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ABSTRACT. This paper studies nonparametric estimation of parameters of multivariate Hawkes processes. We consider the Bayesian setting and derive posterior concentration rates. First rates are derived for $\mathbb{L}_1$-metrics for stochastic intensities of the Hawkes process. We then deduce rates for the $\mathbb{L}_1$-norm of interactions functions of the process. Our results are exemplified by using priors based on piecewise constant functions, with regular or random partitions and priors based on mixtures of Betas distributions. Numerical illustrations are then proposed with in mind applications for inferring functional connectivity graphs of neurons.

1. INTRODUCTION

In this paper we study the properties of Bayesian nonparametric procedures in the context of multivariate Hawkes processes. The aim of this paper is to give some general results on posterior concentration rates for such models and to study some families of nonparametric priors.

1.1. **Hawkes processes.** Hawkes processes, introduced by Hawkes (1971), are specific point processes which are extensively used to model data whose occurrences depend on previous occurrences of the same process. To describe them, we first consider N a point process on $\mathbb{R}$. We denote by $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra on $\mathbb{R}$ and for any Borel set $A \in \mathcal{B}(\mathbb{R})$, we denote by $N(A)$ the number of occurrences of N in A. For short, for any $t \geq 0$, N_t denotes the number of occurrences in $[0, t]$. We assume that for any $t \geq 0$, $N_t < \infty$ almost surely. If $\mathcal{G}_t$ is the history of N until t, then, λ_t, the predictable intensity of N at time t, which represents the probability to observe a new occurrence at time t given previous occurrences, is defined by

$$\lambda_t dt = \mathbb{P}(dN_t = 1 \,|\, \mathcal{G}_{t^-}),$$

where dt denotes an arbitrary small increment of t and $dN_t = N([t, t + dt])$. For the case of *univariate Hawkes processes*, we have

$$\lambda_t = \phi\left(\int_{-\infty}^{t^-} h(t-s)dN_s\right),$$

for $\phi : \mathbb{R} \mapsto \mathbb{R}_+$ and $h : \mathbb{R} \mapsto \mathbb{R}$. We recall that the last integral means

$$\int_{-\infty}^{t^-} h(t-s)dN_s = \sum_{T_i \in N :\, T_i < t} h(t - T_i).$$

The case of *linear Hawkes processes* corresponds to $\phi(x) = \nu + x$ and $h(t) \geq 0$ for any t. The parameter $\nu \in \mathbb{R}_+^*$ is the *spontaneous rate* and h is the *self-exciting function*. We now assume that N is a *marked* point process, meaning that each occurrence T_i of N is associated to a mark $m_i \in \{1, \dots, K\}$, see Daley and Vere-Jones (2003). In this case, we can identify N with a *multivariate point process* and for any $k \in \{1, \dots, K\}$, $N^k(A)$ denotes the number of occurrences of N in A with mark k. In the sequel, we only consider linear multivariate Hawkes processes, so we assume that λ_t^k, the intensity of N^k, is

$$\lambda_t^k = \nu_k + \sum_{\ell=1}^{K} \int_{-\infty}^{t^-} h_{\ell,k}(t-u)dN_u^\ell, \tag{1.1}$$

where $\nu_k > 0$ and $h_{\ell,k}$, which is assumed to be non-negative and supported by $\mathbb{R}_+$, is the interaction function of N^ℓ on N^k. Theorem 7 of Brémaud and Massoulié (1996) shows that if the $K \times K$ matrix ρ, with

$$\rho_{\ell,k} = \int_0^{+\infty} h_{\ell,k}(t)dt, \quad \ell, k = 1, \dots, K, \tag{1.2}$$

has a spectral radius strictly smaller than 1, then there exists a unique stationary distribution for the multivariate process $N = (N^k)_{k=1,\dots,K}$ with the previous dynamics and finite average intensity.

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Hawkes processes have been extensively used in a wide range of applications. They are used to model earthquakes Vere-Jones and Ozaki (1982); Ogata (1988); Zhuang et al. (2002), interactions in social networks Simma and Jordan (2010); Zhou et al. (2013); Li and Zha (2014); Bacry et al. (2015a); Crane and Sornette (2008); Mitchell and Cates (2009); Yang and Zha (2013), financial data Embrechts et al. (2011); Bacry et al. (2015b, 2016, 2013); Aït-Sahalia et al. (2015), violence rates Mohler et al. (2011); Porter et al. (2012), genomes Gusto et al. (2005); Carstensen et al. (2010); Reynaud-Bouret and Schbath (2010) or neuronal activities Brillinger (1988); Chornoboy et al. (1988); Okatan et al. (2005); Paninski et al. (2007); Pillow et al. (2008); Hansen et al. (2015); Reynaud-Bouret et al. (2014, 2013), to name but a few.

Parametric inference for Hawkes models based on the likelihood is the most common in the literature and we refer the reader to Ogata (1988); Carstensen et al. (2010) for instance. Non-parametric estimation has first been considered by Reynaud-Bouret and Schbath Reynaud-Bouret and Schbath (2010) who proposed a procedure based on minimization of an ℓ_2-criterion penalized by an ℓ_0-penalty for univariate Hawkes processes. Their results have been extended to the multivariate setting by Hansen, Reynaud-Bouret and Rivoirard Hansen et al. (2015) where the ℓ_0-penalty is replaced with an ℓ_1-penalty. The resulting Lasso-type estimate leads to an easily implementable procedure providing sparse estimation of the structure of the underlying connectivity graph. To generalize this procedure to the high-dimensional setting, Chen, Witten and Shojaie Chen et al. (2017) proposed a simple and computationally inexpensive edge screening approach, whereas Bacry, Gaïffas and Muzy Bacry et al. (2015a) combine ℓ_1 and trace norm penalizations to take into account the low rank property of their self-excitement matrix. Other alternatives based on spectral methods Bacry et al. (2012) or estimation through the resolution of a Wiener-Hopf system Bacry and Muzy (2016) can also been found in the literature. These are all frequentist methods; Bayesian approaches for Hawkes models have received much less attention. To the best of our knowledge, the only contributions for the Bayesian inference are due to Rasmussen Rasmussen (2013) and Blundell, Beck and Heller Blundell et al. (2012) who explored parametric approaches and used MCMC to approximate the posterior distribution of the parameters.

1.2. **Our contribution.** In this paper, we study nonparametric posterior concentration rates when $T \to +\infty$, for estimating the parameter $f = ((\nu_k)_{k=1,\dots,K}, (h_{\ell,k})_{k,\ell=1,\dots,K})$ by using realizations of the multivariate process $(N_t^k)_{k=1,\dots,K}$ for $t \in [0,T]$. Analyzing asymptotic properties in the setting where $T \to +\infty$ means that the observation time becomes very large hence providing a large number of observations. Note that along the paper, K, the number of observed processes, is assumed to be fixed and can be viewed as a constant. Considering $K \to +\infty$ is a very challenging problem beyond the scope of this paper. Using the general theory of Ghosal and van der Vaart (2007a), we express the posterior concentration rates in terms of simple and usual quantities associated to the prior on f and under mild conditions on the true parameter. Two types of posterior concentration rates are provided: the first one is in terms of the $\mathbb{L}_1$-distance on the stochastic intensity functions $(\lambda^k)_{k=1,\dots,K}$ and the second one is in terms of the $\mathbb{L}_1$-distance on the parameter f (see precise notations below). To the best of our knowledge, these are the first theoretical results on Bayesian nonparametric inference in Hawkes models. Moreover, these are the first results on $\mathbb{L}_1$-convergence rates for the interaction functions $h_{\ell,k}$. In the frequentist literature, theoretical results are given in terms of either the $\mathbb{L}_2$-error of the stochastic intensity, as in Bacry et al. (2015a) and Bacry and Muzy (2016), or in terms of the $\mathbb{L}_2$-error on the interaction functions themselves, the latter being much more involved, as in Reynaud-Bouret and Schbath (2010) and Hansen et al. (2015). In Reynaud-Bouret and Schbath (2010), the estimator is constructed using a frequentist model selection procedure with a specific family of models based on piecewise constant functions. In the multivariate setting of Hansen et al. (2015), more generic families of approximation models are considered (wavelets of Fourier dictionaries) and then combined with a Lasso procedure, but under a somewhat restrictive assumption on the size of models that can be used to construct the estimators (see Section 5.2 of Hansen et al. (2015)). Our general results do not involve such strong conditions and therefore allows us to work with approximating families of models that are quite general. In particular, we can apply them to two families of prior models on the interaction functions $h_{\ell,k}$: priors based on piecewise constant functions, with regular or random partitions and priors based on mixtures of Betas distributions. From the posterior concentration rates, we also deduce a frequentist convergence rate for the posterior mean, seen as a point estimator. We finally propose an MCMC algorithm to simulate from the posterior distribution for the priors constructed from piecewise constant functions and a simulation study is conducted to illustrate our results.

1.3. **Overview of the paper.** In Section 2, Theorem 1 first states the posterior convergence rates obtained for stochastic intensities. Theorem 2 constitues a variation of this first result. From these results, we derive $\mathbb{L}_1$-rates for the parameter f (see Theorem 3) and for the posterior mean (see Corollary 1). Examples of prior models satisfying conditions of these theorems are given in Section 2.3. In Section **??**, numerical results are provided.

1.4. **Notations and assumptions.** We denote by $f_0 = ((\nu_k^0)_{k=1,\dots,K}, (h_{\ell,k}^0)_{k,\ell=1,\dots,K})$ the true parameter and assume that the interaction functions $h_{\ell,k}^0$ are supported by a compact interval $[0,A]$, with A assumed to be known. Given a parameter $f = ((\nu_k)_{k=1,\dots,K}, (h_{\ell,k})_{k,\ell=1,\dots,K})$, we denote by $\|\rho\|$ the spectral norm of the matrix ρ

associated with f and defined in (1.2). We recall that $\|\rho\|$ provides an upper bound of the spectral radius of ρ and we set

$$\mathcal{H} = \{(h_{\ell,k})_{k,\ell=1,\dots,K};\ h_{\ell,k} \geq 0,\ \text{support}(h_{\ell,k}) \subset [0,A],\ \rho_{\ell,k} < \infty,\ \forall\, k,\ell = 1,\dots,K,\ \|\rho\| < 1\}$$

and

$$\mathcal{F} = \{f = ((\nu_k)_{k=1,\dots,K}, (h_{\ell,k})_{k,\ell=1,\dots,K});\ 0 < \nu_k < \infty,\ \forall\, k = 1,\dots,K,\ (h_{\ell,k})_{k,\ell=1,\dots,K} \in \mathcal{H}\}.$$

We assume that $f_0 \in \mathcal{F}$ and denote by ρ^0 the matrix such that $\rho^0_{\ell,k} = \int_0^A h^0_{\ell,k}(t)dt$.

For any function $h : \mathbb{R} \mapsto \mathbb{R}$, we denote by $\|h\|_p$ the $\mathbb{L}_p$-norm of h. With a slight abuse of notations, we also use for $f = ((\nu_k)_{k=1,\dots,K}, (h_{\ell,k})_{k,\ell=1,\dots,K})$ and $f' = ((\nu_k)_{k=1,\dots,K}, (h'_{\ell,k})_{k,\ell=1,\dots,K})$ belonging to $\mathcal{F}$

$$\|f - f'\|_1 = \sum_{k=1}^K |\nu_k - \nu'_k| + \sum_{k=1}^K \sum_{\ell=1}^K \|h_{\ell,k} - h'_{\ell,k}\|_1. \tag{1.3}$$

Finally, we consider stochastic distances on $\mathcal{F}$:

$$d^2_{2,T}(f,f') = \frac{1}{T}\sum_{k=1}^K \int_0^T (\lambda^k_t(f) - \lambda^k_t(f'))^2 dt \quad \text{and} \quad d_{1,T}(f,f') = \frac{1}{T}\sum_{k=1}^K \int_0^T |\lambda^k_t(f) - \lambda^k_t(f')| dt,$$

where $\lambda^k_t(f)$ and $\lambda^k_t(f')$ denote the stochastic intensity (introduced in (1.1)) associated with f and f' respectively. We denote by $\mathcal{N}(u, \mathcal{H}_0, d)$ the covering number of a set $\mathcal{H}_0$ by balls with respect to the metric d with radius u. We set for any ℓ, μ^0_ℓ the mean of $\lambda^\ell_t(f_0)$ under $\mathbb{P}_0$

$$\mu^0_\ell = \mathbb{E}_0[\lambda^\ell_t(f_0)],$$

where $\mathbb{P}_0$ denotes the stationary distribution associated with f_0 and $\mathbb{E}_0$ is the expectation associated with $\mathbb{P}_0$. We also write $u_T \lesssim v_T$ if $|u_T/v_T|$ is bounded when $T \to +\infty$ and similarly $u_T \gtrsim v_T$ if $|v_T/u_T|$ is bounded.

2. Main results

This section contains main results of the paper. We first provide an expression for the posterior distribution.

2.1. **Posterior distribution.** Using Proposition 7.3.III of Daley and Vere-Jones (2003), and identifying a multivariate Hawkes process as a specific marked Hawkes process, we can write the log-likelihood function of the process observed on the interval $[0,T]$, conditional on $\mathcal{G}_{0^-} = \sigma\left(N^k_t,\ t < 0, 1 \leq k \leq K\right)$, as

$$L_T(f) \quad := \quad \sum_{k=1}^K \left[\int_0^T \log(\lambda^k_t(f))dN^k_t - \int_0^T \lambda^k_t(f)dt\right]. \tag{2.1}$$

With a slight abuse of notation, we shall also denote $L_T(\lambda)$ instead of $L_T(f)$.

Recall that we restrict ourselves to the setup where for all ℓ, k, $h_{\ell,k}$ has support included in $[0,A]$ for some fixed $A > 0$. This hypothesis is very common in the context of Hawkes processes, see Hansen et al. (2015). Note that in this case the conditional distribution of $(N^k)_{k=1,\dots,K}$ observed on the interval $[0,T]$ given $\mathcal{G}_{0^-}$ is equal to its conditional distribution given $\mathcal{G}_{[-A,0[} = \sigma\left(N^k_t, -A \leq t < 0, 1 \leq k \leq K\right)$.

Hence, in the following, we assume that we observe the process $(N^k)_{k=1,\dots,K}$ on $[-A,T]$, but we base our inference on the log-likelihood (2.1), which is associated to the observation of $(N^k)_{k=1,\dots,K}$ on $[0,T]$. We consider a Bayesian nonparametric approach and denote by Π the prior distribution on the parameter $f = ((\nu_k)_{k=1,\dots,K}, (h_{\ell,k})_{k,\ell=1,\dots,K})$. The posterior distribution is then formally equal to

$$\Pi\left(B|N, \mathcal{G}_{0^-}\right) = \frac{\int_B \exp(L_T(f))d\Pi(f|\mathcal{G}_{0^-})}{\int_{\mathcal{F}} \exp(L_T(f))d\Pi(f|\mathcal{G}_{0^-})}.$$

We approximate it by the following pseudo-posterior distribution, which we write $\Pi\left(\cdot|N\right)$

$$\Pi\left(B|N\right) = \frac{\int_B \exp(L_T(f))d\Pi(f)}{\int_{\mathcal{F}} \exp(L_T(f))d\Pi(f)}, \tag{2.2}$$

which thus corresponds to choosing $d\Pi(f) = d\Pi(f|\mathcal{G}^-_0)$.

2.2. **Posterior convergence rates for $d_{1,T}$ and $\mathbb{L}_1$-metrics.** In this section we give two results of posterior concentration rates, one in terms of the stochastic distance $d_{1,T}$ and another one in terms of the $\mathbb{L}_1$-distance, which constitutes the main result of this paper. We define

$$\Omega_T = \left\{ \max_{\ell \in \{1,\dots,K\}} \sup_{t\in[0,T]} N^\ell[t-A,t) \leq C_\alpha \log T \right\} \cap \left\{ \sum_{\ell=1}^K \left| \frac{N^\ell[-A,T]}{T} - \mu_\ell^0 \right| \leq \delta_T \right\}$$

with $\delta_T = \delta_0 (\log T)^{3/2}/\sqrt{T}$ and $\delta_0 > 0$ and C_α two positive constants not depending on T. From Lemmas 4 and 5 in Section 4.7, we have that for all $\alpha > 0$ there exist $C_\alpha > 0$ and $\delta_0 > 0$ only depending on α and f_0 such that

$$\mathbb{P}_0\left(\Omega_T^c\right) \leq T^{-\alpha}, \tag{2.3}$$

when T is large enough. In the sequel, we take $\alpha > 1$ and C_α accordingly. Note in particular that, on Ω_T,

$$\sum_{\ell=1}^K N^\ell([-A,T]) \leq N_0 T,$$

with $N_0 = 1 + \sum_{\ell=1}^K \mu_\ell^0$, when T is large enough. We then have the following theorem.

Theorem 1. *Consider the multivariate Hawkes process $(N^k)_{k=1,\dots,K}$ observed on $[-A,T]$, with likelihood given by* (2.1)*. Let Π be a prior distribution on $\mathcal{F}$. Let ϵ_T be a positive sequence such that $\epsilon_T = o(1)$ and*

$$\log\log(T)\log^3(T) = o(T\epsilon_T^2).$$

For $B > 0$, we consider

$$B(\epsilon_T, B) := \left\{ (\nu_k, (h_{\ell,k})_\ell)_k : \quad \max_k |\nu_k - \nu_k^0| \leq \epsilon_T,\ \max_{\ell,k} \|h_{\ell,k} - h_{\ell,k}^0\|_2 \leq \epsilon_T,\ \max_{\ell,k} \|h_{\ell,k}\|_\infty \leq B \right\}$$

and assume following conditions are satisfied for T large enough.

(i) *There exists $c_1 > 0$ and $B > 0$ such that*

$$\Pi\left(B(\epsilon_T, B)\right) \geq e^{-c_1 T \epsilon_T^2}.$$

(ii) *There exists a subset $\mathcal{H}_T \subset \mathcal{H}$, such that*

$$\frac{\Pi\left(\mathcal{H}_T^c\right)}{\Pi\left(B(\epsilon_T, B)\right)} \leq e^{-(2\kappa_T+3)T\epsilon_T^2},$$

where $\kappa_T = \kappa \log(r_T^{-1}) \asymp \log\log T$, with $r_T \gtrsim \log T$ defined in (4.12) *and κ defined in* (4.10)*.*

(iii) *There exist $\zeta_0 > 0$ and $x_0 > 0$ such that*

$$\log \mathcal{N}(\zeta_0 \epsilon_T, \mathcal{H}_T, \|.\|_1) \leq x_0 T \epsilon_T^2.$$

Then, there exist $M > 0$ and $C > 0$ such that

$$\mathbb{E}_0\left[\Pi\left(d_{1,T}(f_0, f) > M\sqrt{\log\log T}\,\epsilon_T | N\right)\right] \leq \frac{C\log\log(T)\log^3(T)}{T\epsilon_T^2} + \mathbb{P}_0(\Omega_T^c) + o(1) = o(1).$$

Assumptions (i), (ii) and (iii) are very common in the literature about posterior convergence rates. As expressed by Assumption (ii), some conditions are required on the prior on $\mathcal{H}_T$ but not on $\mathcal{F}_T$. Except the usual concentration property of ν around ν^0 expressed in the definition of $B(\epsilon_T, B)$, which is in particular satisfied if ν has a positive continuous density with respect to Lebesgue measure, we have no further condition on the tails of the distribution of ν.

Remark 1. *As appears in the proof of Theorem 1, the term $\sqrt{\log\log T}$ appearing in the posterior concentration rate can be dropped if $B(\epsilon_T, B)$ is replaced by*

$$B_\infty(\epsilon_T, B) = \left\{ (\nu_k, (h_{\ell,k})_\ell)_k : \quad \max_k |\nu_k - \nu_k^0| \leq \epsilon_T,\ \max_{\ell,k} \|h_{\ell,k} - h_{\ell,k}^0\|_\infty \leq \epsilon_T \right\},$$

in Assumption (i). This is used for instance in Section 2.3.1 to study random histograms priors whereas mixtures of Beta priors are controlled using the $\mathbb{L}_2$-norm.

Similarly to other general theorems on posterior concentration rates, we can consider some variants. Since the metric $d_{1,T}$ is stochastic, we cannot use slices in the form $d_{1,T}(f_0, f) \in (j\epsilon_T, (j+1)\epsilon_T)$ as in Theorem 1 of Ghosal and van der Vaart (2007a), however we can consider other forms of slices, using a similar idea as in Theorem 5 of Ghosal and van der Vaart (2007b). This is presented in the following theorem.

Theorem 2. *Consider the setting and assumptions of Theorem 1 except that assumption (iii) is replaced by the following one: There exists a sequence of sets* $(\mathcal{H}_{T,i})_{i\geq 1} \subset \mathcal{H}$ *with* $\cup_i \mathcal{H}_{T,i} = \mathcal{H}_T$ *and* $\zeta_0 > 0$ *such that*

$$\sum_{i=1}^{\infty} \mathcal{N}(\zeta_0 \epsilon_T, \mathcal{H}_{T,i}, \|.\|_1)\sqrt{\Pi(\mathcal{H}_{T,i})}e^{-x_0 T\epsilon_T^2} = o(1), \tag{2.4}$$

for some positive constant $x_0 > 0$*. Then, there exists* $M > 0$ *such that*

$$\mathbb{E}_0\left[\Pi\left(d_{1,T}(f_0, f) > M\sqrt{\log\log T}\epsilon_T|N\right)\right] = o(1).$$

The posterior concentration rates of Theorems 1 and 2 are in terms of the metric $d_{1,T}$ on the intensity functions, which are data dependent and therefore not completely satisfying to understand concentration around the objects of interest namely f_0. We now use Theorem 1 to provide a general result to derive a posterior concentration rate in terms of the $\mathbb{L}_1$-norm.

Theorem 3. *Assume that the prior* Π *satisfies following assumptions.*

(i) *There exists* $\varepsilon_T = o(1)$ *such that* $\varepsilon_T \geq \delta_T$ *(see the definition of* Ω_T*) and* $c_1 > 0$ *such that*

$$\mathbb{E}_0\left[\Pi\left(A_{\varepsilon_T}^c|N\right)\right] = o(1) \quad \& \quad \mathbb{P}_0\left(D_T < e^{-c_1 T\varepsilon_T^2}\right) = o(1),$$

where $D_T = \int_{\mathcal{F}} e^{L_T(f)-L_T(f_0)}d\Pi(f)$ *and* $A_{\varepsilon_T} = \{f; d_{1,T}(f_0, f) \leq \varepsilon_T\}$.

(ii) *The prior on* ρ *satisfies : for all* $u_0 > 0$*, when* T *is large enough,*

$$\Pi(\|\rho\| > 1 - u_0(\log T)^{1/6}\varepsilon_T^{1/3}) \leq e^{-2c_1 T\varepsilon_T^2}. \tag{2.5}$$

Then, for any $w_T \to +\infty$*,*

$$\mathbb{E}_0\left[\Pi\left(\|f - f_0\|_1 > w_T\varepsilon_T|N\right)\right] = o(1). \tag{2.6}$$

Remark 2. *Condition (i) of Theorem 3 is in particular verified under the assumptions of Theorem 1, with* $\varepsilon_T = M\epsilon_T\sqrt{\log\log T}$ *for* M *a constant.*

Remark 3. *Compared to Theorem 1, we also assume (ii), i.e. that the prior distribution puts very little mass near the boundary of space* $\{f; \|\rho\| < 1\}$*. In particular, if under* Π*,* $\|\rho\|$ *has its support included in* $[0, 1-\epsilon]$ *for a fixed small* $\epsilon > 0$ *then* (2.5) *is verified.*

A consequence of previous theorems is that the posterior mean $\hat{f} = \mathbb{E}^{\pi}[f|N]$ is converging to f_0 at the rate ε_T, which is described by the following Corollary.

Corollary 1. *Under the assumptions of Theorem 1 or Theorem 2, together with* (2.5) *with* $\varepsilon_T = \sqrt{\log\log T}\epsilon_T$ *and if* $\int_{\mathcal{F}} \|f\|_1 d\Pi(f) < +\infty$*, then for any* $w_T \to +\infty$

$$\mathbb{P}_0\left(\|\hat{f} - f_0\|_1 > w_T\varepsilon_T\right) = o(1).$$

The proof of Corollary 1 is given in Section 4.6. We now illustrate these general results on specific prior models.

2.3. **Examples of prior models.** The advantage of Theorems 1 and 3 is that the conditions required on the priors on the functions $h_{k,\ell}$ are quite standard, in particular if the functions $h_{k,\ell}$ are parameterized in the following way

$$h_{k,\ell} = \rho_{k,\ell}\bar{h}_{k,\ell}, \qquad \int_0^A \bar{h}_{k,\ell}(u)du = 1$$

We thus consider priors on $\theta = (\nu_\ell, \rho_{k,\ell}, \bar{h}_{k,\ell}, k, \ell \leq K)$ following the scheme

$$\nu_\ell \overset{iid}{\sim} \Pi_\nu, \quad \rho = (\rho_{k,\ell})_{k,\ell\leq K} \sim \Pi_\rho, \quad \bar{h}_{k,\ell} \overset{iid}{\sim} \Pi_h \tag{2.7}$$

We consider Π_ν absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^+$ with positive and continuous density π_ν, Π_ρ a probability distribution on the set of matrices with positive entries and spectral radius $\|\rho\| < 1$, with positive density with respect to Lebesgue measures . We now concentrate on the nonparametric part, namely the prior distribution Π_h. Then from Theorems 1 and 3 it is enough that Π_h satisfies for each $k, \ell \leq K$,

$$\Pi_h\left(\|h - \bar{h}_{k,\ell}^0\|_2 \leq \epsilon_T, \quad \|h\|_\infty \leq B\right) \geq e^{-cT\epsilon_T^2},$$

for some $B > 0$ and such that there exists $\mathcal{F}_{1,T}$ with

$$\mathcal{F}_{1,T} \subset \left\{h : [0, A] \to \mathbb{R}^+, \int_0^A h(x)dx = 1\right\}$$

satisfying

$$\Pi_h\left(\mathcal{F}_{1,T}^c\right) = o(e^{-(cK+\kappa+1)T\epsilon_T^2}\epsilon_T^{K+K^2}), \quad N(\zeta\epsilon_T; \mathcal{F}_{1,T}; \|.\|_1) \leq x_0 T\epsilon_T^2,$$

for some $\zeta > 0$. In other words it is enough to have the bound $\Pi_h\left(\mathcal{F}_{1,T}^c\right) \le e^{-CT\epsilon_T^2}$ for some C large enough and to control the L_1 entropy of $\mathcal{F}_{1,T}$.

These conditions have been checked for a large selection of types of priors on the set of densities. We discuss here two cases: one based on random histograms, these priors make sense in particular in the context of modeling neuronal interactions and the second based on mixtures of Betas, because it leads to adaptive posterior concentration rates over a large collection of functional classes. To simplify the presentation we assume that $A = 1$ but generalization to any $A > 0$ is straightforward.

2.3.1. *Random histogram prior.* These priors are motivated by the neuronal application, where one is interested in characterizing time zones when neurons are or are not interracting. Random histograms have been studied quite a lot recently for density estimation, both in semi and non parametric problems. We consider two types of random histograms: regular partitions and random partitions histograms. The regular partition histogram prior is defined by

$$\bar{h}_{w,J}(x) = \delta\sum_{j=1}^{J} J w_j \mathbb{1}_{I_j}, \quad I_j = ((j-1)/J, j/J), \quad \sum_{j=1}^{J} w_j = 1, \quad \delta \sim \text{Be}(p) \tag{2.8}$$

Let for all J

$$\begin{aligned} J \sim \Pi_J, \quad e^{-c_1 x L_1(x)} \lesssim \Pi_J(J = x), \quad \Pi_J(J > x) \lesssim e^{-c_2 x L_1(x)}, \\ L_1(x) = 1 \text{ or } L_1(x) = \log x \\ (w_1, \dots, w_J)|J \sim \Pi_J \end{aligned} \tag{2.9}$$

We assume that the prior on $(w_1, \cdots, w_J)$ satisfies : for all $M > 0$, there exists $c > 0$ such that for all $\underline{w_0} \in \mathcal{S}_J$ with $w_{0j} \le M/J$

$$\Pi_w[\cap_j (w_{0j} - u/J^2, w_{0j} + u/J^2)] > e^{-cJ\log J}. \tag{2.10}$$

Many probability distributions on $\mathcal{S}_J$ satisfy (2.10). For instance if Π_J is the Dirichlet distribution $\mathcal{D}(\alpha_{1,J}, \cdots, \alpha_{J,J})$ with $c_1 J^{-a} \le \alpha_{i,J} \le c_2$, then (2.10) holds. Also, consider the following hierarchical prior allowing some the of w_j's to be equal to 0. Set

$$Z_j \overset{iid}{\sim} \text{Be}(p), \quad j \le J, \quad s_z = \sum_{j=1}^{J} Z_j$$

and $(j_1, \cdots, j_{s_z})$ the indices corresponding to $Z_j = 1$. Then

$$\begin{aligned} (w_{j_1}, \cdots w_{j_{s_z}}) \sim \mathcal{D}(\alpha_1, \cdots, \alpha_{s_z}), \quad c_1 J^{-a} \le \alpha_{i,J} \le c_2 \\ w_j = 0 \quad \text{if } Z_j = 0 \end{aligned}$$

From Remark 1, we use the version of assumption (i) based on $B_\infty(\epsilon_T, B) = \Big\{(\nu_k, (h_{\ell,k})_\ell)_k : \quad \max_k |\nu_k - \nu_k^0| \le \epsilon_T, \max_{\ell,k} \|h_{\ell,$
so that the posterior concentration rate in ϵ_T instead of $\epsilon_T\sqrt{\log\log T}$.

Then applying Lemma 4 of the supplementary material of Castillo and Rousseau (2015), we obtain for all $\bar{h}_0 \in \mathcal{H}(\beta, L_0)$ the set of β Hölder functions with constant L_0, $\beta \le 1$ and if $\bar{h}_0$ is not the null function

$$\Pi\left(\|\bar{h}_{w,J} - \bar{h}_0\|_\infty \le \epsilon_T + L_0 J^{-\beta} | J\right) \gtrsim p e^{-cJ\log T}$$

for some $c > 0$ and $\Pi_J(J = J_0\lfloor (T/\log T)\rfloor^{1/(2\beta+1)}) \gtrsim e^{-c_1 (T/\log T)^{1/(2\beta+1)} L_1(T)}$. If $h_0 = 0$ then

$$\Pi\left(\|h_{w,J} - h_0\|_\infty = 0\right) = 1 - p.$$

Moreover set $\mathcal{F}_{1,T} = \{h_{w,J}, J \le J_1 (T/\log T)^{1/(2\beta+1)}\}$, then for all $\zeta > 0$

$$N(\zeta\epsilon_T, \mathcal{F}_{1,T}, \|.\|_1) \lesssim J_1 (T/\log T)^{1/(2\beta+1)} \log T.$$

We finally obtain the following corollary

Corollary 2 (regular partition)**.** *Under the prior defined by* (2.8)*,* (2.9) *and* (2.10) *and if*

$$\Pi_\rho\left(\|\rho\| > 1 - u\right) \le e^{-c_0 T\sqrt{\log T} u^{1/3}}$$

for all u small enough, then if $\forall k, \ell\ h_{k,\ell}^0 \in \mathcal{H}(\beta, L)$ for some $\beta > 0$,

$$\mathbb{E}_0\Pi\left(\|f - f_0\|_1 > M(T/\log T)^{-\beta/(2\beta+1)} | N\right) = o(1).$$

To extend this result to the case of random partition histogram priors we parameterize the normalized interaction function as

$$\bar{h}_{w,t,J}(x) = \delta \sum_{j=1}^{J} \frac{w_j}{t_j - t_{j-1}} \mathbb{1}_{I_j}, \quad I_j = (t_{j-1}, t_j), \quad \sum_{j=1}^{J} w_j = 1, \quad \delta \sim \text{Be}(p) \tag{2.11}$$

and

$$T_0 = 0 < t_1 < \cdots < t_J = 1.$$

We consider the same prior on $(J, w_1, \cdots, w_J)$ as in (2.9) and the following condition on the prior on $\underline{t} = (t_1, \cdots, t_K)$. Writing $u_1 = t_1$, $u_j = t_j - t_{j-1}$, we have that $\underline{u} = (u_1, \cdots, u_J)$ belongs to the J-dimensional simplex $\mathcal{S}_J$ and we consider a Dirichlet distribution on $(u_1, \cdots, u_J)$, $\mathcal{D}(\alpha'_1, \cdots, \alpha'_J)$ with $0 < c_1 \leq \alpha'_j \leq c_2$.

The main difference with the regular partition - random histogram prior is the control of the L_1 entropy. Let $J \leq J_1(T/\log T)^{1/(2\beta+1)}$ and $(\underline{w}, \underline{u})$ and $(\underline{w'}, \underline{u'})$ belonging to $\mathcal{S}_J^2$. Then if $\delta = \delta' = 1$, $|t'_j - t_j| \leq \epsilon_T \min(|t_j - t_{j-1}|, |t_j - t_{j+1}|)$ for all j and $\sum_j |w_j - w'_j| \leq \epsilon_T$ then

$$\begin{aligned} \|h_{w,J} - h_{w',J'}\|_1 &\leq \|h_{w,J} - h_{w',J}\|_1 + \|h_{w',J} - h_{w',J'}\|_1 \\ &\leq \sum_{j=1}^{J} |w_j - w'_j| + \sum_{j=1}^{J} \epsilon_T (w'_j + w'_{j+1}) \leq 3\epsilon_T \end{aligned}$$

We thus need to compute the covering number of $\{(\underline{w}, \underline{u}) \in \mathcal{S}_J^2;\ \min_j u_j \geq \delta_T\}$ by balls of radius ϵ under the pseudo norm

$$\ell((\underline{w}, \underline{u}), (\underline{w'}, \underline{u'})) = \|w - w'\| + \sum_j \frac{|u_j - u'_j|}{u_j \wedge u_{j+1}}$$

Let $\mathcal{U}_{J,T} = \{\underline{u} \in \mathcal{S}_J, \min_j u_j \geq \delta_T\}$, under the Dirichlet prior on $\underline{u}$

$$\Pi_u(\mathcal{U}_{J,T}^c | J) \lesssim J \delta_T^{c_1} = o(e^{-Tc\epsilon_T^2})$$

if $\log \delta_T \leq -(c/c_1 + 1)T\epsilon_T^2$. The covering number is then bounded by the covering number associated with $\underline{w}$ times $N_{u,J}(\zeta\epsilon_T/2)$, where $N_{u,J}$ is the number of points $\underline{u}^s \in \mathcal{U}_{J,T}$ such that for all $\underline{u} \in \mathcal{U}_{J,T}$ there exists a $\underline{u}^s$ satisfying $|u_j^s - u_j| \leq \zeta\epsilon_T/2(|u_j^s| \wedge |u_{j+1}^s|)$. It can be shown that

$$N_{u,J} \leq \left(\frac{C\log(1/\delta_T)}{\zeta\epsilon_T}\right)^J \delta_T^{-1} \lesssim \exp\left(B[J\log T + T\epsilon_T^2]\right),$$

for some $B > 0$. Hence setting $\mathcal{F}_{1,T} = \{h_{w,t,J},\ w \in \mathcal{S}_J, \min_j |t_{j-1} - t_j| > \delta_T, J \leq J_1(T/\log T)^{1/(2\beta+1)}\}$ then

$$N(\zeta\epsilon_T, \mathcal{F}_{1,T}, \|.\|_1) \lesssim T^{1/(2\beta+1)}(\log T)^{2\beta/(2\beta+1)}.$$

We then have the following corollary

Corollary 3. *Under the prior* (2.11)*, if the prior on* $\underline{w}$ *satisfies* (2.10)*, under a Dirichlet prior on* $\underline{u} = (t_j - t_{j-1}, j \leq J)$*, and if*

$$\Pi_\rho(\|\rho\| > 1 - u) \leq e^{-c_0 T\sqrt{\log T} u^{1/3}}$$

for all u *small enough, then if* $\forall k, \ell$ $h_{k,\ell}^0 \in \mathcal{H}(\beta, L)$ *for some* $1 \geq \beta > 0$,

$$\mathbb{E}_0 \Pi\left(\|f - f_0\|_1 > M(T/\log T)^{-\beta/(2\beta+1)} | N\right) = o(1).$$

In the following section we consider another family of priors suited for smooth functions $h_{k,\ell}$ and based on mixtures of Beta distributions.

2.3.2. *Mixtures of Betas.* The following family of prior distributions is inspired by Rousseau (2010). Consider functions

$$h_{k,\ell} = \rho_{k,\ell}\left(\int_0^1 g_{\alpha_{k,\ell},\epsilon} dM_{k,\ell}(\epsilon)\right)_+, \quad g_{\alpha,\epsilon}(x) = \frac{\Gamma(\alpha/(\epsilon(1-\epsilon)))}{\Gamma(\alpha/\epsilon)\Gamma(\alpha/(1-\epsilon))} x^{\frac{\alpha}{1-\epsilon}-1}(1-x)^{\frac{\alpha}{\epsilon}-1}$$

where $M_{k,\ell}$ are bounded signed measures on $[0,1]$ such that $|M_{k,\ell}| = 1$. In other words the above functions are the positive parts of mixtures of Betas distributions with parameterization $(\alpha/\epsilon, \alpha/(1-\epsilon))$ so that ϵ is the mean parameter. The mixing random measures $M_{k,\ell}$ are allowed to be negative. The reason for allowing $M_{k,\ell}$ to be negative is that $h_{k,\ell}$ is then allowed to be null on sets with positive Lebesgue measure. The prior is then constructed in the following way, writing $h_{k,\ell} = \rho_{k,\ell}\tilde{h}_{k,\ell}$ we define a prior on $\tilde{h}_{k,\ell}$ via a prior on $M_{k,\ell}$ and on $\alpha_{k,\ell}$. In particular we assume that $M_{k,\ell} \overset{iid}{\sim} \Pi_M$ and $\alpha_{k,\ell} \overset{iid}{\sim} \pi_\alpha$. As in Rousseau (2010) we consider a prior on α absolutely continuous with respect to Lebesgue measure and with density satisfying: there exists $b_1, c_1, c_2, c_3, A, C > 0$ such that for all u large enough,

$$\begin{aligned} \pi_\alpha(c_1 u < \alpha < c_2 u) &\geq C e^{-b_1 u^{1/2}} \\ \pi_\alpha(\alpha < e^{-Au}) + \pi_\alpha(\alpha > c_3 u) &\leq C e^{-b_1 u^{1/2}}. \end{aligned} \tag{2.12}$$

There are many ways to construct discrete signed measures on $[0, 1]$, for instance, writing

$$M = \sum_{j=1}^{J} e_j p_j \delta_{\epsilon_j}, \tag{2.13}$$

the prior on M is then defined by $J \sim \Pi_J$ and conditionally on J,

$$e_j \overset{iid}{\sim} \text{Ra}(1/2), \quad \epsilon_j \overset{iid}{\sim} G_\epsilon, \quad (p_1, \cdots, p_J) \sim \mathcal{D}(a_1, \cdots, a_J),$$

where Ra denotes the Rademacher distribution taking values $\{-1, 1\}$ each with probability $1/2$. Assume that G_ϵ has positive continuous density on $[0, 1]$ and that there exists $A_0 > 0$ such that $\sum_{j=1}^{J} a_j \leq A_0$. We have the following proposition

Proposition 1. *Consider a prior as described above. Assume that for all $k, \ell \leq K$ $h^0_{k,\ell} = (g^0_{k,\ell})_+$ for some functions $g^0_{k,\ell} \in \mathcal{H}(\beta, L_0)$ with $\beta > 0$. If*

$$\Pi_\rho\left(\|\rho\| > 1 - u\right) \leq e^{-c_0 T \sqrt{\log T} \log\log T u^{1/3}}$$

for all u small enough, and if G_ϵ has density with respect to Lebesgue measure verifying

$$x^{A_1}(1-x)^{A_1} \lesssim g_\epsilon(x) \lesssim x^3(1-x)^3, \quad \textit{for some} \quad A_1 \geq 3,$$

then there exists $M_0 > 0$ such that

$$\mathbb{E}_0 \Pi(\|f - f_0\|_1 > M_0 T^{-\beta/(2\beta+1)} (\log T)^{5\beta/(4\beta+2)} \log\log T | \mathbf{N}) = o(1).$$

Note that in the context of density estimation, $T^{-\beta/(2\beta+1)}$ is the minimax rate and we expect that it is the same for Hawkes processes.

Proof of Proposition 1. The proof is based on Rousseau (2010), where mixtures of Beta densities are studied for density estimation, and using Theorem 2. Note that for all h_1, h_2

$$|(h_1(x))_+ - (h_2(x))_+| \leq |h_1(x) - h_2(x)|$$

so that Proposition 1 is proved by studying

$$\tilde{B}(\epsilon_T, B) = \{\tilde{f} = (\nu_\ell, g_{k,\ell}, k, \ell \leq K); \sum_\ell |\nu_\ell - \nu^0_\ell| + \sum_{\ell, k \leq K} \|g_{k,\ell} - g^0_{k,\ell}\|_2 \leq \epsilon_T; \|g_{k,\ell}\|_\infty \leq B\}$$

in the place of $B(\epsilon_T, B)$ and by controling the L_1 entropy associated to

$$\mathcal{G}_{1,T} = \{g_{\alpha,P}; P = \sum_{j=1}^{k} p_j \delta_{(\epsilon_j)},\ \epsilon_j \in [e_0, 1 - e_0];\ \alpha \in [\alpha_{0T}, \alpha_{1T}];\ \sum_j |p_j| = 1,\ k \leq k_{1,T}\}$$

where

$$e_0 = e^{-a_0 T \epsilon_T^2}, \quad \alpha_{0T} = \exp\left(-T c_0 \epsilon_T^2\right); \quad \alpha_{1T} = \alpha_1 T^2 \epsilon_T^4, \quad c_0, \alpha_1, t_0 > 0, k_{1,T} = k_1 T^{1/(2\beta+1)} (\log T)^{(\beta-2)/(4\beta+2)}$$

From the proof of Theorem 2.1 in Rousseau (2010), we have that for all $c_2 > 0$ we can choose $a_0, c_0, \alpha_1 > 0$ such that $\Pi_g\left(\mathcal{G}^c_{1,T}\right) \leq e^{-c_2 T \epsilon_T^2}$ and $\mathcal{G}_{1,T}$ can be cut into the following slices : we group the components into the intervals $[e_j, e_{j+1}]$ or $[1 - e_{j+1}, 1 - e_j]$ with $e_j = e_0^{1/j}$ and $e_{J_T} = n^{-t}$, for some $t > 0$, and the interval $[e_{J_T}, 1 - e_{J_T}]$. For each of these intervals we denote $N(j)$ the number of components which fall into the said interval, $N(j) = \sum_{i=1}^{k} \mathbb{1}_{\epsilon_i \in (e_j, e_{j+1})}$. Let $\mathcal{G}_{1,\sigma} = \{g_{\alpha,P} \in \mathcal{G}_{1,T};\ N(j) = k_j,\ \sum_{j=1}^{2J_T+1} k_j = k \leq k_{1,T}\}$ with σ denoting the configuration $(k_1, \cdots, k_{2J_T+1})$. From Rousseau (2010) for all $\zeta > 0$, we have

$$\sum_\sigma \sqrt{\Pi(G_{1,\sigma})} N(\zeta\epsilon_T, \mathcal{G}_{1,\sigma}) \lesssim e^{Ck_{1,T}\log n} \max_\sigma e_0^{\sum_j k_j[(T+1)/(2j+2) - 1/j]} \lesssim e^{Ck_{1,T}\log n}$$

for some constant $C > 0$ so that choosing x_0 large enough

$$\sum_\sigma \sqrt{\Pi(G_{1,\sigma})} N(\zeta\epsilon_T, \mathcal{G}_{1,\sigma}) e^{-x_0 T \epsilon_T^2} = o(1).$$

We now study the Kullback-Leibler condition (i). Again, we use Theorem 3.1 in Rousseau (2010), so that for all $f_0 \in \mathcal{H}(\beta, L)$ and all $\beta > 0$ there exists f_1 such that $\|f_0 - g_{\alpha,f_1}\|_\infty \leq \alpha^{-\beta/2}$, where f_1 is either equal to f_0 if $\beta \leq 2$ or $f_1 = f_0 \sum_{j=1}^{\lceil\beta\rceil-1} w_j/\alpha^{j/2}$, with w_j a polynomial function with coefficient depending on $f_0^{(l)}$ $l \leq j$. From that we construct a finite mixture approximation of g_{α,f_1}. Note that even if f_0 is positive, f_1 is not necessarily so. Hence to use the convexity argument of Lemma A1 of Ghosal and van der Vaart (2001) we write f_1 as $m_+ f_{1,+} - m_- f_{1,-}$ with $f_{1,+}, f_{1,-} \geq 0$ and probability densities. In the case where $m_- = 0$ then $f_{1,-} = 0$. We approximate $g_{\alpha,f_{1,+}}$ and $g_{\alpha,f_{1,-}}$ separately. Contrarywise to what happens in Rousseau (2010), here we want to allow f_0 to be null in some sub-intervals of $[0,1]$. Hence we adapt the proof of Theorem 3.2 of Rousseau (2010) to this set up. Let f be a probability density on $[0,1]$ we construct a discrete approximation of $g_{\alpha,f}$. Let $\epsilon_0 = \alpha^{-H_0}$ for some $H_0 > 0$ and define $\epsilon_j = \epsilon_0(1 + B\sqrt{\log\alpha/\alpha})^j$ for $j = 1, \cdots, J_\alpha$ with $J_\alpha = O(\sqrt{\alpha \log \alpha})$. We then have, from Lemma 1 below that there exists a signed measure P_0 with at most $N = O(\sqrt{\alpha}(\log\alpha)^{3/2})$ supporting points on $[\epsilon_1, 1-\epsilon_1]$, such that

$$\|g_{\alpha,P_0} - f_0\|_2 \lesssim \alpha^{-\beta/2}; \quad \|g_{\alpha,P_0}\|_\infty \leq \|f_0\|_\infty + o(1), \quad P_0 = \sum_{i=1}^N p_i \delta_{(\epsilon_i)}.$$

Also, as in Rousseau (2010), we can assume that $|p_i| \geq \alpha^{-A}$ for some fixed A large enough and all P satisfying $\max_i |P(U_i) - p_i| \leq \alpha^{-A'}|p_i|$, with $U_i = [\epsilon_i(1-\epsilon_i)(1-\alpha^{-A'}), \epsilon_i(1-\epsilon_i)(1+\alpha^{-A'})]$ then

$$\|g_{\alpha,P_0} - g_{\alpha,P}\|_2 \leq \alpha^{-\beta/2}, \quad \|g_{\alpha,P}\|_\infty \leq \|f_0\|_\infty + o(1)$$

if A' is chosen large enough. As in Rousseau (2010), if $\epsilon_T = \epsilon_0 T^{-\beta/(2\beta+1)}(\log T)^{5\beta/(4\beta+2)}$, then

$$\Pi\left(B(\epsilon_T, \|f_0\|_\infty + 1)\right) \geq e^{-c_1 T \epsilon_T^2}$$

for some $c_1 > 0$, which terminates the proof of Proposition 1.

□

Lemma 1. *Assume that f is a bounded probability density on $[0,1]$, then for all $B_0 > 0$ there exists $\tilde{N}_0 > 0$ and a signed measure P_0 with at most $N \leq \tilde{N}_0\sqrt{\alpha}(\log\alpha)^{3/2}$ on $[\epsilon_1, 1-\epsilon_1]$ such that*

$$\|g_{\alpha,f} - g_{\alpha,P}\|_2 \lesssim \alpha^{-B_0}, \quad \|g_{\alpha,P_0}\|_\infty \lesssim \|f_0\|_\infty + o(1)$$

Proof of Lemma 1. On each of the intervals $(\epsilon_{j-1}, \epsilon_j)$ we construct a probability P_j having support on $(\epsilon_{j-1}, \epsilon_j)$ with cardinality smaller than $N_j \leq N_0 \log\alpha$ and such that

$$\|g_{\alpha,f_j} - g_{\alpha,P_j}\|_2^2 \lesssim \alpha^{-B_0}, \quad f_j = \frac{f \mathbb{1}_{(\epsilon_{j-1},\epsilon_j)}}{\int_{\epsilon_{j-1}}^{\epsilon_j} f(\epsilon)d\epsilon} \tag{2.14}$$

where B_0 can be chosen arbitrarily large by choosing N_0 large enough. To prove (2.14) we use the same ideas as in the proof of Theorem 3.2 of Rousseau (2010). For all For all $j = 2, \cdots, J-2$ on $(\epsilon_{j-1}, \epsilon_j)$, there exists P_j with at most $N_1 \log\alpha$ terms such that if $x \in [0,1]$,

$$\left|g_{\alpha,f_j} - g_{\alpha,P_j}\right|(x) \leq \frac{\alpha^{-H}}{x(1-x)}$$

where H can be chosen as large as need be, by choosing N_1 large enough. Moreover, let $x \leq \epsilon_0$ or $x > 1-\epsilon_0$, then for all $\epsilon \in (\epsilon_1, 1-\epsilon_1)$, if $x < \epsilon_0$ then $x/\epsilon \leq \delta_\alpha = (1 + B\sqrt{\log\alpha/\alpha})^{-1}$ and

$$g_{\alpha,\epsilon}(x) \lesssim \sqrt{\alpha} \exp\left(\alpha\left[\frac{\log(x/\epsilon)}{1-\epsilon} - (\log x)/\alpha + \frac{\log((1-x)/(1-\epsilon))}{\epsilon}\right]\right)$$

If $\epsilon_1 \leq \epsilon < 1/4$ then the function $\epsilon \to \frac{\log(\epsilon/x)}{1-\epsilon} - \log(\epsilon/x)/\alpha + \frac{\log((1-\epsilon)/(1-x))}{\epsilon}$ is increasing and

$$\begin{aligned} g_{\alpha,\epsilon}(x) &\lesssim \frac{\sqrt{\alpha}}{\epsilon} \exp\left(\alpha\left[\log(\delta_\alpha)\left(1 + x\delta_\alpha + \delta_\alpha^2 x^2\right) + O(x^3)\right) - 1 + \delta_\alpha^{-1}\right]\right) \\ &\lesssim \alpha^{-B^2/3+H_0} \lesssim \alpha^{-B^2/4}, \end{aligned}$$

by choosing $B^2 \geq 12H_0$. The same reasoning can be applied to $x > 1-\epsilon_0$, which terminates the proof. □

3. Numerical illustration in the neurosciences context

Hawkes processes are used in neurosciences to model action potentials trains of neurons. In a few words, neurons communicate through sequences of action potentials. Contemporary models assume that the information is conveyed by the action potentials' times of occurrence rather than by the action potentials' waveforms. The series of occurrences times are assumed to be the realization of a non homogeneous point process. The multivariate Hawkes processes allow to take into account the dependences/interactions between neurons (namely excitation or inhibition) (Hansen et al., 2015). In this section, we conduct a simulation study, choosing the parameters so that the simulated data mimic action potentials trains.

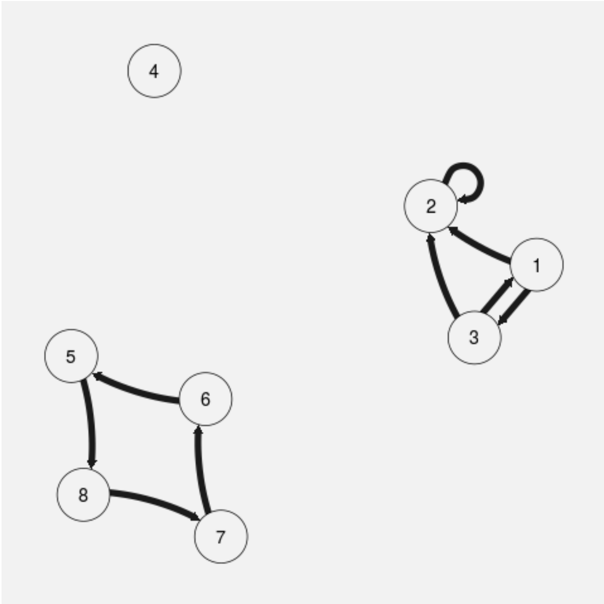

FIGURE 1. **Scenario 2**. True interaction graph between the $K = 8$ neurones. A directed edge is plotted from vertex ℓ to vertex k if the interaction functions $h_{k,\ell}$ is non-null.

3.1. **Simulation scenarios.** We consider three simulation scenarios involving respectively $K = 2$ and $K = 8$ neurons. The scenarios are roughly similar to the one tested in Hansen et al. (2015). Following the notations introduced in the previous sections, for any $(k, \ell) \in \{1, \dots K\}^2$, $h_{k,\ell}$ denotes the interaction function of neuron ℓ over neuron k. We now describe the three scenarii. The upper bound of each $h_{k,\ell}$'s support, denoted $[0, A]$ is set equal to $A = 0.04$ seconds.

- *Scenario 1*: We first consider $K = 2$ neurons and piecewise constant interactions:
$$h_{1,1} = 30 \cdot \mathbb{1}_{(0,0.02]}, \quad h_{2,1} = 30 \cdot \mathbb{1}_{(0,0.01]}, \quad h_{1,2} = 30 \cdot \mathbb{1}_{(0.01,0.02]}, \quad h_{2,2} = 0.$$
- *Scenario 2*: In this scenario, we mimic $K = 8$ neurons belonging to three independent groups. The non-null interactions are the piecewise constant functions defined as:
$$h_{2,1} = h_{3,1} = h_{2,2} = h_{1,3} = h_{2,3} = h_{8,5} = h_{5,6} = h_{6,7} = h_{7,8} = 30 \cdot \mathbb{1}_{(0,0.02]}.$$
In Figure 1, we plot the subsequent interactions directed graph between the 8 neurons: the vertices represent the K neurons and an oriented edge is plotted from vertex ℓ to vertex k if the interaction function $h_{k,\ell}$ is non-null.
- *Scenario 3*: Setting $K = 2$, we consider non piecewise constant interactions functions defined as:
$$\begin{array}{rclrcl} h_{1,1}(t) & = & 100 \cdot e^{-100t} \mathbb{1}_{(0,0.04]}(t), & h_{2,1}(t) & = & 30 \cdot \mathbb{1}_{(0,0.02]}(t) \\ h_{1,2}(t) & = & \frac{1}{2 \times 0.004\sqrt{2\pi}} e^{-\frac{(t-0.02)^2}{2 \times 0.004^2}} \cdot \mathbb{1}_{(0,0.04]}(t), & h_{2,2}(t) & = & 0 \end{array}$$

In all the scenarios, we consider $\nu_\ell = 20, \forall k = 1 \dots K$.
For each scenario, we simulate 25 datasets on the time interval $[0, 22]$ seconds. The Bayesian inference is performed considering recordings on three possible periods of length $T = 5$ seconds, $T = 10$ seconds and $T = 20$ seconds. For any dataset, we remove the initial period of 2 seconds –corresponding to 50 times the length of the support of the $h_{k,\ell}$– assuming that, after this period, the Hawkes processes have reached their stationary distribution.

3.2. **Prior distribution on** $f = (\nu_\ell, h_{k,\ell})_{l,k \in \{1,\dots,K\}}$**.** We use the prior distribution in $f = (\nu_\ell, h_{k,\ell})_{l,k \in \{1,\dots,K\}}$ described in Section 2.3 setting a logÃ¨prior distribution on ν of parameter μ_ν, s_ν^2. About the interaction functions $(h_{k,\ell})_{k,\ell \in \{1,\dots,K\}}$, the prior distribution is defined on the set of piecewise constant functions, $h_{k,\ell}$ being written as follows:

$$h_{k,\ell}(t) = \delta^{(k,\ell)} \sum_{j=1}^{J^{(k,\ell)}} \beta_j^{(k,\ell)} \mathbb{1}_{[t_{j-1}^{(k,\ell)}, t_j^{(k,\ell)}]}(t) \tag{3.1}$$

with $t_0^{(k,\ell)} = 0$ and $t_{J^{(k,\ell)}}^{(k,\ell)} = A$. Using the notations in Section 2.3, we have $\beta_j^{(k,\ell)} = \rho^{(k,\ell)}\omega_j^{(k,\ell)}$. $\delta^{(k,\ell)}$ is a global parameter of nullity for $h_{k,\ell}$: for all $(k,\ell) \in \{1,\dots,K\}^2$,

$$\delta^{(k,\ell)} \sim_{i.i.d} \mathcal{B}ern(p). \tag{3.2}$$

For all $(k,\ell) \in \{1,\dots,K\}^2$, the number of steps $(J^{(k,\ell)})$ follows a translated Poisson prior distribution:

$$J^{(k,\ell)}|\{\delta^{(k,\ell)} = 1\} \sim_{i.i.d.} 1 + \mathcal{P}(\lambda). \tag{3.3}$$

To minimize the influence of λ on the posterior distribution, we consider an hyperprior distribution on the hyperparameter λ:

$$\lambda \sim \Gamma(a_\lambda, b_\lambda). \tag{3.4}$$

Given $J^{(k,\ell)}$, we consider a spike and slab prior distribution on $(\beta_j^{(k,\ell)})_{j=1\dots J^{(k,\ell)}}$. Let $Z_j^{(k,\ell)} \in \{0,1\}$ denote a sign indicator for each step, we set: $\forall j \in \{1,\dots,J^{(k,\ell)}\}$:

$$\begin{array}{rcl} \mathbb{P}\left(Z_j^{(k,\ell)} = z|\delta^{(k,\ell)} = 1\right) & = & \pi_z, \quad \forall z \in \{0,1\} \\ \beta_j^{(k,\ell)}|\{\delta^{(k,\ell)} = 1\} & \sim & Z_j^{(k,\ell)} \times \log\mathcal{N}(\mu_\beta, s_\beta^2) \end{array} \tag{3.5}$$

We consider two prior distributions on $(t_j^{(k,\ell)})_{j=1\dots J^{(k,\ell)}}$. The first one (refered as the Regular histogram prior) is a regular partition of $[0,A]$:

$$t_j^{(k,\ell)} = \frac{j}{J^{(k,\ell)}}A \qquad \forall j = 0,\dots,J^{(k,\ell)}. \tag{3.6}$$

The second prior distribution is refered as random histogram prior and specifies:

$$\begin{array}{rcl} (u_1,\dots,u_{J^{(k,\ell)}}) & \sim & \mathcal{D}(\alpha'_1,\dots\alpha'_{J^{(k,\ell)}}) \\ t_0^{(k,\ell)} & = & 0 \\ t_j^{(k,\ell)} & = & A\sum_{r=1}^{j} u_r, \quad \forall j = 1,\dots,J^{(k,\ell)} \end{array} \tag{3.7}$$

In the simulations studies, we set the following hyperparameters:

$$\begin{array}{rclrcl} \mu_\beta & = & 3.5, & s_\beta & = & 1 \\ \mu_\nu & = & 3.5, & s_\nu & = & 1 \\ \mathbb{P}(Z_j^{(k,\ell)} = 1) & = & 1/2, & \mathbb{P}(\delta^{(k,\ell)} = 1) = p & = & 1/2 \\ \alpha'_j & = & 2, \quad \forall j & & & \end{array}$$

3.3. **Posterior sampling.** The posterior distribution is sampled using a standard Reversible-jump Markov chain Monte Carlo. Considering the current parameter $(\boldsymbol{\nu},\boldsymbol{h})$, $\boldsymbol{\nu}^{(c)}$ is proposed using a Metropolis-adjusted Langevin proposalFor a fixed $J^{(k,\ell)}$, the heights $\beta_j^{(k,\ell)}$ are proposed using a random walk proposing null or non-null candidates. Changes in the number of steps $J^{(k,\ell)}$ are proposed by standard birth and death moves (Green, 1995). In this simulation study, we generate chains of length 30000 removing the first 10000 burn-in iterations. The algorithm is implemented in R on an Intel(R) Xeon(R) CPU E5-1650 v3 @ 3.50GHz.

The computation times (mean over the 25 datasets) are given in Table 1. First note that the computation time increases roughly as a linear function of T. This is due to the fact that the heavier task in the algorithm is the integration of the conditional likelihood and the computation time of this operation is roughly a linear function of the length of the integration (observation) time interval. Besides, because we implemented a Reversible Jumps algorithm, the computation time is a stochastic quantity: the algorithm can explore parts of the domain where the number of bins $J_{\ell k}$ is large, thus increasing the computation time. This point can explain the unexpected computation times for $K = 2$. Moreover, we remark that the computation time explodes as K increases (due to the fact that K^2 intensity functions have to be estimated), reaching computation times greater than a day.

	K=2		K=8	K=2 with smooth $h_{k,\ell}$
Prior on t	Regular	Random	Regular	Random
T=5	1508.44	1002.45		823.84
T=10	1383.72	1459.55	37225.19	1284.93
T=20	2529.19	2602.48	49580.18	1897.17

TABLE 1. Mean computation time (in seconds) of the MCMC algorithms as a function of the scenario, the observation time interval and the prior distribution on s. The mean is computed over the 25 simulated datasets

3.4. **Results.** We describe here the results for each scenario. We first present the L_1 distances on λ^k and $h_{k,\ell}$ for all 3 the scenarios, all three length observation time T and the two prior distributions. In Table 2, we show the estimated L_1 distances on λ^k and $h_{k,\ell}$. More precisely, we evaluate the L^1 distances on the interactions functions

$$D^{(1)} = \frac{1}{25} \sum_{sim=1}^{25} \widehat{\mathbb{E}} \left[\frac{1}{K^2} \sum_{k,\ell=1}^{K} \left\| h_{k,\ell} - h^0_{k,\ell} \right\|_1 \middle| (N_t^{sim})_{t\in[0,T]} \right]$$

and the following stocahstic distance :

$$D^{(2)} = \frac{1}{25} \sum_{sim=1}^{25} \widehat{\mathbb{E}} \left[d_{1,T}(f, f^0)) \middle| (N_t^{sim})_{t\in[0,T]} \right]$$

where f^0 is the true set of parameters, $d_{1,T}(f, f^0)$ has been defined in subsection 1.4 and the posterior expectations are approximated by Monte Carlo method using the outputs of the Reversible Jumps algorithm.

As expected, the error decreases as T increases. As we will detail later, the random histogram prior on s gives better results that the regular prior. Finally, we perform better when the true interaction function $(h_{k,\ell})$ are step functions (due to the form of the prior distribution).

		K=2		K=8	K=2 with smooth $h_{k,\ell}$
	Prior	Regular	random	Regular	random
	T=5	11.59	9.59		11.75
$D^(1)$: stochastic distances	T=10	7.49	6.32	5.65	9.48
	T=20	5.40	4.11	3.17	7.9
	T=5	0.1423	0.0996		0.1431
$D^(2)$: distances on $h_{k,\ell}$	T=10	0.0844	0.0578	0.1199	0.1131
	T=20	0.0564	0.0336	0.0616	0.0945

TABLE 2. L1 distances on $h_{k,\ell}$ and λ^k

3.4.1. *Results for scenario 1 : $K = 2$ with step functions.* When $K = 2$, we estimate the parameters using both *regular* and *random* prior distributions on $(t_j^{(k,\ell)})$ (equations (3.6) and (3.7)). One typical posterior distribution of ν^k is given in Figure 2 (left), for a randomly chosen dataset, clearly showing a smaller variance when the length of the observation interval increases. We also present the global estimation results, over the 25 simulated datasets. The distribution of the posterior mean estimators for (ν_1, ν_2) computed for the 25 simulated datasets $\left(\widehat{\mathbb{E}}\left[\nu_\ell|(N_t^{sim})_{t\in[0,T]}\right]\right)_{sim=1\dots 25}$ is given in Figure 2 on the right panel, showing an expected decreasing variance for the estimator as T increases. On the top panels the posterior is based on the regular grid prior while on the bottom the posterior is based on the random (grid) histogram prior: the results are equivalent.

About the estimation of the interaction functions, for the same given dataset, the estimation of the $h_{k,\ell}$ is plotted in Figure 3 (upper panel) for the regular prior, with its credible interval. Its corresponding estimation with the random prior is given in Figure 3 (bottom panel). For, both prior distributions, the functions are globally well estimated, showing a clear concentration when T increases. The regions where the interaction functions are null are also well identified. The estimation given with the random histogram prior is in general better than the one supplied by the regular prior. This may be due to several factors. First the random histogram prior leads to a sparser estimation than the regular one. Secondly, it is easier to design a proposal move in the Reversible Jump algorithm in the former case than in the latter context.
Moreover, the interaction graph is perfectly inferred since the posterior probability for $\delta^{(2,2)}$ to be 0 is almost 1. For the 25 dataset, we estimate the posterior probabilities $\widehat{\mathbb{P}}(\delta^{(k,\ell)} = 1|(N_t^{sim})_{t\in[0,T]})$ for $k, \ell = 1, 2$ and $sim = 1 \dots 25$. In Table 3, we display the mean of these posterior quantities. Even for the shorter observation time interval ($T = 5$) these quantities –defining completely the connexion graph– are well recovered. These results are improved when T increases. Once again, the random histogram prior (3.7) gives better results.

Finally, we also have a look at the conditional intensities $\lambda_k^*(t) = \nu_\ell + \sum_{\ell=1}^K \int_{-\infty}^{t-} h_{\ell,k}(t-u)dN_u^{(\ell)}$. On Figure 4, we plot 50 realizations of the conditional intensity from the posterior distributions. More precisely, for one given dataset, for 50 parameters $\theta^{(i)} = \left((h_{k,\ell}^{(i)})_{k,\ell}, (\nu_k^{(i)})_{k=1\dots K}\right)$ sampled from the posterior distribution (obtained at the end of the MCMC chain), we compute the corresponding $(\lambda_k^{*(i)}(t))$ and plot them. For the sake of clarity, only the conditional intensity of the first process ($k = 1$) is plotted and we restrict the graph to a short time

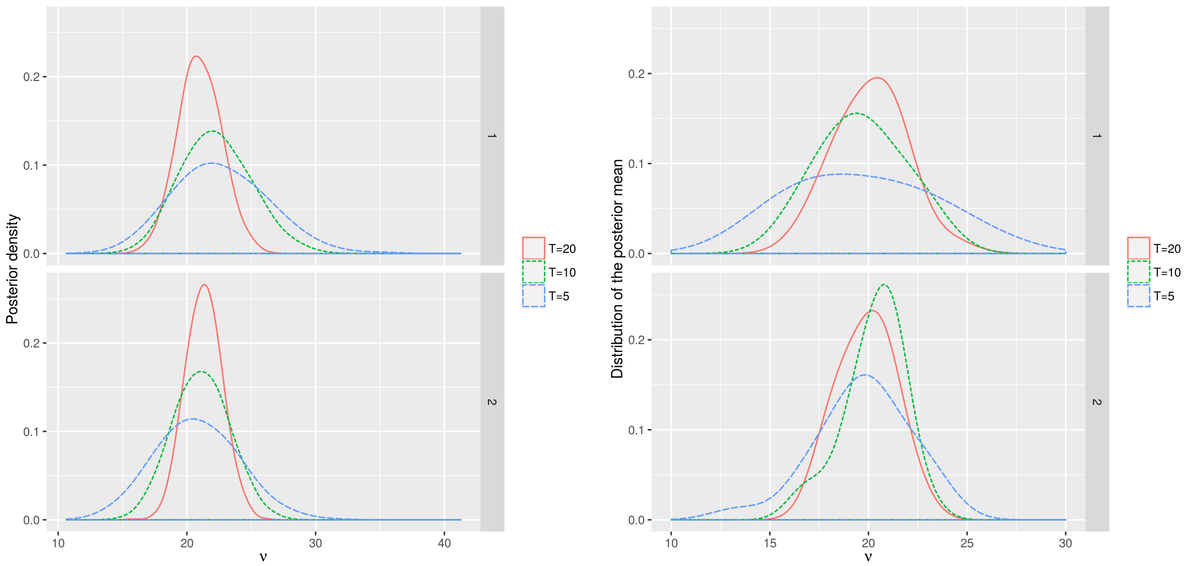

FIGURE 2. **Results for scenario 1**. *On the left*, posterior distribution of (ν_1, ν_2) with $T = 5$, $T = 10$ and $T = 20$ for one dataset. *On the right*, distribution of the posterior mean of (ν_1, ν_2) $\left(\widehat{\mathbb{E}}\left[\nu_\ell | (N_t^{sim})_{t\in[0,T]}\right]\right)_{sim=1\dots25}$ over the 25 simulated datasets. Top : regular histogram; bottom random histogram

ℓ over k		1 over 1	1 over 2	2 over 1	2 over 2
True value of $\delta^{(k,\ell)}$		1	1	1	0
	Prior				
$T = 5$	Regular	1.0000	0.8970	1.0000	0.0071
	Continous	1.0000	0.9812	1.0000	0.0196
$T = 10$	Regular	1.0000	0.9954	1.0000	0.0047
	Continous	1.0000	1.0000	1.0000	0.0102
$T = 20$	Regular	1.0000	1.0000	1.0000	0.0099
	random	1.0000	1.0000	1.0000	0.0102

TABLE 3. **Scenario 1, K=2**. Mean of the posterior estimations: $\frac{1}{25}\sum_{sim=1}^{25}\widehat{\mathbb{P}}(\delta^{(k,\ell)} = 1|(N_t^{sim})_{t\in[0,T]})$, for the three observation time intervals and the two prior distributions.

interval $[3.2, 3.6]$. As noticed before, the conditional intensity is well reconstructed, with a clear improvement of the precision as the length of the observation time T increases.

3.4.2. *Results for scenario 2:* $K = 8$. In this scenario, we perform the Bayesian inference using only the regular prior distribution on $(\mathbf{t}^{(k,\ell)})_{(k,\ell)\in\{1,\dots,K\}^2}$ and two lengths of observation interval ($T = 10$ and $T = 20$). Here we set $a_\lambda = 3$ and $b_\lambda = 1$.

The posterior distribution of the $(\nu_k)_{k=1\dots K}$ for a randomly chosen dataset is plotted in Figure 5. The prior distribution is in dotted line and is flat. The posterior distribution concentrates around the true value (here 20) with a smaller variance when T increases.

In the context of neurosciences, we are especially interested in recovering the interaction graph of the $K = 8$ neurons. In Figure 6, we consider the same dataset as the one used in Figure 5 and plot the posterior estimation of the interaction graph, for respectively $T = 10$ on the left and $T = 20$ on the right. The width and the gray level of the edges are proportional to the estimated posterior probability $\widehat{\mathbb{P}}(\delta^{(k,\ell)} = 1|(N_t)_{t\in[0,T]})$. The global structure of the graph is recovered (to be compared to the true graph plotted in Figure 1). We observe that the false positive edges appearing when $T = 10$ disappear when $T = 20$. In Figure 7, we consider the mean of the estimates of the graph over the 25 datasets. The resulting graph for $T = 10$ is on the left and for $T = 20$ on the right.
Note that, in this example, for any (k, ℓ) such that the true $\delta^{(k,\ell)} = 1$, the estimated posterior probability $\widehat{\mathbb{P}}(\delta^{(k,\ell)} = 1|(N_t^{sim})_{t\in[0,T]})$ is equal to 1, for any dataset and any length of observation interval. In other words, the non-null

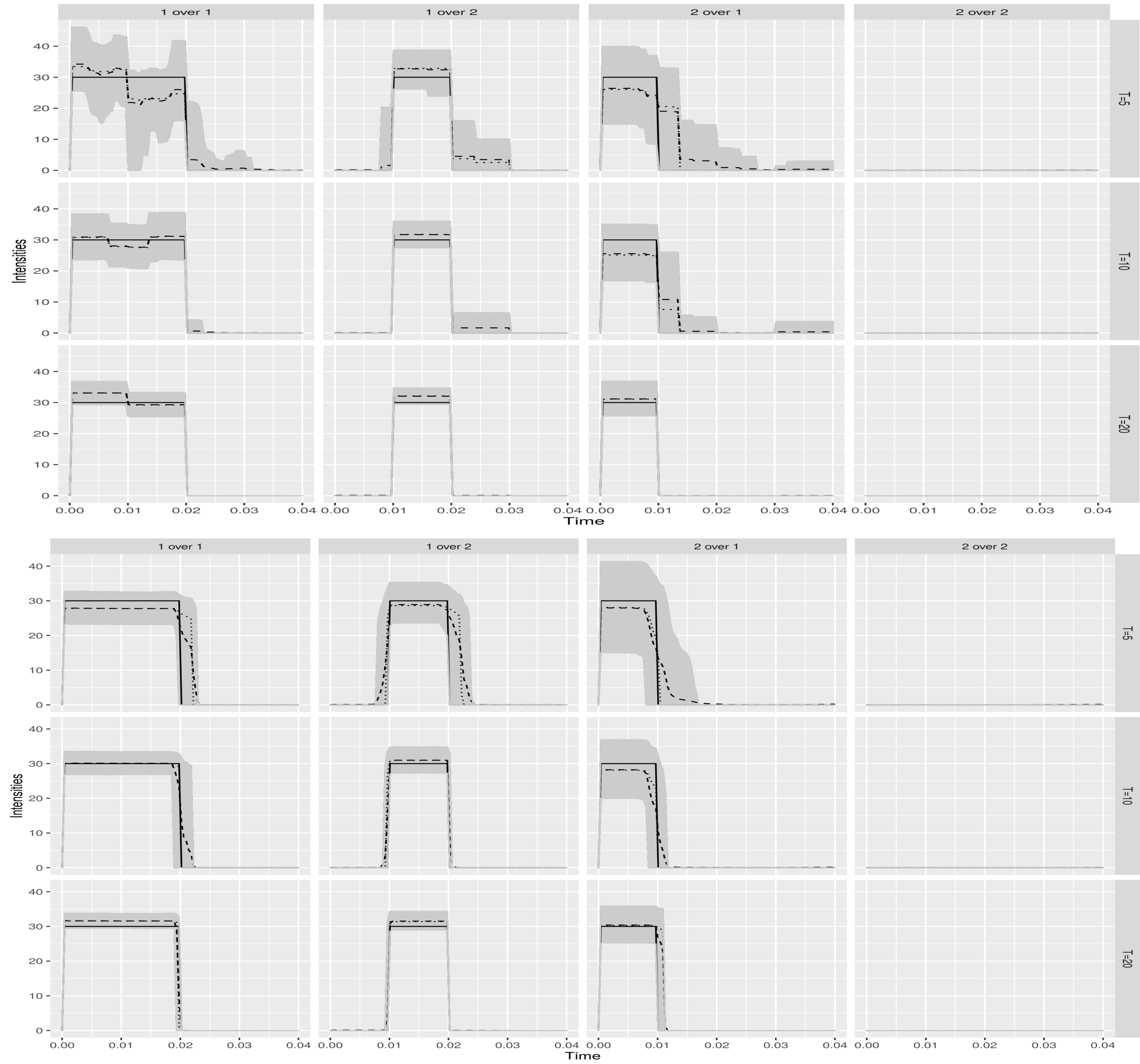

FIGURE 3. **Scenario 1, K=2**. Estimation of the $(h_{k,\ell})_{k,\ell=1,2}$ using the regular prior (upper panel) random histogram prior (bottom panel). The gray region indicates the credible region for $h_{k,\ell}(t)$ (delimited by the 5% and 95% percentiles of the posterior distribution). The true $h_{k,\ell}$ is in plain line, the posterior expectation and posterior median for $h_{k,\ell}(t)$ are in dotted and dashed lines respectively.

interactions are perfectly recovered. In a simulation scenario with other interaction functions, the results could have been different.
In Figure 8, we plot the posterior means (with credible regions) of the non-null interaction functions for the same simulated dataset as in Figure 6. The time intervals where the interaction functions are null are again perfectly recovered. The posterior incertainty around the non-null functions $h_{k,\ell}$ decreases when T increases.

3.4.3. *Results for scenario 3 : $K = 2$ with smooth functions.* In this context, we perform the inference using the random histogram prior distribution (3.7). In this case, we set $a_\lambda = 10$ and $b_\lambda = 1$. thus encouraging a greater number of step in the interactions functions. The behavior of the posterior distribution of ν^k is the same as in the other examples. In Figure 9, we plot the distribution of $\left(\mathbb{E}\left[\nu^k | (N_t^{sim})_{t\in[0,T]}\right]\right)_{sim=1\dots 25}$ for $T = 5, 10, 20$ seconds and clearly observe a decrease of the biais and the variance as the length of the observation period increases. Some estimation of the interaction functions are given in Figure 11. Due to the choice of the prior distribution of these quantities, we get a sparse posterior inference.

3.5. **Discussion.** Note that, in order to stick to the theoretical results, we restrict this simulation study to null or positive interaction functions, setting $Z_{k,\ell}^{(m)} \in \{0, 1\}$ in (3.5). In practice, the methodology presented below could

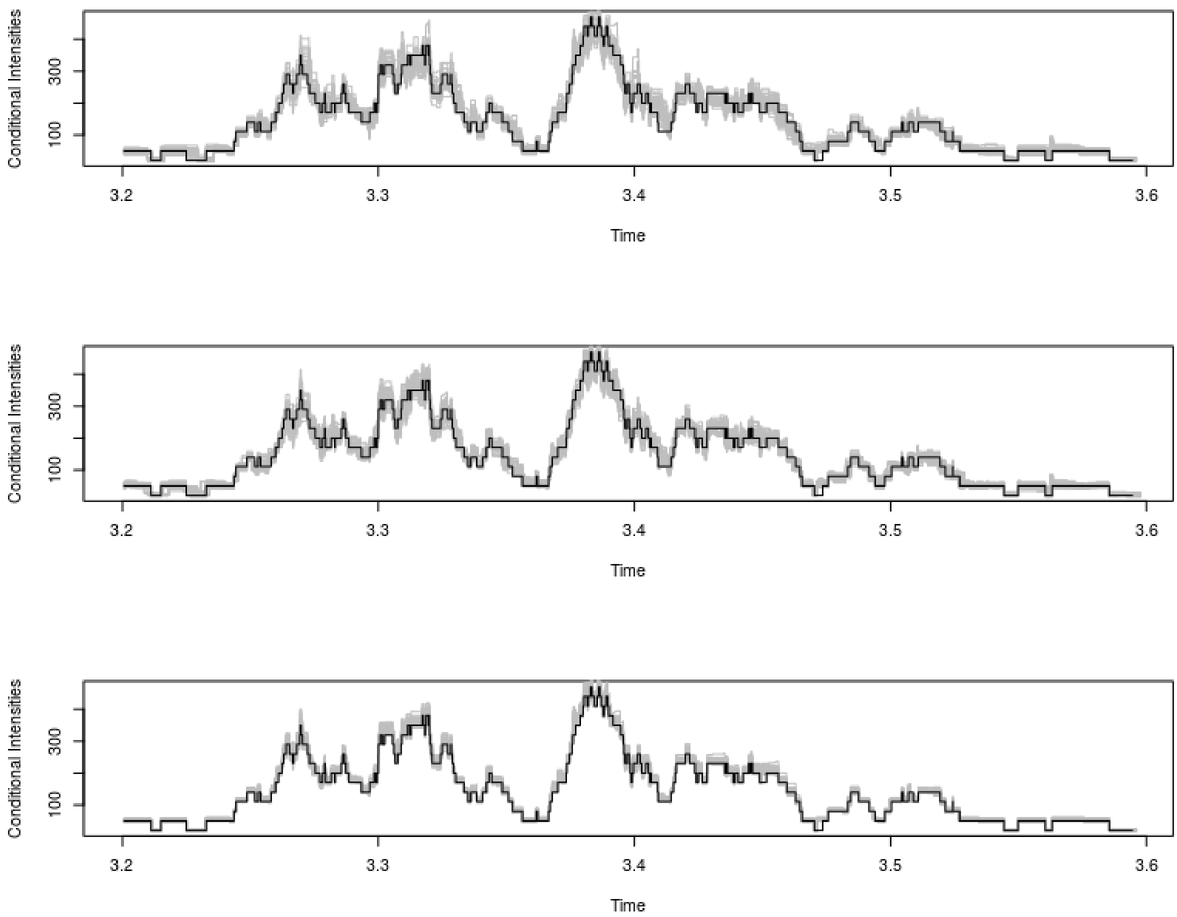

FIGURE 4. **Scenario 1**. Conditional intensity $\lambda_1^*(t)$: 50 realizations of $\lambda_1^*(t)$ from the posterior distribution for one particular dataset and 3 lengths of observation interval ($T = 5$ on the first line, $T = 10$ on the second line, and $T = 20$ on the third line). True conditional intensity in black plain line.

be easily extended to any type of interactions, i.e. $Z_{k,\ell}^{(m)} \in \{-1, 0, 1\}$. However, this extension could lead to additional non-negligible computation time.
Indeed, when the interaction functions are non-negative, the conditional intensity for neuron k defined as :

$$\lambda^\star(t,k) = \nu_\ell + \sum_{\ell=1}^{K} \int_{-\infty}^{t-} h_{\ell,k}(t-u) dN_u^{(\ell)}.$$

has to be integrated over $[0, T]$ to get the likelihood function. Since $t \mapsto \lambda^\star(t,k)$ is not regular, its integration can not be performed with a standard numerical solver. When $(h_{\ell,k})_{k,\ell\in\{1,\dots K\}}$ are piecewise constant functions, the integral can be computed in a close form with a complexity linear in the number of step sizes ($J_{\ell k}$) and the number of occurrence times.
However, when considering non-positive interactions, the conditional intensity has to be modified to guarantee its positivity. A standard modification is the following one:

$$\lambda^\star(t,k) = \phi\left\{\nu_\ell + \sum_{\ell=1}^{K} \int_{-\infty}^{t-} h_{\ell,k}(t-u) dN_u^{(\ell)}.\right\}$$

where $\phi : \mathbb{R} \mapsto \mathbb{R}^+$ is a non linear function ($\phi(x) = x^2$ or $\{x\}_+$). In this context, the integration of $t \mapsto \lambda^\star(t,k)$ is an operation of larger complexity, thus implying, in practice, a possibly substantial increase of the computation time.

4. PROOFS OF THEOREMS

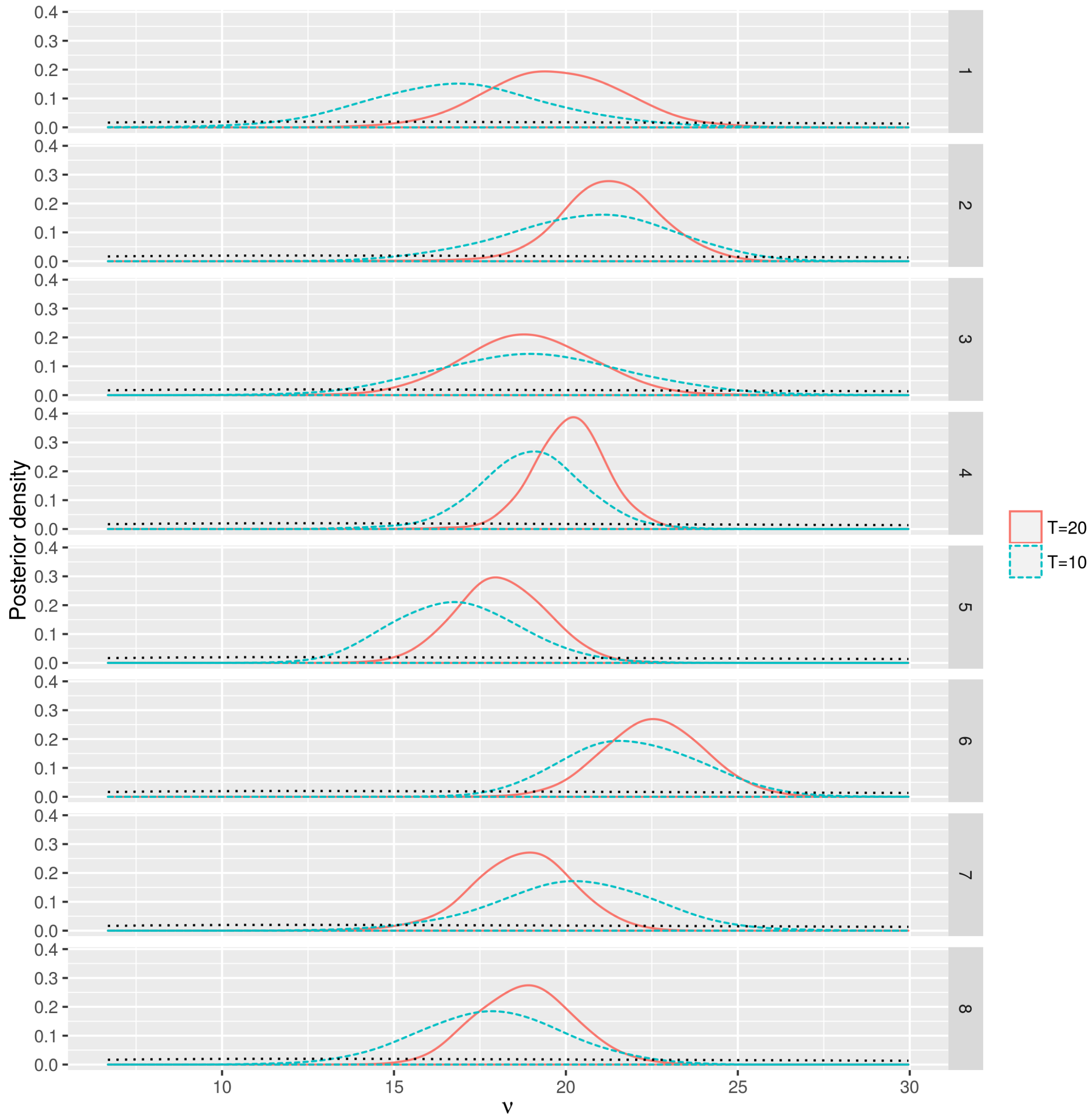

FIGURE 5. **Scenario 2**. Results on $(\nu_\ell)_{k=1\dots K}$ for a particular dataset: Prior distribution (dotted line), Posterior distributions for $T = 10$ (dashed line) and $T = 20$ (plain line).

4.1. **Proof of Theorem 1.** To prove Theorem 1, we apply the general methodology of Ghosal and van der Vaart (2007a), with modifications due to the fact that $\exp(L_T(f))$ is the likelihood of the distribution of $(N^k)_{k=1,\dots,K}$ on $[0,T]$ conditional on $\mathcal{G}_{0^-}$ and that the metric $d_{1,T}$ depends on the observations. We set $M_T = M\sqrt{\log\log T}$, for M a positive constant. Let

$$A_\epsilon = \{f \in \mathcal{F};\ d_{1,T}(f_0, f) \le K\epsilon\}$$

and for $j \ge 1$, we set

$$S_j = \{f \in \mathcal{F}_T;\ d_{1,T}(f, f_0) \in (Kj\epsilon_T, K(j+1)\epsilon_T]\}\,, \tag{4.1}$$

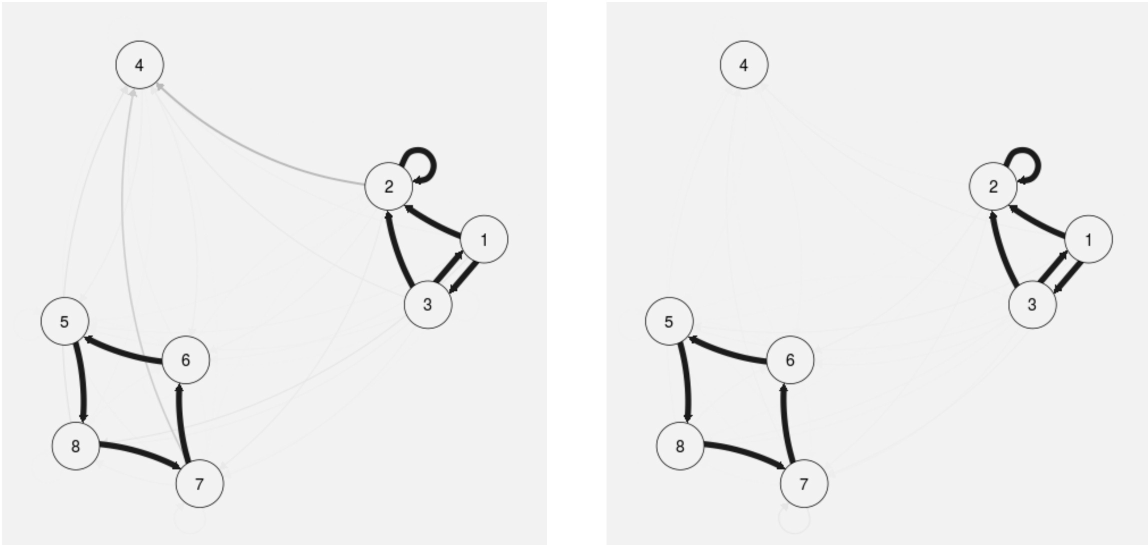

FIGURE 6. **Results for scenario 2 for one given dataset**. Posterior estimation of the interaction graph for $T = 10$ *on the left* and $T = 20$ *on the right*, for one randomly chosen dataset. Level of grey and width of the edges proportional to the posterior estimated probability of $\widehat{\mathbb{P}}(\delta^{(k,\ell)} = 1|(N_t^{sim})_{t\in[0,T]})$.

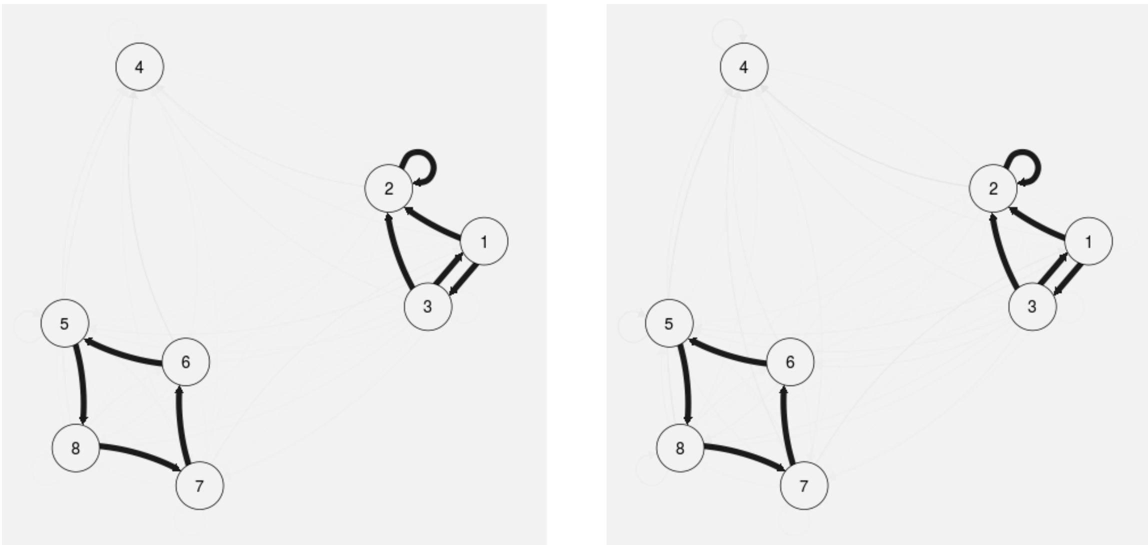

FIGURE 7. **Results for scenario 2 over the** 25 **simulated datasets**. Posterior estimation of the interaction graph for $T = 10$ *on the left* and $T = 20$ *on the right*. Level of grey and width of the edges are proportional to the posterior estimated probability of $\frac{1}{25}\sum_{sim=1}^{25}\widehat{\mathbb{P}}(\delta^{(k,\ell)} = 1|(N_t^{sim})_{t\in[0,T]})$.

where $\mathcal{F}_T = \{f = ((\nu_k)_k, (h_{k,\ell})_{k,\ell}) \in \mathcal{F};\ (h_{k,\ell})_{k,\ell}) \in \mathcal{H}_T\}$. So that, for any test function ϕ,

$$\begin{aligned}
\Pi\left(A^c_{M_T\epsilon_T}|N\right) &= \frac{\int_{A^c_{M_T\epsilon_T}} e^{L_T(f)-L_T(f_0)}d\Pi(f)}{\int_{\mathcal{F}} e^{L_T(f)-L_T(f_0)}d\Pi(f)} =: \frac{\bar{N}_T}{D_T} \\
&\leq \mathbb{1}_{\Omega_T^c} + \mathbb{1}_{\left\{D_T < \frac{\Pi(B(\epsilon_T,T))}{\exp(2(\kappa_T+1)T\epsilon_T^2)}\right\}} + \phi\mathbb{1}_{\Omega_T} + \frac{e^{2(\kappa_T+1)T\epsilon_T^2}}{\Pi(B(\epsilon_T,T))}\int_{\mathcal{F}_T^c} e^{L_T(f)-L_T(f_0)}d\Pi(f) \\
&+ \mathbb{1}_{\Omega_T}\frac{e^{2(\kappa_T+1)T\epsilon_T^2}}{\Pi(B(\epsilon_T,T))}\sum_{j=M_T}^{\infty}\int_{\mathcal{F}_T}\mathbb{1}_{f\in S_j}e^{L_T(f)-L_T(f_0)}(1-\phi)d\Pi(f)
\end{aligned}$$

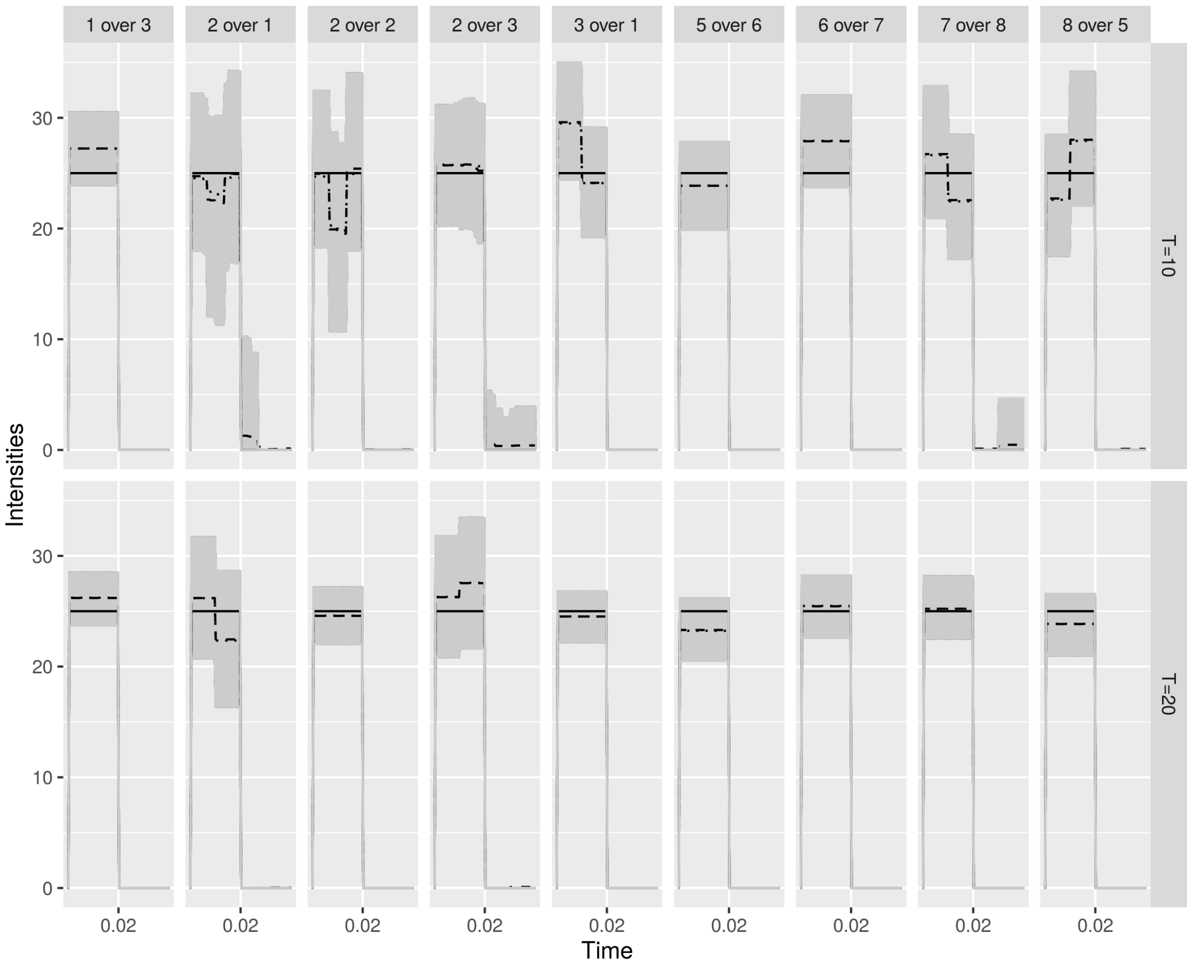

FIGURE 8. **Results for scenario 2 for one given dataset**. Estimation of the non null interaction functions $(h_{k,\ell})_{k,\ell=1,\dots,8}$ using the regular prior for $T = 10$ (upper panel) and $T = 20$ (bottom). The gray region indicates the credible region for $h_{k,\ell}(t)$ (delimited by the 5% and 95% percentiles of the posterior distribution). The true $h_{k,\ell}$ is in plain line, the posterior expectation and posterior median for $h_{k,\ell}(t)$ are in dotted and dashed lines respectively (often undistinguishable).

and

$$\mathbb{E}_0\left[\Pi\left(A^c_{M_T\epsilon_T}|N\right)\right] \leq \mathbb{P}_0(\Omega_T^c) + \mathbb{P}_0\left(D_T < e^{-2(\kappa_T+1)T\epsilon_T^2}\Pi(B(\epsilon_T, B))\right) + \mathbb{E}_0[\phi \mathbb{1}_{\Omega_T}] \\ + \frac{e^{2(\kappa_T+1)T\epsilon_T^2}}{\Pi(B(\epsilon_T, B))}\left(\Pi(\mathcal{F}_T^c) + \sum_{j=M_T}^{\infty}\int_{\mathcal{F}_T}\mathbb{E}_0\left[\mathbb{E}_f\left[\mathbb{1}_{\Omega_T}\mathbb{1}_{f\in S_j}(1-\phi)|\mathcal{G}_{0^-}\right]\right]d\Pi(f)\right),$$

since

$$\mathbb{E}_0\left[\int_{\mathcal{F}_T^c} e^{L_T(f)-L_T(f_0)}d\Pi(f)\right] = \mathbb{E}_0\left[\mathbb{E}_0\left[\int_{\mathcal{F}_T^c} e^{L_T(f)-L_T(f_0)}d\Pi(f)|\mathcal{G}_{0^-}\right]\right] = \mathbb{E}_0\left[\mathbb{E}_f\left[\int_{\mathcal{F}_T^c} d\Pi(f)|\mathcal{G}_{0^-}\right]\right] = \Pi(\mathcal{F}_T^c).$$

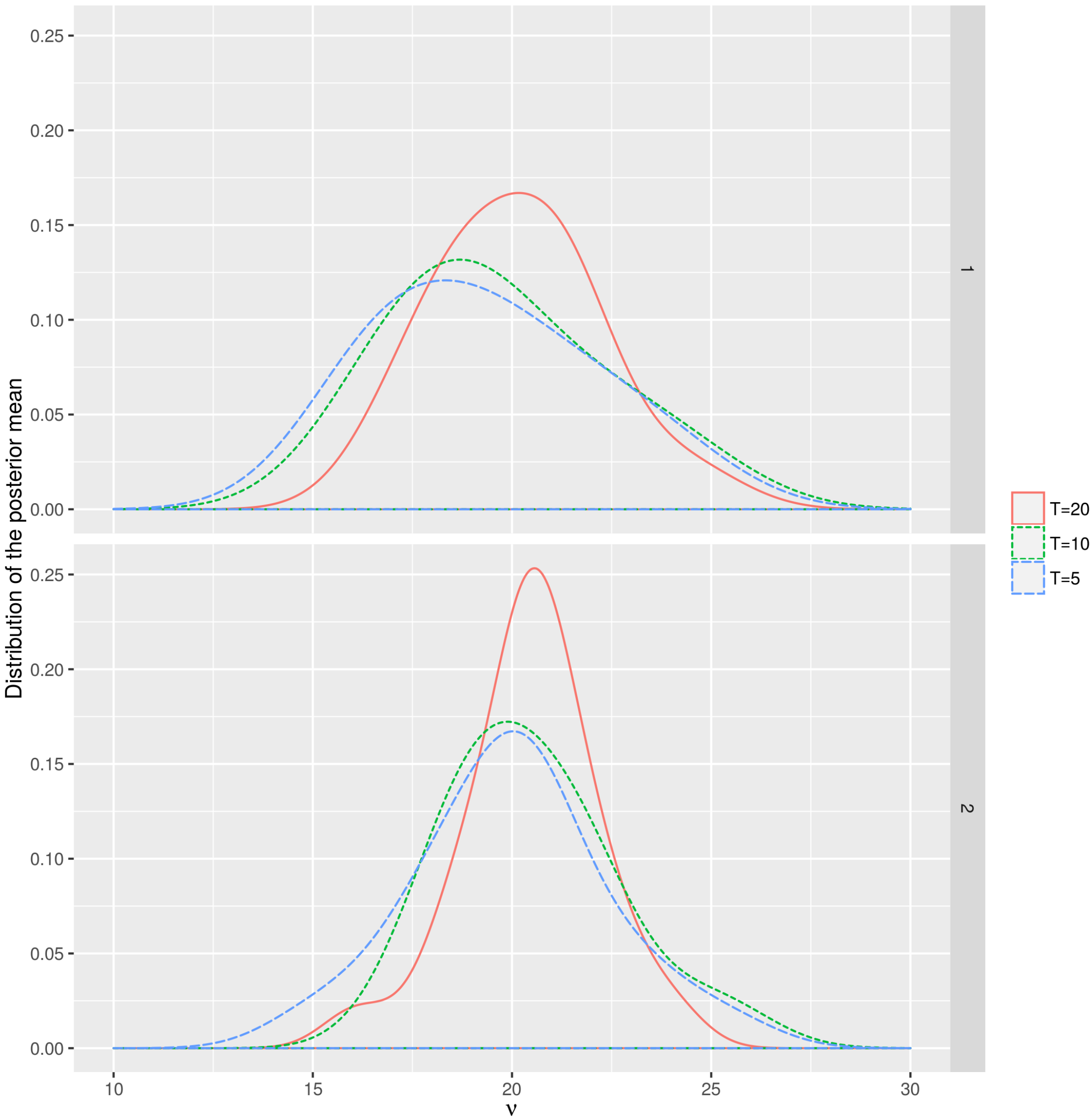

FIGURE 9. **Results for scenario 3 : smooth interaction functions**. Distributions of $\left(\mathbb{E}\left[\nu^k|(N_t^{sim})_{t\in[0,T]}\right]\right)_{sim=1\dots 25}$ for $T = 5, 10, 20$ seconds (long dashed, short dashed and plain line respectively).

Since $e^{(\kappa_T+1)T\epsilon_T^2}e^{L_T(f)-L_T(f_0)} \geq \mathbb{1}_{\{L_T(f)-L_T(f_0)\geq -(\kappa_T+1)T\epsilon_T^2\}}$,

$$
\begin{aligned}
\mathbb{P}_0\left(D_T \leq e^{-2(\kappa_T+1)T\epsilon_T^2}\Pi(B(\epsilon_T,B))\right) &\leq \mathbb{P}_0\left(\int_{B(\epsilon_T,B)} e^{L_T(f)-L_T(f_0)}\frac{d\Pi(f)}{\Pi(B(\epsilon_T,B))} \leq e^{-2(\kappa_T+1)T\epsilon_T^2}\right)\\
&\leq \mathbb{P}_0\left(\int_{B(\epsilon_T,B)} \mathbb{1}_{\{L_T(f)-L_T(f_0)\geq -(\kappa_T+1)T\epsilon_T^2\}}\frac{d\Pi(f)}{\Pi(B(\epsilon_T,B))} \leq e^{-(\kappa_T+1)T\epsilon_T^2}\right)\\
&\leq \frac{\mathbb{E}_0\left[\int_{B(\epsilon_T,B)} \mathbb{1}_{\{L_T(f)-L_T(f_0)<-(\kappa_T+1)T\epsilon_T^2\}}\frac{d\Pi(f)}{\Pi(B(\epsilon_T,B))}\right]}{\left(1-e^{-(\kappa_T+1)T\epsilon_T^2}\right)}\\
&\leq \frac{\int_{B(\epsilon_T,B)}\mathbb{P}_0\left(L_T(f_0)-L_T(f)>(\kappa_T+1)T\epsilon_T^2\right)d\Pi(f)}{\Pi(B(\epsilon_T,B))\left(1-e^{-(\kappa_T+1)T\epsilon_T^2}\right)}\\
&\lesssim \frac{\log\log(T)\log^3(T)}{T\epsilon_T^2},
\end{aligned}
$$

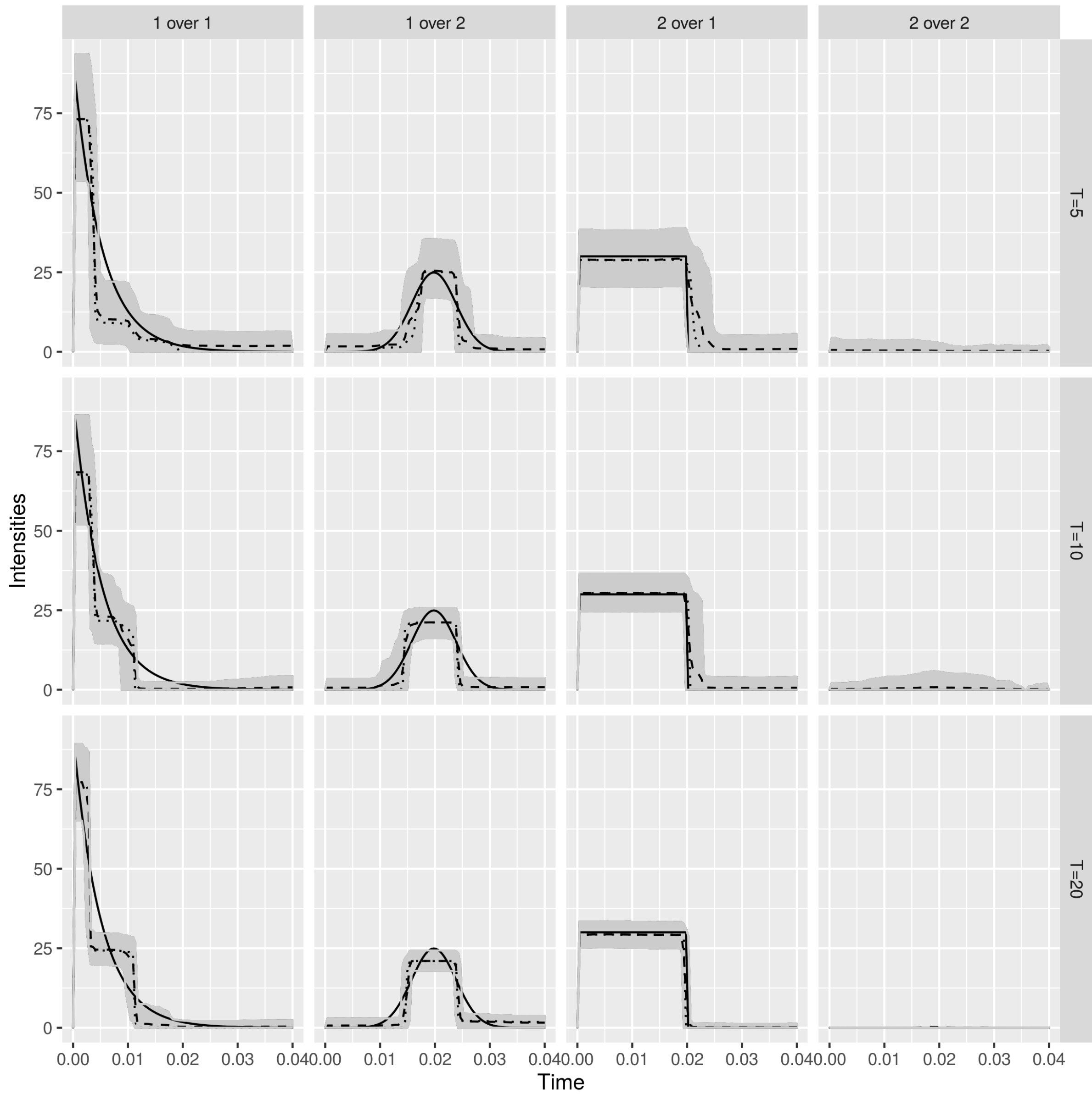

FIGURE 10. **Results for scenario 3 : smooth interaction functions**. Estimation of the interaction functions $(h_{k,\ell})_{k,\ell=1,2}$ using the regular prior for $T = 10$ (upper panel) and $T = 10$ (bottom). The gray region indicates the credible region for $h_{k,\ell}(t)$ (delimited by the 5% and 95% percentiles of the posterior distribution). The true $h_{k,\ell}$ is in plain line, the posterior expectation and posterior median for $h_{k,\ell}(t)$ are in dotted and dashed lines respectively (often undistinguishable).

by using Lemma 3 of Section 4.4. Remember we have set $\rho^0_{k,\ell} := \|h^0_{k,\ell}\|_1$ and $\rho_{k,\ell} := \|h_{k,\ell}\|_1$. Since $h_{k,\ell}$ and $h^0_{k,\ell}$ are non-negative functions, $\int_{-s}^{A} h^0_{k,\ell}(u)du \leq \rho^0_{k,\ell}$, $\int_0^{T-s} h^0_{k,\ell}(u)du \leq \rho^0_{k,\ell}$, and note that

$$\begin{aligned}
Td_{1,T}(f, f_0) &= \sum_{\ell=1}^{K} \int_0^T \left| \nu_\ell - \nu^0_\ell + \sum_{k=1}^{K} \int_{t-A}^{t^-} (h_{k,\ell} - h^0_{k,\ell})(t-s) dN^k_s \right| dt \\
&\geq \sum_{\ell=1}^{K} \left| \int_0^T \left(\nu_\ell - \nu^0_\ell + \sum_{k=1}^{K} \int_{t-A}^{t^-} (h_{k,\ell} - h^0_{k,\ell})(t-s) dN^k_s \right) dt \right| \\
&\geq \sum_{\ell=1}^{K} \left| T(\nu_\ell - \nu^0_\ell) + \int_0^T \left(\sum_{k=1}^{K} \int_{t-A}^{t^-} (h_{k,\ell} - h^0_{k,\ell})(t-s) dN^k_s \right) dt \right|,
\end{aligned}$$

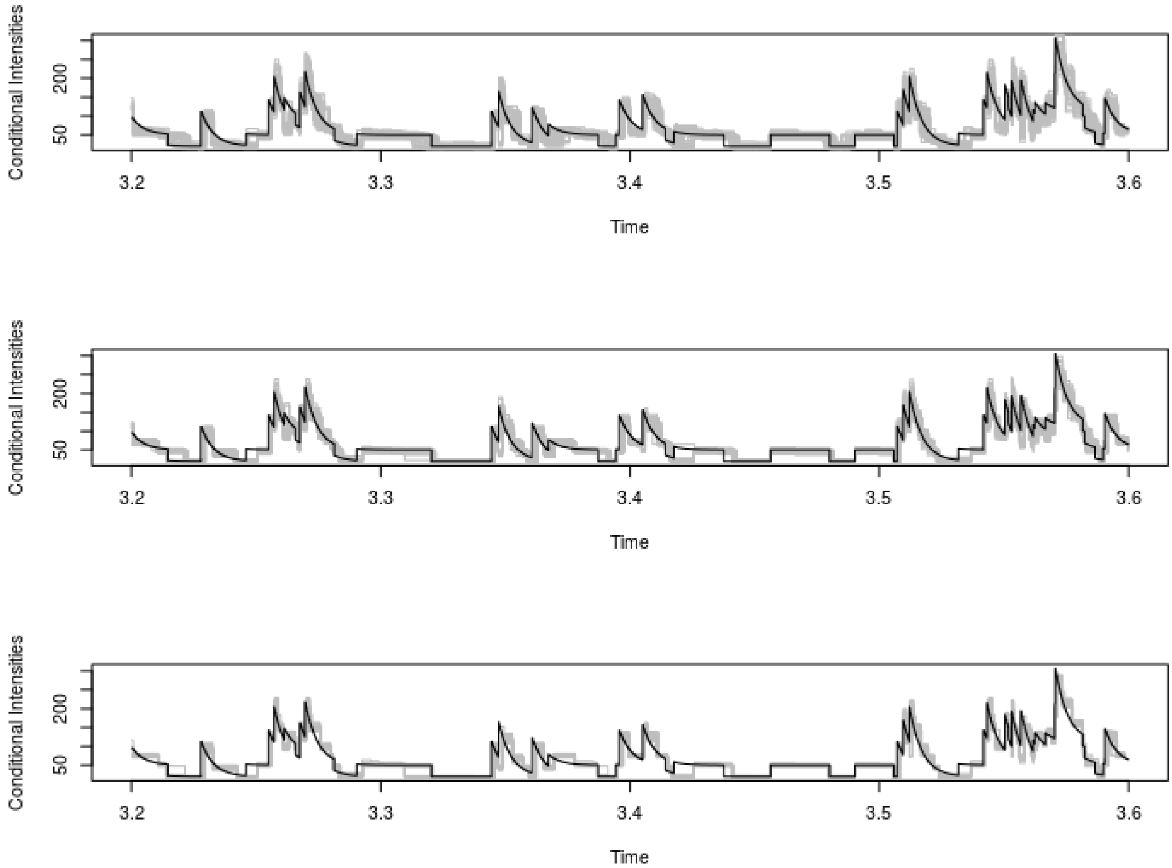

FIGURE 11. **Results for scenario 3 : smooth interaction functions**. Estimation of the interaction functions $(h_{k,\ell})_{k,\ell=1,2}$ using the regular prior for $T = 10$ (upper panel) and $T = 10$ (bottom). The gray region indicates the credible region for $h_{k,\ell}(t)$ (delimited by the 5% and 95% percentiles of the posterior distribution). The true $h_{k,\ell}$ is in plain line, the posterior expectation and posterior median for $h_{k,\ell}(t)$ are in dotted and dashed lines respectively (often undistinguishable).

then for any $\ell = 1, \dots, K$,

$$\begin{aligned}
d_{1,T}(f, f_0) &\geq \left| \nu_\ell - \nu_\ell^0 + \frac{1}{T} \sum_{k=1}^K \int_0^T \int_{t-A}^{t^-} (h_{k,\ell} - h_{k,\ell}^0)(t-s) dN_s^k dt \right| \\
&= \Bigg| \nu_\ell - \nu_\ell^0 + \sum_{k=1}^K (\rho_{k,\ell} - \rho_{k,\ell}^0) \frac{N^k[0, T-A]}{T} \\
&\quad + \frac{1}{T} \int_{-A}^0 \int_{-s}^A (h_{k,\ell} - h_{k,\ell}^0)(u) du dN_s^k + \frac{1}{T} \int_{T-A}^{T^-} \int_0^{T-s} (h_{k,\ell} - h_{k,\ell}^0)(u) du dN_s^k \Bigg| \\
&= \Bigg| \nu_\ell + \sum_{k=1}^K \rho_{k,\ell} \frac{N^k[0, T-A]}{T} + \frac{1}{T} \int_{-A}^0 \int_{-s}^A h_{k,\ell}(u) du dN_s^k + \frac{1}{T} \int_{T-A}^{T^-} \int_0^{T-s} h_{k,\ell}(u) du dN_s^k \\
&\quad - \left(\nu_\ell^0 + \sum_{k=1}^K \rho_{k,\ell}^0 \frac{N^k[0, T-A]}{T} + \frac{1}{T} \int_{-A}^0 \int_{-s}^A h_{k,\ell}^0(u) du dN_s^k + \frac{1}{T} \int_{T-A}^{T^-} \int_0^{T-s} h_{k,\ell}^0(u) du dN_s^k \right) \Bigg| .
\end{aligned}$$

This implies for $f \in S_j$ that

$$\begin{aligned}
&\nu_\ell + \sum_{k=1}^K \rho_{k,\ell} \frac{N^k[0,T-A]}{T} \le \nu_\ell^0 + \sum_{k=1}^K \rho_{k,\ell}^0 \frac{N^k[-A,T]}{T} + K(j+1)\epsilon_T \\
&\nu_\ell + \sum_{k=1}^K \rho_{k,\ell} \frac{N^k[-A,T]}{T} \ge \nu_\ell^0 + \sum_{k=1}^K \rho_{k,\ell}^0 \frac{N^k[0,T-A]}{T} - K(j+1)\epsilon_T.
\end{aligned} \tag{4.2}$$

On Ω_T,

$$\sum_{k=1}^K \rho_{k,\ell}^0 \frac{N^k[-A,T]}{T} \le \sum_{k=1}^K \rho_{k,\ell}^0 (\mu_k^0 + \delta_T),$$

so that, for T large enough, for all $j \ge 1$ $S_j \subset \mathcal{F}_j$ with

$$\mathcal{F}_j := \{f \in \mathcal{F}_T;\ \nu_\ell \le \mu_\ell^0 + 1 + Kj\epsilon_T, \forall \ell \le K\},$$

since

$$\mu_\ell^0 = \nu_\ell^0 + \sum_{k=1}^K \rho_{k,\ell}^0 \mu_k^0. \tag{4.3}$$

Let $(f_i)_{i=1,\dots,\mathcal{N}_j}$ be the centering points of a minimal $\mathbb{L}_1$-covering of $\mathcal{F}_j$ by balls of radius $\zeta j \epsilon_T$ with $\zeta = 1/(6N_0)$ (with N_0 defined in Section 2) and define $\phi_{(j)} = \max_{i=1,\dots,\mathcal{N}_j} \phi_{f_i,j}$ where $\phi_{f_i,j}$ is the individual test defined in Lemma 2 associated to f_i and j (see Section 4.3). Note also that there exists a constant C_0 such that

$$\mathcal{N}_j \le (C_0(1+j\epsilon_T)/j\epsilon_T)^K \mathcal{N}(\zeta j\epsilon_T/2, \mathcal{H}_T, \|.\|_1)$$

where $\mathcal{N}(\zeta j\epsilon_T/2, \mathcal{H}_T, \|.\|_1)$ is the covering number of $\mathcal{H}_T$ by $\mathbb{L}_1$-balls with radius $\zeta j\epsilon_T/2$. There exists C_K such that if $j\epsilon_T \le 1$ then $\mathcal{N}_j \le C_K e^{-K\log(j\epsilon_T)} \mathcal{N}(\zeta j\epsilon_T/2, \mathcal{H}_T, \|.\|_1)$ and if $j\epsilon_T > 1$ then $\mathcal{N}_j \le C_K N(\zeta j\epsilon_T/2, \mathcal{H}_T, \|.\|_1)$. Moreover $j \mapsto \mathcal{N}(\zeta j\epsilon_T/2, \mathcal{H}_T, \|.\|_1)$ is monotone non-increasing, choosing $j \ge 2\zeta_0/\zeta$, we obtain that

$$\mathcal{N}_j \le C_K(\zeta/\zeta_0)^K e^{K\log T} e^{x_0 T\epsilon_T^2},$$

from hypothesis (iii) in Theorem 1. Combining this with Lemma 2, we have for all $j \ge 2\zeta_0/\zeta$,

$$\begin{aligned}
\mathbb{E}_0[\mathbb{1}_{\Omega_T}\phi_{(j)}] &\lesssim \mathcal{N}_j e^{-Tx_2(j\epsilon_T \wedge j^2\epsilon_T^2)} \lesssim e^{K\log T} e^{x_0 T\epsilon_T^2} e^{-x_2 T(j\epsilon_T \wedge j^2\epsilon_T^2)} \\
\sup_{f\in\mathcal{F}_j} \mathbb{E}_0\left[\mathbb{E}_f[\mathbb{1}_{\Omega_T}\mathbb{1}_{f\in S_j}(1-\phi_{(j)})|\mathcal{G}_{0^-}]\right] &\lesssim e^{-x_2 T(j\epsilon_T\wedge j^2\epsilon_T^2)},
\end{aligned}$$

for x_2 a constant. Set $\phi = \max_{j\ge M_T}\phi_{(j)}$ with $M_T > 2\zeta_0/\zeta$, then

$$\mathbb{E}_0[\mathbb{1}_{\Omega_T}\phi] \lesssim e^{K\log T} e^{x_0 T\epsilon_T^2}\left[\sum_{j=M_T}^{\lfloor \epsilon_T^{-1}\rfloor} e^{-x_2 T\epsilon_T^2 j^2} + \sum_{j\ge\epsilon_T^{-1}} e^{-Tx_2\epsilon_T j}\right] \lesssim e^{-x_2 T\epsilon_T^2 M_T^2/2}$$

and

$$\sum_{j=M_T}^\infty \int_{\mathcal{F}_T} \mathbb{E}_0\left[\mathbb{E}_f\left[\mathbb{1}_{\Omega_T}\mathbb{1}_{f\in S_j}(1-\phi)|\mathcal{G}_{0^-}\right]\right] d\Pi(f) \lesssim e^{-x_2 T\epsilon_T^2 M_T^2/2}.$$

Therefore,

$$\frac{e^{2(\kappa_T+1)T\epsilon_T^2}}{\Pi(B(\epsilon_T,B))}\sum_{j=M_T}^\infty \int_{\mathcal{F}_T} \mathbb{E}_0\left[\mathbb{E}_f\left[\mathbb{1}_{\Omega_T}\mathbb{1}_{f\in S_j}(1-\phi)|\mathcal{G}_{0^-}\right]\right] d\Pi(f) = o(1)$$

if M is a constant large enough, which terminates the proof of Theorem 1.

4.2. **Proof of Theorem 2.** The proof of Theorem 2 follows the same lines as for Theorem 1, except that the decomposition of $\mathcal{F}_T$ is based on the sets $\mathcal{F}_j$ and $\mathcal{H}_{T,i}$, $i \ge 1$ and $j \ge M_T$ for some $M_T > 0$. For each $i \ge 1$, $j \ge M_T$, consider $S'_{i,j}$ a maximal set of $\zeta j\epsilon_T$-separated points in $\mathcal{F}_j \cap \mathcal{H}_{T,i}$ (with a slight abuse of notations) and $\phi_{i,j} = \max_{f_1\in S'_{i,j}} \phi_{f_1}$ with ϕ_{f_1} defined in Lemma 2. Then,

$$|S'_{i,j}| \le C_K(\zeta/\zeta_0)^K e^{K\log(T)} \mathcal{N}(\zeta j\epsilon_T/2, \mathcal{H}_{T,i}, \|.\|_1).$$

Setting $\bar N_{T,ij} := \int_{\mathcal{F}_T\cap\mathcal{H}_{T,i}} \mathbb{1}_{f\in S_j} e^{L_T(f)-L_T(f_0)} d\Pi(f)$, using similar computations as for the proof of Theorem 1, we have:

$$\mathbb{E}_0\left[\Pi\left(A^c_{M_T\epsilon_T}|N\right)\right] \le \mathbb{P}_0(\Omega_T^c) + \mathbb{P}_0\left(D_T < e^{-2(\kappa_T+1)T\epsilon_T^2}\Pi(B(\epsilon_T,B))\right) + \frac{e^{2(\kappa_T+1)T\epsilon_T^2}}{\Pi(B(\epsilon_T,B))}\Pi(\mathcal{F}_T^c)$$
$$+ \mathbb{E}_0\left[\mathbb{1}_{\Omega_T}\sum_{i=1}^{+\infty}\sum_{j=M_T}^{+\infty}\phi_{ij}\frac{\bar N_{T,ij}}{D_T}\right] + \frac{e^{2(\kappa_T+1)T\epsilon_T^2}}{\Pi(B(\epsilon_T,B))}\mathbb{E}_0\left[\mathbb{1}_{\Omega_T}\sum_{i=1}^{+\infty}\sum_{j=M_T}^{+\infty}(1-\phi_{ij})\bar N_{T,ij}\right].$$

Assumptions of the theorem allow us to deal with the first three terms. So, we just have to bound the last two ones. Using the same arguments and the same notations as for Theorem 1,

$$\begin{aligned}\mathbb{E}_0\left[\mathbb{1}_{\Omega_T}\sum_{i=1}^{+\infty}\sum_{j=M_T}^{+\infty}(1-\phi_{ij})\bar N_{T,ij}\right] &= \sum_{i=1}^{+\infty}\int_{\mathcal{F}_T\cap\mathcal{H}_{T,i}}\sum_{j=M_T}^{+\infty}\mathbb{E}_0\left[\mathbb{1}_{\Omega_T}\mathbb{1}_{f\in S_j}(1-\phi_{ij})e^{L_T(f)-L_T(f_0)}\right]d\Pi(f)\\ &= \sum_{i=1}^{+\infty}\int_{\mathcal{F}_T\cap\mathcal{H}_{T,i}}\sum_{j=M_T}^{+\infty}\mathbb{E}_0\left[\mathbb{E}_f[\mathbb{1}_{\Omega_T}\mathbb{1}_{f\in S_j}(1-\phi_{ij})|\mathcal{G}_{0^-}]\right]d\Pi(f)\\ &\lesssim \sum_{i=1}^{+\infty}\int_{\mathcal{F}_T\cap\mathcal{H}_{T,i}}d\Pi(f)\sum_{j=M_T}^{+\infty}e^{-x_2T(j\epsilon_T\wedge j^2\epsilon_T^2)} \lesssim e^{-x_2T\epsilon_T^2M_T^2/2}.\end{aligned}$$

Now, for γ a fixed positive constant smaller than x_2, setting $\pi_{T,i} = \Pi(\mathcal{H}_{T,i})$, we have

$$\mathbb{E}_0\left[\mathbb{1}_{\Omega_T}\sum_{i=1}^{+\infty}\sum_{j=M_T}^{+\infty}\phi_{ij}\frac{\bar N_{T,ij}}{D_T}\right] \le \mathbb{P}_0\left(D_T < e^{-2(\kappa_T+1)T\epsilon_T^2}\Pi(B(\epsilon_T,B))\right) + \mathbb{P}_0\left(\exists(i,j);\sqrt{\pi_{T,i}}\phi_{i,j} > e^{-\gamma T(j\epsilon_T\wedge j^2\epsilon_T^2)}\cap\Omega_T\right)$$
$$+\sum_{i=1}^{+\infty}\sum_{j=M_T}^{+\infty}e^{-\gamma T(j\epsilon_T\wedge j^2\epsilon_T^2)}\sqrt{\pi_{T,i}}\frac{e^{2(\kappa_T+1)T\epsilon_T^2}}{\Pi(B(\epsilon_T,B))}\mathbb{E}_0\left[\mathbb{1}_{\Omega_T}\int_{\mathcal{F}_T}\mathbb{1}_{f\in S_j}e^{L_T(f)-L_T(f_0)}d\Pi(f|\mathcal{H}_{T,i})\right].$$

Now,

$$\begin{aligned}\mathbb{P}_0\left(\exists(i,j);\sqrt{\pi_{T,i}}\phi_{i,j} > e^{-\gamma T(j\epsilon_T\wedge j^2\epsilon_T^2)}\cap\Omega_T\right) &\le \sum_{i=1}^{+\infty}\sqrt{\pi_{T,i}}\sum_{j=M_T}^{+\infty}e^{\gamma T(j\epsilon_T\wedge j^2\epsilon_T^2)}\mathbb{E}_0[\mathbb{1}_{\Omega_T}\phi_{i,j}]\\ &\lesssim \sum_{i=1}^{+\infty}\sqrt{\pi_{T,i}}\sum_{j=M_T}^{+\infty}e^{(\gamma-x_2)T(j\epsilon_T\wedge j^2\epsilon_T^2)+K\log(T)}\mathcal{N}(\zeta j\epsilon_T/2,\mathcal{H}_{T,i},\|.\|_1)\\ &\lesssim e^{(\gamma-x_2)T\epsilon_T^2M_T^2/2}\sum_{i=1}^{+\infty}\sqrt{\pi_{T,i}}\mathcal{N}(\zeta_0\epsilon_T,\mathcal{H}_{T,i},\|.\|_1) = o(1).\end{aligned}$$

But, we have

$$\mathbb{E}_0\left[\mathbb{1}_{\Omega_T}\int_{\mathcal{F}_T}\mathbb{1}_{f\in S_j}e^{L_T(f)-L_T(f_0)}d\Pi(f|\mathcal{H}_{T,i})\right] \le 1$$

and

$$\mathbb{E}_0\left[\mathbb{1}_{\Omega_T}\sum_{i=1}^{+\infty}\sum_{j=M_T}^{+\infty}\phi_{ij}\frac{\bar N_{T,ij}}{D_T}\right] \lesssim \sum_{i=1}^{+\infty}\sqrt{\pi_{T,i}}e^{-\gamma T\epsilon_T^2M_T^2}\frac{e^{2(\kappa_T+1)T\epsilon_T^2}}{\Pi(B(\epsilon_T,B))} + o(1) = o(1),$$

for M a contant large enough. This terminates the proof of Theorem 2.

4.3. **Construction of tests.** As usual, the control of the posterior distributions is based on specific tests. We build them in the following lemma.

Lemma 2. *Let $j \ge 1$, $f_1 \in \mathcal{F}_j$ and define the test*

$$\phi_{f_1,j} = \max_{\ell=1,\dots,K}\left(\mathbb{1}_{\{N^\ell(A_{1,\ell})-\Lambda^\ell(A_{1,\ell};f_0)\ge jT\epsilon_T/8\}}\vee\mathbb{1}_{\{N^\ell(A_{1,\ell}^c)-\Lambda^\ell(A_{1,\ell}^c;f_0)\ge jT\epsilon_T/8\}}\right),$$

with for all $\ell \le K$, $A_{1,\ell} = \{t\in[0,T];\ \lambda_t^\ell(f_1)\ge\lambda_t^\ell(f_0)\}$, $\Lambda^\ell(A_{1,\ell};f_0) = \int_0^T \mathbb{1}_{A_{1,\ell}}(t)\lambda_t^\ell(f_0)dt$ and $\Lambda^\ell(A_{1,\ell}^c;f_0) = \int_0^T \mathbb{1}_{A_{1,\ell}^c}(t)\lambda_t^\ell(f_0)dt$. Then

$$\mathbb{E}_0\left[\mathbb{1}_{\Omega_T}\phi_{f_1,j}\right] + \sup_{\|f-f_1\|_1\le j\epsilon_T/(6N_0)}\mathbb{E}_0\left[\mathbb{E}_f\left[\mathbb{1}_{\Omega_T}\mathbb{1}_{f\in S_j}(1-\phi_{f_1,j})|\mathcal{G}_{0^-}\right]\right] \le (2K+1)\max_\ell e^{-x_{1,\ell}Tj\epsilon_T(\sqrt{\mu_\ell^0}\wedge j\epsilon_T)},$$

with N_0 is defined in Section 2 and

$$x_{1,\ell} = \min\left(36, 1/(4096\mu_\ell^0), 1/\left(1024K\sqrt{\mu_\ell^0}\right)\right).$$

Proof of Lemma 2. Let $j \geq 1$ and $f_1 = ((\nu_k^1)_{k=1,\dots,K}, (h_{\ell,k}^1)_{k,\ell=1,\dots,K}) \in \mathcal{F}_j$. Let $\ell \in \{1,\dots,K\}$ and let

$$\phi_{j,A_{1,\ell}} = \mathbb{1}_{\{N^\ell(A_{1,\ell}) - \Lambda^\ell(A_{1,\ell};f_0) \geq jT\epsilon_T/8\}}.$$

By using (4.3), observe that on the event Ω_T,

$$\begin{aligned}\int_0^T \lambda_s^\ell(f_0)ds &= \nu_\ell^0 T + \sum_{k=1}^K \int_0^T \int_{s-A}^{s^-} h_{k,\ell}^0(s-u)dN_u^k ds \\ &\leq \nu_\ell^0 T + \sum_{k=1}^K \int_{-A}^{T^-} \int_0^T \mathbb{1}_{u<s\leq A+u} h_{k,\ell}^0(s-u)ds dN_u^k\end{aligned}$$

and for T large enough,

$$\int_0^T \lambda_s^\ell(f_0)ds \leq \nu_\ell^0 T + \sum_{k=1}^K \rho_{k,\ell}^0 N^k[-A,T] \leq 2T\mu_\ell^0. \tag{4.4}$$

Let $j \leq \sqrt{\mu_\ell^0}\epsilon_T^{-1}$ and $x = x_1 j^2 T\epsilon_T^2$, for x_1 a constant. We use inequality (7.7) of Hansen et al. (2015), with $\tau = T$, $H_t = 1_{A_{1,\ell}}(t)$, $v = 2T\mu_\ell^0$ and $M_T = N^\ell(A_{1,\ell}) - \Lambda^\ell(A_{1,\ell};f_0)$. So,

$$\mathbb{P}_0\left(\left\{N^\ell(A_{1,\ell}) - \Lambda^\ell(A_{1,\ell};f_0) \geq \sqrt{2vx} + \frac{x}{3}\right\} \cap \Omega_T\right) \leq e^{-x_1 j^2 T\epsilon_T^2}.$$

If $x_1 \leq 1/(1024\mu_\ell^0)$ and $x_1 \leq 36$, we have that

$$\sqrt{2vx} + \frac{x}{3} = 2\sqrt{\mu_\ell^0 x_1} jT\epsilon_T + \frac{x_1 j^2 T\epsilon_T^2}{3} \leq 2\sqrt{\mu_\ell^0 x_1}\left(1 + \frac{\sqrt{x_1}}{6}\right) jT\epsilon_T \leq \frac{jT\epsilon_T}{8}. \tag{4.5}$$

Then

$$\mathbb{P}_0\left(\left\{N^\ell(A_{1,\ell}) - \Lambda^\ell(A_{1,\ell};f_0) \geq \frac{jT\epsilon_T}{8}\right\} \cap \Omega_T\right) \leq e^{-x_1 j^2 T\epsilon_T^2}.$$

If $j \geq \sqrt{\mu_\ell^0}\epsilon_T^{-1}$, we apply the same inequality but with $x = x_0 jT\epsilon_T$ with $x_0 = \sqrt{\mu_\ell^0} \times x_1$. Then,

$$\sqrt{2vx} + \frac{x}{3} = 2\sqrt{\mu_\ell^0 x_1 \sqrt{\mu_\ell^0} j\epsilon_T T} + \frac{x_1\sqrt{\mu_\ell^0} jT\epsilon_T}{3} \leq 2\sqrt{\mu_\ell^0 x_1} jT\epsilon_T + \frac{x_1\sqrt{\mu_\ell^0} jT\epsilon_T}{3} \leq \frac{jT\epsilon_T}{8},$$

where we have used (4.5). It implies

$$\mathbb{P}_0\left(\left\{N^\ell(A_{1,\ell}) - \Lambda^\ell(A_{1,\ell};f_0) \geq \frac{jT\epsilon_T}{8}\right\} \cap \Omega_T\right) \leq e^{-x_0 jT\epsilon_T}.$$

Finally $\mathbb{E}_0\left[\mathbb{1}_{\Omega_T}\phi_{j,A_{1,\ell}}\right] \leq e^{-x_1 Tj\epsilon_T(\sqrt{\mu_\ell^0}\wedge j\epsilon_T)}$. Now, assume that

$$\int_{A_{1,\ell}} (\lambda_t^\ell(f_1) - \lambda_t^\ell(f_0))dt \geq \int_{A_{1,\ell}^c} (\lambda_t^\ell(f_0) - \lambda_t^\ell(f_1))dt.$$

Then

$$\frac{\|\lambda^\ell(f_1) - \lambda^\ell(f_0)\|_1}{2} := \frac{\int_0^T |\lambda_t^\ell(f_1) - \lambda_t^\ell(f_0)|dt}{2} \leq \int_{A_{1,\ell}} (\lambda_t^\ell(f_1) - \lambda_t^\ell(f_0))dt. \tag{4.6}$$

Let $f = ((\nu_k)_{k=1,\dots,K}, (h_{\ell,k})_{k,\ell=1,\dots,K}) \in S_j$ satisyfing $\|f - f_1\|_1 \leq \zeta j\epsilon_T$ for some $\zeta > 0$. Then,

$$\begin{aligned}\|\lambda^\ell(f) - \lambda^\ell(f_1)\|_1 &\leq T|\nu_\ell - \nu_\ell^1| + \int_0^T \left|\int_{t-A}^{t^-} \sum_k (h_{k,\ell} - h_{k,\ell}^1)(t-u)dN_u^k\right| dt \\ &\leq T|\nu_\ell - \nu_\ell^1| + \sum_k \int_0^T \int_{t-A}^{t^-} |(h_{k,\ell} - h_{k,\ell}^1)(t-u)|dN_u^k dt \\ &\leq T|\nu_\ell - \nu_\ell^1| + \max_k N^k[-A,T] \sum_k \|h_{k,\ell} - h_{k,\ell}^1\|_1 \leq TN_0\|f - f_1\|_1\end{aligned} \tag{4.7}$$

and $\|\lambda^\ell(f) - \lambda^\ell(f_1)\|_1 \leq TN_0\zeta j\epsilon_T$. Since $f \in S_j$, there exists ℓ (depending on f) such that

$$\|\lambda^\ell(f) - \lambda^\ell(f_0)\|_1 \geq jT\epsilon_T.$$

This implies in particular that if $N_0\zeta < 1$,

$$\|\lambda^\ell(f_1) - \lambda^\ell(f_0)\|_1 \geq \|\lambda^\ell(f) - \lambda^\ell(f_0)\|_1 - TN_0\zeta j\epsilon_T \geq (1-N_0\zeta)Tj\epsilon_T.$$

We then have

$$\begin{aligned}
\Lambda^\ell(A_{1,\ell}; f) - \Lambda^\ell(A_{1,\ell}; f_0) &= \Lambda^\ell(A_{1,\ell}; f) - \Lambda^\ell(A_{1,\ell}; f_1) + \Lambda^\ell(A_{1,\ell}; f_1) - \Lambda^\ell(A_{1,\ell}; f_0) \\
&\geq -\|\lambda^\ell(f) - \lambda^\ell(f_1)\|_1 + \int_{A_{1,\ell}} (\lambda_t^\ell(f_1) - \lambda_t^\ell(f_0))dt \\
&\geq -\|\lambda^\ell(f) - \lambda^\ell(f_1)\|_1 + \frac{\|\lambda^\ell(f_1) - \lambda^\ell(f_0)\|_1}{2} \\
&\geq -TN_0\zeta j\epsilon_T + \frac{(1-N_0\zeta)Tj\epsilon_T}{2} = (1/2 - 3N_0\zeta/2)Tj\epsilon_T.
\end{aligned}$$

Taking $\zeta = 1/(6N_0)$ leads to

$$\begin{aligned}
\mathbb{E}_f\left[\mathbb{1}_{f\in S_j}(1-\phi_{j,A_{1,\ell}})\mathbb{1}_{\Omega_T}|\mathcal{G}_{0^-}\right] &= \mathbb{E}_f\left[\mathbb{1}_{f\in S_j}\mathbb{1}_{\{N^\ell(A_{1,\ell})-\Lambda^\ell(A_{1,\ell};f_0)<jT\epsilon_T/8\}}\mathbb{1}_{\Omega_T}|\mathcal{G}_{0^-}\right] \\
&\leq \mathbb{E}_f\left[\mathbb{1}_{f\in S_j}\mathbb{1}_{\{N^\ell(A_{1,\ell})-\Lambda^\ell(A_{1,\ell};f)\leq -jT\epsilon_T/8\}}\mathbb{1}_{\Omega_T}|\mathcal{G}_{0^-}\right] \\
&\leq \mathbb{E}_f\left[\mathbb{1}_{\{N^\ell(A_{1,\ell})-\Lambda^\ell(A_{1,\ell};f)\leq -jT\epsilon_T/8\}}\mathbb{1}_{\Omega_T}|\mathcal{G}_{0^-}\right].
\end{aligned}$$

Note that we can adapt inequality (7.7) of Hansen et al. (2015), with $H_t = \mathbb{1}_{A_{1,\ell}}(t)$ to the case of conditional probability given $\mathcal{G}_{0^-}$ since the process E_t defined in the proof of Theorem 3 of Hansen et al. (2015), being a supermartingale, satisfies $\mathbb{E}_f[E_t|\mathcal{G}_{0^-}] \leq E_0 = 1$ and, given that from (4.2) and (4.4),

$$\int_0^T \lambda_s^\ell(f)ds \leq \nu_\ell T + \sum_{k=1}^K \rho_{k,\ell} N^k[-A,T] \leq 2T\mu_\ell^0 + K(j+1)T\epsilon_T =: \tilde{v}$$

for T large enough, we obtain:

$$\mathbb{E}_f\left[\mathbb{1}_{\left\{N^\ell(A_{1,\ell})-\Lambda^\ell(A_{1,\ell};f)\leq -\sqrt{2\tilde{v}x}-\frac{x}{3}\right\}}\mathbb{1}_{\Omega_T}|\mathcal{G}_{0^-}\right] \leq e^{-x}.$$

We use the same computations as before, observing that $\tilde{v} = v + K(j+1)T\epsilon_T$.
If $j \leq \sqrt{\mu_\ell^0}\epsilon_T^{-1}$ we set $x = x_1 j^2 T\epsilon_T^2$, for x_1 a constant. Then,

$$\begin{aligned}
\sqrt{2\tilde{v}x} + \frac{x}{3} &\leq \sqrt{2vx} + \frac{x}{3} + \sqrt{2K(j+1)T\epsilon_T x} \\
&\leq 2\sqrt{\mu_\ell^0 x_1} jT\epsilon_T + \frac{x_1 j^2 T\epsilon_T^2}{3} + \sqrt{2K(j+1)\epsilon_T x_1} jT\epsilon_T \\
&\leq 2\sqrt{\mu_\ell^0 x_1}\left(1 + \frac{\sqrt{x_1}}{6}\right) jT\epsilon_T + 2\sqrt{Kj\epsilon_T x_1} jT\epsilon_T \\
&\leq \left(2\sqrt{\mu_\ell^0 x_1}\left(1+\frac{\sqrt{x_1}}{6}\right) + 2\sqrt{K\sqrt{\mu_\ell^0}x_1}\right) jT\epsilon_T.
\end{aligned}$$

Therefore, if $x_1 \leq \min\left(36, 1/(4096\mu_\ell^0), 1/\left(1024K\sqrt{\mu_\ell^0}\right)\right)$, then

$$\sqrt{2\tilde{v}x} + \frac{x}{3} \leq \frac{jT\epsilon_T}{8}.$$

If $j \geq \sqrt{\mu_\ell^0}\epsilon_T^{-1}$, we set $x = x_0 jT\epsilon_T$ with $x_0 = \sqrt{\mu_\ell^0} \times x_1$. Then,

$$\begin{aligned}
\sqrt{2\tilde{v}x} + \frac{x}{3} &\leq \sqrt{2vx} + \frac{x}{3} + \sqrt{2K(j+1)T\epsilon_T x} \\
&\leq 2\sqrt{\mu_\ell^0 x_1\sqrt{\mu_\ell^0} j\epsilon_T T} + \frac{x_1\sqrt{\mu_\ell^0} jT\epsilon_T}{3} + \sqrt{2K(j+1)T\epsilon_T\sqrt{\mu_\ell^0}x_1 jT\epsilon_T} \\
&\leq 2\sqrt{\mu_\ell^0 x_1} jT\epsilon_T + \frac{x_1\sqrt{\mu_\ell^0} jT\epsilon_T}{3} + 2\sqrt{K\sqrt{\mu_\ell^0}x_1} jT\epsilon_T \leq \frac{jT\epsilon_T}{8}.
\end{aligned}$$

Therefore,

$$\mathbb{E}_f\left[\mathbb{1}_{\{N^\ell(A_{1,\ell})-\Lambda^\ell(A_{1,\ell};f)\leq -jT\epsilon_T/8\}}\mathbb{1}_{\Omega_T}|\mathcal{G}_{0^-}\right] \leq e^{-x_1 Tj\epsilon_T(\sqrt{\mu_\ell^0}\wedge j\epsilon_T)}.$$

Now, if

$$\int_{A_{1,\ell}} (\lambda_t^\ell(f_1) - \lambda_t^\ell(f_0))dt < \int_{A_{1,\ell}^c} (\lambda_t^\ell(f_0) - \lambda_t^\ell(f_1))dt,$$

then

$$\int_{A_{1,\ell}^c} (\lambda_t^\ell(f_1) - \lambda_t^\ell(f_0))dt \geq \frac{\|\lambda^\ell(f_1) - \lambda^\ell(f_0)\|_1}{2} \tag{4.8}$$

and the same computations are run with $A_{1,\ell}$ playing the role of $A_{1,\ell}^c$. This ends the proof of Lemma 2. □

4.4. **Control of the denominator.** The following lemma gives a control of D_T.

Lemma 3. *Let*

$$KL(f_0, f) = \mathbb{E}_0[L_T(f_0) - L_T(f)].$$

On $B(\epsilon_T, B)$,

$$0 \leq KL(f_0, f) \leq \kappa \log(r_T^{-1}) T\epsilon_T^2, \tag{4.9}$$

for T larger than T_0, with T_0 some constant depending on f_0, with

$$\kappa = 4\sum_{k=1}^K (\nu_k^0)^{-1} \left(3 + 4K \sum_{\ell=1}^K \left(A\mathbb{E}_0[(\lambda_0^\ell(f_0))^2] + \mathbb{E}_0[\lambda_0^\ell(f_0)]\right)\right) \tag{4.10}$$

and r_T is defined in (4.12)*.*

$$\mathbb{P}_0\left(L_T(f_0) - L_T(f) \geq (\kappa \log(r_T^{-1}) + 1)T\epsilon_T^2\right) \leq \frac{C \log\log(T) \log^3(T)}{T\epsilon_T^2}, \tag{4.11}$$

for C a constant only depending on f_0 and B.

Proof. We consider the set $\tilde{\Omega}_T$ defined in Lemma 4 and we set $\mathcal{N}_T = C_\alpha \log T$. We have:

$$\begin{aligned}
KL(f^0, f) &= \sum_{k=1}^K \mathbb{E}_0\left[\int_0^T \log\left(\frac{\lambda_t^k(f_0)}{\lambda_t^k(f)}\right) dN_t^k - \int_0^T \left(\lambda_t^k(f_0) - \lambda_t^k(f)\right) dt\right] \\
&= \sum_{k=1}^K \mathbb{E}_0\left[\int_0^T \log\left(\frac{\lambda_t^k(f_0)}{\lambda_t^k(f)}\right) \lambda_t^k(f_0)dt - \int_0^T \left(\lambda_t^k(f_0) - \lambda_t^k(f)\right) dt\right] \\
&= \sum_{k=1}^K \mathbb{E}_0\left[\int_0^T \Psi\left(\frac{\lambda_t^k(f)}{\lambda_t^k(f_0)}\right) \lambda_t^k(f_0)dt\right],
\end{aligned}$$

where for $u > 0$, $\Psi(u) := -\log(u) - 1 + u \geq 0$. First, observe that on $\tilde{\Omega}_T \cap B(\epsilon_T, B)$,

$$\frac{\lambda_t^k(f)}{\lambda_t^k(f_0)} \geq \frac{\nu_k}{\nu_k^0 + \sum_{\ell=1}^K \int_{t-A}^{t-} h_{\ell,k}^0(t-u)dN^\ell(u)} \geq \frac{\min_k \nu_k^0 - \epsilon_T}{\max_k \nu_k^0 + \max_{\ell,k} \|h_{\ell,k}^0\|_\infty K\mathcal{N}_T} =: r_T. \tag{4.12}$$

Furthermore, observe that for $u \in [r_T, 1/2)$, $\Psi(u) \leq \log(r_T^{-1})$, since $r_T = o(1)$. And for all $u \geq 1/2$, $\Psi(u) \leq (u-1)^2$. Finally, for any $u \geq r_T$,

$$\Psi(u) \leq 4\log(r_T^{-1})(u-1)^2.$$

Therefore, on $B(\epsilon_T, B)$, we have

$$\begin{aligned}
0 \leq KL(f^0, f) &\leq 4\log(r_T^{-1}) \sum_{k=1}^K \mathbb{E}_0\left[\int_0^T \frac{(\lambda_t^k(f_0) - \lambda_t^k(f))^2}{\lambda_t^k(f_0)} \mathbb{1}_{\tilde{\Omega}_T} dt\right] + R_T \\
&\leq 4\log(r_T^{-1}) \sum_{k=1}^K (\nu_k^0)^{-1} \mathbb{E}_0\left[\int_0^T (\lambda_t^k(f_0) - \lambda_t^k(f))^2 dt\right] + R_T
\end{aligned}$$

where

$$R_T = \sum_{k=1}^K \mathbb{E}_0\left[\mathbb{1}_{\tilde{\Omega}_T^c} \int_0^T \left(-\log\left(\frac{\lambda_t^k(f)}{\lambda_t^k(f_0)}\right) - 1 + \frac{\lambda_t^k(f)}{\lambda_t^k(f_0)}\right) \lambda_t^k(f_0)dt\right].$$

We first deal with the first term. Using stationarity of the process and Proposition 2 of Hansen et al. (2015)

$$\begin{aligned}
\mathbb{E}_0\left[\int_0^T (\lambda_t^k(f_0)-\lambda_t^k(f))^2 dt\right] &\le 2T(\nu_k^0-\nu_k)^2 + 2\int_0^T \mathbb{E}_0\left[\left(\sum_{\ell=1}^K \int_{t-A}^{t^-}(h_{\ell,k}-h_{\ell,k}^0)(t-u)dN^\ell(u)\right)^2\right]dt\\
&\le 2T\epsilon_T^2 + 4K\int_0^T \mathbb{E}_0\left[\sum_{\ell=1}^K\left(\int_{t-A}^{t^-}(h_{\ell,k}-h_{\ell,k}^0)(t-u)\lambda_u^\ell(f_0)du\right)^2\right]dt\\
&\quad + 4K\int_0^T \mathbb{E}_0\left[\sum_{\ell=1}^K\left(\int_{t-A}^{t^-}(h_{\ell,k}-h_{\ell,k}^0)(t-u)\left(dN_u^\ell-\lambda_u^\ell(f_0)du\right)\right)^2\right]dt\\
&\le 2T\epsilon_T^2 + 4K\sum_{\ell=1}^K \|h_{\ell,k}-h_{\ell,k}^0\|_2^2\int_0^T\int_{t-A}^{t^-}\mathbb{E}_0[(\lambda_u^\ell(f_0))^2]dudt\\
&\quad + 4K\int_0^T\sum_{\ell=1}^K\int_{t-A}^{t^-}(h_{\ell,k}-h_{\ell,k}^0)^2(t-u)\mathbb{E}_0\left[\lambda_u^\ell(f_0)\right]dudt\\
&\le 2T\epsilon_T^2 + 4KT\sum_{\ell=1}^K\|h_{\ell,k}-h_{\ell,k}^0\|_2^2\left(A\mathbb{E}_0[(\lambda_0^\ell(f_0))^2]+\mathbb{E}_0[\lambda_0^\ell(f_0)]\right)\\
&\le T\epsilon_T^2\left(2+4K\sum_{\ell=1}^K\left(A\mathbb{E}_0[(\lambda_0^\ell(f_0))^2]+\mathbb{E}_0[\lambda_0^\ell(f_0)]\right)\right).
\end{aligned}$$

We now deal with R_T. We have, on $B(\epsilon_T, B)$,

$$\begin{aligned}
(4.13)\qquad \frac{\lambda_t^k(f)}{\lambda_t^k(f_0)} &\le (\nu_k^0)^{-1}\left(\nu_k+\sum_{\ell=1}^K\|h_{\ell,k}\|_\infty\sup_{t\in[0,T]}N^\ell([t-A,t))\right)\\
(4.14)\qquad &\le (\nu_k^0)^{-1}\left(\nu_k^0+\epsilon_T+B\sum_{\ell=1}^K\sup_{t\in[0,T]}N^\ell([t-A,t))\right).
\end{aligned}$$

Conversely,

$$(4.15)\qquad \frac{\lambda_t^k(f)}{\lambda_t^k(f_0)} \ge (\nu_k^0-\epsilon_T)\left(\nu_k^0+\sum_{\ell=1}^K\|h_{\ell,k}^0\|_\infty\sup_{t\in[0,T]}N^\ell([t-A,t))\right)^{-1}.$$

So, using Lemma 4, if α is an absolute constant large enough, $R_T = o(1)$ and

$$R_T = o(T\epsilon_T^2).$$

Choosing $\kappa = 4\sum_{k=1}^K(\nu_k^0)^{-1}\left(3+4K\sum_{\ell=1}^K\left(A\mathbb{E}_0[(\lambda_0^\ell(f_0))^2]+\mathbb{E}_0[\lambda_0^\ell(f_0)]\right)\right)$ terminates the proof of (4.9). Note that if $B(\epsilon_T, B)$ is replaced with $B_\infty(\epsilon_T, B)$ (see Remark 1) then

$$\frac{\lambda_t^k(f)}{\lambda_t^k(f_0)} \le 1+\frac{|\nu_k-\nu_k^0|+\sum_\ell\|h_{\ell,k}-h_{\ell,k}\|_\infty\mathcal{N}_T}{\nu_k^0}$$

and

$$\frac{\lambda_t^k(f)}{\lambda_t^k(f_0)} \ge 1-\frac{|\nu_k-\nu_k^0|+\sum_\ell\|h_{\ell,k}-h_{\ell,k}\|_\infty\mathcal{N}_T}{\nu_k^0}$$

so that we can take $r_T = 1/2$ and $R_T = o(T\epsilon_T^2)$.

We now study

$$\mathcal{L}_T := L_T(f_0) - L_T(f) - \mathbb{E}_0[L_T(f_0)-L_T(f)].$$

We have for any integer Q_T such that $x := T/(2Q_T) > A$,

$$\begin{aligned}
L_T(f_0) - L_T(f) &= \sum_{k=1}^{K}\left(\int_0^T \log\left(\frac{\lambda_t^k(f_0)}{\lambda_t^k(f)}\right) dN_t^k - \int_0^T \left(\lambda_t^k(f_0) - \lambda_t^k(f)\right) dt\right)\\
&= \sum_{q=0}^{Q_T-1}\int_{2qx}^{2qx+x}\sum_{k=1}^{K}\left(\log\left(\frac{\lambda_t^k(f_0)}{\lambda_t^k(f)}\right) dN_t^k - \left(\lambda_t^k(f_0) - \lambda_t^k(f)\right) dt\right)\\
&\quad + \sum_{q=0}^{Q_T-1}\int_{2qx+x}^{2qx+2x}\sum_{k=1}^{K}\left(\log\left(\frac{\lambda_t^k(f_0)}{\lambda_t^k(f)}\right) dN_t^k - \left(\lambda_t^k(f_0) - \lambda_t^k(f)\right) dt\right)\\
&=: \sum_{q=0}^{Q_T-1} F_q + \sum_{q=0}^{Q_T-1}\tilde{F}_q.
\end{aligned}$$

Note that F_q is a measurable function of the points of N appearing in $[2qx-A; 2qx+x)$ denoted by $\mathcal{F}(N_{|[2qx-A;2qx+x)})$. Using Proposition 3.1 of Reynaud-Bouret and Roy (2006), we consider an i.i.d. sequence $(M_q^x)_{q=0,\dots,Q_T-1}$ of Hawkes processes with the same distribution as N but restricted to $[2qx-A; 2qx+x)$ and such that for all q, the variation distance between M_q^x and $N_{|[2qx-A;2qx+x)}$ is less than $2\mathbb{P}_0(T_e > x-A)$, where T_e is the extinction time of the process. We then set for any q,

$$G_q = \mathcal{F}(M_q^x).$$

We have built an i.i.d. sequence $(G_q)_{q=0,\dots,Q_T-1}$ with the same distributions as the F_q's. Furthermore, for any q,

$$\mathbb{P}_0(F_q \neq G_q) \leq 2\mathbb{P}_0(T_e > x-A).$$

We now have, by stationarity

$$\begin{aligned}
\mathbb{P}_0(\mathcal{L}_T \geq T\epsilon_T^2) &= \mathbb{P}_0\left(L_T(f_0) - L_T(f) - \mathbb{E}_0[L_T(f_0) - L_T(f)] \geq T\epsilon_T^2\right)\\
&= \mathbb{P}_0\left(\sum_{q=0}^{Q_T-1}(F_q - \mathbb{E}_0[F_q]) + \sum_{q=0}^{Q_T-1}(\tilde{F}_q - \mathbb{E}_0[\tilde{F}_q]) \geq T\epsilon_T^2\right)\\
&\leq 2\mathbb{P}_0\left(\sum_{q=0}^{Q_T-1}(F_q - \mathbb{E}_0[F_q]) \geq T\epsilon_T^2/2\right)\\
&\leq 2\mathbb{P}_0\left(\sum_{q=0}^{Q_T-1}(G_q - \mathbb{E}_0[G_q]) \geq T\epsilon_T^2/2\right) + 2\mathbb{P}_0\left(\exists q;\ F_q \neq G_q\right)\\
&\leq 2\mathbb{P}_0\left(\sum_{q=0}^{Q_T-1}(G_q - \mathbb{E}_0[G_q]) \geq T\epsilon_T^2/2\right) + 4Q_T\mathbb{P}_0(T_e > x-A).
\end{aligned}$$

We first deal with the first term of the previous expression:

$$\begin{aligned}
\mathbb{P}_0\left(\sum_{q=0}^{Q_T-1}(G_q - \mathbb{E}_0[G_q]) \geq T\epsilon_T^2/2\right) &\leq \frac{4}{T^2\epsilon_T^4}\mathrm{Var}_0\left(\sum_{q=0}^{Q_T-1} G_q\right)\\
&\leq \frac{4}{T^2\epsilon_T^4}\sum_{q=0}^{Q_T-1}\mathrm{Var}_0\left(G_q\right)\\
&\leq \frac{4Q_T}{T^2\epsilon_T^4}\mathrm{Var}_0\left(G_0\right) = \frac{4Q_T}{T^2\epsilon_T^4}\mathrm{Var}_0\left(F_0\right).
\end{aligned}$$

Now, by setting $d\mathcal{M}_t^{(k)} = dN_t^k - \lambda_t^k(f_0)dt$,

$$\begin{aligned}
\mathrm{Var}_0\left(F_0\right) &\leq \mathbb{E}_0\left[F_0^2\right] \\
&\leq \mathbb{E}_0\left[\left(\sum_{k=1}^K\int_0^{\frac{T}{2Q_T}}\log\left(\frac{\lambda_t^k(f_0)}{\lambda_t^k(f)}\right)dN_t^k - \sum_{k=1}^K\int_0^{\frac{T}{2Q_T}}(\lambda_t^k(f_0)-\lambda_t^k(f))dt\right)^2\right] \\
&\lesssim \sum_{k=1}^K\mathbb{E}_0\left[\left(\int_0^{\frac{T}{2Q_T}}\Psi\left(\frac{\lambda_t^k(f)}{\lambda_t^k(f_0)}\right)\lambda_t^k(f_0)dt + \int_0^{\frac{T}{2Q_T}}\log\left(\frac{\lambda_t^k(f_0)}{\lambda_t^k(f)}\right)d\mathcal{M}_t^{(k)}\right)^2\right] \\
&\lesssim \sum_{k=1}^K\mathbb{E}_0\left[\left(\int_0^{\frac{T}{2Q_T}}\Psi\left(\frac{\lambda_t^k(f)}{\lambda_t^k(f_0)}\right)\lambda_t^k(f_0)dt\right)^2\right] + \mathbb{E}_0\left[\left(\int_0^{\frac{T}{2Q_T}}\log\left(\frac{\lambda_t^k(f_0)}{\lambda_t^k(f)}\right)d\mathcal{M}_t^{(k)}\right)^2\right] \\
&\lesssim \sum_{k=1}^K\frac{T}{Q_T}\mathbb{E}_0\left[\int_0^{\frac{T}{2Q_T}}\Psi^2\left(\frac{\lambda_t^k(f)}{\lambda_t^k(f_0)}\right)(\lambda_t^k(f_0))^2dt\right] + \mathbb{E}_0\left[\int_0^{\frac{T}{2Q_T}}\log^2\left(\frac{\lambda_t^k(f_0)}{\lambda_t^k(f)}\right)\lambda_t^k(f_0)dt\right].
\end{aligned}$$

Note that on $\tilde{\Omega}_T$, for any $t \in [0; T/(2Q_T)]$,

$$0 \leq \Psi\left(\frac{\lambda_t^k(f)}{\lambda_t^k(f_0)}\right)\lambda_t^k(f_0) \leq C_1(B, f_0)\mathcal{N}_T^2,$$

where $C_1(B, f_0)$ only depends on B and f_0. Then,

$$\mathbb{E}_0\left[\mathbb{1}_{\tilde{\Omega}_T}\int_0^{\frac{T}{2Q_T}}\Psi^2\left(\frac{\lambda_t^k(f)}{\lambda_t^k(f_0)}\right)(\lambda_t^k(f_0))^2dt\right] \leq C_1(B, f_0)\mathcal{N}_T^2 \times \mathbb{E}_0\left[\mathbb{1}_{\tilde{\Omega}_T}\int_0^{\frac{T}{2Q_T}}\Psi\left(\frac{\lambda_t^k(f)}{\lambda_t^k(f_0)}\right)\lambda_t^k(f_0)dt\right]$$

and using same arguments as for the bound of $KL(f^0, f)$, the previous term is bounded by $\log(r_T^{-1})\mathcal{N}_T^2 \times (T/Q_T)\epsilon_T^2$ up to a constant. Since for any $u \geq 1/2$, we have $|\log(u)| \leq 2|u-1|$, we have for any $u \geq r_T$,

$$|\log(u)| \leq 2\log(r_T^{-1})|u-1|$$

and

$$\begin{aligned}
\mathbb{E}_0\left[\mathbb{1}_{\tilde{\Omega}_T}\int_0^{\frac{T}{2Q_T}}\log^2\left(\frac{\lambda_t^k(f_0)}{\lambda_t^k(f)}\right)\lambda_t^k(f_0)dt\right] &\leq 4\log^2(r_T^{-1})(\nu_k^0)^{-1}\mathbb{E}_0\left[\mathbb{1}_{\tilde{\Omega}_T}\int_0^{\frac{T}{2Q_T}}(\lambda_t^k(f_0)-\lambda_t^k(f))^2dt\right] \\
&\lesssim \log^2(r_T^{-1})(T/Q_T)\epsilon_T^2.
\end{aligned}$$

By taking $\alpha \geq 2$ and using Lemma 4, we obtain:

$$\mathbb{E}_0\left[\mathbb{1}_{\tilde{\Omega}_T^c}\int_0^{\frac{T}{2Q_T}}\Psi^2\left(\frac{\lambda_t^k(f)}{\lambda_t^k(f_0)}\right)(\lambda_t^k(f_0))^2dt\right] + \mathbb{E}_0\left[\mathbb{1}_{\tilde{\Omega}_T^c}\int_0^{\frac{T}{2Q_T}}\log^2\left(\frac{\lambda_t^k(f_0)}{\lambda_t^k(f)}\right)\lambda_t^k(f_0)dt\right] = o(TQ_T^{-1}\epsilon_T^2).$$

Finally,

$$\mathrm{Var}_0\left(F_0\right) \leq C_2(B, f_0)\log(r_T^{-1})\mathcal{N}_T^2 \times (T/Q_T)^2\epsilon_T^2.$$

for $C_2(B, f_0)$ a constant only depending on B and f_0, and

$$\mathbb{P}_0(\mathcal{L}_T \geq T\epsilon_T^2) \leq 8C_2(B, f_0)\log(r_T^{-1})\mathcal{N}_T^2 \times (T/Q_T) \times (1/(T\epsilon_T^2) + 4Q_T\mathbb{P}_0(T_e > x - A).$$

It remains to deal with the last term of the previous expression. The proof of Proposition 3 of Hansen et al. (2015) shows that there exists a constant D only depending on f_0 such that if we take $x = D\log T$, which is larger than A for T large enough, then

$$4Q_T\mathbb{P}_0(T_e > x - A) = o(T^{-1}).$$

We now have

$$\log(r_T^{-1})\mathcal{N}_T^2 \times (T/Q_T) = O(\log\log(T)\log^3(T)),$$

which ends the proof of the lemma. □

4.5. **Proof of Theorem 3.** Define

$$A_{L_1}(w_T\varepsilon_T) = \{f \in \mathcal{F};\ \|f - f_0\|_1 \le w_T\varepsilon_T\},$$

then

$$\Pi\left(A_{L_1}(w_T\varepsilon_T)^c|N\right) \le \Pi(A^c_{\varepsilon_T}|N) + \Pi\left(A_{L_1}(w_T\varepsilon_T)^c \cap A_{\varepsilon_T}|N\right).$$

Using Assumption (i), we just need to prove that

$$\mathbb{E}_0\left[\mathbb{1}_{\Omega_{1,T}}\Pi\left(A_{L_1}(w_T\varepsilon_T)^c \cap A_{\varepsilon_T}|N\right)\right] = o(1) \tag{4.16}$$

for some well chosen set $\Omega_{1,T} \subset \Omega_T$ such that

$$\mathbb{P}_0(\Omega^c_{1,T} \cap \Omega_T) = o(1). \tag{4.17}$$

Using (4.2), there exists C_0 such that for all $f \in A_{\varepsilon_T}$, on Ω_T,

$$\sum_\ell \nu_\ell + \sum_{\ell,k}\rho_{\ell,k} \le C_0.$$

Therefore, on Ω_T,

$$A_{L_1}(w_T\varepsilon_T)^c \cap A_{\varepsilon_T} \subset \{f \in \mathcal{F};\ \|f - f_0\|_1 > w_T\varepsilon_T; \sum_\ell(\nu_\ell + \sum_k \rho_{\ell,k}) \le C_0\}.$$

We set $u_T := u_0(\log T)^{1/6}\varepsilon_T^{1/3}$ with u_0 a large constant to be chosen later. Let $\mathcal{F}_T = \{f \in \mathcal{F}; \|\rho\| \le 1 - u_T\}$. From Assumption (ii),

$$\Pi(\mathcal{F}^c_T) \le e^{-2c_1T\varepsilon_T^2}$$

for T large enough. Following the same lines as in the proof of Theorem 1, we then have

$$\begin{aligned}
&\mathbb{E}_0\left[\mathbb{1}_{\Omega_{1,T}}\Pi\left(A_{L_1}(w_T\varepsilon_T)^c \cap A_{\varepsilon_T}|N\right)\right] \le \mathbb{P}_0(D_T < e^{-c_1T\varepsilon_T^2}) \\
&+ e^{c_1T\varepsilon_T^2}\int_{A_{L_1}(w_T\varepsilon_T)^c\cap\mathcal{F}_T} \mathbb{E}_0\left[\mathbb{P}_f\left(\Omega_{1,T} \cap \{d_{1,T}(f,f_0) \le \varepsilon_T\}|\mathcal{G}_{0-}\right)\right]d\Pi(f) + e^{-c_1T\varepsilon_T^2},
\end{aligned} \tag{4.18}$$

where $\mathbb{P}_f$ denotes the stationary distribution when the true parameter is f. We will now prove that $\mathbb{P}_f\left(\Omega_{1,T} \cap \{d_{1,T}(f,f_0) \le \varepsilon_T\}|\mathcal{G}_{0}\right.$ $o_{P_0}(1)$ for all $f \in A_{L_1}(w_T\varepsilon_T)^c \cap \mathcal{F}_T$. Let $Z_{m,\ell}$ be defined by

$$Z_{m,\ell} = \int_{2mT/(2J_T)}^{(2m+1)T/(2J_T)}\left|\nu_\ell - \nu^0_\ell + \sum_{k=1}^K\int_{t-A}^{t^-}(h_{k,\ell} - h^0_{k,\ell})(t-s)dN^k_s\right|dt$$

with J_T such that $J_T = \lfloor\kappa_0(\log T)^{-1}Tu_T^2\rfloor$ and κ_0 a constant chosen later. Note that $J_T \to +\infty$ and $T/J_T \to +\infty$ when $T \to +\infty$. Since $Td_{1,T}(f,f_0) \ge \max_{1\le\ell\le K}\sum_{m=1}^{J_T-1}Z_{m,\ell}$ we have that

$$\begin{aligned}
\mathbb{P}_f\left(\Omega_{1,T} \cap \{d_{1,T}(f,f_0) \le \varepsilon_T\}|\mathcal{G}_{0-}\right) &\le \min_{1\le\ell\le K}\mathbb{P}_f\left(\Omega_{1,T} \cap \left\{\sum_{m=1}^{J_T-1}Z_{m,\ell} \le \varepsilon_TT\right\}|\mathcal{G}_{0-}\right) \\
&\le \min_{1\le\ell\le K}\mathbb{P}_f\left(\Omega_{1,T} \cap \left\{\sum_{m=1}^{J_T-1}(Z_{m,\ell} - \mathbb{E}_f[Z_{m,\ell}]) \le \varepsilon_TT - (J_T-1)\mathbb{E}_f[Z_{1,\ell}]\right\}\middle|\mathcal{G}_{0-}\right).
\end{aligned}$$

From Lemma 6 we have that there exists ℓ (depending on f and f^0) such that $\mathbb{E}_f[Z_{1,\ell}] \ge CT\|f - f_0\|_1/J_T$ for some $C > 0$ so that if $f \in A_{L_1}(w_T\varepsilon_T)^c$ then, since $w_T \to +\infty$,

$$\mathbb{P}_f\left(\Omega_{1,T} \cap \{d_{1,T}(f,f_0) \le T\varepsilon_T\}|\mathcal{G}_{0-}\right) \le \max_\ell\mathbb{P}_f\left(\Omega_{1,T} \cap \left\{\sum_{m=1}^{J_T-1}[Z_{m,\ell} - \mathbb{E}_f[Z_{m,\ell}]] \le -\frac{CT\|f-f_0\|_1}{2}\right\}\middle|\mathcal{G}_{0-}\right).$$

The problem in dealing with the right hand side of the above inequality is that the $Z_{m,\ell}$'s are not independent. We therefore show that we can construct independent random variables $\tilde{Z}_{m,\ell}$ such that, conditionally on $\mathcal{G}_{0-}$, $\sum_{m=1}^{J_T-1}(Z_{m,\ell} - \mathbb{E}_f[Z_{m,\ell}])$ is close to $\sum_{m=1}^{J_T-1}(\tilde{Z}_{m,\ell} - \mathbb{E}_f[\tilde{Z}_{m,\ell}])$ on $\Omega_{1,T}$. For all $1 \le m \le J_T - 1$, define $N^{0,m}$ the sub-counting measure of N generated from the ancestors of any type born on $[(2m-1)T/(2J_T), (2m+1)T/(2J_T)]$ and the K-multivariate point process $\bar{N}^m$ defined by

$$\bar{N}^m = N - N^{0,m}.$$

Denote

$$\tilde{Z}_{m,\ell} = \int_{2mT/(2J_T)}^{(2m+1)T/(2J_T)}\left|\nu_\ell - \nu^0_\ell + \sum_{k=1}^K\int_{t-A}^{t^-}(h_{k,\ell} - h^0_{k,\ell})(t-s)dN^{0,m,k}_s\right|dt,$$

where $N^{0,m,k}$ if the kth coordinate of $N^{0,m}$. Observe that if $I_m = [2mT/(2J_T) - A, (2m+1)T/(2J_T)]$, then $\bar{N}^m(I_m)$ is the number of points of $\bar{N}^m$ lying in I_m. We have:

$$
\begin{aligned}
|Z_{m,\ell} - \tilde{Z}_{m,\ell}| &= \left| \int_{2mT/(2J_T)}^{(2m+1)T/(2J_T)} \left(\left| \nu_\ell - \nu_\ell^0 + \sum_{k=1}^K \int_{t-A}^{t^-} (h_{k,\ell} - h_{k,\ell}^0)(t-s) dN_s^k \right| \right.\right. \\
&\qquad \left.\left. - \left| \nu_\ell - \nu_\ell^0 + \sum_{k=1}^K \int_{t-A}^{t^-} (h_{k,\ell} - h_{k,\ell}^0)(t-s) dN_s^{0,m,k} \right| \right) dt \right| \\
&\leq \mathbb{1}_{\bar{N}^m(I_m) \neq 0} \sum_{k=1}^K \int_{2mT/(2J_T)}^{(2m+1)T/(2J_T)} \int_{t-A}^{t^-} |(h_{k,\ell} - h_{k,\ell}^0)(t-s)| d\bar{N}_s^{m,k} dt \\
&\leq \mathbb{1}_{\bar{N}^m(I_m) \neq 0} \sum_{k=1}^K \|h_{k,\ell} - h_{k,\ell}^0\|_1 \bar{N}^{m,k}(I_m) \leq \|f - f_0\|_1 \bar{N}^m(I_m).
\end{aligned}
\tag{4.19}
$$

Let $\Omega_{1,T} = \Omega_T \cap \{\sum_{m=1}^{J_T-1} \bar{N}^m(I_m) \leq CT/8\}$. In Lemma 8, we prove that there exists $\tilde{c}_0$ such that

$$\mathbb{P}_0 \left(\Omega_{1,T}^c \cap \Omega_T \right) \leq e^{-C\tilde{c}_0 T}.$$

and (4.17) is satisfied. Using (4.19), we have on $\Omega_{1,T}$

$$|Z_{m,\ell} - \tilde{Z}_{m,\ell}| \leq \|f - f_0\|_1 CT/8. \tag{4.20}$$

Lemma 8 proves that there exists a constant $\kappa_0 > 0$ (see the definition of J_T) such that

$$\sum_{m=1}^{J_T-1} \mathbb{E}_f[\bar{N}^m(I_m)] \leq CT/8,$$

so that

$$\sum_{m=1}^{J_T-1} |\mathbb{E}_f[Z_{m,\ell}] - \mathbb{E}_f[\tilde{Z}_{m,\ell}]| \leq \sum_{m=1}^{J_T-1} \mathbb{E}_f|Z_{m,\ell} - \tilde{Z}_{m,\ell}| \leq \|f - f_0\|_1 \sum_{m=1}^{J_T-1} \mathbb{E}_f[\bar{N}^m(I_m)] \leq C\|f - f_0\|_1 T/8$$

and

$$
\begin{aligned}
\mathbb{P}_f \left(\Omega_{1,T} \cap \{d_{1,T}(f, f_0) \leq T\varepsilon_T\} | \mathcal{G}_{0^-} \right) &\leq \max_\ell \mathbb{P}_f \left(\Omega_{1,T} \cap \left\{ \sum_{m=1}^{J_T-1} [Z_{m,\ell} - \mathbb{E}_f[Z_{m,\ell}]] \leq -\frac{CT\|f - f_0\|_1}{2} \right\} \middle| \mathcal{G}_{0^-} \right) \\
&\leq \mathbb{P}_f \left(\sum_{m=1}^{J_T-1} (-\tilde{Z}_{m,\ell} + \mathbb{E}_f(\tilde{Z}_{m,\ell})) \geq CT\|f - f_0\|_1/4 \middle| \mathcal{G}_{0^-} \right).
\end{aligned}
$$

Since by construction the $\tilde{Z}_{m,\ell}$ are positive, independent, identically distributed and independent of $\mathcal{G}_{0-}$, the Bernstein inequality gives

$$\mathbb{P}_f \left(\sum_{m=1}^{J_T-1} (-\tilde{Z}_{m,\ell} + \mathbb{E}_f(\tilde{Z}_{m,\ell})) \geq CT\|f - f_0\|_1/4 \middle| \mathcal{G}_{0^-} \right) \leq e^{-\frac{C^2T^2\|f-f_0\|_1^2}{32(J_T-1)\mathbb{E}_f(\tilde{Z}_{1,\ell}^2)}}.$$

We have to bound $\mathbb{E}_f(\tilde{Z}_{1,\ell}^2)$. Observe that

$$
\begin{aligned}
\tilde{Z}_{m,\ell} &\leq \int_{2mT/(2J_T)}^{(2m+1)T/(2J_T)} \left|\nu_\ell - \nu_\ell^0\right| dt + \int_{2mT/(2J_T)}^{(2m+1)T/(2J_T)} \sum_{k=1}^K \int_{t-A}^{t^-} \left|(h_{k,\ell} - h_{k,\ell}^0)(t-s)\right| dN_s^{0,m,k} dt \\
&\leq \frac{T}{2J_T} \left|\nu_\ell - \nu_\ell^0\right| + \sum_{k=1}^K \|h_{k,\ell} - h_{k,\ell}^0\|_1 N^{0,m,k}(I_m)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}_f \left[\tilde{Z}_{1,\ell}^2 \right] &\leq \frac{T^2}{2J_T^2} |\nu_\ell - \nu_\ell^0|^2 + 2K \sum_{k=1}^K \|h_{k,\ell} - h_{k,\ell}^0\|_1^2 \mathbb{E}_f[N^{0,1,k}(I_1)^2] \\
&\leq \frac{T^2}{J_T^2} \|f - f_0\|_1^2 \left(\frac{1}{2} + \frac{2K \max_k \mathbb{E}_f[N^{0,1,k}(I_1)^2] J_T^2}{T^2} \right).
\end{aligned}
$$

We then have to bound $T^{-2}J_T^2 \max_k \mathbb{E}_f[N^{0,1,k}(I_1)^2]$. Using notations of Lemma 8, we have:

$$\begin{aligned}
\mathbb{E}_f[N^{0,1,k}(I_1)^2] &\leq \mathbb{E}_f\left[\left(\sum_{\ell=1}^{K}\sum_{T/(2J_T)\leq p\leq 3T/(2J_T)}\sum_{k=1}^{B_{p,\ell}} W_{k,p}^{\ell}\right)^2\right] \\
&\leq \frac{KT}{J_T}\sum_{\ell=1}^{K}\sum_{T/(2J_T)\leq p\leq 3T/(2J_T)} \mathbb{E}_f\left[\left(\sum_{k=1}^{B_{p,\ell}} W_{k,p}^{\ell}\right)^2\right] \\
&\leq \frac{KT}{J_T}\sum_{\ell=1}^{K}\sum_{T/(2J_T)\leq p\leq 3T/(2J_T)} \mathbb{E}_f\left[\mathbb{E}_f\left[\left(\sum_{k=1}^{B_{p,\ell}} W_{k,p}^{\ell}\right)^2 |B_{p,\ell}\right]\right] \\
&\leq \frac{KT^2}{J_T^2}\sum_{\ell=1}^{K}(\nu_\ell^2+\nu_\ell)\mathbb{E}_f[(W^\ell)^2].
\end{aligned}$$

We now bound $\mathbb{E}_f[(W^\ell)^2]$ by using Lemma 7. Without loss of generality, we can assume that $\|\rho\| > 1/2$. We take $t = \frac{1-\|\rho\|}{2\sqrt{K}}\log\left(\frac{1+\|\rho\|}{2\|\rho\|}\right)$ and

$$\mathbb{E}_f[(W^\ell)^2] \leq 2t^{-2}\mathbb{E}_f[\exp(tW^\ell)] \lesssim t^{-2} \lesssim (1-\|\rho\|)^{-4}$$

and

$$T^{-2}J_T^2\max_k \mathbb{E}_f[N^{0,1,k}(I_1)^2] \lesssim (1-\|\rho\|)^{-4}.$$

Therefore, since $f \in \mathcal{F}_T$, there exists a constant C'_K only depending on K such that

$$\begin{aligned}
\mathbb{P}_f\left(\sum_{m=1}^{J_T-1}(-\tilde{Z}_{m,\ell}+\mathbb{E}_f(\tilde{Z}_{m,\ell})) \geq CT\|f-f_0\|_1/4\,\middle|\,\mathcal{G}_{0^-}\right) &\leq e^{-C'_K J_T(1-\|\rho\|)^4} \leq e^{-C'_K J_T u_T^4} \\
&\lesssim e^{-C'_K\kappa_0(\log T)^{-1}Tu_T^6} \lesssim e^{-C'_K\kappa_0 u_0^6 T\epsilon_T^2}
\end{aligned}$$

where the last inequality follows from the definition of u_T and J_T. We obtain the desired bound as soon as u_0 is large enough.

$$\mathbb{P}_f\left(\Omega_{1,T}\cap\{d_{1,T}\leq T\varepsilon_T\}|\mathcal{G}_{0^-}\right) = o(e^{-c_1T\varepsilon_T^2}).$$

Using (4.18) and Assumption (i), we then have that (4.16) is true, which proves the theorem.

4.6. **Proof of Corollary 1.** Let $w_T \to +\infty$. The proof of Corollary 1 follows from the usual convexity argument, so that

$$\|\hat{f}-f_0\|_1 \leq w_T\varepsilon_T + \mathbb{E}^\pi\left[\|f-f_0\|_1 \mathbb{1}_{\|f-f_0\|_1>w_T\varepsilon_T}|N\right],$$

together with a control of the second term of the right hand side similar to the proof of Theorem 3. We write

$$\mathbb{E}^\pi\left[\|f-f_0\|_1\mathbb{1}_{\|f-f_0\|_1>w_T\varepsilon_T}|N\right] \leq \mathbb{E}^\pi\left[\|f-f_0\|_1\mathbb{1}_{A_{L_1}(w_T\varepsilon_T)^c}\mathbb{1}_{A_{\varepsilon_T}}|N\right] + \mathbb{E}^\pi\left[\|f-f_0\|_1\mathbb{1}_{A_{\varepsilon_T}^c}|N\right]$$

and since $\int\|f-f_0\|_1 d\Pi(f) \leq \|f_0\|_1 + \int\|f\|_1 d\Pi(f) < \infty$,

$$\begin{aligned}
&\mathbb{P}_0\left(\mathbb{E}^\pi\left[\|f-f_0\|_1\mathbb{1}_{A_{L_1}(w_T\varepsilon_T)^c}\mathbb{1}_{A_{\varepsilon_T}}|N\right] > w_T\varepsilon_T\right) \leq \mathbb{P}_0\left(\Omega_{1,T}^c\right) + \mathbb{P}_0\left(D_T < e^{-c_1T\epsilon_T^2}\right) \\
&\quad + \frac{e^{c_1T\epsilon_T^2}}{w_T\varepsilon_T}\int_{A_{L_1}(w_T\varepsilon_T)^c}\|f-f_0\|_1\mathbb{E}_0\left[\mathbb{P}_f\left(\Omega_{1,T}\cap\{d_{1,T}(f_0,f)\leq\varepsilon_T\}\right)|\mathcal{G}_{0^-}\right]d\Pi(f) \\
&\leq o(1) + o(1)\int\|f-f_0\|_1 d\Pi(f) = o(1),
\end{aligned}$$

where the last inequality comes from the proof of Theorem 3. Similarly, using the proof of Theorem 1,

$$\begin{aligned}
&\mathbb{P}_0\left(\mathbb{E}^\pi\left[\|f-f_0\|_1\mathbb{1}_{A_{\varepsilon_T}^c}|N\right] > w_T\varepsilon_T\right) \leq \mathbb{P}_0\left(\Omega_T^c\right) + \mathbb{P}_0\left(D_T < e^{-c_1T\epsilon_T^2}\right) + \mathbb{E}_0[\mathbb{1}_{\Omega_T}\phi] \\
&\quad + \frac{e^{c_1T\epsilon_T^2}}{w_T\varepsilon_T}\int_{A_{L_1}(w_T\varepsilon_T)^c}\|f-f_0\|_1\mathbb{E}_0\left[\mathbb{E}_f\left[(1-\phi)\mathbb{1}_{\Omega_T}\mathbb{1}_{\{d_{1,T}(f_0,f)>\varepsilon_T\}}\right]|\mathcal{G}_{0^-}\right]d\Pi(f) \\
&\leq o(1) + o(1)\int\|f-f_0\|_1 d\Pi(f),
\end{aligned}$$

and $\mathbb{P}_0(\|\hat{f}-f_0\|_1 > 3w_T\varepsilon_T) = o(1)$. Since this is true for any $w_T \to +\infty$, this terminates the proof.

4.7. **Technical lemmas.**

4.7.1. *Control of the number of occurrences of the process on a fixed interval.*

Lemma 4. *For any $M \geq 1$, for any $\alpha > 0$, there exists a constant C_α only depending on f_0 such that for any $T > 0$, the set*

$$\tilde{\Omega}_T = \left\{ \max_{\ell \in \{1,\dots,K\}} \sup_{t\in[0,T]} N^\ell([t-A,t)) \leq C_\alpha \log T \right\}$$

satisfies

$$\mathbb{P}_0(\tilde{\Omega}_T^c) \leq T^{-\alpha}$$

and for any $1 \leq m \leq M$

$$\mathbb{E}_0 \left[\max_{\ell \in \{1,\dots,K\}} \sup_{t\in[0,T]} \left(N^\ell([t-A,t))\right)^m \times 1_{\tilde{\Omega}_T^c} \right] \leq 2T^{-\alpha/2},$$

for T large enough.

Proof. For the first part, we split the interval $[-A;T]$ into disjoint intervals of length A and we use Proposition 2 of Hansen et al. (2015). For the second part, we set

$$X := \max_{\ell \in \{1,\dots,K\}} \sup_{t\in[0,T]} \left(N^\ell([t-A,t))\right) \times 1_{\tilde{\Omega}_T^c} \geq 0$$

and the equality

$$\begin{aligned}
\mathbb{E}_0[X^m] &= \int_0^{+\infty} mx^{m-1}\mathbb{P}_0(X > x)dx \\
&= \int_0^{C_\alpha \log T} mx^{m-1}\mathbb{P}_0\left(X > x\right)dx + \int_{C_\alpha \log T}^{+\infty} mx^{m-1}\mathbb{P}_0\left(X > x\right)dx \\
&\leq m(C_\alpha \log T)^{m-1} \int_0^{C_\alpha \log T} \mathbb{P}_0(\tilde{\Omega}_T^c)dx + \int_{C_\alpha \log T}^{+\infty} mx^{m-1}\mathbb{P}_0\left(X > x\right)dx \\
&\leq m(C_\alpha \log T)^m T^{-\alpha} + \int_{C_\alpha \log T}^{+\infty} mx^{m-1}\mathbb{P}_0\left(X > x\right)dx.
\end{aligned}$$

Furthermore, for T large enough,

$$\begin{aligned}
\int_{C_\alpha \log T}^{+\infty} mx^{m-1}\mathbb{P}_0(X > x)dx &\leq \int_{C_\alpha \log T}^{+\infty} mx^{m-1}\mathbb{P}_0\left(\max_{\ell \in \{1,\dots,K\}} \sup_{t\in[0,T]} \left(N^\ell([t-A,t))\right) > x \right)dx \\
&\leq \int_{C_\alpha \log T}^{+\infty} mx^{m-1}\mathbb{P}_0\left(\max_{\ell \in \{1,\dots,K\}} \sup_{t\in[0,e^{x/C_\alpha}]} \left(N^\ell([t-A,t))\right) > x \right)dx \\
&\leq \int_{C_\alpha \log T}^{+\infty} mx^{m-1}\exp(-\alpha x/C_\alpha)dx \leq T^{-\alpha/2}.
\end{aligned}$$

□

4.7.2. *Control of $N[0,T]$.* Let $k \in \{1,\dots,K\}$. We have the following result.

Lemma 5. *For any $k \in \{1,\dots,K\}$, for all $\alpha > 0$ there exists $\delta_0 > 0$ such that*

$$\mathbb{P}_0\left(\left| \frac{N^k[0,T]}{T} - \mu_k^0 \right| \geq \delta_0 \sqrt{\frac{(\log T)^3}{T}} \right) = O(T^{-\alpha}).$$

Proof of Lemma 5. We use Proposition 3 of Hansen et al. (2015) and notations introduced for this result. We denote $N[-A,0)$ the total number of points of N in $[-A,0)$, all marks included. Let $\delta_T := \delta_0\sqrt{(\log T)^3/T}$, with δ_0 a constant. We have:

(4.21)
$$\mathbb{P}_0\left(\left| \frac{N^k[0,T]}{T} - \mu_k^0 \right| > \delta_T \right) \leq \mathbb{P}_0\left(\left| N^k[0,T] - \int_0^T \lambda_t^k(f_0)dt \right| > \frac{T\delta_T}{2} \right) + \mathbb{P}_0\left(\left| \int_0^T [\lambda_t^k(f_0) - \mu_k^0]dt \right| > \frac{T\delta_T}{2} \right)$$

and we observe that

$$\lambda_t^k(f_0) = \nu_k^0 + \int_{t-A}^{t^-} \sum_{\ell=1}^K h_{\ell,k}^0(t-s)dN_s^\ell = Z \circ \mathfrak{S}_t(N),$$

with $Z(N) = \lambda_0^k(f_0)$, where $\mathfrak{S}$ is the shift operator introduced in Proposition 3 of Hansen et al. (2015). We then have

$$Z(N) \leq b(1 + N[-A,0))$$

with

$$b = \max_k \max\{\nu_k^0, \max_\ell \|h_{\ell,k}^0\|_\infty\}.$$

So, for any $\alpha > 0$, the second term of (4.21) is $O(T^{-\alpha})$ for δ_0 large enough depending on α and f_0. The first term is controlled by using Inequality (7.7) of Hansen et al. (2015) with $\tau = T$, $x = x_0 T \delta_T^2$, $H_t = 1$, $v = \mu_k^0 T + T\delta_T/2$ and

$$M_T = N^k[0,T] - \int_0^T \lambda_t^k(f_0)dt.$$

We take x_0 a positive constant such that $\sqrt{8\mu_k^0 x_0} < 1$, so that, for T large enough

$$\frac{T\delta_T}{2} \geq \sqrt{2vx} + x/3.$$

Therefore, we have

$$\begin{aligned}
\mathbb{P}_0\left(|M_T| > \frac{T\delta_T}{2}\right) &\leq \mathbb{P}_0\left(|M_T| \geq \sqrt{2vx} + x/3 \text{ and } \int_0^T \lambda_t^k(f_0)dt \leq v\right) + \mathbb{P}_0\left(\int_0^T \lambda_t^k(f_0)dt > v\right) \\
&\leq 2\exp(-x) + \mathbb{P}_0\left(\left|\int_0^T [\lambda_t^k(f_0) - \mu_k^0]dt\right| > \frac{T\delta_T}{2}\right) \\
&\leq 2\exp(-x_0\delta_0^2(\log T)^3) + O(T^{-\alpha}) = O(T^{-\alpha}),
\end{aligned}$$

which terminates the proof. $\square$

4.7.3. *Lemma on* $\mathbb{E}_f[Z_{1,\ell}]$. We have the following result which is useful to prove Theorem 3.

Lemma 6. *For for all $f \in \mathcal{F}_T$ such that $d_{1,T}(f, f_0) \leq \varepsilon_T$, there exists ℓ (depending on f and f^0) such that on Ω_T,*

$$\mathbb{E}_f[Z_{1,\ell}] \geq C\frac{T}{J_T}\|f - f^0\|_1,$$

where C is a constant depending on f^0.

Proof. By using the first bound of (4.2), we observe that on Ω_T, for any ℓ, since $\inf_\ell \nu_\ell^0 > 0$, then $\inf_\ell \mu_\ell^0 > 0$ (by using (4.3)) and we obtain that $\sum_{k=1}^K \rho_{k,\ell}$ and $\sum_{k=1}^K \nu_k$ are bounded. Therefore $\|f\|_1$ is bounded. On Ω_T, since $\varepsilon_T \geq \delta_T$, still using (4.2), for any ℓ,

$$\nu_\ell + \sum_{k=1}^K \rho_{k,\ell}\mu_k^0 - M\varepsilon_T \leq \nu_\ell^0 + \sum_{k=1}^K \rho_{k,\ell}^0\mu_k^0 \leq \nu_\ell + \sum_{k=1}^K \rho_{k,\ell}\mu_k^0 + M\varepsilon_T$$

for M a constant large enough. By using the formula

$$\nu_\ell + \sum_{k=1}^K \rho_{k,\ell}\mu_k = \mu_\ell, \quad \nu_\ell^0 + \sum_{k=1}^K \rho_{k,\ell}^0\mu_k^0 = \mu_\ell^0,$$

we obtain

$$\left|(\mu_\ell - \mu_\ell^0) - \sum_k \rho_{k,\ell}(\mu_k - \mu_k^0)\right| \leq M\varepsilon_T,$$

which means that

$$\|(I_d - \rho^T)(\mu - \mu^0)\|_\infty \leq M\varepsilon_T.$$

Therefore, since $\|\rho\| = \|\rho^T\|$ ($\rho\rho^T$ and $\rho^T\rho$ have the same eigenvalues),

$$\begin{aligned}
\|\mu - \mu_0\|_2 &= \|(I_d - \rho^T)^{-1}(I_d - \rho^T)(\mu - \mu_0)\|_2 \\
&\leq (1 - \|\rho\|)^{-1}\sqrt{K}\|(I_d - \rho^T)(\mu - \mu^0)\|_\infty \\
&\leq (1 - \|\rho\|)^{-1}\sqrt{K}M\varepsilon_T.
\end{aligned}$$

Since $f \in \mathcal{F}_T$, $1 - \|\rho\| \geq u_T \gtrsim \varepsilon_T^{1/3}(\log T)^{1/6}$. Therefore, μ is bounded. As in Hansen et al. (2015), we denote $\mathbb{Q}_f$ a measure such that under $\mathbb{Q}_f$ the distribution of the full point process restricted to $(-\infty, 0]$ is identical to the distribution under $\mathbb{P}_f$ and such that on $(0, \infty)$ the process consists of independent components each being a homogeneous Poisson process with rate 1. Furthermore, the Poisson processes should be independent of the process on $(-\infty, 0]$. From Corollary 5.1.2 in Jacobsen (2006) the likelihood process is given by

$$\mathcal{L}_t(f) = \exp\left(Kt - \sum_{k=1}^K \int_0^t \lambda_u^k(f)du + \sum_{k=1}^K \int_0^t \log(\lambda_u^k(f))dN_u^k\right).$$

Let $\tau > 0$ satisfying
$$0 < \frac{A\tau K^2}{1-\tau K} < \frac{1}{2} \quad \text{and} \quad \tau \leq \frac{\min_{\ell'} \nu^0_{\ell'}}{2C'_0},$$
with C'_0 an upper bound of $\|f - f_0\|_1$.

- Assume that for any ℓ', $|\nu_{\ell'} - \nu^0_{\ell'}| < \tau \|f - f_0\|_1$. Then, for any ℓ',
$$|\nu_{\ell'} - \nu^0_{\ell'}| < \tau \|f - f_0\|_1 = \tau \left(\sum_k |\nu_k - \nu^0_k| + \sum_{k,\ell} \|h_{k,\ell} - h^0_{k,\ell}\|_1 \right)$$
and
$$|\nu_{\ell'} - \nu^0_{\ell'}| \leq \sum_\ell |\nu_\ell - \nu^0_\ell| < \frac{\tau K}{1-\tau K} \sum_{k,\ell} \|h_{k,\ell} - h^0_{k,\ell}\|_1.$$
Let ℓ such that
$$\sum_k \|h_{k,\ell} - h^0_{k,\ell}\|_1 = \max_{\ell'} \left\{ \sum_k \|h_{k,\ell'} - h^0_{k,\ell'}\|_1 \right\}.$$
Then, for any ℓ',
$$|\nu_{\ell'} - \nu^0_{\ell'}| < \frac{\tau K^2}{1-\tau K} \sum_k \|h_{k,\ell} - h^0_{k,\ell}\|_1, \tag{4.22}$$
and
$$\begin{aligned} \|f - f^0\|_1 &= \sum_{\ell'} |\nu_{\ell'} - \nu^0_{\ell'}| + \sum_{\ell'} \sum_k \|h_{k,\ell'} - h^0_{k,\ell'}\|_1 \\ &\leq \left(\frac{\tau K^2}{1-\tau K} + K \right) \sum_k \|h_{k,\ell} - h^0_{k,\ell}\|_1. \end{aligned} \tag{4.23}$$
We denote
$$\Omega_k = \left\{ \max_{k' \neq k} N^{k'}[0,A] = 0, \quad N^k[0,A] = 1, \quad N^{k'}[-A,0] \leq aA\mu_{k'} \; \forall k' \right\},$$
where a is a fixed constant chosen later. We then have
$$\begin{aligned} \mathbb{E}_f[Z_{m,\ell}] &= \frac{T}{2J_T} \mathbb{E}_f \left[\left| \nu_\ell - \nu^0_\ell + \sum_{k=1}^K \int_0^{A^-} (h_{k,\ell} - h^0_{k,\ell})(A-s) dN^k_s \right| \right] \\ &\geq \frac{T}{2J_T} \sum_k \mathbb{E}_f \left[\mathbb{1}_{\max_{k' \neq k} N^{k'}[0,A]=0} \mathbb{1}_{N^k[0,A]=1} \left| \nu_\ell - \nu^0_\ell + \int_0^{A^-} (h_{k,\ell} - h^0_{k,\ell})(A-s) dN^k_s \right| \right] \\ &\geq \frac{T}{2J_T} \sum_k \mathbb{E}_{\mathbb{Q}_f} \left[\mathcal{L}_A(f) \mathbb{1}_{\max_{k' \neq k} N^{k'}[0,A]=0} \mathbb{1}_{N^k[0,A]=1} \left| \nu_\ell - \nu^0_\ell + \int_0^{A^-} (h_{k,\ell} - h^0_{k,\ell})(A-s) dN^k_s \right| \right] \\ &\geq \frac{T}{2J_T} \sum_k \mathbb{E}_{\mathbb{Q}_f} \left[\mathcal{L}_A(f) \mathbb{1}_{\Omega_k} \left| \nu_\ell - \nu^0_\ell + \int_0^{A^-} (h_{k,\ell} - h^0_{k,\ell})(A-s) dN^k_s \right| \right]. \end{aligned}$$
Note that on Ω_k,
$$\begin{aligned} \mathcal{L}_A(f) &:= \exp\left(KA - \sum_{k'} \int_0^A \lambda^{k'}_t(f) dt + \sum_{k'} \int_0^A \log(\lambda^{k'}_t(f)) dN^{k'}_t \right) \\ &\geq \nu_k \exp(KA) \exp\left(-\sum_{k'} \int_0^A \lambda^{k'}_t(f) dt \right) \\ &\geq \nu_k \exp(KA) \exp\left(-\sum_{k'} \int_0^A \left(\nu_{k'} + \int_{t-A}^{t-} \sum_{k''} h_{k''k'}(t-u) dN^{k''}_u \right) dt \right) \\ &\geq \nu_k \exp\left(KA - A \sum_{k'} \nu_{k'} \right) \exp\left(-\int_{-A}^{A^-} \sum_{k',k''} \rho_{k''k'} dN^{k''}_u \right) \\ &\geq \nu_k \exp\left(KA - A \sum_{k'} \nu_{k'} \right) \exp\left(-aA \sum_{k''} \mu_{k''} \sum_{k'} \rho_{k''k'} - \sum_{k'} \rho_{kk'} \right). \end{aligned}$$

Since on $\mathcal{F}_T$,

$$\nu_k \exp\left(KA - A\sum_{k'}\nu_{k'}\right)\exp\left(-aA\sum_{k''}\mu_{k''}\sum_{k'}\rho_{k''k'} - \sum_{k'}\rho_{kk'}\right) \geq \nu_k e^{-KaAC_1} \geq \nu_k^0 e^{-KaAC_1}/2 \geq C(f_0),$$

where C_1 and $C(f_0)$ are some constants, we have, by definition of $\mathbb{Q}_f$,

$$\begin{aligned} I_k &:= \mathbb{E}_{\mathbb{Q}_f}\left[\mathcal{L}_A(f)\mathbb{1}_{\Omega_k}\left|\nu_\ell - \nu_\ell^0 + \int_0^{A^-}(h_{k,\ell} - h_{k,\ell}^0)(A-s)dN_s^k\right|\right] \\ &\geq C(f_0)\mathbb{E}_{\mathbb{Q}_f}\left[\mathbb{1}_{N^k[0,A]=1}\left|\nu_\ell - \nu_\ell^0 + \int_0^{A^-}(h_{k,\ell} - h_{k,\ell}^0)(A-s)dN_s^k\right|\right] \\ &\quad \times \mathbb{Q}_f(N^{k'}[-A,0] \leq aA\mu_{k'}\,\forall k') \times \mathbb{Q}_f(\max_{k'\neq k} N^{k'}[0,A] = 0). \end{aligned}$$

Under $\mathbb{Q}_f$, $N^k[0,A] \sim \text{Poisson}(A)$. If $U \sim Unif([0,A])$,

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_f}\left[\mathbb{1}_{N^k[0,A]=1}\left|\int_0^{A^-}(h_{k,\ell} - h_{k,\ell}^0)(A-s)dN_s^k\right|\right] &= \mathbb{E}\left[\left|(h_{k,\ell} - h_{k,\ell}^0)(A-U)\right|\right]\mathbb{Q}_f(N^k[0,A]=1) \\ &= \frac{1}{A}\int_0^A \left|(h_{k,\ell} - h_{k,\ell}^0)(A-s)\right| ds \times Ae^{-A} \\ &= e^{-A}\|h_{k,\ell} - h_{k,\ell}^0\|_1. \end{aligned}$$

We also have, using (4.22),

$$\mathbb{E}_{\mathbb{Q}_f}\left[\mathbb{1}_{N^k[0,A]=1}\left|\nu_\ell - \nu_\ell^0\right|\right] = Ae^{-A}\left|\nu_\ell - \nu_\ell^0\right| \leq Ae^{-A}\frac{\tau K^2}{1-\tau K}\sum_k \|h_{k,\ell} - h_{k,\ell}^0\|_1.$$

Furthermore,

$$\mathbb{Q}_f(\max_{k'\neq k} N^{k'}[0,A] = 0) = \exp(-(K-1)A),$$

and

$$\begin{aligned} \mathbb{Q}_f(N^{k'}[-A,0] \leq aA\mu_{k'}\,\forall k') &\geq 1 - \sum_{k'}\mathbb{Q}_f\left(N^{k'}[-A,0] > aA\mu_{k'}\right) \\ &\geq 1 - \sum_{k'}\frac{\mu_{k'}A}{aA\mu_{k'}} = 1 - \frac{K}{a} = \frac{1}{2}, \end{aligned}$$

with $a = 2K$. Finally,

$$I_k \geq \frac{1}{2}C(f_0)\exp(-KA)\left(1 - \frac{A\tau K^2}{1-\tau K}\right)\|h_{k,\ell} - h_{k,\ell}^0\|_1$$

and using (4.23),

$$\begin{aligned} \mathbb{E}_f[Z_{m,\ell}] &\geq \frac{T}{2J_T}\sum_k I_k \\ &\geq \frac{T}{2J_T}\frac{1}{2}C(f_0)\exp(-KA)\left(1 - \frac{A\tau K^2}{1-\tau K}\right)\sum_k \|h_{k,\ell} - h_{k,\ell}^0\|_1 \\ &\geq C\frac{T}{J_T}\|f - f^0\|_1, \end{aligned}$$

where C depends on f_0.

- We now assume that there exists ℓ such that

$$\left|\nu_\ell - \nu_\ell^0\right| \geq \tau\|f - f_0\|_1.$$

In this case, using similar arguments, still with $a = 2K$,

$$\begin{aligned}
\mathbb{E}_f[Z_{m,\ell}] &\geq \frac{T}{2J_T}\mathbb{P}_f[\{\max_k N^k[0,A]=0\}]\left|\nu_\ell - \nu_\ell^0\right| \\
&\geq \frac{\tau T}{2J_T}\|f-f_0\|_1 \mathbb{E}_{\mathbb{Q}_f}\left[\mathcal{L}_A(f)\mathbb{1}_{\{\max_k N^k[0,A]=0\}}\right] \\
&\geq \frac{\tau T}{2J_T}\|f-f_0\|_1 \mathbb{E}_{\mathbb{Q}_f}\left[\mathcal{L}_A(f)\mathbb{1}_{\{\max_k N^k[0,A]=0\}}\mathbb{1}_{\{N^k[-A,0]\leq aA\mu_k,\,\forall k\}}\right] \\
&\geq \frac{\tau T}{2J_T}\|f-f_0\|_1 \exp\left(KA - A\sum_{k'}\nu_{k'} - aA\sum_{k''}\mu_{k''}\sum_{k'}\rho_{k''k'}\right)\mathbb{E}_{\mathbb{Q}_f}\left[\mathbb{1}_{\{N[0,A]=0\}}\mathbb{1}_{\{\forall k\, N^k[-A,0]\leq aA\mu_k\}}\right] \\
&\geq \frac{\tau T}{4J_T}\|f-f_0\|_1 \exp\left(-A\sum_{k'}\nu_{k'} - aA\sum_{k'}(\mu_{k'}-\nu_{k'})\right) \geq C\frac{T}{J_T}\|f-f_0\|_1
\end{aligned}$$

for C depending on f_0. Lemma 6 is proved.

□

4.7.4. *Upper bound for the Laplace transform of the number of points in a cluster.* In the next lemma, we refine the proof of Lemma 1 of Hansen et al. (2015). Given an ancestor of type ℓ, we denote W^ℓ the number of points in its cluster. We have the following result.

Lemma 7. *Assume $\|\rho\| < 1$ and consider t such that $0 \leq t \leq \frac{1-\|\rho\|}{2\sqrt{K}}\log\left(\frac{1+\|\rho\|}{2\|\rho\|}\right)$. Then, we have for any $\ell \in \{1,\dots,K\}$,*

$$\mathbb{E}_f[\exp(tW^\ell)] \leq \frac{1+\|\rho\|}{2\|\rho\|}.$$

Moreover, if $\|\rho\| \leq 1/2$, then there exist two absolute constants c_0 and C_0 such that if $\sqrt{K}t \leq c_0$, then $\mathbb{E}_f[\exp(tW^\ell)] \leq C_0$. Finally,

$$\mathbb{E}_f[W^\ell] = \mathbb{1}^T(I-\rho^T)^{-1}\mathbf{e}_\ell.$$

Proof of Lemma 7. We introduce $K^\ell(n) \in \mathbb{R}^K$ the vector of the number of descendants of the nth generation from a single ancestral point of type ℓ, with $K^\ell(0) = \mathbf{e}_\ell$, where $(\mathbf{e}_\ell)_k = \mathbb{1}_{k=\ell}$. More precisely, $(K^\ell(n))_k$ is the number of descendants of the nth generation and of the type k from a single ancestral point of type ℓ. Then,

$$W^\ell = \mathbb{1}^T \times \sum_{n=0}^{\infty} K^\ell(n).$$

We now set for any $\theta \in \mathbb{R}^K$,

$$\phi_\ell(\theta) = \log\left(\mathbb{E}_f[\exp(\theta^T K^\ell(1))]\right)$$

and

$$\phi(\theta) = (\phi_1(\theta),\dots,\phi_K(\theta))^T.$$

Note that

$$K^\ell(1)_j \sim \mathcal{P}(\rho_{\ell,j}), \quad \forall j \leq K$$

and

$$\phi_\ell(\theta) = \sum_{j=1}^K \log\left(\mathbb{E}_f[\exp(\theta_j K^\ell(1)_j)]\right) = \sum_{j=1}^K \rho_{\ell,j}(\exp(\theta_j)-1).$$

Therefore,

$$(D\phi(\theta))_{\ell,j} := \frac{\partial\phi_\ell(\theta)}{\partial\theta_j} = \rho_{\ell,j}\exp(\theta_j)$$

and for any $x \in \mathbb{R}^K$, since $\|\rho\| := \sup_{x,\|x\|_2=1}\|\rho x\|_2$,

$$\begin{aligned}
\|D\phi(\theta)x\|_2^2 &= \sum_{\ell=1}^K\left(\sum_{j=1}^K \rho_{\ell,j}\exp(\theta_j)x_j\right)^2 \\
&= \sum_j\sum_{j'}(\rho^T\rho)_{j,j'}\exp(\theta_j)x_j\exp(\theta_{j'})x_{j'} \\
&= v^T\rho^T\rho v \\
&\leq \|\rho\|^2\|v\|_2^2 = \|\rho\|^2\sum_{j=1}^K x_j^2\exp(2\theta_j)
\end{aligned}$$

with v the vector of $\mathbb{R}^K$ such that $v_j = \exp(\theta_j)x_j$. So,

$$|||D\phi(\theta)||| \quad \leq \quad \|\rho\| \max_j \exp(|\theta_j|) \leq \|\rho\| e^{\|\theta\|_2}.$$

So, by applying the mean value theorem,

$$\|\phi(\theta)\|_2 = \|\phi(\theta) - \phi(0)\|_2 \leq \|\rho\| e^{\|\theta\|_2} \|\theta\|_2.$$

We use a modification of the arguments in the proof of Lemma 1 of Hansen et al. (2015). Writing $g_1(\theta) = \theta + \phi(\theta)$, we have for $n \geq 3$:

$$\begin{aligned}
\mathbb{E}_f\left[e^{\theta^T(\sum_{k=0}^n K^\ell(k))}\right] &= \mathbb{E}_f\left[e^{\theta^T(\sum_{k=0}^{n-1} K^\ell(k))}\mathbb{E}_f\left[e^{\theta^T K^\ell(n)}|K^\ell(n-1),\dots,K^\ell(1)\right]\right] \\
&= \mathbb{E}_f\left[e^{\theta^T(\sum_{k=0}^{n-2} K^\ell(k))}e^{(\theta+\phi(\theta))^T K^\ell(n-1)}\right] = \mathbb{E}_f\left[e^{\theta^T(\sum_{k=0}^{n-2} K^\ell(k))}e^{g_1(\theta)^T K^\ell(n-1)}\right] \\
&= \mathbb{E}_f\left[e^{\theta^T(\sum_{k=0}^{n-3} K^\ell(k))}e^{(\theta+\phi(g_1(\theta)))^T K^\ell(n-2)}\right] = \mathbb{E}_f\left[e^{\theta^T(\sum_{k=0}^{n-3} K^\ell(k))}e^{g_2(\theta)^T K^\ell(n-2)}\right] \\
&= \mathbb{E}_f\left[e^{\theta^T K^\ell(0)}e^{g_{n-1}(\theta)^T K^\ell(1)}\right] = e^{(g_n(\theta)_\ell)},
\end{aligned}$$

with the induction formula: $g_n(\theta) = \theta + \phi(g_{n-1}(\theta))$ for $n \geq 2$. In particular,

$$\|g_1(\theta)\|_2 \leq \|\theta\|_2(1 + \|\rho\| e^{\|\theta\|_2}) \quad \text{and} \quad \|g_n(\theta)\|_2 \leq \|\theta\|_2 + \|\rho\| e^{\|g_{n-1}(\theta)\|_2} \|g_{n-1}(\theta)\|_2.$$

We now set $C := (1 + \|\rho\|)/(1 - \|\rho\|) > 1$. Then, if $\|g_{n-1}(\theta)\|_2 \leq \|\theta\|_2(1 + C)$,

$$\|g_n(\theta)\|_2 \leq \|\theta\|_2(1 + \|\rho\|(1 + C)e^{\|\theta\|_2(1+C)}) \leq \|\theta\|_2(1 + C)$$

as soon as

$$\|\theta\|_2 \leq (1 + C)^{-1} \log(C/(\|\rho\|(1 + C))) = \frac{1 - \|\rho\|}{2} \log\left(\frac{1 + \|\rho\|}{2\|\rho\|}\right). \tag{4.24}$$

Since $\|\rho\| < 1$, the previous upper bound is positive. Note that under (4.24), $\|\theta\|_2 \leq \log(C/\|\rho\|)$, and

$$\|g_1(\theta)\|_2 \leq \|\theta\|_2(1 + \|\rho\| e^{\|\theta\|_2}) \leq \|\theta\|_2(1 + \|\rho\| e^{\log(C/\|\rho\|)}) \leq \|\theta\|_2(1 + C).$$

We finally obtain that under (4.24),

$$\|g_n(\theta)\|_2 \leq \|\theta\|_2(1 + C), \quad \forall\, n \geq 1.$$

Since for any m, $n \mapsto \sum_{k=0}^n (K^\ell(k))_m$ is increasing and $W^\ell = \mathbb{1}^T \times \sum_{n=0}^\infty K^\ell(n)$, we have by monotone convergence that for $t > 0$,

$$\mathbb{E}_f[\exp(tW^\ell)] = \lim_{n\to\infty} \exp(g_n(t\mathbf{1})_\ell).$$

By the previous result, the right hand side is bounded if t is small enough. More precisely, for all $0 < t \leq (1 + C)^{-1} \log(C/(\|\rho\|(1 + C)))/\sqrt{K}$,

$$\mathbb{E}_f[\exp(tW^\ell)] \leq \exp(t\sqrt{K}(1 + C)) \leq \frac{C}{\|\rho\|(1 + C)} = \frac{1 + \|\rho\|}{2\|\rho\|}.$$

The second point is obvious in view of previous computations. Moreover, since $\mathbb{E}_f[W^\ell] = \sum_{n=0}^\infty \mathbb{E}_f[\mathbb{1}^T K^\ell(n)]$ and since for any $v \in \mathbb{R}^K$

$$\mathbb{E}_f[v^T K^\ell(n)|K^\ell(0),\dots,K^\ell(n-1)] = \sum_{j=1}^K \sum_{k=1}^K K^\ell(n-1)_j v_k \rho_{j,k} = v^T \rho^T K^\ell(n-1).$$

We obtain by induction that $\mathbb{E}_f[\mathbb{1}^T K^\ell(n)] = \mathbb{1}^T(\rho^T)^n \mathbf{e}_\ell$ and taking the limit, since $\|\rho\| < 1$,

$$\mathbb{E}_f[W^\ell] = \mathbb{1}^T(I - \rho^T)^{-1}\mathbf{e}_\ell.$$

□

4.7.5. *Lemma on $\bar{N}^m$.*

Lemma 8. *There exists $\tilde{c}_0$ such that for all $c_0 > 0$ such that for T large enough,*

$$\mathbb{P}_0\left(\sum_{m=1}^{J_T-1} \bar{N}^m(I_m) > c_0 T\right) \le e^{-\tilde{c}_0 c_0 T}.$$

Furthermore, there exists a constant $\kappa_0 > 0$ (see the definition of J_T) such that

$$\sum_{m=1}^{J_T-1} \mathbb{E}_f[\bar{N}^m(I_m)] = o(T).$$

Proof of Lemma 8. We use computations of the proof of Proposition 2 of Hansen et al. (2015). To bound $\bar{N}^m(I_m)$, first observe that we only consider points of N whose ancestors are born before $(2m-1)T/(2J_T)$, i.e. the distance between the occurrence of an ancestor and I_m is at least $2mT/(2J_T) - A - (2m-1)T/(2J_T) = T/(2J_T) - A$ since

$$I_m = \left[\frac{2mT}{2J_T} - A, \frac{(2m+1)T}{2J_T}\right].$$

Using the cluster representations of the process, for any $p \in \mathbb{Z}$ and for any $\ell \in \{1, \dots, K\}$, we consider $B_{p,\ell}$ the number of ancestors of type ℓ born in the interval $[p, p+1]$. The $B_{p,\ell}$'s are iid Poisson random variables with parameter ν_ℓ. We have

$$\sum_{m=1}^{J_T-1} \bar{N}^m(I_m) \le \sum_{\ell=1}^{K} \sum_{p\in\mathcal{J}_T^+} \sum_{k=1}^{B_{p,\ell}} \left(W_{p,k}^\ell - \frac{1}{A}\left(\frac{T}{2J_T} - A\right)\right)_+ + \sum_{\ell=1}^{K} \sum_{p=-\infty}^{0} \sum_{k=1}^{B_{p,\ell}} \left(W_{p,k}^\ell - \frac{1}{A}\left(-p-1+\frac{T}{J_T} - A\right)\right)_+,$$

where $W_{p,k}^\ell$ is the number of points in the cluster generated by the ancestor k which is of type ℓ and

$$\mathcal{J}_T^+ = \{p :\ 1 \le p \le T - T/(2J_T)\}$$

since

$$\bigcup_{m=1}^{J_T-1} I_m \subset \left[\frac{T}{J_T} - A, T - \frac{T}{2J_T}\right].$$

For the first term of the previous right hand side, we have used same arguments as Hansen et al. (2015) and the lower bound of the distance determined previously. For the second term of the right hand side, since $p \le 0$, this lower bound is at least $-p - 1 + \frac{T}{J_T} - A$. Conditioned on the $B_{p,\ell}$'s, the variables $(W_{p,k}^\ell)_k$ are iid with same distribution as W^ℓ introduced in Lemma 7. Furthermore, by Lemma 7 applied with $f = f_0$, since $\|\rho_0\| < 1$, we know that for $t_0 > 0$ small enough (only depending on $\|\rho_0\|$ and K),

$$\mathbb{E}_0[\exp(t_0 W^\ell)] \le C_0,$$

where C_0 is a constant. So, for any $c > 0$,

$$\begin{aligned}
\mathcal{P}_{T,1} &:= \mathbb{P}_0\left(\sum_{\ell=1}^{K} \sum_{p\in\mathcal{J}_T^+} \sum_{k=1}^{B_{p,\ell}} \left(W_{p,k}^\ell - \frac{1}{A}\left(\frac{T}{2J_T} - A\right)\right)_+ \ge cT\right)\\
&\le \exp(-t_0 cT) \prod_{\ell=1}^{K} \prod_{p\in\mathcal{J}_T^+} \mathbb{E}_0\left[\prod_{k=1}^{B_{p,\ell}} \mathbb{E}_0\left[\exp\left(t_0\left(W_{p,k}^\ell - \frac{T}{2AJ_T} + 1\right)_+\right) |B_{p,\ell}\right]\right]\\
&\le \exp(-t_0 cT) \prod_{\ell=1}^{K} \prod_{p\in\mathcal{J}_T^+} \mathbb{E}_0\left[(H_\ell(t_0))^{B_{p,\ell}}\right] = \exp\Big(-t_0 cT + \sum_{\ell=1}^{K} \sum_{p\in\mathcal{J}_T^+} \nu_\ell^0 (H_\ell(t_0) - 1)\Big),
\end{aligned}$$

where

$$H_\ell(t_0) := \mathbb{E}_0\left[\exp\left(t_0\left(W^\ell - \frac{T}{2AJ_T} + 1\right)_+\right)\right],$$

satisfying

$$\begin{aligned}
H_\ell(t_0) &\le \mathbb{P}_0\left(W^\ell \le \frac{T}{2AJ_T} - 1\right) + \exp(t_0 - Tt_0/(2AJ_T))\mathbb{E}_0\left[\exp\left(t_0 W^\ell\right)\right]\\
&\le 1 + C_0 \exp(t_0 - Tt_0/(2AJ_T)).
\end{aligned}$$

Therefore,

$$\sum_{\ell=1}^{K}\sum_{p\in\mathcal{J}_T^+}\nu_\ell^0(H_\ell(t_0)-1)\lesssim(T-T/(2J_T)\exp(-Tt_0/(2AJ_T))\lesssim e^{-C'\kappa_0\log T}=o(t_0cT)$$

by choosing κ_0 large enough and then

$$\mathcal{P}_{T,1}\lesssim\exp(-t_0cT/2).$$

Similarly,

$$\begin{aligned}\mathcal{P}_{T,2}&:=\mathbb{P}_0\left(\sum_{\ell=1}^{K}\sum_{p=-\infty}^{0}\sum_{k=1}^{B_{p,\ell}}\left(W_{p,k}^{\ell}-\frac{1}{A}\left(-p-1+\frac{T}{J_T}-A\right)\right)_+\geq cT\right)\\&\leq\exp\Big(-t_0cT+\sum_{\ell=1}^{K}\sum_{p=-\infty}^{0}\nu_\ell^0(\tilde H_{\ell,p}(t_0)-1)\Big),\end{aligned}$$

where

$$\tilde H_{\ell,p}(t_0):=\mathbb{E}_0\left[\exp\left(t_0\left(W^\ell-\frac{T}{AJ_T}+1+\frac{1}{A}+\frac{p}{A}\right)_+\right)\right],$$

satisfying

$$\tilde H_{\ell,p}(t_0)\leq 1+C_0\exp(t_0+t_0/A-Tt_0/(AJ_T)+t_0p/A).$$

Therefore,

$$\sum_{\ell=1}^{K}\sum_{p=-\infty}^{0}\nu_\ell^0(\tilde H_{\ell,p}(t_0)-1)\lesssim\exp(-Tt_0/(AJ_T))=o(t_0cT)$$

and then

$$\mathcal{P}_{T,2}\lesssim\exp(-t_0cT/2).$$

Finally, there exists $\tilde c_0$ (only depending on t_0, so only depending on $\|\rho_0\|$ and K) such that for all $c_0>0$ such that for T large enough

$$\mathbb{P}_f\left(\sum_{m=0}^{J_T-1}\bar N^m(I_m)>c_0T\right)\leq e^{-\tilde c_0c_0T}$$

and the first part of the lemma is proved.

For the second part, we only consider the case $1/2\leq\|\rho\|<1$. The case $\|\rho\|<1/2$ can be derived easily using following computations. We have:

$$\sum_{m=1}^{J_T-1}\mathbb{E}_f[\bar N^m(I_m)]=\mathcal{E}_{T,1}+\mathcal{E}_{T,2},$$

with

$$\mathcal{E}_{T,1}:=\mathbb{E}_f\left[\sum_{\ell=1}^{K}\sum_{p\in\mathcal{J}_T^+}\sum_{k=1}^{B_{p,\ell}}\left(W_{p,k}^{\ell}-\frac{1}{A}\left(\frac{T}{2J_T}-A\right)\right)_+\right]$$

and, with $t=\frac{1-\|\rho\|}{2\sqrt{K}}\log\left(\frac{1+\|\rho\|}{2\|\rho\|}\right)\gtrsim(1-\|\rho\|)^2\gtrsim u_T^2$ on $\mathcal{F}_T$, since for $x>0$, $x\leq e^x$, by using Lemma 7,

$$\begin{aligned}\mathcal{E}_{T,2}&:=\mathbb{E}_f\left[\sum_{\ell=1}^{K}\sum_{p=-\infty}^{0}\sum_{k=1}^{B_{p,\ell}}\left(W_{p,k}^{\ell}-\frac{1}{A}\left(-p-1+\frac{T}{J_T}-A\right)\right)_+\right]\\&=\sum_{\ell=1}^{K}\nu_\ell\sum_{p=-\infty}^{0}\mathbb{E}_f\left[\left(W_{p,k}^{\ell}-\frac{1}{A}\left(-p-1+\frac{T}{J_T}-A\right)\right)_+\right]\\&=t^{-1}\sum_{\ell=1}^{K}\nu_\ell\sum_{p=-\infty}^{0}\mathbb{E}_f\left[e^{t\left(W_{p,k}^{\ell}-\frac{1}{A}\left(-p-1+\frac{T}{J_T}-A\right)\right)}\right]\\&\lesssim t^{-1}e^{-\frac{tT}{AJ_T}}(1-e^{-\frac{t}{A}})^{-1}\mathbb{E}_f\left[e^{tW^\ell}\right]\sum_{\ell=1}^{K}\nu_\ell\\&\lesssim(1-\|\rho\|)^{-4}e^{-\frac{(1-\|\rho\|)^2T}{AJ_T}}\sum_{\ell=1}^{K}\nu_\ell\lesssim e^{-\kappa_0^{-1}C''\log T}(\log T)^{-2/3}\varepsilon_T^{-4/3},\end{aligned}$$

for C'' depending on A and K. Similarly,

$$\mathcal{E}_{T,1} \lesssim T(1 - \|\rho\|)^{-2} e^{-\frac{(1-\|\rho\|)^2 T}{2AJ_T}} \sum_{\ell=1}^{K} \nu_\ell.$$

Choosing κ_0 small enough,

$$\sum_{m=1}^{J_T - 1} \mathbb{E}_f[\bar{N}^m(I_m)] = o(T).$$

□

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