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HAJER BAHOURI, JEAN-YVES CHEMIN, AND ISABELLE GALLAGHER

Abstract. We prove a weak stability result for the three-dimensional homogeneous incompressible Navier-Stokes system. More precisely, we investigate the following problem : if a sequence $(u_{0,n})_{n\in\mathbb{N}}$ of initial data, bounded in some scaling invariant space, converges weakly to an initial data u_0 which generates a global smooth solution, does $u_{0,n}$ generate a global smooth solution ? A positive answer in general to this question would imply global regularity for any data, through the following examples $u_{0,n} = n\varphi_0(n\cdot)$ or $u_{0,n} = \varphi_0(\cdot - x_n)$ with $|x_n| \to \infty$. We therefore introduce a new concept of weak convergence (rescaled weak convergence) under which we are able to give a positive answer. The proof relies on profile decompositions in anisotropic spaces and their propagation by the Navier-Stokes equations.

1. Introduction and statement of the main result

1.1. The Navier-Stokes equations. We are interested in the Cauchy problem for the three dimensional, homogeneous, incompressible Navier-Stokes system

$$\text{(NS)} \qquad \begin{cases} \partial_t u + u\cdot\nabla u - \Delta u = -\nabla p \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^3 \\ \operatorname{div} u = 0 \\ u_{|t=0} = u_0\,, \end{cases}$$

where $p = p(t,x)$ and $u = (u^1,u^2,u^3)(t,x)$ are respectively the pressure and velocity of an incompressible, viscous fluid.

As is well-known, the Navier-Stokes system enjoys two important features. First it formally conserves the energy, in the sense that smooth and decaying solutions satisfy the following energy equality for all times $t\geq 0$:

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^3)}\,dt' = \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^3)}\,. \tag{1.1}$$

Second, (NS) enjoys a scaling invariance property: defining the scaling operators, for any positive real number λ and any point x_0 of $\mathbb{R}^3$,

$$\Lambda_{\lambda,x_0}\phi(t,x) \stackrel{\text{def}}{=} \frac{1}{\lambda}\phi\Big(\frac{t}{\lambda^2},\frac{x-x_0}{\lambda}\Big) \quad \text{and} \quad \Lambda_{\lambda}\phi(t,x) \stackrel{\text{def}}{=} \frac{1}{\lambda}\phi\Big(\frac{t}{\lambda^2},\frac{x}{\lambda}\Big)\,, \tag{1.2}$$

if u solves (NS) with data u_0, then $\Lambda_{\lambda,x_0}u$ solves (NS) with data $\Lambda_{\lambda,x_0}u_0$.

1.2. The Cauchy problem. We shall say that $u\in L^2_{\text{loc}}([0,T]\times\mathbb{R}^3)$ is a *weak solution* of (NS) associated with the data u_0 if for any compactly supported, divergence free vector field ϕ belonging to $\mathcal{C}^\infty([0,T]\times\mathbb{R}^3)$ the following identity holds for all $t\leq T$:

$$\int_{\mathbb{R}^3} u\cdot\phi(t,x)dx = \int_{\mathbb{R}^3} u_0(x)\cdot\phi(0,x)dx + \int_0^t\int_{\mathbb{R}^3}(u\cdot\Delta\phi + u\otimes u:\nabla\phi + u\cdot\partial_t\phi)(t',x)dxdt'\,,$$

with

$$u\otimes u:\nabla\phi \stackrel{\text{def}}{=} \sum_{1\leq j,k\leq 3} u^j u^k\partial_k\phi^j\,.$$

Key words and phrases. Navier-Stokes equations; anisotropy; Besov spaces; profile decomposition.

Weak solutions satisfying the energy inequality

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^3)} + \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^3)}\, dt' \leq \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^3)} \tag{1.3}$$

are said to be *turbulent solutions*, following the terminology of J. Leray [45].

We shall say that a familly $(X_T)_{T>0}$ of spaces of distributions defined over $[0,T]\times\mathbb{R}^3$ is *scaling invariant* if for all $T>0$ one has, with notation (1.2),

$$\forall \lambda>0\,,\forall x_0\in\mathbb{R}^3\,,\; u\in X_T \Longleftrightarrow \Lambda_{\lambda,x_0}u\in X_{\lambda^{-2}T} \quad\text{with}\quad \|u\|_{X_T}=\|\Lambda_{\lambda,x_0}u\|_{X_{\lambda^{-2}T}}\,.$$

Similarly a space X_0 of distributions defined on $\mathbb{R}^3$ will be said to be scaling invariant if

$$\forall \lambda>0\,,\forall x_0\in\mathbb{R}^3\,,\; u_0\in X_0 \Longleftrightarrow \Lambda_{\lambda,x_0}u_0\in X_0 \quad\text{with}\quad \|u_0\|_{X_0}=\|\Lambda_{\lambda,x_0}u_0\|_{X_0}\,.$$

This leads to the following definition of a solution, which will be the notion of solution we shall consider throughout this work.

Definition 1.1. *A vector field u is a (scaled) solution to* (NS) *on* $[0,T]$, *associated with the data u_0 if it is a weak solution in X_T, where X_T belongs to a family of scaling invariant spaces.*

The energy conservation (1.1) is the main ingredient which enabled J. Leray to prove in [45] that any initial data in $L^2(\mathbb{R}^3)$ gives rise to (at least) one global turbulent solution to (NS). The result is the following.

Theorem 1 ([45, 46])**.** *Associated with any divergence free vector field in $L^2(\mathbb{R}^d)$ there is a global in time turbulent solution. Moreover if $d=2$ then this solution is unique.*

Uniqueness in space dimension 2, which is proved in [46], is linked to the fact that $L^2(\mathbb{R}^2)$ is scale invariant. In dimensions three and more, the question of the uniqueness of Leray's solutions is still an open problem; we refer to the recent work [34] for some numerical evidence of non uniqueness. Related to that problem, a number of results have been proved concerning the uniqueness, and global in time existence of solutions under a scaling invariant smallness assumption on the data – note that smallness has to be measured in a scale invariant space to have any relevance. Without such a smallness assumption, existence and uniqueness often holds in a scale invariant space for a short time but nothing is known beyond that time, at which some scale-invariant norms of the solution could blow up. The question of the possible blow up in finite time of solutions to (NS) is actually one of the Millenium Prize Problems in Mathematics. We shall not recall all the results existing in the literature concerning the Cauchy problem in scale invariant spaces for the Navier-Stokes system; we refer for instance to [2], [44], [49] and the references therein, for surveys on the subject. Let us nevertheless recall that along with the fundamental Theorem 1, J. Leray also proved that if u_0 is a divergence free vector field satisfying

$$\|u_0\|_{L^2(\mathbb{R}^3)}\|\nabla u_0\|_{L^2(\mathbb{R}^3)} \leq c \tag{1.4}$$

for a small enough c, then there is only one turbulent solution associated with u_0, and the bound (1.4) still holds for future times. Notice that the quantity $\|u_0\|_{L^2(\mathbb{R}^3)}\|\nabla u_0\|_{L^2(\mathbb{R}^3)}$ is invariant by the scaling operator Λ_{λ,x_0}. Without the smallness assumption (1.4), the uniqueness property holds at least for a short time, time at which the solution ceases to belong to H^1: we recall the definition of (homogeneous) Sobolev spaces, given by the (semi-)norm

$$\|f\|_{H^s} \stackrel{\text{def}}{=} \left(\int |\widehat{f}(\xi)|^2|\xi|^{2s}\, d\xi\right)^{\frac{1}{2}}\,.$$

Note that in d space dimensions, $H^s(\mathbb{R}^d)$ is a normed space only if $s < d/2$. Homogeneous spaces are usually denoted by $\dot H^s(\mathbb{R}^d)$ but since this paper is only concerned with homogeneous spaces we choose to drop the dot in the notation. J. Leray also proved that if one turbulent solution u lies in $L^2([0,T];L^\infty(\mathbb{R}^d))$, then all turbulent solutions associated with the same initial data as u coincide with u on $[0,T]$. Thus $L^2([0,T];L^\infty(\mathbb{R}^d))$ is a uniqueness class for the Navier-Stokes system. Let us now recall the following slightly more general statement than the one described above: it is due to H. Fujita and T. Kato [21], who proved that if $u_0 \in H^{\frac12}(\mathbb{R}^3)$ is a divergence free vector field satisfying $\|u_0\|_{H^{\frac12}(\mathbb{R}^3)} \leq c$ for a small enough constant c, then there is only one turbulent solution associated with u_0. It satisfies

$$\|u(t)\|^2_{H^{\frac12}(\mathbb{R}^3)} + \int_0^t \|\nabla u(t')\|^2_{H^{\frac12}(\mathbb{R}^3)}\, dt' \leq \|u_0\|^2_{H^{\frac12}(\mathbb{R}^3)}\,.$$

Without the smallness assumption, the uniqueness property holds at least for a short time, time at which the solution ceases to belong to $L^2([0,T];H^{\frac32}(\mathbb{R}^3))$. Note that this generalizes the Leray result since by interpolation

$$\|u_0\|^2_{H^{\frac12}(\mathbb{R}^3)} \leq \|u_0\|_{L^2(\mathbb{R}^3)}\|\nabla u_0\|_{L^2(\mathbb{R}^3)}\,. \tag{1.5}$$

Many results of this type are known to hold, for instance replacing $H^{\frac12}(\mathbb{R}^3)$ by the larger Lebesgue space $L^3(\mathbb{R}^3)$ (see [33, 38, 62]). The best result known to this day on the uniqueness of solutions to (NS) is due to H. Koch and D. Tataru [43]. It is proved, as most results of the type, by a fixed point theorem in an appropriate Banach space. The smallness condition is the following:

$$\begin{aligned}\|u_0\|_{\mathrm{BMO}^{-1}(\mathbb{R}^3)} &\stackrel{\mathrm{def}}{=} \sup_{t>0} t^{\frac12}\|e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^3)}\\ &\qquad + \sup_{\substack{x\in\mathbb{R}^3\\ R>0}} \frac{1}{R^{\frac32}}\Big(\int_{[0,R^2]\times B(x,R)} |(e^{t\Delta}u_0)(t,y)|^2\,dydt\Big)^{\frac12} \leq c\,.\end{aligned}$$

Note that the space BMO^{-1} is again invariant by the scaling operators Λ_{λ,x_0}. In the definition of BMO^{-1} norm above, the norm $\sup\limits_{t>0} t^{\frac12}\|e^{t\Delta}u_0\|_{L^\infty(\mathbb{R}^3)}$ is equivalent to the Besov norm $\|u_0\|_{B^{-1}_{\infty,\infty}(\mathbb{R}^3)}$. The Besov space $B^{-1}_{\infty,\infty}(\mathbb{R}^3)$ is actually the largest space in which any scale and translation invariant Banach space of tempered distributions embeds; it is in fact known that (NS) is illposed for initial data in $B^{-1}_{\infty,\infty}(\mathbb{R}^3)$ (see [10] and [28]), but for small data in $B^{-1+\frac3p}_{p,\infty}$ for finite p global existence and uniqueness are known to hold (see [52]). More on Besov spaces is provided in Appendix A, let us recall their definition here.

Definition 1.2. *Let $\widehat\chi$ be a radial function in $\mathcal{D}(\mathbb{R})$ such that $\widehat\chi(t) = 1$ for $|t| \leq 1$ and $\widehat\chi(t) = 0$ for $|t| > 2$. For $j \in \mathbb{Z}$, the truncation operators are defined by*

$$\widehat{S_j f}(\xi) \stackrel{\mathrm{def}}{=} \widehat\chi\big(2^{-j}|\xi|\big)\hat f(\xi) \quad \textit{and} \quad \Delta_j \stackrel{\mathrm{def}}{=} S_{j+1} - S_j\,.$$

For all p in $[1,\infty]$ and q in $]0,\infty]$, and all s in $\mathbb{R}$, with $s < 3/p$ (or $s \leq 3/p$ if $q = 1$), the homogeneous Besov space $B^s_{p,q}$ is defined as the space of tempered distributions f such that

$$\|f\|_{B^s_{p,q}} \stackrel{\mathrm{def}}{=} \Big\|2^{js}\|\Delta_j f\|_{L^p}\Big\|_{\ell^q} < \infty\,.$$

In all other cases of indexes s, the Besov space is defined similarly, up to taking the quotient with polynomials.

The results recalled above tend to suggest that the initial data should satisfy some sort of smallness assumption if one is to prove global existence and uniqueness of solutions. Actually this turns out not to be the case: there are situations where global unique solutions are known to exist despite the fact that the initial data is not small in $B^{-1}_{\infty,\infty}$. That is the case in two space dimensions as recalled above, as well as under some geometric assumptions (helicity, axisymmetry without swirl...). Let us describe a result of that type, whose main interest is that its proof gives an idea of the methods used in this work in a simple framework.

Theorem 2 ([15, 17]). *Consider the sequence of divergence free vector fields*

$$u_{0,n}(x) = u_0(x) + \big(v_0^1, v_0^2, 0\big)\big(x_1, x_2, \frac{x_3}{n}\big) \tag{1.6}$$

with (v_0^1, v_0^2) a smooth, two-component, divergence free vector field, satisfying

$$(v_0^1, v_0^2)(x_1, x_2, 0) \equiv 0 \quad \textit{if } u_0 \textit{ is not identically zero}.$$

If u_0 gives rise to a unique, global solution to the Navier-Stokes equations, then so does $u_{0,n}$ as soon as n is large enough.

The case when $u_0 \equiv 0$ is proved in [15]. It consists in looking for the solution u_n as

$$u_n(t,x) = \big(v^1, v^2, 0\big)\big(t, x_1, x_2, \frac{x_3}{n}\big) + r_n(t,x)$$

where for all y_3, $v(\cdot, y_3) \stackrel{\text{def}}{=} (v^1, v^2)(\cdot, y_3)$ solves the two-dimensional Navier-Stokes equations with data $(v_0^1, v_0^2)(\cdot, y_3)$. We know that v is unique, and globally defined thanks to Theorem 1. Then the key to the proof is that r_n solves a perturbed Navier-Stokes equation of the type

$$\partial_t r_n + r_n \cdot \nabla r_n + v \cdot \nabla r_n + r_n \cdot \nabla v - \Delta r_n = -\nabla p + f_n\,, \quad \operatorname{div} r_n = 0\,,$$

where the error term f_n contains derivatives in x_3 of $\big(v^1, v^2, 0\big)\big(t, x_1, x_2, \frac{x_3}{n}\big)$, which are of size roughly n^{-1}, hence small. One can therefore solve the equation satisfied by r_n using the same methods as solving globally (NS) with small data and small force. In the case when u_0 is not identically zero, the proof consists in looking for the solution under the form

$$u_n(t,x) = u(t,x) + \big(v^1, v^2, 0\big)\big(t, x_1, x_2, \frac{x_3}{n}\big) + \widetilde{r}_n(t,x)$$

with u the global solution associated with u_0. Then the rough idea is that u decays at infinity in x_3 whereas due to the fact that $(v_0^1, v_0^2)(x_1, x_2, 0) \equiv 0$, the vector field

$$\big(v^1, v^2, 0\big)\Big(t, x_1, x_2, \frac{x_3}{n}\Big)$$

has a support roughly in $x_3 \sim n$. So those two functions do not interact one with the other, and the perturbed equation satisfied by $\widetilde{r}_n$ can again be solved globally.

It should be noted that the sequence $u_{0,n}$ of Theorem 2 converges in the sense of distributions to u_0. The goal of this work is to try to understand if such a property, which we can call "weak stability", holds more generally: we would like to address the question of *weak stability*:

If $(u_{0,n})_{n\in\mathbb{N}}$, bounded in some scale invariant space X_0, converges to u_0 in the sense of distributions, with u_0 giving rise to a global smooth solution, is it the case for $u_{0,n}$ when n is large enough ?

1.3. **Strong stability results.** Let us recall that it is proved in [1] (see [24] for the case of Besov spaces $B^{-1+3/p}_{p,q}$) that the set of initial data generating a global solution is open in BMO^{-1}. More precisely, denoting by VMO^{-1} the closure of smooth functions in BMO^{-1}, it is proved in [1] that if u_0 belongs to VMO^{-1} and generates a global, smooth solution to (NS), then any sequence $(u_{0,n})_{n\in\mathbb{N}}$ converging to u_0 in the BMO^{-1} norm also generates a global smooth solution as soon as n is large enough. The question asked above addresses the case when the sequence converges non longer strongly, but in the sense of distributions.

1.4. **Weak stability results.**

1.4.1. *A stability result for weak convergence up to rescaling in $B^{-1+3/p}_{p,q}(\mathbb{R}^3)$.* To answer the above question, the first example that may come to mind is the case when $u_0 \equiv 0$ (which gives rise to the unique, global solution which is identically zero), and

$$u_{0,n}(x) = \frac{1}{\lambda_n}\Phi_0(\frac{x}{\lambda_n}) = \Lambda_{\lambda_n}\Phi_0(x) \quad \text{with} \quad \lim_{n\to\infty}\Big(\lambda_n + \frac{1}{\lambda_n}\Big) = \infty\,, \tag{1.7}$$

with Φ_0 an arbitrary divergence-free vector field. If the weak stability result we are after were true, then since the weak limit of $(u_{0,n})_{n\in\mathbb{N}}$ is zero then for n large enough $u_{0,n}$ would give rise to a unique, global solution. By scale invariance then so would Φ_0, and this for any Φ_0, so that would solve the global regularity problem for (NS). Another natural example is the sequence

$$u_{0,n} = \Phi_0(\cdot - x_n) = \Lambda_{1,x_n}\Phi_0\,, \tag{1.8}$$

with $(x_n)_{n\in\mathbb{N}}$ a sequence of $\mathbb{R}^3$ going to infinity. Thus sequences built by rescaling fixed divergence free vector fields according to the invariances of the equation have to be excluded from our analysis, since solving (NS) for any smooth initial data seems out of reach. This naturally leads to the following definition.

Definition 1.3 (Convergence up to rescaling)**.** *We say that a sequence $(\varphi_n)_{n\in\mathbb{N}}$ defined on $\mathbb{R}^3$ converges up to rescaling to φ if φ_n converges to φ in the sense of distributions and if for all sequences $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers and for all sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^3$ satisfying*

$$\lambda_n + \frac{1}{\lambda_n} + |x_n| \to \infty \quad n\to\infty\,, \tag{1.9}$$

the sequence $(\Lambda_{\lambda_n,x_n}\varphi_n)_{n\in\mathbb{N}}$ converges to 0 in the sense of distributions, as n goes to infinity.

The following result is a first answer to our question. Its proof is straightforward. We choose to present it for pedagogical reasons, to relate the notion of convergence up to rescaling to strong convergence in a larger scale invariant space.

Proposition 1.4. *Let p and q be two real numbers in $[1,\infty[$ and consider $(u_{0,n})_{n\in\mathbb{N}}$ a sequence of divergence-free vector fields bounded in $B^{-1+3/p}_{p,q}(\mathbb{R}^3)$, converging up to rescaling to u_0, with u_0 giving rise to a global unique solution. Then the same holds for $u_{0,n}$ as soon as n is large enough.*

Note that the same theorem actually holds in any scale invariant space strictly embedded in BMO^{-1}.

Proof. The proof of Proposition 1.4 relies on the following "profile decomposition" theorem, which describes the lack of compactness of the embedding $B^{-1+3/p}_{p,q}$ into $B^{-1+3/\tilde p}_{\tilde p,\tilde q}$ for indices $p<\tilde p$ and $q<\tilde q$. The proof of that result can be found in [3], following the pioneering work of [27] in the framework of Sobolev spaces H^s and [35] for Sobolev spaces $W^{s,p}$. More on profile decompositions is to be found in Section 2.

Proposition 1.5 ([3]). *Fix $p<\tilde p$ and $q<\tilde q$ four real numbers in $[1,\infty[$ and consider $(\varphi_n)_{n\in\mathbb{N}}$ a sequence of functions, bounded in $B^{-1+3/p}_{p,q}(\mathbb{R}^3)$ and converging weakly to some function φ^0. Then up to extracting a subsequence (which we denote in the same way), there is a family of functions $(\varphi^j)_{j\geq 1}$ in $B^{-1+3/p}_{p,q}(\mathbb{R}^3)$, and a family $(x^j_n)_{j\geq1}$ of sequences of points in $\mathbb{R}^3$, as well as a family of sequences of positive real numbers $(h^j_n)_{j\geq 1}$, orthogonal in the sense that if $j\neq k$ then*

$$\text{either}\quad \frac{h^j_n}{h^k_n}+\frac{h^k_n}{h^j_n}\to\infty \quad \text{as } n\to\infty\,,\quad \text{or}\quad h^j_n=h^k_n \quad\text{and}\quad \frac{|x^k_n-x^j_n|}{h^j_n}\to\infty \quad \text{as } n\to\infty$$

such that for all integers $L\geq 1$ the function $\psi^L_n \stackrel{\text{def}}{=} \varphi_n-\varphi^0-\sum_{j=1}^{L}\Lambda_{h^j_n,x^j_n}\varphi^j$ satisfies

$$\limsup_{n\to\infty}\|\psi^L_n\|_{B^{-1+3/\tilde p}_{\tilde p,\tilde q}(\mathbb{R}^3)}\to 0 \quad \text{as } L\to\infty\,.$$

Moreover one has

$$\Lambda_{(h^j_n)^{-1},-(h^j_n)^{-1}x^j_n}\varphi_n \rightharpoonup \varphi^j\,,\quad \text{as } n\to\infty\,. \tag{1.10}$$

Note that the result (1.10) is not explicitly stated in [3] but is easy to check. Proposition 1.4 is then an immediate consequence of Proposition 1.5. Indeed if $(u_{0,n})$ is bounded in $B^{-1+3/p}_{p,q}(\mathbb{R}^3)$, then one can decompose each of its components using Proposition 1.5, and the convergence up to rescaling assumption, joint with (1.10), implies that all profiles are zero. The sequence $(u_{0,n})$ therefore converges strongly in $B^{-1+3/\tilde p}_{\tilde p,\tilde q}(\mathbb{R}^3)$ and the result follows from the strong stability in $B^{-1+3/\tilde p}_{\tilde p,\tilde q}(\mathbb{R}^3)$ proved in [24] and recalled in Section 1.3. $\square$

1.4.2. *Stability under rescaled weak convergence.* Considering Theorem 2, it is natural to try to extend Proposition 1.4 to more general situations. Indeed the sequences

$$u_{0,n}(x)=(v^1_0,v^2_0,0)\big(x_1,x_2,\frac{x_3}{n}\big) \tag{1.11}$$

and

$$\widetilde u_{0,n}(x)=u_0(x)+(\widetilde v^1_0,\widetilde v^2_0,0)\big(x_1,x_2,\frac{x_3}{n}\big)\,,\quad \text{with}\quad \widetilde v_0(x_1,x_2,0)\equiv 0 \tag{1.12}$$

are *not* bounded in $B^{-1+3/p}_{p,q}$ (or in any such scale invariant space) but we do know that they converge weakly to a vector field giving rise to a global solution, and that the same holds for each term of the sequence as soon as soon as n is large enough. In order to understand in what direction one can generalize Proposition 1.4 to take into account such examples, there are two points to clarify on the sequences (1.11) and (1.12):

(1) what function spaces they are bounded in;
(2) what type of weak convergence (possibly after rescaling as in Definition 1.3) holds for those sequences.

The main feature of the sequences defined in (1.11) and (1.12) is that they are not bounded in any space $B^{-1+3/p}_{p,q}$, but rather in *anisotropic* spaces where the regularity in the third variable scales like L^∞: for instance $L^2(\mathbb{R}^2;H^{\frac12}(\mathbb{R}))$, or $L^2(\mathbb{R}^2;L^\infty(\mathbb{R}))$. Notice that those spaces are scaling invariant by the scaling operator Λ_{λ,x_0} and satisfy the additional invariance for the change of variable

$$(x_1,x_2,x_3)\longmapsto(x_1,x_2,\lambda x_3)$$

for any positive λ. It seems therefore natural to work in those function spaces, or others having the same scaling properties. Unfortunately $H^{\frac12}(\mathbb{R})$ is not a Banach space, and that fact makes analysis in $H^{\frac12}(\mathbb{R})$ rather awkward. We shall therefore trade $H^{\frac12}(\mathbb{R})$ off for the slightly smaller Besov space $B^{\frac{1}{2}}_{2,1}$: we define anisotropic Besov spaces as follows. These spaces generalize the more usual isotropic Besov spaces seen in Definition 1.2, which are studied for instance in [2, 9, 54, 59, 60].

Definition 1.6. *With the notation of Definition 1.2, for $(j,k)\in\mathbb{Z}^2$, the horizontal truncations are defined by*

$$\widehat{S^{\rm h}_k f}(\xi)\stackrel{\rm def}{=}\widehat{\chi}\big(2^{-k}|(\xi_1,\xi_2)|\big)\hat f(\xi)\quad\text{and}\quad \Delta^{\rm h}_k\stackrel{\rm def}{=}S^{\rm h}_{k+1}-S^{\rm h}_k\,,$$

and the vertical truncations by

$$\widehat{S^{\rm v}_j f}\stackrel{\rm def}{=}\widehat{\chi}(2^{-j}|\xi_3|)\hat f(\xi)\quad\text{and}\quad \Delta^{\rm v}_j\stackrel{\rm def}{=}S^{\rm v}_{j+1}-S^{\rm v}_j\,.$$

For all p in $[1,\infty]$ and q in $]0,\infty]$, and all (s,s') in $\mathbb{R}^2$, with $s<2/p, s'<1/p$ (or $s\leq 2/p$ and $s'\leq 1/p$ if $q=1$), the anisotropic homogeneous Besov space $B^{s,s'}_{p,q}$ is defined as the space of tempered distributions f such that

$$\|f\|_{B^{s,s'}_{p,q}}\stackrel{\rm def}{=}\Big\|2^{ks+js'}\|\Delta^{\rm h}_k\Delta^{\rm v}_j f\|_{L^p}\Big\|_{\ell^q}<\infty\,.$$

In all other cases of indexes s and s', the Besov space is defined similarly, up to taking the quotient with polynomials.

Notation. We shall in what follows use the following shorthand notation:
(1.13)

$$\mathcal{B}^{s,s'}_p\stackrel{\rm def}{=}B^{s,s'}_{p,1}\,,\quad \mathcal{B}^{s,s'}\stackrel{\rm def}{=}\mathcal{B}^{s,s'}_2\,,\quad \mathcal{B}^{s}_{p,q}\stackrel{\rm def}{=}B^{-1+\frac{2}{p}+s,\frac{1}{p}}_{p,q}\,,\quad \mathcal{B}^{s}_p\stackrel{\rm def}{=}\mathcal{B}^{s}_{p,1}\,,\quad\text{and}\quad \mathcal{B}^{s}\stackrel{\rm def}{=}\mathcal{B}^{s}_2\,.$$

Let us point out that the scaling operators (1.2) satisfy

$$\|\Lambda_{\lambda,x_0}\varphi\|_{\mathcal{B}^0_{p,q}}=\|\varphi\|_{\mathcal{B}^0_{p,q}}\,.$$

The Navier-Stokes equations in anisotropic spaces have been studied in a number of frameworks. We refer for instance, among others, to [4], [19], [30], [32], [51]. In particular in [4] it is proved that if u_0 belongs to $\mathcal{B}^0$, then there is a unique solution (global in time if the data is small enough) in $L^2([0,T];\mathcal{B}^1)$. That norm controls the equation, in the sense that as soon as the solution belongs to $L^2([0,T];\mathcal{B}^1)$, then it lies in fact in $L^r([0,T];\mathcal{B}^{\frac{2}{r}})$ for all $1\leq r\leq\infty$. The space $\mathcal{B}^1$ is included in L^∞ and since the seminal work [45] of J. Leray recalled above, it is known that the $L^2([0,T];L^\infty(\mathbb{R}^3))$ norm controls the propagation of regularity and also ensures weak uniqueness among turbulent solutions. Thus the space $\mathcal{B}^0$ is natural in this context.

The initial data defined in (1.11) converges in the sense of distributions to the two-dimensional vector field $(u_0^1,u_0^2,0)\big(x_1,x_2,0\big)$, whereas the one defined in (1.12) converges in the sense of distributions to u_0. This leads naturally to a stronger notion of weak convergence, denoted by *rescaled weak convergence*, which we shall call *R-convergence*.

Definition 1.7 (R-convergence)**.** *We say that a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of tempered distributions defined on $\mathbb{R}^3$ R-converges to φ if φ_n converges to φ in the sense of distributions, and if for all sequences $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers and for all sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^3$ satisfying (1.9), up to extracting a subsequence there is a tempered distribution ψ of (x_1,x_2) such that $(\Lambda_{\lambda_n,x_n}\varphi_n)_{n\in\mathbb{N}}$ converges to ψ in the sense of distributions, as n goes to infinity.*

The following examples give some insight into the type of sequences that can be considered with Definition 1.7.

Proposition 1.8. *Let μ_n be a sequence of positive real numbers converging to infinity. Then*

(1) The sequence $\varphi_n^{(1)}(x) \stackrel{\rm def}{=} \dfrac{1}{\mu_n}\varphi^{(1)}\Big(\dfrac{x}{\mu_n}\Big)$*, with* $\varphi^{(1)}$ *a smooth function, R-converges weakly to 0 if and only if* $\varphi^{(1)}$ *only depends on* (x_1, x_2)*.*

(2) The sequence $\varphi_n^{(2)}(x) \stackrel{\rm def}{=} \varphi^{(2)}\Big(x_1, x_2, \dfrac{x_3}{\mu_n}\Big)$*, with* $\varphi^{(2)}$ *a smooth function, R-converges weakly to* $\varphi^{(2)}(x_1, x_2, 0)$*.*

Proof. (1) Obviously the sequence $\varphi_n^{(1)}$ converges to zero in the sense of distributions, and the same goes for $\Lambda_{1,x_n}\varphi_n^{(1)}$ if $|x_n| \to \infty$. Now let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers going to zero or infinity, and for any $(x_n)_{n\in\mathbb{N}}$, consider the sequence $(\Lambda_{\lambda_n,x_n}\varphi_n^{(1)}(x))_{n\in\mathbb{N}}$, which is given by

$$\Lambda_{\lambda_n,x_n}\varphi_n^{(1)}(x) = \frac{1}{\lambda_n\mu_n}\varphi^{(1)}\Big(\frac{x-x_n}{\lambda_n\mu_n}\Big).$$

This sequence goes to zero in the sense of distributions as long as $\lambda_n\mu_n$ does not converge to a constant. So assume now $\lambda_n\mu_n \to 1$. We notice that

$$\Lambda_{\frac{1}{\mu_n},x_n}\varphi_n^{(1)}(x) = \varphi^{(1)}(x - x_n)$$

which again goes to zero in the sense of distributions if $|x_n| \to \infty$. Finally if $|x_n|$ is bounded, then up to a subsequence we may assume that $x_n \to a \in \mathbb{R}^3$ in which case $\Lambda_{\frac{1}{\mu_n},x_n}\varphi_n^{(1)}$ converges in the sense of distributions to $\varphi^{(1)}(x-a)$, and the assumption requires that $\varphi^{(1)}$ is a function of (x_1, x_2) only.

(2). Next consider the sequence $\varphi_n^{(2)}$. Clearly it converges to $\varphi^{(2)}(x_1, x_2, 0)$ in the sense of distributions, so let us check the R-convergence property. We have

$$\Lambda_{\lambda_n,x_n}\varphi_n^{(2)}(x) = \frac{1}{\lambda_n}\varphi\Big(\frac{x_1 - x_{1,n}}{\lambda_n}, \frac{x_2 - x_{2,n}}{\lambda_n}, \frac{x_3 - x_{3,n}}{\lambda_n\mu_n}\Big),$$

which clearly goes to zero in the sense of distributions when $(\lambda_n)_{n\in\mathbb{N}}$ goes to zero or infinity. The same goes when $\lambda_n = 1$ and $(x_{1,n}, x_{2,n}) \to \infty$, so let us finally assume that $\lambda_n = 1$ and $(x_{1,n}, x_{2,n})$ is bounded. In that case we write

$$\Lambda_{1,x_n}\varphi_n^{(2)}(x) = \varphi\Big(x_1 - x_{1,n}, x_2 - x_{2,n}, \frac{x_3 - x_{3,n}}{\mu_n}\Big),$$

which, up to a subsequence, converges to zero or to a function of (x_1, x_2) depending on the behaviour of the sequence $x_{3,n}/\mu_n$ and on the limit of $(x_{1,n}, x_{2,n})$. This ends the proof of Proposition 1.8. □

Our main result is the following.

Theorem 3. *Let* q *be given in* $]0, 1[$ *and let* u_0 *in* $\mathcal{B}^0_{2,q}$ *generate a unique global solution to* (NS) *in* $L^2(\mathbb{R}^+; \mathcal{B}^1)$*. Let* $(u_{0,n})_{n\in\mathbb{N}}$ *be a sequence of divergence free vector fields bounded in* $\mathcal{B}^0_{2,q}$*, such that* $u_{0,n}$ R-converges *to* u_0*. Then for* n *large enough,* $u_{0,n}$ *generates a unique, global solution to* (NS) *in the space* $L^2(\mathbb{R}^+; \mathcal{B}^1)$*.*

Acknowledgments. We want to thank very warmly Pierre Germain for suggesting the concept of rescaled weak convergence.

1.5. **Main steps of the proof of Theorem 3.**

1.5.1. *Anisotropic profile decomposition of the initial data.* To prove Theorem 3, the first step consists in the proof of an anisotropic profile decomposition of the sequence of initial data, in the spirit of Proposition 1.5. Let us start by introducing some definitions and notations.

Definition 1.9. *We say that two sequences of positive real numbers $(\lambda_n^1)_{n\in\mathbb{N}}$ and $(\lambda_n^2)_{n\in\mathbb{N}}$ are orthogonal if*

$$\frac{\lambda_n^1}{\lambda_n^2}+\frac{\lambda_n^2}{\lambda_n^1}\to\infty\,,\quad n\to\infty\,.$$

A family of sequences $\big((\lambda_n^j)_{n\in\mathbb{N}}\big)_j$ is said to be a family of scales if $\lambda_n^0\equiv 1$ and if λ_n^j and λ_n^k are orthogonal when $j\neq k$.

Notation. For all points $x=(x_1,x_2,x_3)$ in $\mathbb{R}^3$ and all vector fields $u=(u^1,u^2,u^3)$, we denote their horizontal projections by

$$x_{\rm h}\stackrel{\rm def}{=}(x_1,x_2)\quad\text{and}\quad u^{\rm h}\stackrel{\rm def}{=}(u^1,u^2)\,.$$

We shall be considering functions which have different types of variations in the x_3 variable and the $x_{\rm h}$ variable. The following notation will be used:

$$\big[f\big]_\beta(x)\stackrel{\rm def}{=}f(x_{\rm h},\beta x_3)\,.$$

Clearly, for any function f, we have the following identity which will be of constant use all along this work:

$$\big\|[f]_\beta\big\|_{\mathcal{B}_p^{s_1,s_2}}\sim\beta^{s_2-\frac1p}\|f\|_{\mathcal{B}_p^{s_1,s_2}}\,. \tag{1.14}$$

In all that follows, θ is a given function in $\mathcal{D}(B_{\mathbb{R}^3}(0,1))$ which has value 1 near $B_{\mathbb{R}^3}(0,1/2)$. For any positive real number η, we denote

$$\theta_\eta(x)\stackrel{\rm def}{=}\theta(\eta x)\quad\text{and}\quad\theta_{{\rm h},\eta}(x_{\rm h})\stackrel{\rm def}{=}\theta_\eta(x_{\rm h},0)\,. \tag{1.15}$$

In order to make notations as light as possible, the letter v (possibly with indices) will always denote a two-component divergence free vector field, which may depend on the vertical variable x_3.

Finally we define horizontal differentiation operators $\nabla^{\rm h}\stackrel{\rm def}{=}(\partial_1,\partial_2)$ and ${\rm div}_{\rm h}\stackrel{\rm def}{=}\nabla^{\rm h}\cdot$, as well as $\Delta_{\rm h}\stackrel{\rm def}{=}\partial_1^2+\partial_2^2$, and we shall use the following shorthand notation: $X_{\rm h}Y_{\rm v}\stackrel{\rm def}{=}X(\mathbb{R}^2;Y(\mathbb{R}))$ where X is a function space defined on $\mathbb{R}^2$ and Y is defined on $\mathbb{R}$.

Definition 1.10. *Let μ be a positive real number less than 1/2, fixed from now on.*
We define $D_\mu\stackrel{\rm def}{=}[-2+\mu,1-\mu]\times[1/2,7/2]$ and $\widetilde D_\mu\stackrel{\rm def}{=}[-1+\mu,1-\mu]\times[1/2,3/2]$. We denote by S_μ the space of functions a belonging to $\displaystyle\bigcap_{(s,s')\in D_\mu}\mathcal{B}^{s,s'}$ such that

$$\|a\|_{S_\mu}\stackrel{\rm def}{=}\sup_{(s,s')\in D_\mu}\|a\|_{\mathcal{B}^{s,s'}}<\infty\,.$$

Remark 1.11. *Everything proved here would work choosing for D_μ any set of the type $[-2+\mu,1-\mu]\times[1/2,A]$, with $A\geq 7/2$. For simplicity we limit ourselves to the case when $A=7/2$.*

Proposition 1.12. *Under the assumptions of Theorem 3 and up to the extraction of a subsequence, the following holds. Let $p>2$ be given. There is a family of scales $\big((\lambda_n^j)_{n\in\mathbb{N}}\big)_{j\in\mathbb{N}}$ and for all $L\geq 1$ there is a family of sequences $\big((h_n^j)_{n\in\mathbb{N}}\big)_{j\in\mathbb{N}}$ going to zero when n goes to ∞ such that for any real number α in $]0,1[$, there are families of sequences of divergence-free vector fields (for $j\in[1,L]$), $(v_{n,\alpha,L}^j)_{n\in\mathbb{N}}$, $(w_{n,\alpha,L}^j)_{n\in\mathbb{N}}$, $(v_{n,\alpha,L}^{0,\infty})_{n\in\mathbb{N}}$, $(w_{0,n,\alpha,L}^{0,\infty})_{n\in\mathbb{N}}$, $(v_{0,n,\alpha,L}^{0,{\rm loc}})_{n\in\mathbb{N}}$*

and $(w^{0,\mathrm{loc}}_{0,n,\alpha,L})_{n\in\mathbb{N}}$ *all belonging to* S_μ*, and a smooth, compactly supported function* $u_{0,\alpha}$ *such that the sequence* $(u_{0,n})_{n\in\mathbb{N}}$ *can be written under the form*

$$u_{0,n} \equiv u_{0,\alpha} + \big[\big(v^{0,\infty}_{0,n,\alpha,L} + h^0_n w^{0,\infty,\mathrm{h}}_{0,n,\alpha,L}, w^{0,\infty,3}_{0,n,\alpha,L}\big)\big]_{h^0_n} + \big[(v^{0,\mathrm{loc}}_{0,n,\alpha,L} + h^0_n w^{0,\mathrm{loc},\mathrm{h}}_{0,n,\alpha,L}, w^{0,\mathrm{loc},3}_{0,n,\alpha,L})\big]_{h^0_n}$$
$$+ \sum_{j=1}^{L} \Lambda_{\lambda^j_n}\big[(v^j_{n,\alpha,L} + h^j_n w^{j,\mathrm{h}}_{n,\alpha,L}, w^{j,3}_{n,\alpha,L})\big]_{h^j_n} + \rho_{n,\alpha,L}$$

where $u_{0,\alpha}$ *approximates* u_0 *in the sense that*

$$\lim_{\alpha\to 0} \|u_{0,\alpha} - u_0\|_{\mathcal{B}^0} = 0\,, \tag{1.16}$$

where the remainder term satisfies

$$\lim_{L\to\infty}\lim_{\alpha\to 0}\limsup_{n\to\infty} \|e^{t\Delta}\rho_{n,\alpha,L}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)} = 0\,, \tag{1.17}$$

while the following uniform bounds hold:

$$\begin{aligned}\mathcal{M} \stackrel{def}{=} \sup_{L\geq 1}\sup_{\alpha\in]0,1[}\sup_{n\in\mathbb{N}}\Big(&\big\|(v^{0,\infty}_{0,n,\alpha,L}, w^{0,\infty,3}_{0,n,\alpha,L})\big\|_{\mathcal{B}^0} + \big\|(v^{0,\mathrm{loc}}_{0,n,\alpha,L}, w^{0,\mathrm{loc},3}_{0,n,\alpha,L})\big\|_{\mathcal{B}^0}\\ &+ \|u_{0,\alpha}\|_{\mathcal{B}^0} + \sum_{j=1}^{L}\big\|(v^j_{n,\alpha,L}, w^{j,3}_{n,\alpha,L})\big\|_{\mathcal{B}^0}\Big) < \infty\end{aligned} \tag{1.18}$$

and for all α *in* $]0,1[$,

$$\begin{aligned}\mathcal{M}_\alpha \stackrel{def}{=} \sup_{L\geq 1}\sup_{\substack{1\leq j\leq L\\ n\in\mathbb{N}}}\Big(&\big\|(v^{0,\infty}_{0,n,\alpha,L}, w^{0,\infty,3}_{0,n,\alpha,L})\big\|_{S_\mu} + \big\|(v^{0,\mathrm{loc}}_{0,n,\alpha,L}, w^{0,\mathrm{loc},3}_{0,n,\alpha,L})\big\|_{S_\mu}\\ &+ \|u_{0,\alpha}\|_{S_\mu} + \big\|(v^j_{n,\alpha,L}, w^{j,3}_{n,\alpha,L})\big\|_{S_\mu}\Big)\end{aligned} \tag{1.19}$$

is finite. Finally, we have

$$\lim_{L\to\infty}\lim_{\alpha\to 0}\limsup_{n\to\infty}\big\|\big(v^{0,\mathrm{loc}}_{0,n,\alpha,L}, w^{0,\mathrm{loc},3}_{0,n,\alpha,L}\big)(\cdot,0)\big\|_{B^0_{2,1}(\mathbb{R}^2)} = 0\,, \tag{1.20}$$

$$\forall\,(\alpha,L)\,,\exists\,\eta(\alpha,L)\,/\,\forall\eta\leq\eta(\alpha,L)\,,\forall n\in\mathbb{N}\,,\ (1-\theta_{\mathrm{h},\eta})(v^{0,\mathrm{loc}}_{0,n,\alpha,L}, w^{0,\mathrm{loc},3}_{0,n,\alpha,L}) \equiv 0\,,\ \textit{and} \tag{1.21}$$

$$\forall\,(\alpha,L,\eta)\,,\ \exists\, n(\alpha,L,\eta)\,/\,\forall n\geq n(\alpha,L,\eta)\,,\ \theta_{\mathrm{h},\eta}(v^{0,\infty}_{0,n,\alpha,L}, w^{0,\infty,3}_{0,n,\alpha,L}) \equiv 0\,. \tag{1.22}$$

The proof of this proposition is the purpose of Section 2.

Proposition 1.12 states that the sequence $u_{0,n}$ is equal, up to a small remainder term, to a finite sum of orthogonal sequences of divergence-free vector fields. These sequences are obtained from the profile decomposition derived in [4] (see Proposition 2.2 in this work) by grouping together all the profiles having the same horizontal scale λ_n, and the form they take depends on whether the scale λ_n is identically equal to one or not. In the case when λ_n goes to 0 or infinity, these sequences are of the type $\Lambda_{\lambda_n}\big[(v^{\mathrm{h}}_n + h_n w^{\mathrm{h}}_n, w^3_n)\big]_{h_n}$, with h_n a sequence going to zero. In the case when λ_n is identically equal to one, we deal with three types of orthogonal sequences: the first one consists in $u_{0,\alpha}$, an approximation of the weak limit u_0, the second one given by $\big[(v^{\mathrm{loc},\mathrm{h}}_{0,n,\alpha,L} + h^0_n w^{\mathrm{loc},\mathrm{h}}_{0,n,\alpha,L}, w^{\mathrm{loc},3}_{0,n,\alpha,L})\big]_{h^0_n}$ is uniformly localized in the horizontal variable and vanishes at $x_3 = 0$, while the horizontal support of the third one $\big[(v^{\infty,\mathrm{h}}_{0,n,\alpha,L} + h^0_n w^{\infty,\mathrm{h}}_{0,n,\alpha,L}, w^{\infty,3}_{0,n,\alpha,L})\big]_{h^0_n}$ goes to infinity.

Note that in contrast with classical profile decompositions (as stated in Proposition 1.5 for instance), cores of concentration do not appear in the profile decomposition given in Proposition 1.12 since all the profiles with the same horizontal scale are grouped together, and thus the decomposition is written in terms of scales only. The price to pay is that the profiles are no longer fixed functions, but bounded sequences.

Let us point out that the R-convergence of $u_{0,n}$ to u_0 arises in a crucial way in the proof of Proposition 1.12. It excludes in the profile decomposition of $u_{0,n}$ sequences of type (1.7) and (1.8).

1.5.2. *Proof of Theorem 3.* Once Proposition 1.12 is known, the main step of the proof of Theorem 3 consists in proving that each individual profile involved in the decomposition of Proposition 1.12 does generate a global solution to (NS) as soon as n is large enough. This is based on the following results concerning respectively profiles $\Lambda_{\lambda_n^j}\big[(v^j_{n,\alpha,L}+h^j_n w^{j,\mathrm{h}}_{n,\alpha,L}, w^{j,3}_{n,\alpha,L})\big]_{h^j_n}$, with λ^j_n going to 0 or infinity, and profiles of horizontal scale one, see respectively Theorems 4 and 5. Then, an orthogonality argument leads to the fact that the sum of the profiles also generates a global regular solution for large enough n.

In order to state the results, let us define the function spaces we shall be working with.

Definition 1.13. – *We define the space* $\mathcal{A}^{s,s'}_p = L^\infty(\mathbb{R}^+;\mathcal{B}^{s,s'}_p)\cap L^2(\mathbb{R}^+;\mathcal{B}^{s+1,s'}_p)$ *equipped with the norm*

$$\|a\|_{\mathcal{A}^{s,s'}_p} \stackrel{\mathrm{def}}{=} \|a\|_{L^\infty(\mathbb{R}^+;\mathcal{B}^{s,s'}_p)} + \|a\|_{L^2(\mathbb{R}^+;\mathcal{B}^{s+1,s'}_p)}\,,$$

and we denote $\mathcal{A}^{s,s'}=\mathcal{A}^{s,s'}_2$ *and* $\mathcal{A}^s=\mathcal{A}^{s,\frac12}$.

– *We denote by* $F^{s,s'}_{p,q}$ *any function space such that*

$$\|L_0 f\|_{L^2(\mathbb{R}^+;B^{s+1,s'}_{p,q})} \lesssim \|f\|_{F^{s,s'}_{p,q}}$$

where, for any non negative real number τ, $L_\tau f$ *is the solution of* $\partial_t L_\tau f - \Delta L_\tau f = f$ *with* $L_\tau f_{|t=\tau}=0$. *We denote* $\mathcal{F}^s_p = F^{-1+\frac2p+s,\frac1p}_{p,1}$ *and* $\mathcal{F}^s=\mathcal{F}^s_2$.

Examples. Using the smoothing effect of the heat flow as described by Lemma A.2, it is easy to prove that the spaces $L^1(\mathbb{R}^+;B^{s,s'}_{p,q})$ and $L^1(\mathbb{R}^+;\mathcal{B}^{s+1,s'-1}_p)$ are continuously embedded in $F^{s,s'}_{p,q}$. We refer to Lemma A.3 for a proof, along with other examples.

In the following we shall denote by $\mathcal{T}_0(A,B)$ a generic constant depending only on the quantities A and B. We shall denote by $\mathcal{T}_1$ a generic non decreasing function from $\mathbb{R}^+$ into $\mathbb{R}^+$ such that

$$\limsup_{r\to0}\frac{\mathcal{T}_1(r)}{r}<\infty\,, \tag{1.23}$$

and by $\mathcal{T}_\infty$ a generic locally bounded function from $\mathbb{R}^+$ into $\mathbb{R}^+$. All those functions may vary from line to line. Let us notice that for any positive sequence $(a_n)_{n\in\mathbb{N}}$ belonging to ℓ^1, we have

$$\sum_n \mathcal{T}_1(a_n)\le \mathcal{T}_\infty\Big(\sum_n a_n\Big)\,. \tag{1.24}$$

The notation $a\lesssim b$ means that an absolute constant C exists such that $a\le Cb$.

Theorem 4. *A locally bounded function* ε_1 *from* $\mathbb{R}^+$ *into* $\mathbb{R}^+$ *exists which satisfies the following. For any* (v_0,w^3_0) *in* S_μ *(see Definition* 1.10*), for any positive real number* β *such that* $\beta\le\varepsilon_1(\|(v_0,w^3_0)\|_{S_\mu})$*, the divergence free vector field*

$$\Phi_0 \stackrel{\mathrm{def}}{=} \big[(v_0-\beta\nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\partial_3 w^3_0, w^3_0)\big]_\beta$$

generates a global solution Φ_β *to* (NS) *which satisfies*

$$\|\Phi_\beta\|_{\mathcal{A}^0}\le \mathcal{T}_1(\|(v_0,w^3_0)\|_{\mathcal{B}^0})+\beta\,\mathcal{T}_\infty(\|(v_0,w^3_0)\|_{S_\mu})\,. \tag{1.25}$$

Moreover, for any (s,s') *in* $[-1+\mu, 1-\mu]\times[1/2,7/2]$, *we have, for any* r *in* $[1,\infty]$,

$$\|\Phi_\beta\|_{L^r(\mathbb{R}^+;\mathcal{B}^{s+\frac{2}{r}})}+\frac{1}{\beta^{s'-\frac12}}\|\Phi_\beta\|_{L^r(\mathbb{R}^+;\mathcal{B}^{\frac{2}{r},s'})}\leq \mathcal{T}_\infty(\|(v_0,w_0^3)\|_{S_\mu}). \tag{1.26}$$

The proof of this theorem is the purpose of Section 3. Let us point out that this theorem is a global existence result for the Navier-Stokes system associated with a new class of arbitrarily large initial data generalizing the example considered in [15], and where the regularity is sharply estimated, in particular in terms of anisotropic norms.

The existence of a global regular solution for the set of profiles associated with the horizontal scale 1 is ensured by the following theorem.

Theorem 5. *Let us consider the initial data, with the notation of Proposition* 1.12,

$$\Phi^0_{0,n,\alpha,L}\stackrel{def}{=}u_{0,\alpha}+\big[\big(v^{0,\infty}_{0,n,\alpha,L}+h^0_nw^{0,\infty,\mathrm{h}}_{0,n,\alpha,L},w^{0,\infty,3}_{0,n,\alpha,L}\big)\big]_{h^0_n}+\big[\big(v^{0,\mathrm{loc}}_{0,n,\alpha,L}+h^0_nw^{0,\mathrm{loc,h}}_{0,n,\alpha,L},w^{0,\mathrm{loc},3}_{0,n,\alpha,L}\big)\big]_{h^0_n}.$$

There is a constant ε_0, *depending only on* u_0 *and on* $\mathcal{M}_\alpha$, *such that if* $h^0_n\leq\varepsilon_0$, *then the initial data* $\Phi^0_{0,n,\alpha,L}$ *generates a global smooth solution* $\Phi^0_{n,\alpha,L}$ *which satisfies for all* s *in* $[-1+\mu,1-\mu]$ *and all* r *in* $[1,\infty]$,

$$\|\Phi^0_{n,\alpha,L}\|_{L^r(\mathbb{R}^+;\mathcal{B}^{s+\frac{2}{r}})}\leq\mathcal{T}_0(u_0,\mathcal{M}_\alpha). \tag{1.27}$$

The proof of this theorem is the object of Section 4. As Theorem 4, this is also a global existence result for the Navier-Stokes system, generalizing Theorem 3 of [16] and Theorem 2 of [17], where we control regularity in a very precise way.

Proof of Theorem 3. Let us consider the profile decomposition given by Proposition 1.12. For a given positive (and small) ε, Assertion (1.17) allows to choose α, L and N_0 (depending of course on ε) such that

$$\forall n\geq N_0\,,\ \|e^{t\Delta}\rho_{n,\alpha,L}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\leq\varepsilon. \tag{1.28}$$

From now on the parameters α and L are fixed so that (1.28) holds. Now let us consider the two functions ε_1, $\mathcal{T}_1$ and $\mathcal{T}_\infty$ (resp. ε_0 and $\mathcal{T}_0$) which appear in the statement of Theorem 4 (resp. Theorem 5). Since each sequence $(h^j_n)_{n\in\mathbb{N}}$, for $0\leq j\leq L$, goes to zero as n goes to infinity, let us choose an integer N_1 greater than or equal to N_0 such that

$$\forall n\geq N_1\,,\ \forall j\in\{0,\dots,L\}\,,\ h^j_n\leq\min\Big\{\varepsilon_1(\mathcal{M}_\alpha),\varepsilon_0,\frac{\varepsilon}{L\mathcal{T}_\infty(\mathcal{M}_\alpha)}\Big\}. \tag{1.29}$$

Then for $1\leq j\leq L$ (resp. $j=0$), let us denote by $\Phi^j_{n,\varepsilon}$ (resp. $\Phi^0_{n,\varepsilon}$) the global solution of (NS) associated with the initial data

$$\big[(v^j_{n,\alpha,L}+h^j_nw^{j,\mathrm{h}}_{n,\alpha,L},w^{j,3}_{n,\alpha,L})\big]_{h^j_n}$$
$$\Big(\text{resp.}\quad u_{0,\alpha}+\big[\big(v^{0,\infty}_{0,n,\alpha,L}+h^0_nw^{0,\infty,\mathrm{h}}_{0,n,\alpha,L},w^{0,\infty,3}_{0,n,\alpha,L}\big)\big]_{h^0_n}+\big[(v^{0,\mathrm{loc}}_{0,n,\alpha,L}+h^0_nw^{0,\mathrm{loc,h}}_{0,n,\alpha,L},w^{0,\mathrm{loc},3}_{0,n,\alpha,L})\big]_{h^0_n}\Big)$$

given by Theorem 4 (resp. Theorem 5). We look for the global solution associated with $u_{0,n}$ under the form

$$u_n=u^{\mathrm{app}}_{n,\varepsilon}+R_{n,\varepsilon}\quad\text{with}\quad u^{\mathrm{app}}_{n,\varepsilon}\stackrel{\mathrm{def}}{=}\sum_{j=0}^L\Lambda_{\lambda^j_n}\Phi^j_{n,\varepsilon}+e^{t\Delta}\rho_{n,\alpha,L}\,,$$

recalling that $\lambda^0_n\equiv1$, see Definition 1.9. As recalled in Section 1, $\Lambda_{\lambda^j_n}\Phi^j_{n,\varepsilon}$ solves (NS) with the initial data $\Lambda_{\lambda^j_n}\big[(v^j_{n,\alpha,L}+h^j_nw^{j,\mathrm{h}}_{n,\alpha,L},w^{j,3}_{n,\alpha,L})\big]_{h^j_n}$ by the scaling invariance of the Navier-Stokes equations. Plugging this decomposition into the Navier-Stokes equation therefore gives the following equation on $R_{n,\varepsilon}$:

$$\partial_t R_{n,\varepsilon} - \Delta R_{n,\varepsilon} + \operatorname{div}\big(R_{n,\varepsilon}\otimes R_{n,\varepsilon} + R_{n,\varepsilon}\otimes u^{\mathrm{app}}_{n,\varepsilon} + u^{\mathrm{app}}_{n,\varepsilon}\otimes R_{n,\varepsilon}\big) + \nabla p_{n,\varepsilon} = F_{n,\varepsilon} = F^1_{n,\varepsilon} + F^2_{n,\varepsilon} + F^3_{n,\varepsilon} \quad \text{with}$$

$$(1.30)\qquad \begin{aligned} F^1_{n,\varepsilon} &\stackrel{\mathrm{def}}{=} \operatorname{div}\big(e^{t\Delta}\rho_{n,\alpha,L}\otimes e^{t\Delta}\rho_{n,\alpha,L}\big)\\ F^2_{n,\varepsilon} &\stackrel{\mathrm{def}}{=} \sum_{j=0}^{L} \operatorname{div}\left(\Lambda_{\lambda_n^j}\Phi^j_{n,\varepsilon}\otimes e^{t\Delta}\rho_{n,\alpha,L} + e^{t\Delta}\rho_{n,\alpha,L}\otimes \Lambda_{\lambda_n^j}\Phi^j_{n,\varepsilon}\right) \quad \text{and}\\ F^3_{n,\varepsilon} &\stackrel{\mathrm{def}}{=} \sum_{\substack{0\le j,k\le L\\ j\neq k}} \operatorname{div}\big(\Lambda_{\lambda_n^j}\Phi^j_{n,\varepsilon}\otimes \Lambda_{\lambda_n^k}\Phi^k_{n,\varepsilon}\big)\,,\end{aligned}$$

and where $\big(\operatorname{div}(u\otimes v)\big)^j = \sum_{k=1}^{3}\partial_k(u^jv^k)$.

We shall prove that there is an integer $N\ge N_1$ such that with the notation of Definition 1.13,

$$(1.31)\qquad \forall n\ge N\,,\quad \|F_{n,\varepsilon}\|_{\mathcal{F}^0_p}\le C\varepsilon\,,$$

where C only depends on L and $\mathcal{M}_\alpha$. In the next estimates we omit the dependence of all constants on α and L, which are fixed.

Let us start with the estimate of $F^1_{n,\varepsilon}$. Using the fact that $\mathcal{B}^1_p$ is an algebra, we have

$$\big\|e^{t\Delta}\rho^{\mathrm{h}}_{n,\alpha,L}\otimes e^{t\Delta}\rho_{n,\alpha,L}\big\|_{L^1(\mathbb{R}^+;\mathcal{B}^1_p)} \lesssim \big\|e^{t\Delta}\rho_{n,\alpha,L}\big\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\,,$$

so

$$\|\operatorname{div}_{\mathrm{h}}\big(e^{t\Delta}\rho^{\mathrm{h}}_{n,\alpha,L}\otimes e^{t\Delta}\rho_{n,\alpha,L}\big)\|_{L^1(\mathbb{R}^+;\mathcal{B}^0_p)} \lesssim \big\|e^{t\Delta}\rho_{n,\alpha,L}\big\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}$$

and

$$\|\partial_3\big(e^{t\Delta}\rho^3_{n,\alpha,L}e^{t\Delta}\rho_{n,\alpha,L}\big)\|_{L^1(\mathbb{R}^+;B^{\frac{2}{p},-1+\frac{1}{p}}_{p,1})} \lesssim \big\|e^{t\Delta}\rho_{n,\alpha,L}\big\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\,.$$

According to the examples page 11, we infer that

$$(1.32)\qquad \|F^1_{n,\varepsilon}\|_{\mathcal{F}^0_p}\lesssim \big\|e^{t\Delta}\rho_{n,\alpha,L}\big\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\,.$$

In view of Inequality (1.28), Estimate (1.32) ensures that

$$(1.33)\qquad \forall n\ge N_1\,,\quad \|F^1_{n,\varepsilon}\|_{\mathcal{F}^0_p}\lesssim \varepsilon^2.$$

Now let us consider $F^2_{n,\varepsilon}$. By the scaling invariance of the operators $\Lambda_{\lambda_n^j}$ in $L^2(\mathbb{R}^+;\mathcal{B}^1_p)$ and again the fact that $B^{\frac{2}{p},\frac{1}{p}}_{p,1}$ is an algebra, we get

$$(1.34)\qquad \begin{aligned} &\big\|\Lambda_{\lambda_n^j}\Phi^j_{n,\varepsilon}\otimes e^{t\Delta}\rho_{n,\alpha,L} + e^{t\Delta}\rho_{n,\alpha,L}\otimes\Lambda_{\lambda_n^j}\Phi^j_{n,\varepsilon}\big\|_{L^1(\mathbb{R}^+;\mathcal{B}^1_p)}\\ &\qquad\lesssim \|\Phi^j_{n,\varepsilon}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\|e^{t\Delta}\rho_{n,\alpha,L}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\\ &\qquad\lesssim \|\Phi^j_{n,\varepsilon}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|e^{t\Delta}\rho_{n,\alpha,L}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\,.\end{aligned}$$

Next we write, thanks to Estimates (1.25) and (1.27),

$$\sum_{j=0}^{L}\|\Phi^j_{n,\varepsilon}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\le \mathcal{T}_0(u_0,\mathcal{M}_\alpha) + \sum_{j=1}^{L}\Big(\mathcal{T}_1\big(\|(v^j_{n,\alpha,L},w^{j,3}_{n,\alpha,L})\|_{\mathcal{B}^0}\big) + h^j_n\mathcal{T}_\infty\big(\|(v^j_{n,\alpha,L},w^{j,3}_{n,\alpha,L})\|_{S_\mu}\big)\Big)\,,$$

which can be written due to (1.24)

$$\sum_{j=0}^{L}\big\|\Phi^j_{n,\varepsilon}\big\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)} \leq \mathcal{T}_0(u_0,\mathcal{M}_\alpha)+\mathcal{T}_\infty(\mathcal{M})+\sum_{j=1}^{L} h^j_n\mathcal{T}_\infty(\mathcal{M}_\alpha)\,.$$

Using Condition (1.29) on the sequences $(h^j_n)_{n\in\mathbb{N}}$ implies that

$$\Big\|\sum_{j=0}^{L}\Phi^j_{n,\varepsilon}\Big\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)} \leq \mathcal{T}_0(u_0,\mathcal{M}_\alpha)+\mathcal{T}_\infty(\mathcal{M})+\varepsilon\,.$$

It follows (of course up to a change of $\mathcal{T}_\infty$) that for small enough ε

$$\Big\|\sum_{j=0}^{L}\Phi^j_{n,\varepsilon}\Big\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)} \leq \mathcal{T}_0(u_0,\mathcal{M}_\alpha)+\mathcal{T}_\infty(\mathcal{M})\,. \tag{1.35}$$

Thanks to (1.28) and (1.34), this gives rise to

$$\forall n\geq N_1\,,\quad \|F^2_{n,\varepsilon}\|_{\mathcal{F}^0_p}\leq \varepsilon\big(\mathcal{T}_0(u_0,\mathcal{M}_\alpha)+\mathcal{T}_\infty(\mathcal{M})\big)\,. \tag{1.36}$$

Finally let us consider $F^3_{n,\varepsilon}$. Recalling that α and L are fixed, it suffices to prove in view of the examples page 11 that there is $N_2\geq N_1$ such that for all $n\geq N_2$ and for all $0\leq j\neq k\leq L$,

$$\big\|\Lambda_{\lambda^j_n}\Phi^j_{n,\varepsilon}\otimes\Lambda_{\lambda^k_n}\Phi^k_{n,\varepsilon}\big\|_{L^1(\mathbb{R}^+;\mathcal{B}^1)}\lesssim\varepsilon\,.$$

Using the fact that $\mathcal{B}^1$ is an algebra along with the Hölder inequality, we infer that for a small enough γ in $]0,1[$,

$$\big\|\Lambda_{\lambda^j_n}\Phi^j_{n,\varepsilon}\otimes\Lambda_{\lambda^k_n}\Phi^k_{n,\varepsilon}\big\|_{L^1(\mathbb{R}^+;\mathcal{B}^1)}\leq\|\Lambda_{\lambda^j_n}\Phi^j_{n,\varepsilon}\|_{L^{\frac{2}{1+\gamma}}(\mathbb{R}^+;\mathcal{B}^1)}\|\Lambda_{\lambda^k_n}\Phi^k_{n,\varepsilon}\|_{L^{\frac{2}{1-\gamma}}(\mathbb{R}^+;\mathcal{B}^1)}\,.$$

Notice that

$$\begin{aligned}\|\Lambda_{\lambda^j_n}\Phi^j_{n,\varepsilon}\|_{L^{\frac{2}{1+\gamma}}(\mathbb{R}^+;\mathcal{B}^1)} &\sim (\lambda^j_n)^\gamma\|\Phi^j_{n,\varepsilon}\|_{L^{\frac{2}{1+\gamma}}(\mathbb{R}^+;\mathcal{B}^1)}\quad\text{and}\\ \|\Lambda_{\lambda^k_n}\Phi^k_{n,\varepsilon}\|_{L^{\frac{2}{1-\gamma}}(\mathbb{R}^+;\mathcal{B}^1)} &\sim \frac{1}{(\lambda^k_n)^\gamma}\|\Phi^k_{n,\varepsilon}\|_{L^{\frac{2}{1-\gamma}}(\mathbb{R}^+;\mathcal{B}^1)}\,.\end{aligned}$$

For small enough γ, Theorems 4 and 5 imply that

$$\big\|\Lambda_{\lambda^j_n}\Phi^j_{n,\varepsilon}\otimes\Lambda_{\lambda^k_n}\Phi^k_{n,\varepsilon}\big\|_{L^1(\mathbb{R}^+;\mathcal{B}^1)}\lesssim\Big(\frac{\lambda^j_n}{\lambda^k_n}\Big)^\gamma\,.$$

We deduce that

$$\|F^3_{n,\varepsilon}\|_{\mathcal{F}^0_p}\leq C\|F^3_{n,\varepsilon}\|_{\mathcal{F}^0}\lesssim\sum_{\substack{0\leq j,k\leq L\\ j\neq k}}\min\Big\{\frac{\lambda^j_n}{\lambda^k_n},\frac{\lambda^k_n}{\lambda^j_n}\Big\}^\gamma\,.$$

As the sequences $(\lambda^j_n)_{n\in\mathbb{N}}$ and $(\lambda^k_n)_{n\in\mathbb{N}}$ are orthogonal (see Definition 1.9), we have for any j and k such that $j\neq k$

$$\lim_{n\to\infty}\min\Big\{\frac{\lambda^j_n}{\lambda^k_n},\frac{\lambda^k_n}{\lambda^j_n}\Big\}=0\,.$$

Thus an integer N_2 greater than or equal to N_1 exists such that

$$\forall n\geq N_2\,,\quad\|F^3_{n,\varepsilon}\|_{\mathcal{F}^0_p}\lesssim\varepsilon\,.$$

Together with (1.33) and (1.36), this implies that

$$n\geq N_2\Longrightarrow\|F_{n,\varepsilon}\|_{\mathcal{F}^0_p}\lesssim\varepsilon\,,$$

which proves (1.31).

Now, in order to conclude the proof of Theorem 3, we need the following results.

Proposition 1.14. *Let p be in the inverval $[2,\infty[$. A constant C_0 exists such that, if U is in $L^2(\mathbb{R}^+;\mathcal{B}^1_p)$, u_0 in $\mathcal{B}^0_p$ and f in $\mathcal{F}^0_p$ such that*

$$\|u_0\|_{\mathcal{B}^0_p}+\|f\|_{\mathcal{F}^0_p}\leq\frac{1}{C_0}\exp\Big(-C_0\int_0^\infty\|U(t)\|^2_{\mathcal{B}^1_p}dt\Big)\,,$$

then the problem

$$(\mathrm{NS}_U)\qquad\left\{\begin{array}{c}\partial_t u+\mathrm{div}(u\otimes u+u\otimes U+U\otimes u)-\Delta u=-\nabla p+f\\ \mathrm{div}\,u=0\quad\text{and}\quad u_{|t=0}=u_0\end{array}\right.$$

has a unique global solution in $L^2(\mathbb{R}^+;\mathcal{B}^1_p)$ which satisfies

$$\|u\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\lesssim\|u_0\|_{\mathcal{B}^0_p}+\|f\|_{\mathcal{F}^0_p}\,.$$

Proposition 1.15. *Let $p\in[2,4[$ be given and let u be a solution of* (NS) *which belongs to $L^2(\mathbb{R}^+;\mathcal{B}^1_p)$ and with initial data u_0 in $\mathcal{B}^0$. Then u belongs to $\mathcal{A}^0$ and satisfies*

$$\forall r\in[1,\infty]\,,\ \|u\|_{L^r(\mathbb{R}^+;\mathcal{B}^{\frac{2}{r}})}+\|u\|_{L^1(\mathbb{R}^+;\mathcal{B}^{1,\frac{3}{2}})}\lesssim\|u_0\|_{\mathcal{B}^0}+\|u\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\,.\tag{1.37}$$

Moreover, if $p=2$ and if the initial data u_0 belongs in addition to $\mathcal{B}^s$ for some s in the interval $[-1+\mu,1-\mu]$, then

$$\forall r\in[1,\infty]\,,\ \|u\|_{L^r(\mathbb{R}^+;\mathcal{B}^{s+\frac{2}{r}})}\leq\mathcal{T}_1(\|u_0\|_{\mathcal{B}^s})\mathcal{T}_0(\|u_0\|_{\mathcal{B}^0},\|u\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)})\,.\tag{1.38}$$

Finally, if $p=2$ and if u_0 belongs to $\mathcal{B}^{0,s'}$ for some s' greater than $1/2$, then

$$\forall r\in[1,\infty]\,,\ \|u\|_{L^r(\mathbb{R}^+;\mathcal{B}^{\frac{2}{r},s'})}\leq\mathcal{T}_1(\|u_0\|_{\mathcal{B}^{0,s'}})\mathcal{T}_0(\|u_0\|_{\mathcal{B}^0},\|u\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)})\,.\tag{1.39}$$

The proof of both propositions can be found in Appendix A.

Conclusion of the proof of Theorem 3. Let us fix $p\in]2,4[$. By definition of $u^{\mathrm{app}}_{n,\varepsilon}$ we have

$$\|u^{\mathrm{app}}_{n,\varepsilon}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\leq\Big\|\sum_{j=0}^L\Lambda_{\lambda^j_n}\Phi^j_{n,\varepsilon}\Big\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}+\|e^{t\Delta}\rho_{n,\alpha,L}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\,.$$

Inequalities (1.28) and (1.35) imply that for n sufficiently large

$$\|u^{\mathrm{app}}_{n,\varepsilon}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\leq\mathcal{T}_0(u_0,\mathcal{M}_\alpha)+\mathcal{T}_\infty(\mathcal{M})+C\varepsilon\,.$$

Because of (1.31), it is clear that, if ε is small enough,

$$\|F_{n,\varepsilon}\|_{\mathcal{F}^0_p}\leq\frac{1}{C_0}\exp\Big(-C_0\|u^{\mathrm{app}}_{n,\varepsilon}\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\Big)$$

which ensures thanks to Proposition 1.14 that $u_{0,n}$ generates a global regular solution in the space $L^2(\mathbb{R}^+;\mathcal{B}^1_p)$. Then the conclusion of the proof of Theorem 3 is a direct consequence of Proposition 1.15. □

The proof of Theorem 3 is structured as follows. In Section 2 we prove Proposition 1.12. Theorems 4 and 5 are proved in Sections 3 and 4 respectively. Appendix A is devoted to the recollection of some material on anisotropic Besov spaces. We also prove in the Appendix Proposition 1.14 and the anisotropic propagation of regularity result for the Navier-Stokes system stated in Proposition 1.15.

2. Profile decompositions

2.1. **An anisotropic profile decomposition.** The study of the defect of compactness in Sobolev embeddings originates in the works of P.-L. Lions (see [47] and [48]), L. Tartar (see [58]) and P. Gérard (see [26]) and earlier decompositions of bounded sequences into a sum of "profiles" can be found in the studies by H. Brézis and J.-M. Coron in [11] and M. Struwe in [57]. Our source of inspiration here is the work [27] of P. Gérard in which the defect of compactness of the critical Sobolev embedding $H^s \subset L^p$ is described in terms of a sum of rescaled and translated orthogonal profiles, up to a small term in L^p. This was generalized to other Sobolev spaces by S. Jaffard in [35], to Besov spaces by G. Koch [42], and finally to general critical embeddings by H. Bahouri, A. Cohen and G. Koch in [3] : see Proposition 1.5 for a statement. We refer also to [6, 7, 8] for Sobolev embeddings in Orlicz spaces and [20] for an abstract, functional analytic presentation.

In the pionneering works [5] (for the critical 3D wave equation) and [50] (for the critical 2D Schrödinger equation), this type of decomposition was introduced in the study of nonlinear partial differential equations. The ideas of [5] were revisited in [41] and [22] in the context of the Schrödinger equations and Navier-Stokes equations respectively, with an aim at describing the structure of bounded sequences of solutions to those equations. These profile decomposition techniques have since then been succesfully used in order to study the possible blow-up of solutions to nonlinear partial differential equations, in various contexts; we refer for instance to [25], [31], [36], [37], [39], [40], [53], [55].

Before stating the result, let us give the definition of anisotropic scaling operators: for any two sequences of positive real numbers $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(\gamma_n)_{n\in\mathbb{N}}$, and for any sequence $(x_n)_{n\in\mathbb{N}}$ of points in $\mathbb{R}^3$, we denote

$$\Lambda_{\varepsilon_n,\gamma_n,x_n}\phi(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon_n}\phi\left(\frac{x_\mathrm{h}-x_{n,\mathrm{h}}}{\varepsilon_n},\frac{x_3-x_{n,3}}{\gamma_n}\right).$$

Observe that the operator $\Lambda_{\varepsilon_n,\gamma_n,x_n}$ is an isometry in the space $\mathcal{B}^0_{p,q}$ for any $1\leq p\leq\infty$ and any $0<q\leq\infty$ – recall the definition of those spaces in (1.13). Notice also that when the sequences (ε_n) and (γ_n) are equal, then the operator $\Lambda_{\varepsilon_n,\gamma_n,x_n}$ reduces to the isotropic scaling operator $\Lambda_{\varepsilon_n,x_n}$ defined in (1.2), and such isotropic profiles will be the ones to disappear in the profile decomposition thanks to the assumption of R-convergence. We also have a definition of orthogonal triplets of sequences, analogous to Definition 1.9.

Definition 2.1. *We say that two triplets of sequences $(\varepsilon_n^\ell,\gamma_n^\ell,x_n^\ell)_{n\in\mathbb{N}}$ with ℓ belonging to $\{1,2\}$, where $(\varepsilon_n^\ell,\gamma_n^\ell)_{n\in\mathbb{N}}$ are two sequences of positive real numbers and x_n^ℓ are sequences in $\mathbb{R}^3$, are orthogonal if, when n tends to infinity,*

$$\begin{aligned}&\textit{either}\quad \frac{\varepsilon_n^1}{\varepsilon_n^2}+\frac{\varepsilon_n^2}{\varepsilon_n^1}+\frac{\gamma_n^1}{\gamma_n^2}+\frac{\gamma_n^2}{\gamma_n^1}\to\infty\\ &\textit{or}\quad (\varepsilon_n^1,\gamma_n^1)\equiv(\varepsilon_n^2,\gamma_n^2)\quad\textit{and}\quad |(x_n^1)^{\varepsilon_n^1,\gamma_n^1}-(x_n^2)^{\varepsilon_n^1,\gamma_n^1}|\to\infty\,,\end{aligned}$$

where we have denoted $(x_n^\ell)^{\varepsilon_n^k,\gamma_n^k}\stackrel{\text{def}}{=}\Big(\dfrac{x_{n,\mathrm{h}}^\ell}{\varepsilon_n^k},\dfrac{x_{n,3}^\ell}{\gamma_n^k}\Big)$.

The cornerstone to the proof of Proposition 1.12 is the following proposition.

Proposition 2.2. *Let $(\varphi_n)_{n\in\mathbb{N}}$ and ϕ_0 belong to $\mathcal{B}^0_{2,q}$ for some $0<q<1$, with (φ_n) converging to ϕ^0 in the sense of distributions as n goes to infinity. Let $p>2$ be given. For all integers $\ell\geq 1$ there is a triplet of orthogonal sequences in the sense of Definition 2.1, denoted by $(\varepsilon_n^\ell,\gamma_n^\ell,x_n^\ell)_{n\in\mathbb{N}}$ and functions ϕ^ℓ in $\mathcal{B}^0_{2,q}$ such that up to extracting a subsequence, one can* *write the sequence* $(\varphi_n)_{n\in\mathbb{N}}$ *under the following form, for each* $L \geq 1$*:*

$$\varphi_n = \phi^0 + \sum_{\ell=1}^{L} \Lambda_{\varepsilon_n^\ell,\gamma_n^\ell,x_n^\ell}\phi^\ell + \psi_n^L \,, \tag{2.1}$$

where ψ_n^L *satisfies*

$$\limsup_{n\to\infty} \|\psi_n^L\|_{\mathcal{B}^0_{p,1}} \to 0\,, \quad L \to \infty\,. \tag{2.2}$$

Moreover the following stability result holds:

$$\sum_{\ell\geq 1} \|\phi^\ell\|_{\mathcal{B}^0} \lesssim \sup_n \|\varphi_n\|_{\mathcal{B}^0} + \|\phi_0\|_{\mathcal{B}^0}\,. \tag{2.3}$$

The proof follows word for word the proof of Theorem 3 in [4], up to straightforward modifications of the indices of the Besov spaces at play.

Remark 2.3. *If two scales appearing in the above decomposition are not orthogonal, then they can be chosen to be equal. We shall therefore assume from now on that is the case: two sequences of scales are either orthogonal, or equal.*

Remark 2.4. *By density of smooth, compactly supported functions in* $\mathcal{B}^0_{2,q}$*, one can write for each integer* ℓ

$$\phi^\ell = \phi^\ell_\alpha + r^\ell_\alpha \quad \textit{with} \quad \|r^\ell_\alpha\|_{B^{0,\frac12}_{2,q}} \leq \alpha$$

where ϕ^ℓ_α *are arbitrarily smooth and compactly supported, and moreover*

$$\sum_{\ell\geq 1} \big(\|\phi^\ell_\alpha\|_{\mathcal{B}^0} + \|r^\ell_\alpha\|_{\mathcal{B}^0}\big) \lesssim \sup_n \|\varphi_n\|_{\mathcal{B}^0} + \|\varphi_0\|_{\mathcal{B}^0}\,. \tag{2.4}$$

2.2. **Proof of Proposition 1.12.** The proof of Proposition 1.12 is structured as follows. First we write down a profile decomposition for any bounded, R-converging sequence of divergence free vector fields, following the results of the previous section. Next we reorganize the profile decomposition by grouping together all profiles having the same horizontal scale and finally we check that all the conclusions of Proposition 1.12 hold.

2.2.1. *Profile decomposition of R-converging divergence free vector fields.* In this section we start with the anisotropic profile decomposition of sequences of $\mathcal{B}^0_{2,q}$ given in Proposition 2.2 and we use the assumption of R-convergence (see Definition 1.7) to eliminate from the profile decomposition all isotropic profiles. Finally we study the particular case of divergence free vector fields. Under this assumption, we are able to restrict our attention to (rescaled) vector fields with slow vertical variations.

Let us first prove the following result.

Proposition 2.5. *Let* (φ_n) *and* ϕ_0 *belong to* $\mathcal{B}^0_{2,q}$ *for some* $0 < q < 1$*, with* (φ_n) *R-converging to* ϕ^0 *as* n *goes to infinity. Then with the notation of Proposition 2.2, the following result holds:*

$$\forall \ell \geq 1\,, \quad \lim_{n\to\infty} (\gamma_n^\ell)^{-1}\varepsilon_n^\ell \in \{0,\infty\}\,. \tag{2.5}$$

Remark 2.6. *This proposition shows that if one assumes that the weak convergence is actually an R-convergence, then the only profiles remaining in the decomposition are those with truly anisotropic horizontal and vertical scales. This eliminates profiles of the type* $n\phi(nx)$ *and* $\varphi(\cdot - x_n)$ *with* $|x_n| \to \infty$*, for which clearly the conclusion of Theorem 3 is unknown in general (see the discussion in Section 1).*

Proof of Proposition 2.5. To prove (2.5) we consider the decomposition provided in Proposition 2.2 and we assume that there is $k \in \mathbb{N}$ such that $(\gamma_n^k)^{-1}\varepsilon_n^k$ goes to 1 as n goes to infinity. We rescale the decomposition (2.1) to find, choosing $L \geq k$,

$$\varepsilon_n^k(\varphi_n - \varphi_0)(\varepsilon_n^k \cdot + x_n^k) = \sum_{\ell=1}^{L} \Lambda_{\frac{\varepsilon_n^\ell}{\varepsilon_n^k}, \frac{\gamma_n^\ell}{\varepsilon_n^k}, x_n^{\ell,k}} \phi^\ell + \Lambda_{\frac{1}{\varepsilon_n^k}, \frac{1}{\varepsilon_n^k}, -\frac{x_n^k}{\varepsilon_n^k}} \psi_n^L$$

where

$$x_n^{\ell,k} \stackrel{\text{def}}{=} \frac{x_n^\ell - x_n^k}{\varepsilon_n^k} \cdot$$

Now let us take the weak limit of both sides of the equality as n goes to infinity. By Definition 1.7 we know that the left-hand side goes weakly to a function depending only on (x_1, x_2) (up to an extraction), denoted by $\tilde{\psi}(x_1, x_2)$. Concerning the right-hand side, we start by noticing that

$$\frac{\varepsilon_n^\ell}{\varepsilon_n^k} \to 0 \ \text{ or } \ \frac{\varepsilon_n^\ell}{\varepsilon_n^k} \to \infty \Longrightarrow \Lambda_{\frac{\varepsilon_n^\ell}{\varepsilon_n^k}, \frac{\gamma_n^\ell}{\varepsilon_n^k}, x_n^{\ell,k}} \phi^\ell \rightharpoonup 0\,,$$

as n tends to infinity, for any value of the sequences γ_n^ℓ, x_n^ℓ, and x_n^k. So we can restrict the sum on the right-hand side to the case when $\varepsilon_n^\ell/\varepsilon_n^k \to 1$. Then we write similarly

$$\frac{\varepsilon_n^\ell}{\gamma_n^\ell} \to \infty \Longrightarrow \Lambda_{1, \frac{\gamma_n^\ell}{\varepsilon_n^\ell}, x_n^{\ell,k}} \phi^\ell \rightharpoonup 0\,,$$

so there only remain indexes ℓ such that $\varepsilon_n^\ell/\gamma_n^\ell \to 0$ or 1. Finally we use the fact that if $\varepsilon_n^\ell/\gamma_n^\ell \to 1$, then the weak limit of $\Lambda_{1, x_n^{\ell,k}} \phi^\ell$ can be other than zero only if $x_n^{\ell,k} \to a^{\ell,k} \in \mathbb{R}^3$, and similarly if $\varepsilon_n^\ell/\gamma_n^\ell \to 0$, then the weak limit of $\Lambda_{1, \frac{\gamma_n^\ell}{\varepsilon_n^\ell}, x_n^{\ell,k}} \phi^\ell$ can be other than zero only if $x_{n,\mathrm{h}}^{\ell,k} \to a_{\mathrm{h}}^{\ell,k} \in \mathbb{R}^2$, and $(x_{n,3}^\ell - x_{n,3}^k)/\gamma_n^\ell \to a_3^{\ell,k} \in \mathbb{R}$. So let us define

$$S^{1,L}(k) \stackrel{\text{def}}{=} \Big\{ 1 \leq \ell \leq L \,/\, \varepsilon_n^\ell = \varepsilon_n^k\,,\ x_n^{\ell,k} \to a^{\ell,k} \in \mathbb{R}^3\,,\ \frac{\varepsilon_n^\ell}{\gamma_n^\ell} \to 1 \Big\} \quad \text{and}$$
$$S^{0,L}(k) \stackrel{\text{def}}{=} \Big\{ 1 \leq \ell \leq L \,/\, \varepsilon_n^\ell = \varepsilon_n^k\,,\ x_{n,\mathrm{h}}^{\ell,k} \to a_{\mathrm{h}}^{\ell,k} \in \mathbb{R}^2\,,\ \frac{x_{n,3}^\ell - x_{n,3}^k}{\gamma_n^\ell} \to a_3^{\ell,k} \in \mathbb{R}\,,\ \frac{\varepsilon_n^\ell}{\gamma_n^\ell} \to 0 \Big\}\,.$$

Actually by orthogonality the set $S^{1,L}(k)$ only contains one element, which is k. So for each $L \geq 1$, as n goes to infinity we have finally

$$-\Lambda_{\frac{1}{\varepsilon_n^k}, \frac{1}{\varepsilon_n^k}, -\frac{x_n^k}{\varepsilon_n^k}} \psi_n^L \rightharpoonup \phi^k + \sum_{\ell \in S^{0,L}(k)} \phi^\ell(\cdot_{\mathrm{h}} - a_{\mathrm{h}}^{\ell,k}, -a_3^{\ell,k}) + \tilde{\psi}(\cdot_{\mathrm{h}})\,.$$

Since the left-hand side tends to 0 in $B_{2,1}^{-1+\frac{2}{p},\frac{1}{p}}$ as L tends to infinity, uniformly in $n \in \mathbb{N}$, we deduce that ϕ^k must be independent of x_3. That is a contradiction since ϕ^k belongs to $\mathcal{B}^0$. It follows that $(\gamma_n^k)^{-1}\varepsilon_n^k$ goes to 0 or infinity as n goes to infinity.

The case of divergence free vector fields. Putting together Propositions 2.2 and 2.5 along with Remark 2.4 and the fact that $u_{0,n}$ is divergence free we obtain the following result.

Proposition 2.7. *Under the assumptions of Theorem* 3*, the following holds. Let* $p > 2$ *be given. For all integers* $\ell \geq 1$ *there is a triplet of orthogonal sequences in the sense of Definition* 1.9*, denoted by* $(\varepsilon_n^\ell, \gamma_n^\ell, x_n^\ell)_{n \in \mathbb{N}}$ *and for all* α *in* $]0,1[$ *there are arbitrarily smooth divergence free vector fields* $(\widetilde{\phi}_\alpha^{h,\ell}, 0)$ *and* $(-\nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\partial_3\phi_\alpha^\ell, \phi_\alpha^\ell)$ *with* $\widetilde{\phi}_\alpha^{h,\ell}$ *and* ϕ_α^ℓ *compactly* *supported, and such that up to extracting a subsequence, one can write the sequence* $(u_{0,n})_{n\in\mathbb{N}}$ *under the following form, for each* $L\geq 1$*:*

$$u_{0,n} = u_0 + \sum_{\ell=1}^{L} \Lambda_{\varepsilon_n^\ell,\gamma_n^\ell,x_n^\ell}\Big(\widetilde{\phi}_\alpha^{\mathrm{h},\ell} + \widetilde{r}_\alpha^{\mathrm{h},\ell} - \frac{\varepsilon_n^\ell}{\gamma_n^\ell}\nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\partial_3(\phi_\alpha^\ell + r_\alpha^\ell), \phi_\alpha^\ell + r_\alpha^\ell\Big) + \big(\widetilde{\psi}_n^{\mathrm{h},L} - \nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\partial_3\psi_n^L, \psi_n^L\big)\,, \tag{2.6}$$

where $\widetilde{\psi}_n^{\mathrm{h},L}$ *and* ψ_n^L *are independent of* α *and satisfy*

$$\limsup_{n\to\infty}\Big(\|\widetilde{\psi}_n^{\mathrm{h},L}\|_{\mathcal{B}^0_{p,1}} + \|\psi_n^L\|_{\mathcal{B}^0_{p,1}}\Big) \to 0\,, \quad L\to\infty\,, \tag{2.7}$$

while $\widetilde{r}_\alpha^{\mathrm{h},\ell}$ *and* r_α^ℓ *are independent of* n *and* L *and satisfy for each* $\ell\in\mathbb{N}$

$$\|\widetilde{r}_\alpha^{\mathrm{h},\ell}\|_{\mathcal{B}^0} + \|r_\alpha^\ell\|_{\mathcal{B}^0} \leq \alpha\,. \tag{2.8}$$

Moreover the following properties hold:

$$\forall \ell\geq 1\,,\quad \lim_{n\to\infty}(\gamma_n^\ell)^{-1}\varepsilon_n^\ell \in \{0,\infty\}\,, \quad \text{and} \quad \lim_{n\to\infty}(\gamma_n^\ell)^{-1}\varepsilon_n^\ell = \infty \implies \phi_\alpha^\ell \equiv r_\alpha^\ell \equiv 0\,, \tag{2.9}$$

as well as the following stability result, which is uniform in α*:*

$$\sum_{\ell\geq 1}\big(\|\widetilde{\phi}_\alpha^{\mathrm{h},\ell}\|_{\mathcal{B}^0} + \|\widetilde{r}_\alpha^{\mathrm{h},\ell}\|_{\mathcal{B}^0} + \|\phi_\alpha^\ell\|_{\mathcal{B}^0} + \|r_\alpha^\ell\|_{\mathcal{B}^0}\big) \lesssim \sup_n\|u_{0,n}\|_{\mathcal{B}^0} + \|u_0\|_{\mathcal{B}^0}\,. \tag{2.10}$$

Proof of Proposition 2.7. First we decompose the third component $u_{0,n}^3$ according to Proposition 2.2 and Remark 2.4: with the above notation, this gives rise to

$$u_{0,n}^3 = u_0^3 + \sum_{\ell=1}^{L}\Lambda_{\varepsilon_n^\ell,\gamma_n^\ell,x_n^\ell}\big(\phi_\alpha^\ell + r_\alpha^\ell\big) + \psi_n^L\,, \tag{2.11}$$

with $\limsup_{n\to\infty}\|\psi_n^L\|_{\mathcal{B}^0_{p,1}} \overset{L\to\infty}{\longrightarrow} 0$. Moreover thanks to Proposition 2.5, we know that for all $\ell\geq 1$,

$$\lim_{n\to\infty}(\gamma_n^\ell)^{-1}\varepsilon_n^\ell \in \{0,\infty\}\,.$$

Next thanks to the divergence-free assumption we recover the profile decomposition for $u_{0,n}^{\mathrm{h}}$. Indeed there is a two-component, divergence-free vector field $\nabla^{\mathrm{h}\perp}C_{0,n}$ such that

$$u_{0,n}^{\mathrm{h}} = \nabla^{\mathrm{h}\perp}C_{0,n} - \nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\partial_3 u_{0,n}^3\,,$$

where $\nabla^{\mathrm{h}\perp} = (-\partial_1,\partial_2)$, and some function φ such that

$$u_0^{\mathrm{h}} = \nabla^{\mathrm{h}\perp}\varphi - \nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\partial_3 u_0^3\,.$$

Now since $\partial_3 u_{0,n}^3 = -\mathrm{div}_{\mathrm{h}}\, u_{0,n}^{\mathrm{h}}$ and $u_{0,n}^{\mathrm{h}}$ is bounded in $\mathcal{B}^0_{2,q}$, we deduce that $\nabla^{\mathrm{h}\perp}C_{0,n}$ is a bounded sequence in $\mathcal{B}^0_{2,q}$ and similarly for $\nabla^{\mathrm{h}\perp}\varphi$. Thus, applying again the profile decomposition of Proposition 2.2 and Remark 2.4, we get

$$\nabla^{\mathrm{h}\perp}C_{0,n} - \nabla_h^\perp\varphi = \sum_{\ell=1}^{L}\Lambda_{\tilde\varepsilon_n^\ell,\tilde\gamma_n^\ell,\tilde x_n^\ell}\big(\widetilde{\phi}_\alpha^{\mathrm{h},\ell} + \widetilde{r}_\alpha^{\mathrm{h},\ell}\big) + \widetilde{\psi}_n^{\mathrm{h},L} \tag{2.12}$$

with $\limsup_{n\to\infty}\|\widetilde{\psi}_n^{\mathrm{h},L}\|_{B_{2,1}^{-1+\frac{2}{p},\frac{1}{p}}} \overset{L\to\infty}{\longrightarrow} 0$ and $\|\widetilde{r}_\alpha^{\mathrm{h},\ell}\|_{\mathcal{B}^0}\leq\alpha$. Moreover Proposition 2.5 ensures that for all $\ell\geq 1$, we have $\lim_{n\to\infty}(\tilde\gamma_n^\ell)^{-1}\tilde\varepsilon_n^\ell \in \{0,\infty\}$.

Finally, by the divergence free assumption, $u^3_{0,n}$ is bounded in $B^{-1,\frac{3}{2}}_{2,q}$ which implies that necessarily $\phi^\ell_\alpha \equiv r^\ell_\alpha \equiv 0$ in the case when $\lim\limits_{n\to\infty} (\gamma^\ell_n)^{-1}\varepsilon^\ell_n = \infty$ (see Lemma 5.3 in [4]). Up to relabelling the various sequences appearing in (2.11) and (2.12), Proposition 2.7 follows. $\square$

2.2.2. *Regrouping of profiles according to horizontal scales.* With the notation of Proposition 2.7, let us define the following scales: $\varepsilon^0_n \equiv \gamma^0_n \equiv 1$, and $x^0_n \equiv 0$, so that $u_0 \equiv \Lambda_{\varepsilon^0_n,\gamma^0_n,x^0_n} u_0$. In order to proceed with the re-organization of the profile decomposition provided in Proposition 2.7, we introduce some more definitions, keeping the notation of Proposition 2.7. For a given $L \geq 1$ we define recursively an increasing (finite) sequence of indexes $\ell_k \in \{1,\dots,L\}$ by

$$(2.13)\qquad \ell_0 \stackrel{\rm def}{=} 0\,, \quad \ell_{k+1} \stackrel{\rm def}{=} \min\Big\{\ell \in \{\ell_k+1,\dots,L\}\,/\, \frac{\varepsilon^\ell_n}{\gamma^\ell_n} \to 0 \quad \text{and} \quad \ell \notin \bigcup_{k'=0}^{k} \Gamma^L(\varepsilon^{\ell_{k'}}_n)\Big\}\,,$$

where for $0 \leq \ell \leq L$, we define $\Gamma^L(\varepsilon^\ell_n)$ as the set of all indices having the same horizontal scale ε^ℓ_n, namely (recalling that by Remark 2.3 if two scales are not orthogonal, then they are equal)

$$(2.14)\qquad \Gamma^L(\varepsilon^\ell_n) \stackrel{\rm def}{=} \Big\{\ell' \in \{1,\dots,L\}\,/\,\varepsilon^{\ell'}_n \equiv \varepsilon^\ell_n \quad \text{and} \quad \varepsilon^{\ell'}_n(\gamma^{\ell'}_n)^{-1} \to 0\,,\, n\to\infty\Big\}\,.$$

We call $\mathcal{L}(L)$ the largest index of the sequence (ℓ_k) and we may then introduce the following partition:

$$(2.15)\qquad \Big\{\ell \in \{1,\dots,L\}\,/\,\varepsilon^\ell_n(\gamma^\ell_n)^{-1} \to 0\Big\} = \bigcup_{k=0}^{\mathcal{L}(L)} \Gamma^L(\varepsilon^{\ell_k}_n)\,.$$

We shall now regroup profiles in the decomposition (2.6) of $u_{0,n}$ according to the value of their horizontal scale. We fix from now on an integer $L \geq 1$.

Construction of the profiles for $\ell = 0$. Before going into the technical details of the construction, let us discuss an example explaining the computations of this paragraph. Consider the particular case when $u_{0,n}$ is given by

$$u_{0,n}(x) = u_0(x) + \Big(v^0_0(x_{\rm h}, 2^{-n}x_3) + w^{0,{\rm h}}_0(x_{\rm h}, 2^{-2n}x_3), 0\Big) + \Big(v^0_0(x_1+n, x_2, 2^{-n}x_3), 0\Big)\,,$$

with v^0_0 and $w^{0,{\rm h}}_0$ smooth (say in $B^{s,s'}_{1,q}$ for all s, s' in $\mathbb{R}$) and compactly supported. Recall that the notation v for a vector field always stands for a two component vector field. Let us assume that $(u_{0,n})_{n\in\mathbb{N}}$ R-converges to u_0, as n tends to infinity. Then we can write

$$u_{0,n}(x) = u_0(x) + \big(v^{0,{\rm loc}}_{0,n}(x_{\rm h}, 2^{-n}x_3), 0\big) + \big(v^{0,\infty}_{0,n}(x_{\rm h}, 2^{-n}x_3), 0\big)\,,$$

with $v^{0,{\rm loc}}_{0,n}(y) \stackrel{\rm def}{=} v^0_0(y) + w^{0,{\rm h}}_0(y_h, 2^{-n}y_3)$ and $v^{0,\infty}_{0,n}(y) = v^0_0(y_1+n, y_2, y_3)$. We notice that $v^{0,{\rm loc}}_{0,n}$ and $v^{0,\infty}_{0,n}$ are uniformly bounded in $\mathcal{B}^0$, but also in $B^{s,s'}_{2,1}$ for any s in $\mathbb{R}$ and $s' \geq 1/2$. Moreover since $u_{0,n} \rightharpoonup u_0$, we have $v^0_0(x_{\rm h}, 0) + w^{\rm h}_0(x_{\rm h}, 0) \equiv 0$, hence $v^{0,{\rm loc}}_{0,n}(x_{\rm h}, 0) = 0$. The initial data $u_{0,n}$ has therefore been re-written as

$$u_{0,n}(x) = u_0(x) + \big(v^{0,{\rm loc}}_{0,n}(x_{\rm h}, 2^{-n}x_3), 0\big) + \big(v^{0,\infty}_{0,n}(x_{\rm h}, 2^{-n}x_3), 0\big) \quad \text{with} \quad v^{0,{\rm loc}}_{0,n}(x_{\rm h}, 0) = 0$$

and where the support in $x_{\rm h}$ of $v^{0,{\rm loc}}_{0,n}(x_{\rm h}, 2^{-n}x_3)$ is in a fixed compact set whereas the support in $x_{\rm h}$ of $v^{0,\infty}_{0,n}(x_{\rm h}, 2^{-n}x_3)$ escapes to infinity. This is of the same form as in the statement of Proposition 1.12.

When considering all the profiles having the same horizontal scale (1 here), the point is therefore to choose the smallest vertical scale (2^n here) and to write the decomposition in terms of that scale only. Of course that implies that contrary to usual profile decompositions, the profiles are no longer fixed functions in $\mathcal{B}^0$, but sequences of functions, bounded in $\mathcal{B}^0$.

In view of the above example, let ℓ_0^- be an integer such that $\gamma_n^{\ell_0^-}$ is the smallest vertical scale going to infinity, associated with profiles for $1 \leq \ell \leq L$, having 1 for horizontal scale. More precisely we ask that

$$\gamma_n^{\ell_0^-} = \min_{\ell \in \Gamma^L(1)} \gamma_n^\ell \, ,$$

where according to (2.14),

$$\Gamma^L(1) = \Big\{ \ell' \in \{1, \dots, L\} \, / \, \varepsilon_n^{\ell'} \equiv 1 \quad \text{and} \quad \gamma_n^{\ell'} \to \infty \, , \, n \to \infty \Big\} \, .$$

Notice that the minimum of the sequences γ_n^ℓ is well defined in our context thanks to the fact that due to Remark 2.3, either two sequences are orthogonal in the sense of Definition 1.9, or they are equal. Remark also that ℓ_0^- is by no means unique, as several profiles may have the same horizontal scale as well as the same vertical scale (in which case the concentration cores must be orthogonal).

Now we denote

$$h_n^0 \stackrel{\text{def}}{=} (\gamma_n^{\ell_0^-})^{-1} \, , \tag{2.16}$$

and we notice that h_n^0 goes to zero as n goes to infinity for each L. Note also that h_n^0 depends on L through the choice of ℓ_0^-, since if L increases then ℓ_0^- may also increase; this dependence is omitted in the notation for simplicity. Let us define (up to a subsequence extraction)

$$a^\ell \stackrel{\text{def}}{=} \lim_{n \to \infty} \Big(x_{n,\mathrm{h}}^\ell, \frac{x_{n,3}^\ell}{\gamma_n^\ell} \Big) \, . \tag{2.17}$$

We then define the divergence-free vector fields

$$v_{0,n,\alpha,L}^{0,\mathrm{loc},\mathrm{h}}(y) \stackrel{\text{def}}{=} \sum_{\substack{\ell \in \Gamma^L(1) \\ a_{\mathrm{h}}^\ell \in \mathbb{R}^2}} \widetilde{\phi}_\alpha^{h,\ell} \Big(y_{\mathrm{h}} - x_{n,\mathrm{h}}^\ell \, , \frac{y_3}{h_n^0 \gamma_n^\ell} - \frac{x_{n,3}^\ell}{\gamma_n^\ell} \Big) \tag{2.18}$$

and

$$w_{0,n,\alpha,L}^{0,\mathrm{loc}}(y) \stackrel{\text{def}}{=} \sum_{\substack{\ell \in \Gamma^L(1) \\ a_{\mathrm{h}}^\ell \in \mathbb{R}^2}} \Big(- \frac{1}{h_n^0 \gamma_n^\ell} \nabla^{\mathrm{h}} \Delta_{\mathrm{h}}^{-1} \partial_3 \phi_\alpha^\ell, \phi_\alpha^\ell \Big) \Big(y_{\mathrm{h}} - x_{n,\mathrm{h}}^\ell \, , \frac{y_3}{h_n^0 \gamma_n^\ell} - \frac{x_{n,3}^\ell}{\gamma_n^\ell} \Big) \, . \tag{2.19}$$

By construction we have

$$w_{0,n,\alpha,L}^{0,\mathrm{loc},\mathrm{h}} = -\nabla^{\mathrm{h}} \Delta_{\mathrm{h}}^{-1} \partial_3 w_{0,n,\alpha,L}^{0,\mathrm{loc},3} \, .$$

Similarly we define

$$v_{0,n,\alpha,L}^{0,\infty,\mathrm{h}}(y) \stackrel{\text{def}}{=} \sum_{\substack{\ell \in \Gamma^L(1) \\ |a_{\mathrm{h}}^\ell| = \infty}} \widetilde{\phi}_\alpha^{h,\ell} \Big(y_{\mathrm{h}} - x_{n,\mathrm{h}}^\ell \, , \frac{y_3}{h_n^0 \gamma_n^\ell} - \frac{x_{n,3}^\ell}{\gamma_n^\ell} \Big) \tag{2.20}$$

and

$$w_{0,n,\alpha,L}^{0,\infty}(y) \stackrel{\text{def}}{=} \sum_{\substack{\ell \in \Gamma^L(1) \\ |a_{\mathrm{h}}^\ell| = \infty}} \Big(- \frac{1}{h_n^0 \gamma_n^\ell} \nabla^{\mathrm{h}} \Delta_{\mathrm{h}}^{-1} \partial_3 \phi_\alpha^\ell, \phi_\alpha^\ell \Big) \Big(y_{\mathrm{h}} - x_{n,\mathrm{h}}^\ell \, , \frac{y_3}{h_n^0 \gamma_n^\ell} - \frac{x_{n,3}^\ell}{\gamma_n^\ell} \Big) \, . \tag{2.21}$$

By construction we have again

$$w^{0,\infty,\mathrm{h}}_{0,n,\alpha,L} = -\nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\partial_3 w^{0,\infty,3}_{0,n,\alpha,L}\,.$$

Moreover recalling the notation

$$[f]_{h^0_n}(x) \stackrel{\mathrm{def}}{=} f(x_{\mathrm{h}}, h^0_n x_3)$$

and

$$\Lambda_{\varepsilon_n,\gamma_n,x_n}\phi(x) \stackrel{\mathrm{def}}{=} \frac{1}{\varepsilon_n}\phi\left(\frac{x_{\mathrm{h}}-x_{n,\mathrm{h}}}{\varepsilon_n}, \frac{x_3-x_{n,3}}{\gamma_n}\right)\,,$$

one can compute that

$$\sum_{\substack{\ell\in\Gamma^L(1)\\ a^\ell_{\mathrm{h}}\in\mathbb{R}^2}} \Lambda_{1,\gamma^\ell_n,x^\ell_n}\left(\widetilde{\phi}^{h,\ell}_\alpha - \frac{1}{\gamma^\ell_n}\nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\partial_3\phi^\ell_\alpha, \phi^\ell_\alpha\right) = \big[(v^{0,\mathrm{loc,h}}_{0,n,\alpha,L} + h^0_n w^{0,\mathrm{loc,h}}_{0,n,\alpha,L}, w^{0,\mathrm{loc},3}_{0,n,\alpha,L})\big]_{h^0_n} \tag{2.22}$$

and

$$\sum_{\substack{\ell\in\Gamma^L(1)\\ |a^\ell_{\mathrm{h}}|=\infty}} \Lambda_{1,\gamma^\ell_n,x^\ell_n}\left(\widetilde{\phi}^{h,\ell}_\alpha - \frac{1}{\gamma^\ell_n}\nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\partial_3\phi^\ell_\alpha, \phi^\ell_\alpha\right) = \big[(v^{0,\infty,\mathrm{h}}_{0,n,\alpha,L} + h^0_n w^{0,\infty,\mathrm{h}}_{0,n,\alpha,L}, w^{0,\infty,3}_{0,n,\alpha,L})\big]_{h^0_n}\,. \tag{2.23}$$

Let us now check that $v^{0,\mathrm{loc,h}}_{0,n,\alpha,L}$, $w^{0,\mathrm{loc}}_{0,n,\alpha,L}$, $v^{0,\infty,\mathrm{h}}_{0,n,\alpha,L}$ and $w^{0,\infty}_{0,n,\alpha,L}$ satisfy the bounds given in the statement of Proposition 1.12. We shall only study $v^{0,\mathrm{loc,h}}_{0,n,\alpha,L}$ and $w^{0,\mathrm{loc}}_{0,n,\alpha,L}$ as the other study is very similar. On the one hand, by translation and scale invariance of $B^{0,\frac12}_{2,1}$ and using definitions (2.18) and (2.19), we get

$$\|v^{0,\mathrm{loc,h}}_{0,n,\alpha,L}\|_{\mathcal{B}^0} \leq \sum_{\ell\geq1}\|\widetilde{\phi}^{h,\ell}_\alpha\|_{\mathcal{B}^0} \quad\text{and}\quad \|w^{0,\mathrm{loc},3}_{0,n,\alpha,L}\|_{\mathcal{B}^0} \leq \sum_{\ell\geq1}\|\phi^\ell_\alpha\|_{\mathcal{B}^0}\,. \tag{2.24}$$

By (2.10), we infer that

$$\|v^{0,\mathrm{loc,h}}_{0,n,\alpha,L}\|_{\mathcal{B}^0} + \|w^{0,\mathrm{loc},3}_{0,n,\alpha,L}\|_{\mathcal{B}^0} \leq C \quad \text{uniformly in } \alpha\,, L\,, n\,. \tag{2.25}$$

Moreover for each given α, the profiles are as smooth as needed, and since in the above sums by construction $\gamma^{\ell_0^-}_{n,L} \leq \gamma^\ell_n$, one gets also after an easy computation

$$\forall s\in\mathbb{R}\,, \forall s'\geq 1/2\,,\quad \|v^{0,\mathrm{loc,h}}_{0,n,\alpha,L}\|_{B^{s,s'}_{2,q}} + \|v^{0,\mathrm{loc}}_{0,n,\alpha,L}\|_{B^{s,s'}_{2,q}} \leq C(\alpha) \quad \text{uniformly in } n\,, L\,. \tag{2.26}$$

Estimates (2.25) and (2.26) give easily (1.18) and (1.19).

Finally let us estimate $v^{0,\mathrm{loc,h}}_{0,n,\alpha,L}(\cdot,0)$ and $w^{0,\mathrm{loc},3}_{0,n,\alpha,L}(\cdot,0)$ in $B^0_{2,1}(\mathbb{R}^2)$ and prove (1.20). On the one hand by assumption we know that $u_{0,n} \rightharpoonup u_0$ in the sense of distributions. On the other hand we can take weak limits in the decomposition of $u_{0,n}$ provided by Proposition 2.7. We recall that by (2.9), if $\varepsilon^\ell_n/\gamma^\ell_n \to \infty$ then $\phi^\ell_\alpha \equiv r^\ell_\alpha \equiv 0$. Then we notice that clearly

$$\varepsilon^\ell_n \to 0 \;\text{ or }\; \varepsilon^\ell_n \to \infty \implies \Lambda_{\varepsilon^\ell_n,\gamma^\ell_n,x^\ell_n} f \rightharpoonup 0$$

for any value of the sequences γ^ℓ_n, x^ℓ_n and any function f. Moreover

$$\gamma^\ell_n \to 0 \implies \Lambda_{1,\gamma^\ell_n,x^\ell_n} f \rightharpoonup 0$$

for any sequence of cores x^ℓ_n and any function f, so we are left with the study of profiles such that $\varepsilon^\ell_n \equiv 1$ and $\gamma^\ell_n \to \infty$. Then we also notice that if $\gamma^\ell_n \to \infty$, then with Notation (2.17),

$$|a^\ell_{\mathrm{h}}| = \infty \implies \Lambda_{1,\gamma^\ell_n,x^\ell_n} f \rightharpoonup 0\,. \tag{2.27}$$

Consequently for each $L \geq 1$ and each α in $]0,1[$, we have in view of (2.11) and (2.12), as n goes to infinity

$$u^3_{0,n} - \psi^L_n - \sum_{\ell\in\Gamma^L(1)} r^\ell_\alpha(\cdot - x^\ell_{n,\mathrm{h}}, \frac{\cdot - x^\ell_{n,3}}{\gamma^\ell_n}) \rightharpoonup u^3_0 + \sum_{\substack{\ell\in\Gamma^L(1)\\ s.t.\, a^\ell_{\mathrm{h}}\in\mathbb{R}^2}} \phi^\ell_\alpha(\cdot - a^\ell_h, 0)$$

$$\nabla^\perp_h C_{0,n} - \tilde\psi^{h,L}_n - \sum_{\ell\in\Gamma^L(1)} \widetilde{r}^{h,\ell}_\alpha(\cdot - x^\ell_{n,\mathrm{h}}, \frac{\cdot - x^\ell_{n,3}}{\gamma^\ell_n}) \rightharpoonup \nabla^\perp_h\varphi + \sum_{\substack{\ell\in\Gamma^L(1)\\ s.t.\, a^\ell_{\mathrm{h}}\in\mathbb{R}^2}} \phi^{\mathrm{h},\ell}_\alpha(\cdot - a^\ell_h, 0)\,.$$

By hypothesis the sequence $(u^3_{0,n})_{n\in\mathbb{N}}$ converges weakly to u^3_0 and the sequence $(\nabla^\perp_h C_{0,n})_{n\in\mathbb{N}}$ converges weakly to $\nabla^\perp_h\varphi$, so for each $L\geq 1$ and all α in $]0,1[$, we have as n goes to infinity

$$(2.28)\qquad \begin{aligned} -\psi^L_n - \sum_{\ell\in\Gamma^L(1)} r^\ell_\alpha(\cdot - x^\ell_{n,\mathrm{h}}, \frac{\cdot - x^\ell_{n,3}}{\gamma^\ell_n}) &\rightharpoonup \sum_{\substack{\ell\in\Gamma^L(1)\\ s.t.\, a^\ell_{\mathrm{h}}\in\mathbb{R}^2}} \phi^\ell_\alpha(\cdot - a^\ell_h, 0)\\ -\tilde\psi^{h,L}_n - \sum_{\ell\in\Gamma^L(1)} \widetilde{r}^{h,\ell}_\alpha(\cdot - x^\ell_{n,\mathrm{h}}, \frac{\cdot - x^\ell_{n,3}}{\gamma^\ell_n}) &\rightharpoonup \sum_{\substack{\ell\in\Gamma^L(1)\\ s.t.\, a^\ell_{\mathrm{h}}\in\mathbb{R}^2}} \widetilde\phi^{h,\ell}_\alpha(\cdot - a^\ell_h, 0)\,. \end{aligned}$$

Now let $\eta > 0$ be given. Then thanks to (2.7) and (2.8), there is $L_0 \geq 1$ such that for all $L \geq L_0$ there is $\alpha_0 \leq 1$ (depending on L) such that for all $L \geq L_0$ and $\alpha \leq \alpha_0$, uniformly in $n\in\mathbb{N}$

$$\big\|(\widetilde\psi^{\mathrm{h},L}_n, \psi^L_n)\big\|_{\mathcal{B}^0} + \Big\|\sum_{\ell\in\Gamma^L(1)} (\widetilde{r}^{h,\ell}_\alpha, r^\ell_\alpha)(\cdot - x^\ell_{n,\mathrm{h}}, \frac{\cdot - x^\ell_{n,3}}{\gamma^\ell_n})\Big\|_{\mathcal{B}^0} \leq \eta\,.$$

Using the fact that $\mathcal{B}^0$ is embedded in $L^\infty(\mathbb{R}; B^0_{2,1}(\mathbb{R}^2))$, we infer from (2.28) that for $L \geq L_0$ and $\alpha \leq \alpha_0$

$$(2.29)\qquad \Big\|\sum_{\substack{\ell\in\Gamma^L(1)\\ s.t.\, a^\ell_{\mathrm{h}}\in\mathbb{R}^2}} \widetilde\phi^{h,\ell}_\alpha(\cdot - a^\ell_h, 0)\Big\|_{B^0_{2,1}(\mathbb{R}^2)} \leq \eta$$

and

$$(2.30)\qquad \Big\|\sum_{\substack{\ell\in\Gamma^L(1)\\ s.t.\, a^\ell_{\mathrm{h}}\in\mathbb{R}^2}} \phi^\ell_\alpha(\cdot - a^\ell_h, 0)\Big\|_{B^0_{2,1}(\mathbb{R}^2)} \leq \eta\,.$$

But by (2.18), we have

$$v^{0,\mathrm{loc,h}}_{0,n,\alpha,L}(\cdot,0) = \sum_{\substack{\ell\in\Gamma^L(1)\\ a^\ell_{\mathrm{h}}\in\mathbb{R}^2}} \widetilde\phi^{h,\ell}_\alpha\Big(\cdot - x^\ell_{n,\mathrm{h}}, -\frac{x^\ell_{n,3}}{\gamma^\ell_n}\Big)$$

and by (2.19) we have also

$$w^{0,\mathrm{loc},3}_{0,n,\alpha,L}(\cdot,0) = \sum_{\substack{\ell\in\Gamma^L(1)\\ a^\ell_{\mathrm{h}}\in\mathbb{R}^2}} \phi^\ell_\alpha\Big(\cdot - x^\ell_{n,\mathrm{h}}, -\frac{x^\ell_{n,3}}{\gamma^\ell_n}\Big)\,.$$

It follows that we can write for all $L \geq L_0$ and $\alpha \leq \alpha_0$,

$$\limsup_{n\to\infty} \|v^{0,\mathrm{loc,h}}_{0,n,\alpha,L}(\cdot,0)\|_{B^0_{2,1}(\mathbb{R}^2)} \leq \Big\| \sum_{\substack{\ell\in\Gamma^L(1)\\ a^\ell_{\mathrm{h}}\in\mathbb{R}^2}} \widetilde{\phi}^{h,\ell}_\alpha(\cdot - a^\ell_h, 0)\Big\|_{B^0_{2,1}(\mathbb{R}^2)} \leq \eta$$

thanks to (2.29). A similar estimate for $w^{0,\mathrm{loc},3}_{0,n,\alpha,L}(\cdot,0)$ using (2.30) gives finally

$$\lim_{L\to\infty}\lim_{\alpha\to 0}\limsup_{n\to\infty}\Big(\|v^{0,\mathrm{loc,h}}_{0,n,\alpha,L}(\cdot,0)\|_{B^0_{2,1}(\mathbb{R}^2)} + \|w^{0,\mathrm{loc},3}_{0,n,\alpha,L}(\cdot,0)\|_{B^0_{2,1}(\mathbb{R}^2)}\Big) = 0\,. \tag{2.31}$$

The results (1.21) and (1.22) involving the cut-off function θ are simply due to the fact that the profiles are compactly supported.

Construction of the profiles for $\ell \geq 1$. The construction is very similar to the previous one. We start by considering a fixed integer $j \in \{1,\dots,\mathcal{L}(L)\}$. Then we define an integer ℓ^-_j so that, up to a sequence extraction,

$$\gamma^{\ell^-_j}_n = \min_{\ell\in\Gamma^L(\varepsilon^{\ell_j}_n)} \gamma^\ell_n\,,$$

where as in (2.14)

$$\Gamma^L(\varepsilon^\ell_n) \stackrel{\mathrm{def}}{=} \Big\{\ell' \in \{1,\dots,L\}\,/\,\varepsilon^{\ell'}_n \equiv \varepsilon^\ell_n \quad\text{and}\quad \varepsilon^{\ell'}_n(\gamma^{\ell'}_n)^{-1} \to 0\,,\, n\to\infty\Big\}\,.$$

Notice that necessarily $\varepsilon^{\ell^-_j} \not\equiv 1$. Finally we define

$$h^j_n \stackrel{\mathrm{def}}{=} \varepsilon^{\ell_j}_n(\gamma^{\ell^-_j}_n)^{-1}\,.$$

By construction we have that $h^j_n \to 0$ as $n\to\infty$ (recall that $\varepsilon^{\ell_j}_n \equiv \varepsilon^{\ell^-_j}_n$). Then we define for $j \leq \mathcal{L}(L)$

$$v^{j,\mathrm{h}}_{n,\alpha,L}(y) \stackrel{\mathrm{def}}{=} \sum_{\ell\in\Gamma^L(\varepsilon^{\ell_j}_n)} \widetilde{\phi}^{h,\ell}_\alpha\Big(y_h - \frac{x^\ell_{n,\mathrm{h}}}{\varepsilon^{\ell_j}_n}, \frac{\varepsilon^{\ell_j}_n}{h^j_n\gamma^\ell_n}y_3 - \frac{x^\ell_{n,3}}{\gamma^\ell_n}\Big) \tag{2.32}$$

and

$$w^j_{n,\alpha,L}(y) \stackrel{\mathrm{def}}{=} \sum_{\ell\in\Gamma^L(\varepsilon^{\ell_j}_n)} \Big(-\frac{\varepsilon^{\ell_j}_n}{h^j_n\gamma^\ell_n}\nabla^{\mathrm{h}}\Delta^{-1}_{\mathrm{h}}\partial_3\phi^\ell_\alpha, \phi^\ell_\alpha\Big)\Big(y_h - \frac{x^\ell_{n,\mathrm{h}}}{\varepsilon^{\ell_j}_n}, \frac{\varepsilon^{\ell_j}_n}{h^j_n\gamma^\ell_n}y_3 - \frac{x^\ell_{n,3}}{\gamma^\ell_n}\Big)$$

and we choose

$$\mathcal{L}(L) < j \leq L \quad\Rightarrow\quad v^{j,\mathrm{h}}_{n,\alpha,L} \equiv 0 \quad\text{and}\quad w^j_{n,\alpha,L} \equiv 0\,. \tag{2.33}$$

We notice that

$$w^{j,\mathrm{h}}_{n,\alpha,L} = -\nabla^{\mathrm{h}}\Delta^{-1}_{\mathrm{h}}\partial_3 w^{j,3}_{n,\alpha,L}\,.$$

Defining

$$\lambda^j_n \stackrel{\mathrm{def}}{=} \varepsilon^{\ell_j}_n\,,$$

a computation, similar to that giving (2.22) implies directly that

$$\begin{aligned}\sum_{\ell\in\Gamma^L(\varepsilon^{\ell_j}_n)} &\Lambda_{\varepsilon^{\ell_j}_n,\gamma^\ell_n,x^\ell_n}\Big(\widetilde{\phi}^{h,\ell}_\alpha - \frac{\lambda^j_n}{\gamma^\ell_n}\nabla^{\mathrm{h}}\Delta^{-1}_{\mathrm{h}}\partial_3\phi^\ell_\alpha, \phi^\ell_\alpha\Big)\\ &= \Lambda_{\lambda^j_n}\big[(v^{j,\mathrm{h}}_{n,\alpha,L} + h^j_n w^{j,\mathrm{h}}_{n,\alpha,L}, w^{j,3}_{n,\alpha,L})\big]_{h^j_n}\,.\end{aligned} \tag{2.34}$$

Notice that since $\varepsilon_n^{\ell_j} \not\equiv 1$ as recalled above, we have that $\lambda_n^j \to 0$ or ∞ as $n \to \infty$.
The a priori bounds for the profiles $(v^{j,\mathrm{h}}_{n,\alpha,L}, w^{j,3}_{n,\alpha,L})_{1\leq j\leq L}$ are obtained exactly as in the previous paragraph: let us prove that

$$
(2.35)\qquad
\begin{aligned}
&\sum_{j\geq 1} \big(\|v^{j,\mathrm{h}}_{n,\alpha,L}\|_{\mathcal{B}^0_{2,q}} + \|w^{j,3}_{n,\alpha,L}\|_{\mathcal{B}^0_{2,q}}\big) \leq C\,, \quad \text{and}\\
\forall s\in\mathbb{R}\,,\quad \forall s'\geq 1/2\,,\quad &\sum_{j\geq 1} \big(\|v^{j,\mathrm{h}}_{n,\alpha,L}\|_{B^{s,s'}_{2,q}} + \|w^{j,3}_{n,\alpha,L}\|_{B^{s,s'}_{2,q}}\big) \leq C(\alpha)\,.
\end{aligned}
$$

We shall detail the argument for the first inequality only, and in the case of $v^{j,\mathrm{h}}_{n,\alpha,L}$ as the study of $w^{j,3}_{n,\alpha,L}$ is similar. We write, using the definition of $v^{j,\mathrm{h}}_{n,\alpha,L}$ in (2.32),

$$
\sum_{j=1}^{L} \|v^{j,\mathrm{h}}_{n,\alpha,L}\|_{B^{0,\frac12}_{2,q}} = \sum_{j=1}^{\mathcal{L}(L)} \Big\| \sum_{\ell\in\Gamma^L(\varepsilon_n^{\ell_j})} \widetilde{\phi}^{h,\ell}_\alpha\Big(y_h - \frac{x^\ell_{n,\mathrm{h}}}{\varepsilon_n^{\ell_j}}, \frac{\varepsilon_n^{\ell_j}}{h^j_n\gamma^\ell_n} y_3 - \frac{x^\ell_{n,3}}{\gamma^\ell_n}\Big)\Big\|_{B^{0,\frac12}_{2,q}}\,,
$$

so by definition of the partition (2.15) and by scale and translation invariance of $B^{0,\frac12}_{2,q}$ we find thanks to (2.10), that there is a constant C independent of L such that

$$
\sum_{j=1}^{L} \|v^{j,\mathrm{h}}_{n,\alpha,L}\|_{B^{0,\frac12}_{2,q}} \leq \sum_{\ell=1}^{L} \|\widetilde{\phi}^{h,\ell}_\alpha\|_{B^{0,\frac12}_{2,q}} \leq C\,.
$$

The result is proved.
Construction of the remainder term. With the notation of Proposition 2.7, let us first define the remainder terms

$$
(2.36)\qquad \tilde\rho^{(1),\mathrm{h}}_{n,\alpha,L} \stackrel{\mathrm{def}}{=} -\sum_{\ell=1}^{L} \frac{\varepsilon^\ell_n}{\gamma^\ell_n} \Lambda_{\varepsilon^\ell_n,\gamma^\ell_n,x^\ell_n} \nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\partial_3 r^\ell_\alpha - \nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\partial_3\psi^L_n
$$

and

$$
(2.37)\qquad \rho^{(2)}_{n,\alpha,L} \stackrel{\mathrm{def}}{=} \sum_{\ell=1}^{L} \Lambda_{\varepsilon^\ell_n,\gamma^\ell_n,x^\ell_n}\big(\tilde r^{h,\ell}_\alpha, 0\big) + \sum_{\ell=1}^{L} \Lambda_{\varepsilon^\ell_n,\gamma^\ell_n,x^\ell_n}(0, r^\ell_\alpha) + \big(\widetilde{\psi}^{h,L}_n, \psi^L_n\big)\,.
$$

Observe that by construction, thanks to (2.2) and (2.8) and to the fact that if $r^\ell_\alpha \not\equiv 0$, then $\varepsilon^\ell_n/\gamma^\ell_n$ goes to zero as n goes to infinity, we have

$$
(2.38)\qquad
\begin{aligned}
&\lim_{L\to\infty}\lim_{\alpha\to 0}\limsup_{n\to\infty} \|\tilde\rho^{(1),\mathrm{h}}_{\alpha,n,L}\|_{B^{\frac2p,-1+\frac1p}_{p,1}} = 0\,,\\
\text{and}\quad &\lim_{L\to\infty}\lim_{\alpha\to 0}\limsup_{n\to\infty} \|\rho^{(2)}_{\alpha,n,L}\|_{B^{-1+\frac2p,\frac1p}_{p,1}} = 0\,.
\end{aligned}
$$

Then we notice that for each $\ell\in\mathbb{N}$ and each $\alpha\in]0,1[$, we have by a direct computation

$$
\Big\|\Lambda_{\varepsilon^\ell_n,\gamma^\ell_n,x^\ell_n}(\widetilde{\phi}^{h,\ell}_\alpha, 0)\Big\|_{\mathcal{B}^{1,-\frac12}} \sim \Big(\frac{\gamma^\ell_n}{\varepsilon^\ell_n}\Big)^{\frac12} \Big\|\widetilde{\phi}^{h,\ell}_\alpha\Big\|_{\mathcal{B}^{1,-\frac12}}\,.
$$

We deduce that if $\varepsilon^\ell_n/\gamma^\ell_n \to \infty$, then $\Lambda_{\varepsilon^\ell_n,\gamma^\ell_n,\tilde x^\ell_n}(\widetilde{\phi}^{h,\ell}_\alpha, 0)$ goes to zero in $\mathcal{B}^{1,-\frac12}$ as n goes to infinity, hence so does the sum over $\ell\in\{1,\dots,L\}$. It follows that for each given α in $]0,1[$ and $L\geq 1$ we may define

$$
\rho^{(1)}_{n,\alpha,L} \stackrel{\mathrm{def}}{=} \tilde\rho^{(1),\mathrm{h}}_{n,\alpha,L} + \sum_{\substack{\ell=1\\ \varepsilon^\ell_n/\gamma^\ell_n\to\infty}}^{L} \Lambda_{\varepsilon^\ell_n,\gamma^\ell_n,x^\ell_n}(\widetilde{\phi}^{h,\ell}_\alpha, 0)
$$

and we have

$$\lim_{L\to\infty}\lim_{\alpha\to 0}\limsup_{n\to\infty}\|\rho^{(1)}_{n,\alpha,L}\|_{B^{\frac{2}{p},-1+\frac{1}{p}}_{p,1}}=0\,. \tag{2.39}$$

Finally, as the space $\mathcal{D}(\mathbb{R}^3)$ is dense in $\mathcal{B}^0$, let us choose a family $(u_{0,\alpha})_\alpha$ of functions in $\mathcal{D}(\mathbb{R}^3)$ such that $\|u_0-u_{0,\alpha}\|_{\mathcal{B}^0}\leq\alpha$ and let us define

$$\rho_{n,\alpha,L}\stackrel{\rm def}{=}\rho^{(1)}_{\alpha,n,L}+\rho^{(2)}_{n,\alpha,L}+u_0-u_{0,\alpha}\,. \tag{2.40}$$

Inequalities (2.38) and (2.39) give

$$\lim_{L\to\infty}\lim_{\alpha\to 0}\limsup_{n\to\infty}\|e^{t\Delta}\rho_{n,\alpha,L}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}=0\,. \tag{2.41}$$

2.2.3. *End of the proof of Proposition* 1.12. Let us return to the decomposition given in Proposition 2.7, and use definitions (2.36), (2.37) and (2.40) which imply that

$$u_{0,n}=u_{0,\alpha}+\sum_{\substack{\ell=1\\ \varepsilon_n^\ell/\gamma_n^\ell\to 0}}^{L}\Lambda_{\varepsilon_n^\ell,\gamma_n^\ell,x_n^\ell}\Big(\widetilde\phi^{h,\ell}_\alpha-\frac{\varepsilon_n^\ell}{\gamma_n^\ell}\nabla^{\rm h}\Delta_{\rm h}^{-1}\partial_3\phi^\ell_\alpha,\phi^\ell_\alpha\Big)+\rho_{n,\alpha,L}\,.$$

We recall that for all ℓ in $\mathbb{N}$, we have $\lim_{n\to\infty}(\gamma_n^\ell)^{-1}\varepsilon_n^\ell\in\{0,\infty\}$ and in the case where the ratio $\varepsilon_n^\ell/\gamma_n^\ell$ goes to infinity then $\phi^\ell_\alpha\equiv 0$. Next we separate the case when the horizontal scale is one, from the others: with the notation (2.14) we write

$$\begin{aligned}u_{0,n}=u_{0,\alpha}&+\sum_{\ell\in\Gamma^L(1)}\Lambda_{1,\gamma_n^\ell,x_n^\ell}\left(\widetilde\phi^{h,\ell}_\alpha-\frac{1}{\gamma_n^\ell}\nabla^{\rm h}\Delta_{\rm h}^{-1}\partial_3\phi^\ell_\alpha,\phi^\ell_\alpha\right)\\&+\sum_{\substack{\ell=1\\ \varepsilon_n^\ell\not\equiv 1\\ \varepsilon_n^\ell/\gamma_n^\ell\to 0}}^{L}\Lambda_{\varepsilon_n^\ell,\gamma_n^\ell,x_n^\ell}\Big(\widetilde\phi^{h,\ell}_\alpha-\frac{\varepsilon_n^\ell}{\gamma_n^\ell}\nabla^{\rm h}\Delta_{\rm h}^{-1}\partial_3\phi^\ell_\alpha,\phi^\ell_\alpha\Big)+\rho_{n,\alpha,L}\,.\end{aligned}$$

With (2.22) this can be written

$$\begin{aligned}u_{0,n}=u_{0,\alpha}&+\big[(v^{0,{\rm loc,h}}_{0,n,\alpha,L}+h_n^0w^{0,{\rm loc,h}}_{0,n,\alpha,L},w^{0,{\rm loc},3}_{0,n,\alpha,L})\big]_{h_n^0}+\big[(v^{0,\infty,{\rm h}}_{0,n,\alpha,L}+h_n^0w^{0,\infty,{\rm h}}_{0,n,\alpha,L},w^{0,\infty,3}_{0,n,\alpha,L})\big]_{h_n^0}\\&+\sum_{\substack{\ell=1\\ \varepsilon_n^\ell\not\equiv 1\\ \varepsilon_n^\ell/\gamma_n^\ell\to 0}}\Lambda_{\varepsilon_n^\ell,\gamma_n^\ell,x_n^\ell}\Big(\widetilde\phi^{h,\ell}_\alpha-\frac{\varepsilon_n^\ell}{\gamma_n^\ell}\nabla^{\rm h}\Delta_{\rm h}^{-1}\partial_3\phi^\ell_\alpha,\phi^\ell_\alpha\Big)+\rho_{n,\alpha,L}\,.\end{aligned}$$

Next we use the partition (2.15), so that with notation (2.13) and (2.14),

$$\begin{aligned}u_{0,n}=u_{0,\alpha}&+\big[(v^{0,{\rm loc,h}}_{0,n,\alpha,L}+h_n^0w^{0,{\rm loc,h}}_{0,n,\alpha,L},w^{0,{\rm loc},3}_{0,n,\alpha,L})\big]_{h_n^0}+\big[(v^{0,\infty,{\rm h}}_{0,n,\alpha,L}+h_n^0w^{0,\infty,{\rm h}}_{0,n,\alpha,L},w^{0,\infty,3}_{0,n,\alpha,L})\big]_{h_n^0}\\&+\sum_{j=1}^{\mathcal{L}(L)}\sum_{\substack{\ell\in\Gamma^L(\varepsilon_n^{\ell_j})\\ \varepsilon_n^{\ell_j}\not\equiv 1}}\Lambda_{\varepsilon_n^{\ell_j},\gamma_n^\ell,x_n^\ell}\left(\widetilde\phi^{h,\ell}_\alpha-\frac{\varepsilon_n^{\ell_j}}{\gamma_n^\ell}\nabla^{\rm h}\Delta_{\rm h}^{-1}\partial_3\phi^\ell_\alpha,\phi^\ell_\alpha\right)+\rho_{n,\alpha,L}\,.\end{aligned}$$

Then we finally use the identity (2.34) which gives

$$\begin{aligned}u_{0,n}=u_{0,\alpha}&+\big[(v^{0,{\rm loc,h}}_{0,n,\alpha,L}+h_n^0w^{0,{\rm loc,h}}_{0,n,\alpha,L},w^{0,{\rm loc},3}_{0,n,\alpha,L})\big]_{h_n^0}+\big[(v^{0,\infty,{\rm h}}_{0,n,\alpha,L}+h_n^0w^{0,\infty,{\rm h}}_{0,n,\alpha,L},w^{0,\infty,3}_{0,n,\alpha,L})\big]_{h_n^0}\\&+\sum_{j=1}^{L}\Lambda_{\lambda_n^j}\big[(v^{j,{\rm h}}_{n,\alpha,L}+h_n^jw^{j,{\rm h}}_{n,\alpha,L},w^{j,3}_{n,\alpha,L})\big]_{h_n^j}+\rho_{n,\alpha,L}\,.\end{aligned}$$

The end of the proof follows from the estimates (2.25), (2.26), (2.31), (2.35), along with (2.41). Proposition 1.12 is proved. □

3. Propagation of profiles: proof of Theorem 4

The goal of this section is the proof of Theorem 4. Let us consider (v_0, w_0^3) satisfying the assumptions of that theorem. In order to prove that the initial data defined by

$$\Phi_0 \stackrel{\text{def}}{=} \big[(v_0 - \beta\nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\partial_3 w_0^3, w_0^3)\big]_\beta$$

generates a global smooth solution for small enough β, let us look for the solution under the form

$$\Phi_\beta = \Phi^{\mathrm{app}} + \psi \quad \text{with} \quad \Phi^{\mathrm{app}} \stackrel{\text{def}}{=} \big[(v + \beta w^{\mathrm{h}}, w^3)\big]_\beta \tag{3.1}$$

where v solves the two-dimensional Navier-Stokes equations

$$(\mathrm{NS2D})_{x_3} \quad \begin{cases} \partial_t v + v\cdot\nabla^{\mathrm{h}} v - \Delta_{\mathrm{h}} v = -\nabla^{\mathrm{h}} p \quad \text{in } \mathbb{R}^+\times\mathbb{R}^2 \\ \mathrm{div}_{\mathrm{h}} v = 0 \\ v_{|t=0} = v_0(\cdot, x_3)\,, \end{cases}$$

while w^3 solves the transport-diffusion equation

$$(T_\beta) \quad \begin{cases} \partial_t w^3 + v\cdot\nabla^{\mathrm{h}} w^3 - \Delta_{\mathrm{h}} w^3 - \beta^2\partial_3^2 w^3 = 0 \quad \text{in } \mathbb{R}^+\times\mathbb{R}^3 \\ w^3_{|t=0} = w_0^3 \end{cases}$$

and w^{h} is determined by the divergence free condition on w which gives $w^{\mathrm{h}} \stackrel{\text{def}}{=} -\nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\partial_3 w^3$. In Section 3.1 (resp. 3.2), we prove a priori estimates on v (resp. w), and Section 3.3 is devoted to the conclusion of the proof of Theorem 4, studying the perturbed Navier-Stokes equation satisfied by ψ.

Before starting the proof we recall the following definitions of space-time norms, first introduced by J.-Y. Chemin and N. Lerner in [18], and which are very useful in the context of the Navier-Stokes equations:

$$\|f\|_{\widetilde{L}^r([0,T];B^{s,s'}_{p,q})} \stackrel{\text{def}}{=} \big\|2^{ks+js'}\|\Delta_k^{\mathrm{h}}\Delta_j^{\mathrm{v}} f\|_{L^r([0,T];L^p)}\big\|_{\ell^q}\,. \tag{3.2}$$

Notice that of course $\widetilde{L}^r([0,T];B^{s,s'}_{p,r}) = L^r([0,T];B^{s,s'}_{p,r})$, and by Minkowski's inequality, we have the embedding $\widetilde{L}^r([0,T];B^{s,s'}_{p,q}) \subset L^r([0,T];B^{s,s'}_{p,q})$ if $r \geq q$.

3.1. **Two dimensional flows with parameter.**

Let us prove the following result on v, the solution of $(\mathrm{NS2D})_{x_3}$. We shall use the notation introduced in Definitions 1.10 and 1.13.

Proposition 3.1. *Let v_0 be a two-component divergence free vector field depending on the vertical variable x_3, and belonging to S_μ. Then the unique, global solution v to $(\mathrm{NS2D})_{x_3}$ belongs to $\mathcal{A}^0$ and satisfies the following estimate:*

$$\|v\|_{\mathcal{A}^0} \leq \mathcal{T}_1(\|v_0\|_{\mathcal{B}^0})\,. \tag{3.3}$$

Moreover, for all (s,s') in D_μ, we have

$$\forall r\in[1,\infty]\,,\ \|v\|_{\widetilde{L}^r(\mathbb{R}^+;\mathcal{B}^{s+\frac{2}{r},s'})} \leq \mathcal{T}_\infty(\|v_0\|_{S_\mu}). \tag{3.4}$$

Proof. This proposition is a result about the regularity of the solution of (NS2D) when the initial data depends on a real parameter x_3, measured in terms of Besov spaces with respect to the variable x_3. Its proof is structured as follows. First, we deduce from the classical energy estimate for the two dimensional Navier-Stokes system, a stability result in the spaces $L^r(\mathbb{R}^+;H^{s+\frac{2}{r}}(\mathbb{R}^2))$ with r in $[2,\infty]$ and s in $]-1,1[$. This is the purpose of

Lemma 3.2, the proof of which uses essentially energy estimates together with paraproduct laws.
Then we have to translate the stability result of Lemma 3.2 in terms of Besov spaces with respect to the third variable (seen before simply as a parameter), namely by propagating the vertical regularity. First of all, this requires to deduce from the stability in the spaces $L^r(\mathbb{R}^+; H^{s+\frac{2}{r}}(\mathbb{R}^2))$ with r in $[2,\infty]$, the fact that the vector field v, now seen as a function of three variables, belongs to $L^r(\mathbb{R}^+; L^\infty_{\rm v}(H^{s+\frac{2}{r}}(\mathbb{R}^2))$ again for r in $[2,\infty]$. This is the purpose of Lemma 3.3, the proof of which relies on the equivalence of two definitions of Besov spaces with regularity index in $]0,1[$: the first one involving the dyadic decomposition of the frequency space, and the other one consisting in estimating integrals in physical space. Finally for s in $]-\frac{1}{2},\frac{1}{2}[$ and $s'>0$ a Gronwall type lemma enables us to propagate the regularities. When $s'\geq \dfrac{1}{2}$ product laws enable us to gain horizontal regularity up to $]-2,1[$ and to conclude the proof of Proposition 3.1.

Let us state and prove the first lemma in this proof.

Lemma 3.2. *For any compact set I included in $]-1,1[$, a constant C exists such that, for any r in $[2,\infty]$ and any s in I, we have for any two solutions v_1 and v_2 of the two-dimensional Navier-Stokes equations*

$$\|v_1-v_2\|_{L^r(\mathbb{R}^+;H^{s+\frac{2}{r}}(\mathbb{R}^2))} \lesssim \|v_1(0)-v_2(0)\|_{H^s(\mathbb{R}^2)}\, E_{12}(0)\,, \tag{3.5}$$

where we define

$$E_{12}(0) \stackrel{def}{=} \exp C\big(\|v_1(0)\|^2_{L^2}+\|v_2(0)\|^2_{L^2}\big)\,.$$

Proof. In the proof of this lemma, all the functional spaces are over $\mathbb{R}^2$ and we no longer mention this fact in notations. Moreover, the constant which appears in the definition of $E_{12}(0)$ can change along the proof. Defining $v_{12}(t) \stackrel{\rm def}{=} v_1(t)-v_2(t)$, we get

$$\partial_t v_{12} + v_2\cdot\nabla^{\rm h} v_{12} - \Delta_{\rm h} v_{12} = -v_{12}\cdot\nabla^{\rm h} v_1 - \nabla^{\rm h} p\,. \tag{3.6}$$

In order to establish (3.5), we shall resort to an energy estimate making use of product laws and of the following estimate proved in [13, Lemma 1.1]:

$$\big(v\cdot\nabla^{\rm h} a|a\big)_{H^s} \lesssim \|\nabla^{\rm h} v\|_{L^2}\|a\|_{H^s}\|\nabla^{\rm h} a\|_{H^s}, \tag{3.7}$$

available uniformly for any s in $[-2+\mu,1-\mu]$.
Let us notice that thanks to the divergence free condition, taking the H^s scalar product with v_{12} in Equation (3.6) implies that

$$\frac{1}{2}\frac{d}{dt}\|v_{12}(t)\|^2_{H^s} + \|\nabla^{\rm h} v_{12}(t)\|^2_{H^s} = -\big(v_2(t)\cdot\nabla^{\rm h} v_{12}(t)|v_{12}(t)\big)_{H^s} - \big(v_{12}(t)\cdot\nabla^{\rm h} v_1(t)|v_{12}(t)\big)_{H^s}\,.$$

Whence, by time integration we get

$$\begin{aligned}\|v_{12}(t)\|^2_{H^s} + 2\int_0^t \|\nabla^{\rm h} v_{12}(t')\|^2_{H^s}dt' = \|v_{12}(0)\|^2_{H^s} &- 2\int_0^t \big(v_2(t')\cdot\nabla^{\rm h} v_{12}(t')|v_{12}(t')\big)_{H^s}\,dt'\\ &-2\int_0^t \big(v_{12}(t')\cdot\nabla^{\rm h} v_1(t')|v_{12}(t')\big)_{H^s}\,dt'\,.\end{aligned}$$

Now using Estimate (3.7), we deduce that there is a positive constant C such that for any s in I, we have

$$
\begin{aligned}
&2\Big|\int_0^t \big(v_2(t')\cdot\nabla^{\rm h} v_{12}(t')|v_{12}(t')\big)_{H^s}dt'\Big|\\
&\qquad\qquad\leq C\int_0^t \|v_{12}(t')\|_{H^s}\|\nabla^{\rm h} v_2(t')\|_{L^2}\|\nabla^{\rm h} v_{12}(t')\|_{H^s}\,dt'\\
&\qquad\qquad\leq \frac12\int_0^t \|\nabla^{\rm h} v_{12}(t')\|_{H^s}^2\,dt'+\frac{C^2}{2}\int_0^t \|v_{12}(t')\|_{H^s}^2\|\nabla^{\rm h} v_2(t')\|_{L^2}^2\,dt'\,.
\end{aligned}
\tag{3.8}
$$

Noticing that

$$
\int_0^t \big(v_{12}(t')\cdot\nabla^{\rm h} v_1(t')|v_{12}(t')\big)_{H^s}dt'\leq \int_0^t \|\nabla^{\rm h} v_{12}(t')\|_{H^s}\|v_{12}(t')\cdot\nabla^{\rm h} v_1(t')\|_{H^{s-1}}\,dt'\,,
$$

we deduce by Cauchy-Schwarz inequality and product laws in Sobolev spaces on $\mathbb{R}^2$ that as long as s is in $]0,1[$,

$$
\begin{aligned}
&2\Big|\int_0^t \big(v_{12}(t')\cdot\nabla^{\rm h} v_1(t')|v_{12}(t')\big)_{H^s}dt'\Big|\\
&\qquad\qquad\leq C\int_0^t \|\nabla^{\rm h} v_{12}(t')\|_{H^s}\|v_{12}(t')\|_{H^s}\|\nabla^{\rm h} v_1(t')\|_{L^2}\,dt'\\
&\qquad\qquad\leq \frac12\int_0^t \|\nabla^{\rm h} v_{12}(t')\|_{H^s}^2\,dt'+\frac{C^2}{2}\int_0^t \|v_{12}(t')\|_{H^s}^2\|\nabla^{\rm h} v_1(t')\|_{L^2}^2\,dt'\,.
\end{aligned}
\tag{3.9}
$$

When $s=0$ we simply write, by product laws and interpolation,

$$
\begin{aligned}
&2\Big|\int_0^t \big(v_{12}(t')\cdot\nabla^{\rm h} v_1(t')|v_{12}(t')\big)_{L^2}dt'\Big|\\
&\qquad\qquad\leq C\int_0^t \|v_{12}(t')\|_{H^{\frac12}}\|v_{12}(t')\cdot\nabla^{\rm h} v_1(t')\|_{H^{-\frac12}}dt'\\
&\qquad\qquad\leq \frac12\int_0^t \|\nabla^{\rm h} v_{12}(t')\|_{L^2}^2\,dt'+\frac{C^2}{2}\int_0^t \|v_{12}(t')\|_{L^2}^2\|\nabla^{\rm h} v_1(t')\|_{L^2}^2\,dt'\,.
\end{aligned}
\tag{3.10}
$$

Finally in the case when s belongs to $]-1,0[$, we have

$$
\begin{aligned}
&2\Big|\int_0^t \big(v_{12}(t')\cdot\nabla^{\rm h} v_1(t')|v_{12}(t')\big)_{H^s}dt'\Big|\\
&\qquad\qquad\leq C\int_0^t \|v_{12}(t')\|_{H^s}\|v_{12}(t')\cdot\nabla^{\rm h} v_1(t')\|_{H^s}dt'\\
&\qquad\qquad\leq \frac12\int_0^t \|\nabla^{\rm h} v_{12}(t')\|_{H^s}^2\,dt'+\frac{C^2}{2}\int_0^t \|v_{12}(t')\|_{H^s}^2\|\nabla^{\rm h} v_1(t')\|_{L^2}^2\,dt'\,.
\end{aligned}
\tag{3.11}
$$

Combining (3.8) and (3.9)-(3.11), we infer that for s in $]-1,1[$,

$$
\begin{aligned}
\|v_{12}(t)\|_{H^s}^2+\int_0^t \|\nabla^{\rm h} v_{12}(t')\|_{H^s}^2dt'&\lesssim \|v_{12}(0)\|_{H^s}^2\\
&\qquad+\int_0^t \|v_{12}(t')\|_{H^s}^2\big(\|\nabla^{\rm h} v_1(t')\|_{L^2}^2+\|\nabla^{\rm h} v_2(t')\|_{L^2}^2\big)\,dt'\,.
\end{aligned}
$$

Gronwall's lemma implies that there exists a positive constant C such that

$$
\|v_{12}(t)\|_{H^s}^2+\int_0^t \|\nabla^{\rm h} v_{12}(t')\|_{H^s}^2dt'\lesssim \|v_{12}(0)\|_{H^s}^2\exp C\int_0^t \big(\|\nabla^{\rm h} v_1(t')\|_{L^2}^2+\|\nabla^{\rm h} v_2(t')\|_{L^2}^2\big)dt'\,.
$$

But for any i in $\{1,2\}$, we have by the classical L^2 energy estimate

$$\int_0^t \|\nabla^{\mathrm{h}} v_i(t')\|_{L^2}^2 dt' \leq \frac{1}{2}\|v_i(0)\|_{L^2}^2 \,. \tag{3.12}$$

Consequently for s in $]-1,1[$,

$$\|v_{12}(t)\|_{H^s}^2 + \int_0^t \|\nabla^{\mathrm{h}} v_{12}(t')\|_{H^s}^2 dt' \lesssim \|v_{12}(0)\|_{H^s}^2 \, E_{12}(0)\,,$$

which leads to the result by interpolation. Lemma 3.2 is proved. □

Using Lemma 3.2, we are going to establish the following result, which will be of great help to control all norms of v of the type $\widetilde{L}^r(\mathbb{R}^+;\mathcal{B}^{\frac{2}{r}})$ for r in $[4,\infty]$ thanks to a Gronwall type argument.

Lemma 3.3. *For any compact set I included in $]-1,1[$, a constant C exists such that, for any r in $[2,\infty]$ and any s in I, we have for any solution v to* $(\mathrm{NS2D})_{x_3}$,

$$\|v\|_{L^r(\mathbb{R}^+;L^\infty_{\mathrm{v}}(H_{\mathrm{h}}^{s+\frac{2}{r}}))} \lesssim \|v_0\|_{\mathcal{B}^s} E(0) \quad \textit{with} \quad E(0) \stackrel{\mathrm{def}}{=} \exp\bigl(C\|v(0)\|^2_{L^\infty_{\mathrm{v}} L^2_{\mathrm{h}}}\bigr).$$

Proof. We shall use the characterization of Besov spaces via differences in physical space: as is well-known (see for instance Theorem 2.36 of [2]), for any Banach space X of distributions one has

$$\bigl\|\bigl(2^{\frac{j}{2}}\|\Delta_j^{\mathrm{v}} u\|_{L^2_{\mathrm{v}}(X)}\bigr)_j\bigr\|_{\ell^1(\mathbb{Z})} \sim \int_{\mathbb{R}} \frac{\|u-(\tau_{-z}u)\|_{L^2_{\mathrm{v}}(X)}}{|z|^{\frac{1}{2}}}\frac{dz}{|z|} \tag{3.13}$$

where the translation operator τ_{-z} is defined by

$$(\tau_{-z}f)(t,x_{\mathrm{h}},x_3) \stackrel{\mathrm{def}}{=} f(t,x_{\mathrm{h}},x_3+z)\,.$$

The above Lemma 3.2 implies in particular that, for any r in $[2,\infty]$, any s in I and any couple (x_3,z) in $\mathbb{R}^2$, if v solves $(\mathrm{NS2D})_{x_3}$ then

$$\|v-\tau_{-z}v\|_{Y_r^s} \lesssim \|v_0-\tau_{-z}v_0\|_{H^s_{\mathrm{h}}} E(0) \quad \text{with} \quad Y_r^s \stackrel{\mathrm{def}}{=} L^r(\mathbb{R}^+;H_{\mathrm{h}}^{s+\frac{2}{r}})\,.$$

Taking the L^2 norm of the above inequality with respect to the x_3 variable and then the L^1 norm with respect to the measure $|z|^{-\frac{3}{2}}dz$ gives

$$\int_{\mathbb{R}} \frac{\|v-\tau_{-z}v\|_{L^2_{\mathrm{v}}(Y_r^s)}}{|z|^{\frac{1}{2}}}\frac{dz}{|z|} \lesssim \int_{\mathbb{R}} \frac{\|v_0-\tau_{-z}v_0\|_{L^2_{\mathrm{v}}(H^s_{\mathrm{h}})}}{|z|^{\frac{1}{2}}}\frac{dz}{|z|}\, E(0)\,. \tag{3.14}$$

Returning to the characterization (3.13) with $X=Y_r^s$, we find that

$$\int_{\mathbb{R}} \frac{\|v-\tau_{-z}v\|_{L^2_{\mathrm{v}}(Y_r^s)}}{|z|^{\frac{1}{2}}}\frac{dz}{|z|} \sim \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \Bigl\|\bigl\|\bigl(2^{k(s+\frac{2}{r})}\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}} v(t,\cdot,z)\bigr)_k\bigr\|_{L^r(\mathbb{R}^+;\ell^2(\mathbb{Z};L^2_{\mathrm{h}}))}\Bigr\|_{L^2_{\mathrm{v}}}\,.$$

Similarly we have

$$\int_{\mathbb{R}} \frac{\|v_0-\tau_{-z}v_0\|_{L^2_{\mathrm{v}}(H^s_{\mathrm{h}})}}{|z|^{\frac{1}{2}}}\frac{dz}{|z|} \sim \sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}}\bigl\|\bigl(2^{ks}\|\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}} v_0\|_{L^2_{\mathrm{h}}}\bigr)_k\bigr\|_{\ell^2(\mathbb{Z};L^2_{\mathrm{v}})}\,,$$

so by the embedding from $\ell^1(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$, we get

$$\int_{\mathbb{R}} \frac{\|v_0-\tau_{-z}v_0\|_{L^2_{\mathrm{v}}(H^s_{\mathrm{h}})}}{|z|^{\frac{1}{2}}}\frac{dz}{|z|} \lesssim \sum_{(j,k)\in\mathbb{Z}^2} 2^{\frac{j}{2}}2^{ks}\|\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}} v_0\|_{L^2(\mathbb{R}^3)}\,.$$

Therefore, we deduce from Estimate (3.14) that

$$\sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \Big\| \big\| \big(2^{k(s+\frac{2}{r})}\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}} v(t,\cdot,z)\big)_k \big\|_{L^r(\mathbb{R}^+;\ell^2(\mathbb{Z};L^2_{\mathrm{h}}))} \Big\|_{L^2_{\mathrm{v}}} \lesssim \|v_0\|_{\mathcal{B}^s}\, E(0)\,.$$

As $r \geq 2$, Minkowski's inequality implies that

$$\sum_{j\in\mathbb{Z}} 2^{\frac{j}{2}} \Big\| \big\| \big(2^{k(s+\frac{2}{r})}\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}} v(t,\cdot)\big)_k \big\|_{\ell^2(\mathbb{Z};L^2)} \Big\|_{L^r(\mathbb{R}^+)} \lesssim \|v_0\|_{\mathcal{B}^s}\, E(0)\,.$$

Bernstein's inequality as stated in Lemma A.1 implies that

$$\|\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}} v(t,\cdot)\|_{L^\infty_{\mathrm{v}}(L^2_{\mathrm{h}})} \lesssim 2^{\frac{j}{2}} \|\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}} v(t,\cdot)\|_{L^2}\,,$$

thus we infer that

$$\Big\| \big\| \big(2^{k(s+\frac{2}{r})}\|\Delta_k^{\mathrm{h}} v\|_{L^\infty_{\mathrm{v}}(L^2_{\mathrm{h}})}\big)_k \big\|_{\ell^2(\mathbb{Z})} \Big\|_{L^r(\mathbb{R}^+)} \lesssim \|v_0\|_{\mathcal{B}^s}\, E(0)\,.$$

Permuting the ℓ^2 norm and the L^∞_{v} norm thanks to Minkowski's inequality again, concludes the proof of Lemma 3.3. $\square$

Remark 3.4. *Let us remark that thanks to the Sobolev embedding of $H^{\frac12}(\mathbb{R}^2)$ into $L^4(\mathbb{R}^2)$, we have, choosing $s=0$ and $r=4$ or $r=2$,*

$$\|v\|_{L^4(\mathbb{R}^+;L^\infty_{\mathrm{v}}(L^4_{\mathrm{h}}))} + \|v\|_{L^2(\mathbb{R}^+;L^\infty_{\mathrm{v}}(H^1_{\mathrm{h}}))} \lesssim \|v_0\|_{\mathcal{B}^0}\, E(0)\,.$$

Now our purpose is the proof of the following inequality: for any v solving $(\mathrm{NS2D})_{x_3}$, for any r in $[4,\infty]$ and any s in $\left]-\frac12,\frac12\right[$ and any positive s',

$$\|v\|_{\widetilde{L}^r(\mathbb{R}^+;\mathcal{B}^{s+\frac{2}{r},s'})} \lesssim \|v_0\|_{\mathcal{B}^{s,s'}} \exp\Big(\int_0^\infty C\big(\|v(t)\|^4_{L^\infty_{\mathrm{v}}(L^4_{\mathrm{h}})} + \|v(t)\|^2_{L^\infty_{\mathrm{v}}(H^1_{\mathrm{h}})}\big)dt\Big)\,. \tag{3.15}$$

The case when r is in $[2,4]$ will be dealt with later. We are going to use a Gronwall-type argument. Let us introduce, for any nonnegative λ, the following notation: for any function F we define

$$F_\lambda(t) \stackrel{\mathrm{def}}{=} F(t)\exp\Big(-\lambda\int_0^t \phi(t')dt'\Big) \quad \text{with} \quad \phi(t) \stackrel{\mathrm{def}}{=} \|v(t)\|^4_{L^\infty_{\mathrm{v}}(L^4_{\mathrm{h}})} + \|v(t)\|^2_{L^\infty_{\mathrm{v}}(H^1_{\mathrm{h}})}\,.$$

Notice that thanks to Remark 3.4, we know that

$$\int_0^t \phi(t')\,dt' \lesssim E(0)(\|v_0\|^2_{\mathcal{B}^0} + \|v_0\|^4_{\mathcal{B}^0})\,. \tag{3.16}$$

Then we write, using the Duhamel formula and the action of the heat flow described in Lemma A.2, that

$$\begin{aligned}\|\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}} v_\lambda(t)\|_{L^2} &\leq Ce^{-c2^{2k}t}\|\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}} v_0\|_{L^2}\\ &\quad + C2^k\int_0^t \exp\Big(-c(t-t')2^{2k} - \lambda\int_{t'}^t \phi(t'')dt''\Big)\|\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}}(v\otimes v)_\lambda(t')\|_{L^2}dt'\,.\end{aligned} \tag{3.17}$$

Notice that $(v\otimes v)_\lambda = v\otimes v_\lambda$. In order to study the term $\|\Delta_j^{\mathrm v}\Delta_k^{\mathrm h}(v\otimes v)_\lambda(t')\|_{L^2}$, we need an anisotropic version of Bony's paraproduct decomposition. Let us write that

$$
\begin{aligned}
ab &= \sum_{\ell=1}^{4} T^\ell(a,b) \quad \text{with}\\
T^1(a,b) &= \sum_{j,k} S_j^{\mathrm v} S_k^{\mathrm h} a \Delta_j^{\mathrm v}\Delta_k^{\mathrm h} b\,,\\
T^2(a,b) &= \sum_{j,k} S_j^{\mathrm v} \Delta_k^{\mathrm h} a \Delta_j^{\mathrm v} S_{k+1}^{\mathrm h} b\,,\\
T^3(a,b) &= \sum_{j,k} \Delta_j^{\mathrm v} S_k^{\mathrm h} a S_{j+1}^{\mathrm v}\Delta_k^{\mathrm h} b\,,\\
T^4(a,b) &= \sum_{j,k} \Delta_j^{\mathrm v} \Delta_k^{\mathrm h} a S_{j+1}^{\mathrm v} S_{k+1}^{\mathrm h} b\,.
\end{aligned}
\tag{3.18}
$$

We shall only estimate T^1 and T^2, the other two terms being strictly analogous. By definition of T^1, using the definition of horizontal and vertical truncations together with the fact that the support of the Fourier transform of the product of two functions is included in the sum of the two supports, and Bernstein's and Hölder's inequalities, there is some fixed nonzero integer N_0 such that

$$
\begin{aligned}
\|\Delta_j^{\mathrm v}\Delta_k^{\mathrm h} T^1(v(t),v_\lambda(t))\|_{L^2} &\lesssim 2^{\frac k2}\|\Delta_j^{\mathrm v}\Delta_k^{\mathrm h} T^1(v(t),v_\lambda(t))\|_{L^2_{\mathrm v}(L^{\frac43}_{\mathrm h})}\\
&\lesssim 2^{\frac k2}\sum_{\substack{j'\geq j-N_0\\ k'\geq k-N_0}} \|S_{j'}^{\mathrm v}S_{k'}^{\mathrm h} v(t)\|_{L^\infty_{\mathrm v}(L^4_{\mathrm h})}\|\Delta_{j'}^{\mathrm v}\Delta_{k'}^{\mathrm h} v_\lambda(t)\|_{L^2}\\
&\lesssim 2^{\frac k2}\|v(t)\|_{L^\infty_{\mathrm v}(L^4_{\mathrm h})}\sum_{\substack{j'\geq j-N_0\\ k'\geq k-N_0}} \|\Delta_{j'}^{\mathrm v}\Delta_{k'}^{\mathrm h} v_\lambda(t)\|_{L^2}\,.
\end{aligned}
$$

By definition of $\widetilde L^4(\mathbb R^+;\mathcal B^{s+\frac12,s'})$ we get

$$
\|\Delta_j^{\mathrm v}\Delta_k^{\mathrm h} T^1(v(t),v_\lambda(t))\|_{L^2}\lesssim 2^{\frac k2}\|v_\lambda\|_{\widetilde L^4(\mathbb R^+;\mathcal B^{s+\frac12,s'})}\|v(t)\|_{L^\infty_{\mathrm v}(L^4_{\mathrm h})}\sum_{\substack{j'\geq j-N_0\\ k'\geq k-N_0}} 2^{-k'(s+\frac12)}2^{-j's'}\widetilde f_{j',k'}(t)
$$

where $\widetilde f_{j',k'}(t)$, defined by

$$
\widetilde f_{j',k'}(t)\stackrel{\rm def}{=}\|v_\lambda\|^{-1}_{\widetilde L^4(\mathbb R^+;\mathcal B^{s+\frac12,s'})}2^{k'(s+\frac12)}2^{j's'}\|\Delta_{j'}^{\mathrm v}\Delta_{k'}^{\mathrm h} v_\lambda(t)\|_{L^2}\,,
$$

is on the sphere of $\ell^1(\mathbb Z^2;L^4(\mathbb R^+))$. This implies that

$$
\begin{aligned}
&2^{js'}2^{ks}\|\Delta_j^{\mathrm v}\Delta_k^{\mathrm h} T^1(v(t),v_\lambda(t))\|_{L^2}\\
&\qquad\lesssim\|v_\lambda\|_{\widetilde L^4(\mathbb R^+;\mathcal B^{s+\frac12,s'})}\|v(t)\|_{L^\infty_{\mathrm v}(L^4_{\mathrm h})}\sum_{\substack{j'\geq j-N_0\\ k'\geq k-N_0}} 2^{-(j'-j)s'}2^{-(k'-k)(s+\frac12)}\widetilde f_{j',k'}(t)\,.
\end{aligned}
$$

Since $s>-\dfrac12$ and $s'>0$, it follows by Young's inequality on series, that

$$
2^{js'}2^{ks}\|\Delta_j^{\mathrm v}\Delta_k^{\mathrm h} T^1(v(t),v_\lambda(t))\|_{L^2}\lesssim\|v_\lambda\|_{\widetilde L^4(\mathbb R^+;\mathcal B^{s+\frac12,s'})}\|v(t)\|_{L^\infty_{\mathrm v}(L^4_{\mathrm h})}f_{j,k}(t)
$$

where $f_{j,k}(t)$ is on the sphere of $\ell^1(\mathbb{Z}^2;L^4(\mathbb{R}^+))$. As $\phi(t)$ is greater than $\|v(t)\|^4_{L^\infty_{\rm v}(L^4_{\rm h})}$, we infer that

$$
\begin{aligned}
\mathcal{T}^1_{j,k,\lambda}(t) &\stackrel{\rm def}{=} 2^k 2^{js'} 2^{ks} \int_0^t \exp\Bigl(-c(t-t')2^{2k} - \lambda \int_{t'}^t \phi(t'')dt''\Bigr) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times \|\Delta^{\rm v}_j \Delta^{\rm h}_k T^1(v(t'),v_\lambda(t'))\|_{L^2} dt' \\
&\lesssim \|v_\lambda\|_{\widetilde L^4(\mathbb{R}^+;\mathcal{B}^{s+\frac12,s'})} \\
&\qquad \times 2^k \int_0^t \exp\Bigl(-c(t-t')2^{2k} - \lambda \int_{t'}^t \phi(t'')dt''\Bigr) \phi^{\frac14}(t') f_{j,k}(t') dt'\,.
\end{aligned} \tag{3.19}
$$

Using Hölder's inequality, we deduce that

$$
\begin{aligned}
\mathcal{T}^1_{j,k,\lambda}(t) \lesssim \|v_\lambda\|_{\widetilde L^4(\mathbb{R}^+;\mathcal{B}^{s+\frac12,s'})} &\biggl(\int_0^t e^{-c(t-t')2^{2k}} f^4_{j,k}(t')dt'\biggr)^{\frac14} \\
&\times 2^k \biggl(\int_0^t \exp\Bigl(-c(t-t')2^{2k} - \lambda \int_{t'}^t \phi(t'')dt''\Bigr) \phi(t')^{\frac13} dt'\biggr)^{\frac34}.
\end{aligned}
$$

Then Hölder's inequality in the last term of the above inequality ensures that

$$
\mathcal{T}^1_{j,k,\lambda}(t) \lesssim \frac{1}{\lambda^{\frac14}} \biggl(\int_0^t e^{-c(t-t')2^{2k}} f^4_{j,k}(t')dt'\biggr)^{\frac14} \|v_\lambda\|_{\widetilde L^4(\mathbb{R}^+;\mathcal{B}^{s+\frac12,s'})}\,. \tag{3.20}
$$

Now let us study the term with T^2. Using again that the support of the Fourier transform of the product of two functions is included in the sum of the two supports, let us write that

$$
\|\Delta^{\rm v}_j \Delta^{\rm h}_k T^2(v(t),v_\lambda(t))\|_{L^2} \lesssim \sum_{\substack{j'\geq j-N_0\\ k'\geq k-N_0}} \|S^{\rm v}_{j'} \Delta^{\rm h}_{k'} v(t)\|_{L^\infty_{\rm v}(L^2_{\rm h})} \|\Delta^{\rm v}_{j'} S^{\rm h}_{k'+1} v_\lambda(t)\|_{L^2_{\rm v}(L^\infty_{\rm h})}\,.
$$

Combining Lemma A.1 with the definition of the function ϕ, we get

$$
\|S^{\rm v}_{j'} \Delta^{\rm h}_{k'} v(t)\|_{L^\infty_{\rm v}(L^2_{\rm h})} \lesssim 2^{-k'} \|v(t)\|_{L^\infty_{\rm v}(H^1_{\rm h})} \lesssim 2^{-k'} \phi^{\frac12}(t)\,. \tag{3.21}
$$

Now let us observe that using again the Bernstein inequality, we have

$$
\begin{aligned}
\|\Delta^{\rm v}_{j'} S^{\rm h}_{k'+1} v_\lambda(t)\|_{L^2_{\rm v}(L^\infty_{\rm h})} &\lesssim \sum_{k''\leq k'} \|\Delta^{\rm v}_{j'} \Delta^{\rm h}_{k''} v_\lambda(t)\|_{L^2_{\rm v}(L^\infty_{\rm h})} \\
&\lesssim \sum_{k''\leq k'} 2^{k''} \|\Delta^{\rm v}_{j'} \Delta^{\rm h}_{k''} v_\lambda(t)\|_{L^2}\,.
\end{aligned}
$$

By definition of the $\widetilde L^4(\mathbb{R}^+;\mathcal{B}^{s+\frac12,s'})$ norm, we have

$$
2^{j's'} 2^{k'(s-\frac12)} \|\Delta^{\rm v}_{j'} S^{\rm h}_{k'+1} v_\lambda(t)\|_{L^2_{\rm v}(L^\infty_{\rm h})} \lesssim \|v_\lambda\|_{\widetilde L^4(\mathbb{R}^+;\mathcal{B}^{s+\frac12,s'})} \sum_{k''\leq k'} 2^{(k'-k'')(s-\frac12)} \underline{f}_{j',k''}(t)
$$

where $\underline{f}_{j',k''}(t)$, on the sphere of $\ell^1(\mathbb{Z}^2;L^4(\mathbb{R}^+))$, is defined by

$$
\underline{f}_{j',k''}(t) \stackrel{\rm def}{=} \|v_\lambda\|^{-1}_{\widetilde L^4(\mathbb{R}^+;\mathcal{B}^{s+\frac12,s'})} 2^{j's'} 2^{k''(s+\frac12)} \|\Delta^{\rm v}_{j'} \Delta^{\rm h}_{k''} v_\lambda(t)\|_{L^2}\,.
$$

Since $s<\frac12$, this ensures by Young's inequality that

$$
\|\Delta^{\rm v}_{j'} S^{\rm h}_{k'+1} v_\lambda(t)\|_{L^2_{\rm v}(L^\infty_{\rm h})} \lesssim 2^{-j's'} 2^{-k'(s-\frac12)} \|v_\lambda\|_{\widetilde L^4(\mathbb{R}^+;\mathcal{B}^{s+\frac12,s'})} \widetilde f_{j',k'}(t)
$$

where $\widetilde f_{j',k'}(t)$ is on the sphere of $\ell^1(\mathbb{Z}^2;L^4(\mathbb{R}^+))$. Together with Inequality (3.21), this gives

$$
2^{js'} 2^{k(s+\frac12)} \|\Delta^{\rm v}_j \Delta^{\rm h}_k T^2(v(t),v_\lambda(t))\|_{L^2} \lesssim \phi(t)^{\frac12} \|v_\lambda\|_{\widetilde L^4(\mathbb{R}^+;\mathcal{B}^{s+\frac12,s'})} f_{j,k}(t)\,,
$$

where $f_{j,k}(t)$ is on the sphere of $\ell^1(\mathbb{Z}^2;L^4(\mathbb{R}^+))$. We deduce that

$$
\begin{aligned}
\mathcal{T}^2_{j,k,\lambda}(t) &\stackrel{\text{def}}{=} 2^k 2^{js'}2^{ks}\int_0^t \exp\Big(-c(t-t')2^{2k}-\lambda\int_{t'}^t\phi(t'')dt''\Big)\\
&\qquad\qquad\qquad\qquad \times\|\Delta^{\rm v}_j\Delta^{\rm h}_k T^2(v(t'),v_\lambda(t'))\|_{L^2}\,dt'\\
&\lesssim \|v_\lambda\|_{\widetilde L^4(\mathbb{R}^+;\mathcal{B}^{s+\frac12,s'})}\\
&\qquad \times 2^{\frac k2}\int_0^t \exp\Big(-c(t-t')2^{2k}-\lambda\int_{t'}^t\phi(t'')dt''\Big)\phi(t')^{\frac12}f_{j,k}(t')dt'\,.
\end{aligned}\tag{3.22}
$$

Using Hölder's inequality twice, we get

$$
\begin{aligned}
\mathcal{T}^2_{j,k,\lambda}(t) &\lesssim \|v_\lambda\|_{\widetilde L^4(\mathbb{R}^+;\mathcal{B}^{s+\frac12,s'})}\Big(\int_0^t e^{-c(t-t')2^{2k}}f^4_{j,k}(t')dt'\Big)^{\frac14}\\
&\qquad\times 2^{\frac k2}\Big(\int_0^t \exp\Big(-c(t-t')2^{2k}-\lambda\int_{t'}^t\phi(t'')dt''\Big)\phi(t')^{\frac23}dt'\Big)^{\frac34}\\
&\lesssim \frac{1}{\lambda^{\frac12}}\|v_\lambda\|_{\widetilde L^4(\mathbb{R}^+;\mathcal{B}^{s+\frac12,s'})}\Big(\int_0^t e^{-c(t-t')2^{2k}}f^4_{j,k}(t')dt'\Big)^{\frac14}.
\end{aligned}\tag{3.23}
$$

As T^3 is estimated like T^1 and T^4 is estimated like T^2, this implies finally that

$$
\begin{aligned}
2^{js'}2^{ks}\|\Delta^{\rm v}_j\Delta^{\rm h}_k v_\lambda(t)\|_{L^2} &\lesssim 2^{js'}2^{ks}e^{-c2^{2k}t}\|\Delta^{\rm v}_j\Delta^{\rm h}_k v_0\|_{L^2}\\
&\quad+\Big(\int_0^t e^{-c(t-t')2^{2k}}f^4_{j,k}(t')dt'\Big)^{\frac14}\Big(\frac{1}{\lambda^{\frac14}}+\frac{1}{\lambda^{\frac12}}\Big)\|v_\lambda\|_{\widetilde L^4(\mathbb{R}^+;\mathcal{B}^{s+\frac12,s'})}\,.
\end{aligned}
$$

As we have

$$
\Big(\int_0^\infty\Big(\int_0^t e^{-c(t-t')2^{2k}}f^4_{j,k}(t')dt'\Big)^{\frac14\times 4}dt\Big)^{\frac14}=c^{-1}d_{j,k}2^{-\frac k2}
$$

$$
\text{and}\ \sup_{t\in\mathbb{R}^+}\Big(\int_0^t e^{-c(t-t')2^{2k}}f^4_{j,k}(t')dt'\Big)^{\frac14}=d_{j,k}\,,\quad \text{with}\, d_{j,k}\in\ell^1(\mathbb{Z}^2)\,,
$$

we infer that

$$
\begin{aligned}
&2^{js'}2^{ks}\big(\|\Delta^{\rm v}_j\Delta^{\rm h}_k v_\lambda\|_{L^\infty(\mathbb{R}^+;L^2)}+2^{\frac k2}\|\Delta^{\rm v}_j\Delta^{\rm h}_k v_\lambda\|_{L^4(\mathbb{R}^+;L^2)}\big)\\
&\qquad\qquad\lesssim 2^{js'}2^{ks}\|\Delta^{\rm v}_j\Delta^{\rm h}_k v_0\|_{L^2}+d_{j,k}\Big(\frac{1}{\lambda^{\frac14}}+\frac{1}{\lambda^{\frac12}}\Big)\|v_\lambda\|_{\widetilde L^4(\mathbb{R}^+;\mathcal{B}^{s+\frac12,s'})}\,.
\end{aligned}
$$

Taking the sum over j and k and choosing λ large enough, we have proved (3.15).

Let us gain L^2-integrability in t. Using (3.19) and (3.22) with $\lambda=0$, we find that

$$
\begin{aligned}
2^{js'}2^{k(s+1)}\|\Delta^{\rm v}_j\Delta^{\rm h}_k v(t)\|_{L^2} &\lesssim 2^{js'}2^{k(s+1)}e^{-c2^{2k}t}\|\Delta^{\rm v}_j\Delta^{\rm h}_k v_0\|_{L^2}\\
&\quad+2^{2k}\|v\|_{\widetilde L^4(\mathbb{R}^+;\mathcal{B}^{s+\frac12,s'})}\int_0^t e^{-c(t-t')2^{2k}}\big((g_{j,k}(t')+2^{-\frac k2}h_{j,k}(t')\big)dt'\,,
\end{aligned}
$$

where $g_{j,k}$ (resp. $h_{j,k}$) are in $\ell^1(\mathbb{Z}^2;L^2(\mathbb{R}^+))$ (resp. $\ell^1(\mathbb{Z}^2;L^{\frac43}(\mathbb{R}^+))$), with

$$
\sum_{(j,k)\in\mathbb{Z}^2}\|g_{j,k}\|_{L^2(\mathbb{R}^+)}\lesssim\|\phi\|^{\frac14}_{L^1}\quad\text{and}\quad\sum_{(j,k)\in\mathbb{Z}^2}\|h_{j,k}\|_{L^{\frac43}(\mathbb{R}^+)}\lesssim\|\phi\|^{\frac12}_{L^1}\,.
$$

Laws of convolution in the time variable, summation over j and k and (3.15) imply that

$$
\|v\|_{\widetilde L^2(\mathbb{R}^+;\mathcal{B}^{s+1,s'})}\lesssim\|v_0\|_{\mathcal{B}^{s,s'}}\exp\Big(C\int_0^\infty\phi(t)dt\Big)\,.
$$

This implies by interpolation in view of (3.15) that for all r in $[2,\infty]$, all s in $]-\frac{1}{2},\frac{1}{2}[$ and all positive s'

$$\|v\|_{\widetilde{L}^r(\mathbb{R}^+;\mathcal{B}^{s+\frac{2}{r},s'})} \lesssim \|v_0\|_{\mathcal{B}^{s,s'}} \exp\Big(C\int_0^\infty \phi(t)dt\Big)\,, \tag{3.24}$$

which in view of (3.16) ensures Inequality (3.3) and achieves the proof of Estimate (3.4) in the case when s belongs to $]-\frac{1}{2},\frac{1}{2}[$.

Now we are going to double the interval, namely prove that for any s in $]-1,1[$, any $s'\geq 1/2$ and any r in $[2,\infty]$ we have

$$\|v\|_{\widetilde{L}^r(\mathbb{R}^+;\mathcal{B}^{s+\frac{2}{r},s'})} \lesssim \|v_0\|_{\mathcal{B}^{s,s'}} + \|v_0\|_{\mathcal{B}^{\frac{s}{2},s'}}\|v_0\|_{\mathcal{B}^{\frac{s}{2}}} \exp(C\|v_0\|_{\mathcal{B}^0}E_0)\,. \tag{3.25}$$

Proposition A.4 implies that for any s in $]-1,1[$ and any $s'\geq 1/2$, we have

$$\|v(t)\otimes v(t)\|_{\mathcal{B}^{s,s'}} \lesssim \|v(t)\|_{\mathcal{B}^{\frac{s+1}{2}}}\|v(t)\|_{\mathcal{B}^{\frac{s+1}{2},s'}}\,.$$

The smoothing effect of the horizontal heat flow described in Lemma A.2 implies therefore that, for any s belonging to $]-1,1[$, any $s'\geq 1/2$ and any r in $[2,\infty]$,

$$\begin{aligned}\|v\|_{\widetilde{L}^r(\mathbb{R}^+;\mathcal{B}^{s+\frac{2}{r},s'})} &\lesssim \|v_0\|_{\mathcal{B}^{s,s'}} + \|v\otimes v\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{s,s'})}\\ &\lesssim \|v_0\|_{\mathcal{B}^{s,s'}} + \|v\|_{\widetilde{L}^4(\mathbb{R}^+;\mathcal{B}^{\frac{s+1}{2}})}\|v\|_{\widetilde{L}^4(\mathbb{R}^+;\mathcal{B}^{\frac{s+1}{2},s'})}\,.\end{aligned}$$

Finally Inequality (3.15) ensures that for any s in $]-1,1[$, any $s'\geq 1/2$ and any r in $[2,\infty]$,

$$\|v\|_{\widetilde{L}^r(\mathbb{R}^+;\mathcal{B}^{s+\frac{2}{r},s'})} \lesssim \|v_0\|_{\mathcal{B}^{s,s'}} + \|v_0\|_{\mathcal{B}^{\frac{s}{2}}}\|v_0\|_{\mathcal{B}^{\frac{s}{2},s'}} \exp(C\|v_0\|_{\mathcal{B}^0}E(0))\,. \tag{3.26}$$

This concludes the proof of Inequality (3.25).

Now let us conclude the proof of Estimate (3.4). Again Proposition A.4 implies that, for any s in $]-2,0]$ and any $s'\geq 1/2$, we have

$$\|v(t)\otimes v(t)\|_{\mathcal{B}^{s+1,s'}} \lesssim \|v(t)\|_{\mathcal{B}^{\frac{s}{2}+1}}\|v(t)\|_{\mathcal{B}^{\frac{s}{2}+1,s'}}\,.$$

This gives rise to

$$\|v\otimes v\|_{L^1(\mathbb{R}^+;\mathcal{B}^{s+1,s'})} \lesssim \|v\|_{L^2(\mathbb{R}^+;\mathcal{B}^{\frac{s}{2}+1})}\|v\|_{L^2(\mathbb{R}^+;\mathcal{B}^{\frac{s}{2}+1,s'})}\,.$$

The smoothing effect of the heat flow gives, for any r in $[1,\infty]$ and any s in $]-2,0]$,

$$\|v\|_{\widetilde{L}^r(\mathbb{R}^+;\mathcal{B}^{s+\frac{2}{r},s'})} \lesssim \|v_0\|_{\mathcal{B}^{s,s'}} + \|v\|_{L^2(\mathbb{R}^+;\mathcal{B}^{\frac{s}{2}+1})}\|v\|_{L^2(\mathbb{R}^+;\mathcal{B}^{\frac{s}{2}+1,s'})}\,.$$

Inequality (3.26) implies that, for any r in $[1,\infty]$ and any s in $]-2,0]$ and $s'\geq 1/2$,

$$\|v\|_{\widetilde{L}^r(\mathbb{R}^+;\mathcal{B}^{s+\frac{2}{r},s'})} \lesssim \|v_0\|_{\mathcal{B}^{s,s'}} + \|v_0\|^3_{\mathcal{B}^{\frac{s}{4}}}\|v_0\|_{\mathcal{B}^{\frac{s}{4},s'}} \exp(C\|v_0\|_{\mathcal{B}^0}E_0)\,. \tag{3.27}$$

This proves the estimate (3.4) and thus Proposition 3.1. □

3.2. **Propagation of regularity by a 2D flow with parameter.** Now let us estimate the norm of the function w^3 defined as the solution of (T_β) defined page 27. This is described in the following proposition.

Proposition 3.5. *Let v_0 and v be as in Proposition 3.1. For any non negative real number β, let us consider w^3 the solution of*

$$(T_\beta)\quad \partial_t w^3 + v\cdot\nabla^{\mathrm{h}} w^3 - \Delta_{\mathrm{h}} w^3 - \beta^2\partial_3^2 w^3 = 0 \quad\text{and}\quad w^3_{|t=0} = w^3_0\,.$$

Then w^3 satisfies the following estimates where all the constants are independent of β:

$$\|w^3\|_{\mathcal{A}^0} \lesssim \|w_0^3\|_{\mathcal{B}^0}\exp\big(\mathcal{T}_1(\|v_0\|_{\mathcal{B}^0})\big)\,, \tag{3.28}$$

and for any s in $[-2+\mu,0]$ and any $s'\geq 1/2$, we have

$$\|w^3\|_{\mathcal{A}^{s,s'}} \lesssim \big(\|w_0^3\|_{\mathcal{B}^{s,s'}} + \|w_0^3\|_{\mathcal{B}^0}\mathcal{T}_\infty(\|v_0\|_{S_\mu})\big)\exp\big(\mathcal{T}_1(\|v_0\|_{\mathcal{B}^0})\big)\,. \tag{3.29}$$

Proof. This is a question of propagating anisotropic regularity by a transport-diffusion equation. This propagation is described by the following lemma, which will easily lead to Proposition 3.5.

Lemma 3.6. *Let us consider (s,s') a couple of real numbers, and $\mathcal{Q}$ a bilinear operator which maps continuously $\mathcal{B}^1\times\mathcal{B}^{s+1,s'}$ into $\mathcal{B}^{s,s'}$. A constant C exists such that for any two-component vector field v in $L^2(\mathbb{R}^+;\mathcal{B}^1)$, any f in $L^1(\mathbb{R}^+;\mathcal{B}^{s,s'})$, any a_0 in $\mathcal{B}^{s,s'}$ and for any non negative β, if $\Delta_\beta\stackrel{\rm def}{=}\Delta_{\rm h}+\beta^2\partial_z^2$ and a is the solution of*

$$\partial_t a-\Delta_\beta a+\mathcal{Q}(v,a)=f\quad\text{and}\quad a_{|t=0}=a_0\,,$$

then a satisfies

$$\forall r\in[1,\infty]\,,\ \|a\|_{\widetilde L^r(\mathbb{R}^+;\mathcal{B}^{s+\frac{2}{r},s'})}\leq C\big(\|a_0\|_{\mathcal{B}^{s,s'}}+\|f\|_{L^1(\mathbb{R}^+;\mathcal{B}^{s,s'})}\big)\exp\Big(C\int_0^\infty\|v(t)\|_{\mathcal{B}^1}^2dt\Big)\,.$$

Proof. This is a Gronwall type estimate. However the fact that the third index of the Besov spaces is one, induces some technical difficulties which lead us to work first on subintervals I of $\mathbb{R}^+$ on which $\|v\|_{L^2(I;\mathcal{B}^1)}$ is small.

Let us first consider any subinterval $I=[\tau_0,\tau_1]$ of $\mathbb{R}^+$. The Duhamel formula and the smoothing effect of the heat flow described in Lemma A.2 imply that

$$\begin{aligned}\|\Delta_k^{\rm h}\Delta_j^{\rm v}a(t)\|_{L^2}&\leq e^{-c2^{2k}(t-\tau_0)}\|\Delta_k^{\rm h}\Delta_j^{\rm v}a(\tau_0)\|_{L^2}\\&\qquad+C\int_{\tau_0}^t e^{-c2^{2k}(t-t')}\big\|\Delta_k^{\rm h}\Delta_j^{\rm v}\big(\mathcal{Q}(v(t'),a(t'))+f(t')\big)\big\|_{L^2}dt'\,.\end{aligned}$$

After multiplication by $2^{ks+js'}$ and using Young's inequality in the time integral, we deduce that

$$\begin{aligned}2^{ks+js'}\big(\|\Delta_k^{\rm h}\Delta_j^{\rm v}a\|_{L^\infty(I;L^2)}+2^{2k}\|\Delta_k^{\rm h}\Delta_j^{\rm v}a\|_{L^1(I;L^2)}\big)&\leq C2^{ks+js'}\|\Delta_k^{\rm h}\Delta_j^{\rm v}a(\tau_0)\|_{L^2}\\&\qquad+C\int_{\tau_0}^t d_{k,j}(t')\big(\|v(t')\|_{\mathcal{B}^1}\|a(t')\|_{\mathcal{B}^{s+1,s'}}+\|f(t')\|_{\mathcal{B}^{s,s'}}\big)dt'\end{aligned}$$

where for any t, $d_{k,j}(t)$ is an element of the sphere of $\ell^1(\mathbb{Z}^2)$. By summation over (k,j) and using the Cauchy-Schwarz inequality, we infer that

$$\begin{aligned}\|a\|_{\widetilde L^\infty(I;\mathcal{B}^{s,s'})}+\|a\|_{L^1(I;\mathcal{B}^{s+2,s'})}&\leq C\|a(\tau_0)\|_{\mathcal{B}^{s,s'}}+C\|f\|_{L^1(I;\mathcal{B}^{s,s'})}\\&\qquad+C\|v\|_{L^2(I;\mathcal{B}^1)}\|a\|_{L^2(I;\mathcal{B}^{s+1,s'})}\,.\end{aligned}\tag{3.30}$$

Let us define the increasing sequence $(T_m)_{0\leq m\leq M+1}$ by induction such that $T_0=0, T_{M+1}=\infty$ and

$$\forall m<M\,,\ \int_{T_m}^{T_{m+1}}\|v(t)\|_{\mathcal{B}^1}^2dt=c_0\quad\text{and}\quad\int_{T_M}^\infty\|v(t)\|_{\mathcal{B}^1}^2dt\leq c_0\,,$$

for some given c_0 which will be chosen later on. Obviously, we have

$$\int_0^\infty\|v(t)\|_{\mathcal{B}^1}^2dt\geq\int_0^{T_M}\|v(t)\|_{\mathcal{B}^1}^2dt=Mc_0\,.\tag{3.31}$$

Thus the number M of $T_m's$ such that T_m is finite is less than $c_0^{-1}\|v\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}^2$. Applying Estimate (3.30) to the interval $[T_m,T_{m+1}]$, we get

$$\begin{aligned}\|a\|_{L^\infty([T_m,T_{m+1}];\mathcal{B}^{s,s'})}+\|a\|_{L^1([T_m,T_{m+1}];\mathcal{B}^{s+2,s'})}&\leq\|a\|_{L^2([T_m,T_{m+1}];\mathcal{B}^{s+1,s'})}\\&\qquad+C\big(\|a(T_m)\|_{\mathcal{B}^{s,s'}}+C\|f\|_{L^1([T_m,T_{m+1}];\mathcal{B}^{s,s'})}\big)\end{aligned}$$

if c_0 is chosen such that $C\sqrt{c_0} \leq 1$. As

$$\|a\|_{L^2([T_m,T_{m+1}];\mathcal{B}^{s+1,s'})} \leq \|a\|^{\frac{1}{2}}_{L^\infty([T_m,T_{m+1}];\mathcal{B}^{s,s'})}\|a\|^{\frac{1}{2}}_{L^1([T_m,T_{m+1}];\mathcal{B}^{s+2,s'})},$$

we infer that

$$\begin{aligned}(3.32)\qquad &\|a\|_{L^\infty([T_m,T_{m+1}];\mathcal{B}^{s,s'})} + \|a\|_{L^1([T_m,T_{m+1}];\mathcal{B}^{s+2,s'})}\\ &\qquad\qquad \leq 2C\big(\|a(T_m)\|_{\mathcal{B}^{s,s'}} + \|f\|_{L^1([T_m,T_{m+1}];\mathcal{B}^{s,s'})}\big)\,.\end{aligned}$$

Now let us us prove by induction that

$$\|a\|_{L^\infty([0,T_m];\mathcal{B}^{s,s'})} \leq (2C)^m\big(\|a_0\|_{\mathcal{B}^{s,s'}} + \|f\|_{L^1([0,T_m],\mathcal{B}^{s,s'})}\big).$$

Using (3.32) and the induction hypothesis we get

$$\begin{aligned}\|a\|_{L^\infty([T_m,T_{m+1}];\mathcal{B}^{s,s'})} &\leq 2C\big(\|a\|_{L^\infty([0,T_m];\mathcal{B}^{s,s'})} + \|f\|_{L^1([T_m,T_{m+1}];\mathcal{B}^{s,s'})}\big)\\ &\leq (2C)^{m+1}\big(\|a_0\|_{\mathcal{B}^{s,s'}} + \|f\|_{L^1([0,T_{m+1}],\mathcal{B}^{s,s'})}\big)\,,\end{aligned}$$

provided that $2C \geq 1$. This proves in view of (3.31) that

$$\|a\|_{L^\infty(\mathbb{R}^+;\mathcal{B}^{s,s'})} \leq C\big(\|a_0\|_{\mathcal{B}^{s,s'}} + \|f\|_{L^1(\mathbb{R}^+;\mathcal{B}^{s,s'})}\big)\exp\Big(C\int_0^\infty \|v(t)\|^2_{\mathcal{B}^1}dt\Big)\,.$$

We deduce from (3.32) that

$$\begin{aligned}\|a\|_{L^1([T_m,T_{m+1}];\mathcal{B}^{s+2,s'})} \leq C\big(\|a_0\|_{\mathcal{B}^{s,s'}} + \|f\|_{L^1(\mathbb{R}^+;\mathcal{B}^{s,s'})}\big)&\exp\Big(C\int_0^\infty \|v(t)\|^2_{\mathcal{B}^1}dt\Big)\\ &+ C\|f\|_{L^1([T_m,T_{m+1}];\mathcal{B}^{s,s'})}\,.\end{aligned}$$

Once noticed that $xe^{Cx^2} \leq e^{C'x^2}$, the result comes by summation over m and the fact that the total number of m's is less than or equal to $c_0^{-1}\|v\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1)}$. Lemma 3.6 is proved. □

We apply Lemma 3.6 with $\mathcal{Q}(v,a) = \operatorname{div}_{\mathrm{h}}(av)$, $f=0$, $a = w^3$, and $(s,s') = (0,1/2)$. Indeed since $\mathcal{B}^1$ is an algebra we have

$$\|\mathcal{Q}(v,a)\|_{\mathcal{B}^0} \lesssim \|av\|_{\mathcal{B}^1} \lesssim \|a\|_{\mathcal{B}^1}\|v\|_{\mathcal{B}^1}.$$

So Lemma 3.6 gives

$$\|w^3\|_{\mathcal{A}^0} \lesssim \|w^3_0\|_{\mathcal{B}^0}\exp\Big(C\int_0^\infty \|v(t)\|^2_{\mathcal{B}^1}dt\Big)\,.$$

Thanks to Estimate (3.3) of Proposition 3.1 we deduce (3.28).

Now for s belonging to $[-2+\mu,0]$, we apply Lemma 3.6 with $a = w^3$, $\mathcal{Q}(v,a) = \operatorname{div}_{\mathrm{h}}(T^{\mathrm{v}}_v a)$, and $f = \operatorname{div}_{\mathrm{h}}(\widetilde{T}^{\mathrm{v}}_a v)$, where with the notations of Definition 1.6

$$(3.33)\quad T^{\mathrm{v}}_v a \stackrel{\mathrm{def}}{=} \sum_j S^{\mathrm{v}}_{j-1}v\Delta^{\mathrm{v}}_j a\,,\ R^{\mathrm{v}}(a,v) \stackrel{\mathrm{def}}{=} \sum_{\substack{j\\ -1\leq\ell\leq 1}} \Delta^{\mathrm{v}}_{j-\ell}a\Delta^{\mathrm{v}}_j v \quad\text{and}\quad \widetilde{T}^{\mathrm{v}}_a v \stackrel{\mathrm{def}}{=} T^{\mathrm{v}}_a v + R^{\mathrm{v}}(a,v)\,.$$

Lemma A.5 implies that for any s in $[-2+\mu,0]$ and any $s' \geq 1/2$,

$$\|T^{\mathrm{v}}_v w^3\|_{\mathcal{B}^{s+1,s'}} \lesssim \|v\|_{\mathcal{B}^1}\|w^3\|_{\mathcal{B}^{s+1,s'}}\,.$$

We infer from Lemma 3.6 that, for any r in $[1,\infty]$,

$$(3.34)\qquad \|w^3\|_{\widetilde{L}^r(\mathbb{R}^+;\mathcal{B}^{s+\frac{2}{r},s'})} \lesssim \big(\|w^3_0\|_{\mathcal{B}^{s,s'}} + \|\operatorname{div}_{\mathrm{h}}(\widetilde{T}^{\mathrm{v}}_a v)\|_{L^1(\mathbb{R}^+;\mathcal{B}^{s,s'})}\big)\exp\big(\mathcal{T}_1(\|v_0\|_{\mathcal{B}^0})\big)\,.$$

But we have, using laws of anisotropic paraproduct given in Lemma A.5,

$$\begin{aligned}\|\operatorname{div}_{\mathrm{h}}(\widetilde{T}^{\mathrm{v}}_{w^3}v)\|_{L^1(\mathbb{R}^+;\mathcal{B}^{s,s'})} &\lesssim \|\widetilde{T}^{\mathrm{v}}_{w^3}v\|_{L^1(\mathbb{R}^+;\mathcal{B}^{s+1,s'})}\\ &\lesssim \|w^3\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|v\|_{L^2(\mathbb{R}^+;\mathcal{B}^{s+1,s'})}\,.\end{aligned}$$

Applying (3.28) and (3.4) gives (3.29). Proposition 3.5 is proved. □

As $w^{\rm h}$ is defined by $w^{\rm h} = -\nabla_{\rm h}\Delta_{\rm h}^{-1}\partial_3 w^3$, we deduce from Proposition 3.5, Lemma A.1 and the scaling property (1.14), the following corollary.

Corollary 3.7. *For any s in $[-2+\mu, 0]$ and any $s' \geq 1/2$,*

$$\|w^{\rm h}\|_{\mathcal{A}^{s+1,s'-1}} \lesssim \big(\|w_0^3\|_{\mathcal{B}^{s,s'}} + \|w_0^3\|_{\mathcal{B}^0}\mathcal{T}_\infty(\|v_0\|_{S_\mu})\big)\exp\big(\mathcal{T}_1(\|v_0\|_{\mathcal{B}^0})\big)\,.$$

3.3. **Conclusion of the proof of Theorem 4.** Using the definition of the approximate solution $\Phi^{\rm app}$ given in (3.1), we infer from Propositions 3.1 and 3.5 and Corollary 3.7 that

$$\|\Phi^{\rm app}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)} \leq \mathcal{T}_1(\|(v_0,w_0^3)\|_{\mathcal{B}^0}) + \beta\mathcal{T}_\infty(\|(v_0,w_0^3)\|_{S_\mu})\,. \tag{3.35}$$

Moreover, the error term ψ satisfies the following modified Navier-Stokes equation, with zero initial data:

$$\partial_t\psi + \mathrm{div}\big(\psi\otimes\psi + \Phi^{\rm app}\otimes\psi + \psi\otimes\Phi^{\rm app}\big) - \Delta\psi = -\nabla q_\beta + \sum_{\ell=1}^4 E_\beta^\ell \quad \text{with}$$

$$\begin{aligned}
&E_\beta^1 \stackrel{\rm def}{=} \partial_3^2[(v,0)]_\beta + \beta(0,[\partial_3 p]_\beta)\,,\\
&E_\beta^2 \stackrel{\rm def}{=} \beta\Big[\Big(w^3\partial_3(v,w^3) + \big(\nabla^{\rm h}\Delta_{\rm h}^{-1}\mathrm{div}_{\rm h}\partial_3(vw^3),0\big)\Big)\Big]_\beta\,,\\
&E_\beta^3 \stackrel{\rm def}{=} \beta\Big[\Big(w^{\rm h}\cdot\nabla_h(v,w^3) + v\cdot\nabla^{\rm h}(w^{\rm h},0)\Big)\Big]_\beta \quad \text{and}\\
&E_\beta^4 \stackrel{\rm def}{=} \beta^2\Big[\Big(w^{\rm h}\cdot\nabla^{\rm h}(w^{\rm h},0) + w^3\partial_3(w^{\rm h},0)\Big)\Big]_\beta\,.
\end{aligned} \tag{3.36}$$

If we prove that

$$\Big\|\sum_{\ell=1}^4 E_\beta^\ell\Big\|_{\mathcal{F}^0} \leq \beta\mathcal{T}_\infty\big(\|(v_0,w_0^3)\|_{S_\mu}\big)\,, \tag{3.37}$$

then according to the fact $\psi_{|t=0} = 0$, Proposition 1.14 implies that ψ exists globally and satisfies

$$\|\psi\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)} \lesssim \beta\,\mathcal{T}_\infty\big(\|(v_0,w_0^3)\|_{S_\mu}\big)\,. \tag{3.38}$$

This in turn implies that Φ_0 generates a global regular solution Φ_β in $L^2(\mathbb{R}^+;\mathcal{B}^1)$ which satisfies

$$\|\Phi_\beta\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)} \leq \mathcal{T}_1\big(\|(v_0,w_0^3)\|_{\mathcal{B}^0}\big) + \beta\,\mathcal{T}_\infty\big(\|(v_0,w_0^3)\|_{S_\mu}\big)\,. \tag{3.39}$$

Once this bound in $L^2(\mathbb{R}^+;\mathcal{B}^1)$ is obtained, the bound in $\mathcal{A}^0$ follows by heat flow estimates, and in $\mathcal{A}^{s,s'}$ by propagation of regularity for the Navier-Stokes equations as stated in Proposition 1.15.

So all we need to do is to prove Inequality (3.37). Let us first estimate the term $\partial_3^2[(v,0)]_\beta$. This requires the use of some $\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{s,s'})$ norms. We get

$$\|\partial_3^2[v]_\beta\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{0,-\frac12})} \lesssim \|[v]_\beta\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{0,\frac32})}.$$

Using the vertical scaling property (1.14) of the space $\mathcal{B}^{0,\frac32}$, this gives

$$\|\partial_3^2[v]_\beta\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{0,-\frac12})} \lesssim \beta\,\|v\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{0,\frac32})}\,.$$

Using Proposition 3.1, we get

$$\|\partial_3^2[v]_\beta\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{0,-\frac12})} \leq \beta\,\mathcal{T}_\infty(\|v_0\|_{S_\mu})\,. \tag{3.40}$$

Now let us study the pressure term. By applying the horizontal divergence to the equation satisfied by v we get, thanks to the fact that $\mathrm{div}_{\mathrm{h}} v = 0$,

$$\partial_3 p = -\partial_3 \Delta_{\mathrm{h}}^{-1} \sum_{\ell,m=1}^{2} \partial_\ell \partial_m (v^\ell v^m)\,.$$

Using the fact that $\Delta_{\mathrm{h}}^{-1}\partial_\ell\partial_m$ is a zero-order horizontal Fourier multiplier (since ℓ and m belong to $\{1,2\}$), we infer that

$$\begin{aligned}\big\|[\partial_3 p]_\beta\big\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)} &= \|\partial_3 p\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)}\\ &\lesssim \|v\partial_3 v\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)}\,.\end{aligned}$$

Laws of product in anisotropic Besov as described by Proposition A.4 imply that

$$\|v(t)\partial_3 v(t)\|_{\mathcal{B}^0} \lesssim \|v(t)\|_{\mathcal{B}^1}\|\partial_3 v(t)\|_{\mathcal{B}^0}\,,$$

which gives rise to

$$\begin{aligned}\big\|[\partial_3 p]_\beta\big\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)} &\lesssim \|v\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|\partial_3 v\|_{L^2(\mathbb{R}^+;\mathcal{B}^0)}\\ &\lesssim \|v\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|v\|_{L^2(\mathbb{R}^+;\mathcal{B}^{0,\frac{3}{2}})}\,.\end{aligned}\tag{3.41}$$

Combining (3.40) and (3.41), we get by Proposition 3.1 and Lemma A.3

$$\|E^1_\beta\|_{\mathcal{F}^0} \leq \beta\,\mathcal{T}_\infty\big(\|v_0\|_{S_\mu}\big)\,.\tag{3.42}$$

Now we estimate E^2_β. Applying again the laws of product in anisotropic Besov spaces (see Proposition A.4) together with the action of vertical derivatives, we obtain

$$\begin{aligned}\|w^3(t)\partial_3(v,w^3)(t)\|_{\mathcal{B}^0} &\lesssim \|w^3(t)\|_{\mathcal{B}^1}\|\partial_3(v,w^3)(t)\|_{\mathcal{B}^0}\\ &\lesssim \|w^3(t)\|_{\mathcal{B}^1}\|(v,w^3)(t)\|_{\mathcal{B}^{0,\frac{3}{2}}}\,.\end{aligned}$$

Thus we infer that

$$\|w^3\partial_3(v,w^3)\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)} \lesssim \|w^3\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|(v,w^3)\|_{L^2(\mathbb{R}^+;\mathcal{B}^{0,\frac{3}{2}})}\,.\tag{3.43}$$

For the other term of E^2_β, using the fact that $\nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\,\mathrm{div}_{\mathrm{h}}$ is an order 0 horizontal Fourier multiplier and the Leibniz formula, we infer from Lemma A.1 that

$$\begin{aligned}\|\nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\,\mathrm{div}_{\mathrm{h}}\,\partial_3(vw^3)(t)\|_{\mathcal{B}^0} &\lesssim \|\partial_3(vw^3)(t)\|_{\mathcal{B}^0}\\ &\lesssim \|v(t)\partial_3 w^3(t)\|_{\mathcal{B}^0} + \|w^3(t)\partial_3 v(t)\|_{\mathcal{B}^0}\,.\end{aligned}$$

In view of laws of product in anisotropic Besov spaces and the action of vertical derivatives, this gives rise to

$$\|\nabla^{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\,\mathrm{div}_{\mathrm{h}}\,\partial_3(vw^3)(t)\|_{\mathcal{B}^0} \lesssim \|v(t)\|_{\mathcal{B}^1}\|w^3(t)\|_{\mathcal{B}^{0,\frac{3}{2}}} + \|w^3(t)\|_{\mathcal{B}^1}\|v(t)\|_{\mathcal{B}^{0,\frac{3}{2}}}\,.$$

Together with (3.43), this leads to

$$\begin{aligned}\|E^2_\beta\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)} \lesssim \beta\,\|w^3\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}&\|(v,w^3)\|_{L^2(\mathbb{R}^+;\mathcal{B}^{0,\frac{3}{2}})}\\ &+\beta\,\|w^3\|_{L^2(\mathbb{R}^+;\mathcal{B}^{0,\frac{3}{2}})}\|v\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\,,\end{aligned}$$

hence by Propositions 3.1 and 3.5 along with Lemma A.3

$$\|E^2_\beta\|_{\mathcal{F}^0} \leq \beta\,\mathcal{T}_\infty\big(\|(v_0,w^3_0)\|_{S_\mu}\big)\,.\tag{3.44}$$

Let us estimate E^3_β. Again by laws of product and the action of horizontal derivatives, we obtain

$$\begin{aligned}\|w^{\mathrm{h}}\cdot\nabla_h(v,w^3)\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)} &\lesssim \|w^{\mathrm{h}}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|\nabla^{\mathrm{h}}(v,w^3)\|_{L^2(\mathbb{R}^+;\mathcal{B}^0)}\\ &\lesssim \|w^{\mathrm{h}}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|(v,w^3)\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\,.\end{aligned}$$

Corollary 3.7 and Propositions 3.1 and 3.5 imply that

$$\|w^{\rm h}\cdot\nabla_h(v,w^3)\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)}\leq\mathcal{T}_\infty\big(\|(v_0,w_0^3)\|_{S_\mu}\big)\,. \tag{3.45}$$

Following the same lines we get

$$\|v\cdot\nabla^{\rm h}(w^{\rm h},0)\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)}\leq\mathcal{T}_\infty\big(\|(v_0,w_0^3)\|_{S_\mu}\big)\,.$$

Together with (3.45), this gives thanks to Lemma A.3

$$\|E^3_\beta\|_{\mathcal{F}^0}\lesssim\|E^3_\beta\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)}\leq\beta\,\mathcal{T}_\infty\big(\|(v_0,w_0^3)\|_{S_\mu}\big)\,. \tag{3.46}$$

Now let us estimate E^4_β. Laws of product and the action of derivations give

$$\begin{aligned}\|w^{\rm h}\cdot\nabla^{\rm h}w^{\rm h}\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)} &\lesssim \|w^{\rm h}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|\nabla^{\rm h}w^{\rm h}(t)\|_{L^2(\mathbb{R}^+;\mathcal{B}^0)}\\ &\lesssim \|w^{\rm h}\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\,.\end{aligned} \tag{3.47}$$

In the same way, we get

$$\|w^3(t)\partial_3w^{\rm h}\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)}\lesssim\|w^3\|_{L^2(\mathbb{R}^2;\mathcal{B}^0)}\|w^{\rm h}\|_{L^2(\mathbb{R}^+;\mathcal{B}^{1,\frac32})}\,.$$

Together with (3.47), this gives thanks to Corollary 3.7 and Propositions 3.5

$$\|E^4_\beta\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)}\leq\beta^2\,\mathcal{T}_\infty\big(\|(v_0,w_0^3)\|_{S_\mu}\big)\,.$$

Lemma A.3 implies that

$$\|E^4_\beta\|_{\mathcal{F}^0}\leq\beta^2\,\mathcal{T}_\infty\big(\|(v_0,w_0^3)\|_{S_\mu}\big)\,.$$

Together with Inequalities (3.42), (3.44) and (3.46), this gives

$$\|E_\beta\|_{\mathcal{F}^0}\leq\beta\,\mathcal{T}_\infty\big(\|(v_0,w_0^3)\|_{S_\mu}\big)\,.$$

Thanks to Proposition 1.14 we obtain that the solution Φ_β of (NS) with intial data

$$\Phi_0=\big[(v_0-\beta\nabla^{\rm h}\Delta_{\rm h}^{-1}\partial_3w_0^3,w_0^3)\big]_\beta$$

is global and belongs to $L^2(\mathbb{R}^+;\mathcal{B}^1)$. The whole Theorem 4 follows from the propagation result Proposition 1.15 proved in Appendix A. □

4. Interaction between profiles of scale 1: proof of Theorem 5

The goal of this section is to prove Theorem 5. In the next paragraph we define an approximate solution, using results proved in the previous section, and Paragraph 4.2 is devoted to the proof of useful localization results on the different parts entering the definition of the approximate solution. Paragraph 4.3 concludes the proof of the theorem, using those localization results.

4.1. **The approximate solution.**

Consider the divergence free vector field

$$\Phi^0_{0,n,\alpha,L}\stackrel{\rm def}{=}u_{0,\alpha}+\big[\big(v^{0,\infty}_{0,n,\alpha,L}+h^0_nw^{0,\infty,{\rm h}}_{0,n,\alpha,L},w^{0,\infty,3}_{0,n,\alpha,L}\big)\big]_{h^0_n}+\big[(v^{0,{\rm loc}}_{0,n,\alpha,L}+h^0_nw^{0,{\rm loc},{\rm h}}_{0,n,\alpha,L},w^{0,{\rm loc},3}_{0,n,\alpha,L})\big]_{h^0_n}\,,$$

with the notation of Proposition 1.12. We want to prove that for h^0_n small enough, depending only on u_0 and on $\big\|(v^{0,\infty}_{0,n,\alpha,L},w^{0,\infty,3}_{0,n,\alpha,L})\big\|_{S_\mu}$ as well as $\big\|(v^{0,{\rm loc}}_{0,n,\alpha,L},w^{0,{\rm loc},3}_{0,n,\alpha,L})\big\|_{S_\mu}$, there is a unique, global smooth solution to (NS) with data $\Phi^0_{0,n,\alpha,L}$.

Let us start by solving globally (NS) with the data $u_{0,\alpha}$. By using the global strong stability of (NS) in $\mathcal{B}^0$ (see [4], Theorems 4, 5 and Corollary 3) and the convergence result (1.16) we deduce that for α small enough there is a unique, global solution to (NS) associated with $u_{0,\alpha}$, which we shall denote by u_α and which lies in $L^2(\mathbb{R}^+;\mathcal{B}^1)$.

Next let us define

$$\Phi^{0,\infty}_{0,n,\alpha,L}\stackrel{\rm def}{=}\big[\big(v^{0,\infty}_{0,n,\alpha,L}+h^0_nw^{0,\infty,{\rm h}}_{0,n,\alpha,L},w^{0,\infty,3}_{0,n,\alpha,L}\big)\big]_{h^0_n}\,.$$

Thanks to Theorem 4, we know that for h_n^0 smaller than $\varepsilon_1\big(\big\|(v_{0,n,\alpha,L}^{0,\infty}, w_{0,n,\alpha,L}^{0,\infty,3})\big\|_{S_\mu}\big)$ there is a unique global smooth solution $\Phi^{0,\infty}_{n,\alpha,L}$ associated with $\Phi^{0,\infty}_{0,n,\alpha,L}$, which belongs to $\mathcal{A}_0$, and using the notation and results of Section 3, in particular (3.1) and (3.38), we can write

$$
\begin{aligned}
&\Phi^{0,\infty}_{n,\alpha,L} \stackrel{\text{def}}{=} \Phi^{0,\infty,\text{app}}_{n,\alpha,L} + \psi^{0,\infty}_{n,\alpha,L} \quad \text{with} \\
&\Phi^{0,\infty,\text{app}}_{n,\alpha,L} \stackrel{\text{def}}{=} \big[v^{0,\infty}_{n,\alpha,L} + h_n^0 w^{0,\infty,\text{h}}_{n,\alpha,L}, w^{0,\infty,3}_{n,\alpha,L}\big]_{h_n^0} \quad \text{and} \\
&\|\psi^{0,\infty}_{n,\alpha,L}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)} \lesssim h_n^0 \mathcal{T}_\infty\big(\big\|(v_{0,n,\alpha,L}^{0,\infty}, w_{0,n,\alpha,L}^{0,\infty,3})\big\|_{S_\mu}\big)\,,
\end{aligned} \tag{4.1}
$$

where $v^{0,\infty}_{n,\alpha,L}$ solves $(\text{NS2D})_{x_3}$ with data $v^{0,\infty}_{0,n,\alpha,L}$ and $w^{0,\infty,3}_{n,\alpha,L}$ solves the transport-diffusion equation $(T_{h_n^0})$ defined page 27 with data $w^{0,\infty,3}_{0,n,\alpha,L}$. Finally we recall that

$$
w^{0,\infty,\text{h}}_{n,\alpha,L} = -\nabla^\text{h}\Delta_\text{h}^{-1}\partial_3 w^{0,\infty,3}_{n,\alpha,L}\,.
$$

Similarly defining

$$
\Phi^{0,\text{loc}}_{0,n,\alpha,L} \stackrel{\text{def}}{=} \big[\big(v^{0,\text{loc}}_{0,n,\alpha,L} + h_n^0 w^{0,\text{loc},\text{h}}_{0,n,\alpha,L}, w^{0,\text{loc},3}_{0,n,\alpha,L}\big)\big]_{h_n^0}\,,
$$

then for h_n^0 smaller than $\varepsilon_1\big(\big\|(v_{0,n,\alpha,L}^{0,\text{loc}}, w_{0,n,\alpha,L}^{0,\text{loc},3})\big\|_{S_\mu}\big)$ there is a unique global smooth solution $\Phi^{0,\text{loc}}_{n,\alpha,L}$ associated with $\Phi^{0,\text{loc}}_{0,n,\alpha,L}$, which belongs to $\mathcal{A}_0$, and

$$
\begin{aligned}
&\Phi^{0,\text{loc}}_{n,\alpha,L} \stackrel{\text{def}}{=} \Phi^{0,\text{loc},\text{app}}_{n,\alpha,L} + \psi^{0,\text{loc}}_{n,\alpha,L} \quad \text{with} \\
&\Phi^{0,\text{loc},\text{app}}_{n,\alpha,L} \stackrel{\text{def}}{=} \big[v^{0,\text{loc}}_{n,\alpha,L} + h_n^0 w^{0,\text{loc},\text{h}}_{n,\alpha,L}, w^{0,\text{loc},3}_{n,\alpha,L}\big]_{h_n^0} \quad \text{and} \\
&\|\psi^{0,\text{loc}}_{n,\alpha,L}\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)} \lesssim h_n^0 \mathcal{T}_\infty\big(\big\|(v_{0,n,\alpha,L}^{0,\text{loc}}, w_{0,n,\alpha,L}^{0,\text{loc},3})\big\|_{S_\mu}\big)\,,
\end{aligned} \tag{4.2}
$$

where $v^{0,\text{loc}}_{n,\alpha,L}$ solves $(\text{NS2D})_{x_3}$ with data $v^{0,\text{loc}}_{0,n,\alpha,L}$ and $w^{0,\text{loc},3}_{n,\alpha,L}$ solves $(T_{h_n^0})$ with data $w^{0,\text{loc},3}_{0,n,\alpha,L}$. Finally we recall that $w^{0,\text{loc},\text{h}}_{n,\alpha,L} = -\nabla^\text{h}\Delta_\text{h}^{-1}\partial_3 w^{0,\text{loc},3}_{n,\alpha,L}$.

Now we look for the solution under the form

$$
\Phi^0_{n,\alpha,L} \stackrel{\text{def}}{=} u_\alpha + \Phi^{0,\infty}_{n,\alpha,L} + \Phi^{0,\text{loc}}_{n,\alpha,L} + \psi_{n,\alpha,L}\,.
$$

In the next section we shall prove localization properties on $\Phi^{0,\infty}_{n,\alpha,L}$ and $\Phi^{0,\text{loc}}_{n,\alpha,L}$, namely the fact that $\Phi^{0,\infty,\text{app}}_{n,\alpha,L}$ escapes to infinity in the space variable, while $\Phi^{0,\text{loc},\text{app}}_{n,\alpha,L}$ remains localized (approximately), and we shall also prove that $\Phi^{0,\text{loc},\text{app}}_{n,\alpha,L}$ remains small near $x_3 = 0$. Let us recall that as claimed by (1.20), (1.21) and (1.22), those properties are true for their respective initial data. Those localization properties will enable us to prove, in Paragraph 4.3, that the function $u_\alpha + \Phi^{0,\infty}_{n,\alpha,L} + \Phi^{0,\text{loc}}_{n,\alpha,L}$ is itself an approximate solution to (NS) for the Cauchy data $u_{0,\alpha} + \Phi^{0,\infty}_{0,n,\alpha,L} + \Phi^{0,\text{loc}}_{0,n,\alpha,L}$.

4.2. **Localization properties of the approximate solution.** One important step in the proof of Theorem 5 consists in the following result.

Proposition 4.1. *Under the assumptions of Proposition* 3.1, *the control of the value of v at the point $x_3 = 0$ is given by*

$$
\forall r \in [1,\infty]\,,\ \|v(\cdot,0)\|_{\widetilde{L}^r(\mathbb{R}^+;B^{\frac{2}{r}}_{2,1}(\mathbb{R}^2))} \lesssim \|v_0(\cdot,0)\|_{B^0_{2,1}(\mathbb{R}^2)} + \|v(\cdot,0)\|^2_{L^2(\mathbb{R}^2)}\,. \tag{4.3}
$$

Moreover we have for all η in $]0,1[$ and γ in $\{0,1\}$,

$$
\|(\gamma - \theta_{\text{h},\eta})v\|_{\mathcal{A}^0} \leq \big\|(\gamma - \theta_{\text{h},\eta})v_0\big\|_{\mathcal{B}^0} \exp \mathcal{T}_1(\|v_0\|_{\mathcal{B}^0}) + \eta\mathcal{T}_\infty(\|v_0\|_{S_\mu})\,, \tag{4.4}
$$

with $\theta_{\text{h},\eta}$ is the truncation function defined by (1.15).

Proof. In this proof we omit for simplicity the dependence of the function spaces on the space $\mathbb{R}^2$. Let us remark that the proof of Lemma 1.1 of [13] claims that for all x_3 in $\mathbb{R}$,

$$
\begin{aligned}
&\big(\Delta_k^{\rm h}(v(t,\cdot,x_3)\cdot\nabla^{\rm h}v(t,\cdot,x_3))\big|\Delta_k^{\rm h}v(t,\cdot,x_3)\big)_{L^2}\\
&\qquad\qquad\lesssim d_k(t,x_3)\|\nabla^{\rm h}v(t,\cdot,x_3)\|_{L^2}^2\|\Delta_k^{\rm h}v(t,\cdot,x_3)\|_{L^2}
\end{aligned}
\tag{4.5}
$$

where $(d_k(t,x_3))_{k\in\mathbb{Z}}$ is a generic element of the sphere of $\ell^1(\mathbb{Z})$. A L^2 energy estimate in $\mathbb{R}^2$ gives therefore, taking $x_3=0$,

$$
\frac12\frac{d}{dt}\|\Delta_k^{\rm h}v(t,\cdot,0)\|_{L^2}^2+c2^{2k}\|\Delta_k^{\rm h}v(t,\cdot,0)\|_{L^2}^2\lesssim d_k(t)\|\nabla^{\rm h}v(t,\cdot,0)\|_{L^2}^2\|\Delta_k^{\rm h}v(t,\cdot,0)\|_{L^2}\,,
$$

where $(d_k(t))_{k\in\mathbb{Z}}$ belongs to the sphere of $\ell^1(\mathbb{Z})$. After division by $\|\Delta_k^{\rm h}v(t,\cdot,0)\|_{L^2}$ and time integration, we get

$$
\begin{aligned}
&\|\Delta_k^{\rm h}v(\cdot,0)\|_{L^\infty(\mathbb{R}^+;L^2)}+c2^{2k}\|\Delta_k^{\rm h}v(\cdot,0)\|_{L^1(\mathbb{R}^+;L^2)}\\
&\qquad\qquad\leq\|\Delta_kv_0(\cdot,0)\|_{L^2}+C\int_0^\infty d_k(t)\|\nabla^{\rm h}v(t,\cdot,0)\|_{L^2}^2dt\,.
\end{aligned}
\tag{4.6}
$$

By summation over k and in view of (3.12), we obtain Inequality (4.3) of Proposition 4.1.

In order to prove Inequality (4.4), let us define $v_{\gamma,\eta}\stackrel{\rm def}{=}(\gamma-\theta_{{\rm h},\eta})v$ and write that

$$
\begin{aligned}
&\partial_tv_{\gamma,\eta}-\Delta_{\rm h}v_{\gamma,\eta}+{\rm div}_{\rm h}\big(v\otimes v_{\gamma,\eta}\big)=E_\eta(v)=\sum_{i=1}^3E_\eta^i(v)\quad\text{with}\\
&E_\eta^1(v)\stackrel{\rm def}{=}-2\eta(\nabla^{\rm h}\theta)_{{\rm h},\eta}\nabla^{\rm h}v-\eta^2(\Delta_{\rm h}\theta)_{{\rm h},\eta}v\,,\\
&E_\eta^2(v)\stackrel{\rm def}{=}\eta\,v\cdot(\nabla^{\rm h}\theta)_{{\rm h},\eta}v\quad\text{and}\\
&E_\eta^3(v)\stackrel{\rm def}{=}-(\gamma-\theta_{{\rm h},\eta})\nabla^{\rm h}\Delta_{\rm h}^{-1}\sum_{1\leq\ell,m\leq2}\partial_\ell\partial_m\big(v^\ell v^m\big)\,.
\end{aligned}
\tag{4.7}
$$

Let us prove that

$$
\|E_\eta(v)\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)}\lesssim\eta\,\mathcal{T}_\infty(\|v_0\|_{S_\mu})\,.
\tag{4.8}
$$

Using Inequality (3.27) applied with $r=1$ and $s=-1$ (resp. $r=2$ and $s=-1/2$) this will follow from

$$
\|E_\eta(v)\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)}\lesssim\eta\big(\|v\|_{L^1(\mathbb{R}^+;\mathcal{B}^1)}+\|v\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^{\frac12})}\big)\,.
\tag{4.9}
$$

Proposition A.6 and the scaling properties of homogeneous Besov spaces give

$$
\begin{aligned}
\|(\nabla^{\rm h}\theta)_{{\rm h},\eta}\nabla^{\rm h}v(t)\|_{\mathcal{B}^0}&\lesssim\|(\nabla^{\rm h}\theta)_{{\rm h},\eta}\|_{B^1_{2,1}(\mathbb{R}^2)}\|\nabla^{\rm h}v(t)\|_{\mathcal{B}^0}\\
&\lesssim\|\nabla^{\rm h}\theta\|_{B^1_{2,1}(\mathbb{R}^2)}\|v(t)\|_{\mathcal{B}^1}\,.
\end{aligned}
$$

Following the same lines, we get

$$
\begin{aligned}
\|(\Delta_{\rm h}\theta)_{{\rm h},\eta}v(t)\|_{\mathcal{B}^0}&\lesssim\|(\Delta_{\rm h}\theta)_{{\rm h},\eta}\|_{B^0_{2,1}(\mathbb{R}^2)}\|v(t)\|_{\mathcal{B}^1}\\
&\lesssim\frac1\eta\|\Delta_{\rm h}\theta\|_{B^0_{2,1}(\mathbb{R}^2)}\|v(t)\|_{\mathcal{B}^1}\,,
\end{aligned}
$$

hence

$$
\|E^1_\eta(v)\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)}\lesssim\eta\|v\|_{L^1(\mathbb{R}^+;\mathcal{B}^1)}\,.
\tag{4.10}
$$

Let us study the term $E^2_\eta(v)$. Proposition A.6 implies

$$
\begin{aligned}
\|v(t)\cdot(\nabla^{\rm h}\theta)_{{\rm h},\eta}v(t)\|_{\mathcal{B}^0} &\lesssim \|(\nabla^{\rm h}\theta)_{{\rm h},\eta}\|_{B^1_{2,1}(\mathbb{R}^2)}\sup_{\ell,m}\|v^\ell(t)v^m(t)\|_{\mathcal{B}^0}\\
&\lesssim \|\nabla^{\rm h}\theta\|_{B^1_{2,1}(\mathbb{R}^2)}\|v(t)\|^2_{\mathcal{B}^{\frac12}}.
\end{aligned}
$$

Thus we get

$$
\|E^2_\eta(v)\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)}\lesssim \eta\|v\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^{\frac12})}\,. \tag{4.11}
$$

Let us study the term $E^3_\eta(v)$ which is related to the pressure. For that purpose, we shall make use of the horizontal paraproduct decomposition:

$$
av=T^{\rm h}_v a+T^{\rm h}_a v+R^{\rm h}(a,b)\quad\text{with}\quad T^{\rm h}_a b\stackrel{\rm def}{=}\sum_k S^{\rm h}_{k-1}a\Delta^{\rm h}_k b\quad\text{and}\quad R^{\rm h}(a,b)\stackrel{\rm def}{=}\sum_k\widetilde\Delta^{\rm h}_k a\Delta^{\rm h}_k b\,.
$$

This allows us to write

$$
\begin{aligned}
&E^3_\eta(v)=\sum_{\ell=1}^3 E^{3,\ell}_\eta(v)\quad\text{with}\\
&E^{3,1}_\eta(v)\stackrel{\rm def}{=}\widetilde T^{\rm h}_{\nabla^{\rm h}p}\theta_{{\rm h},\eta}\quad\text{with}\quad\nabla^{\rm h}p=\nabla^{\rm h}\Delta_{\rm h}^{-1}\sum_{1\leq\ell,m\leq2}\partial_\ell\partial_m(v^\ell v^m)\,,\\
&E^{3,2}_\eta(v)\stackrel{\rm def}{=}-\sum_{1\leq\ell,m\leq2}\big[T^{\rm h}_{\gamma-\theta_{{\rm h},\eta}},\nabla^{\rm h}\Delta_{\rm h}^{-1}\partial_\ell\partial_m\big]v^\ell v^m\quad\text{and}\\
&E^{3,3}_\eta(v)\stackrel{\rm def}{=}\sum_{1\leq\ell,m\leq2}\nabla^{\rm h}\Delta_{\rm h}^{-1}\partial_\ell\partial_m\widetilde T^{\rm h}_{v^\ell v^m}\theta_{{\rm h},\eta}.
\end{aligned}
\tag{4.12}
$$

Laws of (para)product, as given in (A.10), and scaling properties of Besov spaces give

$$
\begin{aligned}
\|\widetilde T^{\rm h}_{\nabla^{\rm h}p(t)}\theta_{{\rm h},\eta}\|_{\mathcal{B}^0} &\lesssim \|\nabla^{\rm h}p(t)\|_{\mathcal{B}^{-1}}\|\theta_{{\rm h},\eta}\|_{B^2_{2,1}(\mathbb{R}^2)}\\
&\lesssim \eta\sup_{1\leq\ell,m\leq2}\|v^\ell(t)v^m(t)\|_{\mathcal{B}^0}\|\theta\|_{B^2_{2,1}(\mathbb{R}^2)}\\
&\lesssim \eta\,\|v(t)\|^2_{\mathcal{B}^{\frac12}}\|\theta\|_{B^2_{2,1}(\mathbb{R}^2)}\,.
\end{aligned}
$$

Along the same lines we get

$$
\begin{aligned}
\|\nabla^{\rm h}\Delta_{\rm h}^{-1}\partial_\ell\partial_m\widetilde T^{\rm h}_{v^\ell(t)v^m(t)}\theta_{{\rm h},\eta}\|_{\mathcal{B}^0} &\lesssim \|\widetilde T^{\rm h}_{v^\ell(t)v^m(t)}\theta_{{\rm h},\eta}\|_{\mathcal{B}^1}\\
&\lesssim \|v^\ell(t)v^m(t)\|_{\mathcal{B}^0}\|\theta_{{\rm h},\eta}\|_{B^2_{2,1}(\mathbb{R}^2)}\\
&\lesssim \eta\,\|v(t)\|^2_{\mathcal{B}^{\frac12}}\|\theta\|_{B^2_{2,1}(\mathbb{R}^2)}\,.
\end{aligned}
$$

This gives

$$
\|E^{3,1}_\eta(v)+E^{3,3}_\eta(v)\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)}\lesssim\eta\,\|v\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^{\frac12})}\,. \tag{4.13}
$$

Now let us estimate $E^{3,2}_\eta(v)$. By definition, we have

$$
\begin{aligned}
&\big[T^{\rm h}_{\gamma-\theta_{{\rm h},\eta}},\nabla^{\rm h}\Delta_{\rm h}^{-1}\partial_\ell\partial_m\big]v^\ell v^m=\sum_k\mathcal{E}_{k,\eta}(v)\quad\text{with}\\
&\mathcal{E}_{k,\eta}(v)\stackrel{\rm def}{=}\big[S^{\rm h}_{k-N_0}(\gamma-\theta_{{\rm h},\eta}),\widetilde\Delta^{\rm h}_k\nabla^{\rm h}\Delta_{\rm h}^{-1}\partial_\ell\partial_m\big]\Delta^{\rm h}_k(v^\ell v^m)
\end{aligned}
$$

where $\widetilde\Delta^{\rm h}_k\stackrel{\rm def}{=}\widetilde\varphi(2^{-k}\xi_{\rm h})$ with $\widetilde\varphi$ is a smooth compactly supported (in $\mathbb{R}^2\setminus\{0\}$) function which has value 1 near $B(0,2^{-N_0})+\mathcal{C}$, where $\mathcal{C}$ is an adequate annulus. Then by commutator estimates (see for instance Lemma 2.97 in [2])

$$\|\Delta_j^{\rm v}\mathcal{E}_{k,\eta}(v(t))\|_{L^2} \lesssim \|\nabla\theta_{{\rm h},\eta}\|_{L^\infty}\|\Delta_k^{\rm h}\Delta_j^{\rm v}(v^\ell(t)v^m(t))\|_{L^2}\,.$$

As $\|\nabla\theta_{{\rm h},\eta}\|_{L^\infty} = \eta\|\nabla\theta\|_{L^\infty}$, by characterization of anisotropic Besov spaces and laws of product, we get

$$\|E_\eta^{3,2}(v)\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)} \lesssim \eta\|v\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^{\frac12})}\,.$$

Together with estimates (4.10)–(4.13), this gives (4.9), hence (4.8).
Applying Lemma 3.6 with $s = 0$, $s' = 1/2$, $a = v_{\gamma,\eta}$, $\mathcal{Q}(v,a) = {\rm div}_{\rm h}(v \otimes a)$, $f = E_\eta(v)$ and $\beta = 0$ allows to conclude the proof of Proposition 4.1. $\square$

A similar result holds for the solution w^3 of

$$(T_\beta) \quad \partial_t w^3 + v\cdot\nabla^{\rm h} w^3 - \Delta_{\rm h} w^3 - \beta^2\partial_3^2 w^3 = 0 \quad \text{and} \quad w^3_{|t=0} = w^3_0\,,$$

where β is any non negative real number. In the following statement, all the constants are independent of β.

Proposition 4.2. *Let v and w_3 be as in Proposition 3.5. The control of the value of w^3 at the point $x_3 = 0$ is given by the following inequality. For any r in $[2,\infty]$,*

$$\|w^3(\cdot,0)\|_{\widetilde L^r(\mathbb{R}^+;B^{\frac2r}_{2,1}(\mathbb{R}^2))} \leq \mathcal{T}_\infty(\|(v_0,w_0^3)\|_{S_\mu})\Big(\|w_0^3(\cdot,0)\|^{\frac{1-2\mu}{4(1-\mu)}}_{B^0_{2,1}(\mathbb{R}^2)} + \beta\Big)\,. \tag{4.14}$$

Moreover, with the notations of Theorem 4, we have for all η in $]0,1[$ and γ in $\{0,1\}$,

$$\|(\gamma-\theta_{{\rm h},\eta})w^3\|_{\mathcal{A}^0} \leq \big\|(\gamma-\theta_{{\rm h},\eta})w_0^3\big\|_{\mathcal{B}^0}\exp\mathcal{T}_1(\|v_0\|_{\mathcal{B}^0}) + \eta\mathcal{T}_\infty(\|(v_0,w_0^3)\|_{S_\mu})\,. \tag{4.15}$$

Proof. The proof is very similar to the proof of Proposition 4.1. The main difference lies in the proof of (4.14) due to the presence of the extra term $\beta^2\partial_3^2 w^3$, so let us detail that estimate: we shall first prove an estimate for $w^3(t,x_h,0)$ in $\widetilde L^r(\mathbb{R}^+;B^{\frac12+\frac2r}_{2,1}(\mathbb{R}^2))$, and then we shall interpolate that estimate with the known a priori estimate (3.29) of w^3 in $\widetilde L^r(\mathbb{R}^+;\mathcal{B}^{-\frac12+\frac2r}_{2,1}(\mathbb{R}^2))$ to find the result.
Let us be more precise, and first obtain a bound for $w^3(t,x_h,0)$ in $\widetilde L^r(\mathbb{R}^+;B^{\frac12+\frac2r}_{2,1}(\mathbb{R}^2))$. Defining

$$\widetilde w^3(t,x_h) \stackrel{\rm def}{=} w^3(t,x_h,0)\,, \quad \widetilde w^3_0(x_h) \stackrel{\rm def}{=} w^3_0(x_h,0) \quad \text{and} \quad \widetilde v(t,x_h) \stackrel{\rm def}{=} v(t,x_h,0)\,,$$

we have

$$\partial_t\widetilde w^3 + \widetilde v\cdot\nabla^{\rm h}\widetilde w^3 - \Delta_{\rm h}\widetilde w^3 = \beta^2(\partial_3^2 w^3)(\cdot,0) \quad \text{and} \quad \widetilde w^3_{|t=0} = \widetilde w^3_0\,. \tag{4.16}$$

Similarly to (4.5) we write (dropping for simplicity the dependence of the spaces on $\mathbb{R}^2$)

$$\big(\Delta_k^{\rm h}(\widetilde v\cdot\nabla^{\rm h}\widetilde w^3)\big|\Delta_k^{\rm h}\widetilde w^3\big)_{L^2} \lesssim d_k(t)\,2^{-\frac k2}\|\nabla^{\rm h}\widetilde v\|_{L^2}\|\nabla^{\rm h}\widetilde w^3\|_{B^{\frac12}_{2,1}}\|\Delta_k^{\rm h}\widetilde w^3\|_{L^2}\,,$$

where $(d_k(t))_{k\in\mathbb{Z}}$ belongs to the sphere of $\ell^1(\mathbb{Z})$. Taking the L^2 scalar product of $\Delta_k^{\rm h}$ of Equation (4.16) with $\Delta_k^{\rm h}\widetilde w^3$ implies that

$$\begin{aligned}\frac12 2^{\frac k2}\frac{d}{dt}\|\Delta_k^{\rm h}\widetilde w^3\|^2_{L^2} + c2^{\frac{5k}{2}}\|\Delta_k^{\rm h}\widetilde w^3\|^2_{L^2} &\lesssim d_k(t)\|\nabla^{\rm h}\widetilde v(t)\|_{L^2}\|\nabla^{\rm h}\widetilde w^3\|_{B^{\frac12}_{2,1}}\|\Delta_k^{\rm h}\widetilde w^3\|_{L^2}\\ &\quad + \beta^2 2^{\frac k2}\|\Delta_k^{\rm h}(\partial_3^2 w^3)(\cdot,0)\|_{L^2}\|\Delta_k^{\rm h}\widetilde w^3\|_{L^2}\,,\end{aligned}$$

so as in (4.6) we find

$$\begin{aligned}&2^{\frac k2}\|\Delta_k^{\rm h}\widetilde w^3\|_{L^\infty(\mathbb{R}^+;L^2)} + c2^{\frac{5k}{2}}\|\Delta_k^{\rm h}\widetilde w^3\|_{L^1(\mathbb{R}^+;L^2)} \leq 2^{\frac k2}\|\Delta_k\widetilde w^3_0\|_{L^2}\\ &\quad + C\int_0^\infty d_k(t)\|\nabla^{\rm h}\widetilde v(t)\|_{L^2}\|\nabla^{\rm h}\widetilde w^3(t)\|_{B^{\frac12}_{2,1}}dt + C\beta^2\int_0^\infty 2^{\frac k2}\|\Delta_k^{\rm h}(\partial_3^2 w^3)(t,\cdot,0)\|_{L^2}dt\,.\end{aligned}$$

After summation we find that

$$\|\widetilde{w}^3\|_{\widetilde{L}^\infty(\mathbb{R}^+;B^{\frac12}_{2,1})} + \|\widetilde{w}^3\|_{L^1(\mathbb{R}^+;B^{\frac52}_{2,1})} \lesssim \|\widetilde{w}^3_0\|_{B^{\frac12}_{2,1}} + \|\widetilde{w}^3\|_{L^2(\mathbb{R}^+;B^{\frac32}_{2,1})}\|\nabla^{\rm h}\widetilde{v}\|_{L^2(\mathbb{R}^+;L^2)} + \beta^2\|(\partial_3^2 w^3)(\cdot,0)\|_{L^1(\mathbb{R}^+;B^{\frac12}_{2,1})}\,.$$

This is exactly an inequality of the type (3.30), up to a harmless localization in time, so by the same arguments we obtain the same conclusion as in Lemma 3.6, namely the fact that for all $r\in[1,\infty]$,

$$\|\widetilde{w}^3\|_{\widetilde{L}^r(\mathbb{R}^+;B^{\frac12+\frac2r})} \lesssim \big(\|\widetilde{w}^3_0\|_{B^{\frac12}_{2,1}} + \beta^2\|(\partial_3^2 w^3)(\cdot,0)\|_{L^1(\mathbb{R}^+;B^{\frac12}_{2,1})}\big)\exp C\|v_0(\cdot,0)\|^2_{L^2}\,.$$

Since we have

$$\|(\partial_3^2 w^3)(\cdot,0)\|_{L^1(\mathbb{R}^+;B^{\frac12}_{2,1}(\mathbb{R}^2))} \lesssim \|w^3\|_{L^1(\mathbb{R}^+;\mathcal{B}^{\frac12,\frac52})}$$

we infer from the a priori bounds (3.34) obtained on w^3 in the previous section that

$$\|(\partial_3^2 w^3)(\cdot,0)\|_{L^1(\mathbb{R}^+;B^{\frac12}_{2,1}(\mathbb{R}^2))} \lesssim \mathcal{T}_\infty(\|(v_0,w_0^3)\|_{S_\mu})\,,$$

so we obtain that for any r in $[1,\infty]$,

$$\|w^3(\cdot,0)\|_{\widetilde{L}^r(\mathbb{R}^+;B^{\frac12+\frac2r}_{2,1}(\mathbb{R}^2))} \leq \big(\|w^3_0(\cdot,0)\|_{B^{\frac12}_{2,1}(\mathbb{R}^2)} + \beta^2\big)\mathcal{T}_\infty(\|(v_0,w_0^3)\|_{S_\mu})\,. \tag{4.17}$$

Recalling that w^3_0 belongs to the space S_μ introduced in Definition 1.10, we find that

$$w^3_0(\cdot,0)\in\bigcap_{s\in[-2+\mu,1-\mu]} B^s_{2,1}(\mathbb{R}^2)\,.$$

Since $0<\mu<\dfrac12$, we get by interpolation and Sobolev embeddings that

$$\|w^3_0(\cdot,0)\|_{B^{\frac12}_{2,1}(\mathbb{R}^2)} \lesssim \|w^3_0(\cdot,0)\|^{\frac{1-2\mu}{2(1-\mu)}}_{B^0_{2,1}(\mathbb{R}^2)}\|w^3_0\|^{\frac{1}{2(1-\mu)}}_{S_\mu}\,,$$

which implies that (4.17) can be written under the form

$$\|w^3(\cdot,0)\|_{\widetilde{L}^r(\mathbb{R}^+;B^{\frac12+\frac2r}_{2,1}(\mathbb{R}^2))} \leq \Big(\|w^3_0(\cdot,0)\|^{\frac{1-2\mu}{2(1-\mu)}}_{B^0_{2,1}(\mathbb{R}^2)} + \beta^2\Big)\mathcal{T}_\infty(\|(v_0,w_0^3)\|_{S_\mu})\,.$$

Now interpolating with the a priori bound obtained in Proposition 3.5, we find

$$\begin{aligned}\|w^3(\cdot,0)\|_{\widetilde{L}^r(\mathbb{R}^+;B^{-\frac12+\frac2r}_{2,1}(\mathbb{R}^2))} &\lesssim \|w^3\|_{\widetilde{L}^r(\mathbb{R}^+;\mathcal{B}^{-\frac12+\frac2r})}\\ &\lesssim \mathcal{T}_\infty(\|(v_0,w_0^3)\|_{S_\mu}\,,\end{aligned}$$

so we obtain finally

$$\|w^3(\cdot,0)\|_{\widetilde{L}^r(\mathbb{R}^+;B^{\frac2r}_{2,1}(\mathbb{R}^2))} \leq \mathcal{T}_\infty(\|(v_0,w_0^3)\|_{S_\mu})\Big(\|w^3_0(\cdot,0)\|^{\frac{1-2\mu}{4(1-\mu)}}_{B^0_{2,1}(\mathbb{R}^2)} + \beta\Big)\,.$$

This ends the proof of (4.14).

We shall not detail the proof of (4.15) as it is very similar to the proof of (4.4). Proposition 4.2 is therefore proved. □

Propositions 4.1 and 4.2 imply easily the following result, using the special form of $\Phi^{0,\infty}_{n,\alpha,L}$ and $\Phi^{0,{\rm loc}}_{n,\alpha,L}$ recalled in (4.1) and (4.2), and thanks to (1.20), (1.21) and (1.22).

Corollary 4.3. *The vector fields $\Phi^{0,\mathrm{loc}}_{n,\alpha,L}$ and $\Phi^{0,\infty}_{n,\alpha,L}$ satisfy the following: $\Phi^{0,\mathrm{loc}}_{n,\alpha,L}$ vanishes at $x_3=0$, in the sense that for all r in $[2,\infty]$,*

$$\lim_{L\to\infty}\lim_{\alpha\to 0}\limsup_{n\to\infty}\|\Phi^{0,\mathrm{loc}}_{n,\alpha,L}(\cdot,0)\|_{\widetilde L^r(\mathbb{R}^+;B^{\frac{2}{r}}_{2,1}(\mathbb{R}^2))}=0\,,$$

and there is a constant $C(\alpha,L)$ such that for all η in $]0,1[$,

$$\limsup_{n\to\infty}\Big(\|(1-\theta_{\mathrm{h},\eta})\Phi^{0,\mathrm{loc}}_{n,\alpha,L}\|_{\mathcal{A}^0}+\|\theta_{\mathrm{h},\eta}\Phi^{0,\infty}_{n,\alpha,L}\|_{\mathcal{A}^0}\Big)\le C(\alpha,L)\eta\,.$$

4.3. **Conclusion of the proof of Theorem 5.** Recall that we look for the solution of (NS) under the form

$$\Phi^0_{n,\alpha,L}=u_\alpha+\Phi^{0,\infty}_{n,\alpha,L}+\Phi^{0,\mathrm{loc}}_{n,\alpha,L}+\psi_{n,\alpha,L}\,,$$

with the notation introduced in Paragraph 4.1. In particular the two vector fields $\Phi^{0,\mathrm{loc}}_{n,\alpha,L}$ and $\Phi^{0,\infty}_{n,\alpha,L}$ satisfy Corollary 4.3, and furthermore thanks to the Lebesgue theorem,

$$\lim_{\eta\to0}\|(1-\theta_\eta)u_\alpha\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}=0\,. \tag{4.18}$$

Given a small number $\varepsilon>0$, to be chosen later, we choose L, α and $\eta=\eta(\alpha,L,u_0)$ so that thanks to Corollary 4.3 and (4.18), for all r in $[2,\infty]$, and for n large enough,

$$\begin{aligned}\|\Phi^{0,\mathrm{loc}}_{n,\alpha,L}(\cdot,0)\|_{L^r(\mathbb{R}^+;B^{\frac{2}{r}}_{2,1}(\mathbb{R}^2))}+\|(1-\theta_{\mathrm{h},\eta})\Phi^{0,\mathrm{loc}}_{n,\alpha,L}\|_{\mathcal{A}^0}+\|(1-\theta_\eta)u_\alpha\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}&\\ +\|\theta_{\mathrm{h},\eta}\Phi^{0,\infty}_{n,\alpha,L}\|_{\mathcal{A}^0}\le\varepsilon\,.&\end{aligned} \tag{4.19}$$

In the following we denote for simplicity

$$(\Phi^{0,\infty}_\varepsilon,\Phi^{0,\mathrm{loc}}_\varepsilon,\psi_\varepsilon)\stackrel{\mathrm{def}}{=}(\Phi^{0,\infty}_{n,\alpha,L},\Phi^{0,\mathrm{loc}}_{n,\alpha,L},\psi_{n,\alpha,L})\quad\text{and}\quad\Phi^{\mathrm{app}}_\varepsilon\stackrel{\mathrm{def}}{=}u_\alpha+\Phi^{0,\infty}_\varepsilon+\Phi^{0,\mathrm{loc}}_\varepsilon\,,$$

so the vector field ψ_ε satisfies the following equation, with zero initial data:

$$\begin{aligned}&\partial_t\psi_\varepsilon-\Delta\psi_\varepsilon+\mathrm{div}\big(\psi_\varepsilon\otimes\psi_\varepsilon+\Phi^{\mathrm{app}}_\varepsilon\otimes\psi_\varepsilon+\psi_\varepsilon\otimes\Phi^{\mathrm{app}}_\varepsilon\big)=-\nabla q_\varepsilon+E_\varepsilon\,,\\ &\qquad\text{with}\quad E_\varepsilon=E^1_\varepsilon+E^2_\varepsilon\quad\text{and}\\ &\qquad\qquad E^1_\varepsilon\stackrel{\mathrm{def}}{=}\mathrm{div}\Big(\Phi^{0,\infty}_\varepsilon\otimes(\Phi^{0,\mathrm{loc}}_\varepsilon+u_\alpha)+(\Phi^{0,\mathrm{loc}}_\varepsilon+u_\alpha)\otimes\Phi^{0,\infty}_\varepsilon\\ &\qquad\qquad\qquad\qquad+\Phi^{0,\mathrm{loc}}\otimes(1-\theta_\eta)u_\alpha+(1-\theta_\eta)u_\alpha\otimes\Phi^{0,\mathrm{loc}}\Big)\,,\\ &\qquad\qquad E^2_\varepsilon\stackrel{\mathrm{def}}{=}\mathrm{div}\left(\Phi^{0,\mathrm{loc}}_\varepsilon\otimes\theta_\eta u_\alpha+\theta_\eta u_\alpha\otimes\Phi^{0,\mathrm{loc}}_\varepsilon\right).\end{aligned} \tag{4.20}$$

If we prove that

$$\lim_{\varepsilon\to0}\|E_\varepsilon\|_{\mathcal{F}^0}=0\,, \tag{4.21}$$

then Proposition 1.14 implies that ψ_ε belongs to $L^2(\mathbb{R}^+;\mathcal{B}^1)$, with

$$\lim_{\varepsilon\to0}\|\psi_\varepsilon\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}=0\,,$$

and we conclude the proof of Theorem 5 exactly as in the proof of Theorem 4, by resorting to Proposition 1.15.

So let us prove (4.21). The term E^1_ε is the easiest, thanks to the separation of the spatial supports. Let us first write $E^1_\varepsilon = E^1_{\varepsilon,\mathrm{h}} + E^1_{\varepsilon,3}$ with

$$
\begin{aligned}
E^1_{\varepsilon,\mathrm{h}} &\stackrel{\mathrm{def}}{=} \mathrm{div}_\mathrm{h}\Big((\Phi^{0,\mathrm{loc}}_\varepsilon + u_\alpha)\otimes \Phi^{0,\infty,\mathrm{h}}_\varepsilon + \Phi^{0,\infty}_\varepsilon \otimes (\Phi^{0,\mathrm{loc},\mathrm{h}}_\varepsilon + u^\mathrm{h}_\alpha)\\
&\qquad + (1-\theta_\eta)u_\alpha \otimes \Phi^{0,\mathrm{loc},\mathrm{h}} + \Phi^{0,\mathrm{loc}}\otimes (1-\theta_\eta)u^\mathrm{h}_\alpha\Big) \quad \text{and}\\
E^1_{\varepsilon,3} &\stackrel{\mathrm{def}}{=} \partial_3\Big((\Phi^{0,\mathrm{loc}}_\varepsilon + u_\alpha)\Phi^{0,\infty,3}_\varepsilon + \Phi^{0,\infty}_\varepsilon(\Phi^{0,\mathrm{loc},3}_\varepsilon + u^3_\alpha)\\
&\qquad + (1-\theta_\eta)u_\alpha \Phi^{0,\mathrm{loc},3} + \Phi^{0,\mathrm{loc}}(1-\theta_\eta)u^3_\alpha\Big)\,.
\end{aligned}
$$

Next let us write, for any two functions a and b,

$$
ab = (\theta_{\mathrm{h},\eta}a)b + a\big((1-\theta_{\mathrm{h},\eta})b\big)\,.
$$

Denoting

$$
u^\infty_\varepsilon \stackrel{\mathrm{def}}{=} (1-\theta_\eta)u_\alpha
$$

and using by now as usual the action of derivatives and the fact that $\mathcal{B}^1$ is an algebra, we infer that

$$
\begin{aligned}
\|E^1_{\varepsilon,\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)} + \|E^1_{\varepsilon,3}\|_{L^1(\mathbb{R}^+;B^{1,-\frac12}_{2,1})} &\le \|\theta_{\mathrm{h},\eta}\Phi^{0,\infty}_\varepsilon\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|\Phi^{0,\mathrm{loc}}_\varepsilon + u_\alpha\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\\
&\quad + \|(1-\theta_{\mathrm{h},\eta})(\Phi^{0,\mathrm{loc}}_\varepsilon + u_\alpha)\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|\Phi^{0,\infty}_\varepsilon\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\\
&\quad + \|\Phi^{0,\mathrm{loc}}_\varepsilon\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|u^\infty_\varepsilon\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\,.
\end{aligned}
$$

Thanks to (4.19) and to the a priori bounds on $\Phi^{0,\infty}_\varepsilon$, $\Phi^{0,\mathrm{loc}}_\varepsilon$ and u_α, we get directly in view of the examples page 11 that

$$
\lim_{\varepsilon\to0}\|E^1_\varepsilon\|_{\mathcal{F}^0} = 0\,.
$$

Next let us turn to E^2_ε. We shall follow the method of [17], and in particular the following lemma will be very useful.

Lemma 4.4. *There is a constant C such that for all functions a and b, we have*

$$
\|ab\|_{\mathcal{B}^1} \le C\|a\|_{\mathcal{B}^1}\|b(\cdot,0)\|_{B^1_{2,1}(\mathbb{R}^2)} + C\|x_3 a\|_{\mathcal{B}^1}\|\partial_3 b\|_{\mathcal{B}^1}\,.
$$

We postpone the proof of that lemma. Let us apply it to estimate E^2_ε. We write, as in the case of E^1_ε and defining $u^\mathrm{loc}_\varepsilon \stackrel{\mathrm{def}}{=} \theta_\eta u_\alpha$,

$$
\begin{aligned}
\|E^2_\varepsilon\|_{\mathcal{F}^0} \lesssim{}& \|u^\mathrm{loc}_\varepsilon\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|\Phi^{0,\mathrm{loc}}_\varepsilon(\cdot,0)\|_{L^2(\mathbb{R}^+;B^1_{2,1}(\mathbb{R}^2))}\\
&+\|x_3 u^\mathrm{loc}_\varepsilon\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|\partial_3\Phi^{0,\mathrm{loc}}_\varepsilon\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\,.
\end{aligned}
$$

Thanks to (4.19) as well as Inequality (1.26) of Theorem 4, we obtain

$$
\lim_{\varepsilon\to0}\|E^2_\varepsilon\|_{\mathcal{F}^0} = 0\,.
$$

This proves (4.21), hence Theorem 5. □

Proof of Lemma 4.4. This is essentially Lemma 3.3 of [17], we recall the proof for the convenience of the reader. Let us decompose b in the following way:

$$
b(x_\mathrm{h},x_3) = b(x_\mathrm{h},0) + \int_0^{x_3}\partial_3 b(x_\mathrm{h},y_3)dy_3\,. \tag{4.22}
$$

Laws of product give directly on the one hand

$$
\|a(b_{|x_3=0})\|_{\mathcal{B}^1} \lesssim \|a\|_{\mathcal{B}^1}\|b_{|x_3=0}\|_{B^1_{2,1}(\mathbb{R}^2)}\,.
$$

On the other hand, observe that

$$\begin{aligned}\Big\|a(\cdot,x_3)\int_0^{x_3}\partial_3 b(\cdot,y_3)dy_3\Big\|_{B^1_{2,1}(\mathbb{R}^2)} &\lesssim \|a(\cdot,x_3)\|_{B^1_{2,1}(\mathbb{R}^2)}\int_0^{x_3}\|\partial_3 b(\cdot,y_3)\|_{B^1_{2,1}(\mathbb{R}^2)}dy_3\\ &\leq C|x_3|\|a(\cdot,x_3)\|_{B^1_{2,1}(\mathbb{R}^2)}\|\partial_3 b\|_{L^\infty_v(B^1_{2,1}(\mathbb{R}^2_{\mathrm{h}}))}\,.\end{aligned}$$

The result follows. □

Appendix A. Some results in anisotropic Besov spaces

A.1. **Anisotropic Besov spaces.** In this section we first recall some basic facts about (anisotropic) Littlewood-Paley theory and then we prove some basic properties of anisotropic Besov spaces introduced in Definition 1.6, in particular laws of product which have used all along this text.

First let us recall the following estimates which are the generalization of the classical Bernstein's inequalities in the context of anisotropic Littlewood-Paley theory (see Lemma 6.10 of [2]) describing the action of horizontal and vertical derivatives on frequency localized distributions:

Lemma A.1. *Let (p_1,p_2,r) be in $[1,\infty]^3$ such that p_1 is less than or equal to p_2. Let m be a real number and σ_{h} (resp. σ_{v}) a smooth homogeneous function of degree m on $\mathbb{R}^2$ (resp. $\mathbb{R}$). Then we have*

$$\begin{aligned}\|\sigma_{\mathrm{h}}(D_{\mathrm{h}})\Delta^{\mathrm{h}}_k f\|_{L^{p_2}_{\mathrm{h}}L^r_{\mathrm{v}}} &\lesssim 2^{k(m+\frac{2}{p_1}-\frac{2}{p_2})}\|\Delta^{\mathrm{h}}_k f\|_{L^{p_1}_{\mathrm{h}}L^r_{\mathrm{v}}} \quad \text{and}\\ \|\sigma_{\mathrm{v}}(D_3)\Delta^{\mathrm{v}}_j f\|_{L^r_{\mathrm{h}}L^{p_2}_{\mathrm{v}}} &\lesssim 2^{j(m+\frac{1}{p_1}-\frac{1}{p_2})}\|\Delta^{\mathrm{v}}_j f\|_{L^r_{\mathrm{h}}L^{p_1}_{\mathrm{v}}}\,.\end{aligned}$$

Now let us recall the action of the heat flow on frequency localized distributions in an anisotropic context.

Lemma A.2. *For any p in $[1,\infty]$, we have*

$$\begin{aligned}\|e^{t\Delta}\Delta^{\mathrm{h}}_k\Delta^{\mathrm{v}}_j f\|_{L^p} &\lesssim e^{-ct(2^{2k}+2^{2j})}\|\Delta^{\mathrm{h}}_k\Delta^{\mathrm{v}}_j f\|_{L^p}\\ \|e^{t\Delta_{\mathrm{h}}}\Delta^{\mathrm{h}}_k\Delta^{\mathrm{v}}_j f\|_{L^p} &\lesssim e^{-ct2^{2k}}\|\Delta^{\mathrm{h}}_k\Delta^{\mathrm{v}}_j f\|_{L^p} \quad \text{and}\\ \|e^{t\partial_3^2}\Delta^{\mathrm{h}}_k\Delta^{\mathrm{v}}_j f\|_{L^p} &\lesssim e^{-ct2^{2j}}\|\Delta^{\mathrm{h}}_k\Delta^{\mathrm{v}}_j f\|_{L^p}\,.\end{aligned}$$

The proof of this lemma consists in a straightforward (omitted) modification of the proof of Lemma 2.3 of [2].

The following result was mentioned in the introduction of this article (see page 11). We refer to (3.2) and to Definition 1.13 for notations.

Lemma A.3. *Let $p\geq 2$ be given. The spaces $\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{s-1,s'}_p)$, $\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{s,s'-1}_p)$ are $\mathcal{F}^{s,s'}$ spaces, as well as the spaces $L^1(\mathbb{R}^+;\mathcal{B}^{s,s'}_p)$ and $L^1(\mathbb{R}^+;\mathcal{B}^{s+1,s'-1}_p)$.*

Proof. Let f be a function in $\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{s-1,s'}_p)$, and let us show that

$$\|L_0 f\|_{\mathcal{A}^{s,s'}_p}\lesssim\|f\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{s-1,s'}_p)}.$$

Applying Lemma A.2 gives

$$\|\Delta^{\mathrm{h}}_k\Delta^{\mathrm{v}}_j L_0 f\|_{L^p}\lesssim\int_0^t e^{-ct'(2^{2k}+2^{2j})}\|\Delta^{\mathrm{h}}_k\Delta^{\mathrm{v}}_j f(t')\|_{L^p}\,dt'$$

so there is a sequence $d_{j,k}(t')$ in the sphere of $\ell^1(\mathbb{Z}\times\mathbb{Z};L^2(\mathbb{R}^+))$ such that

$$\|\Delta^{\mathrm{h}}_k\Delta^{\mathrm{v}}_j L_0 f\|_{L^p}\lesssim\|f\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}^{s-1,s'}_p)}2^{-k(s-1)}2^{-js'}\int_0^t e^{-ct'(2^{2k}+2^{2j})}d_{j,k}(t')\,dt'\,.$$

Young's inequality in time therefore gives

$$\|\Delta_k^{\mathrm{h}}\Delta_j^{\mathrm{v}} L_0 f\|_{L^2(\mathbb{R}^+;L^p)} \lesssim \|f\|_{\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}_p^{s-1,s'})} 2^{-k(s-1)-js'} d_{j,k}\,,$$

where $d_{j,k}$ is a generic sequence in the sphere of $\ell^1(\mathbb{Z}\times\mathbb{Z})$, which proves the result in the case when f belongs to $\widetilde{L}^2(\mathbb{R}^+;\mathcal{B}_p^{s-1,s'})$. The argument is similar in the other cases. □

Now let us study laws of product.

Proposition A.4. *Let us consider $p\in[2,4[$, and let $(\sigma,\sigma',\widetilde{\sigma},\widetilde{\sigma}')$ be in $]1-4/p,-1+4/p]^4$ such that*

$$\sigma+\sigma' = \widetilde{\sigma}+\widetilde{\sigma}' \stackrel{\mathrm{def}}{=} \overline{\sigma} > 0\,.$$

If s' is in $]1/2-2/p,-1/2+2/p]$, we have

$$\|ab\|_{\mathcal{B}^{\overline{\sigma}-1,s'}} \lesssim \|a\|_{\mathcal{B}_p^{\sigma}}\|b\|_{\mathcal{B}_p^{\sigma',s'}}\,. \tag{A.1}$$

If s' is greater than $1/2$, then we have

$$\|ab\|_{\mathcal{B}^{\overline{\sigma}-1,s'}} \lesssim \|a\|_{\mathcal{B}^{\sigma}}\|b\|_{\mathcal{B}^{\sigma',s'}} + \|a\|_{\mathcal{B}^{\widetilde{\sigma}',s'}}\|b\|_{\mathcal{B}^{\widetilde{\sigma}}}\,. \tag{A.2}$$

Proof. Let us use Bony's decomposition in the vertical variable introduced in (3.33), namely

$$ab = T_a^{\mathrm{v}} b + T_b^{\mathrm{v}} a + R^{\mathrm{v}}(a,b).$$

The first two terms are almost the same (up to the interchanging of a and b). Thus we only estimate $T_a^{\mathrm{v}}b$. This is done through the following lemma.

Lemma A.5. *Let us consider $p\in[2,4[$, (σ,σ') in $]1-4/p,-1+4/p]^2$ such that $\sigma+\sigma'$ is positive, and (s,s') in $\mathbb{R}^2$. If $s\leq -1/2+2/p$, then we have*

$$\|T_a^{\mathrm{v}}b\|_{\mathcal{B}_2^{\sigma+\sigma'+1-\frac4p,s+s'+\frac12-\frac2p}} \lesssim \|a\|_{\mathcal{B}_p^{\sigma,s}}\|b\|_{\mathcal{B}_p^{\sigma',s'}}\,. \tag{A.3}$$

If $s+s'$ is positive, then we have

$$\|R^{\mathrm{v}}(a,b)\|_{\mathcal{B}_2^{\sigma+\sigma'+1-\frac4p,s+s'+\frac12-\frac2p}} \lesssim \|a\|_{\mathcal{B}_p^{\sigma,s}}\|b\|_{\mathcal{B}_p^{\sigma',s'}}\,. \tag{A.4}$$

Proof. Let us use Bony's decomposition of $T_a^{\mathrm{v}}b$ with respect to the horizontal variable.

$$\begin{aligned}
T_a^{\mathrm{v}}b &= T^{\mathrm{v}}T_a^{\mathrm{h}}b + T^{\mathrm{v}}\widetilde{T}_b^{\mathrm{h}}a + T^{\mathrm{v}}R^{\mathrm{h}}(a,b) \quad\text{with}\\
T^{\mathrm{v}}T_a^{\mathrm{h}}b &\stackrel{\mathrm{def}}{=} \sum_{j,k} S_{j-1}^{\mathrm{v}}S_{k-1}^{\mathrm{h}}a\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}}b\,,\\
T^{\mathrm{v}}\widetilde{T}_b^{\mathrm{h}}a &\stackrel{\mathrm{def}}{=} \sum_{j,k} S_{j-1}^{\mathrm{v}}\Delta_k^{\mathrm{h}}a\Delta_j^{\mathrm{v}}S_{k-1}^{\mathrm{h}}b \quad\text{and}\\
T^{\mathrm{v}}R^{\mathrm{h}}(a,b) &\stackrel{\mathrm{def}}{=} \sum_{\substack{j,k\\ -1\leq\ell\leq1}} S_{j-1}^{\mathrm{v}}\Delta_{k-\ell}^{\mathrm{h}}a\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}}b\,.
\end{aligned}$$

Following the same lines as in the proof of Proposition 3.1 (see the lines following decomposition (3.18)) we have for some large enough integer N_0

$$\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}}T^{\mathrm{v}}T_a^{\mathrm{h}}b = \sum_{\substack{|j'-j|\leq N_0\\ |k'-k|\leq N_0}} \Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}}\big(S_{j'-1}^{\mathrm{v}}S_{k'-1}^{\mathrm{h}}a\Delta_{j'}^{\mathrm{v}}\Delta_{k'}^{\mathrm{h}}b\big).$$

By definition of the $\mathcal{B}_p^{\sigma,s'}$ norms, this gives, denoting

$$\frac1p+\frac1{\bar p}=\frac12$$

$$
\begin{aligned}
\mathcal{T}^{\mathrm{v,h}}_{j,k} &\stackrel{\mathrm{def}}{=} 2^{j(s+s'+\frac12-\frac2p)+k(\sigma+\sigma'+1-\frac4p)}\|\Delta^{\mathrm{v}}_j\Delta^{\mathrm{h}}_k T^{\mathrm{v}}T^{\mathrm{h}}_a b\|_{L^2}\\
&\lesssim \sum_{\substack{|j'-j|\leq N_0\\ |k'-k|\leq N_0}} 2^{-(j'-j)(s+s'+\frac12-\frac2p)-(k'-k)(\sigma+\sigma'+1-\frac4p)}\\
&\qquad\qquad \times 2^{j'(s+\frac12-\frac2p)+k'(\sigma+1-\frac4p)}\|S^{\mathrm{v}}_{j'-1}S^{\mathrm{h}}_{k'-1}a\|_{L^{\bar p}}2^{j's'+k'\sigma'}\|\Delta^{\mathrm{v}}_{j'}\Delta^{\mathrm{h}}_{k'}b\|_{L^p}\\
&\lesssim \|b\|_{\mathcal{B}^{\sigma',s'}_p}\sum_{\substack{|j'-j|\leq N_0\\ |k'-k|\leq N_0}} 2^{-(j'-j)(s+s'+\frac12-\frac2p)-(k'-k)(\sigma+\sigma'+1-\frac4p)}\\
&\qquad\qquad\qquad\qquad \times d_{j',k'}2^{j'(s+\frac12-\frac2p)+k'(\sigma+1-\frac4p)}\|S^{\mathrm{v}}_{j'-1}S^{\mathrm{h}}_{k'-1}a\|_{L^{\bar p}}
\end{aligned}
$$

where, as in all that follows, $(d_{j,k})_{(j,k)\in\mathbb{Z}^2}$ lies on the sphere of $\ell^1(\mathbb{Z}^2)$. Using anisotropic Bernstein inequalities given by Lemma A.1 and the definition of the $\mathcal{B}^{\sigma,s}_p$ norm, we get

$$
\begin{aligned}
2^{j'(s+\frac12-\frac2p)+k'(\sigma+1-\frac4p)}\|S^{\mathrm{v}}_{j'-1}S^{\mathrm{h}}_{k'-1}a\|_{L^{\bar p}} &\lesssim \sum_{\substack{j''\leq j'-2\\ k''\leq k'-2}} 2^{(j'-j'')(s+\frac12-\frac2p)+(k'-k'')(\sigma+1-\frac4p)}\\
&\qquad\quad \times 2^{j''(s+\frac12-\frac2p)+k''(\sigma+1-\frac4p)}\|\Delta^{\mathrm{v}}_{j''}\Delta^{\mathrm{h}}_{k''}a\|_{L^{\bar p}}\\
&\lesssim \sum_{\substack{j''\leq j'-2\\ k''\leq k'-2}} 2^{(j'-j'')(s+\frac12-\frac2p)+(k'-k'')(\sigma+1-\frac4p)}\\
&\qquad\quad \times 2^{j''s+k''\sigma}\|\Delta^{\mathrm{v}}_{j''}\Delta^{\mathrm{h}}_{k''}a\|_{L^p}\\
&\lesssim \|a\|_{\mathcal{B}^{\sigma,s}_p}\sum_{\substack{j''\leq j'-2\\ k''\leq k'-2}} 2^{(j'-j'')(s+\frac12-\frac2p)+(k'-k'')(\sigma+1-\frac4p)}d_{j''k''}.
\end{aligned}
$$

As $s\leq -1/2+2/p$ and $\sigma\leq -1+4/p$, we get

$$
2^{j'(s+\frac12-\frac2p)+k'(\sigma+1-\frac4p)}\|S^{\mathrm{v}}_{j'-1}S^{\mathrm{h}}_{k'-1}a\|_{L^{\bar p}}\lesssim \|a\|_{\mathcal{B}^{\sigma,s}_p}\,.
$$

Young's inequality on series leads to

$$
\|T^{\mathrm{v}}T^{\mathrm{h}}_a b\|_{\mathcal{B}_2^{\sigma+\sigma'+1-\frac4p,s+s'+\frac12-\frac2p}}\lesssim \|a\|_{\mathcal{B}^{\sigma,s}_p}\|b\|_{\mathcal{B}^{\sigma',s'}_p}\,. \tag{A.5}
$$

Following exactly the same lines, we can prove

$$
\|T^{\mathrm{v}}\widetilde{T}^{\mathrm{h}}_b a\|_{\mathcal{B}_2^{\sigma+\sigma'+1-\frac4p,s+s'+\frac12-\frac2p}}\lesssim \|a\|_{\mathcal{B}^{\sigma,s}_p}\|b\|_{\mathcal{B}^{\sigma',s'}_p}\,. \tag{A.6}
$$

The estimate of $T^{\mathrm{v}}R^{\mathrm{h}}(a,b)$ is a little bit different. Let us write that

$$
\Delta^{\mathrm{v}}_j\Delta^{\mathrm{h}}_k T^{\mathrm{v}}R^{\mathrm{h}}(a,b)=\sum_{\substack{j',k'\\ -1\leq\ell\leq 1}}\Delta^{\mathrm{v}}_j\Delta^{\mathrm{h}}_k\big(S^{\mathrm{v}}_{j'-1}\Delta^{\mathrm{h}}_{k'-\ell}a\Delta^{\mathrm{v}}_{j'}\Delta^{\mathrm{h}}_{k'}b\big)\,.
$$

Arguing as in the proof of Proposition 3.1 we have for some large enough integer N_0

$$
\Delta^{\mathrm{v}}_j\Delta^{\mathrm{h}}_k T^{\mathrm{v}}R^{\mathrm{h}}(a,b)=\sum_{\substack{|j'-j|\leq N_0\\ k'\geq k-N_0}}\sum_{-1\leq\ell\leq 1}\Delta^{\mathrm{v}}_j\Delta^{\mathrm{h}}_k\big(S^{\mathrm{v}}_{j'-1}\Delta^{\mathrm{h}}_{k'-\ell}a\Delta^{\mathrm{v}}_{j'}\Delta^{\mathrm{h}}_{k'}b\big).
$$

Anisotropic Bernstein inequalities given by Lemma A.1 imply that

$$\begin{aligned}\big\|\Delta_j^{\rm v}\Delta_k^{\rm h}\big(S_{j'-1}^{\rm v}\Delta_{k'-\ell}^{\rm h}a\Delta_{j'}^{\rm v}\Delta_{k'}^{\rm h}b\big)\big\|_{L^2} &\lesssim 2^{2k(\frac{2}{p}-\frac{1}{2})}\big\|S_{j'-1}^{\rm v}\Delta_{k'-\ell}^{\rm h}a\Delta_{j'}^{\rm v}\Delta_{k'}^{\rm h}b\big\|_{L_{\rm h}^{\frac{p}{2}}(L_{\rm v}^2)}\\ &\lesssim 2^{2k(\frac{2}{p}-\frac{1}{2})}\|S_{j'-1}^{\rm v}\Delta_{k'-\ell}^{\rm h}a\|_{L_{\rm h}^p(L_{\rm v}^{\bar p})}\|\Delta_{j'}^{\rm v}\Delta_{k'}^{\rm h}b\|_{L^p}\,.\end{aligned}$$

Thus we infer that

$$\begin{aligned}&2^{k(\sigma+\sigma'+1-\frac{4}{p})+j(s+s'+\frac{1}{2}-\frac{2}{p})}\|\Delta_j^{\rm v}\Delta_k^{\rm h}T^{\rm v}R^{\rm h}(a,b)\|_{L^2}\\ &\qquad\lesssim\sum_{\substack{|j'-j|\leq N_0\\ k'\geq k-N_0}}\sum_{-1\leq\ell\leq1}2^{-(k'-k)(\sigma+\sigma')-(j-j')(s+s'+\frac{1}{2}-\frac{2}{p})}\\ &\qquad\qquad\times 2^{j'(s+\frac{1}{2}-\frac{2}{p})+k'\sigma}\|S_{j'-1}^{\rm v}\Delta_{k'-\ell}^{\rm h}a\|_{L_{\rm h}^p(L_{\rm v}^{\bar p})}2^{k'\sigma'+j's'}\|\Delta_{j'}^{\rm v}\Delta_{k'}^{\rm h}b\|_{L^p}\,.\end{aligned}$$

Using again anisotropic Bernstein inequalities and by definition of the $\mathcal{B}^{\sigma,s}$ norm, we get

$$\begin{aligned}2^{j'(s+\frac{1}{2}-\frac{2}{p})+k'\sigma}\|S_{j'-1}^{\rm v}\Delta_{k'-\ell}^{\rm h}a\|_{L_{\rm h}^p(L_{\rm v}^{\bar p})} &\lesssim \sum_{j''\leq j'-2}2^{(j'-j'')(s+\frac{1}{2}-\frac{2}{p})}2^{j''s+k'\sigma}\|\Delta_{j''}^{\rm v}\Delta_{k'-\ell}^{\rm h}a\|_{L^p}\\ &\lesssim \|a\|_{\mathcal{B}_p^{\sigma,s}}\sum_{j''\leq j'-2}2^{(j'-j'')(s+\frac{1}{2}-\frac{2}{p})}d_{j'',k'}\,.\end{aligned}$$

As s is less than or equal to $-\frac{1}{2}+\frac{2}{p}$, we get

$$2^{j'(s+\frac{1}{2}-\frac{2}{p})+k'\sigma}\|S_{j'-1}^{\rm v}\Delta_{k'-\ell}^{\rm h}a\|_{L_{\rm h}^p(L_{\rm v}^{\bar p})}\lesssim\|a\|_{\mathcal{B}_p^{\sigma,s}}\,.$$

By definition of the $\mathcal{B}_p^{\sigma',s'}$ norm, this gives

$$\begin{aligned}2^{k(\sigma+\sigma'+1-\frac{4}{p})+j(s+s'+\frac{1}{2}-\frac{2}{p})}\|\Delta_j^{\rm v}\Delta_k^{\rm h}T^{\rm v}R^{\rm h}(a,b)\|_{L^2} &\lesssim\|a\|_{\mathcal{B}_p^{\sigma,s}}\|b\|_{\mathcal{B}_p^{\sigma',s'}}\\ &\qquad\times\sum_{\substack{|j'-j|\leq N_0\\ k'\geq k-N_0}}\sum_{-1\leq\ell\leq1}2^{-(k'-k)(\sigma+\sigma')-(j'-j)(s+s'+\frac{1}{2}-\frac{2}{p})}d_{j',k'}\,.\end{aligned}$$

As $\sigma+\sigma'$ is positive, we get that

$$2^{k(\sigma+\sigma'+1-\frac{4}{p})+j(s+s'+\frac{1}{2}-\frac{2}{p})}\|\Delta_j^{\rm v}\Delta_k^{\rm h}T^{\rm v}R^{\rm h}(a,b)\|_{L^2}\lesssim d_{j,k}\|a\|_{\mathcal{B}_p^{\sigma,s}}\|b\|_{\mathcal{B}_p^{\sigma',s'}}\,.$$

Together with (A.5) and (A.6) this concludes the proof of Inequality (A.3).

In order to prove Inequality (A.4), let us use again the horizontal Bony decomposition. Defining

$$\widetilde{\Delta}_j^{\rm v}\ (\text{resp. }\widetilde{\Delta}_k^{\rm h})\stackrel{\rm def}{=}\sum_{\ell=-1}^{1}\Delta_{j-\ell}^{\rm v}\ (\text{resp. }\Delta_{k-\ell}^{\rm h})$$

let us write that

$$\begin{aligned}R_a^{\rm v}b &= R^{\rm v}T_a^{\rm h}b+R^{\rm v}T_b^{\rm h}a+R^{\rm v}R^{\rm h}(a,b)\quad\text{with}\\ R^{\rm v}T_a^{\rm h}b &\stackrel{\rm def}{=} \sum_{j,k}\widetilde{\Delta}_j^{\rm v}S_{k-1}^{\rm h}a\Delta_j^{\rm v}\Delta_k^{\rm h}b\quad\text{and}\\ R^{\rm v}R^{\rm h}(a,b) &\stackrel{\rm def}{=} \sum_{j,k}\widetilde{\Delta}_j^{\rm v}\widetilde{\Delta}_{k-\ell}^{\rm h}a\Delta_j^{\rm v}\Delta_k^{\rm h}b\,.\end{aligned}$$

We have for a large enough integer N_0,

$$\Delta_j^{\rm v}\Delta_k^{\rm h}R^{\rm v}T_a^{\rm h}b=\sum_{\substack{j'\geq j-N_0\\ |k'-k|\leq N_0}}\Delta_j^{\rm v}\Delta_k^{\rm h}\bigl(\widetilde\Delta_{j'}^{\rm v}S_{k'-1}^{\rm h}a\Delta_{j'}^{\rm v}\Delta_{k'}^{\rm h}b\bigr).$$

Using anisotropic Bernstein inequalities, this gives by definition of the $\mathcal{B}_p^{\sigma,s'}$ norm,

$$\begin{aligned}\mathcal{RT}_{j,k}^{\rm v,h}(a,b) &\stackrel{\rm def}{=} 2^{j(s+s'+\frac12-\frac2p)+k(\sigma+\sigma'+1-\frac4p)}\|\Delta_j^{\rm v}\Delta_k^{\rm h}R^{\rm v}T_a^{\rm h}b\|_{L^2}\\
&\lesssim 2^{j(s+s')+k(\sigma+\sigma'+1-\frac4p)}\|\Delta_j^{\rm v}\Delta_k^{\rm h}R^{\rm v}T_a^{\rm h}b\|_{L^2_{\rm h}(L^{\frac p2}_{\rm v})}\\
&\lesssim \sum_{\substack{j'\geq j-N_0\\ |k'-k|\leq N_0}} 2^{-(j'-j)(s+s')-(k'-k)(\sigma+\sigma'+1-\frac4p)}\\
&\qquad\qquad \times 2^{j's+k'(\sigma+1-\frac4p)}\|\widetilde\Delta_{j'}^{\rm v}S_{k'-1}^{\rm h}a\|_{L^{\bar p}_{\rm h}(L^p_{\rm v})}2^{j's'+k'\sigma'}\|\Delta_{j'}^{\rm v}\Delta_{k'}^{\rm h}b\|_{L^p}\\
&\lesssim \|b\|_{\mathcal{B}_p^{\sigma',s'}}\sum_{\substack{j'\geq j-N_0\\ |k'-k|\leq N_0}} 2^{-(j'-j)(s+s')-(k'-k)(\sigma+\sigma'+1-\frac4p)}\\
&\qquad\qquad \times d_{j',k'}2^{j's+k'(\sigma+1-\frac4p)}\|\widetilde\Delta_{j'}^{\rm v}S_{k'-1}^{\rm h}a\|_{L^{\bar p}_{\rm h}(L^p_{\rm v})}\,.\end{aligned}$$

Using anisotropic Bernstein inequalities and the definition of the $\mathcal{B}^{\sigma,s}$ norm, we get

$$\begin{aligned}2^{j's+k'(\sigma+1-\frac4p)}\|\widetilde\Delta_{j'}^{\rm v}S_{k'-1}^{\rm h}a\|_{L^{\bar p}_{\rm h}(L^p_{\rm v})} &\lesssim \sum_{\substack{j'-1\leq j''\leq j'+1\\ k''\leq k'-2}} 2^{(j'-j'')s+(k'-k'')(\sigma+1-\frac4p)}\\
&\qquad\qquad \times 2^{j''s+k''(\sigma+1-\frac4p)}\|\Delta_{j''}^{\rm v}\Delta_{k''}^{\rm h}a\|_{L^{\bar p}_{\rm h}(L^p_{\rm v})}\\
&\lesssim \sum_{\substack{j'-1\leq j''\leq j'+1\\ k''\leq k'-2}} 2^{(j'-j'')s+(k'-k'')(\sigma+1-\frac4p)}\\
&\qquad\qquad \times 2^{j''s+k''\sigma}\|\Delta_{j''}^{\rm v}\Delta_{k''}^{\rm h}a\|_{L^p}\\
&\lesssim \|a\|_{\mathcal{B}_p^{\sigma,s}}\sum_{\substack{j'-1\leq j''\leq j'+1\\ k''\leq k'-2}} 2^{(j'-j'')s+(k'-k'')(\sigma+1-\frac4p)}d_{j''k''}\,.\end{aligned}$$

As σ is less than or equal to $-1+4/p$, we get

$$2^{j's+k'(\sigma+1-\frac4p)}\|\widetilde\Delta_{j'}^{\rm v}S_{k'-1}^{\rm h}a\|_{L^{\bar p}}\lesssim\|a\|_{\mathcal{B}_p^{\sigma,s}}\,.$$

Since $s+s'$ is positive, Young's inequality on series leads to

$$\|R^{\rm v}T_a^{\rm h}b\|_{\mathcal{B}_2^{\sigma+\sigma'+1-\frac4p,s+s'+\frac12-\frac2p}}\lesssim\|a\|_{\mathcal{B}_p^{\sigma,s}}\|b\|_{\mathcal{B}_p^{\sigma',s'}}\,. \tag{A.7}$$

By symmetry, we get

$$\|R^{\rm v}T_b^{\rm h}a\|_{\mathcal{B}_2^{\sigma+\sigma'+1-\frac4p,s+s'+\frac12-\frac2p}}\lesssim\|a\|_{\mathcal{B}_p^{\sigma,s}}\|b\|_{\mathcal{B}_p^{\sigma',s'}}\,. \tag{A.8}$$

The estimate of $R^{\rm v}R^{\rm h}(a,b)$ is a little bit different. Arguing as in the proof of Proposition 3.1, we obtain

$$\Delta_j^{\rm v}\Delta_k^{\rm h}R^{\rm v}R^{\rm h}(a,b)=\sum_{\substack{j'>j-N_0\\ k'\geq k-N_0}}\Delta_j^{\rm v}\Delta_k^{\rm h}\bigl(\widetilde\Delta_{j'}^{\rm v}\widetilde\Delta_{k'-\ell}^{\rm h}a\Delta_{j'}^{\rm v}\Delta_{k'}^{\rm h}b\bigr).$$

Anisotropic Bernstein inequalities given by Lemma A.1 imply that

$$\begin{aligned}\big\|\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}}\big(\widetilde\Delta_{j'}^{\mathrm{v}}\widetilde\Delta_{k'}^{\mathrm{h}}a\Delta_{j'}^{\mathrm{v}}\Delta_{k'}^{\mathrm{h}}b\big)\big\|_{L^2} &\lesssim 2^{(2k+j)(\frac{2}{p}-\frac12)}\big\|\widetilde\Delta_{j'}^{\mathrm{v}}\widetilde\Delta_{k'}^{\mathrm{h}}a\Delta_{j'}^{\mathrm{v}}\Delta_{k'}^{\mathrm{h}}b\big\|_{L^{\frac{p}{2}}}\\ &\lesssim 2^{(2k+j)(\frac{2}{p}-\frac12)}\big\|\widetilde\Delta_{j'}^{\mathrm{v}}\widetilde\Delta_{k'}^{\mathrm{h}}a\|_{L^p}\|\Delta_{j'}^{\mathrm{v}}\Delta_{k'}^{\mathrm{h}}b\big\|_{L^p}\,.\end{aligned}$$

Thus we infer that

$$\begin{aligned}2^{k(\sigma+\sigma'+1-\frac4p)+j(s+s'+\frac12-\frac2p)}\|\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}}R^{\mathrm{v}}R^{\mathrm{h}}(a,b)\|_{L^2} &\lesssim \sum_{\substack{j'>j-N_0\\ k'\geq k-N_0}} 2^{-(k'-k)(\sigma+\sigma')-(j'-j)(s+s')}\\ &\quad\times 2^{j's+k'\sigma}\|\widetilde\Delta_{j'}^{\mathrm{v}}\Delta_{k'-\ell}^{\mathrm{h}}a\|_{L^p}2^{k'\sigma'+j's'}\|\Delta_{j'}^{\mathrm{v}}\Delta_{k'}^{\mathrm{h}}b\|_{L^p}\,.\end{aligned}$$

By definition of the $\mathcal{B}_p^{\sigma',s'}$ norm, this gives

$$\begin{aligned}2^{k(\sigma+\sigma'+1-\frac4p)+j(s+s'+\frac12-\frac2p)}\|\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}}R^{\mathrm{v}}R^{\mathrm{h}}(a,b)\|_{L^2} &\lesssim \|a\|_{\mathcal{B}_p^{\sigma,s}}\|b\|_{\mathcal{B}_p^{\sigma',s'}}\\ &\quad\times\sum_{\substack{j'>j-N_0\\ k'\geq k-N_0}} 2^{-(k'-k)(\sigma+\sigma')-(j'-j)(s+s')}d_{j',k'}\,.\end{aligned}$$

As $\sigma+\sigma'$ and $s+s'$ are positive, we get that

$$2^{k(\sigma+\sigma'+1-\frac4p)+j(s+s'+\frac12-\frac2p)}\|\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}}R^{\mathrm{v}}R^{\mathrm{h}}(a,b)\|_{L^2}\lesssim d_{j,k}\|a\|_{\mathcal{B}_p^{\sigma,s}}\|b\|_{\mathcal{B}_p^{\sigma',s'}}\,.$$

Together with (A.7) and (A.8) this concludes the proof of Inequality (A.3). □

In order to conclude the proof of Proposition A.4, it is enough to apply Lemma A.5 with (σ,σ') to $T_a^{\mathrm{v}}b$ and with $(\widetilde\sigma',\widetilde\sigma)$ to $\widetilde T_b^{\mathrm{v}}a$. □

Now let us prove laws of product in the case when one of the functions does not depend on the vertical variable x_3. We have the following proposition.

Proposition A.6. *Let a be in $B^\sigma_{2,1}(\mathbb{R}^2)$ and b in $\mathcal{B}^{s,s'}$ with (s,σ) in $]-1,1]^2$ such that $s+\sigma$ is positive and s' greater than or equal to $1/2$. We have*

$$\|ab\|_{\mathcal{B}^{s+\sigma-1,s'}}\lesssim\|a\|_{B^\sigma_{2,1}(\mathbb{R}^2_{\mathrm{h}})}\|b\|_{\mathcal{B}^{s,s'}}\,. \tag{A.9}$$

Proof. Using Bony's decomposition in the horizontal variable gives

$$ab=T_a^{\mathrm{h}}b+T_b^{\mathrm{h}}a+R^{\mathrm{h}}(a,b).$$

As a does not depend on the vertical variable, we have

$$\Delta_j^{\mathrm{v}}T_a^{\mathrm{h}}b=T_a^{\mathrm{h}}\Delta_j^{\mathrm{v}}b\,,\ \Delta_j^{\mathrm{v}}T_b^{\mathrm{h}}a=T_{\Delta_j^{\mathrm{v}}b}^{\mathrm{h}}a\quad\text{and}\quad\Delta_j^{\mathrm{v}}R^{\mathrm{h}}(a,b)=R^{\mathrm{h}}(a,\Delta_j^{\mathrm{v}}b).$$

Then, the result follows from the classical proofs of mappings of paraproduct and remainder operators (see for instance Theorem 2.47 and Theorem 2.52 of [2]). We give a short sketch of the proof for the reader's convenience in the case of T^{h}. Let us write

$$\begin{aligned}2^{k(s+\sigma-1)+js'}\|\Delta_j^{\mathrm{v}}\Delta_k^{\mathrm{h}}T_a^{\mathrm{h}}b\|_{L^2} &\lesssim \sum_{|k'-k|\leq N_0}2^{k'(\sigma-1)}\|S_{k'-1}^{\mathrm{h}}a\|_{L^\infty_{\mathrm{h}}}2^{k's+js'}\|\Delta_j^{\mathrm{v}}\Delta_{k'}^{\mathrm{h}}b\|_{L^2}\\ &\lesssim \|b\|_{\mathcal{B}^{s,s'}}\sum_{|k'-k|\leq N_0}2^{k'(\sigma-1)}\|S_{k'-1}^{\mathrm{h}}a\|_{L^\infty_{\mathrm{h}}}d_{k',j}\,.\end{aligned}$$

Bernstein inequalities imply that

$$\begin{aligned}2^{-k(1-\sigma)}\|S_{k-1}^{\mathrm{h}}a\|_{L^\infty_{\mathrm{h}}} &\lesssim \sum_{k'\leq k-1}2^{(k'-k)(1-\sigma)}2^{k'\sigma}\|\Delta_{k'}^{\mathrm{h}}a\|_{L^2_{\mathrm{h}}}\\ &\lesssim \|a\|_{B^\sigma_{2,1}(\mathbb{R}^2_{\mathrm{h}})}\sum_{k'\leq k-1}2^{(k'-k)(1-\sigma)}d_{k'}\,.\end{aligned}$$

This gives, with no restriction on the parameter s and with σ less than or equal to 1 and s' greater than or equal to 1/2,

$$\|T^{\mathrm{h}}_a b\|_{\mathcal{B}^{s+\sigma-1,s'}} \lesssim \|a\|_{B^\sigma_{2,1}(\mathbb{R}^2_{\mathrm{h}})} \|b\|_{\mathcal{B}^{s,s'}} \,. \tag{A.10}$$

For the other (horizontal) paraproduct term, let us write

$$\begin{aligned} 2^{k(s+\sigma-1)+js'} \|\Delta^{\mathrm{v}}_j \Delta^{\mathrm{h}}_k T^{\mathrm{h}}_b a\|_{L^2} &\lesssim \sum_{|k'-k|\leq N_0} 2^{k'(s-1)+js'} \|S^{\mathrm{h}}_{k'-1}\Delta^{\mathrm{v}}_j b\|_{L^\infty_{\mathrm{h}}(L^2_{\mathrm{v}})} 2^{k'\sigma} \|\Delta^{\mathrm{h}}_{k'} a\|_{L^2_{\mathrm{h}}} \\ &\lesssim \|a\|_{B^\sigma_{2,1}(\mathbb{R}^2)} \sum_{|k'-k|\leq N_0} 2^{k'(s-1)+js'} \|S^{\mathrm{h}}_{k'-1}\Delta^{\mathrm{v}}_j b\|_{L^\infty_{\mathrm{h}}(L^2_{\mathrm{v}})} d_{k'} \,. \end{aligned} \tag{A.11}$$

Using Lemma A.1, we get

$$\begin{aligned} 2^{-k(1-s)+js'} \|S^{\mathrm{h}}_{k-1}\Delta^{\mathrm{v}}_j b\|_{L^\infty_{\mathrm{h}}(L^2_{\mathrm{v}})} &\lesssim \sum_{k'\leq k-1} 2^{(k'-k)(1-s)} 2^{-k'(1-s)+js'} \|\Delta^{\mathrm{h}}_{k'}\Delta^{\mathrm{v}}_j b\|_{L^\infty_{\mathrm{h}}(L^2_{\mathrm{v}})} \\ &\lesssim \sum_{k'\leq k-1} 2^{(k'-k)(1-s)} 2^{k's+js'} \|\Delta^{\mathrm{h}}_{k'}\Delta^{\mathrm{v}}_j b\|_{L^2} \,. \end{aligned}$$

By definition of the $\mathcal{B}^{s,s'}$ norm and using the fact that $s \leq 1$, we infer that

$$2^{js'-k(1-s)} \|S^{\mathrm{h}}_{k-1}\Delta^{\mathrm{v}}_j b\|_{L^\infty_{\mathrm{h}}(L^2_{\mathrm{v}})} \leq d_j \|b\|_{\mathcal{B}^{s,s'}} \,.$$

Together with (A.11), this gives

$$\|T^{\mathrm{h}}_b a\|_{\mathcal{B}^{s+\sigma-1,s'}} \lesssim \|a\|_{B^\sigma_{2,1}(\mathbb{R}^2)} \|b\|_{\mathcal{B}^{s,s'}} \,. \tag{A.12}$$

Now let us study the (horizontal) remainder term. Using Lemma A.1, let us write that

$$\begin{aligned} 2^{k(s+\sigma-1)+js'} \|\Delta^{\mathrm{v}}_j \Delta^{\mathrm{h}}_k R^{\mathrm{h}}(a,b)\|_{L^2} &\lesssim 2^{k(s+\sigma)+js'} \|\Delta^{\mathrm{v}}_j \Delta^{\mathrm{h}}_k R^{\mathrm{h}}(a,b)\|_{L^2_{\mathrm{v}}(L^1_{\mathrm{h}})} \\ &\lesssim \sum_{k'\geq k-N_0} 2^{-(k'-k)(s+\sigma)} 2^{k'\sigma} \|\Delta^{\mathrm{h}}_{k'} a\|_{L^2_{\mathrm{h}}} 2^{k's+js'} \|\Delta^{\mathrm{v}}_j \Delta^{\mathrm{h}}_{k'} b\|_{L^2} \,. \end{aligned}$$

By definition of the $B^\sigma_{2,1}(\mathbb{R}^2)$ and $\mathcal{B}^{s,s'}$ norms, we get

$$2^{k(s+\sigma-1)+js'} \|\Delta^{\mathrm{v}}_j \Delta^{\mathrm{h}}_k R^{\mathrm{h}}(a,b)\|_{L^2} \lesssim \|a\|_{B^\sigma_{2,1}(\mathbb{R}^2)} \|b\|_{\mathcal{B}^{s,s'}} d_j \sum_{k'\geq k-N_0} 2^{-(k'-k)(s+\sigma)} d_{k'} \,.$$

Together with (A.10) and (A.12), this gives the result thanks to the fact that $s+\sigma$ is positive. Proposition A.6 is proved. □

A.2. **Proof of Proposition 1.14.** The proof of Proposition 1.14 is reminiscent of that of Lemma 3.6, and we shall be using arguments of that proof here.

Let us recall that we want to prove that if U is in $L^2(\mathbb{R}^+;\mathcal{B}^1_p)$, if u_0 is in $\mathcal{B}^0_p$ and f in $\mathcal{F}^0_p$, such that

$$\|u_0\|_{\mathcal{B}^0_p} + \|f\|_{\mathcal{F}^0_p} \leq \frac{1}{C_0} \exp\Bigl(-C_0 \int_0^\infty \|U(t)\|^2_{\mathcal{B}^1_p} dt\Bigr) \,, \tag{A.13}$$

then the problem

$$(NS_U) \qquad \left\{ \begin{array}{c} \partial_t u + \mathrm{div}(u\otimes u + u\otimes U + U\otimes u) - \Delta u = -\nabla p + f \\ \mathrm{div}\, u = 0 \quad \text{and} \quad u_{|t=0} = u_0 \end{array} \right.$$

has a unique global solution in $L^2(\mathbb{R}^+;\mathcal{B}^1_p)$ which satisfies

$$\|u\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)} \lesssim \|u_0\|_{\mathcal{B}^0_p} + \|f\|_{\mathcal{F}^0_p} \,.$$

Let us first prove that the system (NS_U) has a unique solution in $L^2([0,T];\mathcal{B}^1_p)$ for some small enough T. Let us introduce some bilinear operators which distinguish the horizontal derivatives from the vertical one, namely for ℓ belonging to $\{1,2,3\}$,

$$\mathcal{Q}_{\rm h}(u,w)^\ell \stackrel{\rm def}{=} \mathrm{div}_{\rm h}(w^\ell u^{\rm h}) \quad\text{and}\quad \mathcal{Q}_{\rm v}(u,w)^\ell \stackrel{\rm def}{=} \partial_3(w^\ell u^3). \tag{A.14}$$

Then we define $B_{{\rm h},\tau} \stackrel{\rm def}{=} L_\tau \mathcal{Q}_{\rm h}$ and $B_{{\rm v},\tau} \stackrel{\rm def}{=} L_\tau \mathcal{Q}_{\rm v}$ where L_τ is defined in Definition 1.13. It is obvious that solving (NS_U) is equivalent to solving

$$u = e^{t\Delta}u_0 + L_0 f + B_{{\rm h},0}(u,u) + B_{{\rm v},0}(u,u) + B_{{\rm h},0}(U,u) + B_{{\rm v},0}(U,u) + B_{{\rm h},0}(u,U) + B_{{\rm v},0}(u,U)\,.$$

Following an idea introduced by G. Gui, J. Huang and P. Zhang in [29], let us define

$$\mathcal{L}_0 \stackrel{\rm def}{=} e^{t\Delta}u_0 + L_0 f$$

and look for the solution u under the form $u = \mathcal{L}_0 + \rho$. As the horizontal and the vertical derivative are not treated exactly in the same way, let us decompose ρ into $\rho = \rho_{\rm h} + \rho_{\rm v}$ with

$$\begin{aligned}
&\rho_{\rm h} \stackrel{\rm def}{=} B_{{\rm h},0}(\rho,\rho) + B_{{\rm h},0}(\mathcal{L}_0+U,\rho) + B_{{\rm h},0}(\rho,\mathcal{L}_0+U) + F_{\rm h}\,,\\
&\rho_{\rm v} \stackrel{\rm def}{=} B_{{\rm v},0}(\rho,\rho) + B_{{\rm v},0}(\mathcal{L}_0+U,\rho) + B_{{\rm v},0}(\rho,\mathcal{L}_0+U) + F_{\rm v} \quad\text{with}\\
&F_{\rm h} \stackrel{\rm def}{=} B_{{\rm h},0}(\mathcal{L}_0,\mathcal{L}_0) + B_{{\rm h},0}(\mathcal{L}_0,U) + B_{{\rm h},0}(U,\mathcal{L}_0) \quad\text{and}\\
&F_{\rm v} \stackrel{\rm def}{=} B_{{\rm v},0}(\mathcal{L}_0,\mathcal{L}_0) + B_{{\rm v},0}(\mathcal{L}_0,U) + B_{{\rm v},0}(U,\mathcal{L}_0)\,.
\end{aligned} \tag{A.15}$$

The main lemma is the following.

Lemma A.7. *For any subinterval $I=[a,b]$ of $\mathbb{R}^+$, we have*

$$\begin{aligned}
\|B_{{\rm h},a}(u,w)\|_{L^\infty(I;\mathcal{B}^0_p)} &+ \|B_{{\rm h},a}(u,w)\|_{L^1(I;\mathcal{B}^2_p\cap B^{\frac2p,1+\frac1p}_{p,1})}\\
&+ \|B_{{\rm v},a}(u,w)\|_{L^\infty(I;B^{\frac2p,-1+\frac1p}_{p,1})} + \|B_{{\rm v},a}(u,w)\|_{L^1(I;\mathcal{B}^2_p\cap B^{\frac2p,1+\frac1p}_{p,1})}\\
&\lesssim \|u\|_{L^2(I;\mathcal{B}^1_p)}\|w\|_{L^2(I;\mathcal{B}^1_p)}\,.
\end{aligned}$$

v

Proof. As $\mathcal{B}^1_p$ is an algebra and using Lemma A.1, we get

$$\begin{aligned}
\mathcal{Q}_{j,k}(u,w)(t) &\stackrel{\rm def}{=} 2^{k(-1+\frac2p)}2^{\frac jp}\|\Delta^{\rm v}_j\Delta^{\rm h}_k\mathcal{Q}_{\rm h}(u,w)(t)\|_{L^p} + 2^{\frac{2k}{p}+j(-1+\frac1p)}\|\Delta^{\rm v}_j\Delta^{\rm h}_k\mathcal{Q}_{\rm v}(u,w)(t)\|_{L^p}\\
&\lesssim d_{j,k}(t)\|u(t)\|_{\mathcal{B}^1_p}\|w(t)\|_{\mathcal{B}^1_p}\,,
\end{aligned}$$

where as usual we have denoted by $d_{j,k}(t)$ a sequence in the unit sphere of $\ell^1(\mathbb{Z}^2)$ for each t. Lemma A.2 implies that, for any t in $[a,b]$, we have with the notation of Definition 1.13

$$\begin{aligned}
\mathcal{L}_{a,j,k}(u,w)(t) &\stackrel{\rm def}{=} 2^{k(-1+\frac2p)}2^{\frac jp}\|L_a\Delta^{\rm v}_j\Delta^{\rm h}_k\mathcal{Q}_{\rm h}(u,w)(t)\|_{L^p}\\
&\qquad\qquad + 2^{\frac{2k}{p}+j(-1+\frac1p)}\|L_a\Delta^{\rm v}_j\Delta^{\rm h}_k\mathcal{Q}_{\rm v}(u,w)(t)\|_{L^p}\\
&\lesssim \int_a^t d_{j,k}(t')e^{-c2^{(2k+2j)}(t-t')}\|u(t')\|_{\mathcal{B}^1_p}\|w(t')\|_{\mathcal{B}^1_p}dt'\,.
\end{aligned}$$

Convolution inequalities imply that

$$\|\mathcal{L}_{a,j,k}(u,w)\|_{L^\infty(I;L^p)} + c2^{2k+2j}\|\mathcal{L}_{a,j,k}(u,w)\|_{L^1(I;L^p)} \lesssim \int_I d_{j,k}(t)\|u(t)\|_{\mathcal{B}^1_p}\|w(t)\|_{\mathcal{B}^1_p}dt\,.$$

This concludes the proof of Lemma A.7. □

As we have by interpolation,

$$\|a\|_{\mathcal{B}^1_p} \leq \|a\|^{\frac12}_{\mathcal{B}^0_p}\|a\|^{\frac12}_{\mathcal{B}^2_p} \quad\text{and}\quad \|a\|_{\mathcal{B}^1_p} \leq \|a\|^{\frac12}_{B^{\frac2p,-1+\frac1p}_{p,1}}\|a\|^{\frac12}_{B^{\frac2p,1+\frac1p}_{p,1}}\,, \tag{A.16}$$

we infer that the bilinear maps $B_{\mathrm{h},a}$ and $B_{\mathrm{v},a}$ map $L^2(I;\mathcal{B}^1_p)\times L^2(I;\mathcal{B}^1_p)$ into $L^2(I;\mathcal{B}^1_p)$. A classical fixed point theorem implies the local wellposedness in the space $L^2(I;\mathcal{B}^1_p)$ for initial data in the space $\mathcal{B}^0_p + B^{\frac2p,-1+\frac1p}_{p,1}$.

Now let us extend this (unique) solution to the whole interval $\mathbb{R}^+$. Given $\varepsilon>0$, to be chosen small enough later on, let us define T_ε as

$$T_\varepsilon \stackrel{\text{def}}{=} \sup\big\{T<T^\star\,,\ \|\rho\|_{L^2([0,T];\mathcal{B}^1_p)}\leq\varepsilon\big\}\,. \tag{A.17}$$

As in the proof of Lemma 3.6, let us consider the increasing sequence $(T_m)_{0\leq m\leq M}$ such that $T_0=0$, $T_M=\infty$ and for some given c_0 which will be chosen later on

$$\forall m<M-1\,,\ \int_{T_m}^{T_{m+1}}\|U(t)\|^2_{\mathcal{B}^1_p}dt = c_0 \quad\text{and}\quad \int_{T_{M-1}}^{\infty}\|U(t)\|^2_{\mathcal{B}^1_p}dt\leq c_0\,. \tag{A.18}$$

Let us recall that from (3.31), we have

$$M\leq\frac{1}{c_0}\int_0^\infty\|U(t)\|^2_{\mathcal{B}^1_p}dt\,. \tag{A.19}$$

Let us define

$$\mathcal{N}_0 \stackrel{\text{def}}{=} \|\mathcal{L}_0\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)} + \|\mathcal{L}_0\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\|U\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}. \tag{A.20}$$

Let us consider any m such that $T_m<T_\varepsilon$. Lemma A.7 implies that for any time T less than $\min\{T_{m+1};T_\varepsilon\}$, we have

$$\begin{aligned}
\mathcal{R}^{\mathrm{h}}_m(T) &\stackrel{\text{def}}{=} \|\rho_{\mathrm{h}}\|_{L^\infty([T_m,T];\mathcal{B}^0)} + \|\rho_{\mathrm{h}}\|_{L^1([T_m,T];\mathcal{B}^2)}\\
&\leq C\|\rho_{\mathrm{h}}(T_m)\|_{\mathcal{B}^0_p} + C\mathcal{N}_0\\
&\qquad + C\big(\|\rho_{\mathrm{h}}\|_{L^2([T_m,T];\mathcal{B}^1_p)} + \|\mathcal{L}_0+U\|_{L^2([T_m,T];\mathcal{B}^1_p)}\big)\|\rho_{\mathrm{h}}\|_{L^2([T_m,T];\mathcal{B}^1_p)}\\
&\leq C\|\rho_{\mathrm{h}}(T_m)\|_{\mathcal{B}^0_p} + C\mathcal{N}_0\\
&\qquad\qquad + C\big(\varepsilon + \|\mathcal{L}_0\|_{L^2([T_m,T];\mathcal{B}^1_p)} + c_0\big)\|\rho_{\mathrm{h}}\|_{L^2([T_m,T];\mathcal{B}^1_p)}\,.
\end{aligned}$$

Choosing C_0 large enough in (A.13), c_0 small enough in (A.18), and ε small enough in (A.17) implies that

$$\mathcal{R}^{\mathrm{h}}_m(T)\leq C\|\rho_{\mathrm{h}}(T_m)\|_{\mathcal{B}^0_p} + C\mathcal{N}_0 + \frac12\|\rho_{\mathrm{h}}\|_{L^2([T_m,T];\mathcal{B}^1_p)}\,. \tag{A.21}$$

Exactly along the same lines, we get

$$\begin{aligned}
\mathcal{R}^{\mathrm{v}}_m(T) &\stackrel{\text{def}}{=} \|\rho_{\mathrm{v}}\|_{L^\infty([T_m,T];\mathcal{B}^{1,-\frac12}_p)} + \|\rho_{\mathrm{v}}\|_{L^1([T_m,T];\mathcal{B}^{1,\frac32}_p)}\\
&\leq C\|\rho_{\mathrm{v}}(T_m)\|_{\mathcal{B}^{1,-\frac12}_p} + C\mathcal{N}_0 + \frac12\|\rho_{\mathrm{v}}\|_{L^2([T_m,T];\mathcal{B}^1_p)}\,.
\end{aligned}$$

We deduce that

$$\|\rho_{\mathrm{h}}\|_{L^2([T_m,T];\mathcal{B}^1_p)}\leq C\big(\|\rho_{\mathrm{h}}(T_m)\|_{\mathcal{B}^0_p}+\mathcal{N}_0\big) \quad\text{and}\quad \|\rho_{\mathrm{v}}\|_{L^2([T_m,T];\mathcal{B}^1_p)}\leq C\big(\|\rho_{\mathrm{v}}(T_m)\|_{\mathcal{B}^{1,-\frac12}_p}+\mathcal{N}_0\big)\,.$$

This gives, for any m such that $T_m<T_\varepsilon$ and for all T in $[T_m;\min\{T_{m+1},T_\varepsilon\}]$,

$$\mathcal{R}^{\mathrm{h}}_m(T)+\mathcal{R}^{\mathrm{v}}_m(T)\leq C_1\big(\|\rho_{\mathrm{v}}(T_m)\|_{\mathcal{B}^{1,-\frac12}_p} + \|\rho_{\mathrm{h}}(T_m)\|_{\mathcal{B}^0_p}+\mathcal{N}_0\big)\,. \tag{A.22}$$

Let us observe that $\rho_{|t=0}=0$. Thus exactly as in the proof of Lemma 3.6, an iteration process gives, for any m such that $T_m<T_\varepsilon$ and any T in $[T_m,\min\{T_{m+1},T_\varepsilon\}]$,

$$\begin{aligned}\mathcal{R}(T) &\stackrel{\rm def}{=} \|\rho_{\rm h}\|_{L^\infty([0,T];\mathcal{B}^0_p)}+\|\rho_{\rm h}\|_{L^1([0,T];\mathcal{B}^2_p)}+\|\rho_{\rm v}\|_{L^\infty([0,T];B^{\frac2p,-1+\frac1p}_{p,1})}+\|\rho_{\rm h}\|_{L^1([0,T];B^{\frac2p,1+\frac1p}_{p,1})}\\ &\leq (C_1)^{m+1}\mathcal{N}_0\,.\end{aligned}$$

By definition of $\mathcal{N}_0$ given in (A.20), we have in view of Definition 1.13

$$\mathcal{N}_0\lesssim\big(\|u_0\|_{\mathcal{B}^0_p}+\|f\|_{\mathcal{F}^0_p}\big)\big(\|U\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}+\|u_0\|_{\mathcal{B}^0_p}+\|f\|_{\mathcal{F}^0_p}\big)\,.$$

As claimed in (A.19) the total number of intervals is less than $\|U\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}$. We infer that, for any $T<T_\varepsilon$

$$\mathcal{R}(T)\leq C_2\big(\|u_0\|_{\mathcal{B}^0_p}+\|f\|_{\mathcal{F}^0_p}\big)\big(\|U\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}+\|u_0\|_{\mathcal{B}^0_p}+\|f\|_{\mathcal{F}^0_p}\big)\exp\big(C_2\|U\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\big)\,.$$

Using the interpolation inequality (A.16) we infer that, for any $T<T_\varepsilon$,

$$\int_0^T\|\rho(t)\|^2_{\mathcal{B}^1_p}dt\leq C_2\big(\|u_0\|_{\mathcal{B}^0_p}+\|f\|_{\mathcal{F}^0_p}\big)\big(\|U\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}+\|u_0\|_{\mathcal{B}^0_p}+\|f\|_{\mathcal{F}^0_p}\big)\exp\big(C_2\|U\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\big)\,.$$

Choosing

$$C_2\big(\|u_0\|_{\mathcal{B}^0_p}+\|f\|_{\mathcal{F}^0_p}\big)\big(\|U\|_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}+\|u_0\|_{\mathcal{B}^0_p}+\|f\|_{\mathcal{F}^0_p}\big)\exp\big(C_2\|U\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\big)\leq\frac{\varepsilon^2}{2}$$

ensures that $\displaystyle\int_0^T\|\rho(t)\|^2_{\mathcal{B}^1_p}dt$ remains less than ε^2, and thus there is no blow up for the solution of (NS_U). This concludes the proof of Proposition 1.14. □

A.3. **Proof of Proposition 1.15.** Thanks to Proposition A.4, we observe that if u belongs to $L^2(\mathbb{R}^+;\mathcal{B}^1_p)$, then $u\otimes u$ belongs to $L^1(\mathbb{R}^+;\mathcal{B}^1)$. Lemma A.1 implies that the operators $\mathcal{Q}_{\rm h}$ and $\mathcal{Q}_{\rm v}$ defined in (A.14) satisfy

$$\|\mathcal{Q}_{\rm h}(u,u)\|_{L^1(\mathbb{R}^+;\mathcal{B}^0)}+\|\mathcal{Q}_{\rm v}(u,u)\|_{L^1(\mathbb{R}^+;\mathcal{B}^{1,-\frac12})}\lesssim\|u\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\,.$$

Using the Duhamel formula and the action of the heat flow described in Lemma A.2, we deduce that

$$\forall r\in[1,\infty]\,,\ \|u\|_{L^r(\mathbb{R}^+;\mathcal{B}^{\frac2r})}+\|u\|_{L^1(\mathbb{R}^+;\mathcal{B}^{1,\frac32})}\lesssim\|u_0\|_{\mathcal{B}^0}+\|u\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1_p)}\,,$$

which proves (1.37). Let us prove the second inequality of the proposition which is a propagation type inequality. Once an appropriate (para)linearization of the terms $\mathcal{Q}_{\rm h}$ and $\mathcal{Q}_{\rm v}$ is done, the proof is quite similar to the proof of Proposition 1.14. Following the method of [14], let us observe that

$$\begin{aligned}\mathrm{div}(u\otimes u)^\ell &= \mathrm{div}_{\rm h}(u^\ell u^{\rm h})+\partial_3(u^\ell u^3)\\ &= (\mathrm{div}_{\rm h}\,u^{\rm h})u^\ell+u^{\rm h}\cdot\nabla_{\rm h}u^\ell+\partial_3\big(T^{\rm v}_{u^3}u^\ell+T^{\rm v}_{u^\ell}u^3+R^{\rm v}(u^3,u^\ell)\big)\,.\end{aligned}$$

Now let us define the bilinear operator $\mathcal{T}$ by

$$(\mathcal{T}_u w)^\ell\stackrel{\rm def}{=}(\mathrm{div}_{\rm h}\,w^{\rm h})u^\ell+u^{\rm h}\cdot\nabla_{\rm h}w^\ell+\partial_3\big(T^{\rm v}_{u^3}w^\ell+T^{\rm v}_{u^\ell}w^3+R^{\rm v}(u^3,w^\ell)\big)\,.$$

Let us observe that $\mathcal{T}_u u=\mathrm{div}(u\otimes u)$. The laws of product of Proposition A.4 imply that, for any s in $[1-4/p+\mu,-1+4/p-\mu]$,

$$\|(\mathrm{div}_{\rm h}\,w^{\rm h})u^\ell+u^{\rm h}\cdot\nabla_{\rm h}w^\ell\|_{\mathcal{B}^s}\lesssim\|w\|_{\mathcal{B}^{s+1}}\|u\|_{\mathcal{B}^1}\,. \tag{A.23}$$

Lemmas A.1 and A.5 imply that, for any s in $[1-4/p+\mu,-1+4/p-\mu]$,

$$\big\|(\partial_3\big(T^{\rm v}_{u^3}w^\ell+T^{\rm v}_{u^\ell}w^3+R^{\rm v}(u^3,w^\ell)\big)\big\|_{\mathcal{B}^s}\lesssim\|w\|_{\mathcal{B}^{s,\frac32}}\|u\|_{\mathcal{B}^1}\,. \tag{A.24}$$

Let us notice that for any non negative a, u is solution of the linear equation

$$w = e^{(t-a)\Delta}u(a) + L_a\mathcal{T}_u w\,. \tag{A.25}$$

The smoothing effect of the heat flow, as described in Lemma A.2, implies that for any non negative a, and any t greater than or equal to a,

$$\begin{aligned} &2^{\frac{j}{2}+ks}\|\Delta_j^{\rm v}\Delta_k^{\rm h}L_a\mathcal{T}_u w(t)\|_{L^2}\\ &\qquad\lesssim \int_a^t d_{j,k}(t')e^{-c2^{(2k+2j)}(t-t')}\|u(t')\|_{\mathcal{B}^1}\big(\|w(t')\|_{\mathcal{B}^{s+1}}+\|w(t')\|_{\mathcal{B}^{s,\frac32}}\big)dt'. \end{aligned}\tag{A.26}$$

This gives, for any b in $]a,\infty]$,

$$\|L_a\mathcal{T}_u w\|_{L^\infty(I;\mathcal{B}^s)} + \|L_a\mathcal{T}_u w\|_{L^1(I;\mathcal{B}^{s+2}\cap\mathcal{B}^{s,\frac52})} \lesssim \|u\|_{L^2(I;\mathcal{B}^1)}\|w\|_{L^2(I;\mathcal{B}^{s+1}\cap\mathcal{B}^{s,\frac32})}$$

with $I=[a,b]$. Now let us consider the increasing sequence $(T_m)_{0\leq m\leq M}$ which satisfies (A.18). If c_0 is choosen small enough, we have that the linear map $L_{T_m}\mathcal{T}_u$ maps the space

$$L^2([T_m,T_{m+1}];\mathcal{B}^1\cap\mathcal{B}^{s+1}\cap\mathcal{B}^{s,\frac32})$$

into itself with a norm less than 1. Thus u is the unique solution of (A.25) and it satisfies, for any m

$$\|u\|_{L^\infty([T_m,T_{m+1}];\mathcal{B}^s)} + \|u\|_{L^2([T_m,T_{m+1}];\mathcal{B}^{s+1}\cap\mathcal{B}^{s,\frac32})} \leq C_1\|u(T_m)\|_{\mathcal{B}^s}\,.$$

Arguing as in the proofs of Lemma 3.6 and Proposition 1.14, we conclude that u belongs to $\mathcal{A}^s$ and that

$$\|u\|_{\mathcal{A}^s} \lesssim \|u_0\|_{\mathcal{B}^s}\exp\big(C\|u\|^2_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\big)\,.$$

Inequality (1.38) is proved.

In order to prove Inequality (1.39), let us observe that using Bony's decomposition in the vertical variable, we get

$$\begin{aligned} \operatorname{div}(u\otimes u)^\ell &= \sum_{m=1}^3 \partial_m(u^\ell u^m)\\ &= \sum_{m=1}^3 \partial_m\Big(T^{\rm v}_{u^\ell}u^m + T^{\rm v}_{u^m}u^\ell + R^{\rm v}(u^\ell,u^m)\Big)\,. \end{aligned}$$

Now let us define

$$(\overline{\mathcal{T}}_u w)^\ell \stackrel{\rm def}{=} \sum_{m=1}^3 \partial_m\Big(T^{\rm v}_{u^\ell}w^m + T^{\rm v}_{u^m}w^\ell + R^{\rm v}(u^\ell,w^m)\Big)\,.$$

Proposition A.4 implies that, if m equals 1 or 2 then for any s' greater than or equal to 1/2

$$\begin{aligned} \big\|\partial_m\big(T^{\rm v}_{u^\ell}w^m + T^{\rm v}_{u^m}w^\ell + R^{\rm v}(u^\ell,w^m)\big)\big\|_{L^1(\mathbb{R}^+;\mathcal{B}^{0,s'})} &\lesssim \|u\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|w\|_{L^2(\mathbb{R}^+;\mathcal{B}^{1,s'})} \quad\text{and}\\ \big\|\partial_3\big(T^{\rm v}_{u^\ell}w^3 + T^{\rm v}_{u^3}w^\ell + R^{\rm v}(u^\ell,w^3)\big)\big\|_{L^1(\mathbb{R}^+;\mathcal{B}^{0,s'})} &\lesssim \|u\|_{L^2(\mathbb{R}^+;\mathcal{B}^1)}\|w\|_{L^2(\mathbb{R}^+;\mathcal{B}^{0,s'+1})}\,. \end{aligned}$$

Thus we get, for any a in $\mathbb{R}^+$, any b in $I=[a,\infty]$ and any r in $[1,\infty]$,

$$\|L_a\overline{\mathcal{T}}_u w\|_{L^r(I;\mathcal{B}^{\sigma,\sigma'+s'})} \lesssim \|u\|_{L^2(I;\mathcal{B}^1)}\big(\|w\|_{L^2(I;\mathcal{B}^{1,s'})}+\|w\|_{L^2(I;\mathcal{B}^{0,s'+1})}\big) \quad\text{with}\quad \sigma+\sigma'=\frac{2}{r}\cdot$$

Then the lines after Inequality (A.26) can be repeated word for word. Proposition 1.15 is proved. $\square$

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