 1$,

$$
\begin{aligned}
&\sup_{n \leq N} \mathbb{E}\Big[\|X(t_n) - X^N(t_n)\|^p\Big] \\
&\leq C(p) \sqrt[\frac{1}{4}]{\mathbb{E}\Big[\exp(4p\int_0^T \|X(s)\|_{\mathcal{E}}^2)\Big]} \sqrt[\frac{1}{4}]{\mathbb{E}\Big[\exp(4p\int_0^T \|Z^N(s)\|_{\mathcal{E}}^2)\Big]} \\
&\quad \Big(\sqrt{\mathbb{E}[II_2^{2p}]} + \sqrt{\mathbb{E}[II_{13}^{2p}]} + \sqrt{\mathbb{E}[II_{121}^{2p}]} + \sqrt{\mathbb{E}[(II_{122} + II_{123})^{2p}]}\Big) \\
&\leq C\delta t^{(\frac{1}{2}+\eta)p} \sqrt{\mathbb{E}\Big[1 + \sup_{s\in[0,T]} \Big\|Z^N(s)\Big\|_{\mathbb{H}^1}^{6p}\Big]} + C\delta t^p \sqrt{\mathbb{E}\Big[\sup_{s\in[0,T]} \Big(1 + \|X(s)\|_{\mathbb{H}^1}^{8p} + \|Z^N(s)\|_{\mathbb{H}^1}^{12p}\Big)\Big]} \\
&\quad + C\delta t^p \sqrt{\mathbb{E}\Big[\sup_{s\in[0,T]} \Big(1 + \|X(s)\|_{\mathbb{H}^1}^{10p} + \|Z^N(s)\|_{\mathbb{H}^1}^{10p}\Big)\Big]} \\
&\quad + C\delta t^p \sum_{j=0}^{N-1} \delta t \sqrt{\mathbb{E}\Big[\|(-A)X(t_j)\|^{2p}(\|X(t_j)\|_{\mathbb{H}^1}^{4p} + \|Z^N(t_j)\|_{\mathbb{H}^1}^{4p})\Big]} \\
&\quad + C\delta t^{\min(1,2\eta)p} \sqrt{\mathbb{E}\Big[1 + \|X_0\|_{\mathbb{H}^2}^{8p} + \sup_{s\in[0,T]} \Big(\|X(s)\|_{\mathbb{H}^1}^{24p} + \|Z^N(s)\|_{\mathbb{H}^1}^{24p}\Big)\Big]} \\
&\leq C(T,p,Q,X_0)\delta t^p \Big(1 + \sum_{j=0}^{N-1} \delta t \sqrt[\frac{1}{4}]{\mathbb{E}\Big[\|(-A)X(t_j)\|^{4p}\Big]} \sqrt[\frac{1}{2}]{\mathbb{E}\Big[\|X(t_j)\|_{\mathbb{H}^1}^{8p} + \|Z^N(t_j)\|_{\mathbb{H}^1}^{8p}\Big]}\Big) \\
&\leq C(T,p,Q,X_0)\delta t^p,
\end{aligned}
$$

which completes the proof. □

As a direct consequence of the Theorem 4.1 above, we have the following stronger error estimation.

COROLLARY 4.1. *Assume that* $d = 1$, $\|(-A)^{\frac{1}{2}}Q\|_{\mathcal{L}_2^0} < \infty$ *and* $X_0 \in \mathbb{H}^2$. *Then for any* $p \geq 1$ *and* $0 < \eta < 1$,

$$\Big\|\sup_{n\leq N} \|X(t_n) - X^N(t_n)\|\Big\|_{L^p(\Omega)} \leq C\delta t^\eta.$$

PROOF. Since for any $q' \geq 1$, based on Theorem 4.1, we have

$$\mathbb{E}\Big[\sup_{n\leq N} \Big\|X(t_n) - X^N(t_n)\Big\|^q\Big] \leq \sum_{n\leq N} \mathbb{E}\Big[\Big\|X(t_n) - X^N(t_n)\Big\|^q\Big] \leq C\delta t^{q-1}.$$

We completes the proof by taking $1 - \frac{1}{q}' \geq \eta$ and $q' \geq p$. □

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