A Duality Approach for a Class of Semivectorial Bilevel Programming Problems

Abdelmalek Aboussoror¹ · Samir Adly² ·
Fatima Ezzahra Saissi^{1,2}

Abstract In this paper, we present a duality approach using conjugacy for a semivectorial bilevel programming problem (S) where the upper and lower levels are vectorial and scalar respectively. This approach uses the Fenchel–Lagrange duality and is given via a regularization of problem (S) and the operation of scalarization. Then, using this approach, we provide necessary optimality conditions for a class of properly efficient solutions of (S) . Finally, sufficient optimality conditions are given for (S) regardless of the duality approach. Examples are given for illustration.

Keywords Bilevel programming · Vectorial programming · Convex analysis · Duality · Stability

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Dedicated to Professor Michel Théra on his 70th birthday.

✉ Abdelmalek Aboussoror
aboussororabdel@hotmail.com

✉ Samir Adly
samir.adly@unilim.fr

Fatima Ezzahra Saissi
saissi-fz@hotmail.fr

¹ Laboratoire LMC, Faculté Polydisciplinaire de Safi, Université Cadi Ayyad, B.P. 4162, Sidi Bouzid, Safi, Morocco

² Faculté des Sciences et Techniques, Laboratoire XLIM UMR-CNRS 6172, Université de Limoges, 123 Avenue Albert Thomas, 87060 Limoges Cedex, France

1 Introduction

In this paper, we are concerned with the following semivectorial minimization problem

$$(S) \quad \text{v-min}_{x \in \mathcal{M}} F(x),$$

where $\mathcal{M}$ is the solution set of the following scalar non parameterized lower level problem

$$(\mathcal{P}) \quad \min_{x \in X} f(x)$$

$F : \mathbb{R}^p \rightarrow \mathbb{R}^k$, $f : \mathbb{R}^p \rightarrow \mathbb{R}$, are convex functions, and X is a nonempty compact convex subset of $\mathbb{R}^p$, and “v-min” stands for vectorial minimization.

The scalar case of problem (S) yields the following bilevel programming problem

$$(\hat{S}) \quad \min_{x \in \mathcal{M}} \hat{F}(x),$$

where $\hat{F} : \mathbb{R}^p \rightarrow \mathbb{R}$ is a function, which was first considered by Solodov in [11]. The author showed that the class of problems having the form of $(\hat{S})$ includes the following large class of standard optimization problems of the form

$$\min_{x \in H} \hat{h}(x),$$

where $H = \{x \in \hat{X} / \hat{g}(x) \leq 0, Ax = a\}$, $\hat{h} : \mathbb{R}^p \rightarrow \mathbb{R}$, $\hat{g} : \mathbb{R}^p \rightarrow \mathbb{R}^m$ are differentiable convex functions, $A \in \mathbb{R}^{q \times p}$, $a \in \mathbb{R}^q$ and $\hat{X}$ is a closed convex subset of $\mathbb{R}^p$. Theoretical and numerical studies of such a class of scalar bilevel programming problems were considered in [1, 4–6, 11, 14]. In [1], a Fenchel–Lagrange duality approach and optimality conditions are given for $(\hat{S})$. In this paper, we will extend this duality approach via scalarization to the vectorial case where the corresponding upper level is vectorial and the lower level is non parameterized and scalar. The duality that we consider is the so-called Fenchel–Lagrange duality (see [12]). We note that in order to establish strong duality, we will need the Slater constraint qualification. Because of the lack of such a property by the scalarized problem of (S) defined according to the definition given in [7], we will consider first a regularized problem (S_ϵ) of (S) ($\epsilon > 0$), whose scalarized problem (S_ϵ^s) satisfies this condition. The regularization is based on the use of ϵ -approximate solutions of the lower level problem $(\mathcal{P})$. As a stability result, we show that any accumulation point of a sequence of regularized properly efficient solutions is a properly efficient solution of the original problem (S). Afterwards, we consider the Fenchel–Lagrange dual $(\mathcal{D}_\epsilon^s)$ of the scalarized problem (S_ϵ^s) . Under appropriate assumptions, we establish strong duality and provide optimality conditions for the primal-dual pair (S_ϵ^s) – $(\mathcal{D}_\epsilon^s)$. Then, via the duality given for the regularized scalarized case, we deduce necessary optimality conditions for a class of properly efficient solutions of problem (S). These properly efficient solutions are accumulation points of sequences composed of some particular feasible points of the scalarized problems $(S_{\epsilon_n}^s)$, $n \in \mathbb{N}$, $\epsilon_n \searrow 0^+$. Finally, sufficient optimality conditions are established for problem (S) without using duality. Illustrative examples are also given.

The outline of the paper is as follows. In Section 2, we recall some definitions and results related to convex analysis and multiobjective optimization. In Section 3, we introduce a regularized problem (S_ϵ) of (S). As a stability result, we show that any accumulation point of regularized properly efficient solutions is a properly efficient solution of (S). In Section 4, we associate the Fenchel–Lagrange dual to the scalarized problem (S_ϵ^s) . We show that strong duality holds and provide necessary and sufficient optimality conditions for the primal and dual problems. In Section 5, based on the results given in the previous section, we give

necessary optimality conditions for a class of properly efficient solutions of problem (S). Finally, in Section 6, we provide sufficient optimality conditions for problem (S).

2 Preliminaries

In this section, we recall some fundamental definitions and results relating essentially to convex analysis and multiobjective optimization that we will use in the sequel.

Let A be a nonempty subset of $\mathbb{R}^n$. We shall denote by ψ_A and σ_A the indicator and the support functions of the set A respectively, defined on $\mathbb{R}^n$ by

$$\psi_A(x) = \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{if } x \notin A, \end{cases} \quad \sigma_A(x) = \sup_{a \in A} \langle x, a \rangle.$$

In the sequel, the following conventions in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ will be adopted

$$\begin{aligned} (+\infty) - (+\infty) &= (-\infty) - (-\infty) = (+\infty) + (-\infty) = +\infty, \\ \begin{cases} 0 \times (+\infty) = +\infty, \\ 0 \times (-\infty) = 0, \end{cases} & \quad \begin{cases} \alpha(-\infty) = -\infty, & \alpha(+\infty) = +\infty \text{ for } \alpha \in \mathbb{R}_+^*, \\ \alpha(-\infty) = +\infty, & \alpha(+\infty) = -\infty \text{ for } \alpha \in \mathbb{R}_-^*. \end{cases} \end{aligned}$$

Definition 1 Let $g : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be a function.

- i) The conjugate function of g relative to the set A is denoted by g_A^* and defined on $\mathbb{R}^n$ by

$$g_A^*(p) = \sup_{x \in A} \{\langle p, x \rangle - g(x)\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product for two vectors in $\mathbb{R}^n$, i.e., for $x = (x_1, \dots, x_n)^\top$, and $y = (y_1, \dots, y_n)^\top$, $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$. When $A = \mathbb{R}^n$, we get the usual Legendre–Fenchel conjugate function of g denoted by g^* .

- ii) The effective domain of g denoted by $\text{dom } g$ is the set defined by

$$\text{dom } g = \{x \in \mathbb{R}^n / g(x) < +\infty\}.$$

We say that g is proper if $g(x) > -\infty$ for all $x \in \mathbb{R}^n$, and $\text{dom } g$ is nonempty.

Definition 2 Let $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Let $\bar{x} \in \text{dom } g$.

- i) The subdifferential in the sense of convex analysis of g at $\bar{x}$ denoted by $\partial g(\bar{x})$ is the set defined by

$$\partial g(\bar{x}) = \{x^* \in \mathbb{R}^n / g(x) \geq g(\bar{x}) + \langle x^*, x - \bar{x} \rangle \quad \forall x \in \mathbb{R}^n\}.$$

An element $x^* \in \partial g(\bar{x})$ is called a subgradient of g at $\bar{x}$.

- ii) Let $\epsilon \geq 0$. The ϵ -subdifferential of g at $\bar{x}$ denoted by $\partial_\epsilon g(\bar{x})$ is the set defined by

$$\partial_\epsilon g(\bar{x}) = \{x^* \in \mathbb{R}^n / g(x) \geq g(\bar{x}) + \langle x^*, x - \bar{x} \rangle - \epsilon \quad \forall x \in \mathbb{R}^n\}.$$

An element $x^* \in \partial_\epsilon g(\bar{x})$ is called an ϵ -subgradient of g at $\bar{x}$. When $\epsilon = 0$, we have $\partial_\epsilon g(\bar{x}) = \partial g(\bar{x})$.

Remark 1 We have the following properties

- i) $x^* \in \partial g(\bar{x}) \iff \langle x^*, \bar{x} \rangle = g(\bar{x}) + g^*(x^*)$.
ii) $g(x) + g^*(x^*) \geq \langle x^*, x \rangle \quad \forall x, x^* \in \mathbb{R}^n$ (the Young–Fenchel inequality).

Definition 3 Let C be a nonempty convex subset of $\mathbb{R}^n$ and $\bar{x} \in C$.

- i) The normal cone to C at $\bar{x}$ denoted by $\mathcal{N}_C(\bar{x})$ is the set defined by

$$\mathcal{N}_C(\bar{x}) = \{x^* \in \mathbb{R}^n / \langle x^*, x - \bar{x} \rangle \leq 0 \quad \forall x \in C\}.$$

It is not difficult to verify that $\partial\psi_C(\bar{x}) = \mathcal{N}_C(\bar{x})$ and if $\bar{x} \in \text{int } C$, then $\mathcal{N}_C(\bar{x}) = \{0_{\mathbb{R}^n}\}$, where $\text{int } C$ denotes the topological interior of C .

- ii) Let $\epsilon > 0$. The set of ϵ -normal directions to C at $\bar{x}$ denoted by $\mathcal{N}_\epsilon(C, \bar{x})$ is defined as follows

$$\mathcal{N}_\epsilon(C, \bar{x}) = \{x^* \in \mathbb{R}^n / \langle x^*, x - \bar{x} \rangle \leq \epsilon \quad \forall x \in C\}.$$

We have $\partial_\epsilon\psi_C(\bar{x}) = \mathcal{N}_\epsilon(C, \bar{x})$.

Theorem 1 [9, Theorem 24.7] Let $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex and lower semicontinuous function, and C be a nonempty compact subset of $\text{int}(\text{dom } g)$. Then, the set $\bigcup_{x \in C} \partial g(x)$ is compact.

Theorem 2 [2, Theorem IV.23] Let $f_1, f_2 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper convex functions. Assume that there exists $x_0 \in \text{dom } f_1$ such that f_2 is continuous at x_0 . Then, for every $x \in \mathbb{R}^n$

$$\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x).$$

Theorem 3 [8, Theorem 3.3.1] Let $\epsilon \geq 0$ and $f_1, f_2 : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper convex functions such that $\text{ri}(\text{dom } f_1) \cap \text{ri}(\text{dom } f_2) \neq \emptyset$. Then, for every $x \in (\text{dom } f_1) \cap (\text{dom } f_2)$, we have

$$\partial_\epsilon(f_1 + f_2)(x) = \bigcup_{\substack{\epsilon_1 \geq 0, \epsilon_2 \geq 0 \\ \epsilon_1 + \epsilon_2 \leq \epsilon}} \{\partial_{\epsilon_1} f_1(x) + \partial_{\epsilon_2} f_2(x)\},$$

where for a nonempty subset A of $\mathbb{R}^n$, $\text{ri } A$ denotes the relative interior of A , i.e., the interior of A relative to the smallest affine set containing A .

Theorem 4 [9, Theorem 16.4] Let $f_1, \dots, f_m : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ be proper convex functions. Assume that $\bigcap_{i=1}^m \text{ri}(\text{dom } f_i) \neq \emptyset$. Then, for any $x \in \mathbb{R}^n$, we have

$$\left(\sum_{i=1}^m f_i \right)^*(x) = \inf_{\substack{x_1, \dots, x_m \in \mathbb{R}^n \\ x_1 + \dots + x_m = x}} \left\{ \sum_{i=1}^m f_i^*(x_i) \right\}$$

and the infimum is attained.

Now, let us recall some definitions and fundamental results relating to multiobjective optimization that we are concerned in our study. Consider the following vectorial optimization problem

$$(Q) \quad \text{v-min} \tilde{g}(x) \\ x \in \tilde{X}$$

where $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_r)^\top : \mathbb{R}^n \rightarrow \mathbb{R}^r$, $\tilde{g}_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, \dots, r$, $r \in \mathbb{N}^*$, are functions and $\tilde{X}$ is a nonempty subset of $\mathbb{R}^n$.

Throughout the paper, the following partial order on $\mathbb{R}^r$ will be adopted. For $y = (y_1, \dots, y_r)^\top$ and $z = (z_1, \dots, z_r)^\top$

$$y \leq z \iff y_i \leq z_i \quad \forall i \in \{1, \dots, r\}.$$

Definition 4 An element $\bar{x} \in \tilde{X}$ is said to be an efficient solution with respect to problem (Q) if $\tilde{g}(x) \leq \tilde{g}(\bar{x})$, for some $x \in \tilde{X}$, then $\tilde{g}(x) = \tilde{g}(\bar{x})$. An efficient solution is also called a Pareto-efficient solution.

In the sequel, we will adopt the following definition of properly efficient solution in the sense of Geoffrion [7].

Definition 5 An element $\bar{x} \in \tilde{X}$ is said to be a properly efficient solution to problem (Q), if it is efficient and if there exists a positive real number M such that for each $i \in \{1, \dots, r\}$ and $x \in \tilde{X}$ satisfying $\tilde{g}_i(x) < \tilde{g}_i(\bar{x})$, there exists $j \in \{1, \dots, r\}$ such that

- i) $\tilde{g}_j(\bar{x}) < \tilde{g}_j(x)$,
- ii) $\frac{\tilde{g}_i(\bar{x}) - \tilde{g}_i(x)}{\tilde{g}_j(x) - \tilde{g}_j(\bar{x})} \leq M$.

The following well-known result characterizes properly efficient solutions via scalarization.

Theorem 5 [7, Theorem 2] Assume that $\tilde{X}$ and $\tilde{g}$ are convex. Let $\tilde{x} \in \tilde{X}$. Then, $\tilde{x}$ is a properly efficient solution of problem (Q) if and only if there exists $\lambda = (\lambda_1, \dots, \lambda_r)^\top \in \text{int}(\mathbb{R}_+^r)$ such that $\tilde{x}$ solves the following scalar minimization problem

$$(Q^s) \quad \min_{x \in \tilde{X}} \sum_{i=1}^r \lambda_i \tilde{g}_i(x).$$

3 Regularization of Problem (S)

In the sequel, for our investigation, we adopt the following equivalent formulation of problem (S)

$$(S) \quad v\text{-}\min_{\substack{x \in X \\ f(x) \leq v}} F(x),$$

where v denotes the infimal value of the lower level problem (P). This formulation will allow us to extend the work given in [1] for the scalar case to the vectorial one. Remark that since X is compact and f is continuous as a finite convex function, then the problem (P) admits at least one solution. So, v is a finite real number. Besides, we note that the problem (S) does not satisfy the Slater constraint qualification.

The aim of this paper is to give a duality approach for problem (S). Our investigation is inspired from the work given in [1], and will use some results established in [3] concerning a duality for multiobjective optimization problems with d.c. constraints. The duality considered by the authors in [3], is given via scalarization, and strong duality is established under a constraint qualification fulfilled by the considered scalarized problem. In our case, the scalarized problem of (S) defined according to the definition given in [7] has the following form

$$(S^s) \quad \min_{\substack{x \in X \\ f(x) \leq v}} \sum_{i=1}^k \lambda_i F_i(x),$$

where $(\lambda_1, \dots, \lambda_k)^\top \in \text{int}(\mathbb{R}_+^k)$ is fixed. Then, the problem (S^s) has the same set of constraints as (S). It follows that it cannot also satisfy the Slater condition which corresponds

to the constraint qualification considered in [3]. Such a property will be used in our investigation to establish strong duality. To overcome this obstacle, in this section, we consider a regularized problem (S_ϵ) of (S) that satisfies this condition. As a consequence, the scalarized problem (S_ϵ^s) of (S_ϵ) that we consider in the next section will also satisfy this condition. On the other hand, as a stability result, we show that any accumulation point of a sequence of regularized properly efficient solutions is a properly efficient solution to (S) .

For $\epsilon > 0$, we consider the following semivectorial regularized problem of (S)

$$(S_\epsilon) \quad v\text{-} \min_{x \in \mathcal{M}_\epsilon} F(x),$$

where $\mathcal{M}_\epsilon$ is the set of ϵ -approximate solutions of problem $(\mathcal{P})$, i.e.,

$$\mathcal{M}_\epsilon = \{x \in X / f(x) \leq v + \epsilon\}.$$

Define on $\mathbb{R}^p$ the following functions

$$g_{1,\epsilon}(x) = \psi_X(x), \quad g_{2,\epsilon}(x) = f(x), \quad h_{1,\epsilon}(x) = 0, \quad h_{2,\epsilon}(x) = v + \epsilon.$$

Then, the regularized semivectorial bilevel programming problem (S_ϵ) can be written as

$$(S_\epsilon) \quad v\text{-} \min_{\substack{x \in \mathbb{R}^p \\ g_{1,\epsilon}(x) - h_{1,\epsilon}(x) \leq 0 \\ g_{2,\epsilon}(x) - h_{2,\epsilon}(x) \leq 0}} F(x).$$

In the sequel, and in order to apply some results given in [3] for optimization problems having d.c. constraints, we will view the convex constraints

$$g_{1,\epsilon}(x) - h_{1,\epsilon}(x) = \psi_X(x) - 0 \text{ and } g_{2,\epsilon}(x) - h_{2,\epsilon}(x) = f(x) - (v + \epsilon)$$

as d.c. constraints. In this case, (S_ϵ) can be viewed as a multiobjective minimization problem with convex vectorial objective function and two inequality d.c. constraints.

Let $\epsilon_n \searrow 0^+$. For simplicity, we shall denote the problem (S_{ϵ_n}) by (S_n) . The following theorem establishes that any accumulation point of a sequence of regularized properly efficient solutions is a properly efficient solution of problem (S) .

Theorem 6 *Let $\epsilon_n \searrow 0^+$. For $n \in \mathbb{N}$, let $\bar{x}_n$ be a properly efficient solution of problem (S_n) . Then, any accumulation point of the sequence $(\bar{x}_n)_n$ is a properly efficient solution of problem (S) .*

Proof Feasibility: Let $\bar{x}$ be an accumulation point of the sequence $(\bar{x}_n)_n$. Let $\mathcal{N}$ be an infinite subset of $\mathbb{N}$ such that $\bar{x}_n \rightarrow \bar{x}$ as $n \rightarrow +\infty$, $n \in \mathcal{N}$. Since X is compact, it follows that $\bar{x} \in X$. Moreover, from the feasibility of $\bar{x}_n$ to problem (S_n) , we have

$$f(\bar{x}_n) \leq v + \epsilon_n.$$

Passing to the limit as $n \rightarrow +\infty$, $n \in \mathcal{N}$, and using the continuity of the function f , we obtain $f(\bar{x}) \leq v$, i.e., $\bar{x} \in \mathcal{M}$.

Optimality: Since the solution set $\mathcal{M}$ and the objective function F are convex, it follows that the semivectorial bilevel programming problem (S) is convex. Accordingly, it is sufficient to show that there exists $\lambda = (\lambda_1, \dots, \lambda_k)^\top \in \text{int}(\mathbb{R}_+^k)$ such that $\bar{x}$ solves the scalar minimization problem

$$(S^s) \quad \min_{x \in \mathcal{M}} \sum_{i=1}^k \lambda_i F_i(x)$$

and the result will follow from Theorem 5. By contradiction, assume that for all $\lambda = (\lambda_1, \dots, \lambda_k)^\top \in \text{int}(\mathbb{R}_+^k)$, there exists $x_\lambda \in \mathcal{M}$ such that

$$\sum_{i=1}^k \lambda_i F_i(x_\lambda) < \sum_{i=1}^k \lambda_i F_i(\bar{x}).$$

We have $\bar{x}_n \rightarrow \bar{x}$ as $n \rightarrow +\infty$, $n \in \mathcal{N}$. Then, using the continuity of the functions F_i , $i = 1, \dots, k$ (as finite convex functions), we deduce that there exists $n_0 \in \mathcal{N}$ such that

$$\sum_{i=1}^k \lambda_i F_i(x_\lambda) < \sum_{i=1}^k \lambda_i F_i(\bar{x}_n) \quad \forall n \geq n_0, n \in \mathcal{N}. \quad (1)$$

Since $x_\lambda \in \mathcal{M} \subset \mathcal{M}_{\epsilon_n}$, then, the strict inequality in (1) yields a contradiction with the fact that $\bar{x}_n$ is a properly efficient solution of problem (S_n) , $n \in \mathcal{N}$, $n \geq n_0$. $\square$

4 Duality for the Scalarized Problem of (S_ϵ)

In this section, we consider the so-called Fenchel–Lagrange dual of a scalarized problem (S_ϵ^s) of (S_ϵ) . Then, we establish strong duality and provide optimality conditions for the scalar primal-dual pair.

Define on $\mathbb{R}^p$ the functions

$$\tilde{g}_{1,\epsilon} = g_{1,\epsilon} - h_{1,\epsilon} \quad \tilde{g}_{2,\epsilon} = g_{2,\epsilon} - h_{2,\epsilon}, \text{ and } \tilde{g}_\epsilon = (\tilde{g}_{1,\epsilon}, \tilde{g}_{2,\epsilon})^T.$$

Then, we have $\tilde{g}_{1,\epsilon} = \psi_X$ and $\tilde{g}_{2,\epsilon} = f(\cdot) - (v + \epsilon)$. For $\epsilon > 0$ and a fixed $\lambda = (\lambda_1, \dots, \lambda_k)^\top \in \text{int}(\mathbb{R}_+^k)$, consider the following well-known scalar optimization problem associated to the regularized problem (S_ϵ) (see [7])

$$(S_\epsilon^s) \quad \min_{\substack{x \in \mathbb{R}^p \\ \tilde{g}_\epsilon(x) \leq 0}} \sum_{i=1}^k \lambda_i F_i(x).$$

Remark 2 Let $\epsilon > 0$. Then, from the characterization of the infimal value of problem $(\mathcal{P})$, there exists $x_\epsilon \in X$ verifying

$$f(x_\epsilon) < v + \epsilon$$

which is the so-called Slater condition relative to the set X , and corresponds to the constraint qualification used in [3] to establish strong duality. So that, such a constraint qualification is always satisfied in our case by the problem (S_ϵ^s) .

To start our duality approach, let us consider the following dual problem of (S_ϵ^s) (see [12])

$$(\widehat{\mathcal{D}}_\epsilon^s) \quad \sup_{\substack{a \in \mathbb{R}^p \\ q \in \mathbb{R}_+^2}} \left\{ - \left(\sum_{i=1}^k \lambda_i F_i \right)^* (a) - (q^\top \tilde{g}_\epsilon)^* (-a) \right\}$$

called the Fenchel–Lagrange dual. Let us give an explicit formulation of the objective function of problem $(\widehat{\mathcal{D}}_\epsilon^s)$. Using Theorem 4, we have

$$\left(\sum_{i=1}^k \lambda_i F_i \right)^* (a) = \min_{\substack{a_i \in \mathbb{R}^p \\ a = \sum_{i=1}^k a_i}} \sum_{i=1}^k (\lambda_i F_i)^*(a_i).$$

Hence

$$\begin{aligned}
-\left(\sum_{i=1}^k \lambda_i F_i\right)^*(a) - (q^\top \tilde{g}_\epsilon)^*(-a) &= -\min_{\substack{a_i \in \mathbb{R}^p \\ a = \sum_{i=1}^k a_i}} \sum_{i=1}^k (\lambda_i F_i)^*(a_i) - (q^\top \tilde{g}_\epsilon)(-a) \\
&= -\min_{a_i \in \mathbb{R}^p} \sum_{i=1}^k \lambda_i F_i^*(a_i) - (q^\top \tilde{g}_\epsilon)^*\left(-\sum_{i=1}^k \lambda_i a_i\right) \\
&= \sup_{a_i \in \mathbb{R}^p} \left\{ -\sum_{i=1}^k \lambda_i F_i^*(a_i) - (q^\top \tilde{g}_\epsilon)^*\left(-\sum_{i=1}^k \lambda_i a_i\right) \right\}.
\end{aligned}$$

Set $q = (q_1, q_2)^\top$. Then

$$\begin{aligned}
(q^\top \tilde{g}_\epsilon)^*\left(-\sum_{i=1}^k \lambda_i a_i\right) &= \sup_{x \in \mathbb{R}^p} \left\{ \left\langle -\sum_{i=1}^k \lambda_i a_i, x \right\rangle - (q^\top \tilde{g}_\epsilon)(x) \right\} \\
&= \sup_{x \in \mathbb{R}^p} \left\{ \left\langle -\sum_{i=1}^k \lambda_i a_i, x \right\rangle - (q_1 \tilde{g}_{1,\epsilon} + q_2 \tilde{g}_{2,\epsilon})(x) \right\}
\end{aligned}$$

Using the conventions introduced in Section 2, we have

$$\begin{aligned}
(q^\top \tilde{g}_\epsilon)^*\left(-\sum_{i=1}^k \lambda_i a_i\right) &= \sup_{x \in \mathbb{R}^p} \left\{ \left\langle -\sum_{i=1}^k \lambda_i a_i, x \right\rangle - \psi_X(x) - q_2 \tilde{g}_{2,\epsilon}(x) \right\} \\
&= \sup_{x \in X} \left\{ \left\langle -\sum_{i=1}^k \lambda_i a_i, x \right\rangle - q_2 \tilde{g}_{2,\epsilon}(x) \right\} \\
&= (q_2 \tilde{g}_{2,\epsilon})_X^*\left(-\sum_{i=1}^k \lambda_i a_i\right).
\end{aligned}$$

Therefore, after changing the notation of the coefficient of $\tilde{g}_{2,\epsilon}$, the problem $(\widehat{\mathcal{D}}_\epsilon^s)$ and the following problem

$$(\mathcal{D}_\epsilon^s) \quad \sup_{\substack{a_i \in \mathbb{R}^p \\ i=1, \dots, k \\ q \in \mathbb{R}_+}} \left\{ -\sum_{i=1}^k \lambda_i F_i^*(a_i) - (q \tilde{g}_{2,\epsilon})_X^*\left(-\sum_{i=1}^k \lambda_i a_i\right) \right\}$$

have the same optimal value. Therefore, in the sequel for our investigation we will use the problem $(\mathcal{D}_\epsilon^s)$ instead of $(\widehat{\mathcal{D}}_\epsilon^s)$ that we will also call it the Fenchel–Lagrange dual of (S_ϵ^s) . Via simple calculations, we obtain the following formulation of $(\mathcal{D}_\epsilon^s)$

$$(\mathcal{D}_\epsilon^s) \quad \sup_{\substack{a_i \in \mathbb{R}^p \\ i=1, \dots, k \\ q \in \mathbb{R}_+}} \inf_{x \in X} \left\{ -\sum_{i=1}^k \lambda_i F_i^*(a_i) + \left\langle \sum_{i=1}^k \lambda_i a_i, x \right\rangle + qf(x) - q(v + \epsilon) \right\}.$$

Then, we have the following result concerning weak duality.

Proposition 1 *Let $\epsilon > 0$. Then*

$$\inf(S_\epsilon^s) \geq \sup(\mathcal{D}_\epsilon^s).$$

Proof The result uses the fact that $\sup(\widehat{\mathcal{D}}_\epsilon^s) = \sup(\mathcal{D}_\epsilon^s)$ and the well-known result of weak Fenchel–Lagrange duality between (S_ϵ^s) and $(\widehat{\mathcal{D}}_\epsilon^s)$ ([12, 13]). $\square$

The following theorem establishes strong duality between (S_ϵ^s) and $(\mathcal{D}_\epsilon^s)$.

Theorem 7 *Let $\epsilon > 0$. Then, the dual problem $(\mathcal{D}_\epsilon^s)$ has a solution and strong duality holds, i.e.,*

$$\inf(S_\epsilon^s) = \max(\mathcal{D}_\epsilon^s).$$

Proof The result follows from [3, Theorem 3.3]. $\square$

As necessary and sufficient optimality conditions for the primal-dual pair (S_ϵ^s) – $(\mathcal{D}_\epsilon^s)$, we have the following.

Theorem 8 *Let $\epsilon > 0$.*

1) *Necessary optimality conditions.*

Let x_ϵ be a solution of problem (S_ϵ^s) . Then, there exists a solution $(a_\epsilon, \alpha_\epsilon)$ of the dual problem $(\mathcal{D}_\epsilon^s)$, with $a_\epsilon = (a_{1\epsilon}, \dots, a_{k\epsilon})$, $a_{i\epsilon} \in \mathbb{R}^p$, $i = 1, \dots, k$, $\alpha_\epsilon \geq 0$, verifying together with x_ϵ the following optimality conditions

- i) $F_i(x_\epsilon) + F_i^*(a_{i\epsilon}) = \langle a_{i\epsilon}, x_\epsilon \rangle$, $i = 1, \dots, k$,
- ii) $\alpha_\epsilon(f(x_\epsilon) - v - \epsilon) = 0$,
- iii) $\inf_{x \in X} \left\{ \left\langle \sum_{i=1}^k \lambda_i a_{i\epsilon}, x \right\rangle + \alpha_\epsilon f(x) \right\} = \left\langle \sum_{i=1}^k \lambda_i a_{i\epsilon}, x_\epsilon \right\rangle + \alpha_\epsilon(v + \epsilon)$.

2) *Sufficient optimality conditions.*

Let x_ϵ and $(a_\epsilon, \alpha_\epsilon)$ be feasible points of (S_ϵ^s) and $(\mathcal{D}_\epsilon^s)$ respectively. Assume that they satisfy together the above conditions i)–iii). Then, x_ϵ and $(a_\epsilon, \alpha_\epsilon)$ solve (S_ϵ^s) and $(\mathcal{D}_\epsilon^s)$ respectively. Moreover, strong duality holds for the primal-dual pair (S_ϵ^s) – $(\mathcal{D}_\epsilon^s)$.

Proof The result follows from [3, Theorem 3.4]. $\square$

Remark 3 In terms of subdifferential and normal cone, the properties i) and iii) in Theorem 8 can be written respectively as follows

- 1) $a_{i\epsilon} \in \partial F_i(x_\epsilon) \quad \forall i = 1, \dots, k$,
- 3) $-\sum_{i=1}^k \lambda_i a_{i\epsilon} \in \partial(\alpha_\epsilon f)(x_\epsilon) + \mathcal{N}_X(x_\epsilon)$.

If moreover $\alpha_\epsilon > 0$, then, in terms of ϵ -subdifferential and the set of ϵ -normal directions to X , the property ii) is equivalent to the following

- 2) $0_{\mathbb{R}^p} \in \partial_\epsilon(f + \psi_X)(x_\epsilon) = \bigcup_{\substack{\epsilon_1 \geq 0, \epsilon_2 \geq 0 \\ \epsilon_1 + \epsilon_2 \leq \epsilon}} \{\partial_{\epsilon_1} f(x_\epsilon) + \mathcal{N}_{\epsilon_2}(X, x_\epsilon)\}.$

The assertion concerning property 1) is obvious. To show property 3), first remark that by using ii), we can write iii) as follows

$$\inf_{x \in X} \left\{ \left\langle \sum_{i=1}^k \lambda_i a_{i\epsilon}, x \right\rangle + \alpha_\epsilon f(x) \right\} = \left\langle \sum_{i=1}^k \lambda_i a_{i\epsilon}, x_\epsilon \right\rangle + \alpha_\epsilon f(x_\epsilon).$$

That is x_ϵ solves the problem

$$\min_{x \in X} \left\{ \left\langle \sum_{i=1}^k \lambda_i a_{i\epsilon}, x \right\rangle + \alpha_\epsilon f(x) \right\}.$$

It follows that

$$0_{\mathbb{R}^p} \in \partial(\alpha_\epsilon f)(x_\epsilon) + \sum_{i=1}^k \lambda_i a_{i\epsilon} + \mathcal{N}_X(x_\epsilon).$$

Hence,

$$-\sum_{i=1}^k \lambda_i a_{i\epsilon} \in \partial(\alpha_\epsilon f)(x_\epsilon) + \mathcal{N}_X(x_\epsilon).$$

For property 2), it suffices to apply Theorem 3.

Let us give the following example in which we apply the results of Theorems 8 and 6.

Example 1 Let F and f be the functions defined on $\mathbb{R}^2$ by

$$\begin{aligned} F(x_1, x_2) &= (F_1(x_1, x_2), F_2(x_1, x_2))^\top = (x_1^2 + x_2^2, -x_1 + x_2^2)^\top, \\ f(x_1, x_2) &= -\frac{3}{2}x_1 - x_2 \end{aligned}$$

and

$$X = \left\{ (x_1, x_2) \in \mathbb{R}^2 / x_1^2 - 4x_1 - x_2 + 4 \leq 0, \frac{3}{2}x_1 + x_2 - 4 \leq 0 \right\}.$$

Then, the functions F and f are convex and X is a convex compact set. It is not difficult to check that

$$\inf \mathcal{P} = v = -4 \quad \text{and} \quad \mathcal{M} = \text{conv} \left\{ (0, 4)^\top, \left(\frac{5}{2}, \frac{1}{4}\right)^\top \right\}.$$

In order to solve the problem (S) , we will apply the duality considered for the regularized scalarized case associated to problem (S) . For $\epsilon > 0$ sufficiently small, we first consider the scalarized problem (S_ϵ^s) of (S_ϵ) with $(\lambda_1, \lambda_2)^\top = (1, 1)^\top \in \text{int}(\mathbb{R}_+^2)$ which is written as

$$(S_\epsilon^s) \quad \min_{(x_1, x_2) \in \mathcal{M}_\epsilon} (F_1 + F_2)(x_1, x_2).$$

Let $x_\epsilon = (x_1^\epsilon, x_2^\epsilon)^\top$, $(a_\epsilon, \alpha_\epsilon)$, $a_\epsilon = (a_{1\epsilon}, a_{2\epsilon})$, $a_{i\epsilon} \in \mathbb{R}^2$, $i \in \{1, 2\}$, $\alpha_\epsilon > 0$, be feasible points of problems (S_ϵ^s) and $(\mathcal{D}_\epsilon^s)$ respectively. Note that from Theorem 8, we have strong duality. Using again this theorem, we deduce that x_ϵ and $(a_\epsilon, \alpha_\epsilon)$ solve (S_ϵ^s) and $(\mathcal{D}_\epsilon^s)$ respectively if and only if the following conditions are satisfied (Remark 3)

- i) $a_{i\epsilon} \in \partial F_i(x_\epsilon) \quad \forall i \in \{1, 2\}$.
- ii) $\alpha_\epsilon(f(x_\epsilon) - v - \epsilon) = 0$.
- iii) $-\sum_{i=1}^2 \lambda_i a_{i\epsilon} \in \partial(\alpha_\epsilon f)(x_\epsilon) + \mathcal{N}_X(x_\epsilon)$.

Assume that $\alpha_\epsilon > 0$. Then, the complementary slackness condition ii) yields

$$\frac{3}{2}x_1^\epsilon + x_2^\epsilon - 4 + \epsilon = 0. \tag{2}$$

From i), we have

$$(a_{1\epsilon}, a_{2\epsilon}) \in \prod_{i=1}^2 \partial F_i(x_\epsilon). \tag{3}$$

So that using (3) and iii), we obtain

$$0 \in \partial F_1(x_\epsilon) + \partial F_2(x_\epsilon) + \partial(\alpha_\epsilon f)(x_\epsilon) + \mathcal{N}_X(x_\epsilon).$$

Assume that $x_\epsilon \in \text{int } X$, if it exists. Then, $\mathcal{N}_X(x_\epsilon) = \{0_{\mathbb{R}^2}\}$ and

$$0 \in \partial F_1(x_\epsilon) + \partial F_2(x_\epsilon) + \alpha_\epsilon \partial f(x_\epsilon)$$

which is explicitly written as

$$\begin{pmatrix} 2x_1^\epsilon - 1 - \frac{3}{2}\alpha_\epsilon \\ 4x_2^\epsilon - \alpha_\epsilon \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Using (2), we obtain

$$x_\epsilon = \left(\frac{25}{11} - \frac{6}{11}\epsilon, \frac{13}{11} - \frac{2}{11}\epsilon \right)^\top \text{ and } \alpha_\epsilon = \frac{26}{11} - \frac{8}{11}\epsilon.$$

Moreover, from (3), we obtain

$$a_{1\epsilon} = \left(\frac{50}{11} - \frac{12}{11}\epsilon, \frac{26}{11} - \frac{4}{11}\epsilon \right)^\top \text{ and } a_{2\epsilon} = \left(-1, \frac{26}{11} - \frac{4}{11}\epsilon \right)^\top.$$

We have $x_\epsilon \rightarrow \bar{x} = \left(\frac{25}{11}, \frac{13}{11} \right)^\top$, as $\epsilon \rightarrow 0^+$. Then, according to Theorem 6, the point $\bar{x} = \left(\frac{25}{11}, \frac{13}{11} \right)^\top$ is a properly efficient solution of the original semivectorial bilevel programming problem (S).

5 Necessary Optimality Conditions for the Semivectorial Bilevel Programming Problem (S)

In this section, from the results obtained in the previous section for the scalarized problem (S_ϵ^s) of (S_ϵ) , we will deduce necessary optimality conditions for problem (S). The following additional assumptions will be needed.

- $(\mathcal{H}_1)$ For every $\epsilon > 0$ sufficiently small, there exists $x_\epsilon \in \text{int } X$ such that $f(x_\epsilon) \leq v + \epsilon$,
- $(\mathcal{H}_2)$ $\inf_{x \in \mathbb{R}^p} F_i(x) < \inf_{x \in X} F_i(x)$, $i = 1, \dots, k$,
- $(\mathcal{H}_3)$ $\inf_{x \in \mathbb{R}^p} f(x) < \inf_{x \in X} f(x)$.

Let us give the following simple example where assumptions $(\mathcal{H}_1)$ – $(\mathcal{H}_3)$ and assumptions of convexity and compactness are satisfied.

Example 2 Let $X = [1, 2]$, F and f be the functions defined on $\mathbb{R}$ by

$$F(x) = (F_1(x), F_2(x)) = (x^2, x + 1), \quad f(x) = |x| + \frac{3}{2}.$$

Then, X is a convex compact set and the functions F and f are convex. Let us verify if assumptions $(\mathcal{H}_1)$ – $(\mathcal{H}_3)$ are satisfied.

- Assumption $(\mathcal{H}_1)$: Let $\epsilon > 0$ sufficiently small. We have $v = \frac{5}{2}$ and $\mathcal{M}_\epsilon = [1, 1 + \epsilon]$. Let $x_\epsilon = 1 + \frac{\epsilon}{2}$. Then, $x_\epsilon \in \text{int } X$ and assumption $(\mathcal{H}_1)$ is satisfied.
- Assumption $(\mathcal{H}_2)$: We have

$$\begin{aligned} \inf_{x \in \mathbb{R}} F_1(x) &= 0 < \inf_{x \in X} F_1(x) = 1, \\ \inf_{x \in \mathbb{R}} F_2(x) &= -\infty < \inf_{x \in X} F_2(x) = 2. \end{aligned}$$

So that, $(\mathcal{H}_2)$ is satisfied.

- Assumption $(\mathcal{H}_3)$: We have

$$\inf_{x \in \mathbb{R}} f(x) = \frac{3}{2} < \inf_{x \in X} f(x) = \frac{5}{2}.$$

So that, $(\mathcal{H}_3)$ is satisfied.

Let $\lambda = (\lambda_1, \dots, \lambda_k)^\top \in \text{int}(\mathbb{R}_+^k)$ be fixed. For simplicity, in the sequel, the problem $(S_{\epsilon_n}^s)$ will be denoted by (S_n^s) .

The following theorem gives necessary optimality conditions for (S) . The necessary optimality conditions are given for the class of properly efficient solutions of (S) which are accumulation points of a sequence of some particular feasible points of problems (S_n^s) , $n \in \mathbb{N}$.

Theorem 9 *Let $\epsilon_n \searrow 0^+$. Assume that assumptions $(\mathcal{H}_1)$ – $(\mathcal{H}_3)$ are satisfied. Let x_n be the feasible point of the scalarized problem (S_n^s) given by assumption $(\mathcal{H}_1)$ corresponding to ϵ_n . Moreover, assume that x_n satisfies together with a certain feasible point (a_n, α_n) of the dual problem $(\mathcal{D}_n^s)$ the conditions i)–iii) in Theorem 8, with $a_n = (a_{1n}, \dots, a_{kn})$, $a_{in} \in \mathbb{R}^p$, $i = 1, \dots, k$, and $\alpha_n > 0$. Let $\bar{x}$ be an accumulation point of the sequence $(x_n)_n$. Then, $\bar{x}$ is a properly efficient solution of (S) and there exists (a, α) , $a = (a_1, \dots, a_k)$, $a_i \in \mathbb{R}^p$, $\alpha \in \mathbb{R}_+^*$, such that the following optimality conditions are satisfied*

- a) $a \in \prod_{i=1}^k \partial F_i(\bar{x})$,
- b) $0 \in \partial f(\bar{x}) + \mathcal{N}_X(\bar{x})$,
- c) $-\sum_{i=1}^k \lambda_i a_i \in \alpha \partial f(\bar{x})$.

Furthermore, $\bar{x}$ solves the scalar unconstrained minimization problem

$$\min_{x \in \mathbb{R}^n} \left(\sum_{i=1}^k \lambda_i F_i + \alpha f \right) (x).$$

Proof Let $\mathcal{N}$ be an infinite subset of $\mathbb{N}$ such that $x_n \rightarrow \bar{x}$, $n \rightarrow +\infty$, $n \in \mathcal{N}$. Theorem 8 implies that x_n solves the problem

$$(S_n^s) \quad \min_{\substack{x \in \mathbb{R}^p \\ \psi_X(x) \leq 0 \\ f(x) - v - \epsilon_n \leq 0}} \sum_{i=1}^k \lambda_i F_i(x).$$

Then, from Theorem 5, x_n is a properly efficient solution of problem (S_n) . Therefore, from Theorem 6, $\bar{x}$ is a properly efficient solution of problem (S) as an accumulation point of the sequence (x_n) . On the other hand, for every $n \in \mathcal{N}$, we have

- i) $F_i(x_n) + F_i^*(a_{in}) = \langle a_{in}, x_n \rangle \quad \forall i \in \{1, \dots, k\}$,
- ii) $\alpha_n (f(x_n) - v - \epsilon_n) = 0$,
- iii) $(\alpha_n f)_X^* \left(-\sum_{i=1}^k \lambda_i a_{in} \right) = \langle -\sum_{i=1}^k \lambda_i a_{in}, x_n \rangle - \alpha_n (v + \epsilon_n)$.

Since $\alpha_n \neq 0$, then, the complementary slackness condition ii) yields

$$f(x_n) = v + \epsilon_n.$$

Letting $n \rightarrow +\infty$, $n \in \mathcal{N}$, we get $f(\bar{x}) = v$. That is $\bar{x}$ solves the lower level problem $(\mathcal{P})$. Hence

$$0 \in \partial f(\bar{x}) + \mathcal{N}_X(\bar{x}).$$

The first condition i) is equivalent to

$$a_{in} \in \partial F_i(x_n) \quad \forall i \in \{1, \dots, k\}.$$

Let $i \in \{1, \dots, k\}$. We have $\partial F_i(x_n) \subset \bigcup_{x \in X} \partial F_i(x)$. Since X is compact, it follows that the set $\partial F_i(X) = \bigcup_{x \in X} \partial F_i(x)$ is compact (Theorem 1). Then, there exists an infinite subset $\mathcal{N}_1 \subset \mathcal{N}$ such that the sequence $(a_{in})_{n \in \mathcal{N}_1}$ converges to a certain point a_i . Since, for any $n \in \mathcal{N}_1$, we have

$$F_i(y) \geq F_i(x_n) + \langle a_{in}, y - x_n \rangle \quad \forall y \in \mathbb{R}^p$$

then, passing to the limit as $n \rightarrow +\infty, n \in \mathcal{N}_1$, we get

$$F_i(y) \geq F_i(x) + \langle a_i, y - \bar{x} \rangle \quad \forall y \in \mathbb{R}^p.$$

That is

$$a_i \in \partial F_i(\bar{x}) \quad \forall i \in \{1, \dots, k\}. \quad (4)$$

It follows that

$$a = (a_1, \dots, a_k) \in \prod_{i=1}^k \partial F_i(\bar{x}).$$

On the other hand, for every $n \in \mathcal{N}_1$, from the condition iii), we have

$$\begin{aligned} (\alpha_n f)_X^* \left(-\sum_{i=1}^k \lambda_i a_{in} \right) &= \left\langle -\sum_{i=1}^k \lambda_i a_{in}, x_n \right\rangle - \alpha_n (v + \epsilon_n) \\ &= \left\langle -\sum_{i=1}^k \lambda_i a_{in}, x_n \right\rangle - \alpha_n f(x_n). \end{aligned}$$

So

$$\left\langle -\sum_{i=1}^k \lambda_i a_{in}, x_n \right\rangle - \alpha_n f(x_n) \geq \left\langle -\sum_{i=1}^k \lambda_i a_{in}, x \right\rangle - \alpha_n f(x) \quad \forall x \in X.$$

Hence

$$\alpha_n f(x) + \left\langle \sum_{i=1}^k \lambda_i a_{in}, x \right\rangle \geq \alpha_n f(x_n) + \left\langle \sum_{i=1}^k \lambda_i a_{in}, x_n \right\rangle \quad \forall x \in X.$$

Therefore, for all $n \in \mathcal{N}_1$, x_n is a solution of the minimization problem

$$\min_{x \in X} \left(\alpha_n f(x) + \left\langle \sum_{i=1}^k \lambda_i a_{in}, x \right\rangle \right).$$

It follows that

$$0 \in \alpha_n \partial f(x_n) + \sum_{i=1}^k \lambda_i a_{in} + \mathcal{N}_X(x_n).$$

Since $x_n \in \text{int } X$, then $\mathcal{N}_X(x_n) = \{0\}$. We deduce that

$$-\frac{1}{\alpha_n} \sum_{i=1}^k \lambda_i a_{in} \in \partial f(x_n) \quad \forall n \in \mathcal{N}_1.$$

Using the fact that $\partial f(x_n) \subset \bigcup_{x \in X} \partial f(x)$ for all $n \in \mathcal{N}_1$, and that $\bigcup_{x \in X} \partial f(x)$ is compact, we deduce that there exist an infinite subset $\mathcal{N}_2$ of $\mathcal{N}_1$ and $x^* \in \mathbb{R}^p$ such that $-\frac{1}{\alpha_n} \sum_{i=1}^k \lambda_i a_{in} \rightarrow x^*$, as $n \rightarrow +\infty, n \in \mathcal{N}_2$. On the other hand, for $n \in \mathcal{N}_2$, we have

$$f(y) \geq f(x_n) + \left\langle -\frac{1}{\alpha_n} \sum_{i=1}^k \lambda_i a_{in}, y - x_n \right\rangle \quad \forall y \in \mathbb{R}^p.$$

Therefore, passing to the limit as $n \rightarrow +\infty, n \in \mathcal{N}_2$, we obtain

$$f(y) \geq f(\bar{x}) + \langle x^*, y - \bar{x} \rangle \quad \forall y \in \mathbb{R}^p.$$

That is $x^* \in \partial f(x)$. Assumptions $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$ imply respectively that $a_i \neq 0 \quad \forall i \in \{1, \dots, k\}$ and $x^* \neq 0$. For $n \in \mathcal{N}_2$, set $x_n^* = -\frac{1}{\alpha_n} \sum_{i=1}^k \lambda_i a_{in}$. Since

$$\|x_n^*\| \rightarrow \|x^*\| > 0 \quad \text{as } n \rightarrow +\infty, n \in \mathcal{N}_2$$

then, there exists $n_0 \in \mathcal{N}_2$ such that

$$\|x_n^*\| > 0 \quad \forall n \geq n_0, n \in \mathcal{N}_2.$$

Hence, for $n \in \mathcal{N}_2, n \geq n_0$, we have $\alpha_n = \frac{\|\sum_{i=1}^k \lambda_i a_{in}\|}{\|x_n^*\|}$ which converges to $\alpha = \frac{\|\sum_{i=1}^k \lambda_i a_i\|}{\|x^*\|}$, with $\alpha \neq 0$. Besides, the sequence $(-\frac{1}{\alpha_n} \sum_{i=1}^k \lambda_i a_{in})_{n \in \mathcal{N}_2}$ converges to $-\frac{\sum_{i=1}^k \lambda_i a_i}{\alpha} = x^*$. Hence

$$-\sum_{i=1}^k \lambda_i a_i \in \alpha \partial f(\bar{x}). \quad (5)$$

Using (4), (5) and Theorem 2, we obtain

$$0 \in \partial \left(\sum_{i=1}^k \lambda_i F_i + \alpha f \right) (\bar{x}).$$

Therefore, $\bar{x}$ solves the scalar minimization problem

$$\min_{x \in \mathbb{R}^p} \left(\sum_{i=1}^k \lambda_i F_i + \alpha f \right) (x).$$

□

6 Sufficient Optimality Conditions for (S)

In this section, we will provide sufficient optimality conditions for the semivectorial bilevel programming problem (S). These optimality conditions are given regardless of the duality approach.

Theorem 10 *Let $\bar{x} \in \mathbb{R}^p$. Assume that there exists (a, α, λ) , $a = (a_1, \dots, a_k)$, $a_i \in \mathbb{R}^p$, $\alpha \in \mathbb{R}_+^*$, $\lambda = (\lambda_1, \dots, \lambda_k)^\top \in \text{int}(\mathbb{R}_+^k)$ such that the following optimality conditions are satisfied*

- i) $a \in \prod_{i=1}^k \partial F_i(\bar{x})$,
- ii) $0 \in \partial f(\bar{x}) + \mathcal{N}_X(\bar{x})$,
- iii) $-\sum_{i=1}^k \lambda_i a_i \in \alpha \partial f(\bar{x})$.

Then, $\bar{x}$ is a properly efficient solution of problem (S).

Proof Feasibility: The property ii) implies that $\bar{x}$ solves the problem $(\mathcal{P})$. So, it is a feasible point of problem (S).

Optimality: From property i), we have

$$\sum_{i=1}^k \lambda_i a_i \in \partial \left(\sum_{i=1}^k \lambda_i F_i \right) (\bar{x}). \quad (6)$$

Hence, from property iii) and (6), we obtain

$$0 \in \partial \left(\sum_{i=1}^k \lambda_i F_i + \alpha f \right) (\bar{x}).$$

Therefore, $\bar{x}$ solves the problem

$$\min_{x \in \mathbb{R}^n} \left(\sum_{i=1}^k \lambda_i F_i + \alpha f \right) (\bar{x}).$$

It follows that

$$\sum_{i=1}^k \lambda_i F_i(\bar{x}) + \alpha f(\bar{x}) \leq \sum_{i=1}^k \lambda_i F_i(x) + \alpha f(x) \quad \forall x \in \mathbb{R}^p.$$

Then

$$\sum_{i=1}^k \lambda_i F_i(\bar{x}) \leq \sum_{i=1}^k \lambda_i F_i(x) + \alpha(f(x) - f(\bar{x})) \quad \forall x \in \mathbb{R}^p.$$

Let $x \in \mathcal{M}$. Since $\bar{x} \in \mathcal{M}$, then $f(x) = f(\bar{x})$. Hence

$$\sum_{i=1}^k \lambda_i F_i(\bar{x}) \leq \sum_{i=1}^k \lambda_i F_i(x) \quad \forall x \in \mathcal{M}.$$

So, $\bar{x}$ is a solution of problem

$$\min_{x \in \mathcal{M}} \sum_{i=1}^k \lambda_i F_i(x).$$

Therefore, from Theorem 5, $\bar{x}$ is a properly efficient solution of (S). □

Theorem 11 Let $\bar{x} \in \mathbb{R}^p$. Assume that $\bar{x}$ satisfies the following conditions

- i) $0 \in \partial \left(\sum_{i=1}^k \partial F_i + \partial f \right) (\bar{x})$,
- ii) $0 \in \partial f(\bar{x}) + \mathcal{N}_X(\bar{x})$.

Then, $\bar{x}$ is a properly efficient solution of problem (S).

Proof Feasibility: The feasibility of $\bar{x}$ to problem (S) follows from the property ii).

Optimality: Property i) implies that $\bar{x}$ solves the problem

$$\min_{x \in \mathbb{R}^p} \left(\sum_{i=1}^k F_i + f \right) (\bar{x}).$$

So

$$\sum_{i=1}^k F_i(x) + f(x) \geq \sum_{i=1}^k F_i(\bar{x}) + f(\bar{x}) \quad \forall x \in \mathbb{R}^p$$

and hence

$$\sum_{i=1}^k F_i(x) \geq \sum_{i=1}^k F_i(\bar{x}) + (f(\bar{x}) - f(x)) \quad \forall x \in \mathbb{R}^p. \quad (7)$$

Let $x \in \mathcal{M}$. Then, $f(x) = f(\bar{x})$. It follows from (7) that

$$\sum_{i=1}^k F_i(x) \geq \sum_{i=1}^k F_i(\bar{x}) \quad \forall x \in \mathcal{M}.$$

That is $\bar{x}$ solves the problem

$$\min_{x \in \mathcal{M}} \sum_{i=1}^k F_i(x).$$

Therefore, Theorem 5 implies that $\bar{x}$ is a properly efficient solution of problem (S). $\square$

We give the following illustrative example for Theorem 10 in a nondifferentiable case.

Example 3 Let $X = [0, 1] \times [1, 2]$, F and f be the functions defined on $\mathbb{R}^2$ by

$$\begin{aligned} F(x_1, x_2) &= (F_1(x_1, x_2), F_2(x_1, x_2), F_3(x_1, x_2), F_4(x_1, x_2))^{\top} \\ &= \left(|x_1| + x_2^2 - x_2, x_1^2 - 4x_1 + 1, 2|x_1| - 3x_2, x_1 + \frac{1}{2}x_2^2 \right)^{\top}, \\ f(x_1, x_2) &= x_1^2 + x_1 + 1. \end{aligned}$$

Then, the functions F and f are convex and X is a convex compact set. We have

$$v = \inf_{(x_1, x_2)^{\top} \in X} f(x_1, x_2) = 1 \quad \text{and} \quad \mathcal{M} = \{0\} \times [1, 2].$$

Note that the function F is not differentiable on the set $\{0\} \times \mathbb{R}$. The semivectorial bilevel programming problem that we consider is

$$(S) \quad v\text{-} \min_{(x_1, x_2)^{\top} \in \mathcal{M}} \left(|x_1| + x_2^2 - x_2, x_1^2 - 4x_1 + 1, 2|x_1| - 3x_2, x_1 + \frac{1}{2}x_2^2 \right)^{\top},$$

where $\mathcal{M}$ is the solution set of problem

$$(P) \quad \min_{(x_1, x_2)^{\top} \in [0, 1] \times [1, 2]} (x_1^2 + x_1 + 1).$$

Let us determine a point $\bar{x} = (\bar{x}_1, \bar{x}_2)^{\top}$ that satisfies the sufficient optimality conditions in Theorem 10. Then, we are led to verify if there exists (a, α, λ) , $a = (a_1, \dots, a_4)$, $a_i \in \mathbb{R}^2$, $i = 1, \dots, 4$, $\alpha > 0$, and $\lambda = (\lambda_1, \dots, \lambda_4)^{\top} \in \text{int}(\mathbb{R}_+^4)$, such that the conditions i)–iii) in Theorem 10 are satisfied. These conditions are written as follows

- i) $(a_1, \dots, a_4)^{\top} \in \prod_{i=1}^4 \partial F_i(\bar{x}_1, \bar{x}_2)$,
- ii) $0 \in \partial f(\bar{x}_1, \bar{x}_2) + \mathcal{N}_X(\bar{x}_1, \bar{x}_2)$,
- iii) $-\sum_{i=1}^4 \lambda_i a_i \in \alpha \partial f(\bar{x}_1, \bar{x}_2)$.

Condition ii) implies that $\bar{x} \in \mathcal{M}$. Then, $\bar{x}_1 = 0$ and $\bar{x}_2 \in [1, 2]$. We have $\partial f(\bar{x}_1, \bar{x}_2) = \{(1, 0)^{\top}\}$ and

$$\begin{aligned} \partial F_1(\bar{x}_1, \bar{x}_2) &= [-1, 1] \times \{2\bar{x}_2 - 1\}, \quad \partial F_2(\bar{x}_1, \bar{x}_2) = \{(-4, 0)^{\top}\} \\ \partial F_3(\bar{x}_1, \bar{x}_2) &= [-2, 2] \times \{-3\}, \quad \partial F_4(\bar{x}_1, \bar{x}_2) = \{(1, \bar{x}_2)^{\top}\}. \end{aligned}$$

Take $\lambda = (1, 1, 1, 1)^\top$ and set $a_i = (a_{i1}, a_{i2})^\top$, $i \in \{1, \dots, 4\}$. From condition i), we obtain

$a_{11} \in [-1, 1]$, $a_{12} = 2\bar{x}_2 - 1$, $a_2 = (-4, 0)^\top$, $a_{31} \in [-2, 2]$, $a_{32} = -3$, $a_4 = (1, \bar{x}_2)^\top$ and condition iii) yields

$$\begin{cases} -a_{11} - a_{31} - \alpha + 3 = 0, \\ \bar{x}_2 = \frac{4}{3}. \end{cases}$$

So $\alpha \in [0, 6]$. We can take for example $\alpha = 1$, $a_{11} = 0$, and $a_{31} = 2$. Therefore, $(0, \frac{4}{3})^\top$ is a properly efficient solution of problem (S). The conditions i)–iii) are satisfied with $a = (a_1, \dots, a_4)^\top$, $a_1 = (0, \frac{5}{3})^\top$, $a_2 = (-4, 0)^\top$, $a_3 = (2, -3)^\top$, $a_4 = (1, \frac{4}{3})^\top$, $\alpha = 1$ and $\lambda = (1, 1, 1, 1)^\top$.

7 Conclusion

In this paper, we have given a duality approach for problem (S). Our investigation which is based on the operations of regularization and scalarization, is inspired from the work given in [1] and uses some results established in [3]. Since the semivectorial bilevel programming problem (S) and its scalarized problem does not satisfy the Slater condition which we use as a sufficient condition to establish strong duality, we have first proceeded to a regularization of problem (S). The obtained regularized problem (S_ϵ) and its scalarized problem (S_ϵ^s) satisfy this condition. Afterwards, under the data, we have established strong Fenchel–Lagrange duality and provided optimality conditions for the scalarized primal-dual pair (S_ϵ^s)-(D $_\epsilon^s$). Then, via the results established for the regularized scalarized case, we have provided necessary optimality conditions for a class of properly efficient solutions of problem (S). These properly efficient solutions are accumulation points of sequences of feasible points of problem ($S_{\epsilon_n}^s$) that satisfy the property considered in $(\mathcal{H}_1)$, $\epsilon_n \searrow 0^+$. Finally, we have provided sufficient optimality conditions for (S) regardless of the duality approach. We emphasize that the obtained results are new in the literature of multiobjective bilevel optimization, and extend the results given in [1] from the scalar case to the vectorial one. This extension corresponds to the case where the upper and lower levels are respectively vectorial and scalar.

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