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To cite this version:

HAL Id: hal-01708659
https://hal.archives-ouvertes.fr/hal-01708659
Submitted on 13 Feb 2018

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ALZHEIMER’S DISEASE AND PRION: AN IN VITRO MATHEMATICAL MODEL

IONEL S.CIUPERCA¹, MATTHIEU DUMONT¹,², ABDELKADER LAKMECHE³, PAULINE MAZZOCCO¹,², LAURENT PUJO-MENJOUET¹,², HUMAN REZAEI⁴ AND LÉON M. TINE¹,²

¹Université de Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208
Institut Camille Jordan
43 boulevard du 11 novembre 1918
F-69622 Villeurbanne cedex, France
²Inria Team Dracula
Inria Grenoble Rhône-Alpes Center
69100 Villeurbanne, France
³University Sidi Bel Abbes
Laboratory of Biomathematics
Sidi Bel Abbes, Algeria
⁴INRA, UR892 Virologie Immunologie Moléculaires
78352 Jouy-en-Josas, France

Abstract. Alzheimer’s disease (AD) is a fatal incurable disease leading to progressive neuron destruction. AD is caused in part by the accumulation in the brain of Aβ monomers aggregating into oligomers and fibrils. Oligomers are amongst the most toxic structures as they can interact with neurons via membrane receptors, including PrPc proteins. This interaction leads to the misconformation of PrPc into pathogenic oligomeric prions, PrPpol. In this work, we develop a model describing in vitro Aβ polymerization process. We include interactions between oligomers and PrPc, causing the misconformation of PrPc into PrPpol. The model consists of nine equations, including size structured transport equations, ordinary differential equations and delayed differential equations. We analyse the well-posedness of the model and prove the existence and uniqueness of solutions of our model using Schauder fixed point theorem and Cauchy-Lipschitz theorem. Numerical simulations are also provided to give an illustration of the profiles that can be obtained with this model.

1. Introduction.

1.1. Alzheimer’s disease and interaction with prions. According to the World Alzheimer Report, in 2015 more than 46 million people were living with dementia worldwide [24]. With 60% to 80% dementia cases, Alzheimer’s disease (AD) is considered as the most common dementia subtype [28]. AD is a fatal incurable disease leading to progressive neuron destruction, with memory impairment, issues to perform daily tasks and behavior changes as main consequences. Alzheimer’s disease is mainly caused by the accumulation of Aβ monomers inside the brain [17]. Aβ monomers are obtained from an abnormal cleavage of amyloid precursor proteins, that leads Aβ to be released outside the neuron [22]. These monomers are composed of 39 to 43 amino

2010 Mathematics Subject Classification. Primary: 35Q92, 82D60; Secondary: 35A02.

Key words and phrases. Alzheimer’s disease, prions, polymerization, advection-reaction equations, delay equations, existence of solutions.

This work has been supported by Association France-Alzheimer, SM 2014.
acids and are present in both normal and diseased brain tissues [16]. Most common forms of monomers are those composed by 40 amino acids (Aβ-40) and by 42 amino acids (Aβ-42) [5], with an increased Aβ-42/Aβ-40 ratio compared to normal brain tissues [16]. Aβ monomers have the ability to aggregate into insoluble oligomers and fibrils that eventually accumulate to form amyloid plaques [2]. Aβ oligomers are the most toxic form of Aβ, as they are able to directly interact with neurons, via membrane receptors, and cause cytotoxic damages [15, 21].

It has been shown that these interactions require the presence of prion protein [7, 18, 11]. This protein can be found on cell membranes in its normal form PrPc. When misfolded in its pathogenic form, called PrPsc, it is responsible of transmissible spongiform encephalopathies, such as Creutzfeldt-Jakob disease (CJD) for humans. Through the interactions with PrPc, Aβ oligomers impact synaptic plasticity, increasing neuron death by apoptosis [7, 18, 26]. Moreover, Aβ oligomers, when binding to PrPc, misfold these latter into PrPpol, which could lead to CJD triggering, in addition to AD. These interactions and the role of each species are difficult to study through biological experiments. This is the reason why mathematical modeling may help to identify and understand the complex mechanisms of AD, in order to bring an overall view for biologists.

1.2. Alzheimer’s disease and prions formation modeling. There exists a variety of mathematical models that study mechanisms of AD, especially aggregation of Aβ monomers and plaque formation (see for instance [20, 29, 8, 1, 4]). These models are usually based on Becker-Döring equations [3] or Smoluchowski equations [27] to describe polymer lengthening. Several mathematical models have also been developed to study PrPc proliferation only ([13, 25, 9, 6, 10], to cite a few). In these models, PrPc monomers are supposed to aggregate and form PrPsc. PrPpol are then able to split in two, increasing their number. However, to the best of our knowledge, only one mathematical model integrates both Aβ oligomers and PrPc. This model, proposed by Helal et al. [14], describes in vivo dynamics of Aβ oligomers and PrPc. Authors assumed that Aβ oligomers can bind to PrPc, providing a death signal to the neuron, or polymerize into fibrils, leading to plaque formation. However they did not consider the whole process of polymerization, the different types of oligomers, nor PrPpol catalysis by Aβ.

1.3. Objectives. Our aim herein is to introduce and study a new model describing the evolution of Aβ polymers and their interactions with PrPc. We study these mechanisms at the protein level and in a in vitro context. We describe Aβ polymerization process and the role of Aβ in the misconformation of PrPsc. We also distinguish Aβ-40 from Aβ-42, as they oligomerize in different ways [16]. Indeed, Aβ-42 monomers tend to aggregate faster than Aβ-40 monomers and to form larger polymers [5]. Moreover, presence of Aβ-42 polymers prevails in amyloid plaques, although Aβ-42 monomer concentration is around 10% of Aβ-40 concentration [5], and that impacts the emergence of AD. These are the reasons why we are interested in modeling distinctly the two dynamics, with different parameter values.

This paper is organized as follows. We first present the mathematical model proposed to describe in vitro dynamics of Aβ and prions. We then investigate its well-posedness. Finally, we present some numerical simulations and discuss our results.

2. Mathematical modeling. We choose to build our model in an in vitro context, as, to the best of our knowledge, only in vitro data seem to be available. And so, to obtain a consistent qualitative behavior in a first step, then to quantitatively estimate parameters, we decide to study only in vitro mechanisms. We therefore consider no source term of monomers or prions and no degradation of any proteins involved either. We study evolution and impact of Aβ seeded at time $t = 0$ in an environment containing PrPc.
Aβ-40 (respectively Aβ-42) monomers are able to aggregate to form small polymers that can polymerize and depolymerize into bigger structures, by attaching or loosing one monomer. These structures as referred to as Aβ-40 (respectively Aβ-42) proto-oligomers. Once these proto-oligomers reach a maximal size $x_0$, they are supposed to become stable structures called oligomers. We also consider that Aβ-40 (respectively Aβ-42) monomers can form Aβ-40 (respectively Aβ-42) fibrils in addition to proto-oligomers. These fibrils can polymerize and depolymerize, and can be carried out to β-amyloid plaques (in vivo by astrocytes). In our model, we assume the existence of one big amyloid plaque in which fibrils can still depolymerize. Therefore oligomers can also be carried out to β-amyloid plaque (in vivo by astrocytes). We assume that they are gathered into the same plaque as fibrils, with the difference that oligomers cannot depolymerize.

2.1. Notations. To study the evolution of different concentrations, defined at time $t \geq 0$, let us denote by:

- $m_i(t)$: concentration of Aβ monomers,
- $u_i(t, x)$: size density of Aβ proto-oligomers, with $0 \leq x < x_0$,
- $f_i(t, x)$: size density of Aβ fibrils, with $x \geq 0$,
- $f_{a,i}(t, x)$: size density of Aβ fibrils inside Aβ plaque, with $x \geq 0$,
- $w_i^0(t)$: concentration of Aβ oligomers,
- $u_{a,i}(t)$: concentration of Aβ oligomers inside Aβ plaque,
- $p_i(t)$: concentration of PrP$c$,
- $p_{sc}(t)$: concentration of PrP$c$,
- $C_i(t)$: concentration of complex Aβ/PrP$c$,

where $i = 1$ (respectively $i = 2$) stands for Aβ-40 (respectively Aβ-42). Definitions of model parameters (rates and growth velocities) are reported in Table 1. Figure 1 displays a schematic representation of the whole model, with all interactions that are taken into account between the different structures.

2.2. Model for Aβ-40 and -42 polymerization. The first submodel describing the process of Aβ-40 and Aβ-42 polymerization formally consists of six partial differential equations and two ordinary differential equations (system (I)). As the equations are similar for both Aβ-40 and Aβ-42, we give the model for $i = 1, 2$, where $i = 1$ refers to the model for Aβ-40. Our model is based on Lifshitz-Slyozov equations [19], describing the growth process of grains, with a continuous size $x$.

\[
\begin{aligned}
&\left\{ \begin{array}{ll}
\partial_t u_i(t, x) + \partial_x (v_i(t, x) u_i(t, x)) = \mu(x) m_i(t), \\
\partial_t f_i(t, x) + \partial_x (v_{f,i}(t, x) f_i(t, x)) = \mu(x) m_i(t) - \gamma_{f,i} f_i(t, x), \\
\end{array}
\right.

&\partial_t f_{a,i}(t, x) - b_{a,i}(t) \partial_x f_{a,i}(t, x) = \gamma_{f,i} f_i(t, x),
\end{aligned}
\]

\[
\begin{aligned}
m_i(t) &= -m_i(t) \left( \int_0^{+\infty} x \mu(x) dx + \int_0^{x_0} \mu(x) dx + \int_0^{+\infty} g_{f,i}(x) f_i(t, x) dx + \int_0^{x_0} g_{u,i}(x) u_i(t, x) dx \right) \\
&\quad + b_{a,i}(t) \int_0^{+\infty} f_{a,i}(t, x) dx + b_{f,i} \int_0^{+\infty} f_i(t, x) dx + b_i \int_0^{x_0} u_i(t, x) dx,
\end{aligned}
\]

with $t \in [0, +\infty)$ and $x \in [0, x_0)$ in equation (1) and $x \in [0, +\infty)$ in equations (2)-(3).
Parameter/Variable | Definition
--- | ---
t | Time
x | Size of fibrils and proto-oligomers
x_0 | Maximal size of Aβ proto-oligomers
μ(x) | Spontaneous creation of proto-oligomers or fibrils
v_i(t, x) | Polymerization/depolymerization rate of Aβ proto-oligomers
v_{f,i}(t, x) | Polymerization/depolymerization rate of Aβ fibrils
g_i(x) | Rate at which Aβ monomers are added to proto-oligomers
g_{f,i}(x) | Rate at which Aβ monomers are added to fibrils
b_i | Rate at which Aβ monomers are lost from proto-oligomers
b_{f,i} | Rate at which Aβ monomers are lost from fibrils
b_{a,i}(t) | Rate of Aβ monomers escaping amyloid plaque
γ_i | Displacement rate of Aβ oligomers into the plaque
γ_{f,i} | Displacement rate of Aβ fibrils into the plaque
δ_i | Reaction rate between Aβ oligomers and PrP^c
τ | Duration of PrP^c catalysis, with Aβ oligomers

Table 1. Description of model parameters. Parameters are given for i = 1, 2, i = 1 corresponding to parameters related to Aβ-40.

Equations (1)–(2) describe Aβ polymerization in proto-oligomers or fibrils, through standard size structured advection-reaction equations. As proposed in [12], the polymerization rates are given by:

\[
\begin{align*}
v_i(t, x) &= g_i(x)m_i(t) - b_i, \quad (t, x) \in [0, +\infty) \times [0, +\infty), \\
v_{f,i}(t, x) &= g_{f,i}(x)m_{f,i}(t) - b_{f,i}, \quad (t, x) \in [0, +\infty) \times [0, +\infty),
\end{align*}
\]
for \( i = 1, 2 \). We further assume that \( g_i \) and \( g_{f,i} \) are increasing functions of \( x \). Thus, these rates express a constant depolymerization of all polymers, while the polymerization process is accelerated by a high concentration of monomers and facilitated for longer polymers. Further assumptions on polymerization rates are given in Hypothesis 1.

**Hypothesis 1. Polymerization rates**

Rates \( v_i \) and \( v_{f,i} \), for \( i = 1, 2 \), are required to satisfy the following conditions:

- \( b_i > 0, \ b_{f,i} > 0, \)
- \( g_i(0) = 0, \ g_{f,i}(0) = 0, \lim_{x \to +\infty} g_i(x) = +\infty, \lim_{x \to +\infty} g_{f,i}(x) = +\infty, \)
- \( g_i \in C^0([0, +\infty)) \cap C^1((0, +\infty)), \ g_{f,i} \in C^0([0, +\infty)) \cap C^1((0, +\infty)), \)
- for all \( \varepsilon_0 > 0 \), there is a constant \( G_i > 0 \), such as for all \( x \geq \varepsilon_0, 0 \leq g_i(x) \leq G_i, \)
- for all \( \varepsilon_0 > 0 \), there is a constant \( G_{f,i} > 0 \), such as for all \( x \geq \varepsilon_0, 0 \leq g_{f,i}(x) \leq G_{f,i}. \)

It is important to note that for each time \( t \), there exists a critical size \( \pi(t) > 0 \), for which polymerization rate is null, this critical size depending of the monomer concentration at time \( t \). Therefore, polymers of size smaller than \( \pi(t) \) depolymerize whereas polymers of size greater than \( \pi(t) \) tend to attach more monomers. This phenomenon is referred to as Ostwald ripening [23]. Let us remark that \( \pi(t) \) can be greater than \( x_0 \), and all proto-oligomers depolymerize in this case.

Finally, the term \( \mu(x) \) in equations (1)-(2) represents the ability of monomers to spontaneously aggregate in polymers smaller than \( x_0 \), to start the polymerization process. In our model, this function allows the creation of small proto-oligomers that could otherwise not exist due to the depolymerization of small polymers.

**Hypothesis 2. Function \( \mu \)**

We assume that \( \mu \) is a positive function with compact support, defined for all \( x \) in \([0, +\infty)\). Moreover, the function \( \mu \) is in \( L^1([0, +\infty), (1 + x)dx) \cap L^\infty([0, +\infty)) \).

Finally, equation (4), describing the evolution of A\( \beta \) monomers, is given by the gain and loss in monomer from every fibrils and proto-oligomers.

To complete the system, initial conditions are given by:

\[
\begin{align*}
\left\{ \begin{array}{l}
 u_i(t = 0, x) = u_i^0(x) \geq 0, \quad x \in [0, x_0),
 f_i(t = 0, x) = f_i^0(x) \geq 0, \quad x \in [0, +\infty),
 f_{a,i}(t = 0, x) = f_{a,i}^0(x) \geq 0, \quad x \in [0, +\infty),
 m_i(0) = m_i^0 > 0,
\end{array} \right.
\end{align*}
\]

We further assume that:

**Hypothesis 3. Initial conditions**

Initial condition \( u_i^0 \) is in \( L^1([0, x_0), (1 + x)dx) \cap L^\infty([0, x_0)) \). Initial conditions \( f_i^0 \) and \( f_{a,i}^0 \) are in \( L^1(\mathbb{R}+, (1 + x)dx) \cap L^\infty(\mathbb{R}+) \).

We also need boundary conditions in \( x = x_0 \), for proto-oligomers:

\[
\lim_{x \to x_0} u_i(t, x) = 0, \quad \text{if} \quad v_i(t, x_0) \leq 0, \quad i = 1, 2.
\]

This condition represents the fact that no oligomers of size \( x_0 \) depolymerize, even if the rate of polymerization is negative. From Hypothesis 1, let mention that no boundary condition is required at \( x = 0 \), for the simple reason that the polymerization/depolymerization rates for proto-oligomers and fibrils are negative when \( x \) tends to 0.
2.3. Model for A\(\beta\)-Prion interaction. We now introduce the second submodel, describing the interactions between A\(\beta\) oligomers and PrP\(c\). Misconformation process of PrP\(c\) into PrP\(o\) takes an incompressible duration, denoted \(\tau\), during which A\(\beta\) oligomer and PrP\(c\) form a complex. The oligomer is then released and can bind to another PrP\(c\). This reaction leads to a system of delayed differential equations (system (II)):

\[
\begin{align*}
    &u_i^0(t) = S_i(t) - \gamma u_i^0(t) - \delta_i p_c(t) u_i^0(t) + \delta_i p_o(t - \tau) u_i^0(t - \tau), \\
    &\dot{u}_{a,i}(t) = \gamma i u_i^0(t), \\
    &\dot{p}_c(t) = -\delta_1 p_c(t) u_i^0(t) - \delta_2 p_c(t) u_i^0(t), \\
    &\dot{p}_{sc}(t) = \delta_1 p_c(t - \tau) u_i^0(t - \tau) + \delta_2 p_c(t - \tau) u_i^0(t - \tau), \\
    &\dot{C}_i(t) = \delta_i p_o(t) u_i^0(t) - \delta_i p_c(t - \tau) u_i^0(t - \tau),
\end{align*}
\]

for \(t \in [\tau, +\infty)\), and \(i = 1, 2\), \(i = 1\) corresponding to equations for A\(\beta\)-40.

Equation (7) describes the evolution of A\(\beta\)-40 and A\(\beta\)-42 oligomers, with time. The first term \(S_i(t)\) stands for the source term of oligomers. It represents the creation rate of A\(\beta\) oligomers from proto-oligomers that reached the maximal size \(x_0\), and is the coupling with the previous system. The last terms describe the interaction between A\(\beta\) oligomers and PrP\(c\), leading to the formation of PrP\(o\) after a duration of \(\tau\) units of time. Equations (8) describe the displacement of A\(\beta\) oligomers into amyloid plaques. Finally, equations (9)–(11) describe the evolution of prions and complexes.

We assume that PrP\(c\) are the only prion proteins initially in the experiment, i.e., all initial conditions at \(t = 0\) are null, except for \(p_c(0)\), which is equal to \(p_o^0\) and is positive. Then, on \([0, \tau)\), the model is described using equations (7)–(11), without any delayed part, as no PrP\(o\) and oligomer are released from a complex during the first \(\tau\) units of time.

We now want to determine the expressions of \(S_i(t)\) and \(S_2(t)\), representing the source terms of A\(\beta\)-40 and A\(\beta\)-42 oligomers, that is the coupling between the first submodel and this second one. To do so, we use the property of mass conservation of the system. Indeed, as we are in an \textit{in vitro} context, the total mass \(Q(t)\) remains the same during the study (no source term and no loss). We first compute the value of \(Q\), denoting \(m_p\) the size of a prion PrP\(c\) or PrP\(o\):

\[
Q(t) = \sum_{i=1}^{2} \left( m_i(t) + \int_{0}^{x_0} f_i(t, x)dx + \int_{0}^{x_0} f_{a,i}(t, x)dx + \int_{0}^{x_0} x u_i(t, x)dx + x_0(u_i^0(t) + u_{a,i}(t)) \right) + m_p(p_c(t) + p_{sc}(t)) + (x_0 + m_p)(C_1(t) + C_2(t)).
\]

We then compute \(\dot{Q}(t)\), using equations (11)–(14) and (7)–(11). We finally obtain:

\[
\dot{Q}(t) = x_0 \left( S_1(t) - v_1(t, x_0) \lim_{x \to x_0} u_1(t, x) + S_2(t) - v_2(t, x_0) \lim_{x \to x_0} u_2(t, x) \right),
\]

which must be equal to zero. This equation gives sufficient conditions on \(S_i\):

\[
S_i(t) = v_i(t, x_0) \lim_{x \to x_0} u_i(t, x), \quad i = 1, 2.
\]

This condition gives an expression for the source term of oligomers, which is exactly the flow of proto-oligomers reaching the size \(x_0\). We can note that these source terms are non-negative, thanks to condition (6), and continuous.

3. Main results.
3.1. Existence of solutions for the system (I). To show the existence of solutions for the system (I), we based our analysis on the notion of “mild” solutions by introducing the characteristic curves associated to the kinetic rates at which monomers are added to or removal from fibrils or proto-oligomers. In the following definition we specify how “mild” solutions to equations (1)-(3) should be understood:

Definition 1. Mild solutions
Let \( L \in (0, \infty] \), \( T > 0 \); \( a, b : [0, T] \times [0, L] \to \mathbb{R} \) and \( u_0 : [0, L] \to \mathbb{R} \). We assume that \( a \) is a continuous function and satisfies

- \( a \) is a \( C^1 \) function in variable \( x \) on \((0, L)\)
- \( a \) is a globally Lipschitz function in \( x \) uniformly in time \( t \) on \([\varepsilon_0, L] \) \( \forall \varepsilon_0 \in (0, L)\)
- \( a(t, 0) < 0 \) \( \forall t \in [0, T] \).

We also assume that \( b \) is a continuous function with respect to \( t \) and \( x \).

Let consider the linear transport problem that consists to find a solution \( U : [0, T] \times [0, L] \to \mathbb{R} \) such that

\[
\begin{align*}
\partial_t U + \partial_x (aU) &= b, \quad (t, x) \in [0, T] \times [0, L), \\
U(t = 0, x) &= U_0(x), \quad x \in [0, L),
\end{align*}
\]

where in the case \( L < \infty \) we add the following boundary condition:

\[ U(t, L) = 0 \text{ if } a(t, L) \leq 0. \]  

Let \( s \to X(s, t, x) \) the characteristic curve defined for \( t \in [0, T] \) and \( x \in (0, L) \) by

\[
\begin{align*}
\frac{d}{ds} X(s, t, x) &= a(s, X(s, t, x)), \\
X(t, x) &= x.
\end{align*}
\]

Considering \( a_1(t, x) = \frac{\partial_x}{\partial t}(t, x) \) the function defined in \([0, T] \times (0, L)\), we denote by \( V_{t,x} \) the largest interval of all \( s \in [0, T] \) such that \( X(s, t, x) \in (0, L) \) \( \forall t \in [0, T] \). We denote also \( \bar{s} = \bar{s}(t, x) = \inf V_{t,x} \).

So, we call \( U \) to be a “mild” solution of (14)-(15) if for all \((t, x) \in [0, T] \times (0, L)\) we have that the function \( s \in V_{t,x} \to U(s, X(s, t, x)) \) satisfies the following system

\[
\begin{align*}
\frac{d}{ds} U &= -a(s, X(s, t, x))U + b(s, X(s, t, x)), \quad \forall s \in V_{t,x}, \\
U(0, X(0, t, x)) &= U_0(X(0, t, x)), \quad \text{if } \bar{s} = 0, \\
U(\bar{s}, X(\bar{s}, t, x)) &= 0, \quad \text{if } \bar{s} > 0.
\end{align*}
\]

With the previous definition, one can remark that \( X(s, t, x) \) is defined as the continuous extension of \( X(s, t, x) \) at \( \bar{s} \in V_{t,x} \). Such extension always exists.

Theorem 1. Existence of solutions for system (I)
Let Hypotheses 1, 2 and 3 hold. Then, for non-negative initial conditions, there exists \( T \) in \((0, +\infty)\) such that the system (I) has a unique non-negative “mild” solution \((u_i, f_i, f_{a,i}, m_i)\) defined for any \( t \) in \([0, T]\). Moreover:

- \( u_i \) is in \( L^\infty([0, T] \times [0, x_0]) \cap L^\infty([0, T]; L^1([0, x_0], (1 + x)dx) \cap C^0([0, T]; C^0([0, x_0]))), \)
- \( f_i \) and \( f_{a,i} \) are in \( L^\infty([0, T] \times \mathbb{R}_+) \cap L^\infty([0, T]; L^1(\mathbb{R}_+, (1 + x)dx) \cap C^0([0, T]; C^0([\mathbb{R}_+, L^1(\mathbb{R}_+)) \)
and \( m_i \) is in \( L^\infty([0, T]) \cap C^0([0, T]). \)

Proof of existence of solutions follows an iterative process which is based on the fact that for a given function \( m_i \), we can compute the mild solutions \( \tilde{u}_i, \tilde{f}_i \) and \( \tilde{f}_{a,i}, i = 1, 2 \). Using these mild solutions, we can now compute \( m_i \) as the solution of equation (4). We build an application
Theorem 2. System (II) admits a unique solution on $[0, \tau]$. Besides, these solutions are non-negative for non-negative initial conditions.

3.2. Existence of solutions for the system (II). We now focus on the system of delayed differential equations. We state our main results for this submodel.

We first prove existence and uniqueness of solutions on $[0, \tau]$ with Cauchy-Lipschitz theorem, and extend this result to well-chosen time intervals, likewise for the non-negativity. The whole proof is given in section 5.

4. System (I)-Proof of the main results.

4.1. Mild solutions.

Lemma 1. Let $m_i$, $i = 1, 2$ be a continuous function defined for all $t \in [0, T]$, with $T > 0$. We assume that Hypotheses 1, 2 and 3 are satisfied. Then, there exist unique mild solutions $u_i$, $f_i$ and $f_{a,i}$, $i = 1, 2$ of equations (1)-(3) and they verify, for all $t \in [0, T]$:

\[
\begin{align*}
\int_0^T u_i(t, x)dx \leq ||u_i^n||_{L^1} + ||\mu||_{L^1} \int_0^T m_i(s)ds,
\int_0^T g_i(x)u_i(t, x)dx \leq ||g_i u_i^n||_{L^1} + ||g_i \mu||_{L^1} \int_0^T m_i(s)ds,
\int_0^{+\infty} f_i(t, x)dx \leq ||f_i^n||_{L^1} + ||\mu||_{L^1} \int_0^{+\infty} m_i(s)ds,
\int_0^{+\infty} g f_i(x) f_i(t, x)dx \leq ||g f_i f_i^n||_{L^1} + ||g f_i \mu||_{L^1} \int_0^{+\infty} m_i(s)ds,
\int_0^{+\infty} f_{a,i}(t, x)dx \leq ||f_{a,i}||_{L^1} + \gamma f_{a,i} ||\mu||_{L^1} \int_0^T m_i(s)ds.
\end{align*}
\]

Proof. Equation (I):

For $i = 1, 2$, we rewrite as follow the equation (1) which models the dynamics of the two family of proto-oligomers (Aβ40 and Aβ42)

\[
\begin{align*}
\frac{\partial u_i}{\partial t} + \frac{\partial (v_i u_i)}{\partial x} = \mu(x)m_i(t), \quad t \in [0, T]; \quad x \in (0, x_0),
\end{align*}
\]

Using the method of characteristics as depicted in Definition 1 of “mild” solution, we obtain

\[
\begin{align*}
u_i(t, x) = \tilde{u}_i^n(s, X_{u_i}(s, t, x))J_{u_i}(s, t, x) + \int_s^t \mu(X_{u_i}(s, t, x))m_i(s)J_{u_i}(s, t, x)ds
\end{align*}
\]

where $\tilde{u}_i^n(\sigma, y) = \begin{cases} 0 & \text{if } \sigma > 0, \\
u_i^n(y) & \text{if } \sigma = 0 \end{cases}$ is defined in the set $\{t = 0\} \cup \{x = x_0\}$ of the boundary of the domain of $(t, x)$, $J_{u_i}(s; t, x) = \exp(-\int_s^t \partial_x v_i(s, X_{u_i}(\sigma, t, x))d\sigma)$ is the Jacobian and $X_{u_i}$ is the characteristic curve associated $v_i$.

For $s$, on can easily check, by using the argument that characteristics not cross each other, that:
i) For all fixed \( t \in (0, T] \), the function \( x \in (0, x_0) \) \( \rightarrow s(t, x) \) is increasing. Therefore, for all \( t \in (0, T] \) the following limit exists: \( \lim_{x \to x_0, x < x_0} s(t, x) \) and we denote it by \( s_0(t) \).

ii) For all fixed \( t \in (0, T] \), \( \forall x_1, x_2 \in (0, x_0) \) with \( x_1 < x_2 \) and \( \forall \sigma \in V_{t,x_1} \cap V_{t,x_2} \) we have \( X(\sigma; t, x_1) < X(\sigma; t, x_2) \).

**Lemma 2.** With the additional assumption: \( u^{i.n} \) continuous on \([0, x_0]\), one obtains for all \( t \in (0, T] \) the existence of the following limit \( \lim_{x \to x_0, x < x_0} u_i(t, x) \) that we denote by \( \bar{u}_i(t) \).

The proof of the lemma 2 stands on two cases:

**case 1:**
Let assume \( \bar{s}_0(t) = 0 \). So, we have \( \bar{s}(t, x) = 0 \ \forall x < x_0 \) and the “mild” solution take the form

\[
u_i(t, x) = u^{i.n}_i(X_{u,i}(0; t, x))J_{u,i}(0; t, x) + \int_0^t \mu(X_{u,i}(s; t, x))m_i(s)J_{u,i}(s; t, x)\,ds.
\]

Let denote by \( X_{u,i}^0(s, t) \) the limit \( \lim_{x \to x_0, x < x_0} X_{u,i}(s; t, x) \) for all \( s \in (0, t] \). Using the dominated convergence theorem of Lebesgue, one has the existence of the limit

\[
\lim_{x \to x_0, x < x_0} J_{u,i}(s; t, x) \quad \text{because} \quad \frac{\partial \nu_i}{\partial x} \quad \text{is bounded on} \quad [0, T] \times [\varepsilon_0, x_0] \quad \forall \varepsilon > 0 \quad \text{and the characteristic} \quad X_{u,i}(\sigma; t, x) \quad \text{is far from} \quad 0.
\]

We denote by \( J_{u,i}^0 \) this limit that means \( \lim_{x \to x_0, x < x_0} J_{u,i}(s; t, x) = J_{u,i}^0(s; t) \).

We apply again the dominated convergence theorem of Lebesgue and deduce from the previous form of the “mild solution” the existence of the limit

\[
\lim_{x \to x_0, x < x_0} u_i(t, x) = u^{i.n}_i(X_{u,i}^0(0, t))J_{u,i}^0(0, t) + \int_0^t \mu(X_{u,i}^0(s, t))m_i(s)J_{u,i}^0(s, t)\,ds.
\]

**case 2:**
Let assume \( \bar{s}_0(t) > 0 \). For this case there exists \( x_t \in (0, x_0) \) such that \( \bar{s}(t, x) > 0 \ \forall x \in (x_t, x_0) \).

So we get the following expression for the “mild” solution

\[
u_i(t, x) = \int_{\bar{s}(t, x)}^t \mu(X_{u,i}(s; t, x))m_i(s)J_{u,i}(s; t, x)\,ds.
\]

Let consider the sequence \( (x_k)_{k \in \mathbb{N}} \to x_0 \), with \( x_k < x_0 \) and let prove the following convergence

\[
u_i(t, x_k) \xrightarrow{k \to +\infty} \int_{\bar{s}_0(t)}^t \mu(X_{u,i}^0(s, t))m_i(s)J_{u,i}^0(s, t)\,ds.
\]

To prove the relation (20) we know that \( \bar{s}(t, x_k) < \bar{s}_0(t) \), so one can compute

\[
|u_i(t, x_k) - \int_{\bar{s}_0(t)}^t \mu(X_{u,i}^0(s, t))m_i(s)J_{u,i}^0(s, t)\,ds| \\
= |\int_{\bar{s}_0(t)}^t m_i(s) \left( \mu(X_{u,i}(s, t, x_k))J_{u,i}(s, t, x_k) - \mu(X_{u,i}^0(s, t))J_{u,i}^0(s, t) \right)\,ds| + |\int_{\bar{s}(t, x_k)}^t \mu(X_{u,i}(s, t, x_k))m_i(s)J_{u,i}(s, t, x_k)\,ds|
\]

The first term converge to 0 thanks to the dominated convergence theorem of Lebesgue and the second term goes to 0 thanks to the fact that \( \bar{s}(t, x_k) \to \bar{s}_0(t) \) and that the term under the integral is bounded. That achieves the proof of the convergence result.
Lemma 3. Under assumptions of Lemma 2, the limit $\bar{u}_i(t) = \lim_{x \rightarrow x_0, x < x_0} u_i(t,x)$ is a measurable and bounded function which means $\bar{u}(t)$ belongs to $L^\infty(0,T)$.

Proof. In this proof we drop the index $i$ for sake of simplicity. Let's prove that the “mild” solution given by (19) is a measurable function at $(\tau, x)$. Proof.

Step 1: Let's prove that $\bar{u}(t)$ is a measurable function at $(\tau, x)$. Let introduce the sets $A_+ = \{(t, x) : \bar{s}(t,x) > 0\}$ and $A_0 = \{(t, x) : \bar{s}(t,x) = 0\}$. We split the solution as follows $u = u^1 + u^2$ where

$$u^1(t, x) = \begin{cases} 0 & \text{if } (t, x) \in A_+, \\ u^{in}(X_u(0,0,x))m(s)J_u(0,0,x) & \text{if } (t, x) \in A_0, \end{cases}$$

(21)

$$u^2(t, x) = \int_0^t \mu(X_u(s,t,x))m(s)J_u(s,t,x)ds.$$  

(22)

From the measurability of $\bar{s}$ we deduce that $A_+$ and $A_0$ are measurable. Knowing that $(t, x) \rightarrow u^{in}(X_u(0,0,x))J_u(0,0,x)$ is a continuous function on $A_0$, so it is also measurable on $A_0$. That achieves the proof of the measurability for $u^1$.

For the measurability of $u^2$, we put $D = \{(t, x, y) \in \mathbb{R}^3 : (t, x) \in [0, T] \times (0, x_0); y \in \mathbb{V}_{t,x} \cap [0, t]\}$ and introduce the function $\phi : D \rightarrow \mathbb{R}$ such that $\phi(t, x, y) = \int_y^t \mu(X_u(s,t,x))m(s)J_u(s,t,x)ds$. Let check the continuity of $\phi$ on $D$. We consider the sequence $(t_k, x_k, y_k)_{k \in \mathbb{N}} \in \mathbb{D}$ such that $(t_k, x_k, y_k) \rightarrow (t, x, y)$. The continuity of $\phi$ requires to prove the convergence to zero when $k \rightarrow +\infty$ of

$$\int_0^T \mu(X_u(s,t_k,x_k))m(s)J_u(s,t_k,x_k)\mathbb{I}_{[y_k,t_k]}(s) - \mu(X_u(s,t,x))m(s)J_u(s,t,x)\mathbb{I}_{[y,t]}(s)ds$$

(23)

The relation of equation (23) is based on the dominated convergence theorem of Lebesgue. The fact that the functions under the integral are bounded, it suffices to prove that $\forall s \in [0, T] - \{y, t\}$ one obtains

$$\mu(X_u(s,t_k,x_k))m(s)J_u(s,t_k,x_k)\mathbb{I}_{[y_k,t_k]}(s) \rightarrow k \rightarrow +\infty \mu(X_u(s,t,x))m(s)J_u(s,t,x)\mathbb{I}_{[y,t]}(s)$$

(24)

Case 1. Let assume $s \notin (y, t)$. In this case the result is straightforward because all terms vanish when $k$ is high.

Case 2. Let assume $s \in (y, t)$. So, one need just to show

$$\mu(X_u(s,t_k,x_k))J_u(s,t_k,x_k) \rightarrow k \rightarrow +\infty \mu(X_u(s,t,x))J_u(s,t,x).$$

Knowing that $y_k \rightarrow y$ and $t_k \rightarrow t$ then for $k$ large enough we have $s \in (y_k, t_k)$ that implies $s$ belongs either to $V_{t,x}$ and to $V_{t,x,y}$. So $X_u(s,t_k,x_k) \rightarrow k \rightarrow +\infty X_u(s,t,x)$ thanks to the continuity of the characteristic equation.

It remains to prove the convergence of the sequence of Jacobian functions and for that we need to prove the following result

$$\int_s^T \left\{ \frac{\partial v}{\partial x}(\sigma, X_u(\sigma,t_k,x_k))\mathbb{I}_{[s,t_k]}(\sigma) - \frac{\partial v}{\partial x}(\sigma, X_u(\sigma,t,x))\mathbb{I}_{[s,t]}(\sigma) \right\} d\sigma(\sigma) \rightarrow k \rightarrow +\infty 0.$$  

Here also we base our reasoning on the Lebesgue’s dominated convergence theorem and achieve the proof by showing the pointwise convergence of the function under the previous integral for almost every $\sigma \in (s, t)$.

Subcase 2.1. If $\sigma \notin (s, t)$ then the result is straightforward because one obtains $0 \rightarrow 0$.

Subcase 2.2. If $\sigma \in (s, t)$ then the fact that $\sigma \in [s, t_k]$ for large $k$ implies that one needs just to prove

$$\frac{\partial v}{\partial x}(\sigma, X_u(\sigma,t_k,x_k)) \rightarrow k \rightarrow +\infty \frac{\partial v}{\partial x}(\sigma, X_u(\sigma,t,x)).$$

The proof stands on the fact that $\sigma$ is
chosen in \( V_{t,x} \cap V_{t,x_0} \) that implies \( X_u(\sigma,t,k,x_0) \rightarrow X_u(\sigma,t,x) \). Then from the continuity of \( \frac{\partial v}{\partial x} \) with respect to \( X_u \) we achieve the proof of the continuity of \( \phi \) on \( D \).

For the measurability of \( u^2 \), one can write \( u^2 = \phi \circ \psi \) with \( \psi : [0,T] \times (0,x_0) \rightarrow \mathbb{R}^3 \) such that \( \psi(t,x) = (t,x,s(t,x)) \). We remark that \( g([0,T] \times (0,x_0)) \subset D \) and is also measurable because \( s \) is measurable. Then from Rudin’s book [Theorem I. 7, page 10] we obtain that \( u^2 \) is a measurable function. What achieves the Step 1 of the proof.

**Step 2**
Knowing that \( u(t,x) \) is measurable, we apply the Fubini theorem and deduce the existence of \( B \subset (0,x_0) \) with \( \text{mes}(B) = 0 \) (the measure of \( B \)) such that \( \forall x \in (0,x_0) B \) the function \( t \in [0,T] \rightarrow u(t,x) \) is measurable. So, \( \forall k \in \mathbb{N}^* \exists z_k \in (x_0 = \frac{1}{k},x_0) \) such that \( t \rightarrow u(t,z_k) \) is a measurable function. We have \( z_k \rightarrow x_0 \) then we deduce from Lemma \( \exists \) that \( u(t,z_k) \rightarrow u(t) \forall t \in [0,T] \). Then \( u(t) \) is measurable as limit of measurable sequence.

Now we easily see that \( u \) is bounded since we integrate bounded function on bounded intervals. Then we have \( u \in L^\infty(0,T) \).

Using the change of variables \( y = X_{u,i}(0,t,x) \) in the expression \( \int_{0}^{x_0} u_i(t,x) dx \), we deduce:

\[
\int_{0}^{x_0} u_i(t,x) dx = \int_{0}^{x_0} \frac{\partial u_i}{\partial x}(X_{u,i}(0,t,x)) J_{u,i}(0,t,x) dx \\
+ \int_{0}^{t} \left( m_i(s) \int_{0}^{x_0} \mu(X_{u,i}(s,t,x)) J_{u,i}(s,t,x) dx \right) ds, \\
= \int_{X_{u,i}(0,t,x_0)} \frac{\partial u_i}{\partial x}(y) dy + \int_{0}^{t} m_i(s) \int_{X_{u,i}(s,t,0)} \mu(y) dy ds, \\
\leq ||\frac{\partial u_i}{\partial x}||_{L^1} + ||\mu||_{L^1} \int_{0}^{t} m_i(s) ds.
\]

**Equation (2)**
For \( i = 1, 2 \), characteristic curves associated to the growth velocity of fibrils \( v_{f,i} \) are defined by:

\[
\begin{pmatrix}
\frac{d}{ds} X_{f,i}(s,t,x) = v_{f,i}(s,X_{f,i}(s,t,x)), \\
X_{f,i}(t,t,x) = x.
\end{pmatrix}
\]

As done previously (here there is no maximal size for the fibrils, \( L = \infty \)), we obtain the unique mild solution:

\[
f_i(t,x) = f_i^{in}(X_{f,i}(0,t,x)) e^{-\gamma f,i t} J_{f,i}(0,t,x) \\
+ \int_{0}^{t} \mu(X_{f,i}(s,t,x)) m_i(s) e^{-\gamma f,i (t-s)} J_{f,i}(s,t,x) ds,
\]

where \( J_{f,i}(s,t,x) = \partial X_{f,i}(s,t,x) = \exp(-\int_{s}^{t} \partial_{\sigma} v_{f,i}(\sigma,X_{f,i}(\sigma,t,x)) d\sigma) \) is the Jacobian.

We then have:

\[
\int_{0}^{+\infty} f_i(t,x) dx \leq \int_{X_{f,i}(0,t,0)}^{+\infty} f_i^{in}(y) dy + \int_{0}^{t} \left( m_i(s) \int_{X_{f,i}(s,t,0)}^{+\infty} \mu(y) dy \right) ds.
\]

The estimations are directly derived from this relation.

**Equation (3)**
For the last equations, characteristic curves are defined as follow, for \( i = 1, 2 \) (no maximal size:
\[ L = +\infty \]

\[
\begin{cases}
\frac{d}{ds} X_{f,a,s}(s; t, x) = -b_{a,i}(s), \\
X_{f,a,s}(t; t, x) = x.
\end{cases}
\]  

(25)

So the unique mild solution reads

\[ f_{a,i}(t, x) = f_{a,i}^{\text{in}}(X_{f,a,s}(0; t, x)) + \gamma_{f,i} \int_0^T f_i(s, X_{f,a,s}(s; t, x)) ds, \]

which gives us the last estimation. \qed

4.2. Proof of theorem [1]. Let denote by \( \Sigma_T \) the subset of \( C([0, T]) \) such as:

\[ \Sigma_T = \{ m_i \in C^0([0, T]) / 0 \leq m_i(t) \leq M_T \text{ and } m_i(0) = m_i^0 \}, \]

where \( T \) is in \((0, +\infty)\) and \( M_T \) is given by the subset above. We build the following mapping \( h \):

\[
h : \Sigma_T \rightarrow C^0([0, T])
\]

(27)

\[
\hat{m}_i(t) = -m_i(t) \tilde{A}_i(t) + \tilde{B}_i(t), \quad i = 1, 2,
\]

(28)

where

\[
\tilde{A}_i(t) = \int_0^\infty \mu(x) dx + \int_0^\infty g_i(x) \tilde{f}_i(t, x) dx + \int_0^\infty \mathcal{G}_{f,i}(x) \tilde{f}_i(t, x) dx,
\]

(29)

\[
\tilde{B}_i(t) = b_{a,i}(t) \int_0^\infty \tilde{f}_i(t, x) dx + b_{f,i} \int_0^\infty \tilde{f}_i(t, x) dx + b_i \int_0^\infty \tilde{u}_i(t, x) dx,
\]

(30)

and functions \((\tilde{u}_i, \tilde{f}_i, \tilde{f}_{a,i})\) are solutions of the following system of PDE:

\[
\begin{cases}
\partial_t \tilde{u}_i(t, x) + \partial_x ((g_i(x) \tilde{m}_i(t) - b_i \tilde{u}_i(t, x)) = \mu(x) \tilde{m}_i(t), \\
\partial_t \tilde{f}_i(t, x) + \partial_x (g_{f,i}(x) \tilde{m}_i(t) - b_{f,i} \tilde{f}_i(t, x)) = \mu(x) \tilde{m}_i(t) - \gamma_{f,i} \tilde{f}_i(t, x), \\
\partial_t \tilde{f}_{a,i}(t, x) - b_{a,i}(t) \partial_x \tilde{f}_{a,i}(t, x) = \gamma_{f,i} \tilde{f}_i(t, x).
\end{cases}
\]

(31)

To prove the existence of solutions, we follow a Schauder fixed point theorem.

Lemma 4. If

\[
0 < T < \frac{1}{\sqrt{||\mu||_{L^1} \max_a (b_{a,i} \gamma_{f,i} + b_{f,i} + b_i)}},
\]

(32)

with \( \bar{b}_{a,i} = \sup_{[0,T]} b_{a,i}(t) \),

then \( h(\Sigma_T) \) is a subset of \( \Sigma_T \).

Proof. Let \((\tilde{u}_i, \tilde{f}_i, \tilde{f}_{a,i})\) be mild solutions of system \([31]\). Then, for all \( t \) in \([0, T]\), \( \tilde{A}_i \) and \( \tilde{B}_i \) are well-defined thanks to lemma [1]. Their non-negativity is obvious as soon as initial data verify condition [5].

Equation \([25]\) is an ordinary differential equation and admits a continuous solution on \([0, T]\). This implies that \( m_i(t), i = 1, 2 \) is bounded by a constant \( M_T \), that can be computed.

\[
m_i(t) = m_i(0) \exp \left( -\int_0^t \tilde{A}_i(s) ds \right) + \int_0^t \tilde{B}_i(s) \exp \left( -\int_s^t \tilde{A}_i(\sigma) d\sigma \right) ds.
\]
As function $\tilde{A}_i$ is non-negative, we obtain
\[ m_i(t) \leq m_i(0) + \int_0^t \tilde{B}_i(s)ds \leq m_i(0) + T \sup_{[0,T]} \tilde{B}_i(t). \] (33)

We have to determine an upper bound for $\tilde{B}_i(t)$, using equation [30]:
\[ \tilde{B}_i(t) \leq \sup_{[0,T]} \left( b_{a,i}(t)||\tilde{f}_{a,i}(t,.)||_{L^1} \right) + b_f \sup_{[0,T]} ||\tilde{f}_i(t,.)||_{L^1} + b_i \sup_{[0,T]} \left( \int_0^x \tilde{a}_i(t,x)dx \right). \]

Estimations [18] provide the needed upper bounds. Moreover, $\tilde{m}_i$ is upper-bounded by $M_T$ for all $t$ lower than $T$, as it is in $\Sigma_T$. We obtain:
\[ m_i(t) \leq m_i(0) + T \left( \tilde{b}_{a,i}||\tilde{f}_{a,i}^m||_{L^1} + (b_f + \tilde{b}_{a,i} \gamma f_i) ||\tilde{f}_i^m||_{L^1} + b_i ||u_i^m||_{L^1} \right) \]

with $\tilde{b}_{a,i} = \sup_{[0,T]} b_{a,i}(t)$. This relation gives us the upper bound $M_T$:
\[ M_T = \max_{i} \left( m_i(0) + T \left( \tilde{b}_{a,i}||\tilde{f}_{a,i}^m||_{L^1} + (b_f + \tilde{b}_{a,i} \gamma f_i) ||\tilde{f}_i^m||_{L^1} + b_i ||u_i^m||_{L^1} \right) \right) \]

\[ M_T [1 - T^2 ||\mu||_{L^1} \max_{i} (\tilde{b}_{a,i} \gamma f_i + b_f + b_i)] = \max_{i} \left[ m_i(0) + T \tilde{b}_{a,i} ||\tilde{f}_{a,i}^m||_{L^1} \right] \]

\[ + T \left( \max_{i} (b_f + \tilde{b}_{a,i} \gamma f_i) ||\tilde{f}_i^m||_{L^1} + b_i ||u_i^m||_{L^1} \right). \] (34)

Because $T$ verifies relation [32], we have:
\[ 1 - T^2 ||\mu||_{L^1} \max_{i} (\tilde{b}_{a,i} \gamma f_i + b_f + b_i) > 0, \]
and the upper bound $M_T$ is well defined.

\[ \square \]

Lemma 5. $h(\Sigma_T)$ is a relatively compact subspace of $C^0([0,T])$.

Proof. We know that $\Sigma_T$ is a bounded subspace of $C^0([0,T])$. To use Ascoli theorem, we have to show the uniform equicontinuity of $h$. Let $\tilde{m}_i$ and $\tilde{n}_i$ be two elements of $\Sigma_T$, such as $m_i = h(\tilde{m}_i)$ and $n_i = h(\tilde{n}_i)$. We want to show that there exists a constant $K > 0$ such as
\[ ||m_i - n_i||_{L^\infty([0,T])} \leq K ||\tilde{m}_i - \tilde{n}_i||_{L^\infty([0,T])}, \quad i = 1, 2. \]

To lighten notations, we drop out subscript $i$ for now. We have
\[ \hat{m}(t) = -\hat{A}_m(t)m(t) + \hat{B}_m(t), \]
\[ \hat{n}(t) = -\hat{A}_n(t)n(t) + \hat{B}_n(t), \]
where $\hat{A}_m, \hat{B}_m, \hat{A}_n$ and $\hat{B}_n$ are obtained from system [31]. We are interested in the following quantity:
\[ \hat{m} - \hat{n} = -\hat{A}_m m + \hat{B}_m + \hat{A}_n n - \hat{B}_n. \]

We can transform this equality:
\[ (\hat{m} - \hat{n})(m - n) = (m - n)( -\hat{A}_m m + \hat{B}_m + \hat{A}_n n - \hat{B}_n), \]
\[ = -(m - n)^2 \hat{A}_n - m(m - n)(\hat{A}_m - \hat{A}_n) + (m - n)(\hat{B}_m - \hat{B}_n). \]
We thus have:
\[
\frac{1}{2} \frac{d}{dt} (m - n)^2 + (m - n)^2 \dot{A}_n = -m(m - n)(\dot{A}_m - \dot{A}_n) + (m - n)(\dot{B}_m - \dot{B}_n),
\]
\[
\frac{1}{2} \frac{d}{dt} (m - n)^2 \leq -m(m - n)(\dot{A}_m - \dot{A}_n) + (m - n)(\dot{B}_m - \dot{B}_n),
\]
\[
\leq (m - n)^2 + \frac{1}{2} M_f^2 (\dot{A}_m - \dot{A}_n)^2 + \frac{1}{2} (\dot{B}_m - \dot{B}_n)^2.
\]

According to Grönwall’s inequality, we obtain:
\[
(m(t) - n(t))^2 \leq \int_0^t \left( M_f^2 (\dot{A}_m(s) - \dot{A}_n(s))^2 + (\dot{B}_m(s) - \dot{B}_n(s))^2 \right) e^{2(t-s)} ds.
\]

Then,
\[
(m(t) - n(t))^2 \leq C_T \left( M_f^2 \sup_{[0,T]} (\dot{A}_m(t) - \dot{A}_n(t))^2 + \sup_{[0,T]} (\dot{B}_m(t) - \dot{B}_n(t))^2 \right),
\]
where \( C_T > 0 \).

**Lemma 6.** There exists \( \alpha \) and \( \beta \) real positive constants, such that
\[
\sup_{[0,T]} (|A_m(t) - A_n(t)|) \leq \alpha \sup_{[0,T]} (|\tilde{m}(t) - \tilde{n}(t)|),
\]
\[
\sup_{[0,T]} (|B_m(t) - B_n(t)|) \leq \beta \sup_{[0,T]} (|\tilde{m}(t) - \tilde{n}(t)|).
\]

Lemma 6 and relation 36 are sufficient to prove the uniform equicontinuity of \( h \). Then Ascoli theorem gives that \( h(\Sigma_T) \) is a relatively compact subspace of \( C^0([0,T]) \). Proof of lemma 6 is given in Appendix B.

**Lemma 7.** The application \( h \) defined in system (27) is a continuous application.

Proof. Let \( (\tilde{m}_{i,n})_{n \in \mathbb{N}} \) a sequence of elements from \( \Sigma_T \) which tends to \( \tilde{m}_i \) in \( \Sigma_T \). Is the limit of \( h(\tilde{m}_{i,n}) \) equal to \( h(\tilde{m}_i) \) when \( n \) tends to infinity?

We define sequences \( (\tilde{u}_{i,n})_{n \in \mathbb{N}}, (\tilde{f}_{i,n})_{n \in \mathbb{N}} \) and \( (\tilde{f}_{a,i,n})_{n \in \mathbb{N}} \), solutions of the following system of equations:
\[
\begin{aligned}
\partial_t \tilde{u}_{i,n}(t, x) + \partial_x ((g_i(x)\tilde{m}_{i,n}(t) - b_i)\tilde{u}_{i,n}(t, x)) &= \mu(x)\tilde{m}_{i,n}(t), \\
\partial_t \tilde{f}_{i,n}(t, x) + \partial_x ((g_{f,i}(x)\tilde{m}_{i,n}(t) - b_{f,i})\tilde{f}_{i,n}(t, x)) &= \mu(x)\tilde{m}_{i,n}(t) - \gamma_{f,i}\tilde{f}_{i,n}(t, x), \\
\partial_t \tilde{f}_{a,i,n}(t, x) - ba_{i}(t)\partial_x \tilde{f}_{a,i,n}(t, x) &= \gamma_{f,i}\tilde{f}_{i,n}(t, x).
\end{aligned}
\]

These sequences are used to compute \( \tilde{A}_{i,n} \) and \( \tilde{B}_{i,n} \) such as:
\[
\tilde{m}_{i,n}(t) = -\dot{A}_{i,n}(t)\tilde{m}_{i,n}(t) + \tilde{B}_{i,n}(t),
\]
where \( m_{i,n} = h(\tilde{m}_{i,n}) \).

Likewise, we define \( m_i = h(\tilde{m}_i) \):
\[
m_i(t) = -\dot{A}_i(t)m_i(t) + \tilde{B}_i(t).
\]

We proceed in the same way as in the proof of lemma 5 to obtain the following relation:
\[
|m_{i,n}(t) - m_i(t)|^2 \leq C_T (M_f^2 \sup_{[0,T]} |\dot{A}_{i,n} - \dot{A}_i|^2 + \sup_{[0,T]} |\dot{B}_{i,n} - \dot{B}_i|^2).
\]

We then apply lemma 6 and show that if \( \tilde{m}_{i,n} \) tends to \( \tilde{m}_i \) when \( n \) tends to infinity, then it implies that \( h(\tilde{m}_{i,n}) \) tends to \( h(\tilde{m}_i) \), which obviously is in \( \Sigma_T \).
Then, according to Schauder fixed point theorem, the application $h$ admits a fixed point $m^*_i = h(m^*_i)$. This implies that system (I) admits at least one solution.

**Uniqueness:** to prove uniqueness of the solution let us assume that $(u_1, f_1, f_{a,1}, m_1)$ and $(u_2, f_2, f_{a,2}, m_2)$ are two solutions of the system (I) with the same initial data $(u^i_0, f^i_0, f^i_{a,0}, m^0)$ as in equation (5).

Using the same arguments as in the proof of lemma 6 (see equation (35)) one deduces
\[
|m_1(t) - m_2(t)|^2 \leq \int_0^t (M_T^2|A_1(s) - A_2(s)|^2 + |B_1(s) - |B_2(s)|^2) e^{2(t-s)} ds,
\]
so,
\[
|m_1(t) - m_2(t)|^2 \leq e^{2T} \int_0^t (M_T^2|A_1(s) - A_2(s)|^2 + |B_1(s) - |B_2(s)|^2) ds.
\]

Now, using the result of lemma 6 to estimate the right hand side of the previous inequality, we have
\[
|m_1(t) - m_2(t)|^2 \leq e^{2T} \int_0^t (M_T^2\alpha^2 + \beta^2) |m_1(s) - m_2(s)|^2 ds
\]
\[
\leq e^{2T} (M_T^2\alpha^2 + \beta^2) \int_0^t |m_1(s) - m_2(s)|^2 ds.
\]

So the Grönnwall lemma gives
\[
|m_1(t) - m_2(t)|^2 \leq |m_1(0) - m_2(0)|^2 e^{\int_0^T (M_T^2\alpha^2 + \beta^2) ds},
\]
then using the fact that we have the same initial data, means $m_1(0) = m_2(0) = m^0$, we deduce $m_1(t) = m_2(t) \quad \text{so} \quad f_1 \equiv f_2, \quad u_1 \equiv u_2 \quad \text{and} \quad f_{a,1} \equiv f_{a,2}$. That concludes the uniqueness of the solution of (1)–(4).

The non-negativity of the unique solution of (1)–(4) is obvious as soon as initial data fulfill relation (5). The reader can easily check this point from explicit relations of mild solutions.

5. **System (II)-Proof of the main results.**

5.1. **Existence and uniqueness of solutions.** We first prove the existence of initial conditions on $[0, \tau)$, defined by the following system, with $i = 1, 2$:
\[
\begin{cases}
\dot{\varphi}_i(t) = S_i(t) - \gamma_i \varphi_i(t) - \delta_i \varphi_i(t) \varphi_{p_i}(t), \\
\dot{\varphi}_{a,i}(t) = \gamma_i \varphi_i(t), \\
\dot{\varphi}_{p_i}(t) = -\delta_1 \varphi_1(t) \varphi_{p_1}(t) - \delta_2 \varphi_2(t) \varphi_{p_2}(t), \\
\varphi_{C_i}(t) = \delta_i \varphi_i(t) \varphi_{p_i}(t), \\
\varphi_{p_i}(0) = p^0_{ci} \geq 0, \quad \varphi_i(0) = \varphi_{a,i}(0) = \varphi_{C_i}(0) = 0
\end{cases}
\]
with $S_i$ given by (13).

Due to the non-continuity of $S_i$ we can’t directly apply the Cauchy-Lipschitz theorem. So, in order to prove the existence result we use the following change of unknown
\[
\psi_i(t) = \varphi_i(t) - \int_0^t S_i(\sigma)d\sigma
\]
which is relevant because $S_i \in L^\infty(0, \tau)$ thanks to Lemma 3. We
rewrite the system (39) as follows:

\[
\begin{align*}
\dot{\psi}_i(t) &= -\gamma_i \psi_i(t) - \delta_i \psi_i(t) \varphi_{p_c}(t) - \delta_i \left( \int_0^t S_i(\sigma) d\sigma \right) \varphi_{p_c}(t) - \gamma_i \int_0^t S_i(\sigma) d\sigma, \\
\dot{\psi}_{a,i}(t) &= \gamma_i \psi_i(t) + \gamma_i \int_0^t S_i(\sigma) d\sigma, \\
\dot{\varphi}_{p_c}(t) &= -\delta_1 \psi_i(t) \varphi_{p_c}(t) - \delta_2 \psi_2(t) \varphi_{p_c}(t) - \left( \int_0^t (\delta_1 S_1(\sigma) + \delta_2 S_2(\sigma)) d\sigma \right) \varphi_{p_c}(t), \\
\dot{\varphi}_{C_i}(t) &= \delta_i \psi_i(t) \varphi_{p_c}(t) + \delta_1 \left( \int_0^t S_i(\sigma) d\sigma \right) \varphi_{p_c}(t), \\
\varphi_{p_c}(0) &= p_{0_c}^0 > 0, \quad \psi_i(0) = \varphi_{a,i}(0) = \varphi_{C_i}(0) = 0.
\end{align*}
\]

(40)

For the existence let us note the vector

\[ X(t) = \psi_1(t), \psi_2(t), \varphi_{a,1}(t), \varphi_{a,2}(t), \varphi_{p_c}(t), \varphi_{C_1}(t), \varphi_{C_2}(t). \]

We have to solve the following Cauchy problem:

\[
\begin{align*}
\dot{X}(t) &= F(t, X(t)), \quad 0 \leq t < \tau, \\
X(0) &= \psi(0, 0, 0, 0, p_{0_c}^0, 0, 0),
\end{align*}
\]

(41)

where \( F(t, X) \) is defined by

\[
F(t, X) = \begin{pmatrix}
-\gamma_1 X_1 - \delta_1 X_1 X_5 - \delta_1 \left( \int_0^t S_1(\sigma) d\sigma \right) X_5 - \gamma_1 \int_0^t S_1(\sigma) d\sigma \\
-\gamma_2 X_2 - \delta_2 X_2 X_5 - \delta_2 \left( \int_0^t S_2(\sigma) d\sigma \right) X_5 - \gamma_2 \int_0^t S_2(\sigma) d\sigma \\
\gamma_1 X_1 + \gamma_1 \int_0^t S_1(\sigma) d\sigma \\
\gamma_2 X_2 + \gamma_2 \int_0^t S_2(\sigma) d\sigma \\
-\delta_1 X_1 X_5 - \delta_2 X_2 X_5 - \left( \int_0^t (\delta_1 S_1 + \delta_2 S_2)(\sigma) d\sigma \right) X_5 \\
\delta_1 X_1 X_5 + \delta_1 \left( \int_0^t S_1(\sigma) d\sigma \right) X_5 \\
\delta_2 X_2 X_5 + \delta_2 \left( \int_0^t S_2(\sigma) d\sigma \right) X_5
\end{pmatrix}
\]

Function \( F \) is continuous for \( t \) and Lipschitz with respect to the second variable \( X \). Indeed components \( F_i \), \( i = 1 \) to \( 7 \), are continuously differentiable with respect to the second variable. Cauchy-Lipschitz theorem gives the local existence and uniqueness of solution for problem (41). Thereby we have the local existence of solution for the system (39). The global existence of the solution of (39) on \([0, \tau]\) requires the solution \( X(t) = \psi_1(t), \varphi_2(t), \varphi_{a,1}(t), \varphi_{a,2}(t), \varphi_{p_c}(t), \varphi_{C_1}(t), \varphi_{C_2}(t) \) to be bounded and non-negative on \([0, \tau]\). To prove that, let us start with the initial conditions defined in system (39). We know that:

\[
\varphi_{p_c}(t) = - (\delta_1 \varphi_1(t) + \delta_2 \varphi_2(t)) \varphi_{p_c}(t).
\]

(42)

This equation can easily be written as: \( \varphi_{p_c}(t) = p_{0_c}^0 e^{- \left( \int_0^t (\delta_1 \varphi_1(s) + \delta_2 \varphi_2(s)) ds \right)} \), which is positive for all \( t \) in \([0, \tau]\), as \( p_{0_c}^0 \) is greater than 0. In addition it is straightforward that \( \varphi_{p_c}(t) \leq p_{0_c}^0 \).
Then, for \( i = 1, 2 \), we have
\[
\dot{\varphi}_i(t) = S_i(t) - \gamma_i \varphi_i(t) - \delta_i \varphi_i(t) \varphi_p(t) - (\gamma_i + \delta_i \varphi_p(t)) \varphi_i(t),
\]
\[
\varphi_i(t) \geq \varphi_i(0) \exp \left( - \int_0^t \gamma_i + \delta_i \varphi_p(s) ds \right).
\]
As \( \varphi_i(0) = 0 \), we have the non-negativity of \( \varphi_i(t) \) for \( t \) in \([0, \tau]\), and \( i = 1, 2 \). In addition, one can deduce that \( \varphi(t) \leq \int_0^t S_i(\sigma) d\sigma \).

As \( \varphi_1(t), \varphi_2(t) \) and \( \varphi_p(t) \) are greater or equal to 0, for all \( t \) in \([0, \tau]\), functions \( \varphi_{a,i} \) and \( \varphi_{C,i} \), \( i = 1, 2 \) are increasing. This implies the non-negativity of these functions for all \( t \) in \([0, \tau]\), as \( \varphi_{a,i}(0) \) and \( \varphi_{C,i} \), \( i = 1, 2 \) are null. One can easily verify that \( \varphi_{a,i} \) and \( \varphi_{C,i} \), \( i = 1, 2 \) are bounded. We further define \( X(\tau) \) as \( X(\tau) = \lim_{t \to \tau} X(t) \).

We prove existence and uniqueness of solutions of system (II) on \([\tau, +\infty)\) with a method of steps. We first study the system (II) on \([\tau, 2\tau]\). We have to solve the following Cauchy problem:
\[
\begin{align*}
\dot{Y}(t) &= G(t, Y(t), Y(t - \tau)), \quad \tau \leq t < 2\tau, \\
Y(t) &= \tilde{X}(t), \quad 0 \leq t \leq \tau,
\end{align*}
\]
where \( Y(t) = \int^t_0 \left( u_1^0(t), u_2^0(t), u_{a,1}(t), u_{a,2}(t), p_c(t), p_{sc}(t), C_1(t), C_2(t) \right) \), \( \tilde{X}(t) = \int^t_0 \left( \varphi_1(t), \varphi_2(t), \varphi_{a,1}(t), \varphi_{a,2}(t), \varphi_{p,c}(t), 0, \varphi_{C_1}(t), \varphi_{C_2}(t) \right) \), and \( G \) is defined by:
\[
G(t, Y, Z) = \begin{pmatrix}
S_1(t) - \gamma_1 Y_1 - \delta_1 Y_1 Y_5 + \delta_1 Z_1 Z_5 \\
S_2(t) - \gamma_2 Y_2 - \delta_2 Y_2 Y_5 + \delta_2 Z_2 Z_5
\end{pmatrix}.
\]
Here, we perform again a change of variable as done previously in order to overcome the non continuity of \( S_i, \ i = 1, 2 \). With the same strategy, we can actually re-write system (43) as follow:
\[
\begin{align*}
\dot{\tilde{Y}}(t) &= \tilde{G}(t, \tilde{Y}(t), \tilde{X}(t - \tau)) = \tilde{G}(t, \tilde{Y}(t)), \quad \tau \leq t < 2\tau, \\
\tilde{Y}(t) &= \tilde{X}(t), \quad 0 \leq t < \tau
\end{align*}
\]
where \( \tilde{Y}(t) \) and \( \tilde{X} \) are respectively the same vectors as \( Y(t) \) and \( \tilde{X}(t) \) when replacing \( u_0^0(t) \) by \( u_0^0(t) - \int^\tau_0 S_i(\sigma) d\sigma \) (respectively \( \varphi_i(t) \) by \( \varphi_i(t) - \int^\tau_0 S_i(\sigma) d\sigma \).

As we did previously, we easily show that \( \tilde{G} \) is a continuous function, and continuously differentiable with respect to the second variable. So, Cauchy-Lipschitz theorem gives the local existence and uniqueness of solutions on \([\tau, 2\tau]\) for the above problem. That implies the local existence of solution to system (43). To prove that the solution is global we investigate again the positivity and the finite bounds of
\[
Y(t) = \int^t_0 \left( u_1^0(t), u_2^0(t), u_{a,1}(t), u_{a,2}(t), p_c(t), p_{sc}(t), C_1(t), C_2(t) \right).
\]
For that, we begin with the relation
\[
p_c(t) = -(\delta_1 u_1^0 + \delta_2 u_2^0) p_c(t) \implies p_c(t) = p_c(\tau) \exp \left( - \int^\tau_\tau \delta_1 u_1^0(s) + \delta_2 u_2^0(s) ds \right),
\]
which is positive or null for all $t$ in $[\tau, 2\tau]$, as $p_c(\tau) = \varphi_{p_c}(\tau)$ is greater or equal to 0. For the non-negativity of $u_i^0(t)$, $i = 1, 2$ for all $t$ in $[\tau, +2\tau]$, we have

$$\dot{u}_i^0(t) \geq -\gamma_i u_i^0(t) - \delta_i u_i^0(t)p_c(t) \implies u_i^0(t) \geq u_i^0(\tau) \exp\left(-\int_0^t [\gamma_i + \delta_i p_c(s)]ds\right)$$

that induces $u_i^0(t) \geq 0$ for all $t \in (\tau, 2\tau)$ because $u_i^0(\tau) \geq 0$.

Knowing that $u_i^0(t) \geq 0$ it’s straightforward that $p_c(t) \leq \varphi_{p_c}(\tau) < +\infty$.

For the upper bound of $u_i^0$ one can remark that $\dot{u}_i^0(t) \leq S_i(t) + \delta_i p_c(t - \tau)u_i^0(t - \tau)$. Knowing that $t \in [\tau, 2\tau]$ that implies $t_\tau = t - \tau \in [0, \tau]$ so $u_i^0(t_\tau)$ is known and correspond to the initial function $\varphi_i$, which is already bounded. Then a simple integration on $[\tau, t]$ with $t < 2\tau$ achieves the proof that $u_i^0$ is bounded. Given the non-negativity of $p_c$, $u_i^0$ and $u_i^2$, we have, for all $t \in [\tau, 2\tau]$:

$$\dot{u}_{a,i}(t) \geq 0 \implies u_{a,i}(t) \geq u_{a,i}(\tau) \geq 0, \quad i = 1, 2,$$

$$p_{sc}(t) \geq 0 \implies p_{sc}(t) \geq p_{sc}(\tau) = 0.$$  

In addition, it is straightforward to verify that $u_{a,i}$ and $p_{sc}$ are bounded.

We finally consider functions $C_i$:

$$\dot{C}_i(t) = \delta_i u_i^0(t)p_c(t) - \delta_i u_i^0(t - \tau)p_c(t - \tau),$$

$$C_i(t) = C_i(\tau) + \delta_i \int_{\tau}^t u_i^0(s)p_c(s)ds - \delta_i \int_{\tau}^t u_i^0(s - \tau)p_c(s - \tau)ds,$$

$$= \delta_i \int_{0}^t u_i^0(s)p_c(s)ds \quad \tau \leq t < 2\tau.$$  

Thanks to non-negativity of $u_i^0$ and $p_c$, this proves that $C_i$, $i = 1, 2$ is non-negative on $[\tau, 2\tau)$ and obviously bounded.

That achieves the global existence solution on $[\tau, 2\tau)$. We then iterate this process on intervals $[n\tau, (n + 1)\tau)$, $n \geq 2$ and $n \in \mathbb{N}^*$, and obtain existence and uniqueness of solutions of system (II) on $[0, T]$.

6. **Numerical simulations.** In this section, we give illustrations of the dynamics of our model, through numerical simulations, using only one type of $\Lambda\beta$. The numerical scheme is based on a finite volumes method for the size discretization of the advection-reaction equations combined with a second order Runge-Kutta time discretization. We use the Van Leer flux limiters for the advection part which is known to be of order two. So, the numerical solutions of our model are TVD (Total Variation Diminishing) and of order two.

We neglect any difficulties due to truncation of the computational domain and introduce the regular mesh with constant size step $\Delta x > 0$: the cells are the intervals $[x_{k-1}, x_k], k \in \mathbb{N}$ with $x_k = (k + 1/2)\Delta x$ and $x_{-1} = 0$. We denote by $F_k^n$ on of the numerical unknown (it can be the fibrils or the proto-oligomers or the fibrils inside the plaque). In the particular case where $F = f$, $f_k^n$ is intended to be an approximation of $\frac{1}{\Delta x} \int_{x_k}^{x_{k+1}} f(t^n, z)dz$, where $t^n = 0 < t^1 < \cdots < t^n < t^{n+1}$ defines the time-discretization, with possibly variable step $\Delta t^n = t^{n+1} - t^n$ in order to adapt the velocity time variation. For instance the numerical scheme for fibrils
size-density (see equation (2)) is defined by the relation

\[ f_k^* = f_k^n + \Delta t(n) \left( -\frac{flux_k^n - flux_k^{n-1}}{\Delta x} + \mu(i)m^n - \gamma_f f_k^n \right), \]  

\[ f_k^{(n+1)} = \frac{1}{2}(f_k^n + f_k^*) + \frac{\Delta t(n)}{2} \left( -\frac{flux_k^{n+1} - flux_k^*}{\Delta x} + \mu(i)m^* - \gamma_f f_k^* \right). \]  

The interface fluxes, \( flux_k^n = (uf)_k^n \) and \( flux_k^* = (u^*f^*)_k^n \) are computed by using Van Leer approximation respectively with \( uf \) evaluated at time \( t^{(n)} \) and at intermediate time \( t^* \) thanks to the second order Runge-Kutta method. Here \( m^{(n)} \) and \( m^* \) are the numerical approximations of the monomers concentration respectively at first and second stage of the Runge-Kutta method based on the equation \( \theta \).

For the flux with Van Leer limiter method, we compute:

\[
\begin{cases}
\text{if } v_k^n > 0 & \theta = \frac{f_{k-1}^n - f_{k-2}^n}{\epsilon + f_k^n - f_{k-1}^n}, \\
& flux_k^n = v_k^n \left( f_{k-1}^n + (f_k^n - f_{k-1}^n)\phi(\theta) \right), \\
\text{else} & \theta = \frac{f_k^n - f_{k-1}^n}{\epsilon + f_{k+1}^n - f_k^n}, \\
& flux_k^n = v_k^n \left( f_k^n - (f_k^n - f_{k-1}^n)\phi\left( \frac{1}{\theta + \epsilon_1} \right) \right),
\end{cases}
\]

where the limiter function \( \phi \) given by \( \phi(\theta) = \frac{1}{2} \left( \frac{|\theta| + \theta}{1 + |\theta|} \right) \).

We apply this scheme for the part of the model dealing with partial differential equations. For the other part of the model dealing with ordinary differential equation, the approximation is done thanks to the second order Runge-Kutta method. Boundaries conditions are taken into account thanks to fictious mesh added at the domain.

For the parameters of the simulations we consider the following:

\( g_f(x) = g(x) = x^{1/3}, b_f = b_a = b = 1, \gamma_f = \gamma = \delta = 0.1, \tau = 3, x_0 = 5. \)

For all the simulations we take initial conditions for the quantities involved in the prion catalysis process as follow: \( u_b(t = 0) = 0, p_c(t = 0) = 1, p_a(t = 0) = 0, C(t = 0) = 0, u_a(t = 0) = 0, \)

\[ \mu(x) = \begin{cases} 
\exp \left( \frac{1}{(x-0.9)^2 - 1.2} \right) \left( 1 - \frac{x}{2} \right)^{10} & \text{if } 0 < x < 1.9, \\
0 & \text{elsewhere}.
\end{cases} \]

6.1. Results with free initial size-density repartition for fibrils, proto-oligomers and plaque. We first consider the case where there are only \( A\beta \) monomers and prions \( PrP^c \) initially, which corresponds to what can be done experimentally. In terms of initial conditions, we therefore have: \( f^{in}(x) = 0, f^{in}_a(x) = 0 \) and \( u^{in}(x) = 0. \) Figure 2 displays the evolution in time of the size density repartition of fibrils, proto-oligomers and fibrils in plaque, as well as the evolution of the total mass, which remains constant as expected. One can observe the creation of fibrils and proto-oligomers is only due to function \( \mu, \) which allows to create small polymers. In this case, there are very few polymers with a large size. Evolutions of concentration of \( A\beta \) monomers, oligomers, oligomers in plaque, \( PrP^c, \) \( PrP^{sol} \) and complexes are presented in Figure 3. One can note that, because there is no polymer initially, few oligomers are created and thus, the emergence of \( PrP^{sol} \) prions remains quite slow.
Figure 2. Evolution of size density repartition of fibrils $f(t, x)$, proto-oligomers $u(t, x)$ and fibrils in plaque $f_a(t, x)$ for different times ($t = 10, 20, 30, 40$). The last figure displays the evolution of the total mass.

6.2. Results with gaussian initial distribution for fibrils, proto-oligomers and plaque. We now assume that proto-oligomers and fibrils are present initially with monomers and PrP$^c$. Initial conditions are given by:

$$f^{in}(x) = \begin{cases} \frac{\exp\left(-\frac{5(x-1.5)^2}{2}\right)}{\sqrt{0.4\pi}}, \\ 0 \end{cases}$$

$$u^{in}(x) = \begin{cases} \frac{\exp\left(-\frac{5(x-1)^2}{2}\right)}{\sqrt{0.4\pi}} \text{ if } 1 \leq x \leq x_0, \\ 0 \text{ elsewhere,} \end{cases}$$

$$f_a^{in}(x) = \frac{\exp\left(-\frac{5(x-1.75)^2}{2}\right)}{\sqrt{0.4\pi}}.$$  

Figure 4 displays the evolutions of monomer concentration, oligomers, oligomers in plaque, prions PrP$^c$ and PrP$^{0\delta}$ and complexes. As expected, the total mass remains constant.
Figure 3. Evolution with time of Aβ monomers, Aβ oligomers, oligomers in plaque, prions PrP\textsuperscript{c}, prions PrP\textsuperscript{sc} and complexes, with only monomers and PrP\textsuperscript{c} initially.

In a first time we observe an increase in monomers, meaning that proto-oligomers and fibrils initially depolymerize. Then monomer concentration decreases, which corresponds to the formation of larger polymers. Oligomers appear after a certain time, and their concentration decreases...
after a while, meaning that proto-oligomers do not reach the size $x_0$. With the increase of $\Lambda \beta$
oligomers, we notice the emergence of $A\beta/PrP^c$ complexes and of $PrP^{sc}$.

Figures 5 and 6 display the evolution of size density repartition of fibrils $f(t, x)$, proto-oligomers $u(t, x)$ and fibrils in plaque $f_a(t, x)$ for given times. One observes that fibrils become larger with time, but after a certain time there are more small fibrils due to the spontaneous term $\mu$ than large ones. Likewise, because fibrils in plaque only depolymerize, we notice a larger concentration of small ones. For proto-oligomers, we observe the impact of $\mu$ function as small proto-oligomers rapidly appear. Some proto-oligomers finally reach the maximal size $x_0$ and become oligomers.

7. Discussion. The role of $A\beta$ oligomers and $PrP^c$ prions in Alzheimer’s disease remains to be fully understood. Recent evidence suggests that $A\beta$ oligomers can interact with $PrP^c$ to induce cytotoxic damages to neurons, increasing their apoptosis. Moreover, this interaction could misfold $PrP^c$ into pathogenic prions $PrP^{sc}$, potentially leading to the emergence of prion diseases such as Creutzfeldt-Jakob disease. Mathematical modeling can help to qualitatively explain polymerization kinetics and evolution of polymer length that are involved in the emergence of AD.

In this work, we propose a mathematical model to describe the polymerization of $A\beta$ monomers, and the interactions between $A\beta$ oligomers and $PrP^{sc}$. Polymerization process is modeled with partial differential equations, based on Lifshitz-Slyozov equations [19]. One can note that in our model, we study the evolution of three different species (proto-oligomers, fibrils and fibrils in plaque) through advection-reaction equations, making the analysis more complex. $PrP^{\text{ol}}$ catalysis, through interactions with $A\beta$ oligomers, is described using ordinary and delayed differential equations. These two submodels are linked through the source term of oligomers coming from proto-oligomers, and can be studied one at a time. For the first one, we use Schauder fixed point theorem to prove existence and uniqueness of mild solutions, even in the case of singular polymerization rates. Existence and uniqueness of solutions for the second submodel are obtained with Cauchy-Lipschitz theorem. Numerical simulations with different initial conditions are given to illustrate the different profiles that can be obtained with this model. Because we have no experimental data available, we only provide simulations with one type of $A\beta$.

To the best of our knowledge, this is the first model describing both $A\beta$ polymerization process and interactions with $PrP^c$. However, because it is developed in an in vitro context, some in vivo processes are not included in the model. For instance, one could add the production of $A\beta$ monomers on diseased neuronal membranes, as proposed in [1, 4]. Neurons could also be damaged due to the binding of $A\beta$ oligomers to $PrP^c$, as done in [14]. Nevertheless, we believe that our model gives insights on $A\beta$ polymerization and on the interactions between $A\beta$ and $PrP^c$. It remains to compare our numerical simulations to experimental data and to find optimal parameter estimates. This can help to highlight differences between $A\beta$-40 and $A\beta$-42 and to identify new possible therapeutic targets to slow down or even avoid the emergence of Alzheimer’s disease or prion diseases.

Appendix A. Measurability of $\bar{s}(t, x)$. In this appendix, we aim to show the measurability of the function $\bar{s}(t, x)$ introduced in the proof of lemma [1]. Let us consider the set $Q = [0, T] \times (0, x_0)$. For all $(t, x)$ in $Q$ let $\bar{s}(t, x)$ be defined as:

$$\bar{s}(t, x) = \sup\{s \in [0, t], X(s; t, x) = x_0\},$$

where we understand that $\bar{s}(t, x) = 0$ if $X(s; t, x) < x_0$ for any $s$ in $[0, t]$.

For $\alpha$ in $\mathbb{R}$, we consider the following set:

$$A_\alpha = \{(t, x) \in Q, \bar{s}(t, x) \geq \alpha\}.$$
We want to show that $A_\alpha$ is measurable. Let us note that if $\alpha$ is lower or equal to 0, then $A_\alpha$ is exactly $Q$ and if $\alpha$ is greater or equal to $T$, $A_\alpha$ is the empty set. We then assume that $\alpha$ is in $(0, T)$.

**Proposition 1.** With these notations, we have $A_\alpha = B_\alpha$, where

$$B_\alpha = \{(t, X(t; s, x_0)), (t, s) \in F_\alpha\} \cap Q,$$

with

$$F_\alpha = \{(t, s) \in \mathbb{R}^2, 0 \leq s \leq t \leq T\}.$$

**Proof.**

1. $A_\alpha \subseteq B_\alpha$

   Let us focus on $(t, x)$ in $A_\alpha$. We have $(t, x)$ in $Q$ and $X(\bar{s}(t, x); t, x) = x_0$, which is equivalent to $x = X(t; \bar{s}(t, x), x_0)$. We also have $\alpha \leq \bar{s}(t, x) \leq t \leq T$, whence $(\bar{s}(t, x))$ is in $F_\alpha$. Therefore $(t, x)$, which is equal to $(t, X(t; \bar{s}(t, x), x_0))$, is in $B_\alpha$.

2. $B_\alpha \subseteq A_\alpha$

   Let us consider $(t, x) = (t, X(t; s_1, x_0))$ in $B_\alpha$. Then we have $0 < x = X(t; s_1, x_0) < x_0$ and $\alpha \leq s_1 \leq t \leq T$. Then necessarily $\bar{s}(t, x) \geq s_1$, so $\bar{s}(t, x) \geq \alpha$, that is $(t, x)$ is in $A_\alpha$. □

We can note that the set $\{(t, X(t; s, x_0)), (t, s) \in F_\alpha\}$ is the image of the compact set $F_\alpha$ by a continuous function, so it is a compact set. It follows that it is a closed set, and then a measurable set. $Q$ is also measurable, and therefore so is $B_\alpha$. As $B_\alpha$ is exactly $A_\alpha$ by proposition 1, $A_\alpha$ is measurable. Finally, the function $\bar{s}$ from $Q$ to $\mathbb{R}$ is measurable.

**Appendix B. Proof of lemma 6.** We provide here the proof of lemma 6 introduced in section 4.2 to prove that $h(\Sigma_T)$ is a relatively compact subspace of $C_0^0([0, T])$.

**Proof.** According to equation (29), we have:

$$|A_m(t) - A_n(t)| \leq \left| \int_0^{\infty} g_f(x)(f_m(t, x) - f_n(t, x))dx \right|_{I_{A1}} + \left| \int_0^{x_0} g(x)(u_m(t, x) - u_n(t, x))dx \right|_{I_{A2}}.$$  \hspace{1cm} (48)

Let us focus on $I_{A1}$. We know that:

$$f_m(t, x) = f_m(X_m(0; t, x))e^{-\gamma t}J_m(0; t, x)$$

$$+ \int_0^t \mu(X_m(s; t, x))\tilde{n}(s)e^{-\gamma(t-s)}J_m(s; t, x)ds,$$

and the same holds for $f_n(t, x)$.

Therefore, we have

$$I_{A1} \leq e^{-\gamma t} \left| \int_0^{\infty} g_f(x) \left( f_m(X_m(0; t, x))J_m(0; t, x) - f_m(X_n(0; t, x))J_n(0; t, x) \right) dx \right|_{K_1}$$

$$+ \left| \int_0^{x_0} g_f(x)e^{-\gamma(t-s)}\mu(X_m(s; t, x))\tilde{n}(s)J_m(s; t, x) - \mu(X_n(s; t, x))\tilde{n}(s)J_n(s; t, x)dsdx \right|_{K_2}. \hspace{1cm} (49)$$
We compute term $K_1$ with integration by substitution, with $y = X_p(0; t, x)$, $p = m, n$. First, let us note that:

$$\lim_{x \to +\infty} X(s; t, x) = +\infty, \quad 0 \leq s \leq t,$$

and:

if $x < +\infty$, then $X(s; t, x) < \infty$, for all $s, 0 \leq s \leq t$.

We therefore have:

$$K_1 = \left| \int_{X_m(0; t, 0)}^{+\infty} g_f(X_m(t; 0, y))f^{in}_n(y)dy - \int_{X_n(0; t, 0)}^{+\infty} g_f(X_n(t; 0, y))f^{in}_n(y)dy \right|,$$

$$= \left| \int_{X_m(0; t, 0)}^{+\infty} g_f(X_m(t; 0, y))f^{in}_n(y)dy + \int_{X_n(0; t, 0)}^{+\infty} f^{in}_n(y) \left( g_f(X_m(t; 0, y)) - g_f(X_n(t; 0, y)) \right)dy \right|,$$

$$\leq \sup_{[X_m(0; t, 0), X_n(0; t, 0)]} (g_f(X_m(t; 0, y))f^{in}_n(y))X_n(0; t, 0) - X_m(0; t, t)\right| + \int_{X_n(0; t, 0)}^{+\infty} G_f|X_m(t; 0, y) - X_n(t; 0, y)|f^{in}_n(y)dy,$$

where $G_f$ is the upper bound for the derivative of $g_f$, as stated in Hypothesis 1.

Let us now compute $|X_m(s; t, x) - X_n(s; t, x)|$.

**Lemma 8.** For all $s, t$ such as $0 \leq s, t \leq T$, there exists a constant $C$ so that:

$$|X_m(s; t, x) - X_n(s; t, x)| \leq C\sup_{[0, T]}|\tilde{m} - \tilde{n}|.$$

According to lemma 8, we obtain the existence of $C_1$ and $C_2$ such as:

$$K_1 \leq \sup_{[0, T]}|\tilde{m} - \tilde{n}| \left( \sup_{[X_m(0; t, 0), X_n(0; t, 0)]} (g_f(X_m(t; 0, y))f^{in}_n(y))C_1 + G_fC_2||f^{in}_n||_{L^1} \right).$$

Let us now study term $K_2$ in equation (49):

$$K_2 = \left| \int_0^t e^{-\gamma_f(t-s)} \left( \tilde{m}(s) \int_0^{+\infty} g_f(x)\mu(X_m(s; t, x))J_m(s; t, x)dx 
- \tilde{n}(s) \int_0^{+\infty} g_f(x)\mu(X_n(s; t, x))J_n(s; t, x)dx \right)ds \right|.$$

Integrating by substitution with $y = X(s; t, x)$ gives us:

$$K_2 = \left| \int_0^t e^{-\gamma_f(t-s)} \left( \tilde{m}(s) \int_0^{+\infty} g_f(X_m(t; s, y))\mu(y)dy - \tilde{n}(s) \int_0^{+\infty} g_f(X_n(t; s, y))\mu(y)dy \right)ds \right|,$$

$$K_2 = \left| \int_0^t e^{-\gamma_f(t-s)} \tilde{m}(s) \int_{X_m(s; t, 0)}^{X_m(s; t, 0)} g_f(X_m(t; s, y))\mu(y)dyds 
+ \int_0^t e^{-\gamma_f(t-s)} \tilde{m}(s) \int_{X_n(s; t, 0)}^{X_n(s; t, 0)} \mu(y)(g_f(X_m(t; s, y)) - g_f(X_n(t; s, y)))dyds 
+ \int_0^t e^{-\gamma_f(t-s)} (\tilde{m}(s) - \tilde{n}(s)) \int_{X_n(s; t, 0)}^{+\infty} g_f(X_n(t; s, y))\mu(y)dyds \right|.$$
and finally

\[
K_2 \leq \int_0^t M_T \sup_{[X_m(s,t,0),X_n(s,t,0)]} (\int_0^s g_f(X_m(t; s, y))\mu(y)) |X_n(s; t, 0) - X_m(s; t, 0)| ds
\]
\[
+ \int_0^t M_T \int_0^\infty G_f \mu(y) |X_m(t; s, y) - X_n(t; s, y)| dy ds
\]
\[
+ \int_0^t |\tilde{n}(s) - \tilde{n}(s)||g\mu||_{L^1} ds.
\]

Lemma [8] provides the existence of constants \( C_1 \) and \( C_2 \) such as:

\[ K_2 \leq \sup_{[0,T]} |\tilde{n}(t) - \tilde{n}(t)| \leq \sup_{[0,T]} (\int_0^t M_T C_1 T \sup_{[0,T]} |\tilde{n}(t) - \tilde{n}(t)| g_f(X_m(t; s, y))\mu(y)) + M_T C_2 T |\mu||_{L^1} + T |\mu||_{L^1} \cdot \]

Combining relations (51) and (52) gives us the existence of a constant \( \alpha_1 \) such as:

\[ I_{A1} \leq \alpha_1 \sup_{[0,T]} |\tilde{n} - \tilde{n}|. \]  

(53)

We perform the same analysis for \( I_{A2} \), the second term in equation (48) and find the existence of a constant \( \alpha_2 \) such as:

\[ I_{A2} \leq \alpha_2 \sup_{[0,T]} |\tilde{n} - \tilde{n}|. \]  

(54)

Relations (53) and (54) implies the existence of a constant \( \alpha \) such as:

\[ \sup_{[0,T]} |A_m(t) - A_n(t)| \leq \alpha \sup_{[0,T]} |\tilde{n}(t) - \tilde{n}(t)|. \]  

(55)

We now focus on \( B_m(t) - B_n(t) \). According to equation (30), we have:

\[
|B_m(t) - B_n(t)| \leq b(t) \left( \int_0^{x_0} f_{a,m}(t, x) - f_{a,n}(t, x) dx \right)_{I_{B1}} + b_f \left( \int_0^{x_0} f_m(t, x) - f_n(t, x) dx \right)_{I_{B2}} + b \left( \int_0^{x_0} u_m(t, x) - u_n(t, x) dx \right)_{I_{B3}}
\]

(56)

We upper-bound \( I_{B2} \) and \( I_{B3} \) as we did previously for \( I_{A1} \) and \( I_{A2} \), and finally find that there exist \( \beta_2 \) and \( \beta_3 \) such as:

\[ I_{B2} \leq \beta_2 \sup_{[0,T]} |\tilde{n}(t) - \tilde{n}(t)|, \]  

(57)

\[ I_{B3} \leq \beta_3 \sup_{[0,T]} |\tilde{n}(t) - \tilde{n}(t)|. \]  

(58)
We now have to study $I_{B1}$, the first term in equation (56):

$$I_{B1} = \int_{0}^{+\infty} |f_{a,m}(t, x) - f_{a,n}(t, x)|dx,$$

$$= \int_{0}^{+\infty} |f_{a,n}(X(0; t, x)) + \gamma f \int_{0}^{t} f_{n}(s, X(s; t, x))ds - f_{a,n}(X(0; t, x)) + \gamma f \int_{0}^{t} f_{n}(s, X(s; t, x))ds|dx,$$

$$= \gamma f \int_{0}^{t} \int_{0}^{+\infty} |f_{m}(s, X(s; t, x) - f_{n}(s, X(s; t, x))|dx ds.$$

In $I_{11}$, we make the following substitution: $y = X(s, t, x)$ which can be written as $x = X(t, s, y)$. We then have:

$$I_{11} = \int_{X(s, t, 0)}^{+\infty} |f_{m}(s, y) - f_{n}(s, y)|J(t; s, y)dy,$$

$$= \int_{X(s, t, 0)}^{+\infty} |f_{m}(s, y) - f_{n}(s, y)|\exp(\int_{s}^{t} \frac{\partial v}{\partial x}(\sigma, X(\sigma; s, y))d\sigma)dy,$$

$$= \int_{X(s, t, 0)}^{+\infty} |f_{m}(s, y) - f_{n}(s, y)|\exp(\int_{s}^{t} m(\sigma)g'(X(\sigma; s, y))d\sigma)dy,$$

$$\leq \int_{X(s, t, 0)}^{+\infty} |f_{m}(s, y) - f_{n}(s, y)|\exp(MT G(t - s))dy,$$

$$\leq e^{TM} \int_{0}^{t} \int_{0}^{+\infty} |f_{m}(s, y) - f_{n}(s, y)|dy.$$

According to (57), there exists $\beta_{2}$ such as:

$$I_{11} \leq e^{TM} \beta_{2} \sup_{[0, T]} |\bar{m}(t) - \bar{n}(t)|.$$

We now go back to $I_{B1}$, and find that:

$$I_{B1} \leq \gamma f \int_{0}^{t} e^{TM} \beta_{2} \sup_{[0, T]} |\bar{m}(t) - \bar{n}(t)|,$$

$$\leq \gamma f T e^{TM} \beta_{2} \sup_{[0, T]} |\bar{m}(t) - \bar{n}(t)|.$$

Therefore, there exists a constant $\beta_{1}$ such as:

$$I_{B1} \leq \beta_{1} \sup_{[0, T]} |\bar{m}(t) - \bar{n}(t)|. \text{(59)}$$

Combining relations (57)–(59) proves the existence of a constant $\beta$ such as:

$$\sup_{[0, T]} |B_{m}(t) - B_{n}(t)| \leq \beta \sup_{[0, T]} |\bar{m}(t) - \bar{n}(t)|.$$
Proof. Proof of lemma 8

Let $s, t$ such as $0 \leq s \leq t \leq T$.

$$|X_m(s; t, x) - X_n(s; t, x)| = \left| \int_s^t (g(X_m(\sigma; t, x))\tilde{n}(\sigma) - b) \, d\sigma \right|$$

$$- \left| \int_s^t (g(X_n(\sigma; t, x))\tilde{n}(\sigma) - b) \, d\sigma \right|,$$

$$\leq \int_s^t |\tilde{n}(\sigma)||g(X_m(\sigma; t, x)) - g(X_n(\sigma; t, x))| \, d\sigma$$

$$+ \int_s^t |g(X_n(\sigma; t, x))||\tilde{n}(\sigma) - \tilde{n}(\sigma)| \, d\sigma,$$

$$\leq GM_T \int_s^t |\tilde{n}(\sigma)||X_m(\sigma; t, x) - X_n(\sigma; t, x)| \, d\sigma$$

$$+ \left( \int_s^t (g(X_n(\sigma; t, x))^2 \, d\sigma \right)^{1/2} \left( \int_s^t (\tilde{n}(\sigma) - \tilde{n}(\sigma))^2 \, d\sigma \right)^{1/2}. \quad (60)$$

Because $g(0) = 0$ and $g'(x) \leq G$ for all $x$ in $[0, +\infty)$, we have:

$$|g(X_n(s; t, x))| \leq GX_n(s; t, x),$$

and

$$|X_n(s; t, x)| = |x + \int_s^t (g(X_n(\sigma; t, x))\tilde{n}(\sigma) - b) \, d\sigma|,$$

$$\leq x + \int_s^t b + GM_T|X_n(\sigma; t, x)| \, d\sigma.$$

Grönwall’s inequality finally gives us the existence of a constant $L_T$ such as

$$|X_n(s; t, x)| \leq L_T(2bT + x).$$

Let us now go back to relation (60):

$$|X_m(s; t, x) - X_n(s; t, x)| \leq GM_T \int_s^t |X_m(\sigma; t, x) - X_n(\sigma; t, x)| \, d\sigma$$

$$+ GL_T(x + 2bT)^{1/2} \left( \int_s^t (\tilde{n}(\sigma) - \tilde{n}(\sigma))^2 \, d\sigma \right)^{1/2}.$$

We use Grönwall’s inequality to obtain the following relation:

$$|X_m(s; t, x) - X_n(s; t, x)| \leq K(2bT + x)\sup_{[0, T]} (\tilde{n}(t) - \tilde{n}(t)). \quad (61)$$

References


Figure 5. Evolution of size density repartition of fibrils $f(t,x)$, proto-oligomers $u(t,x)$ and fibrils in plaque $f_a(t,x)$ for different times ($t = 0, 10, 20$)
Figure 6. Evolution of size density repartition of fibrils $f(t, x)$, proto-oligomers $u(t, x)$ and fibrils in plaque $f_a(t, x)$ for different times ($t = 20, 30, 40$).