Reduction and Introducer Concepts in d-Dimensional Contexts

Alexandre Bazin, Giacomo Kahn

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Alexandre Bazin[1] Giacomo Kahn[2,*]

[1] LORIA, Nancy, France
[2] LIFO, Orléans, France

Abstract

Concept lattices are well-known conceptual structures that organise interesting patterns – the concepts – extracted from data. In some applications, the size of the lattice can be a problem, as it is often too large to be efficiently computed and too complex to be browsed. In others, redundant information produces noise that makes understanding the data difficult. In classical FCA, those two problems can be attenuated by, respectively, computing a substructure of the lattice – such as the AOC-poset – and reducing the context. These solutions have not been studied in d-dimensional contexts for $d > 3$. In this paper, we generalise the notions of AOC-poset and reduction to d-lattices, the structures that are obtained from multidimensional data in the same way that concept lattices are obtained from binary relations.

Introduction

Formal concept analysis (FCA) is a mathematical framework introduced in the 1980s that allows for the application of lattice theory to data analysis. While it is now widely used, its main drawback, scalability, remains. Many ways to improve it have been proposed over the years. Some focus on reducing the size of the dataset by grouping or removing attributes or objects while others prefer to consider only a substructure of the lattice.

The end result of two of those approaches – respectively reduced contexts and AOC-posets – are of interest in our work. The first is the state of the dataset in which no element can be considered *redundant*. The second is the substructure of the lattice composed of its elements that best describe each individual object or attribute. Both have been extensively studied in traditional, bidimensional FCA. However, such is not the case in Polyadic Concept Analysis (PCA), the multidimensional generalization of FCA. The notion of reducibility of polyadic contexts have only proposed in the 3-dimensional case [1] and, to the best of our knowledge, no work on introducer multidimensional concepts exist.

PCA gives rise to an even greater number of patterns and so scalability is even more of an issue. Additionally, we believe that introducer concepts can be of use to better

*Corresponding author

analyse d-dimensional data as each element is associated to multiple concepts instead of one. For these reasons, in this work, we define and study both the reducibility and the introducer d-ordered set of d-contexts.

The article is divided as follows: in the next section, Section 1, we give the necessary definitions and preliminaries in order to ensure a smooth reading of the paper. Section 2 is focused on dataset reduction while Section 3 introduces introducers. We conclude by highlighting some problems that may – or may not – be of interest in the future.

1 Formal and Polyadic Concept Analysis

1.1 Formal Concept Analysis

Formal Concept Analysis (FCA) is a mathematical framework that revolves around *formal contexts* as a condensed representation of lattices. This framework allows one to inject the powerful mathematical machinery of lattice theory into data analysis. FCA has been introduced in the 1980's by a research team led by Rudolph Wille in Darmstadt. It is based on previous work by Garrett Birkhoff on lattice theory [2] and by Marc Barbut and Bernard Monjardet [3].

In this section, we give the basic definitions of FCA. For an informative book, the reader can refer to [4]. From now on, we will freely alternate between notations ab and $\{ab\}$ to denote the set $\{a, b\}$.

Definition 1. *A* (formal) context *is a triple* $(\mathcal{O}, \mathcal{A}, \mathcal{R})$ *where* $\mathcal{O}$ *and* $\mathcal{A}$ *are finite sets and* $\mathcal{R} \subseteq \mathcal{O} \times \mathcal{A}$ *is a relation between them. We call* $\mathcal{O}$ *the set of (formal) objects and* $\mathcal{A}$ *the set of (formal) attributes.*

A formal context naturally can be naturally represented as a cross table, as shown in Figure 1. A pair $(o, a) \in \mathcal{R}$ corresponds to a cross in cell (o, a) of the cross table. Such a pair is read "object o has attribute a". Since many datasets can be represented as binary relation such as the one in Figure 1, FCA finds natural applications in data analysis.

	a_1	a_2	a_3	a_4	a_5
o_1	×		×		×
o_2		×		×	×
o_3	×	×	×		
o_4			×	×	
o_5	×	×			×
o_6	×		×	×	
o_7	×	×		×	×

Figure 1: An example context with $\mathcal{O} = \{o_1, o_2, o_3, o_4, o_5, o_6, o_7\}$ and $\mathcal{A} = \{a_1, a_2, a_3, a_4, a_5\}$. A cross in a cell (o, a) is read "object a has attribute a". A maximal rectangle of crosses is highlighted.

To allow one to efficiently jump from a set of objects to the set of attributes that describes it, and vice versa, two derivation operators are defined. For a set O of objects and a set A of attributes, they are defined as follows:

$$\cdot' : 2^{\mathcal{O}} \mapsto 2^{\mathcal{A}}$$

$$O' = \{a \in \mathcal{A} | \forall o \in O, (o, a) \in \mathcal{R}\}$$

and

$$\cdot' : 2^{\mathcal{A}} \mapsto 2^{\mathcal{O}}$$

$$A' = \{o \in \mathcal{O} | \forall a \in A, (o, a) \in \mathcal{R}\}.$$

The $\cdot'$ derivation operator maps a set of objects (resp. attributes) to the set of attributes (resp. objects) that they share. The composition of the two derivation operators ($\cdot''$) forms a Galois connection. As such, it forms a closure operator (an extensive, increasing and idempotent operator). Depending on which set the composition of operators is applied, we can have two closure operators : $\cdot'' : 2^{\mathcal{O}} \mapsto 2^{\mathcal{O}}$ or $\cdot'' : 2^{\mathcal{A}} \mapsto 2^{\mathcal{A}}$.

A set X such that $X = X''$ is said to be closed.

Definition 2. *A pair (O, A) where $O \subseteq \mathcal{O}$ and $A \subseteq \mathcal{A}$ are closed, $A = O'$ and $O = A'$ is called a* concept*. O is called the* extent *of the concept while A is called the* intent *of the concept.*

A concept corresponds to a maximal rectangle of crosses in the cross table that represents a context, up to permutation on rows and columns. In Figure 1, the concept $(o_2o_7, a_2a_4a_5)$ is highlighted. We denote by $\mathcal{T}(\mathcal{C})$ the set of all concepts of a context $\mathcal{C}$.

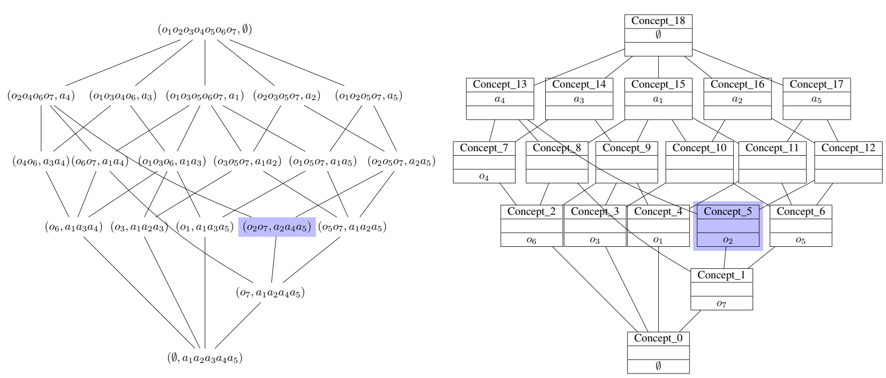

Figure 2: Concept lattice corresponding to the example in Figure 1 (left) and with simplified labels (right).

The concepts of $\mathcal{T}(\mathcal{C})$ can be ordered. Let (O_1, A_1) and (O_2, A_2) be concepts of a context. We say that (O_1, A_1) is a subconcept of (O_2, A_2) (denoted $(O_1, A_1) \leq (O_2, A_2)$) if $O_1 \subseteq O_2$. As the Galois connection that rises from the derivation is antitone, this is equivalent to $A_2 \subseteq A_1$. The concept (O_2, A_2) is a superconcept of (O_1, A_1) if $O_2 \supseteq O_1$ (and then $A_1 \supseteq A_2$).

The set of concepts from a context ordered by inclusion on the extents forms a complete lattice called the *concept lattice* of the context. Additionally, every complete lattice is the concept lattice of some context, as stated in the basic theorem of formal concept analysis [4]. Figure 2 (left) shows the concept lattice corresponding to the example in Figure 1. One can check that the concepts of Figure 2 (left) correspond to maximal boxes of incidence in Figure 1. The highlighted concept $(o_2o_7, a_2a_4a_5)$ corresponds to the highlighted concept of Figure 1. The concepts that are located directly above (upper cover) and directly below (lower cover) a concept X form the *conceptual neighbourhood* of X.

Two types of concepts can be emphasised.

Definition 3. *Let o be an object of a formal context. Then, the concept (o'', o') is called an* object-concept. *It is also called the* introducer *of the object o.*

Definition 4. *Let a be an attribute of a formal context. Then, the concept (a', a'') is called an* attribute-concept. *It is also called the* introducer *of the attribute a.*

For example, in Figure 2, the emphasised concept corresponds to the concept (o_2'', o_2'), and it introduces o_2 in the sense that it is the least concept that contains o_2. In [4], such concepts are denoted respectively $\tilde{\gamma}o$ for object-concepts and $\tilde{\mu}a$ for attribute-concepts. A concept can be both an object-concept and an attribute-concept. Concepts that are neither attribute-concepts nor object-concepts are called plain-concepts. Those concepts will be the main character in Section 3 of this article.

The simplified representation of a concept lattice is a representation where the labels of the concepts are limited, in order to avoid redundancy. The label for a particular object appears only in the smallest concept that contains it (its introducer). Reversely, the label for a particular attribute appears only in the greatest concept that contains it (its introducer). The other labels are inferred using the inheritance property. Figure 2 (right) corresponds to the concept lattice from Figure 2 (left), with simplified labels. The concept named `Concept_10` has both labels empty. By applying the inheritance, we can retrieve that `Concept_10` is in fact the concept $(o_3o_5o_7, a_1a_2)$.

This representation allows for a better reading into the concepts (with a clear separation between the extent and the intent of a concept). It allows to see at first glance which concepts are objects-concepts and attribute-concepts (by definition, the introducers have non-empty labels). The plain-concepts are, quite literally, plain.

1.2 Polyadic Concept Analysis

Polyadic Concept Analysis is a natural generalisation of FCA. It has been introduced firstly by Lehmann and Wille [5, 6] in the triadic case, and then generalised by Voutsadakis [7].

It deals with d-ary relations between sets instead of binary ones. More formally, a d-context can be defined in the following way.

Definition 5. *A* d-context *is a $(d+1)$-tuple $\mathcal{C} = (\mathcal{S}_1, \ldots, \mathcal{S}_d, \mathcal{R})$ where the $\mathcal{S}_i, i \in \{1, \ldots, d\}$, are sets called the* dimensions *and $\mathcal{R}$ is a d-ary relation between them.*

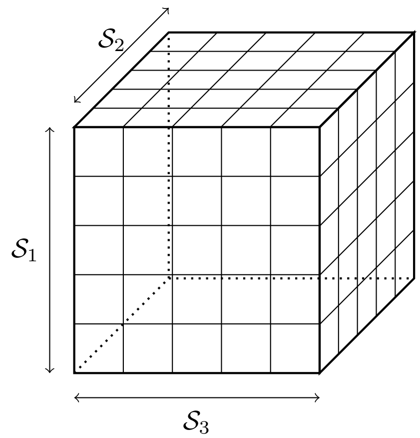

Figure 3: Visual representation of a 3-context without its crosses.

A d-context can be represented as a $|\mathcal{S}_1| \times \cdots \times |\mathcal{S}_d|$ cross table, as shown in Figure 3. For technical reasons, most of our examples figures will be drawn in two or three dimensions.

When needed, one can represent a d-context by separating the "slices" of the cross table. For instance, Figure 4 shows a 3-context which dimensions are $Numbers$, $Latin$, and $Greek$.

	a	b	c	a	b	c	a	b	c
α	$\times$	$\times$		$\times$			$\times$		
β	$\times$			$\times$			$\times$	$\times$	
γ	$\times$			$\times$		$\times$			$\times$
		1			2			3	

Figure 4: A 3-context $\mathcal{C} = (Numbers, Greek, Latin, \mathcal{R})$ where $Numbers = \{1, 2, 3\}$, $Greek = \{\alpha, \beta, \gamma\}$ and $Latin = \{a, b, c\}$. The relation $\mathcal{R}$ is the set of crosses, that is $\{(1, \alpha, a), (1, \alpha, b), (1, \beta, a), (1, \gamma, a), (2, \alpha, a), (2, \beta, a), (2, \gamma, a), (2, \gamma, c), (3, \alpha, a),$ $(3, \beta, a), (3, \beta, b), (3, \gamma, c)\}$.

In the same way as in the 2 dimensional case, d-dimensional maximal boxes of incidence have an important role.

Definition 6. *A d-concept of $\mathcal{C} = (\mathcal{S}_1, \ldots, \mathcal{S}_d, \mathcal{R})$ is a d-tuple $(X_1, \ldots, X_d)$ such that $\prod_{i\in\{1,\ldots,d\}} X_i \subseteq \mathcal{R}$ and, for all $i \in \{1, \ldots, d\}$, there is no $k \in \mathcal{S}_i \setminus X_i$ such that $\{k\} \times \prod_{j\in\{1,\ldots,d\}\setminus\{i\}} X_j \subseteq R$.*

When the dimensionality is clear from the context, we will simply call d-concepts concepts.

Definition 7. *Let $\lesssim_i$, $i \in \{1, \ldots, d\}$, be quasi-orders on a set P. Then, $\mathcal{P} = (P, \lesssim_1, \ldots, \lesssim_d)$ is a d-ordered set if, for A and B in P:*

1. $A \sim_i B$*,* $\forall i \in \{1, \ldots, d\}$ *implies* $A = B$ *(Uniqueness Condition)*

2. $A \lesssim_{i_1} B, \ldots, A \lesssim_{i_{d-1}} B$ *implies* $B \lesssim_{i_d} A$ *(Antiordinal Dependency)*

Let us now define d quasi-orders on a set of d-concepts of a d-context $\mathcal{C}$:

$$(A_1, \ldots, A_d) \lesssim_i (B_1, \ldots, B_d) \Leftrightarrow A_i \subseteq B_i$$

The resulting equivalence relation $\sim_i, i \in \{1, \ldots, d\}$ is then:

$$(A_1, \ldots, A_n) \sim_i (B_1, \ldots, B_n) \Leftrightarrow A_i = B_i$$

We can see that the set of concepts of a d-context, together with the quasi-orders and equivalence relations defined here, forms a d-ordered set. Additionally, the existence of some particular joins makes this d-ordered set a d-lattice. For a more detailed definition, we refer the reader to Voutsadakis' seminal paper on Polyadic Concept Analysis [7].

We can see that concept lattices are in fact 2-ordered sets that satisfy both the uniqueness condition and the antiordinal dependency.

In order to fully understand d-lattices, let us illustrate the definition with a small digression about graphical representation. In 2 dimensions – i.e. concept lattices – the two orders (on the extent and on the intent) are dual and only one is usually mentioned (the set of concepts ordered by inclusion on the extent, for example). Thus, their representation is possible with Hasse diagrams. From dimension 3 and up, the representation of d-ordered sets is harder. For example, in dimension 3, as of the time of writing, no good (in the sense that it allows to represent any 3-ordered set) graphical representation exists. However there is still a possible representation in the form of triadic diagrams.

Figure 5 shows an example of a *triadic diagram*, a representation of a 3-ordered set. Let us explain how this diagram should be read. The white circles in the central triangle are the concepts. The lines of the triangle represent the equivalence relation between concepts: the horizontal lines represent $\sim_1$, the north-west to south-east lines represent $\sim_2$ and the south-west to north-east lines represent $\sim_3$. Recall that two concepts are equivalent with respect to $\sim_i$ if they have the same coordinate on dimension i. This coordinate can be read by following the dotted line until the diagrams outside the triangle. In [5], Lehmann and Wille call the external diagrams the *extent diagram*, *intent diagram* and *modi diagram* depending on which dimension they represent. Here, to pursue the permutability of the dimensions further, we simply denote them by *dimension diagram* for dimension 1, 2 or 3.

If we position ourselves on the red concept of Figure 5, we can follow the dotted line up to the dimension 1 diagram and read its coordinate on the first dimension. It is $\{13\}$. Following the same logic for dimension 2 and 3, we can see that the red concept is $(13, 12, 23)$. The concept on the south-east of the red concept is on the same equivalence line, with respect to $\sim_2$. We know that its second dimension will be $\{12\}$ too. When we follow the lines to the other dimension diagrams, we see that it is $(23, 12, 13)$.

Here, every dimension diagram is a powerset on 3 elements. It is not always the case, as the three dimension diagrams are not always isomorphic.

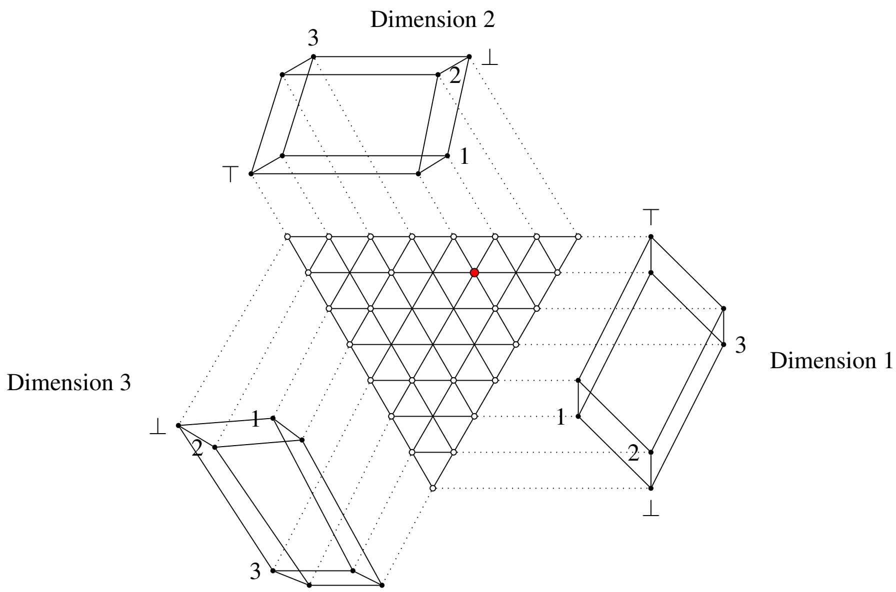

Figure 5: Representation of a powerset 3-lattice. The red concept is $(13, 13, 23)$.

We mentioned earlier that this representation does not allow to draw any 3-lattice (with straight lines). In [5], Lehmann and Wille explain this by saying that a violation of the so-called 'Thomsen Conditions [8]' and their ordinal generalisation [9] may appear in triadic contexts.[1]

2 Reduction in d-contexts

2.1 Reduction in 2-contexts

Two concept lattices from different contexts can be isomorphic. Reduction is a way of reaching a canonical context, the standard context, for any finite lattice. Reduction can be described as a two steps process. The first step of reduction in 2-dimension is the fusion of identical rows or columns (clarification).

Definition 8 (Reformulated from [4, Definition 23])**.** *A context $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{R})$ is called* ***clarified*** *if for any objects o_1 and o_2 in S_1, $o'_1 = o'_2$ implies that $o_1 = o_2$ and for any attributes a_1 and a_2 in $\mathcal{S}_2$, $a'_1 = a'_2$ implies $a_1 = a_2$.*

In [4, Definition 23], the authors use the Figure 6 example of a context that represents the service offers of an office supplies business and the associated clarified context.

[1] I was unable to put my hands on those two books in order to learn more about that. If you have a copy, I am most interested.

	Furniture	Computers	Copy-machine	Typewriters	Specialised machines
Consulting	×	×	×	×	×
Planning	×	×			
Installation	×	×	×	×	×
Instruction		×	×	×	×
Training		×			
Spare parts	×	×	×	×	×
Repairs	×	×	×	×	×
Service contracts		×	×	×	

	Furniture	Computers	Copy-machine and Typewriters	Specialised machines
Consulting, Installation, Spare parts, repairs	×	×	×	×
Planning	×	×		
Instruction		×	×	×
Training		×		
Service contracts		×	×	

Figure 6: Context and clarified context (from [4])

Another possible action is the removal of attributes (resp. objects) that can be written as combination of other attributes (resp. objects). If a is an attribute and A is a set of attributes that does not contain a but have the same extent, then a is **reducible**. Full rows and full columns are always reducible.

In a lattice, an element x is $\vee$-irreducible if $x = a \vee b$ implies $x = a$ or $x = b$ with a and b two elements of the lattice. An element x is $\wedge$-irreducible if $x = a \wedge b$ implies $x = a$ or $x = b$, with a and b two elements of the lattice.

Definition 9 (Reformulated from [4, Definition 24])**.** *A context $(\mathcal{S}_1, \mathcal{S}_2, \mathcal{R})$ is called **row reduced** if every object-concept is $\vee$-irreducible and **column reduced** if every attribute-concept is $\wedge$-irreducible. A context that is both row reduced and column reduced is **reduced**.*

This yields that for every finite lattice L, there is a unique (up to isomorphism) reduced context such that L is the concept lattice of this context. This context is called the standard context of L. This standard context can be obtained from any finite context by first clarifying the context and then deleting all the objects that can be represented as intersection of other objects and attributes that can be represented as intersection of other attributes.

2.2 Generalisation to d-dimensions

A first generalisation to 3-dimensional contexts was made in [1] by Rudolph, Sacarea and Troanca. Here, we generalise further to the d-dimensional case.

To define reduction in multidimensional contexts, we need to recall some definitions. The following notations are borrowed from [7]. A d-context gives rise to numerous[2] k-contexts, with $k \in \{2, \dots, d\}$. Those k-contexts correspond to partitions $\pi = (\pi_1, \dots, \pi_k)$ of $\{1, \dots, d\}$ into k disjoint subsets. The k-context corresponding to π is then $\mathcal{C}^\pi = (\prod_{i \in \pi_1} \mathcal{S}_i, \dots, \prod_{i \in \pi_k} \mathcal{S}_i, \mathcal{R}^\pi)$, where $(s^{(1)}, \dots, s^{(k)}) \in \mathcal{R}^\pi$ if and only if $(s_1, \dots, s_d) \in R$ with $s_i \in s^{(j)} \Leftrightarrow i \in \pi_j$. The contexts $\mathcal{C}^\pi$ are essentially the context $\mathcal{C}$ flattened by merging dimensions with the Cartesian product.

In the following, we consider only binary partitions with a singleton on one side and all the other dimensions on the other. Let $\pi = (i, \{1, \dots, d\} \setminus i)$ be such a partition of $\{1, \dots, d\}$. Then, from the context $\mathcal{C} = (\mathcal{S}_1, \dots, \mathcal{S}_d, \mathcal{R})$ we obtain the dyadic context $\mathcal{C}^\pi = (\mathcal{S}_i, \prod_{j \in \{1,\dots,d\} \setminus \{i\}} \mathcal{S}_j, \mathcal{R}^\pi)$ where $(a, b) \in \mathcal{R}^\pi$ if and only if $a \in \mathcal{S}_i$ and $b = b_1 \times \dots \times b_{d-1} \in \prod_{j \in \{1,\dots,d\} \setminus i} \mathcal{S}_j$ and $(a, b_1, \dots, b_{d-1}) \in \mathcal{R}$.

We refer the reader to figures 4 and 7 for a graphical representation of this transformation. Such a binary partition gives rise to the dyadic derivation operators $X \mapsto X^{(\pi)}$ on the 2-context $\mathcal{C}^\pi$.

	(1,a)	(1,b)	(1,c)	(2,a)	(2,b)	(2,c)	(3,a)	(3,b)	(3,c)
α	×	×		×			×		
β	×			×			×	×	
γ	×			×		×			×

Figure 7: Let $\mathcal{C}$ be our context from Figure 4. Let $\pi = (\{Greek\}, \{Number, Latin\})$. This figure represents the 2-context $\mathcal{C}^\pi$.

Let x be an element of a dimension i of a d-context $\mathcal{C}$. Then we denote by C_x the $(d-1)$-context $C_x = (\mathcal{S}_1, \dots, \mathcal{S}_{i-1}, \mathcal{S}_{i+1}, \dots, \mathcal{S}_d, \mathcal{R}_x)$ with $R_x = \{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d) \mid (x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_d) \in \mathcal{R}\}$.

We first define clarified d-contexts. Just as in the 2-dimensional case, our definition is equivalent to the fusion of identical $(d-1)$-dimensional layers.

Definition 10. *A d-context $(\mathcal{S}_1, \dots, \mathcal{S}_d, \mathcal{R})$ is called **clarified** if, for all i in $\{1, \dots, d\}$ and any x_1, x_2 in $\mathcal{S}_i$, $x_1^{(\pi)} = x_2^{(\pi)}$ implies $x_1 = x_2$, with $\pi = (i, \{1, \dots, d\} \setminus i)$.*

As with the 2-dimensional case, we provide an example in Figure 8.

Definition 11. *A clarified d-context $\mathcal{C} = (\mathcal{S}_1, \dots, \mathcal{S}_d, \mathcal{R})$ is called i-reduced if every object-concept from $\mathcal{C}^{(\pi)}$, with $\pi = (i, \{1, \dots, d\} \setminus \{i\})$, is $\vee$-irreducible. A d-context is **reduced** if it is i-reduced for all i in $\{1, \dots, d\}$.*

The most important property of reduction in the two dimensional setting is that the lattice structure obtained from the concepts of a given context is isomorphic to the one of its reduced context. The following proposition proves that the same holds for our definition of reduction in d dimensions.

Proposition 12. *Let $\mathcal{C} = (\mathcal{S}_1, \dots, \mathcal{S}_d, \mathcal{R})$ be a d context. Let π be a binary partition $(i, \{1, \dots, d\} \setminus \{i\})$. Let y_i be an element of $\mathcal{S}_i$ and Y_i be a subset of $\mathcal{S}_i$ such that y_i is*

[2] Stirling number of the second kind, or number of ways of arranging d dimensions into k slots.

	a	b	c	a	b	c	a	b	c
α	×	×		×	×				
β			×	×	×	×			×
γ	×	×					×	×	×
		1			2			3	

	$(\alpha,1)$	$(\alpha,2)$	$(\alpha,3)$	$(\beta,1)$	$(\beta,2)$	$(\beta,3)$	$(\gamma,1)$	$(\gamma,2)$	$(\gamma,3)$
a	×	×			×		×		×
b	×	×			×		×		×
c				×	×	×			×

	a and b	c	a and b	c	a and b	c
α	×		×			
β		×	×	×		×
γ	×				×	×
	1		2		3	

Figure 8: A 3-context $\mathcal{C} = (Numbers, Greek, Latin, \mathcal{R})$ (above) and the 2-context $\mathcal{C}^{\pi}$ with $\pi = (Latin, \{Greek, Numbers\})$. We can see that $a^{(\pi)} = b^{(\pi)} = \{(\alpha,1),(\alpha,2),(\beta,2),(\gamma,1),(\gamma,3)\}$, which means that a and b can be aggregated into a new attribute "a and b". Below, the corresponding clarified 3-context.

not in Y_i and $y_i^{(\pi)} = Y_i^{(\pi)}$. Then,

$$\mathcal{T}(\mathcal{C}) \cong \mathcal{T}\left(\left(\mathcal{S}_1, \ldots, \mathcal{S}_i \setminus \{y_i\}, \ldots, \mathcal{S}_d, \mathcal{R} \cap \left(\mathcal{S}_i \setminus \{y_i\} \times \prod_{j \in \{1,\ldots,d\} \setminus \{i\}} \mathcal{S}_j\right)\right)\right).$$

Proof. Without loss of generality, let us assume that y_i is an element of $\mathcal{S}_1$. We have to ensure that if $(X_1, \ldots, X_d)$ is a concept of $\mathcal{C}$, then $(X_1 \setminus \{y_i\}, \ldots, X_d)$ is a concept in the reduced context. In $\mathcal{C}$, $(X_1 \setminus \{y_i\}, \ldots, X_d)$ is a d-dimensional box full of crosses. We have to show that removing y_i from X_1 does not allow it to be extended on any other dimension. As $(X_1, \ldots, X_d)$ is a d-concept, its components X_j, $j \neq i$ form a $(d-1)$-concept in the intersection of all the layers induced by the elements of X_i. As $\mathcal{C}$ is not reduced because of y_i, $\mathcal{C}_{y_i}$ is the intersection of at least two layers $\mathcal{C}_a$ and $\mathcal{C}_b$. Obviously, a and b are elements of X_1. If $(X_2, \ldots, X_d)$ is not a $(d-1)$-concept in the intersection of the layers $\mathcal{C}_x$, $x \in X_1 \setminus \{y_i\}$, then it can be extended using crosses that both $\mathcal{C}_a$ and $\mathcal{C}_b$ share. This means that $\mathcal{C}_{y_i}$ should have them too, preventing $(\mathcal{S}_1, \ldots, \mathcal{S}_d, \mathcal{R})$ from being a concept in the first place. This ensures that $(X_1 \setminus \{y_i\}, \ldots, X_d)$ is indeed a concept. □

This proposition states that removing a reducible element from a d-context does not change the structure of the underlying d-lattice. If we keep track of the reduced and clarified elements during the process, it is possible not to lose information (by creating aggregate attributes or objects for example). It is still a deletion from the dataset. In the next section we speak about introducer concepts in a multidimensional setting.

3 Introducer concepts

3.1 Introducer d-concepts

Reduction induces a loss of information in a dataset since reducible elements are erased. Lots of applications cannot afford this loss of information and have to use other ways of reducing the complexity. Another structure, smaller than the concept lattice, has been introduced by Godin and Mili [10] in 1993. This structure consists in the restriction of the lattice to the set of introducer concepts. In the general case, the properties that make a concept lattice a lattice are lost and we are left with a simple poset. Since dyadic FCA deals with objects and attributes, such a poset is also called an Attribute-Object-Concept poset, or AOC-poset for short. In this section, we introduce the introducer concepts in a d-dimensional setting.

Due to the unicity of the dyadic closure (one component of a concept leaves only one choice for the other), each element of a dimension has only one introducer. This bounds the size of an AOC-poset by the number of objects plus the number of attributes of a context, when a concept lattice can have an exponential number of objects or attributes. As we will see in the following, this property is lost when we go multidimensional.

Definition 13. *Let i be a dimension called the height while all other dimensions are called the width. Let x be an element of dimension i. The concepts with maximal width such that x is in the height are the introducer concepts of x.*

	a	b	c	a	b	c	a	b	c	a	b	c
α	×	×		×			×			×	×	
β	×			×			×	×			×	×
γ	×			×		×		×	×		×	×
		1			2			3			4	

Figure 9: This 3-context shall serve as an example of our definitions.

Let us consider the 3-context from Figure 9 as an example. The set of introducers of element a, denoted I_a is $I_a = \{(12, \alpha\beta\gamma, a), (123, \alpha\beta, a), (1234, \alpha, a)\}$. For the element 3, we have $I_3 = \{(123, \alpha\beta, a), (3, \beta, ab), (34, \beta\gamma, b), (34, \gamma, bc)\}$.

We denote by $\mathcal{I}(\mathcal{S}_i)$ the union of the introducer concepts of all the elements of a dimension i and by $\mathcal{I}(\mathcal{C})$ the set of all the introducer concepts of a context $\mathcal{C}$.

Proposition 14. *$(\mathcal{I}(\mathcal{C}), \lesssim_1, \ldots, \lesssim_d)$ is a d-ordered set.*

Proof. Let $A = (A_1, \ldots, A_d)$ and $B = (B_1, \ldots, B_d)$ be elements of $\mathcal{I}(\mathcal{C})$. We recall that $A_i \subseteq B_i \Leftrightarrow A \lesssim_i B$ and that $A_i = B_i \Leftrightarrow A \sim_i B$. Without loss of generality, A is an introducer for an element of dimension i and B for an element of dimension j. If, for all k between 1 and d, $A \sim_k B$, then for all k, $A_k = B_k$ and $A = B$ (Uniqueness Condition).

Let us suppose that A and B are such that $A \lesssim_i B, \forall i \in \{1, \ldots, d\} \setminus \{k\}$. Then, necessarily, $B_k \subseteq A_k$ or A would not be a d-concept. Hence, $B \lesssim_k A$ (Antiordinal Dependency). □

From now on we can mirror the terminology of the 2-dimensional case, where we have complete lattices and attribute-objects-concepts partially ordered set, and use complete d-lattices and introducers d-ordered sets.

The following proposition links introducer d-concepts with the $(d-1)$-concepts that arise on a layer of a d-context.

Proposition 15. *Let x be an element of $\mathcal{S}_i$. If $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_d)$ is a $(d-1)$-concept of $\mathcal{C}_x$, then there exists some X_i such that $(X_1, \ldots, \{x\} \cup X_i, \ldots, X_d)$ is an introducer of x. If $(X_1, \ldots, \{x\} \cup X_i, \ldots, X_d)$ is an introducer of x, then there exists a $(d-1)$-concept $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_d)$ in $\mathcal{C}_x$.*

Proof. We suppose, without loss of generality, that x is in $\mathcal{S}_1$. The $(d-1)$-concepts of $\mathcal{C}_x$ are of the form $(X_2, \ldots, X_d)$. If $(x, X_2, \ldots, X_d)$ is a d-concept of $\mathcal{C}$, then it is maximal in width and has x in its height, so it is an introducer of x.

If $(x, X_2, \ldots, X_d)$ is not a d-concept of $\mathcal{C}$, it means that it can be augmented only on the first dimension (since $(X_2, \ldots, X_d)$ is maximal in $\mathcal{C}_x$). Thus, there exists a d-concept $(\{x\} \cup X_1, X_2, \ldots, X_d)$ that is maximal in width and has x in its height and is, as such, an introducer for x.

Suppose that there is a $X = (X_1, \ldots, X_d)$ that is an introducer of x but that is not obtained from a $(d-1)$-concept of $\mathcal{C}_x$ by extending X_1. It means that $(X_2, \ldots, X_d)$ is not maximal in $\mathcal{C}_x$ (or else it would be a $(d-1)$-concept). Then there exists a d-concept $Y = (Y_1, Y_2, \ldots, Y_d)$ with $x \in Y_1 \subseteq X_1$ and $X_i \subseteq Y_i$ for all i between 1 and d. This is in contradiction with the fact that X is an introducer of x. □

Proposition 15 states that every $(d-1)$-concept of a layer $\mathcal{C}_x$ maps to an introducer of x in $\mathcal{C}$ and that every introducer of x is the image of a $(d-1)$-concept of $\mathcal{C}_x$. This proposition results in a naive algorithm to compute the set of introducer concepts of a d-context. It is sufficient to compute the $(d-1)$-concepts of the $(d-1)$-contexts obtained by fixing an element of a dimension.

Algorithm 1 computes the introducers for each element of a dimension i. For a given element $x \in \mathcal{S}_i$, we compute $\mathcal{T}(\mathcal{C}_x)$. Then, for each $(d-1)$-concept $X \in \mathcal{T}(\mathcal{C}_x)$, we build the set X_i needed to extend X into a d-concept. An element y is added to X_i when $y \times \prod_{j \neq i} X_j \subseteq \mathcal{R}$, that is if there exists a $(d-1)$-dimensional box full of crosses (but not necessarily maximal) in $\mathcal{R}$, at level y. The final set X_i always contains at least x. To compute the set introducer concepts for a d-context, one needs to call Algorithm 1 on each dimension of a d-context. In some applications, it may be useful to compute the introducer concepts with respect to a given quasi-order $\lesssim_i$.

3.2 Combinatorial Intuition : powerset d-lattices

Unlike the 2-dimensional case, where introducers are unique and the size of the AOC-poset is thus bounded by $|\mathcal{S}_1| + |\mathcal{S}_2|$, in the general case, it is bounded by $\mathbb{K}_{d-1} \times \sum_{i \in \{1,\ldots,d\}} |\mathcal{S}_i|$, with $\mathbb{K}_d$ the maximal number of d-concepts in a d-context. Since a 1-context has only one 1-concept, this bound is reached in the 2-dimensional case.

Let us consider a powerset 3-lattice $\mathfrak{T}(5)$ on a ground set of size 5. It is well known[3] that $\mathfrak{T}(5)$ has $3^5 = 243$ concepts. However, the size of the introducer set of

[3] not that well known, but it is said in this paper [11].

Algorithm 1: INTRODUCERDIM($\mathcal{C}, i$)

Input: $\mathcal{C}$ a d-context, $i \in \{1, \ldots, d\}$ a dimension
Output: $\mathcal{I}(\mathcal{S}_i)$ the set of introducer concepts of elements of dimension i

$I \leftarrow \emptyset$
foreach $x \in \mathcal{S}_i$ **do**
  $C \leftarrow \emptyset$
  foreach $X = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_d) \in \mathcal{T}(\mathcal{C}^x)$ **do**
    $X_i \leftarrow \emptyset$
    foreach $y \in \mathcal{S}_i$ **do**
      if $\prod_{j \neq i} X_j \times y \subseteq \mathcal{R}$ **then**
        $X_i \leftarrow X_i \cup y$
    $C \leftarrow C \cup (X_1, \ldots, X_i, \ldots, X_d)$
  $I \leftarrow I \cup C$
return I

the powerset trilattice $\underline{\mathfrak{T}}(5)$ is 30. Indeed, by Proposition 15 we know that there exists a mapping between the 2-concepts of each layer induced by fixing an element of a dimension and the introducers. As, by definition, every layer of the context inducing a powerset trilattice has two 2-concepts, the number of introducer concepts in a powerset 3-lattice on a ground set of five elements is then bounded by $3 \times 5 \times 2$. This number is reached as the unique "hole" in each layer intersects all the concepts of the other layers. A more formal proof is given for a more general proposition below (Proposition 16).

In fact, for any powerset 3-lattice on a ground set of size n, the corresponding introducer d-ordered set has $3 \times 2 \times n$ elements. Figure 10 shows a 4-adic contranominal scale on three elements. Figure 11 shows the introducers of the powerset 3-lattice on a ground set of 3 elements.

		a	b	c	a	b	c	a	b	c
A	α		×	×	×	×	×	×	×	×
	β	×	×	×	×	×	×	×	×	×
	γ	×	×	×	×	×	×	×	×	×
B	α	×	×	×	×	×	×	×	×	×
	β	×	×	×	×		×	×	×	×
	γ	×	×	×	×	×	×	×	×	×
C	α	×	×	×	×	×	×	×	×	×
	β	×	×	×	×	×	×	×	×	×
	γ	×	×	×	×	×	×	×	×	
			1			2			3	

Figure 10: This is a 4-adic contranominal scale where the empty cells have been framed. Every layer induced by fixing an element (for example A) has three 3-concepts (in $\mathcal{C}^A$ we have $(123, \alpha\beta\gamma, bc)$, $(123, \beta\gamma, abc)$ and $(23, \alpha\beta\gamma, abc)$).

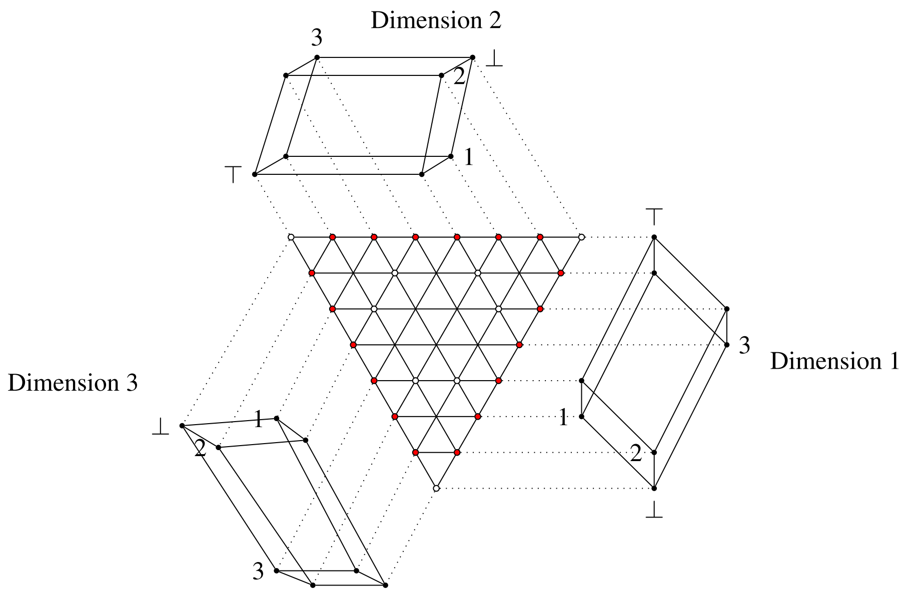

Figure 11: This represents a powerset 3-lattice. Its introducer concepts are filled in red.

Proposition 16. *Let d be an integer. A powerset d-lattice $\mathfrak{T}^d(n)$ on a ground set of n elements has d^n elements. Its corresponding introducer d-ordered set has $d \times (d-1) \times n$ elements.*

Proof. Let $\mathcal{C}$ be a d-dimensional contranominal scale, that gives rise to $\mathfrak{T}^d(n)$. Let x be an element of a dimension. The $(d-1)$-context $\mathcal{C}_x$ has only one hole (by definition of a contranominal scale). This implies that each $(d-1)$-layer induced by fixing an element of a dimension has $d-1$ concepts. By Proposition 15, we know that there exists a mapping between the $(d-1)$-concepts of the layers and the introducer concepts of the context. Since we have d dimensions and n layers by dimension, the number of introducer concepts is bounded by $d \times (d-1) \times n$.

Moreover, let $X = (X_1, \ldots, X_d)$ be an introducer concept for element x of dimension i. Then $X_i = x$. Indeed, by definition of a contranominal scale, there will be a 'hole' per layer of the context $\mathcal{C}$ that will be in the width of X.

This ensures that the $d \times (d-1) \times n$ introducer concepts that arise from Proposition 15 are distinct and that this number is reached in the case of powerset d-lattices. □

4 Open problems

With respect to reduction, it would be interesting to investigate the relation of irreducible elements to the Dedekind-MacNeille completion of d-ordered sets presented by Voutsadakis [12].

Some interesting questions remain open regarding the number of introducer concepts compared to the number of concepts, both experimentally and theoretically. Furthermore, it would be interesting to use introducer concepts in visualisation or explainable approaches in d-dimensions, as they give insight on the less general concepts containing an element.

References

[1] Sebastian Rudolph, Christian Sacarea, and Diana Troanca. Reduction in triadic data sets. In *Proceedings of the 4th International Workshop "What can FCA do for Artificial Intelligence?", FCA4AI 2015, co-located with the International Joint Conference on Artificial Intelligence (IJCAI 2015), Buenos Aires, Argentina, July 25, 2015.*, pages 55–62, 2015.

[2] Garrett Birkhoff. *Lattice theory*, volume 25. American Mathematical Soc., 1940.

[3] Marc Barbut and Bernard Monjardet. Ordre et classification. *Algebre et Combinatoire*, Volumes 1 and 2, 1970.

[4] Bernhard Ganter and Rudolf Wille. *Formal concept analysis - mathematical foundations*. Springer, 1999.

[5] Fritz Lehmann and Rudolf Wille. A triadic approach to formal concept analysis. In *Conceptual Structures: Applications, Implementation and Theory, Third International Conference on Conceptual Structures, ICCS '95, Santa Cruz, California, USA, August 14-18, 1995, Proceedings*, pages 32–43, 1995.

[6] Rudolf Wille. The basic theorem of triadic concept analysis. *Order*, 12(2):149–158, 1995.

[7] George Voutsadakis. Polyadic concept analysis. *Order*, 19(3):295–304, 2002.

[8] David H. Krantz, Patrick Suppes, and Robert Duncan Luce. *Foundations of measurement*. academic press, 1971.

[9] Utta Wille. *Geometric representation of ordinal contexts*. PhD thesis, University Gießen, 1995.

[10] Robert Godin and Hafedh Mili. Building and Maintaining Analysis-Level Class Hierarchies Using Galois Lattices. In *8th Conference on Object-Oriented Programming Systems, Languages, and Applications (OOPSLA)*, pages 394–410, 1993.

[11] Klaus Biedermann. Powerset trilattices. In *Conceptual Structures: Theory, Tools and Applications, 6th International Conference on Conceptual Structures, ICCS '98, Montpellier, France, August 10-12, 1998, Proceedings*, pages 209–224, 1998.

[12] George Voutsadakis. Dedekind-macneille completion of *n*-ordered sets. *Order*, 24(1):15–29, 2007.