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Abstract. In the paper, the authors derive, from the Rodrigues formulas for the Chebyshev polynomials of the first and second kinds and by virtue of the Faà di Bruno formula and two identities for the Bell polynomials of the second kind, two explicit formulas for the Chebyshev polynomials of the first and second kinds, find, by virtue of an inversion formula for combinatorial coefficients, two inversion formulas for explicit formulas of the Chebyshev polynomials of the first and second kinds, and collect variants of the Rodrigues formulas for the Chebyshev polynomials of the first and second kinds.

Contents

1. Introduction and main results

It is well known [2, 3, 4] that the Chebyshev polynomials of the first and second kinds T_n and $U_n(x)$ are very important in mathematical sciences and that, in the study of ordinary differential equations [2, pp. xxxv and 1004], they arise as solutions to the Chebyshev differential equations

$$(1-x^2)y''-xy'+n^2y=0 \quad \text{and} \quad (1-x^2)y''-3xy'+n(n+2)y=0$$

for the Chebyshev polynomials of the first and second kinds T_n and U_n respectively. In [3, Eqs. (4.30) and (4.31)], the Rodrigues formulas for the Chebyshev polynomials of the first and second kinds T_n and U_n read that

$$T_n(x)=(-1)^n\frac{2^nn!}{(2n)!}\bigl(1-x^2\bigr)^{1/2}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\bigl[\bigl(1-x^2\bigr)^{n-1/2}\bigr] \tag{1}$$

and

$$U_n(x)=(-1)^n\frac{2^n(n+1)!}{(2n+1)!}\bigl(1-x^2\bigr)^{-1/2}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\bigl[\bigl(1-x^2\bigr)^{n+1/2}\bigr]. \tag{2}$$

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In [10, pp. 432–433],

$$T_n(x)=\frac{n}{2}\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\frac{(n-m-1)!}{m!(n-2m)!}(2x)^{n-2m} \tag{3}$$

and

$$U_n(x)=\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\frac{(n-m)!}{m!(n-2m)!}(2x)^{n-2m}, \tag{4}$$

where $n\in\mathbb{N}$ and $\lfloor t\rfloor$ denotes the floor function whose value equals the largest integer less than or equal to t.

We notice that the formulas (3) and (4) can be rearranged as

$$T_n(x)=\frac{n}{2}\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\binom{n-m}{m}\frac{(2x)^{n-2m}}{n-m} \tag{5}$$

and

$$U_n(x)=\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\binom{n-m}{m}(2x)^{n-2m}. \tag{6}$$

In this paper, starting out from (1) and (2), by virtue of the Faà di Bruno formula, two identities for the Bell polynomials of the second kind, and an inversion theorem of combinatorial numbers, we will derive two explicit formulas for the Chebyshev polynomials T_n and U_n below.

Theorem 1. *For $n\ge0$, the Chebyshev polynomials T_n and U_n can be explicitly computed by*

$$T_n(x)=x^n\sum_{\ell=0}^{\lfloor n/2\rfloor}\binom{n}{2\ell}\left(1-\frac{1}{x^2}\right)^{\ell} \tag{7}$$

and

$$U_n(x)=x^n\sum_{\ell=0}^{\lfloor n/2\rfloor}\binom{n+1}{2\ell+1}\left(1-\frac{1}{x^2}\right)^{\ell}. \tag{8}$$

Employing an inversion theorem for combinatorial numbers in [9, Theorem 4.3], we will find inversion formulas of the formulas (5) and (6).

Theorem 2. *For $n\in\mathbb{N}$, we have*

$$\sum_{k=1}^{n}\binom{2n-k-1}{n-1}(2x)^kT_k(x)=2^{2n-1}x^{2n} \tag{9}$$

and

$$\sum_{k=1}^{n}k\binom{2n-k-1}{n-1}(2x)^kU_k(x)=n(2x)^{2n}. \tag{10}$$

In the last section, we will collect variants of the Rodrigues formulas for the Chebyshev polynomials of the first and second kinds T_n and U_n.

2. Lemmas

In this paper, we need the following lemmas.

Lemma 1 ([1, pp. 134 and 139]). *For $n \ge k \ge 0$, the Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$, are defined by*

$$\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1\le i\le n,\ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^n i\ell_i=n\\ \sum_{i=1}^n \ell_i=k}} \frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!} \prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

Faà di Bruno's formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n} f\circ h(t) = \sum_{k=0}^{n} f^{(k)}(h(t))\,\mathrm{B}_{n,k}\bigl(h'(t), h''(t), \dots, h^{(n-k+1)}(t)\bigr). \tag{11}$$

Lemma 2 ([1, p. 135]). *For $n \ge k \ge 0$, we have*

$$\mathrm{B}_{n,k}\bigl(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}\bigr) = a^k b^n\,\mathrm{B}_{n,k}(x_1, x_2, \dots, x_{n-k+1}), \tag{12}$$

where a and b are any complex numbers.

Lemma 3 ([6, Theorem 4.1]). *For $n \ge k \ge 0$, the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ satisfy*

$$\mathrm{B}_{n,k}(x, 1, 0, \dots, 0) = \frac{1}{2^{n-k}}\frac{n!}{k!}\binom{k}{n-k}x^{2k-n}, \tag{13}$$

where $\binom{0}{0} = 1$ and $\binom{p}{q} = 0$ for $q > p \ge 0$.

Lemma 4 ([9, Theorem 4.3]). *For $k \ge 1$, let s_k and S_k be two sequences independent of n such that $n \ge k \ge 1$. Then*

$$\frac{s_n}{n!} = \sum_{k=1}^{n}(-1)^k\binom{k}{n-k}S_k \quad \textit{if and only if} \quad nS_n = \sum_{k=1}^{n}\frac{(-1)^k}{(k-1)!}\binom{2n-k-1}{n-1}s_k.$$

3. Proofs of main results

Now we begin to prove our main results stated in the first section.

Proof of Theorem 1. By virtue of the formuals (11), (12), and (13), we have

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\bigl[(1-x^2)^{n-1/2}\bigr] &= \sum_{k=1}^{n}\frac{\mathrm{d}^k\,u^{n-1/2}}{\mathrm{d}\,u^k}\,\mathrm{B}_{n,k}(-2x, -2, 0\dots, 0)\\
&= \sum_{k=1}^{n}\prod_{\ell=0}^{k-1}\left(n-\ell-\frac{1}{2}\right)u^{n-k-1/2}(-2)^k\,\mathrm{B}_{n,k}(x, 1, 0\dots, 0)\\
&= \sum_{k=1}^{n}\frac{1}{2^k}\prod_{\ell=0}^{k-1}(2n-2\ell-1)\bigl(1-x^2\bigr)^{n-k-1/2}(-2)^k\frac{1}{2^{n-k}}\frac{n!}{k!}\binom{k}{n-k}x^{2k-n}\\
&= \frac{n!}{(2x)^n}\bigl(1-x^2\bigr)^{n-1/2}\sum_{k=1}^{n}(-1)^k\binom{k}{n-k}\frac{(2n-1)!!}{[2(n-k)-1]!!}\frac{2^k}{k!}\left(\frac{x^2}{1-x^2}\right)^k\\
&= \frac{n!(2n-1)!!}{(2x)^n}\bigl(1-x^2\bigr)^{n-1/2}\sum_{k=1}^{n}(-1)^k\binom{k}{n-k}\frac{2^k}{k![2(n-k)-1]!!}\left(\frac{x^2}{1-x^2}\right)^k,
\end{aligned}$$

where $n \in \mathbb{N}$, $u = u(x) = 1 - x^2$, and the double factorial of negative odd integers $-2n-1$ is defined by

$$(-2n-1)!! = \frac{(-1)^n}{(2n-1)!!} = (-1)^n\frac{2^n n!}{(2n)!}, \quad n \ge 0.$$

Substituting the above established equality into (1) and simplifying lead to

$$T_n(x)=\sum_{k=1}^{n}\frac{(-1)^{n-k}}{4^{n-k}}\binom{k}{n-k}\binom{n}{k}\frac{[2(n-k)]!!}{[2(n-k)-1]!!}x^{2k-n}\bigl(1-x^2\bigr)^{n-k}$$

which can be rearranged, by replacing $n-k$ by ℓ, as

$$T_n(x)=x^n\sum_{\ell=0}^{n-1}\frac{(-1)^{\ell}}{4^{\ell}}\binom{n}{n-\ell}\binom{n-\ell}{\ell}\frac{(2\ell)!!}{(2\ell-1)!!}\biggl(\frac{1}{x^2}-1\biggr)^{\ell}.$$

Since

$$\frac{1}{4^{\ell}}\binom{n}{n-\ell}\binom{n-\ell}{\ell}\frac{(2\ell)!!}{(2\ell-1)!!}=\binom{n}{2\ell},$$

we arrives at the identity (7).

Repeating the above process, we can obtain

$$\begin{aligned}
\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\bigl[\bigl(1-x^2\bigr)^{n+1/2}\bigr]&=\sum_{k=1}^{n}\frac{\mathrm{d}^k\,u^{n+1/2}}{\mathrm{d}\,u^k}\,\mathrm{B}_{n,k}(-2x,-2,0\dots,0)\\
&=\sum_{k=1}^{n}\prod_{\ell=0}^{k-1}\biggl(n-\ell+\frac12\biggr)u^{n-k+1/2}(-2)^k\,\mathrm{B}_{n,k}(x,1,0\dots,0)\\
&=\sum_{k=1}^{n}\frac{1}{2^k}\prod_{\ell=0}^{k-1}(2n-2\ell+1)\bigl(1-x^2\bigr)^{n-k+1/2}(-2)^k\frac{1}{2^{n-k}}\frac{n!}{k!}\binom{k}{n-k}x^{2k-n}\\
&=\frac{n!}{(2x)^n}\bigl(1-x^2\bigr)^{n+1/2}\sum_{k=1}^{n}(-1)^k\binom{k}{n-k}\frac{(2n+1)!!}{[2(n-k)+1]!!}\frac{2^k}{k!}\biggl(\frac{x^2}{1-x^2}\biggr)^k\\
&=\frac{n!(2n+1)!!}{(2x)^n}\bigl(1-x^2\bigr)^{n+1/2}\sum_{k=1}^{n}(-1)^k\binom{k}{n-k}\frac{2^k}{k![2(n-k)+1]!!}\biggl(\frac{x^2}{1-x^2}\biggr)^k.
\end{aligned}$$

Substituting this into (2) and simplifying lead to

$$U_n(x)=\sum_{k=1}^{n}\frac{(-1)^{n-k}}{2^{2n-2k+1}}\binom{k}{n-k}\binom{n+1}{k}\frac{[2(n-k+1)]!!}{[2(n-k)+1]!!}x^{2k-n}\bigl(1-x^2\bigr)^{n-k}.$$

Replacing $n-k$ by ℓ reveals that

$$U_n(x)=\sum_{\ell=0}^{n-1}\frac{(-1)^{\ell}}{2^{2\ell+1}}\binom{n+1}{n-\ell}\binom{n-\ell}{\ell}\frac{[2(\ell+1)]!!}{(2\ell+1)!!}x^{n-2\ell}\bigl(1-x^2\bigr)^{\ell}.$$

Due to

$$\frac{1}{2^{2\ell+1}}\binom{n+1}{n-\ell}\binom{n-\ell}{\ell}\frac{[2(\ell+1)]!!}{(2\ell+1)!!}=\binom{n+1}{2\ell+1},$$

we derive (8). The proof of Theorem 1 is complete. □

Proof of Theorem 2. The inversion theorem in Lemma 4 can be restated as

$$(-1)^n\frac{s_n}{n!}=\sum_{\ell=0}^{n-1}(-1)^{\ell}\binom{n-\ell}{\ell}S_{n-\ell}=\sum_{\ell=0}^{\lfloor n/2\rfloor}(-1)^{\ell}\binom{n-\ell}{\ell}S_{n-\ell}$$

if and only if

$$nS_n=\sum_{\ell=1}^{n}\frac{(-1)^{\ell}}{(\ell-1)!}\binom{2n-\ell-1}{n-1}s_{\ell}.$$

The formulas (5) and (6) can be rearranged as

$$\frac{2}{n}(2x)^nT_n(x)=\sum_{\ell=0}^{\lfloor n/2\rfloor}(-1)^{\ell}\binom{n-\ell}{\ell}\frac{(2x)^{2(n-\ell)}}{n-\ell}$$

and

$$(2x)^n U_n(x) = \sum_{\ell=0}^{\lfloor n/2 \rfloor} (-1)^\ell \binom{n-\ell}{\ell} (2x)^{2(n-\ell)}.$$

Consequently, we obtain

$$n\frac{(2x)^{2n}}{n} = \sum_{k=1}^{n} \frac{(-1)^k}{(k-1)!} \binom{2n-k-1}{n-1} (-1)^k 2(k-1)!(2x)^k T_k(x)$$

and

$$n(2x)^{2n} = \sum_{k=1}^{n} \frac{(-1)^k}{(k-1)!} \binom{2n-k-1}{n-1} (-1)^k k!(2x)^k U_k(x)$$

which can be simplified as (9) and (10). The proof of Theorem 2 is complete. □

4. Variants of Rodrigues formulas for Chebyshev polynomials

In [2, p. 1003], the Rodrigues formulas for $T_n(x)$ and $U_n(x)$ are written in the forms

$$T_n(x) = (-1)^n \frac{\sqrt{1-x^2}}{(2n-1)!!} \frac{\mathrm{d}^n}{\mathrm{d}\, x^n} \Big[\big(1-x^2\big)^{n-1/2}\Big] \tag{14}$$

and

$$U_n(x) = \frac{(-1)^n (n+1)}{\sqrt{1-x^2}\,(2n+1)!!} \frac{\mathrm{d}^n}{\mathrm{d}\, x^n} \Big[\big(1-x^2\big)^{n+1/2}\Big]. \tag{15}$$

In [2, p. 1004] and [10, pp. 432–433], the Rodrigues formulas for $T_n(x)$ and $U_n(x)$ are formulated as

$$T_n(x) = \frac{(-1)^n \sqrt{\pi}}{2^n \Gamma(n+1/2)} \big(1-x^2\big)^{1/2} \frac{\mathrm{d}^n}{\mathrm{d}\, x^n} \Big[\big(1-x^2\big)^{n-1/2}\Big] \tag{16}$$

and

$$U_n(x) = \frac{(-1)^n \sqrt{\pi}\,(n+1)}{2^{n+1} \Gamma(n+3/2)} \big(1-x^2\big)^{-1/2} \frac{\mathrm{d}^n}{\mathrm{d}\, x^n} \Big[\big(1-x^2\big)^{n+1/2}\Big], \tag{17}$$

where $\Gamma(z)$ stands for the classical gamma function which can be defined [5, 7] by

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{\prod_{k=0}^{n} (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$$

or by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \,\mathrm{d}\, t, \quad \Re(z) > 0.$$

In [4, p. 442], the Rodrigues formulas for $T_n(x)$ and $U_n(x)$ are arranged as

$$T_n(x) = \frac{\big(1-x^2\big)^{1/2}}{(-2)^n (1/2)_n} \frac{\mathrm{d}^n}{\mathrm{d}\, x^n} \Big[\big(1-x^2\big)^{n-1/2}\Big] \tag{18}$$

and

$$U_n(x) = \frac{(n+1)\big(1-x^2\big)^{-1/2}}{(-2)^n (3/2)_n} \frac{\mathrm{d}^n}{\mathrm{d}\, x^n} \Big[\big(1-x^2\big)^{n+1/2}\Big], \tag{19}$$

where $(x)_n$ for $n \ge 0$ and $x \in \mathbb{R}$ denotes the rising factorial which can be defined [8] by

$$(x)_n = \begin{cases} x(x+1)\cdots(x+n-1), & n \ge 1 \\ 1, & n = 0 \end{cases} = \prod_{\ell=0}^{n-1} (x+\ell) = \frac{\Gamma(x+n)}{\Gamma(x)}.$$

By virtue of the recurrence relation $\Gamma(x+1) = x\Gamma(x)$, we have

$$\Gamma\left(n+\frac{1}{2}\right) = \prod_{\ell=0}^{n-1} \left(n-\ell-\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)_n \sqrt{\pi} = \frac{(2n-1)!!}{2^n} \sqrt{\pi}$$

and

$$\Gamma\left(n+\frac{3}{2}\right)=\prod_{\ell=0}^{n}\left(n-\ell+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)=\left(\frac{3}{2}\right)_n\frac{\sqrt{\pi}}{2}=\frac{(2n+1)!!}{2^{n+1}}\sqrt{\pi}\,.$$

Substituting these into (16) and (17) respectively leads to (14), (15), (18), and (19) which are equivalent to (1) and (2) respectively.

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