Dynamics of a stochastically perturbed prey-predator system with modified Leslie-Gower and Holling type II schemes incorporating a prey refuge

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Abstract

We study a modified version of a prey-predator system with modified Leslie-Gower and Holling type II functional response studied by M.A. Aziz-Alaoui and M. Daher-Okiye. The modification consists in incorporating a refuge for preys, and substantially complicates the dynamics of the system. We also investigate a stochastic perturbation of the system.

Contents

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1 Introduction

We study a two-dimensional prey-predator system with modified Leslie-Gowers and Holling type II functional response. This system is a generalization of the system investigated in the papers by M.A. Aziz-Alaoui and M. Daher-Okiye [1, 5].

Aziz-Alaoui and M. Daher-Okiye's model has been studied and generalized in numerous papers: models with spatial diffusion term [3], with time delay [21], with stochastic perturbations [17, 16, 20, 14], or incorportaing a refuge for the prey [4], to cite but a few.

The novelty of the present paper is that we add a refuge and a stochastic perturbation, in a way which is different from [4], since the density of prey in our refuge is not proportional to the total density of prey. This kind of refuge entails a qualitatively different behavior of the solutions, even for a small refuge, contrarily to the type of refuge investigated in [4]. Let us emphasize that, even in the case without refuge, our study provides new results.

In the first part of the paper, we study the system of [1, 5] with refuge, but without stochastic perturbation:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{x}(\rho_1 - \beta\mathbf{x}) - \dfrac{\alpha_1 \mathbf{y}(\mathbf{x}-\mu)_+}{\kappa_1 + (\mathbf{x}-\mu)_+} \\ \dot{\mathbf{y}} = \mathbf{y}\left(\rho_2 - \dfrac{\alpha_2 \mathbf{y}}{\kappa_2 + (\mathbf{x}-\mu)_+}\right). \end{cases} \tag{1.1}$$

In this system,

- $\mathbf{x} \geq 0$ is the density of prey,

- $\mathbf{y} \geq 0$ is the density of predator,
- $\mu \geq 0$ models a refuge for the prey, i.e, the quantity $(\mathbf{x} - \mu)_+ := \max(0, \mathbf{x} - \mu)$ is the density of prey which is accessible to the predator,
- $\rho_1 > 0$ (resp. $\rho_2 > 0$) is the growth rate of prey (resp. of predator),
- $\beta > 0$ measures the strength of competition among individuals of the prey species,
- $\alpha_1 > 0$ (resp. $\alpha_2 > 0$) is the rate of reduction of preys (resp. of predators)
- $\kappa_1 > 0$ (resp. $\kappa_2 > 0$) measures the extent to which the environment provides protection to the prey (resp. to the predator).

When the predator is absent, $\mathbf{x}$ satisfies a logistic equation and converges to $\frac{\rho_1}{\beta}$, so we assume that

$$0 \leq \mu < \frac{\rho_1}{\beta}.$$

Setting, for $i = 1, 2$,

$$x(t) = \frac{\beta}{\rho_1} \mathbf{x}\left(\frac{t}{\rho_1}\right), \; y(t) = \frac{\beta}{\rho_1} \mathbf{y}\left(\frac{t}{\rho_1}\right),$$
$$m = \frac{\mu\beta}{\rho_1}, \; a = \frac{\alpha_1\rho_2}{\alpha_2\rho_1}, \; k_i = \frac{\kappa_i\beta}{\rho_1}, \; b = \frac{\rho_2}{\rho_1}.$$

we get the simpler equivalent system

$$\begin{cases} \dot{x} = x(1-x) - \dfrac{ay(x-m)_+}{k_1 + (x-m)_+} \\ \dot{y} = by\left(1 - \dfrac{y}{k_2 + (x-m)_+}\right), \end{cases} \tag{1.2}$$

where $0 \leq m < 1$, all other parameters are positive, and (x, y) takes its values in the quadrant $\mathbb{R}^+ \times \mathbb{R}^+$.

In a second part, we study the stochastically perturbed system

$$\begin{cases} dx(t) = \left(x(t)(1-x(t)) - \dfrac{ay(t)(x(t)-m)_+}{k_1 + (x(t)-m)_+}\right) dt + \sigma_1 x(t) dw_1(t) \\ dy(t) = by(t)\left(1 - \dfrac{y(t)}{k_2 + (x(t)-m)_+}\right) dt + \sigma_2 y(t) dw_2(t), \end{cases} \tag{1.3}$$

where $w = (w_1, w_2)$ is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathrm{P})$, and σ_1 and σ_2 are constant real numbers. The paper is arranged as following. First, we study the dynamics of the deterministic system, where we show the persistence and the existence of a compact attracting set, local study of equilibrium points, the existence of a globally asymptotically stable equilibrium, and finally, we give some sufficient conditions for the absence of periodic orbits, and some cases of existence of limit cycles. In the second part, we show the existence and uniqueness of the global positive solution with any initial positive value of the stochastic system, and the boundedness of its pth moment. Finally, we make numerical simulation to illustrate our results.

2 Dynamics of the deterministic system

In this section, we study the dynamics of (1.2).

Throughout, we denote by $\boldsymbol{v}$ the vector field associated with (1.2), and

$$\boldsymbol{v} = v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y},$$

so that (1.2) reduces to $\big(\dot{x} = v_1 \text{ and } \dot{y} = v_2\big)$.

The right hand side of (1.2) is locally Lipschitz, thus, for any initial condition, (1.2) has a unique solution defined on a maximal time interval.

Furthermore, the axes are invariant manifolds of (1.2):

- If $x(0) = 0$, then $x(t) = 0$ for every t, and $\dot{y} = by(1 - y/k_2)$ yields

  $$y(t) = \frac{y(0)k_2}{k_2 + y(0)(e^{bt} - 1)},$$

  thus $\lim_{t\to+\infty} y(t) = k_2$ if $y(0) > 0$.

- If $y(0) = 0$, then $y(t) = 0$ for every t, and $\dot{x} = x(1 - x)$ yields

  $$x(t) = \frac{x(0)}{1 + x(0)(e^{t} - 1)},$$

  thus $\lim_{t\to+\infty} x(t) = 1$ if $x(0) > 0$.

From the uniqueness theorem for ODEs, we deduce that the open quadrant $]0, +\infty[\times]0, +\infty[$ is stable, thus there is no extinction of any species in finite time.

2.1 Persistence and compact attracting set

The next result shows that there is no explosion of the system (1.2). It also shows a qualitative difference brought by the refuge: when $m = 0$, the density of prey may converge to 0, whereas, when $m > 0$, the system (1.2) is always uniformly persistent.

Let

$$\mathcal{A} = \left\{(x, y) \in \mathbb{R}^2;\, m \leq x \leq 1,\ k_2 \leq y < L\right\},$$

where $L = 1 + k_2 - m$.

Theorem 2.1 *(a) The set $\mathcal{A}$ is invariant for* (1.2)*. Furthermore, if the initial condition $(x(0), y(0))$ is in the open quadrant $]0, +\infty[\times]0, +\infty[$, we have*

$$\begin{cases} m \leq \liminf\limits_{t\to+\infty} x(t) \leq \limsup\limits_{t\to+\infty} x(t) \leq 1 \\ k_2 \leq \liminf\limits_{t\to+\infty} y(t) \leq \limsup\limits_{t\to+\infty} y(t) \leq L. \end{cases} \tag{2.1}$$

(b) In the case when $m > 0$, for any initial condition $(x(0), y(0))$ in the open quadrant $]0, +\infty[\times]0, +\infty[$, the point $(x(t), y(t))$ enters $\mathcal{A}$ in finite time. In particular, the system (1.2) *is uniformly persistent.*

(c) In the case when $m = 0$, for any $\epsilon > 0$ such that $k_2 - \epsilon > 0$, the compact set $[0, 1] \times [k_2 - \epsilon, L]$ is invariant, and, for any initial condition $(x(0), y(0))$ in the open quadrant $]0, +\infty[\times]0, +\infty[$, the point $(x(t), y(t))$ enters $[0, 1] \times [k_2 - \epsilon, L]$ in finite time. Furthermore :

(i) If $aL < k_1$, the system (1.2) *is uniformly persistent. More precisely, if $(x(0), y(0) \in]0, +\infty[\times]0, +\infty[$, we have*

$$\liminf_{t\to+\infty} x(t) \geq \frac{k_1 - aL}{k_1}. \tag{2.2}$$

(ii) If $ak_2 < k_1 \leq aL$, the system (1.2) *is uniformly weakly persistent. More precisely, if $(x(0), y(0) \in]0, +\infty[\times]0, +\infty[$, we have*

$$\begin{aligned} &\limsup_{t\to+\infty} x(t) \\ &\geq \min\left(\frac{k_1}{a} - k_2, \frac{1 - k_1 - a + \sqrt{(1 - k_1 - a)^2 + 4(k_1 - ak_2)}}{2}\right). \end{aligned} \tag{2.3}$$

(iii) If $k_1 = ak_2$, then

- *if* $1 - k_1 - a > 0$, *the system* (1.2) *is uniformly weakly persistent. More precisely, if* $(x(0), y(0) \in]0,+\infty[\times]0,+\infty[$, *we have*

$$\limsup_{t\to+\infty} x(t) \geq 1 - k_1 - a. \tag{2.4}$$

- *if* $1 - k_1 - a \leq 0$, *the point* $E_2 = (0, k_2)$ *is globally attracting, thus the prey becomes extinct in infinite time for any initial condition in* $]0,+\infty[\times]0,+\infty[$.

(iv) If $k_1 < ak_2$, *the point* $E_2 = (0, k_2)$ *is globally attracting, thus the prey becomes extinct in infinite time for any initial condition in* $]0,+\infty[\times]0,+\infty[$.

Remark 2.2 A more general sufficient condition of global attractivity of E_2 is provided by Theorem 2.7 (see Remark 2.8).

Proof of Theorem 2.1. (a) When $m = 0$, the first inequality in (2.1) is trivial. In the case when $m > 0$, we need to prove that $\liminf_{t\to+\infty} x(t) \geq m$, provided that $x(0) > 0$. Actually we have a better result, since, if $x(0) \leq m$, then x coincides with the solution to the logistic equation $\dot{x} = x(1 - x)$ as long as x does not reach the value m, that is,

$$x(t) = \frac{x(0)}{1 + x(0)(e^t - 1)}.$$

If $x(0) > 0$, this function converges to 1, thus there exists $t_m > 0$ such that

$$t \geq t_m \Rightarrow x(t) \geq m. \tag{2.5}$$

Note that, when $m > 0$, if $x(t) = m$, we have $\dot{x} = m(1 - m) > 0$. Thus

$$\Big(x(0) \geq m\Big) \Rightarrow \Big(x(t) \geq m,\ \forall t \geq 0\Big), \tag{2.6}$$

which implies the first inequality in (2.1). Now, from the first equation of (1.2), we have

$$\dot{x} \leq x(1 - x),$$

which implies that, for every $t \geq 0$,

$$x(t) \leq \frac{x(0)e^t}{1 + x(0)(e^t - 1)}. \tag{2.7}$$

In particular, we have

$$\limsup_{t\to+\infty} x(t) \le 1 \text{ and } \Big(x(0) \le 1 \Rightarrow x(t) \le 1,\ \forall t \ge 0\Big)\ . \tag{2.8}$$

This implies that, for any $\epsilon > 0$, and for t large enough (depending on $x(0)$), we have $x(t) \le 1+\epsilon$. We deduce that, for any $\epsilon > 0$, and for t large enough, we have

$$by\left(1-\frac{y}{k_2}\right) \le \dot{y}(t) \le by\left(1-\frac{y}{k_2+1+\epsilon-m}\right) = by\left(1-\frac{y}{L+\epsilon}\right), \tag{2.9}$$

which implies that, for t large enough, say, $t \ge t_0$,

$$\frac{y(0)k_2e^{bt}}{k_2+y(0)(e^{bt}-1)} \le y(t) \le \frac{y(t_0)(L+\epsilon)e^{b(t-t_0)}}{L+\epsilon+y(t_0)(e^{b(t-t_0)}-1)}. \tag{2.10}$$

Of course, if $x(0) \le 1$, we can drop ϵ in (2.9) and (2.10). Thus, we have

$$\Big(x(0) \le 1 \text{ and } k_2 \le y(0) \le L\Big) \Rightarrow \Big(k_2 \le y(t) \le L,\ \forall t \ge 0\Big)\ . \tag{2.11}$$

We deduce from (2.6), (2.8), and (2.11) that $\mathcal{A}$ is invariant.

As ϵ is arbitrary in (2.10), we have also, when $y(0) > 0$,

$$k_2 \le \liminf_{t\to+\infty} y(t) \le \limsup_{t\to+\infty} y(t) \le L. \tag{2.12}$$

From (2.5), (2.8), and (2.12), we deduce (2.1).

(b) We have already seen that $x(t) \ge m$ for t large enough, let us now check that $x(t) \le 1$ for t large enough. Since $\mathcal{A}$ is invariant, we only need to prove this for $x(0) > 1$. Let $\epsilon > 0$ such that $k_2 - \epsilon > 0$. Let $\delta > 0$ such that $\delta + m < 1$ and such that

$$(x \ge 1-\delta) \Rightarrow x(1-x) < \frac{a(k_2-\epsilon)(1-m)}{1+\epsilon-m}. \tag{2.13}$$

From the first inequality in (2.12), we have $y(t) \ge k_2 - \epsilon$ for t large enough, say $t \ge t_0$. From (2.8), we can take t_0 large enough such that, for $t \ge t_0$, we have also $x(t) \le 1+\epsilon$. Using (2.13), we deduce, for $t \ge t_0$ and $x(t) \ge 1-\delta$,

$$\begin{aligned}\dot{x}(t) &\le x(t)\big(1-x(t)\big) - \frac{a(k_2-\epsilon)(1-\delta-m)}{1+\epsilon-m}\\ &\le -\frac{a\delta(k_2-\epsilon)}{1+\epsilon-m}\end{aligned}$$

Thus x decreases with speed less than $-\frac{a\delta(k_2-\epsilon)}{1+\epsilon-m} < 0$. Thus $x(t) \le 1-\delta$ for t large enough.

We can now repeat the reasoning of (2.9) and (2.10), replacing ϵ by $-\delta$, which yields that $\limsup_{t\to\infty} y(t) \le L-\delta$. In particular, $y(t) < L$ for t large enough.

To prove that $y(t) > k_2$ for t large enough, let us first sharpen the result of (2.5). This is where we use that $m > 0$. Let $\delta > 0$, with $m+\delta < 1$. If $|x-m| < \delta$, we have

$$|x(1-x) - m(1-m)| = |(x-m)\,(1-(x+m))| \le |x-m| < \delta.$$

For t large enough, using (2.10), we deduce that

$$\dot{x} \ge x(1-x) - \frac{a(L+\epsilon)\delta}{k_1} \ge D := m(1-m) - \delta - \frac{a(L+\epsilon)\delta}{k_1}.$$

(we do not write t here for the sake of simplicity). For δ small enough, we have $D > 0$. Thus, if $m > 0$, we can find $\delta > 0$ small enough (depending on m), such that, when $x(t)$ is in the interval $[m, m+\delta]$, it reaches the value $m+\delta$ in finite time (at most $D\delta$), and then it stays in $[m+\delta, 1]$. Using (2.5), we deduce that there exists $t_{m+\delta} > 0$ such that

$$t \ge t_{m+\delta} \Rightarrow x(t) \ge m+\delta. \tag{2.14}$$

Using (2.14) in (1.2), we obtain, for $t \ge t_{m+\delta}$,

$$\dot{y} \ge by\left(1 - \frac{y}{k_2+\delta}\right),$$

which yields, if $y(0) > 0$,

$$y(t) \ge \frac{y(t_{m+\delta})(k_2+\delta)e^{b(t-t_{m+\delta})}}{k_2+\delta+y(t_{m+\delta})(e^{b(t-t_{m+\delta})}-1)}.$$

This proves that

$$\liminf_{t\to+\infty} y(t) \ge k_2+\delta,$$

and that $y > k_2$ for t large enough.

(c) Assume now that $m = 0$. Since the first part of the proof of (b) is valid for all $m \ge 0$, we have already proved that $x(t) < 1$ and $y(t) < L$ for t large enough. Let $\epsilon > 0$ such that $k_2 - \epsilon > 0$. For $y < k_2$, we have $\dot{y} > 0$, thus $[0,1] \times [k_2-\epsilon, L]$ is invariant. Furthermore, for any initial condition $(x(0), y(0) \in]0, +\infty[\times]0, +\infty[$, since $\liminf_{t\to+\infty} y(t) \geq k_2$, we have $y(t) > k_2 - \epsilon$ for t large enough, thus $(x(t), y(t))$ enters $[0,1]\times[k_2-\epsilon, L]$ in finite time.

(ci) Assume that $aL < k_1$, and let $\epsilon > 0$ such that $a(L+\epsilon) < k_1$. Let $K_\epsilon = \frac{k_1 - a(L+\epsilon)}{k_1}$. By the second inequality in (2.12), we have, for t large enough

$$\dot{x} \geq x(1-x) - \frac{ax(L+\epsilon)}{k_1} = K_\epsilon x\left(1 - \frac{x}{K_\epsilon}\right). \tag{2.15}$$

Thus $\liminf x(t) \geq K_\epsilon$. As ϵ is arbitrary, this proves (2.2). From (2.2) and the first inequality in (2.12), we deduce that (1.2) is uniformly persistent.

(cii) Assume now that $ak_2 < k_1 \leq aL$. Observe first that, if $\limsup_{t\to\infty} x(t) < l$ for some $l > 0$, then, for t large enough, we have $x(t) < l$, thus $\dot{y}(t) < by(1 - y/(k_2 + l))$. We deduce that

$$\limsup_{t\to\infty} x(t) < l \Rightarrow \limsup_{t\to\infty} y(t) < k_2 + l. \tag{2.16}$$

Let us now rewrite the first equation of (1.2) as

$$\dot{x} = x\left(1 - x - \frac{ay}{k_1 + x}\right) = \frac{x}{k_1 + x}\Big(-(x-1)(x+k_1) - ay\Big)$$

that is,

$$\dot{x} = \frac{ax}{k_1 + x}\Big(U(x) - y\Big) \tag{2.17}$$

where $U(x) = (-1/a)(x-1)(x+k_1)$. Since $ak_2 < k_1$, the point E_2 lies below the parabola $y = U(x)$, thus in the neighborhood of E_2, for $x > 0$, we have $\dot{x} > 0$.

By (2.16), if $\limsup_{t\to\infty} x(t) < l$ for some $l > 0$, then for t large enough, the point $(x(t), y(t))$ remains in the rectangle $\mathcal{R} = [0, l] \times [0, k_2 + l]$. But if, furthermore, l is small enough such that $\mathcal{R}$ lies entirely below the parabola $y = U(x)$, then, when $(x(t), y(t)) \in \mathcal{R}$, we have $\dot{x}(t) > 0$, which entails that $x(t)$ is eventually greater than l, a contradiction. This shows that, for $l > 0$ small enough, we have necessarily

$$\limsup_{t\to\infty} x(t) \geq l.$$

Let us now calculate the largest value of l such that $(x, y) \in \mathcal{R}$ implies $y < U(x)$, that is, the largest l such that

$$\min_{x\in[0,l]} U(x) \geq k_2 + l.$$

From the concavity of U, the minimum of U on the interval $[0, l]$ is attained at 0 or l. Thus the optimal value of l is the minimum of $U(0) - k_2 = \frac{k_1}{a} - k_2$ and the positive solution to $U(x) - k_2 = x$, which is

$$\frac{1 - k_1 - a + \sqrt{(1 - k_1 - a)^2 + 4(k_1 - ak_2)}}{2}.$$

This proves (2.3).

(ciii) Assume that $k_1 = ak_2$. With the change of variable $\tilde{y} = y - k_2$, the system (1.2) becomes

$$\begin{cases} \dot{x} = \dfrac{ax}{k_1 + x}\Big(V(x) - \tilde{y}\Big) \\ \dot{\tilde{y}} = b\,\dfrac{\tilde{y} + k_2}{x + k_2}(x - \tilde{y}), \end{cases}$$

where $V(x) = \frac{1}{a}\big((1 - k_1)x - x^2\big)$. The second equation shows that $\dot{\tilde{y}} > 0$ when $\tilde{y} < x$, and $\dot{\tilde{y}} < 0$ when $\tilde{y} > x$. The first equation shows that $\dot{x} > 0$ when $(x, \tilde{y})$ is above the parabola $\tilde{y} = V(x)$, and $\dot{x} < 0$ when $(x, \tilde{y})$ is below the parabola $\tilde{y} = V(x)$.

- Assume that $1 - k_1 - a > 0$, that is, $V'(0) = (1 - k_1)/a > 1$. Then, the parabola $\tilde{y} = V(x)$ is above the line $\tilde{y} = x$ for all x in the interval $]0, l[$, where l is the non-zero solution to $V(x) = x$, that is,

$$l = 1 - k_1 - a.$$

Let us show that $\limsup x(t) \geq l$. Assume the contrary, that is, $\limsup x(t) < \delta$ for some $\delta < l$. For t large enough, say, $t \geq t_\delta$, we have $x(t) < \delta$. Let us first prove that $|\tilde{y}(t)| < \delta$ for t large enough. If $\tilde{y}(t_\delta) < \delta$, we have, for all $t \geq t_\delta$, as long as $\tilde{y}(t) < \delta$,

$$\dot{\tilde{y}}(t) < b\,\frac{l + k_2}{k_2}(\delta - \tilde{y}(t)).$$

Since the constant function $\tilde{y} = \delta$ is a solution to $\dot{\tilde{y}} = b\,\frac{l+k_2}{k_2}(\delta - \tilde{y})$, we deduce that $\tilde{y}(t)$ remains in $[-k_2, \delta]$ for all $t \geq t_\delta$. Furthermore, if $\tilde{y}(t) < -\delta$, for $t \geq t_\delta$, we have $\dot{\tilde{y}}(t) > 0$, thus

$$\dot{\tilde{y}}(t) > b\,\frac{\tilde{y}(t_\delta) + k_2}{k_2 + \delta}(-\tilde{y}(t)).$$

Thus

$$\tilde{y}(t) \geq y(t_\delta)\exp\left(-b\,\frac{\tilde{y}(t_\delta) + k_2}{k_2 + \delta}\,(t - t_\delta)\right),$$

which proves that $\tilde{y}(t)$ enters $]-\delta, \delta[$ in finite time. Similarly, if $\tilde{y}(t_\delta) > \delta$, then, for all $t \geq t_\delta$ such that $\tilde{y}(s) > \delta$ for all $s \in [t_\delta, t]$, we have

$$\dot{\tilde{y}}(t) < b\frac{\tilde{y}(t_\delta) + k_2}{k_2}(\delta - \tilde{y}(t)),$$

thus

$$\tilde{y}(t) < \delta + (\tilde{y}(t_\delta) - \delta)\exp\left(-b\frac{\tilde{y}(t_\delta) + k_2}{k_2}(t - t_\delta)\right),$$

which proves that $\tilde{y}(t) < \delta$ after a finite time.

We have proved that, for t large enough, $(x(t), \tilde{y}(t))$ stays in the box $[0, \delta[\times]-\delta, \delta[$. Since $V(x) > x$ for all $x \in]0, l[$, we deduce that, for t large enough, we have

$$\dot{x}(t) > x(t)\frac{V(\delta) - \delta}{k_1 + \delta},$$

which shows that $x(t) > \delta$ for t large enough, a contradiction. This proves (2.4).

• Assume that $1 - k_1 - a \leq 0$, that is, $V'(0) = (1 - k_1)/a \leq 1$. Then, the portion of the parabola $\tilde{y} = V(x)$ which lies in $]0, +\infty[\times] - k_2, +\infty[$, is below the line $\tilde{y} = x$. This means that, for any $\epsilon > 0$ such that $k_2 - \epsilon > 0$, the system (1.2) has no other equilibrium point than E_2 in the invariant attracting compact set $[0, 1] \times [k_2 - \epsilon, L]$. Since there cannot be any periodic orbit around E_2 (because E_2 is on the boundary of $[0, 1] \times [k_2 - \epsilon, L]$), this entails that E_2 is attracting for all inital conditions in $[0, 1] \times [k_2 - \epsilon, L]$, thus for all inital conditions in $]0, +\infty[\times]0, +\infty[$.

(civ) If $k_1 < ak_2$, we can use exactly the same arguments as in the case when $k_1 = ak_2$ with $1 - k_1 - a \leq 0$. □

2.2 Local study of equilibrium points

2.2.1 Trivial critical points

The right hand side of (1.2) has continuous partial derivatives in the first quadrant $\mathbb{R}^+ \times \mathbb{R}^+$, except on the line $x = m$ if $m > 0$. The Jacobian matrix of the right hand side of (1.2) (for $x \neq m$ if $m > 0$), is

$$\mathcal{J}(x, y) = \begin{pmatrix} 1 - 2x - \frac{ayk_1}{(k_1 + (x-m)_+)^2}\mathbb{1}_{x \geq m} & \frac{-a(x-m)_+}{k_1 + (x-m)_+} \\ \frac{by^2}{(k_2 + (x-m)_+)^2}\mathbb{1}_{x \geq m} & b - \frac{2by}{k_2 + (x-m)_+} \end{pmatrix}, \tag{2.18}$$

where $\mathbb{1}_{x \geq m} = 1$ if $x \geq m$ and $\mathbb{1}_{x \geq m} = 0$ if $x < m$.

We start with a result on the obvious critical points of (1.2) which lie on the axes.

Proposition 2.3 *The system* (1.2) *has three trivial critical points on the axes:*

- $E_0 = (0,0)$*, which is an hyperbolic unstable node,*
- $E_1 = (1,0)$*, which is an hyperbolic saddle point whose stable manifold is the* x *axis, and with an unstable manifold which is tangent to the line* $(b+1)(x-1) + \frac{a(1-m)}{k_1+1-m} y = 0$*,*
- $E_2 = (0, k_2)$*, which is*
  - *an hyperbolic saddle point whose stable manifold is the* y *axis, with an unstable manifold which is tangent to the line* $bx + \Big(b + 1 - \frac{ak_2}{k_1} \mathbb{1}_{m=0}\Big)(y - k_2) = 0$ *if* $m > 0$ *or if* $ak_2 < k_1$*,*
  - *an hyperbolic stable node if* $m = 0$ *with* $ak_2 > k_1$*,*
  - *a semi-hyperbolic point if* $m = 0$ *and* $ak_2 = k_1$*, which is*
    - *an attracting topological node if* $1 - k_1 - a \le 0$*,*
    - *a topological saddle point if* $1 - k_1 - a > 0$*. In this case, the* y *axis is the stable manifold, and there is a center manifold which is tangent to the line* $y - k_2 = x$*.*

(Compare with the case (c) *of Theorem 2.1).*

Proof. The nature of E_0, E_1, and E_2, is obvious since

$$\mathcal{J}(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \; \mathcal{J}(1,0) = \begin{pmatrix} -1 & \frac{-a(1-m)}{k_1+1-m} \\ 0 & b \end{pmatrix}, \; \mathcal{J}(0,k_2) = \begin{pmatrix} 1 - \frac{ak_2}{k_1} \mathbb{1}_{m=0} & 0 \\ b & -b \end{pmatrix}.$$

The results on stable and unstable manifolds of hyperbolic saddles are straightforward. In the case when E_2 is semi-hyperbolic, since it is either a topological node or a topological saddle (see [7, Theorem 2.19]), the nature of E_2 follows from Part (ciii) of Theorem 2.1. In the topological saddle case, that is, when $m = 0$ with $ak_2 = k_1$ and $1 - k_1 - a > 0$, the eigen values of $\mathcal{J}(0, k_2)$ are $-b$ and 1, with corresponding eigenvectors $(0,1)$ and $(1,1)$. Clearly, the y axis is the stable manifold. the change of variables

$$X = x, \quad Y = (y - k_2) - x$$

yields the normal form

$$\dot{X} = \dot{x} = \frac{X}{X + k_1}\Big((1-k_1)X - X^2 - a(X+Y)\Big)$$

$$
\begin{aligned}
&= \frac{X}{X+k_1}\Big((1-k_1-a)X - X^2 - aY\Big),\\
\dot{Y} &= \dot{x} - \dot{y} = \dot{x} - b\frac{X+Y+k_2}{X+k_2}(-Y) = \dot{X} - b\left(1+\frac{Y}{X+k_2}\right)Y\\
&= -\,bY + \dot{X} - b\frac{Y^2}{X+k_2}.
\end{aligned}
$$

We can thus write

$$
\begin{aligned}
\dot{X} &= A(X,Y)\\
\dot{Y} &= -\,bY + B(X,Y),
\end{aligned}
\tag{2.19}
$$

where A and B are analytic and their jacobian matrix at $(0,0)$ is 0. In the neighborhood of $(0,0)$, the equation $0 = -Yb + B(X,Y)$ has the unique solution $Y = f(X)$, where

$$
f(X) = \frac{k_2 a}{bk_2} X + O(X),
$$

and $g(X) = A(X, f(X))$ has the form

$$
g(X) = \frac{X^2}{k_2}\left(1 + k_1 - a - \frac{a^2 k_2}{bk_1}\right) + O(X).
$$

From [7, Theorem 2.19], we deduce that there exists an unstable center manifold which is infinitely tangent to the line $Y = 0$. $\square$

2.2.2 Counting and localizing equilibrium points

Let us now look for critical points outside the axes, i.e., critical points $E = (x, y)$ with $x > 0$ and $y > 0$. From the results of Subsection 2.1, such points are necessarily in $\mathcal{A}$, in particular they satisfy $x \geq m$. We have, obviously:

Lemma 2.4 *The set of equilibrium points of* (1.2) *which lie in the open quadrant* $]0,+\infty[\times]0,+\infty[$ *consists of the intersection points of the curves*

$$
x(1-x)\,(k_1 + x - m) = a\,(k_2 + x - m)\,(x - m), \tag{2.20}
$$

$$
k_2 + x - m = y. \tag{2.21}
$$

Furthermore, these points lie in $\mathcal{A}$.

We shall see that, when $m > 0$, the system (1.2) has always at least one equilibrium point in $]0, +\infty[\times]0, +\infty[$, whereas, for $m = 0$, some condition is necessary for the existence of such a point.

• When $m > 0$, the solutions to (2.20) lie at the abscissa of the intersection of the parabola $z = P(x) := a\,(k_2 + x - m)\,(x - m)$ and of the third degree curve $z = Q(x) := x(1 - x)\,(k_1 + x - m)$. We have $P(m) - Q(m) = -Q(m) = -k_1 m(1-m) < 0$ and, for $x > 1$, we have $P(x) < 0$ and $Q(x) > 0$, thus $P(x) - Q(x) > 0$. This implies that the curves of P and Q have at least one intersection whose abscissa is greater than m, and that the abscissa of any such intersection lies necessarily in the interval $]m, 1[$. The change of variable $X = x - m$ leads to

$$R(X) := P(x) - Q(x) = X^3 + \alpha_2 X^2 + \alpha_1 X + \alpha_0 \tag{2.22}$$

with
(2.23)
$$\alpha_2 = a + k_1 - 1 + 2m,\ \alpha_1 = m^2 + m(2k_1 - 1) + ak_2 - k_1,\ \alpha_0 = -k_1 m(1 - m).$$

By Routh's scheme (see [9]), the number $\mathfrak{p}$ of roots of (2.22) with positive real part, counted with multiplicities, is equal to the number of changes of sign of the sequence

$$V := \left(1, \alpha_2, \alpha_1 - \frac{\alpha_0}{\alpha_2}, \alpha_0\right), \tag{2.24}$$

provided that all terms of V are non zero. Thus $\mathfrak{p} = 3$ when

$$\alpha_2 < 0 \text{ and } \alpha_1\alpha_2 < \alpha_0, \tag{2.25}$$

and, in all other cases, $\mathfrak{p} = 1$. When $\mathfrak{p} = 1$, we know that the number $\mathfrak{n}$ of real positive roots of R is exactly 1. When $\mathfrak{p} = 3$, we have either $\mathfrak{n} = 1$ if R has two complex conjugate roots, or $\mathfrak{n} = 3$. So, we need to examine when all roots of R are real numbers. A very simple method to do that for cubic polynomials is described by Tong [24]: a necessary and sufficient condition for R to have three distinct real roots is that R has a local maximum and a local minimum, and that these extrema have opposite signs. The abscissa of these extrema are the roots of the derivative $R'(X) = 3X^2 + 2\alpha_2 X + \alpha_1$, thus R has three distinct real roots if, and only if, the following conditions are simultaneously satisfied:

(i) The discriminant $\Delta_{R'}$ of R' is positive,

(ii) $R(\underline{x})R(\overline{x}) < 0$, where $\underline{x}$ and $\overline{x}$ are the distinct roots of R'.

If $R(\underline{x})R(\overline{x}) = 0$ with $\Delta_{R'} > 0$, the polynomial R still has three real roots, two of which coincide and differ from the third one. If $R(\underline{x})R(\overline{x}) = 0$ with $\Delta_{R'} = 0$, it has a real root with multiplicity 3, which is $\underline{x} = \overline{x}$, and if $\Delta_{R'} = 0$ with $R(\underline{x})R(\overline{x}) \neq 0$, it has only one real root. Fortunately, all radicals disappear in the calculation of $R(\underline{x})R(\overline{x})$:

$$R(\underline{x})R(\overline{x}) = \frac{1}{27}\left(4\alpha_2^3\alpha_0 - \alpha_2^2\alpha_1^2 + 4\alpha_1^3 - 18\alpha_2\alpha_1\alpha_0 + 27\alpha_0^2\right).$$

In particular, Conditions (i) and (ii) can be summarized as

$$\alpha_2^2 - 3\alpha_1 > 0 \text{ and } 4\alpha_2^3\alpha_0 - \alpha_2^2\alpha_1^2 + 4\alpha_1^3 - 18\alpha_2\alpha_1\alpha_0 + 27\alpha_0^2 < 0. \tag{2.26}$$

Let us now examine what happens when one term of the sequence V in (2.24) is zero. We skip temporarily the case $\alpha_0 = 0$, which is equivalent to $m = 0$.

- If $\alpha_2\alpha_1 = \alpha_0$, we have

$$R(X) = (X + \alpha_2)(X^2 + \alpha_1),$$

  and α_2 and α_1 have opposite signs, because $\alpha_0 < 0$. Thus, in that case, R has a unique positive root, which is $\sqrt{-\alpha_1}$ if $\alpha_2 > 0$, and $-\alpha_2$ if $\alpha_2 < 0$.

- If $\alpha_2 = 0$, the derivative of R becomes $R'(X) = 3X^2 + \alpha_1$. If $\alpha_1 > 0$, R is increasing on $]-\infty, \infty[$, thus it has only one (necessarily positive) real root. If $\alpha_1 = 0$, we have $R(X) = X^3 + \alpha_0$, thus R has only one real root, which is $\sqrt[3]{-\alpha_0} > 0$. If $\alpha_1 < 0$, R is decreasing in the interval $[-\sqrt{-\alpha_1}, \sqrt{-\alpha_1}]$, and increasing in $[\sqrt{-\alpha_1}, +\infty[$. Since $R(0) < 0$, R has only one positive root. Thus, in that case too, R has a unique positive root.

From the preceding discussion, we deduce the following theorem:

Theorem 2.5 *Assume that $m > 0$. With the notations of* (2.23)*, the number $\mathfrak{n}$ of distinct equilibrium points of the system* (1.2) *which lie in the open quadrant* $]0, +\infty[\times]0, +\infty[$ *is*

(a) $\mathfrak{n} = 3$ *if* $\Big(\alpha_2 < 0,\ \alpha_1\alpha_2 < \alpha_0,\ \alpha_2^2 - 3\alpha_1 > 0,$ *and* $4\alpha_2^3\alpha_0 - \alpha_2^2\alpha_1^2 + 4\alpha_1^3 - 18\alpha_2\alpha_1\alpha_0 + 27\alpha_0^2 < 0\Big)$,

(b) $\mathfrak{n} = 2$ *if* $\Big(\alpha_2 < 0,\ \alpha_1\alpha_2 < \alpha_0,\ \alpha_2^2 - 3\alpha_1 > 0$ *and* $4\alpha_2^3\alpha_0 - \alpha_2^2\alpha_1^2 + 4\alpha_1^3 - 18\alpha_2\alpha_1\alpha_0 + 27\alpha_0^2 = 0\Big)$,

(c) $\mathfrak{n} = 1$ *in all other cases, i.e., if* $\Big(\alpha_2 \geq 0$ *or* $\alpha_1\alpha_2 \geq \alpha_0$ *or* $\alpha_2^2 - 3\alpha_1 \leq 0$ *or* $4\alpha_2^3\alpha_0 - \alpha_2^2\alpha_1^2 + 4\alpha_1^3 - 18\alpha_2\alpha_1\alpha_0 + 27\alpha_0^2 > 0\Big)$.

Remark 2.6 Numerical computations show that all cases considered in Theorem 2.5 are nonempty. See Figure 1 for an example of positive numbers (a, k_1, k_2, m) satisfying (2.25) and (2.26).

• When $m = 0$, the system (1.2) is exactly the system studied by M.A. Aziz-Alaoui and M. Daher-Okiye [1, 5]. As x is assumed to be positive, (2.20) is equivalent to the quadratic equation

$$(1 - x)(k_1 + x) = a(k_2 + x), \tag{2.27}$$

which can be written

$$x^2 + \alpha_2 x + \alpha_1 = 0,$$

where $\alpha_2 = a + k_1 - 1$ and $\alpha_1 = ak_2 - k_1$ as in (2.23). The associated discriminant is

$$\Delta = \alpha_2^2 - 4\alpha_1 = (a + k_1 - 1)^2 - 4ak_2 + 4k_1, \tag{2.28}$$

thus a sufficient and necessary condition for the existence of solutions to (2.27) in $\mathbb{R}$ is $\Delta \geq 0$, i.e., k_2 must not be too large:

$$4ak_2 \leq (1 - k_1 - a)^2 + 4k_1. \tag{2.29}$$

Since the sum of the solutions to (2.27) is $-\alpha_2$ and their product is α_1, we deduce the following result:

Theorem 2.7 *Assume that* $m = 0$. *With the notations of* (2.23), *the number* $\mathfrak{n}$ *of distinct equilibrium points of the system* (1.2) *which lie in the open quadrant* $]0, +\infty[\times]0, +\infty[$ *is*

(a) $\mathfrak{n} = 2$ *if* $\Delta > 0$ *and* $\alpha_1 > 0$ *and* $\alpha_2 < 0$, *i.e., if*

$$4ak_2 < (1 - k_1 - a)^2 + 4k_1 \text{ and } ak_2 > k_1 \text{ and } 1 - k_1 - a > 0. \tag{2.30}$$

(b) $\mathfrak{n} = 1$ *if* $\Big(\Delta > 0$ *and* $\big(\alpha_1 < 0$ *or* $(\alpha_1 = 0$ *and* $\alpha_2 < 0)\big)\Big)$, *or* $\Big(\Delta = 0$ *and* $\alpha_2 < 0\Big)$ *i.e., if*

$$\Big(\Big(4ak_2 < (1-k_1-a)^2+4k_1\Big) \text{ and } \Big(ak_2 < k_1 \text{ or } \big(ak_2 = k_1 \text{ and } 1-k_1-a > 0\big)\Big)\Big),$$

$$\text{or } \Big(4ak_2 = (1-k_1-a)^2 + 4k_1 \text{ and } 1-k_1-a > 0\Big),$$

(c) $\mathfrak{n} = 0$ *if* $\Delta < 0$, *or if* $\Big(\alpha_1 \geq 0$ *and* $\alpha_2 \geq 0\Big)$, *i.e., if*

$$\Big(4ak_2 > (1-k_1-a)^2 + 4k_1\Big) \text{ or } \Big(ak_2 \geq k_1 \text{ and } 1-k_1-a \leq 0\Big),$$

Remark 2.8 If $m = 0$ and $\mathfrak{n} = 0$, the point E_2 is the only equilibrium point in the compact invariant attracting set $[0,1] \times [k_2 - \epsilon, L]$, for any $\epsilon > 0$ such that $k_2 - \epsilon > 0$, thus E_2 is globally attractive, because there is no cycle around E_2 (since E_2 is on the boundary of $[0,1] \times [k_2 - \epsilon, L]$). This gives a more general condition of global attractivity of E_2 than the result given in Parts (ciii) and (civ) of Theorem 2.1.

Remark 2.9 Since the roots of the polynomial R defined by (2.22) depend continuously on its coefficients, Theorem 2.7 expresses the limiting localization of the equilibrium points of (1.2) when m goes to 0. In particular, the case (a) of Theorem 2.7 is the limiting case of (a) in Theorem 2.5. Indeed, it is easy to check that Condition (2.30), with $m = 0$, is a limit case of (2.25) and (2.26). This means that, in the case (a) of Theorem 2.5, when m goes to 0, one of the equilibrium points in the open quadrant $]0,+\infty[\times]0,+\infty[$ goes to E_2 and leaves the open quadrant $]0,+\infty[\times]0,+\infty[$. (Note that, when $m = 0$, the equilibrium point $E_2 = (0, k_2)$ is in $\mathcal{A}$.)

Remark 2.10 When $k_1 = k_2 := k$, since $x > m$, Equation (2.20) is equivalent to $x(1-x) = a(x-m)$, i.e.,

$$x^2 + x(a-1) - am,$$

thus it has at most one positive solution. In that case, the coordinates of the unique non trivial equilibrium point E^* can be explicited in a simple way, and we have

$$E^* = \left(\frac{1 - a + \sqrt{(1-a)^2 + 4am}}{2}, k + x^* - m\right).$$

If $a \geq 1$, the point E^* converges to E_2 when m goes to 0. If $a > 1$, it converges to $(1-a, 1-a+k)$.

2.2.3 Local stability

Let $E^* = (x^*, y^*)$ be an equilibrium point of (1.2) in the open quadrant $]0, +\infty[\times]0, +\infty[$. Since E^* is necessarily in $\mathcal{A}$, we get, using (2.18) and (2.21),

$$\mathcal{J}(x^*, y^*) = \begin{pmatrix} 1 - 2x^* - \frac{ay^*k_1}{(k_1+x^*-m)^2} & \frac{-a(x^*-m)}{k_1+x^*-m} \\ b & -b \end{pmatrix}. \tag{2.31}$$

The characteristic polynomial of $\mathcal{J}(x^*, y^*)$ is

$$\chi(\lambda) = \lambda^2 + s\lambda + p,$$

where

(2.32)

$$s = -\operatorname{Trace}(\mathcal{J}(x^*, y^*)) = -1 + 2x^* + \frac{ay^*k_1}{(k_1 + x^* - m)^2} + b,$$

(2.33)

$$p = \det(\mathcal{J}(x^*, y^*)) = b\left(-1 + 2x^* + \frac{ay^*k_1}{(k_1 + x^* - m)^2} + \frac{a(x^* - m)}{k_1 + x^* - m}\right).$$

The roots of χ are real if, and only if, $\Delta_\chi \geq 0$, where

$$\Delta_\chi = s^2 - 4p = \left(-1 + 2x^* + \frac{ay^*k_1}{(k_1 + x^* - m)^2} - b\right)^2 - 4b\frac{a(x^* - m)}{k_1 + x^* - m}.$$

The point E^* is non-hyperbolic if one of the roots of χ is zero (that is, if $p = 0$), or if χ has two conjugate purely imaginary roots (that is, if $s = 0$ with $p > 0$). If only one root of χ is zero, that is, if $p = 0$ with $s \neq 0$, the point E^* is semi-hyperbolic.

a- Hyperbolic equilibria When E^* is hyperbolic, we get, using the Routh-Hurwitz criterion, that E^* is

- a saddle point if $p < 0$,
- an unstable node if $s < 0$ and $p > 0$ with $\Delta_\chi > 0$,
- an unstable focus if $s < 0$ and $p > 0$ with $\Delta_\chi < 0$,
- an unstable degenerated node if $s < 0$ and $p > 0$ with $\Delta_\chi = 0$,
- a stable node if $s > 0$ and $p > 0$ with $\Delta_\chi > 0$,

- a stable degenerated node if $s > 0$ and $p > 0$ with $\Delta_\chi = 0$,
- a stable focus if $s > 0$ and $p > 0$ with $\Delta_\chi < 0$.

Remark 2.11 An obvious sufficient condition for any equilibrium point $E^* \in \mathcal{A}$ to be stable hyperbolic is $m \geq 1/2$, since $x^* > m$. This condition can be slightly improved, as we shall see in the study of global stability (see Theorem 2.19).

Application of the Poincaré index theorem When E^* is an hyperbolic equilibrium, its index is either 1 (if it is a node or focus) or -1 (if it is a saddle). Let $\mathfrak{n}$ be the number of distinct equilibrium points, which we denote by $E_1^*, ..., E_\mathfrak{n}^*$, and let $I_1, ..., I_\mathfrak{n}$ their respective indices. As we shall see in the proof of the next theorem, by a generalized version of the Poincaré index theorem, we have $I_1 + ... + I_\mathfrak{n} = 1$. When all equilibrium points are hyperbolic, this allows us to count the number of nodes or foci and of saddles.

Theorem 2.12 *Assume that all equilibrium points of the system* (1.2) *which lie in the open quadrant* $]0, +\infty[\times]0, +\infty[$ *(equivalently, in the interior of* $\mathcal{A}$*) are hyperbolic, and let* $\mathfrak{n}$ *be their number.*

1. *Assume that* $m > 0$*. Then* $\mathfrak{n}$ *is equal to 3 or 1.*
   - *If* $\mathfrak{n} = 1$*, the unique equilibrium point in the interior of* $\mathcal{A}$ *is a node or a focus.*
   - *If* $\mathfrak{n} = 3$*, the system* (1.2) *has one saddle point and two nodes or foci in the interior of* $\mathcal{A}$*.*
2. *Assume now that* $m = 0$*. Then* $\mathfrak{n}$ *is equal to 2, 1, or 0.*
   - *If* $\mathfrak{n} = 2$*, one equilibrium point is a node or focus, and the other is a saddle.*
   - *If* $\mathfrak{n} = 1$*, the unique equilibrium point in the interior of* $\mathcal{A}$ *is a node or a focus.*

Proof. Let N (respectively S) denote the number of nodes or foci (respectively of saddles) among the hyperbolic singular points which lie in $\mathcal{A}$.

1. Assume that $m > 0$. By Theorem 2.1, the vector field $\boldsymbol{v} = v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y}$ generated by (1.2) is directed inward along the boundary of $\mathcal{A}$. By continuity

of $\boldsymbol{v}$, we can round the corners of $\mathcal{A}$ and define a compact domain $\mathcal{A}' \subset \mathcal{A}$ with smooth boundary which contains all critical points of $\mathcal{A}$, and such that $\boldsymbol{v}$ is directed inward along the boundary of $\mathcal{A}'$. Applying a generalized version of the Poincaré index theorem (see e.g. [15, 10, 22]) to $\boldsymbol{v}$ in $\mathcal{A}'$, we get $N - S = 1$. Since $1 \leq N + S \leq 3$, the only possibilities are ($N = 1$ and $S = 0$) or ($N = 2$ and $S = 1$).

2. Assume now that $m = 0$. We use the same reasoning as for $m > 0$, but with a different domain. Instead of $\mathcal{A}$, we consider the domain

$$\mathcal{B} = [-\epsilon, 1] \times [k_2 - \epsilon, L]$$

for a small $\epsilon > 0$. Thus $\mathcal{B}$ contains E_2.

• With the notations of (2.17), if $ak_2 > k_1$, we have $y > U(x)$ for $x = 0$ and for all $y \in [k_2, L]$. We have

$$v_1 = \frac{ax}{k_1 + x}\Big(U(x) - y\Big).$$

By continuity of $\boldsymbol{v}$, we can choose $\epsilon > 0$, with $\epsilon < k_1$, such that the inequality $y > U(x)$ remains true on the rectangle $[-\epsilon, 0] \times [k_2 - \epsilon, L]$. We then have $v_1 > 0$ on the segment $\{-\epsilon\} \times [k_2 - \epsilon, L]$. Since $v_2 > 0$ for $y = k_2 - \epsilon$ and $v_2 < 0$ for $y = L$, the field $\boldsymbol{v}$ is directed inward along the boundary of $\mathcal{B}$. Again, by rounding the corners, we can modify $\mathcal{B}$ into a a compact domain $\mathcal{B}'$ with smooth boundary which contains the same critical points as $\mathcal{B}$ and such that $\boldsymbol{v}$ is directed inward along the boundary of $\mathcal{B}'$. By the Poincaré Index Theorem, we have $N' - S' = 1$, where N' (respectively S') is the number of nodes or foci (respectively of saddles) in the interior of $\mathcal{B}'$. If we have chosen ϵ small enough, the singularities of $\boldsymbol{v}$ in $\mathcal{B}'$ are those which are in the interior of $\mathcal{A}$, with the addition of the point E_2, which is a node by Proposition 2.3. Thus $N = N' - 1$ and $S = S'$ which entails $N - S = 0$. Thus, taking into account Theorem 2.7, we have $N = S = 1$ (if $\mathfrak{n} = 2$), or $N = S = 0$ (if $\mathfrak{n} = 0$).

• If $ak_2 < k_1$, E_2 is a saddle point, thus, constructing $\mathcal{B}$ and $\mathcal{B}'$ as precedingly, we have now $S = S' - 1$ and $N = N'$. Furthermore, the vector field $\boldsymbol{v}$ is no more outward directed along the whole boundary of $\mathcal{B}'$.We use Pugh's algorithm [22] to compute $N' - S'$: taking ϵ small enough such that the vector field $\boldsymbol{v}$ does not vanish on $\partial\mathcal{B}'$, we have

$$N' - S' = \chi(\mathcal{B}') - \chi(\partial\mathcal{B}') + \chi(R^1_-) - \chi(\partial R^1_-) + \chi(R^2_-) - \chi(\partial R^2_-), \tag{2.34}$$

where χ denotes the Euler characteristic, R^1_- is the part of the boundary of $\mathcal{B}'$ where $\boldsymbol{v}$ is directed outward, and R^2_- is the part of ∂R^1_- where $\boldsymbol{v}$ points to the exterior of R^1_-. Since $k_2 < k_1/a$, we see that the parabola $y = U(x)$ crosses the line $\{x = -\epsilon; y > k_2\}$ at some point $(-\epsilon, r)$, so that the part of the boundary of $\mathcal{B}$ where $\boldsymbol{v}$ points outward is the segment $\{-\epsilon\} \times [k_2 - \epsilon, \min(r, L)]$. Thus, for small ϵ, R^1_- is an arc whose extremities are tangency points. Observe also that, since $v_1 < 0$ for $x < 0$ and $v_2 < 0$ for $y > k_2 + x > 0$, the field $\boldsymbol{v}$ points toward the interior of R^1_- at those tangency points, thus R^2_- is empty. Formula (2.34) becomes

$$\Sigma(\boldsymbol{v}) = 1 - 0 + 1 - 2 + 0 - 0 = 0,$$

that is, $N - S = N' - (S' - 1) = 1$. Since, by Theorem 2.7, we have $N + S = 1$, we deduce that $N = 1$ and $S = 0$. $\square$

b. Semi hyperbolic equilibria This is when $p = 0$ and $s \neq 0$. The set of parameters such that $p \neq 0$ is nonempty. Indeed, the values $a = 0,5$, $b = 0,01$, $m = 0,001$, $k_2 = 0,25$, $k_1 = 0,08$ lead to $p = -0.1003032464$ with $\alpha_2 = 0.044161 > 0$ and $a = 0,5$, $b = 0,01$, $m = 0,001$, $k_2 = 0,25$, $k_1 = 0,112$ lead to $p = 0.002422466814$ with $\alpha_2 = 0.012225 > 0$. Since α_2 is linear function of k_1, this shows that $\alpha_2 > 0$ for $a = 0,5$, $b = 0,01$, $m = 0,001$, $k_2 = 0,25$ and $0,08 \leq k_1 \leq 0,112$. Thus, by Theorem(2.5), for all these values, the number $\mathfrak{n}$ of equilibrium points remains equal to 1. By the intermediate value theorem, we deduce that there exists a value k_1, with $0,08 \leq k_1 \leq 0,112$, such that, for $a = 0,5$, $b = 0,01$, $m = 0,001$, $k_2 = 0,25$, the unique equilibrium point satisfies $p = 0$.

From (2.32), (2.33) and (2.23), it is obvious that we can chose b such that $s \neq 0$ without changing $p = 0$ nor the coefficients α_0, α_1, α_2.

For $p = 0$, the Jacobian matrix $\mathcal{J}(x^*, y^*)$ is

$$\mathcal{J}(x^*, y^*) = \begin{pmatrix} a\rho & -a\rho \\ b & -b \end{pmatrix},$$

The change of variables

$$\mathfrak{u} = \frac{a\rho Y - bX}{a\rho - b}, \quad \mathfrak{v} = \frac{X - Y}{a\rho - b}$$

yields

$$v_1 = a\rho(a\rho - b)\mathfrak{v} + a^2\rho\frac{k_1(y^*a\rho - b\kappa) - \rho\kappa^3}{\kappa^3}\mathfrak{v}^2 - \frac{ak_1(\kappa - y^*) + \kappa^3}{\kappa^3}\mathfrak{u}^2$$

$$
\begin{aligned}
&-a\frac{k_1\kappa(b+a\rho)+\rho(2\kappa^3-y^*k_1a)}{\kappa^3}\mathfrak{v}\mathfrak{u}+a^3k_1\rho^2\frac{b\kappa-y^*a\rho}{\kappa^3(\kappa+\mathfrak{u}+\rho\, a\mathfrak{v})}\mathfrak{v}^3\\
&-ak_1\frac{y^*-\kappa}{\kappa^3(\kappa+\mathfrak{u}+\rho a\mathfrak{v})}\mathfrak{u}^3+ak_1\frac{b\kappa+2a\rho\kappa-3y^*a\rho}{\kappa^3(\kappa+\mathfrak{u}+\rho a\mathfrak{v})}\mathfrak{u}^2\mathfrak{v}+a^2k_1\rho\frac{2b\kappa+a\rho\kappa-3y^*a\rho}{\kappa^3(\kappa+\mathfrak{u}+\rho a\mathfrak{v})}\mathfrak{v}^2\mathfrak{u}\\
v_2 =&b(\rho\, a-b)\mathfrak{v}+b\frac{-b^2+2b\rho\, a-\rho^2a^2}{\mathfrak{u}+\rho\, a\mathfrak{v}+y^*}\mathfrak{v}^2
\end{aligned}
$$

The coordinates of $\boldsymbol{v}$ are, in the basis $(\frac{\partial}{\partial\mathfrak{u}},\frac{\partial}{\partial\mathfrak{v}})$,

$$
\begin{aligned}
\dot{\mathfrak{u}} &= \frac{1}{a\rho-b}(a\rho\dot{Y}-b\dot{X})=\frac{1}{a\rho-b}(a\rho v_2-bv_1)\\
&=\frac{b}{b-\rho\, a}\Bigg(-\frac{(-k_1y^*a+\kappa^3+ak_1)\mathfrak{u}^2}{\kappa^2}+\frac{ak_1(\kappa-y^*)}{\kappa^3(\kappa+\mathfrak{u}+\rho\, a\mathfrak{v})}\mathfrak{u}^3+\frac{a\rho(-\kappa^3\rho a+y^*k_1a^2\rho-\kappa ak_1b)}{\kappa^3}\mathfrak{v}^2\\
&\quad+\frac{a^3k_1\rho^2(\kappa b-y^*a\rho)}{\kappa^3(\kappa+\mathfrak{u}+\rho\, a\mathfrak{v})}\mathfrak{v}^3-\frac{a\kappa(k_1b+ak_1\rho\,+2\kappa^2\rho\,)-2y^*k_1a^2\rho}{\kappa^3}\mathfrak{v}\mathfrak{u}-\rho\, a\frac{-b^2+2b\rho\, a-\rho^2a^2}{\mathfrak{u}+\rho\, a\mathfrak{v}+y^*}\mathfrak{v}^2\\
&\quad+\frac{ak_1(b\kappa+2a\kappa\rho\,-3y^*a\rho\,)}{\kappa^3(\kappa+\mathfrak{u}+\rho\, a\mathfrak{v})}\mathfrak{u}^2\mathfrak{v}+\frac{a^2k_1\rho(2b\kappa+a\kappa\rho-3y^*a\rho)}{\kappa^3(\kappa+\mathfrak{u}+\rho\, a\mathfrak{v})}\mathfrak{v}^2\mathfrak{u}\Bigg)\\
\dot{\mathfrak{v}} &= -\frac{1}{a\rho-b}(\dot{X}-\dot{Y})=\frac{1}{a\rho-b}(v_1-v_2)\\
&=(a\rho-b)\mathfrak{v}+\frac{1}{b-\rho\, a}\Bigg(a^2\rho\frac{k_1b\kappa+\rho\kappa^3-y^*k_1a\rho}{\kappa^3}\mathfrak{v}^2+\frac{ak_1\kappa-k_1y^*a+\kappa^3}{\kappa^3}\mathfrak{u}^2\\
&\quad+a\frac{k_1b\kappa+ak_1\rho\kappa+2\rho\,\kappa^3-2y^*k_1a\rho}{\kappa^3}\mathfrak{u}\mathfrak{v}+k_1a^3\rho^2\frac{y^*a\rho-b\kappa}{\kappa^3(\kappa+\mathfrak{u}+\rho\, a\mathfrak{v})}\mathfrak{v}^3+ak_1\frac{y^*-\kappa}{\kappa^3(\kappa+\mathfrak{u}+\rho\, a\mathfrak{v})}\mathfrak{u}^3\\
&\quad+b\frac{(a\rho-b)^2}{\mathfrak{u}+\rho\, a\mathfrak{v}+y^*}\mathfrak{v}^2+ak_1\frac{3y^*a\rho\,-\kappa(b+2a\rho\,)}{\kappa^3(\kappa+\mathfrak{u}+\rho\, a\mathfrak{v})}\mathfrak{u}^2\mathfrak{v}+k_1a^2\rho\frac{3y^*a\rho-a\rho\kappa-2b\kappa}{\kappa^3(\kappa+\mathfrak{u}+\rho\, a\mathfrak{v})}\mathfrak{v}^2\mathfrak{u}\Bigg)
\end{aligned}
$$

We can thus write

$$
\begin{aligned}
\dot{\mathfrak{u}} &= A(\mathfrak{u},\mathfrak{v})\\
\dot{\mathfrak{v}} &= \lambda\mathfrak{v}+B(\mathfrak{u},\mathfrak{v}),
\end{aligned}
\tag{2.35}
$$

where A and B are analytic and their jacobian matrix at $(0,0)$ is 0 and $\lambda>0$. It is not easy to determine $\mathfrak{v}=f(\mathfrak{u})$ the solution to the equation $\lambda\mathfrak{v}+B(\mathfrak{u},\mathfrak{v})=0$ in a neighborhood of the point $(0,0)$, for that we use implicit function theorem. We find :

Case 1: if $\kappa^3-ky^*a+ak_1\kappa\neq 0$,

$$
f(\mathfrak{u})=-\frac{\kappa^3-ky^*a+ak_1\kappa}{\kappa^3(b+\rho^2a^2-\rho\, ab)}\mathfrak{u}^2
$$

and $g(\mathfrak{u}) = A(\mathfrak{u}, f(\mathfrak{u}))$ has the form

$$g(\mathfrak{u}) = \frac{b}{b - a\rho}\left(\frac{\kappa^3 - ky^*a + ak_1\kappa}{\kappa^3}\right)\mathfrak{u}^2.$$

We apply [7, Theorem 2.19] to System (2.35). Since the power of $\mathfrak{u}$ in $f(\mathfrak{u})$ is even, we deduce from Part (iii) of [7, Theorem 2.19] :

Lemma 2.13 *If E^* is a semi-hyperbolic equilibrium of* (1.2) *in the positive quadrant $]0, +\infty[\times]0, +\infty[$, and if $\kappa^3 - ky^*a + ak_1\kappa \neq 0$, then E^* is a saddle-node, that is, its phase portrait is the union of one parabolic and two hyperbolic sectors. In this case, the index of E^* is 0.*

Case 2: if $\kappa^3 - ky^*a + ak_1\kappa = 0$

$$f(\mathfrak{u}) = \frac{ak_1(\kappa - y^*)}{\kappa^4(a\rho - b)^2}\mathfrak{u}^3.$$

And $g(\mathfrak{u}) = A(\mathfrak{u}, f(\mathfrak{u}))$ has the form

$$g(\mathfrak{u}) = \frac{bak_1(\kappa - y^*)}{\kappa^4(a\rho - b)^2}\mathfrak{u}^3.$$

Again, we apply [7, Theorem 2.19] to System (2.35). Since the power of $\mathfrak{u}$ in $f(\mathfrak{u})$ is odd, we look at the cofficient of $\mathfrak{u}^3$ and we have two possibilities :
P1: If $k_1 > k_2$, we deduce from Part (ii) of [7, Theorem 2.19] :

Lemma 2.14 *If E^* is a semi-hyperbolic equilibrium of* (1.2) *in the positive quadrant $]0, +\infty[\times]0, +\infty[$, and if $\kappa^3 - ky^*a + ak_1\kappa = 0$ with $k_1 > k_2$, then E^* is a unstable node. In this case, the index of E^* is 1.*

P2: If $k_1 < k_2$, we deduce from Part (i) of [7, Theorem 2.19] :

Lemma 2.15 *If E^* is a semi-hyperbolic equilibrium of* (1.2) *in the positive quadrant $]0, +\infty[\times]0, +\infty[$, and if $\kappa^3 - ky^*a + ak_1\kappa = 0$ with $k_1 < k_2$, then E^* is a saddle. In this case, the index of E^* is -1.*

Remark 2.16 From Theorem 2.12, when the system 1.2 has one equilibrium point, this point cannot be a saddle.

Hopf bifurcation When $\Delta_\chi < 0$, the roots of χ are $\frac{-s \pm i\sqrt{4p-s^2}}{2}$. The values of x^*, y^* and p do not depend on the parameter b, whereas s is an affine function of b, so that the eigenvalues of χ cross the imaginary axis at speed $-1/2$ when b passes through the value

$$b_0 = 1 - 2x^* + \frac{ay^*k_1}{\kappa^2}.$$

Let us check the genericity condition for Hopf bifurcations. We use the condition of Guckenheimer and Holmes [11, Formula (3.4.11)]. Let us denote

$$\dot{\mathfrak{u}} = f(\mathfrak{u}, \mathfrak{v}), \quad \dot{\mathfrak{v}} = g(\mathfrak{u}, \mathfrak{v}),$$

and $f_{\mathfrak{u}\mathfrak{v}} = \frac{\partial f}{\partial \mathfrak{u} \partial \mathfrak{v}}$, etc. We have

$$\begin{aligned}\lambda = f_{\mathfrak{uuu}} + f_{\mathfrak{uvv}} + g_{\mathfrak{uuv}} + g_{\mathfrak{vvv}} \\ + \frac{1}{\delta}\left(f_{\mathfrak{uv}}(f_{\mathfrak{uu}} + f_{\mathfrak{vv}}) - g_{\mathfrak{uv}}(g_{\mathfrak{uu}} + g_{\mathfrak{vv}}) - f_{\mathfrak{uu}}g_{\mathfrak{uu}} + f_{\mathfrak{vv}}g_{\mathfrak{vv}}\right)\end{aligned}$$

$$\begin{aligned}\lambda =\; & ak_1\kappa^3(-2y^* + \kappa)b_0^2 + \kappa\ (2c\kappa^5 + 2k_1y^*a\kappa^3 - \kappa^3ck_1a + 3\kappa ck_1{y^*}^2a + \kappa a^2k_1^2y^* - 2a^2k_1^2{y^*}^2)b_0 \\ & - 2c(c-y^*)\kappa^6 + ak_1\kappa^4cy^* + 2acy^*k_1(-2y^*+c)\kappa^3 - 3\kappa^2k_1{y^*}^2ac^2 - a^2k_1^2\kappa c{y^*}^2 + 2k_1^2{y^*}^3a^2c\end{aligned}$$

If $\lambda < 0$, then the periodic solutions are stable limit cycles, while if $\lambda > 0$, the periodic solutions are repelling. See Figure (3) for a numerical exemple.

c- Non-elementery equilibria Let us rewrite the vector field $\boldsymbol{v} = v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}$ associated with (1.2) in the neighborhood of an equilibrium point $E^* = (x^*, y^*) \in \mathcal{A}$. Let $X = x - x^*$ and $Y = y - y^*$. Since E^* is a critical point of $\boldsymbol{v}$, we have

$$\begin{aligned}v_1 =\; & x(1-x) - \frac{ay(x-m)}{k_1 + (x-m)} \\ =\; & (X + x^*)(1 - x^* - X) - \frac{a(Y+y^*)(X + x^* - m)}{X + x^* + k_1 - m} \\ =\; & x^*(1-x^*) + X(1 - 2x^* - X) - \frac{ay^*(x^*-m)}{X + x^* + k_1 - m} - \frac{a(Y(X + x^* - m) + Xy^*)}{X + x^* + k_1 - m} \\ =\; & x^*(1-x^*) - \frac{ay^*(x^*-m)}{x^* + k_1 - m} + X(1 - 2x^* - X) \\ & + \frac{ay^*(x^*-m)}{x^* + k_1 - m} - \frac{ay^*(x^*-m)}{X + x^* + k_1 - m} - \frac{a(Y(X + x^* - m) + Xy^*)}{X + x^* + k_1 - m}\end{aligned}$$

$$
\begin{aligned}
&= X(1-2x^*-X) + \frac{ay^*(x^*-m)}{x^*+k_1-m} - \frac{ay^*(x^*-m)}{X+x^*+k_1-m} - \frac{a(Y(X+x^*-m)+Xy^*)}{X+x^*+k_1-m} \\
&= X(1-2x^*-X) + ay^*(x^*-m)\left(\frac{1}{x^*+k_1-m} - \frac{1}{X+x^*+k_1-m}\right) \\
&\quad - \frac{a\Big(Y(X+x^*-m)+Xy^*\Big)}{X+x^*+k_1-m} \\
&= X(1-2x^*-X) + \frac{ay^*(x^*-m)X}{(x^*+k_1-m)(X+x^*+k_1-m)} - \frac{a\Big(Y(X+x^*-m)+Xy^*\Big)}{X+x^*+k_1-m}.
\end{aligned}
$$

For simplification, we denote

$$
\kappa = x^*+k_1-m, \quad \rho = \frac{x^*-m}{x^*+k_1-m}, \tag{2.36}
$$

thus

$$
v_1 = X(1-2x^*-X) + \frac{a\Big(Xy^*(\rho-1) - Y(x^*-m) - YX\Big)}{X+\kappa}.
$$

Using the equality

$$
\frac{1}{x+K} = \frac{1}{K}\left(1 - \frac{x}{K} + \cdots + (-1)^n\frac{x^n}{K^n} + (-1)^{n+1}\frac{x^{n+1}}{K^n(x+K)}\right), \quad n \geq 1,
$$

we get

$$
\begin{aligned}
v_1 &= X(1-2x^*-X) \\
&\quad + \frac{a}{\kappa}\left(1 - \frac{X}{\kappa} + \frac{X^2}{\kappa(X+\kappa)}\right)\Big(Xy^*(\rho-1) - Y(x^*-m) - YX\Big) \\
&= X(1-2x^*-X) \\
&\quad + \frac{a}{\kappa}\Big(Xy^*(\rho-1) - Y(x^*-m) - YX\Big) \\
&\quad - \frac{aX}{\kappa^2}\Big(Xy^*(\rho-1) - Y(x^*-m) - YX\Big) \\
&\quad + \frac{aX^2}{\kappa^2(X+\kappa)}\Big(Xy^*(\rho-1) - Y(x^*-m) - YX\Big) \\
&= X\left(1 - 2x^* + \frac{a}{\kappa}y^*(\rho-1)\right) - Y\frac{a}{\kappa}(x^*-m) \\
&\quad - X^2\left(1 + \frac{a}{\kappa^2}y^*(\rho-1)\right) + XY\left(-\frac{a}{\kappa} + \frac{a}{\kappa^2}(x^*-m)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{aX^2Y}{\kappa^2} + \frac{aX^2}{\kappa^2(X+\kappa)}\Big(Xy^*(\rho-1) - Y(x^*-m+X)\Big)\\
&= X\left(1 - 2x^* - \frac{ay^*k_1}{\kappa^2}\right) - Ya\rho - X^2\left(1 - \frac{ay^*k_1}{\kappa^3}\right) - XY\frac{ak_1}{\kappa^2}\\
&- X^3\frac{ay^*k_1}{\kappa^3(X+\kappa)} + X^2Y\frac{a}{\kappa^2}\left(1 - \frac{x^*-m+X}{X+\kappa}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= X\left(1 - 2x^* - \frac{ay^*k_1}{\kappa^2}\right) - Ya\rho - X^2\left(1 - \frac{ay^*k_1}{\kappa^3}\right) - XY\frac{ak_1}{\kappa^2}\\
&- X^3\frac{ay^*k_1}{\kappa^3(X+\kappa)} + X^2Y\frac{ak_1}{\kappa^2(X+\kappa)}.
\end{aligned}
\tag{2.37}
$$

Since $y^* = x^* + k_2 - m$, we have also

$$
\begin{aligned}
v_2 &= b(Y+y^*)\left(1 - \frac{Y+y^*}{k_2+x-m}\right) = b(Y+y^*)\left(1 - \frac{Y+y^*}{X+y^*}\right)\\
&= b(X-Y)\frac{Y+y^*}{X+y^*}\\
&= b(X-Y)\left(1 - (X-Y)\frac{1}{X+y^*}\right)
\end{aligned}
$$

$$
= b(X-Y) - \frac{b}{y^*}(X-Y)^2\left(1 - \frac{X}{X+y^*}\right). \tag{2.38}
$$

This shows in particular that the linear part of $\boldsymbol{v}$ is never zero. Thus the only non-hyperbolic cases are the nilpotent case and the case when E^* is a center for the linear part of $\boldsymbol{v}$. Let us now investigate these cases :

$\mathbf{c}_1$. Nilpotent case

This is when $p = 0 = s$. From the discussion at the beginning of Case b, it is clear that this case is nonempty.

In this case, the Jacobian matrix $\mathcal{J}(x^*, y^*)$ is

$$
\mathcal{J}(x^*, y^*) = \begin{pmatrix} b & -b \\ b & -b \end{pmatrix},
$$

With the preceding notations, we thus have

$$
v_1 = b(X-Y) - X^2\left(1 - \frac{ay^*k_1}{\kappa^3}\right) - XY\frac{ak_1}{\kappa^2} - X^3\frac{ay^*k_1}{\kappa^3(X+\kappa)} + X^2Y\frac{ak_1}{\kappa^2(X+\kappa)}.
$$

The change of variables

$$
\mathfrak{u} = X, \quad \mathfrak{v} = Y - X
$$

yields

$$
\begin{aligned}
v_1 = & -\mathfrak{v}b - \mathfrak{u}^2\left(1 - \frac{ay^*k_1}{\kappa^3}\right) - \mathfrak{u}(\mathfrak{u}+\mathfrak{v})\frac{ak_1}{\kappa^2} - \mathfrak{u}^3\frac{ay^*k_1}{\kappa^3(\mathfrak{u}+\kappa)} + \mathfrak{u}^2(\mathfrak{u}+\mathfrak{v})\frac{ak_1}{\kappa^2(\mathfrak{u}+\kappa)}\\
= & -\mathfrak{v}b - \mathfrak{u}^2\left(1 - \frac{ay^*k_1}{\kappa^3} + \frac{ak_1}{\kappa^2}\right) - \mathfrak{u}\mathfrak{v}\frac{ak_1}{\kappa^2}\\
& + \mathfrak{u}^3\frac{ak_1}{\kappa^2(\mathfrak{u}+\kappa)}\left(-\frac{y^*}{\kappa}+1\right) + \mathfrak{u}^2\mathfrak{v}\frac{ak_1^2}{\kappa^2(\mathfrak{u}+\kappa)}\\
v_2 = & -\mathfrak{v}b - \mathfrak{v}^2\frac{b}{y^*}\left(1 - \frac{\mathfrak{u}}{\mathfrak{u}+y^*}\right).
\end{aligned}
$$

The coordinates of $\boldsymbol{v}$ are, in the basis $(\frac{\partial}{\partial \mathfrak{u}}, \frac{\partial}{\partial \mathfrak{v}})$,

$$
\begin{aligned}
\dot{\mathfrak{u}} = & \dot{X} = v_1\\
\dot{\mathfrak{v}} = & \dot{Y} - \dot{X} = v_2 - v_1\\
= & -\mathfrak{v}b - \mathfrak{v}^2\frac{b}{y^*}\left(1 - \frac{\mathfrak{u}}{\mathfrak{u}+y^*}\right) + \mathfrak{v}b + \mathfrak{u}^2\left(1 - \frac{ay^*k_1}{\kappa^3} + \frac{ak_1}{\kappa^2}\right) + \mathfrak{u}\mathfrak{v}\frac{ak_1}{\kappa^2}\\
& - \mathfrak{u}^3\frac{ak_1}{\kappa^2(\mathfrak{u}+\kappa)}\left(-\frac{y^*}{\kappa}+1\right) - \mathfrak{u}^2\mathfrak{v}\frac{ak_1^2}{\kappa^2(\mathfrak{u}+\kappa)}\\
= & \mathfrak{u}^2\left(1 - \frac{ay^*k_1}{\kappa^3} + \frac{ak_1}{\kappa^2}\right) + \mathfrak{u}\mathfrak{v}\frac{ak_1}{\kappa^2} - \mathfrak{v}^2\frac{b}{y^*} + \mathfrak{u}^3\frac{ak_1}{\kappa^2(\mathfrak{u}+\kappa)}\left(\frac{y^*}{\kappa}-1\right)\\
& - \mathfrak{u}^2\mathfrak{v}\frac{ak_1^2}{\kappa^2(\mathfrak{u}+\kappa)} + \mathfrak{u}\mathfrak{v}^2\frac{b}{y^*(\mathfrak{u}+y^*)}.
\end{aligned}
$$

We can thus write

$$
\begin{aligned}
\dot{\mathfrak{u}} = & -\mathfrak{v}b + A(\mathfrak{u},\mathfrak{v})\\
\dot{\mathfrak{v}} = & B(\mathfrak{u},\mathfrak{v}),
\end{aligned}
\tag{2.39}
$$

where A and B are analytic and their jacobian matrix at $(0,0)$ is 0. In the neighborhood of $(0,0)$, the equation $0 = -\mathfrak{v}b + A(\mathfrak{u},\mathfrak{v})$ has the unique solution $\mathfrak{v} = f(\mathfrak{u})$, where

$$
\begin{aligned}
f(\mathfrak{u}) = & \frac{-\mathfrak{u}^2\left(1 - \frac{ay^*k_1}{\kappa^3} + \frac{ak_1}{\kappa^2}\right) - \mathfrak{u}^3\frac{ak_1}{\kappa^2(\mathfrak{u}+\kappa)}\left(\frac{y^*}{\kappa} - k_1\right)}{b + \mathfrak{u}\frac{ak_1}{\kappa^2} - \mathfrak{u}^2\frac{ak_1^2}{\kappa^2(\mathfrak{u}+\kappa)}}\\
= & -\frac{1}{b}(1 - \frac{ay^*k_1}{\kappa^3} + \frac{ak_1}{\kappa^2})u^2 + \frac{ak1\left(-k1y^*a + \kappa^3 + ak1\kappa + b\kappa^2 - y^*\kappa b\right)}{b^2\kappa^5}u^3 + O\left(u^4\right).
\end{aligned}
$$

Let $F(\mathfrak{u}) = B(\mathfrak{u}, f(\mathfrak{u}))$. Since $A(\mathfrak{u}, f(\mathfrak{u})) = bf(\mathfrak{u})$ and $B(\mathfrak{u}, \mathfrak{v})$ has the form

$$B(\mathfrak{u}, \mathfrak{v}) = \mathfrak{v}b - A(\mathfrak{u}, \mathfrak{v}) - \mathfrak{v}b - \mathfrak{v}^2 \frac{b}{y^*}\left(1 - \frac{\mathfrak{u}}{\mathfrak{u} + y^*}\right),$$

we have

$$\begin{aligned}
F(\mathfrak{u}) =& - bf(\mathfrak{u}) \ - f^2(\mathfrak{u})\frac{b}{y^*}\left(1 - \frac{\mathfrak{u}}{\mathfrak{u} + y^*}\right) \\
=& \frac{\left(-k_1\, y^*\, a\kappa\, b - {k_1}^2 y^*\, a^2 + \kappa^3 a k_1 + a k_1\, \kappa^2 b + a^2 {k_1}^2 \kappa\right) u^3}{\kappa^5 b^2} \\
&+ \frac{\left(b\kappa^2 k_1\, y^*\, a - \kappa^5 b - b\kappa^3 a k_1\right) u^2}{\kappa^5 b^2} + o(\mathfrak{u}^3).
\end{aligned}$$

Let also $G(\mathfrak{u}) = (\partial A/\partial \mathfrak{u} + \partial B/\partial \mathfrak{v})(\mathfrak{u}, f(\mathfrak{u}))$. We have

$$\begin{aligned}
\partial A/\partial \mathfrak{u} =& - \mathfrak{v}\frac{a k_1}{\kappa^2} - 2\mathfrak{u}\left(1 - \frac{a y^* k_1}{\kappa^3} + \frac{a k_1}{\kappa^2}\right) + 2\mathfrak{u}\mathfrak{v}\frac{a k_1}{\kappa^2(\mathfrak{u} + \kappa)} \\
&+ 3\mathfrak{u}^2 \frac{a k_1}{\kappa^2(\mathfrak{u} + \kappa)}\left(-\frac{y^*}{\kappa} + 1\right) - \mathfrak{u}^3 \frac{a k_1}{\kappa^2(\mathfrak{u} + \kappa)^2}\left(-\frac{y^*}{\kappa} + 1\right) - \mathfrak{u}^2\mathfrak{v}\frac{a k_1}{\kappa^2(\mathfrak{u} + \kappa)^2}, \\
\partial B/\partial \mathfrak{v} =& - 2\mathfrak{v}\frac{b}{y^*}\left(1 - \frac{\mathfrak{u}}{\mathfrak{u} + y^*}\right) + \mathfrak{u}\frac{a k_1}{\kappa^2} - \mathfrak{u}^2\frac{a k_1}{\kappa^2}.
\end{aligned}$$

Replacing $\mathfrak{v}$ by $f(\mathfrak{u})$ yields

$$\begin{aligned}
G(\mathfrak{u}) =& \mathfrak{u}\left[-2\left(1 - \frac{a y^* k_1}{\kappa^3} + \frac{a k_1}{\kappa^2}\right) + \frac{a k_1}{\kappa^2}\right] \\
&+ \mathfrak{u}^2\left[\frac{1}{b}\left(1 - \frac{a y^* k_1}{\kappa^3} + \frac{a k_1}{\kappa^2}\right)\frac{a k_1}{\kappa^2} + 3\frac{a k_1}{\kappa^3}\left(-\frac{y^*}{\kappa} + k_1\right)\right] + o(\mathfrak{u}^2)
\end{aligned}$$

Case 1: $1 - \frac{a y^* k_1}{\kappa^3} + \frac{a k_1}{\kappa^2} \neq 0$, then

$$F(\mathfrak{u}) = u^2(1 - \frac{a y^* k_1}{\kappa^3} + \frac{a k_1}{\kappa^2}) + o(\mathfrak{u}^2).$$

and

$$\begin{aligned}
G(\mathfrak{u}) =& \mathfrak{u}\left[-2\left(1 - \frac{a y^* k_1}{\kappa^3} + \frac{a k_1}{\kappa^2}\right) + \frac{a k_1}{\kappa^2}\right] \\
&+ \mathfrak{u}^2\left[\frac{1}{b}\left(1 - \frac{a y^* k_1}{\kappa^3} + \frac{a k_1}{\kappa^2}\right)\frac{a k_1}{\kappa^2} + 3\frac{a k_1}{\kappa^3}\left(-\frac{y^*}{\kappa} + k_1\right)\right] + o(\mathfrak{u}^2)
\end{aligned}$$

We can now apply [7, Theorem 3.5] to system (2.39). Since the coefficient of $\mathfrak{u}^2$ in $F(\mathfrak{u})$ is nonzero, we deduce from Part (4)-(i1) of [7, Theorem 3.5] :

Lemma 2.17 *If E^* is a nilpotent equilibrium of* (1.2) *in the positive quadrant* $]0,+\infty[\times]0,+\infty[$, *and if* $1-\frac{ay^*k_1}{\kappa^3}+\frac{ak_1}{\kappa^2}\neq 0$, *then E^* is a cusp, that is, its phase portrait consists of two hyperbolic sectors and two separatrices. In this case, the index of E^* is 0.*

Case 2: if $1-\frac{ay^*k_1}{\kappa^3}+\frac{ak_1}{\kappa^2}=0$, then

$$\begin{aligned} f(\mathfrak{u}) =& \frac{ak1\left(-k1y^*a+\kappa^3+ak1\kappa+b\kappa^2-y^*\kappa b\right)}{b^2\kappa^5}u^3+O\left(u^4\right) \\ =& -\frac{1}{b\kappa}u^3+O\left(u^3\right) \end{aligned}$$

and

$$F(\mathfrak{u})=\frac{1}{\kappa}u^3+o(\mathfrak{u}^3).$$

and

$$G(\mathfrak{u})=\mathfrak{u}\left(\frac{ak_1}{\kappa^2}\right)+\mathfrak{u}^2\left(3\frac{ak_1}{\kappa^3}\left(-\frac{y^*}{\kappa}+k_1\right)\right)+o(\mathfrak{u}^2)$$

Again, we apply [7, Theorem 3.5] to System (2.39). Since the coefficient of $\mathfrak{u}^3$ in $F(\mathfrak{u})$ is positive, we deduce from Part (4)-(ii) of [7, Theorem 3.5] :

Lemma 2.18 *If E^* is a nilpotent equilibrium of* (1.2) *in the positive quadrant* $]0,+\infty[\times]0,+\infty[$, *and if* $1-\frac{ay^*k_1}{\kappa^3}+\frac{ak_1}{\kappa^2}=0$, *then E^* is a saddle point. In this case, the index of E^* is -1.*

c_2. The case of a center of the linearized vector field

The point E^* is a center of the linear part of $\boldsymbol{v}$ if the Jacobian $\mathcal{J}(x^*,y^*)$ has purely imaginary eigenvalues $\pm i\sqrt{p}$, that is, when $p>0$ and $s=0$. Again, this case is nonempty. Let us denote

$$b_0=1-2x^*+\frac{ay^*k_1}{\kappa^2}. \tag{2.40}$$

With the notations of (2.36), we have $p>0$ and $s=0$ if, and only if,

$$b=b_0<a\rho. \tag{2.41}$$

Note that x^*, y^*, as well as b_0, a, ρ, and the sign of p do not depend on the parameter b, and that $s=b-b_0$. Let us fix all parameters except b, and assume that $\Delta_\chi<0$, that is, the eigenvalues of $\mathcal{J}(x^*,y^*)$ are

$$\frac{-s\pm i\sqrt{4p-s^2}}{2}.$$

These eigenvalues cross the imaginary axis at speed $-1/2$ when b passes through the value b_0. Let us denote $c = a\rho$. By (2.37) and (2.38), we have

$$\begin{aligned} v_1 &= Xb_0 - Yc - X^2\left(1 - \frac{ay^*k_1}{\kappa^3}\right) - XY\frac{ak_1}{\kappa^2} - X^3\frac{ay^*k_1}{\kappa^3(X+\kappa)} + X^2Y\frac{ak_1}{\kappa^2}, \\ v_2 &= (X-Y)b - \frac{b}{y^*}(X-Y)^2\left(1 - \frac{X}{X+y^*}\right). \end{aligned}$$

Let us denote by $(\boldsymbol{i}, \boldsymbol{j})$ the standard basis of $\mathbb{R}^2$. In this basis, the matrix of the linear part φ of $(X, Y) \mapsto (v_1, v_2)$ is

$$A(b) = \begin{pmatrix} b_0 & -c \\ b & -b \end{pmatrix}.$$

Let

$$\begin{aligned} \delta &= \sqrt{\det A(b_0)} = \sqrt{b_0(c-b_0)}, & \gamma &= \frac{c-b_0}{\delta} = \sqrt{\frac{c-b_0}{b_0}}, \\ \boldsymbol{u} &= \boldsymbol{i} + \boldsymbol{j}, & \boldsymbol{v} &= \frac{1}{\delta}\varphi(\boldsymbol{u}) = -\gamma\boldsymbol{i}. \end{aligned}$$

The matrix of φ in the basis $(\boldsymbol{u}, \boldsymbol{v})$ is

$$\tilde{A}(b) = \begin{pmatrix} 0 & -\frac{b}{b_0}\delta \\ \delta & b_0 - b \end{pmatrix}.$$

The coordinates $(\mathfrak{u}, \mathfrak{v})$ in the basis $(\boldsymbol{u}, \boldsymbol{v})$ satisfy $\mathfrak{u} = Y$, $\quad \mathfrak{v} = \frac{1}{\gamma}(Y - X)$, $X = \mathfrak{u} - \mathfrak{v}\gamma$, $Y = \mathfrak{u}$. The coordinates of $\boldsymbol{v}$ in the basis $(\frac{\partial}{\partial\mathfrak{u}}, \frac{\partial}{\partial\mathfrak{v}})$ are

$$\begin{aligned} \dot{\mathfrak{u}} &= v_2 = -b\gamma\mathfrak{v} + \frac{b}{y^*}\gamma^2\mathfrak{v}^2\left(1 - \frac{\mathfrak{u} - \mathfrak{v}\gamma}{\mathfrak{u} - \mathfrak{v}\gamma + y^*}\right), \\ \dot{\mathfrak{v}} &= \frac{1}{\gamma}(v_2 - v_1) \\ &= -b\mathfrak{v} + \frac{b}{y^*}\gamma\mathfrak{v}^2\left(1 - \frac{\mathfrak{u} - \mathfrak{v}\gamma}{\mathfrak{u} - \mathfrak{v}\gamma + y^*}\right) \\ &\quad - \frac{1}{\gamma}\Bigg((\mathfrak{u} - \mathfrak{v}\gamma)b_0 - \mathfrak{u}c - (\mathfrak{u} - \mathfrak{v}\gamma)^2\left(1 - \frac{ay^*k_1}{\kappa^3}\right) - \mathfrak{u}(\mathfrak{u} - \mathfrak{v}\gamma)\frac{ak_1}{\kappa^2} \\ &\qquad - (\mathfrak{u} - \mathfrak{v}\gamma)^3\frac{ay^*k_1}{\kappa^3(\mathfrak{u} - \mathfrak{v}\gamma + \kappa)} + \mathfrak{u}(\mathfrak{u} - \mathfrak{v}\gamma)^2\frac{ak_1}{\kappa^2}\Bigg). \end{aligned}$$

In particular, for $b = b_0$,

$$\begin{aligned}
\dot{\mathfrak{u}} =& -\delta\mathfrak{v} + \frac{c-b_0}{y^*}\mathfrak{v}^2\left(1 - \frac{\mathfrak{u}-\mathfrak{v}\gamma}{\mathfrak{u}-\mathfrak{v}\gamma+y^*}\right),\\
\dot{\mathfrak{v}} =& \delta\mathfrak{u} + \frac{c-b_0}{y^*}\mathfrak{v}^2\left(1 - \frac{\mathfrak{u}-\mathfrak{v}\gamma}{\mathfrak{u}-\mathfrak{v}\gamma+y^*}\right)\\
&+\frac{1}{\gamma}\Bigg[+(\mathfrak{u}-\mathfrak{v}\gamma)^2\left(1-\frac{ay^*k_1}{\kappa^3}\right)+\mathfrak{u}(\mathfrak{u}-\mathfrak{v}\gamma)\frac{ak_1}{\kappa^2}\\
&\qquad+(\mathfrak{u}-\mathfrak{v}\gamma)^3\frac{ay^*k_1}{\kappa^3(\mathfrak{u}-\mathfrak{v}\gamma+\kappa)}-\mathfrak{u}(\mathfrak{u}-\mathfrak{v}\gamma)^2\frac{ak_1}{\kappa^2}\Bigg].
\end{aligned}$$

2.3 Existence of a globally asymptotically stable equilibrium point

When $m = 0$, in the case (c) of Theorem 2.7, we have seen that (1.2) has no cycle, because the compact set delimited by a cycle would contain a critical point, see [2, Theorem V.3.8]. As the compact set $\mathcal{A}$ is invariant and contains all equilibrium points of the open quadrant $]0, +\infty[\times]0, +\infty[$, all trajectories starting in the quadrant $\mathbb{R}^+ \times \mathbb{R}^+$ converge to E_1 or E_2 (E_0 is excluded because it is an unstable node). On the x axis, we have $\dot{y} = 0$ and x satisfies the logistic equation $\dot{x} = x(1 - x)$, thus, for $x(0) > 0$, $x(t)$ converges to 1, i.e., $(x(t), y(t))$ converges to E_1. On the other hand, for $0 < y < k_2 + x$, we have $\dot{y} > 0$, thus, if $y(0) > 0$, $(x(t), y(t))$ cannot converge to E_1, it converges necessarily to E_2.

Theorem 2.19 *A sufficient condition for the existence of a globally asymptotically stable equilibrium point $E^* = (x^*, y^*)$ in the open quadrant $]0, +\infty[\times]0, +\infty[$ (equivalently, in the interior of $\mathcal{A}$) is that*

$$(2.42)\quad \Big(2m + k_1 \geq 1\Big) \ \textit{and}\ \Big((m > 0)\ \textit{or}\ \big(4ak_2 \leq (1 - k_1 - a)^2 + 4k_1\big)\Big).$$

Proof. Let $E^* = (x^*, y^*) \in \mathcal{A}$ be an equilibrium point in the interior of $\mathcal{A}$. Let us denote

$$\rho(x) = \frac{a(x-m)}{k_1 + x - m},$$

and let us set

$$V(x, y) = \int_{x^*}^{x} \frac{u - x^*}{(k_2 + u - m)\rho(u)}\,\mathrm{d}u + \frac{1}{b}\int_{y^*}^{y} \frac{v - y^*}{v}\,\mathrm{d}v.$$

Then, using (2.20) and (2.21), we have

$$
\begin{aligned}
\dot{V} &= \frac{x-x^*}{(k_2+x-m)\rho(x)}\,\dot{x} + \frac{1}{b}\frac{y-y^*}{y}\,\dot{y} \\
&= \frac{x-x^*}{k_2+x-m}\left(\frac{x(1-x)}{\rho(x)} - \frac{a(x-m)}{k_1+x-m}\frac{1}{\rho(x)}y\right) + \frac{1}{b}(y-y^*)b\left(1-\frac{y}{k_2+x-m}\right) \\
&= \frac{x-x^*}{a(k_2+x-m)}\left(\frac{x(1-x)(k_1+x-m)}{x-m} - y^*\right) - \frac{(x-x^*)(y-y^*)}{k_2+x-m} \\
&\quad + (y-y^*)\left(\frac{y^*}{k_2+x^*-m} - \frac{y}{k_2+x-m}\right) \\
&= \frac{x-x^*}{a(k_2+x-m)}\left(\frac{x(1-x)(k_1+x-m)}{x-m} - \frac{x^*(1-x^*)(k_1+x^*-m)}{x^*-m}\right) \\
&\quad - \frac{(x-x^*)(y-y^*)}{k_2+x-m} \\
&\quad + (y-y^*)\,\frac{y^*(k_2+x-m)-y(k_2+x^*-m)}{(k_2+x^*-m)(k_2+x-m)}.
\end{aligned}
$$

Let us denote $g(x) = x(1-x)(k_1+x-m)/(x-m)$. Then

$$
\begin{aligned}
\dot{V} &= \frac{x-x^*}{a(k_2+x-m)}\,(g(x)-g(x^*)) - \frac{(x-x^*)(y-y^*)}{k_2+x-m} \\
&\quad + (y-y^*)\,\frac{(y^*-y)(k_2-m)+y^*x-yx^*}{(k_2+x^*-m)(k_2+x-m)} \\
&= \frac{x-x^*}{a(k_2+x-m)}\,(g(x)-g(x^*)) - \frac{(x-x^*)(y-y^*)}{k_2+x-m} \\
&\quad + \frac{y-y^*}{y^*}\,\frac{(y^*-y)(x^*+k_2-m)+y^*(x-x^*)}{k_2+x-m} \\
&= \frac{x-x^*}{a(k_2+x-m)}\,(g(x)-g(x^*)) + \frac{y-y^*}{y^*}\,\frac{(y^*-y)(x^*+k_2-m)}{k_2+x-m} \\
&= \frac{1}{k_2+x-m}\left(\frac{x-x^*}{a}\,(g(x)-g(x^*)) - (y-y^*)^2\right).
\end{aligned}
$$

For $x \geq m$, a sufficient condition for $\dot{V}$ to be negative when $(x,y) \neq (x^*,y^*)$ is that g be nonincreasing. Let us make the change of variable $X = x-m$. We have

$$g(x) = \frac{(X+m)(1-X-m)(X+k_1)}{X},$$

which leads to

$$g'(x) = \frac{-2X^3 + (1-2m-k_1)X^2 - k_1(m-m^2)}{X^2}.$$

Thus, if $2m + k_1 \geq 1$, $g'(X)$ remains negative for $X > 0$, i.e., for $x > m$. Thus, for $x > m$, under the assumption (2.42), $\dot{V}$ is negative.

We have seen that the first part of (2.42) implies that the equilibrium point E^*, if it exists, is globally asymptotically stable. Note that Condition (2.42) is independent of the coordinates of E^*, and the global stability implies that the equilibrium point E^*, if it exists, is unique.

The second part of (2.42) is a necessary and sufficient condition for the existence of such an equilibrium point.

When $m > 0$, we already know that there exists at least one equilibrium point in $\mathcal{A}$. Actually, Condition (2.42) implies that the coefficient $\alpha_2 = a + k_1 - 1 + 2m$ of (2.23) is positive. Thus, when $m > 0$, (2.42) is a particular case of (c) in Theorem 2.5.

When $m = 0$, by Theorem 2.7-(c), since $\alpha_2 > 0$, there exists an equilibrium point in the interior of $\mathcal{A}$ if, and only if, (2.29) is satisfied. □

2.4 Cycles

Let us investigate the existence of periodic orbits of (1.2). By Theorem 2.1 such orbits can take place only in $\mathcal{A}$.

2.4.1 Refuge free case (m=0)

This case has been studied by M.A. Aziz-Alaoui and M. Daher-Okiye [5], but we add some new results.

Lemma 2.20 *In the cases* (c) *and* (a) *of Theorem 2.7, that is, when* (1.2) *has 0 or 2 equilibrium points in the open quadrant* $]0, +\infty[\times]0, +\infty[$*, the system* (1.2) *has no limit cycle. On the other hand, in the case* (b) *of Theorem 2.7, that is, when* (1.2) *has 1 equilibrium point in the open quadrant* $]0, +\infty[\times]0, +\infty[$*, if furthermore* $s < 0$ *and* $p > 0$*, the system* (1.2) *has at least one limit cycle.*

Proof. In the case (c), the only equilibrium points of (1.2) in $\mathbb{R}^+ \times \mathbb{R}^+$ are the trivial points E_0, E_1, and E_2, on the axes. Thus (1.2) has no cycle, because the compact set delimited by a cycle would contain a critical point, see [2, Theorem V.3.8].

In the case (a), if there was a cycle inside $\mathcal{A}$, we could apply the Poincaré-Hopf Index Theorem to the compact manifold whose boundary is delineated by this cycle (see [18] for a version of this theorem when the vector field is tangent to the boundary). Denoting N the number of nodes or foci and

S the number of saddles in the open quadrant $]0,+\infty[\times]0,+\infty[$, we would have $N-S=1$. But Theorem 2.12 shows that $N-S=0$, a contradiction.

In the case (b), if $s<0$ and $p>0$, the system (1.2) has an unstable equilibrium point. From Theorem (2.1) and Poincare-Bendixson Theorem, there exists at least one limit cycle around this equilibrium. $\square$

Note that the conditions of Lemma 2.20 do not involve the value of b. Using Bendixson-Dulac criterion, M.A. Aziz-Alaoui and M. Daher-Okiye obtain another criterion:

Lemma 2.21 *[5, Theorem 7] if $b+k_1\geq 1$, then the system* (1.2) *has no limit cycle.*

2.4.2 Case with refuge (m>0)

By Theorem 2.19, if Condition (2.42) is satisfied, there can be no periodic orbits.

Let us now give some sufficient conditions for the absence of periodic orbits, using Bendixson-Dulac criterion. Let us denote by $f(x,y)$ and $g(x,y)$ the coordinates of the vector field in (1.2). For a Dulac function, we choose

$$D(x,y)=x+k_1-m.$$

Let us look for conditions that ensure that $\frac{\partial(fD)}{\partial x}+\frac{\partial(gD)}{\partial y}<0$ in $\mathcal{A}$. We have

$$\begin{aligned}\frac{\partial(fD)}{\partial x}(x,y)&=-3x^2+2(1-k_1+m)x+k_1-m-ay,\\ \frac{\partial(gD)}{\partial y}(x,y)&=\frac{b(x+k_1-m)(x+k_2-m-2y)}{x+k_2-m}.\end{aligned}$$

For $(x,y)\in\mathcal{A}$, we have

$$\frac{\partial(fD)}{\partial x}(x,y)<-3m^2+2(1-k_1+m)x+k_1-m-ak_2.$$

Since the maximum of $-3m^2+m$ is $1/12$ and the maximum of $-m^2+m$ is $1/4$, we deduce:

$$\begin{aligned}1-k_1+m>0\Rightarrow\frac{\partial(fD)}{\partial x}(x,y)&<-3m^2+m-k_1-ak_2+2\\ &\leq 2+\frac{1}{12}-k_1-ak_2,\\ 1-k_1+m<0\Rightarrow\frac{\partial(fD)}{\partial x}(x,y)&<-m^2+m(1-2k_1)-k_1-ak_2\end{aligned}$$

$$< -m^2 - m - k_1 - ak_2 < 0.$$

In particular, a condition that ensures that $\frac{\partial(fD)}{\partial x} < 0$ in $\mathcal{A}$ is

$$(k_1 > 1 + m) \text{ or } (ak_2 + k_1 > 2 + \frac{1}{12}). \tag{2.43}$$

On the other hand, for $(x, y) \in \mathcal{A}$, $\frac{\partial(gD)}{\partial y}(x, y)$ has the same sign as $x + k_2 - m - 2y$, and we have $x + k_2 - m - 2y < 1 - m - k_2$. Thus a sufficient condition for $\frac{\partial(gD)}{\partial x} < 0$ in $\mathcal{A}$ is

$$k_2 > 1 - m. \tag{2.44}$$

The same technique does not provide any sufficient condition for $\frac{\partial(fD)}{\partial x} + \frac{\partial(gD)}{\partial y} > 0$ in $\mathcal{A}$. So, our next result concerning the absence of cycles is:

Lemma 2.22 *A sufficient condition for* (1.2) *to have no periodic solution is*

$$\Big(k_2 > 1 - m\Big) \;\; \textit{and} \;\; \Big((k_1 > 1 + m) \text{ or } (ak_2 + k_1 > 2 + \frac{1}{12})\Big).$$

Now, we consider the existence of limit cycles which are not occuring from a Hopf bifurcation. The special configuration of the existence of a limit cycle enclosing three equilibrium points is numerically investigated. In particular, when the system parameters satisfy $a = 0.5, k1 = 0.08, k2 = 0.2, b = 0.1, m = 0.0025$, then three hyperbolic equilibrium points exist, namely, $E_1^* = (0.0222589; 0.2197589), E_2^* = (0.0299525; 0.2274525), E_3^* = (0.3702886; 0.5677886)$. They define respectively a stable focus, a saddle point and an unstable focus. Accordingly to the Poincaré index theorem, the sum of the corresponding indexes is equal to 1. The numerical simulations show that there exists a limit cycle, which is hyperbolic and stable, see Figure (1).

3 Stochastic model

We now study the dynamics of the system (1.3), with initial conditions $x_0 > 0$ and $y_0 > 0$.

3.1 Existence and uniqueness of the positive global solution

Since $x(t)$ and $y(t)$ represent prey and predator densities, we are only intesrted by positive solutions, so first we show existence and uniqueness of solutions then we will discuss the boundedness of these solutions.

Theorem 3.1 *For any initial condition* $(x_0, y_0) \in \mathfrak{Int}(\mathbb{R}^2_+)$, *the system* (1.3) *admits a unique solution* $(x(t), y(t))$, *defined for all* $t \geq 0$ *a.s, and this solution remains in* $\mathfrak{Int}(\mathbb{R}^2_+)$.

Proof of theorem 3.1. Since $(x, y) = (0, 0)$ is a solution to (1.3), we deduce by the comparaison theorem for SDEs (see [8, Theorem 1] for a more general comparison theorem) that, with the initial condition $(x_0, y_0) \in \mathfrak{Int}(\mathbb{R}^2_+)$ the solution to (1.3) remains in $\mathfrak{Int}(\mathbb{R}^2_+)$ until its explosion time.

Let τ_e be the explosion time of the solution to (1.3). We have to prouve that $(x(t), y(t) \in \mathfrak{Int}(\mathbb{R}^2_+)$ for every $t \in [0, \tau_e[$ and that $\tau_e = \infty$ a.s. Let us first prove the first assertion and we adapte the proof of [6]. The coefficients in system (1.3) are locally Lipschitz, so there is exists a unique local solution for all $t \in [0, \tau_e[$ and for all $(x(0), y(0)) \in \mathfrak{Int}(\mathbb{R}^2_+)$. To show that this solution is global, it suffices to show $\tau_e = \infty$. For that, let $k_0 > 0$ be large enough, such that $(x_0, y_0) \in [\frac{1}{k_0}, k_0] \times [\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$ we define the stopping time

$$\tau_k = \inf\Big\{t \in [0, \tau_e) : x \notin (\frac{1}{k}, k) \text{ ou } \ y \notin (\frac{1}{k}, k)\Big\}.$$

The sequence (τ_k) is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, whence $\tau_\infty \leq \tau_e$, (in fact, as $(x(t), y(t)) > 0$ a.s., we have $\tau_\infty = \tau_e$). It suffices to prove that $\tau_\infty = \infty$ a.s.. Assume that this statement is false, then there exist $T > 0$ and $\varepsilon \in]0, 1[$ such that $\mathrm{P}(\{\tau_\infty \leq T\}) > \varepsilon$. Since (τ_k) is increasing we have

$$\mathrm{P}(\{\tau_k \leq T\}) > \varepsilon.$$

Now, we consider the positive definite function $V : \mathfrak{Int}(\mathbb{R}^2_+) \to \mathfrak{Int}(\mathbb{R}^2_+)$ given by

$$V(x, y) = (x + 1 - \log x) + (y + 1 - \log y).$$

Applying Itô's formula, we get

$$\begin{aligned} dV(x, y) =& \Big[(x-1)(1 - x - \frac{ay(x-m)}{k_1 + x - m}) + \frac{\sigma_1^2}{2} + b(y-1)(1 - \frac{y}{k_2 + x - m}) + \frac{\sigma_2^2}{2}\Big]dt \\ &+\sigma_1(x-1)dW_1 + \sigma_2(y-1)dW_2. \end{aligned}$$

The positivity of $x(t)$ and $y(t)$ implies

$$\begin{aligned} dV(x, y) &\leq \Big(2x + ay + \frac{\sigma_1^2 + \sigma_2^2}{2} + by + \frac{y}{k_2}\Big)dt + \sigma_1(x-1)dW_1 + \sigma_2(y-1)dW_2 \\ &\leq \Big(2x + (a + b + \frac{1}{k_2})y + \frac{\sigma_1^2 + \sigma_2^2}{2}\Big)dt + \sigma_1(x-1)dW_1 + \sigma_2(y-1)dW_2. \end{aligned}$$

Denote $c_1 = a + b + \frac{1}{k_2}$, $c_2 = \frac{\sigma_1^2+\sigma_2^2}{2}$. Using [6, lemma 4.1], we can write

$$\begin{aligned} 2x + c_1 y \leq& 4(x + 1 - \log x) + 2c_1(y + 1 - \log y) \\ \leq& c_3 V(x, y), \text{ where } c_3 = \max(4, 2c_1). \end{aligned}$$

Hence

$$\begin{aligned} dV(x, y) \leq& (c_2 + c_3 V(x, y))dt + \sigma_1(x - 1)dW_1 + \sigma_2(y - 1)dW_2 \\ \leq& c_4(1 + V(x, y))dt + \sigma_1(x - 1)dW_1 + \sigma_2(y - 1)dW_2, \text{ where } c_4 = \max(c_2, c_3). \end{aligned}$$

Integrating both sides from 0 to $\tau_k \wedge T$, and taking expectations, we get

$$\mathrm{E}\, V(x(\tau_k \wedge T), y(\tau_k \wedge T)) \leq V(x_0, y_0) + c_4 T + c_4 \int_0^T \mathrm{E}\, V(x(\tau_k \wedge t), y(\tau_k \wedge t)dt.$$

By Gronwall's inequality, this yields

$$\mathrm{E}\, V(x(\tau_k \wedge T), y(\tau_k \wedge T)) \leq c_5, \tag{3.1}$$

where c_5 is the finite constant given by

$$c_5 = (V(x_0, y_0) + c_4 T)e^{c_4 T}. \tag{3.2}$$

Let $\Omega_k = \{\tau_k \leq T\}$. We have $\mathrm{P}(\Omega_k) \geq \varepsilon$, and for all $\omega \in \Omega_k$, there exists at least one element of $x(\tau_k, \omega), y(\tau_k, \omega)$ which is equal either to k or to $\frac{1}{k}$, hence

$$V(x(\tau_k), y(\tau_k)) \geq (k + 1 - \log k) \wedge (\frac{1}{k} + 1 + \log k).$$

Therefore, by (3.1),

$$c_5 \geq E[1_{\Omega_k}(\omega) V(x(\tau_k, \omega), y(\tau_k, \omega)] \geq \varepsilon \Big[(k + 1 - \log k) \wedge (\frac{1}{k} + 1 + \log k)\Big],$$

where 1_{Ω_k} is the indicator function of Ω_k,. Letting $k \to \infty$, we get $c_5 = \infty$, contradicts (3.2), So we must have $\tau_\infty = \infty$ a.s. $\square$

3.2 Boundedness of solutions and persistence

First we recall the definition of stochastically ultimate boundedness and stochastic permanence, see e.g. [23].

Definition 3.2 The solutions to the system (1.3) are said to be *stochastically ultimately bounded* if, for all $\epsilon \in]0,1[$, there are positive constants $\chi_1(\epsilon)$,$\chi_2(\epsilon)$, such that the solutions to (1.3) with any initial value have the property that

$$\limsup_{t\to\infty} \mathrm{P}\{x(t) > \chi_1\} < \epsilon, \quad \limsup_{t\to\infty} \mathrm{P}\{y(t) > \chi_2\} < \epsilon.$$

Definition 3.3 The system (1.3) is said to be *stochastically permanent* if, for any $\epsilon \in]0,1[$, there exists a pair of positive constants $\delta(\epsilon)$ and $\gamma(\epsilon)$ such that the solution to (1.3) with any initial value $X(0) = (x(0), y(0)) \in]0,+\infty[\times]0,+\infty[$, has the property that

$$\liminf_{t\to\infty} \mathrm{P}\{|X(t)| \le \delta\} \ge 1-\epsilon, \quad \liminf_{t\to\infty} \mathrm{P}\{|X(t)| \ge \gamma\} \ge 1-\epsilon.$$

Let us now investigate the boundedness of moments. From the comparaison theorem for SDEs (see [8, Theorem 1]), we have that $0 \le x(t) \le z(t) \le u(t)$ a.e., where z is the solution starting from x_0 to the stochastic logistic equation (also called stochastic Verhulst equation)

$$dz(t) = z(t)(1-z(t))dt + \sigma_1 z(t) dz_1(t) \tag{3.3}$$

and u is the geometric Brownian motion starting from x_0 satisfying

$$du(t) = u(t)dt + \sigma_1 u(t) dw_1(t). \tag{3.4}$$

The process z is well known. It can be written explicitely, see [12, page 125]. In particular, z has moments of all orders, see [19]

By [13, Theorem 5.1], Z converges to 0 if $\sigma_1^2 \ge 2$, whereas it converges to a nondegenerate stationary distribution if $\sigma_1^2 < 2$.

From [13, Lemma 2.2], comparing $x(t)$ and $y(t)$ with the solution $z(t)$ to (3.3), we deduce that, for any $p > 0$, there exists a constant K_p such that

$$\limsup_{t\to\infty} \mathrm{E}\,(x(t))^p < K_p. \tag{3.5}$$

By Chebyshev's inequality and (3.5), the following result is straightforward.

Theorem 3.4 *The solutions to System* (1.3) *are stochastically ultimately bounded.*

Now, for the study of persistence, we impose a new hypothesis:

Assumption 3.5 $\alpha_1 = \max\{1, b\} > \frac{\check{\sigma}^2}{2}$.

Lemma 3.6 *Under Assumption* (3.5)*, for any initial value* $X_0 = (x_0, y_0) \in]0, +\infty[\times]0, +\infty[$*, the solution* $X(t) = ((x(t), y(t))$ *satisfies*

$$\limsup_{t \longrightarrow \infty} E\left[\frac{1}{|X(t)|^\theta}\right] \leq H,$$

where θ *is any arbitrary positive constant satisfying*

$$\alpha_1 > (\theta + 1)\frac{\check{\sigma}^2}{2}, \tag{3.6}$$

and

$$H = \frac{n^\theta(a_2 + 4ka_1)}{4ka_1} \max\left\{1, \left(\frac{2a_1 + a_2 + \sqrt{a_2^2 + 4a_1a_2}}{2a_1}\right)^{\theta - 2}\right\},$$

in which k *is an arbitrary positive constant satisfying*

$$0 < \frac{k}{\theta} < \alpha_1 - \check{\sigma}^2\frac{\theta + 1}{2}, \tag{3.7}$$

while

$$a_1 = \alpha_1 - \check{\sigma}^2\frac{\theta + 1}{2} - \frac{k}{\theta} > 0, \quad a_2 = \alpha_1 + \check{\sigma}^2 + \frac{2k}{\theta} > 0.$$

Proof. Let us define $V(X) = (x+y)$ for $X = (x, y) \in]0, +\infty[\times]0, +\infty[$, and

$$U(t) = \frac{1}{V(X(t))} \text{ on } t \geq 0.$$

By Itô formula, we get

$$\begin{aligned}
dU(t) = -U(t)^2 \Bigg[& \left(x(t)(1 - x(t)) - \frac{ay(t)(x(t) - m)}{k_1 + x(t) - m} + by(t)\left(1 - \frac{y(t)}{k_2 + x(t) - m}\right)\right) dt \\
& + \sigma_1 x(t) dW_1 + \sigma_2 y(t) dW_2 \Bigg] + U(t)^3[(\sigma_1 x(t))^2 + (\sigma_2 y(t))^2]dt \\
= {} & LU(t)dt - U(t)^2[\sigma_1 x(t)dW_1 + \sigma_2 y(t)dW_2],
\end{aligned}$$

where

$$LU(t) = -U(t)^2 \left(x(t)(1 - x(t)) - \frac{ay(t)(x(t) - m)}{k_1 + x(t) - m} + by(t)\left(1 - \frac{y(t)}{k_2 + x(t) - m}\right)\right)$$

$$+ U(t)^3[(\sigma_1 x(t))^2 + (\sigma_2 y(t))^2].$$

Let θ be any positive constant satisfying (3.6). By Itô formula again, we have

$$\begin{aligned}
& d[(1+U(t))^\theta] \\
= & \Bigg(-\theta(1+U(t))^{\theta-1}U(t)^2 \Big(x(t)(1-x(t)) - \frac{ay(t)(x(t)-m)}{k_1 + x(t) - m}) \\
& + by(t)\left(1 - \frac{y(t)}{k_2 + x(t) - m}\right)\Big) \\
& + \left(\frac{\theta(\theta-1)}{2}(1+U(t))^{\theta-2}U(t)^4 + \theta(1+U(t))^{\theta-1}U(t)^3\right)[(\sigma_1 x(t))^2 + (\sigma_2 y(t))^2]\Bigg) dt \\
& - \theta(1+U(t))^{\theta-1}U(t)^2\sigma_1 x(t)dW_1 - \theta(1+U(t))^{\theta-1}U(t)^2\sigma_2 y(t)dW_2.
\end{aligned}$$

Now, choose $k > 0$ sufficiently small satisfying (3.7). By Itô formula,

$$\begin{aligned}
d[e^{kt}(1+U(t))^\theta] &= ke^{kt}(1+U(t))^\theta + e^{kt}d(1+U(t))^\theta \\
&= e^{kt}(1+U(t))^{\theta-2}\Bigg(\left(k(1+U(t))^2 + J(t)\right) dt \\
&\quad - \theta(1+U(t))U(t)^2\sigma_1 x(t)dW_1 - \theta(1+U(t))U(t)^2\sigma_2 y(t)dW_2\Bigg). \qquad (3.8)
\end{aligned}$$

where

$$\begin{aligned}
J(t) = & -\theta(1+U(t))U(t)^2 \Big(x(t)(1-x(t)) - \frac{ay(t)(x(t)-m)}{k_1 + x(t) - m} \\
& + by(t)\left(1 - \frac{y(t)}{k_2 + x(t) - m}\right)\Big) \\
& + \left(\frac{\theta(\theta-1)}{2}U(t)^4 + \theta(1+U(t))U(t)^3\right)[(\sigma_1 x(t))^2 + (\sigma_2 y(t))^2].
\end{aligned}$$

We thus obtain

$$J(t) \le -\theta\left(\alpha_1 - \check{\sigma}^2\frac{\theta+1}{2}\right)U(t)^2 + \theta(\alpha_1 + \check{\sigma}^2)U(t),$$

where α_1 has been defined in the statement of the lemma. Substituting this into (3.8) yields

$$d[e^{kt}(1+U(t))^\theta] \le e^{kt}(1+U(t))^{\theta-2}\Bigg((k(1+U(t))^2 - \theta\left(\alpha_1 - \check{\sigma}^2\frac{\theta+1}{2}\right)U(t)^2$$

$$+\theta(\alpha_1+\check{\sigma}^2)U(t)\Big)\,dt-\theta e^{kt}(1+U(t))^{\theta-1}U(t)^2\sum_{i=1}^n\sigma_{1i}x_idW_{1i}$$
$$-e^{kt}\theta(1+U(t))^{\theta-1}U(t)^2\sum_{i=1}^n\sigma_{2i}y_idW_{2i}$$
$$=e^{kt}(1+U(t))^{\theta-2}\Bigg(-\theta\left(\alpha_1-\check{\sigma}^2\frac{\theta+1}{2}-\frac{k}{\theta}\right)U(t)^2$$
$$+\theta(\alpha_1+\check{\sigma}^2+\frac{2k}{\theta})U(t)+k\Bigg)\,dt$$
$$-\theta e^{kt}(1+U(t))^{\theta-1}U(t)^2\sum_{i=1}^n(\sigma_{1i}x_idW_{1i}+\sigma_{2i}y_idW_{2i})$$

by a simple calculation, we get

$$(1+U(t))^{\theta-2}\left(-\theta\left(\alpha_1-\check{\sigma}^2\frac{\theta+1}{2}-\frac{k}{\theta}\right)U(t)^2+\theta(\alpha_1+\check{\sigma}^2+\frac{2k}{\theta})U(t)+k\right)\le H_1,$$

where

$$H_1=\frac{a_2+4ka_1}{4a_1}\max\left\{1,\left(\frac{2a_1+a_2+\sqrt{a_2^2+4a_1a_2}}{2a_1}\right)^{\theta-2}\right\}$$

and a_1, a_2 have been defined in the lemma. Thus

$$d[e^{kt}(1+U(t))^\theta]\le H_1e^{kt}dt-\theta e^{kt}(1+U(t))^{\theta-1}U(t)^2\left(\sigma_1x(t)dW_1+\sigma_2y(t)dW_2\right).$$

This implies

$$E[e^{kt}(1+U(t))^\theta]\le(1+U(0))^\theta+\frac{H_1}{k}e^{kt}.$$

Then

$$\limsup_{t\longrightarrow\infty}E[U^\theta(t)]\le\limsup_{t\longrightarrow\infty}E[(1+U(t))^\theta]\le\frac{H_1}{k}.$$

For $X\in\mathbb{R}_+^2$, note that

$$(x(t)+y(t))^\theta\le2^{\frac{\theta}{2}}\left(x(t)^2+y(t)^2\right)^{\frac{\theta}{2}}\le2^\theta|X|^\theta.$$

Consequently,

$$\limsup_{t\longrightarrow\infty}E\left[\frac{1}{|X(t)|^\theta}\right]\le2^{\frac{\theta}{2}}\frac{H_1}{k}\le H.$$

□

Theorem 3.7 *Under condition* (3.6), *the System* (1.3) *is stochastically permanent.*

Proof. Let $X(t)$ be the solution to System (1.3) with any given positive initial value $X(0)$ in the open quadrant $]0,+\infty[\times]0,+\infty[$. For any $\epsilon > 0$, let $\delta = (\frac{\epsilon_1}{H})^{\frac{1}{\theta}}$. We get, using Lemma 3.6,

$$\begin{aligned} P\{X(t) < \delta\} &= P\left\{\frac{1}{|X(t)|^\theta} > \frac{1}{\delta^\theta}\right\} \\ &\leq \frac{E[\frac{1}{|X(t)|^\theta}]}{\frac{1}{\delta^\theta}} \\ &\leq \delta^\theta H = \epsilon. \end{aligned}$$

Hence

$$\limsup_{t\longrightarrow\infty} P\{|X(t)| < \delta\} \leq \epsilon,$$

and this implies

$$\limsup_{t\longrightarrow\infty} P\{|X(t)| \geq \delta\} \geq 1-\epsilon.$$

The other inequality of Definition 3.3 follows from Theorem 3.4. □

4 Numerical simulations and figures

All simulations and pictures of this section are obtained using Scilab.

4.1 Deterministic system

We numerically simulate solutions to System (1.2). Using the Euler scheme, we consider the following discretized system:

$$\begin{aligned} x_{k+1} &= x_k + \left[x_k(1-x_k) - \frac{a y_k(x_k - m)}{k_1 + x_k - m}\right] h, \\ y_{k+1} &= y_k + b y_k \left[1 - \frac{y_k}{k_2 + x_k - m}\right] h. \end{aligned} \tag{4.1}$$

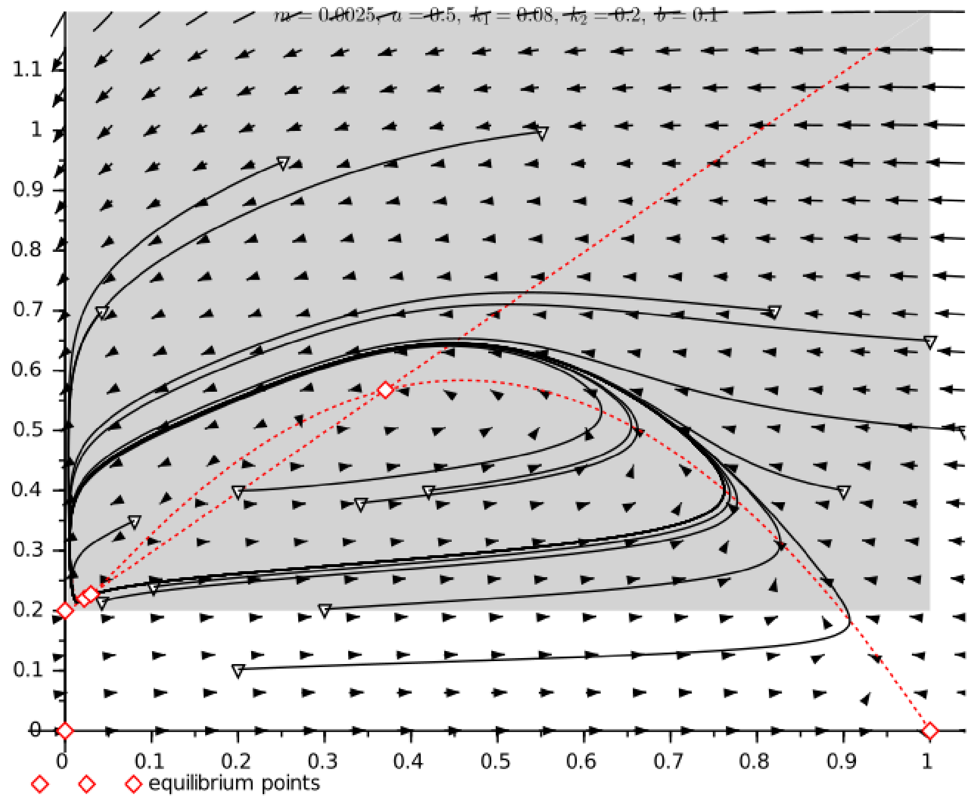


Figure 1: A phase portrait of (1.2) with three equilibrium points and a cycle in the interior of $\mathcal{A}$. The dashed lines are isoclines $y = \frac{x(1-x)(k_1+x-m)}{a(x-m)}$ and $y = k_2 + x - m$. The grey region is the invariant attracting domain $\mathcal{A}$.

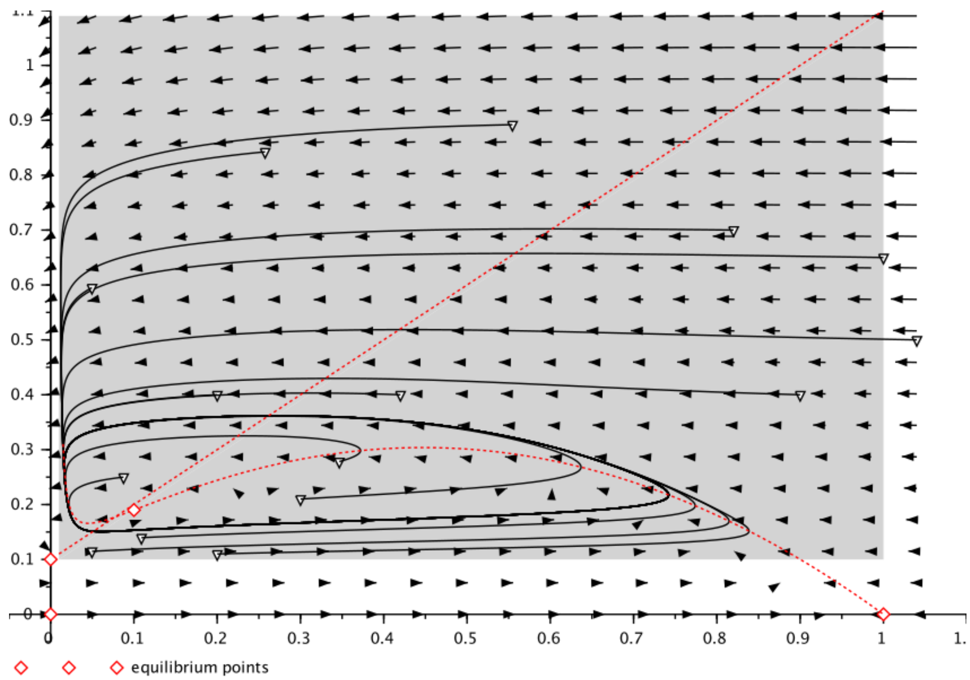
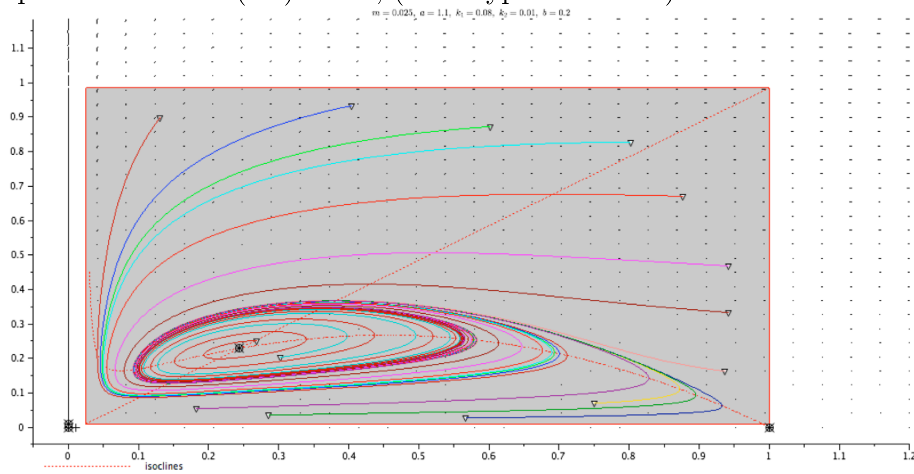


Figure 2: A phase portrait of (1.2) with an unstable equilibrium and a stable limite cycle

Hopf bifurcation of (1.2) $\lambda < 0$, (semi hyperbolic case).



Hopf bifurcation of (1.2) $\lambda > 0$, (semi hyperbolic case).

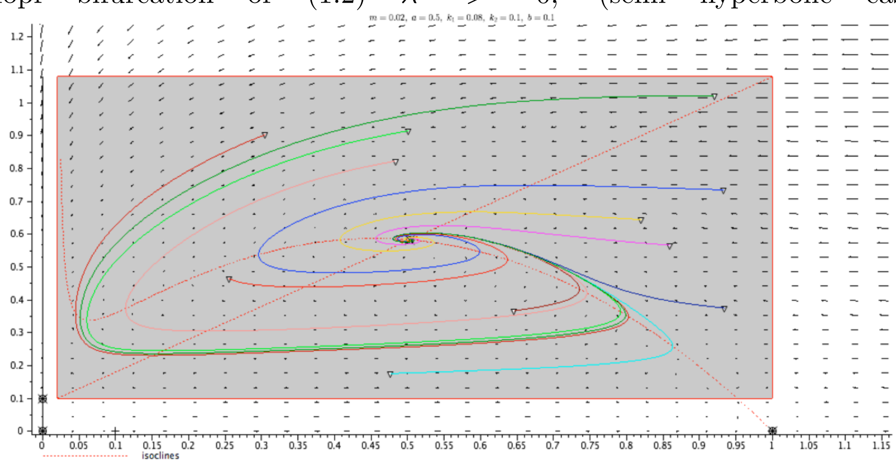


Figure 3: Hopf bifurcation. In the upper subgraph we have a stable limit cycle with $m = 0.0025, k_1 = 0.08, k_2 = 0.01, a = 1.1, b = 0.2$ and in the lower subgraph we have an unstable limite cycle with $m = 0.02, k_1 = 0.08, k_2 = 0.1, a = 0.5, b = 0.1$.

4.2 Stochastically perturbated system

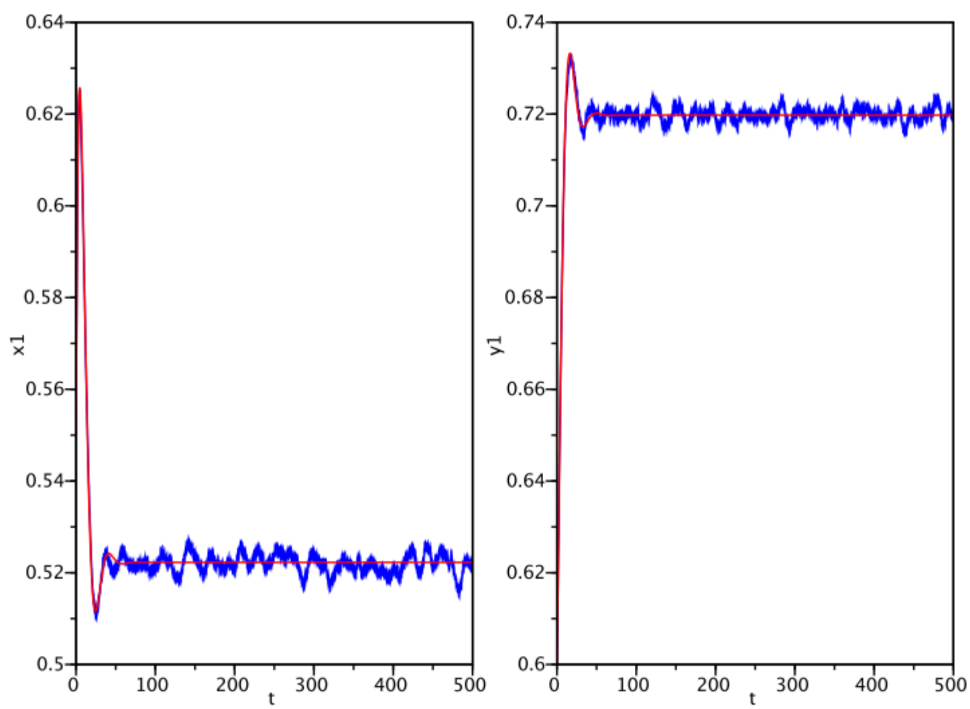
We numerically simulate the solution to System (1.3). Using the Milstein scheme (see [12]), we consider the discretized system

(4.2)

$$x_{k+1} = x_k + \left[x_k(1 - x_k - \frac{ay_k}{k_1 + x_k - m})\right] h + \sigma_1\, x_k \sqrt{h}\, \xi_k^2 + \frac{1}{2}\sigma_1^2 x_k (h\, \xi_k^2 - h),$$

$$y_{k+1} = y_k + by_k \left[1 - \frac{y}{k_2 + x}\right] h + \sigma_2 y_k \sqrt{h}\, \xi_k^2 + \frac{1}{2}\sigma_2^2 y_k (h\, \xi_k^2 - h).$$

In Figure 1, we choose $a = 0.4, k_1 = 0.08, k_2 = 0.2, b = 0.1, m = 0.0025, \sigma_1 = 0.01, \sigma_2 = 0.01$, the initial value $(x(0), y(0)) = (0.55, 0.6)$, and the time step $h = 0.01$. The deterministic model has a globally stable equilibrium point $(x^*, y^*) = (0.55, 0.75)$. The simulations show the permanence of the system (1.3).



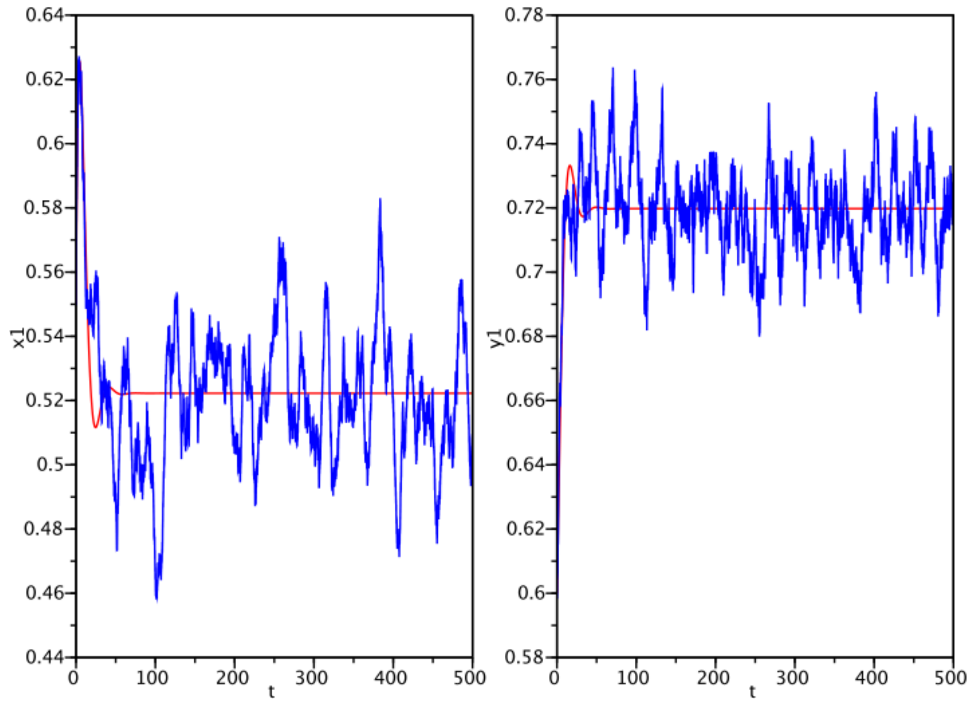


Figure 4: Solutions to the stochastic system (1.3) and the corresponding deterministic system, represented respectively by the blue line and the red line. First subgraph: $\sigma_1 = 0.01, \sigma_2 = 0.01$. Second subgraph: $\sigma_1 = 0.3$, $\sigma_2 = 0.2$.

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