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Implementing the microscopic self-shadowing microflake model

Supplemental material of
"A new microflake model with microscopic self-shadowing for accurate volume downsampling"

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In the main paper, we have introduced a new participating medium model that generalizes the standard microflake model [JAM*10]. It has new parameters that characterize self-shadowing effects at the microscopic scale. This model is more complex than the standard model, and its implementation is difficult unless normal distributions and self-shadowing functions are carefully chosen, so that closed-form expressions and sampling procedures can be derived. In this document, we present all the details, proofs and derivations for implementing our self-shadowing model. We first recall the general expressions of this model in Sec. 2 and of the simplified version in which self-shadowing is isotropic (Sec. 2.5). Then, we discuss the implementation of the model in Sec. 4, and the implementation of the simplified model based on SGGX distribution [HDCD15] in Sec. 5. We briefly recall the theory of rejection sampling in Sec. 3 because we use it several times in this document. We provide code that implements our model along with this document. We tested our sampling procedures using chi-square tests, and verified that our phase functions are normalized using numerical integration.

Contents

1. Notations

In this document, we use often use spherical coordinates and we call $\phi \in (0, 2\pi)$ the azimuth angle and $\theta \in (0, \pi)$ the polar angle, so that a normalized vector ω writes

$$\omega = \begin{bmatrix} \cos(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\theta) \end{bmatrix}.$$

Other commonly used symbols can be found in the following table:

Symbol	Meaning	Unit
$D(\omega)$	Microflake normal distribution	st^{-1}
$A(\omega)$	Microscopic self-shadowing function	1
ρ	Area of microflake per unit volume	m^{-1}
σ_t	Attenuation coefficient	m^{-1}
σ_s, σ_{ss}, σ_{ms}	Scattering coefficients (wavelength dependent)	m^{-1}
f, f_{ss}, f_{ms}	Normalized phase functions	st^{-1}
α, α_{ss}, α_{ms}	Albedos (wavelength dependent)	1
$\langle\ \cdot\ \rangle$	Clamped dot product	-
$\chi_+(x)$	Indicator function of positive reals	-

2. The microscopic self-shadowing model

We recall here the main expressions of our self-shadowing model. Motivations for this model can be found in the main paper.

2.1. Anisotropic RTE

The anisotropic radiative transfer equation (RTE) introduced by Jakob et al. [JAM*10] writes:

$$(\omega \cdot \nabla)L(\omega) + \sigma_t(\omega)L(\omega) = \sigma_s(\omega) \int_{S^2} f(\omega \to \omega')L(\omega')d\omega' + Q(\omega) \tag{2.1}$$

with $\sigma_t(\omega)$ the anisotropic attenuation coefficient, $\sigma_s(\omega)$ the anisotropic scattering coefficient (which is usually wavelength dependent), and f the anisotropic phase function, in the sense that it depends on ω and ω', and not just on the angle between ω and ω'. In this document, phase function are considered normalized in the second parameter:

$$\int_{S^2} f(\omega \to \omega')d\omega' = 1, \quad \forall \omega. \tag{2.2}$$

2.2. Helmholtz's law of reciprocity

The Helmholtz's law of reciprocity for anisotropic media writes

$$\sigma_s(\omega) f(\omega \to \omega') = \sigma_s(\omega') f(\omega' \to \omega). \tag{2.3}$$

This law means that the amount of scattered energy from direction ω in direction ω', given by the scattering coefficient $\sigma_s(\omega)$ and the phase function, should be equal to the amount of energy scattered from direction ω' in direction ω.

2.3. Self-shadowing function

Before introducing our self-shadowing model, we introduce a directional self-shadowing function A with the following properties:

$$\forall \omega \in S^2,\ \ 0 < A(\omega) \leq 1 \quad \text{and} \quad A(\omega) = A(-\omega). \tag{2.4}$$

This function represents the probability of self-shadowing at a microscopic scale, depending on the direction ω.

2.4. Self-shadowing model

The anisotropic attenuation coefficient in our self-shadowing model writes

$$\sigma_t(\omega) = A(\omega)\rho \int D(m)\langle \omega \cdot m\rangle \, \mathrm{d}m. \tag{2.5}$$

This is the attenuation coefficient in the standard model times the factor $A(\omega)$ which takes into account self-shadowing effects. The anisotropic single scattering coefficient is given by

$$\sigma_{ss}(\omega) = \rho \alpha_{ss}(\lambda) A(\omega) \int A(\omega') D(m) \langle \omega \cdot m \rangle \, \mathrm{d}m \tag{2.6}$$

where $\omega' = 2m(m \cdot \omega) - \omega$ is the reflected direction, given an input direction ω and a microflake normal m. Using the Jacobian of the transformation from normals ω_h to reflected directions provided by Walter et al. [WMLT07],

$$\left\| \frac{\partial \omega_h}{\partial \omega'} \right\| = \frac{1}{4|\omega_h \cdot \omega'|},$$

we can also write

$$\sigma_{ss}(\omega) = \frac{\rho \alpha_{ss}(\lambda) A(\omega)}{4} \int A(\omega') D(\omega_h) \, \mathrm{d}\omega' \tag{2.7}$$

with $\omega_h = \dfrac{\omega + \omega'}{\|\omega + \omega'\|}$ the half vector. The single scattering phase function f_{ss} is the standard phase function attenuated in some directions due to microscopic shadowing, and re-normalized:

$$f_{ss}(\omega \to \omega') = \frac{A(\omega') D(\omega_h)}{\displaystyle\int A(\omega'') D\left(\frac{\omega + \omega''}{\|\omega + \omega''\|}\right) \mathrm{d}\omega''} \tag{2.8}$$

$$= \frac{\rho \alpha_{ss}(\lambda) A(\omega) A(\omega') D(\omega_h)}{4 \sigma_{ss}(\omega)} \tag{2.9}$$

For energy conservation, we introduce a local multiple scattering coefficient:

$$\sigma_{ms}(\omega) = \alpha_{ms}(\lambda) \rho A(\omega) \int \left(1 - A(\omega')\right) D(m) \langle \omega \cdot m \rangle dm \tag{2.10}$$

$$= \alpha_{ms}(\lambda) \left(\sigma_t(\omega) - \frac{\sigma_{ss}(\omega)}{\alpha_{ss}(\lambda)} \right) \tag{2.11}$$

and its associated phase function

$$f_{ms}(\omega \to \omega') = f_{ms}(\omega') = \frac{\sigma_{ms}(\omega')}{\int \sigma_{ms}(\omega'') \, \mathrm{d}\omega''} = \frac{\sigma_{ms}(\omega')}{\alpha_{ms}(\lambda) \int \sigma_t(\omega'') \, \mathrm{d}\omega'' - \dfrac{\alpha_{ms}(\lambda)}{\alpha_{ss}(\lambda)} \int \sigma_{ss}(\omega'') \, \mathrm{d}\omega''} \tag{2.12}$$

2.5. Simplified model with isotropic self-shadowing

Our model greatly simplifies when the self-shadowing function is isotropic, meaning that $A(\omega) = A,\ \forall\omega$. In the case of isotropic self-shadowing, expressions reduce to

$$\sigma_t(\omega) = A\rho \int D(m)\langle\omega\cdot m\rangle\,\mathrm{d}m \tag{2.13}$$

$$\sigma_{ss}(\omega) = \alpha_{ss}(\lambda)A\sigma_t(\omega) \tag{2.14}$$

$$f_{ss}(\omega\to\omega') = \frac{\rho\alpha_{ss}(\lambda)A^2D(\omega_h)}{4\sigma_{ss}(\omega)} \tag{2.15}$$

$$\sigma_{ms}(\omega) = \alpha_{ms}(\lambda)\sigma_t(\omega)\,(1-A) \tag{2.16}$$

$$f_{ms}(\omega') = \frac{\sigma_{ms}(\omega')}{\int\sigma_{ms}(\omega'')\,\mathrm{d}\omega''} = \frac{\sigma_t(\omega')}{\int\sigma_t(\omega'')\,\mathrm{d}\omega''} \tag{2.17}$$

Note that $\sigma_t(\omega)$ is equal to the projected area of microflakes times A, and f_{ss} is exactly the specular phase function of the standard microflake model.

3. Rejection sampling

Probability distribution functions (*pdf*) are ubiquitous in physically-based rendering, and procedures that generate samples from such distribution are often needed. In particular, generating samples from phase functions tends to decrease noise in volume path tracing.

Unfortunately, it is not always possible to find an efficient sampling procedure for given a *pdf*. The most common method, known as *inverse transform sampling*, requires an expression of the inverse F^{-1} of the cumulative distribution function of the *pdf*:

$$F(u) = \int_{x=-\infty}^{u} pdf(x)\,\mathrm{d}x,$$

but it is not always possible to find closed-form expressions for F^{-1}. Some authors have found sampling procedures using the fact that their *pdf* comes from a geometric problem – for instance, sampling visible normals of an ellipsoid viewed from a given direction [HDCD15, DWMG15]. This strategy cannot be applied to all *pdf*.

Rejection sampling is a very general method that can generate samples from any normalized *pdf*. For sampling the distribution pdf_1, it is possible to generate samples from any other distribution pdf_2 and to accept or reject these samples with a probability such that accepted samples follow exactly the distribution pdf_1. Rejection sampling is very powerful in the sense that it can be used for sampling arbitrary distributions, but the algorithm is only efficient if pdf_2 is *close enough* to pdf_1. The more pdf_2 is different from pdf_1, the more samples will be rejected by the algorithm, leading to a very inefficient sampling procedure. For sampling pdf_1 using pdf_2, rejection sampling needs a scalar M such that, for all x,

$$pdf_1(x) \leq M\,pdf_2(x)$$

Then, the sampling procedure is the following:

1. Generate a sample x from pdf_2
2. Generate a random number U in $(0,1)$.
   If $U \leq \dfrac{pdf_1(x)}{M\,pdf_2(x)}$, accept the sample x and the procedure is finished.
   Else, go back to step 1.

The value M is the average number of iterations needed before accepting one sample, and $1/M$ is the average probability of accepting one sample from pdf_2. It is thus possible to predict how efficient is a sampling procedure – in average.

4. Implementing the self-shadowing model with trigonometric lobes

In this section, we derive closed-form expressions and sampling procedures for implementing our self-shadowing model, using an anisotropic self-shadowing distribution (Sec. 4.1) and microflake normal distributions based on trigonometric lobes (Sec. 4.2). The derivations include:

- Normalization factors for trigonometric lobes (Sec.4.3).
- Expressions for the attenuation coefficient σ_t (Sec. 4.5).
- Expressions for the single scattering coefficient σ_{ss} (Sec. 4.6).
- Expressions for $\int\sigma_t$ and $\int\sigma_{ss}$, involved in the multiple scattering albedo σ_{ms} (Sec. 4.7).

- Sampling procedures for the single scattering phase function f_{ss} (Sec. 4.10 and 4.11).
- Sampling procedures for the multiple scattering phase function f_{ms} (Sec. 4.12).

In the next section (Sec. 5), we discuss an implementation of our simplified self-shadowing model (when self-shadowing is anisotropic) based on the SGGX distribution.

4.1. Choice of a self-shadowing function

We introduce an anisotropic self-shadowing function A for which we can find closed-form expressions for the spherical convolutions and integrals involved in our self-shadowing model (Eq. 2.6, 2.12). This anisotropic self-shadowing function writes

$$A(\omega) = a_1(\xi_{A_1}\cdot\omega)^2 + a_2(\xi_{A_2}\cdot\omega)^2 + a_3(\xi_{A_3}\cdot\omega)^2. \tag{4.1}$$

It can also be written as the square projected area of an ellipsoid in direction ω [HDCD15] with axes ξ_{A_i}:

$$A(\omega) = \omega^T \begin{bmatrix} \xi_{A_1} & \xi_{A_2} & \xi_{A_3} \end{bmatrix} \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \begin{bmatrix} \xi_{A_1}^T \\ \xi_{A_2}^T \\ \xi_{A_3}^T \end{bmatrix} \omega = \omega^T S_A\, \omega. \tag{4.2}$$

4.2. Trigonometric lobes: $D_{\cos}(\omega,\xi,n)$ and $D_{\sin}(\omega,\xi,n)$

We use distributions based on trigonometric lobes:

$$D_{\cos}(\omega,\xi,n) = \frac{\cos^{2n}(\omega,\xi)}{N_{\cos}(n)} = \frac{(\omega\cdot\xi)^{2n}}{N_{\cos}(n)}$$

$$D_{\sin}(\omega,\xi,n) = \frac{\sin^{2n}(\omega,\xi)}{N_{\sin}(n)} = \frac{(1-(\omega\cdot\xi)^2)^n}{N_{\sin}(n)}$$

$$D_{\mathrm{iso}}(\omega) = \frac{1}{4\pi}$$

with $n \in \mathbb{N}$ and $N_{\cos}(n)$ and $N_{\sin}(n)$ the normalization factors.

4.3. Normalization factors

For the cosine lobe, the normalization factor writes

$$N_{\cos}(n) = \int (m\cdot\xi_D)^{2n}\, \mathrm{d}m = \frac{4\pi}{2n+1}.$$

The normalization factor for the sine lobe writes

$$N_{\sin}(n) = \int \left(1-(m\cdot\xi_D)^2\right)^n \mathrm{d}m = \frac{2\pi^{\frac{3}{2}}\Gamma(1+n)}{\Gamma\left(\frac{3}{2}+n\right)}.$$

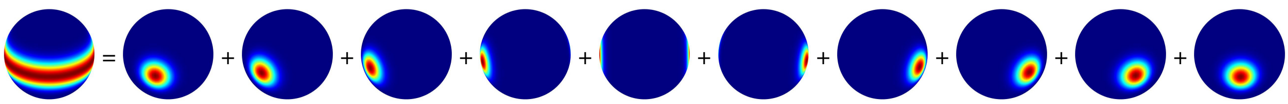
In our implementation, we pre-computed the normalization factors for the sine lobe for $n \in [1..20]$.

4.4. Proof that distributions $D_{\sin}(\omega,\xi,n)$ can be written as a sum of distributions $D_{\cos}(\omega,\xi,n)$

Sine lobes $D_{\sin}(\omega,\xi,n)$ have an interesting property: they can be written as a sum of $2n$ cosine lobes:

$$D_{\sin}(\omega,\xi,n) = \frac{1}{2n}\sum_{p=0}^{2n-1} D_{\cos}(\omega,\xi_p,n) \tag{4.3}$$

with ξ_p being $2n$ evenly spaced axes in the plane orthogonal to ξ. For example, a $D_{\sin}(\omega,\xi,5)$ distribution is equivalent to a sum of 10 $D_{\cos}(\omega,\xi_p,5)$ distributions:



Without loss of generality, we derive the proof in spherical coordinates in the case $\xi_D^{\sin} = [0,0,1]^T$. The sum of cosine lobes writes

$$\frac{1}{2n}\frac{2n+1}{4\pi}\sum_{p=0}^{2n-1}\left(\begin{bmatrix}\cos\left(\frac{p\pi}{2n}\right)\\ \sin\left(\frac{p\pi}{2n}\right)\\ 0\end{bmatrix}\cdot\begin{bmatrix}\cos(\phi)\sin(\theta)\\ \sin(\phi)\sin(\theta)\\ \cos(\theta)\end{bmatrix}\right)^{2n} = \frac{2n+1}{8n\pi}\sum_{p=0}^{2n-1}\left(\cos\left(\frac{p\pi}{2n}\right)\cos(\phi)\sin(\theta)+\sin\left(\frac{p\pi}{2n}\right)\sin(\phi)\sin(\theta)\right)^{2n} \tag{4.4}$$

$$= \frac{2n+1}{8n\pi}\sin(\theta)^{2n}\sum_{p=0}^{2n-1}\left(\cos\left(\frac{p\pi}{2n}\right)\cos(\phi)+\sin\left(\frac{p\pi}{2n}\right)\sin(\phi)\right)^{2n} \tag{4.5}$$

$$= \frac{2n+1}{8n\pi}\sin(\theta)^{2n}\sum_{p=0}^{2n-1}\cos\left(\frac{p\pi}{2n}-\phi\right)^{2n}. \tag{4.6}$$

We can expand $\cos(x)^{2n}$ [Wei17] as follows:

$$\cos(x)^{2n} = \frac{1}{2^{2n}}\binom{2n}{n}+\frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n}{k}\cos(2(n-k)x)$$

so we can write

$$\sum_{p=0}^{2n-1}\cos\left(\frac{p\pi}{2n}-\phi\right)^{2n} = \sum_{p=0}^{2n-1}\left(\frac{1}{2^{2n}}\binom{2n}{n}+\frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}\binom{2n}{k}\cos\left(2(n-k)\left(\frac{p\pi}{2n}-\phi\right)\right)\right).$$

We can check that terms involving ϕ cancel to 0:

$$\sum_{p=0}^{2n-1}\sum_{k=0}^{n-1}\binom{2n}{k}\cos\left(2(n-k)\left(\frac{p\pi}{2n}-\phi\right)\right) = \sum_{k=0}^{n-1}\binom{2n}{k}\sum_{p=0}^{2n-1}\cos\left(2(n-k)\left(\frac{p\pi}{2n}-\phi\right)\right) \tag{4.7}$$

$$= \sum_{k=0}^{n-1}\binom{2n}{k}\Re\left(\sum_{p=0}^{2n-1}\exp\left(2i(n-k)\left(\frac{p\pi}{2n}-\phi\right)\right)\right) \tag{4.8}$$

$$= \sum_{k=0}^{n-1}\binom{2n}{k}\Re\left(\exp(-2i(n-k)\phi)\sum_{p=0}^{2n-1}\exp\left(i(n-k)\frac{2\pi}{2n}\right)^{p}\right) \tag{4.9}$$

$$= \sum_{k=0}^{n-1}\binom{2n}{k}\Re\left(\exp(-2i(n-k)\phi)\,\frac{1-\exp(i(n-k)2\pi)}{1-\exp\left(i(n-k)\frac{2\pi}{2n}\right)}\right) \tag{4.10}$$

$$= 0. \tag{4.11}$$

Finally, we get

$$\frac{2n+1}{8n\pi}\sum_{p=0}^{2n-1}\left(\begin{bmatrix}\cos\left(\frac{p\pi}{2n}\right)\\ \sin\left(\frac{p\pi}{2n}\right)\\ 0\end{bmatrix}\cdot\begin{bmatrix}\cos(\phi)\sin(\theta)\\ \sin(\phi)\sin(\theta)\\ \cos(\theta)\end{bmatrix}\right)^{2n} = \frac{2n+1}{8n\pi}\sin(\theta)^{2n}\sum_{p=0}^{2n-1}\frac{1}{2^{2n}}\binom{2n}{n} \tag{4.12}$$

$$= \sin(\theta)^{2n}\frac{2n+1}{8n\pi}\frac{2n}{2^{2n}}\binom{2n}{n} \tag{4.13}$$

$$= \sin(\theta)^{2n}\frac{2n+1}{\pi 4^{n+1}}\frac{(2n)!}{2n!} \tag{4.14}$$

$$= \sin(\theta)^{2n}\frac{\sqrt{\pi}(2n+2)!}{4^{n+1}(n+1)!}\frac{(n+1)}{(2n+2)\pi^{3/2}n!} \tag{4.15}$$

$$= \sin(\theta)^{2n}\frac{\Gamma(n+3/2)}{2\pi^{3/2}n!} \tag{4.16}$$

$$= D_{\sin}(\omega,\xi_D^{\sin},n). \tag{4.17}$$

4.5. Closed-form expression of the attenuation coefficient $\sigma_t(\omega)$

4.5.1. Derivation of general closed-from expressions for $\sigma_t(\omega)$

A general closed-form expression can be found for $\sigma_t(\omega)$ but this expression is not convenient for evaluation because it involves costly evaluations of Gauss hypergeometric functions ${}_2F_1$. For information, we provide a derivation of this expression for the cosine lobe (Eq. 4.41). For efficient implementation, we have pre-computed simpler closed-form expressions for each $n \in [1..20]$ as explained in Section 4.5.2. Note that a general closed-form expression could also be derived for the sine lobe, using the fact that a sine lobe can be written as a sum of cosine lobes (Sec. 4.4).

In the case of a $\cos^{2n}$ distribution, $\sigma_t(\omega)$ is given by

$$\sigma_t(\omega) = \rho A(\omega) \int D_{\cos}(m, \xi_D, n) \langle m \cdot \omega \rangle \, \mathrm{d}m = \rho \frac{A(\omega)}{N_{\cos}(n)} \int (m \cdot \xi_D)^{2n} \langle m \cdot \omega \rangle \, \mathrm{d}m.$$

Without loss of generality, we choose a convenient spherical parametrization such that $\omega = [\cos(\phi_i), \sin(\phi_i), 0]$ and $\xi_D = [1,0,0]$. We have

$$\sigma_t(\omega) = \rho \frac{A(\omega)}{N_{\cos}(n)} \int \left(\begin{bmatrix} \cos(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)^{2n} \left| \begin{bmatrix} \cos(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\theta) \end{bmatrix} \cdot \begin{bmatrix} \cos(\phi_i) \\ \sin(\phi_i) \\ 0 \end{bmatrix} \right| \sin(\theta) \, \mathrm{d}\phi \, \mathrm{d}\theta \tag{4.18}$$

$$= \rho \frac{A(\omega)}{N_{\cos}(n)} \int_{\phi=-\pi/2-\phi_i}^{\pi/2-\phi_i} \int_{\theta=0}^{\pi} (\cos(\phi)\sin(\theta))^{2n} (\cos(\phi)\sin(\theta)\cos(\phi_i) + \sin(\phi)\sin(\theta)\sin(\phi_i)) \sin(\theta) \, \mathrm{d}\phi \, \mathrm{d}\theta \tag{4.19}$$

$$= \rho \frac{A(\omega)}{N_{\cos}(n)} \int_{\phi=-\pi/2-\phi_i}^{\pi/2-\phi_i} \int_{\theta=0}^{\pi} \cos(\phi)^{2n} (\cos(\phi)\cos(\phi_i) + \sin(\phi)\sin(\phi_i)) \sin(\theta)^{2n+2} \, \mathrm{d}\phi \, \mathrm{d}\theta \tag{4.20}$$

$$= \rho \frac{A(\omega)}{N_{\cos}(n)} \frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+n\right)}{\Gamma(2+n)} \int_{\phi=-\pi/2-\phi_i}^{\pi/2-\phi_i} \cos(\phi)^{2n} (\cos(\phi)\cos(\phi_i) + \sin(\phi)\sin(\phi_i)) \, \mathrm{d}\phi \tag{4.21}$$

$$= \rho \frac{A(\omega)}{N_{\cos}(n)} \frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+n\right)}{\Gamma(2+n)} \left(\cos(\phi_i) \int_{\phi=-\pi/2-\phi_i}^{\pi/2-\phi_i} \cos(\phi)^{2n+1} \, \mathrm{d}\phi + \sin(\phi_i) \int_{\phi=-\pi/2-\phi_i}^{\pi/2-\phi_i} \cos(\phi)^{2n} \sin(\phi) \, \mathrm{d}\phi \right) \tag{4.22}$$

$$= \rho \frac{A(\omega)}{N_{\cos}(n)} \frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+n\right)}{\Gamma(2+n)} \left(\cos(\phi_i) \int_{\phi=-\pi/2-\phi_i}^{\pi/2-\phi_i} \cos(\phi)^{2n+1} \, \mathrm{d}\phi - \frac{2\sin(\phi_i)^{2n+2}}{2n+1} \right). \tag{4.23}$$

Now integral $\int \cos(x)^m \, \mathrm{d}x$ can be written as a finite sum using

$$\int \cos(x)^m \, \mathrm{d}x = \frac{1}{m} \cos(x)^{m-1} \sin(x) + \frac{m+1}{m} \int \cos(x)^{m-2} \, \mathrm{d}x.$$

It is also possible to write it with the Gauss hypergeometric function. Given that $0 \leq \phi_i \leq \pi/2$, on the interval $(-\frac{\pi}{2} - \phi_i, -\frac{\pi}{2})$ we perform a change of variable with $X = \cos(\phi)$, and get

$$\int_{\phi=-\pi/2-\phi_i}^{-\pi/2} \cos(\phi)^{2n+1} \, \mathrm{d}\phi = \int_{X=\cos(-\pi/2-\phi_i)}^{0} \frac{X^{2n+1}}{\sqrt{1-X^2}} \, \mathrm{d}X \tag{4.24}$$

$$= \int_{X=-\sin(\phi_i)}^{0} \frac{X^{2n+1}}{\sqrt{1-X^2}} \, \mathrm{d}X. \tag{4.25}$$

Using the change of variable $Y = \frac{X^2}{-\sin(\phi_i)}$, with $\left\|\frac{\partial X}{\partial Y}\right\| = \frac{\sin(\phi_i)}{2\sqrt{Y}}$ we get

$$\int_{X=-\sin(\phi_i)}^{0} \frac{X^{2n+1}}{\sqrt{1-X^2}}\ \mathrm{d}X = \int_{0}^{1} \frac{Y^{n+\frac{1}{2}}\sin(\phi_i)^{2n+1}}{\sqrt{1-Y\sin(\phi_i)^2}}\frac{\sin(\phi_i)}{2\sqrt{Y}}\ \mathrm{d}Y \tag{4.26}$$

$$= -\frac{\sin(\phi_i)^{2n+2}}{2}\int_{1}^{0}\frac{Y^n}{\sqrt{1-Y\sin(\phi_i)^2}}\ \mathrm{d}Y. \tag{4.27}$$

Using

$$\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b,c,z) = \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\ \mathrm{d}t$$

we can write

$$-\frac{\sin(\phi_i)^{2n+2}}{2}\int_{1}^{0}\frac{Y^{n+\frac{1}{2}}}{\sqrt{1-Y\sin(\phi_i)^2}}\ \mathrm{d}Y = -\frac{\sin(\phi_i)^{2n+2}}{2}\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\ {}_2F_1(a,b,c,z) \tag{4.28}$$

with $a = \frac{1}{2}$, $b = n+1$, $c = n+2$ and $z = \sin(\phi_i)^2$. We have

$$\int_{\phi=-\pi/2-\phi_i}^{-\pi/2} \cos(\phi)^{2n+1}\ \mathrm{d}\phi = -\frac{\sin(\phi_i)^{2n+2}}{2}\frac{\Gamma(n+1)\Gamma(1)}{\Gamma(n+2)}\ {}_2F_1\left(\frac{1}{2},n+1,n+2,\sin(\phi_i)^2\right) \tag{4.29}$$

$$= -\frac{\sin(\phi_i)^{2n+2}}{2n+2}\ {}_2F_1\left(\frac{1}{2},n+1,n+2,\sin(\phi_i)^2\right). \tag{4.30}$$

Now, we integrate between $-\pi/2$ and 0:

$$\int_{\phi=-\pi/2}^{0} \cos(\phi)^{2n+1}\ \mathrm{d}\phi = \int_0^1 \frac{X^{2n+1}}{\sqrt{1-X^2}}\ \mathrm{d}X \tag{4.31}$$

$$= \int_0^1 \frac{Y^{n+\frac{1}{2}}}{\sqrt{1-Y}}\frac{1}{2\sqrt{Y}}\ \mathrm{d}Y \tag{4.32}$$

$$= \frac{1}{2}\int_0^1 \frac{Y^n}{\sqrt{1-Y}}\ \mathrm{d}Y \tag{4.33}$$

$$= \frac{\Gamma(b)\Gamma(c-b)}{2\Gamma(c)}\ {}_2F_1(a,b,c,1) \tag{4.34}$$

$$\tag{4.35}$$

with $a = \frac{1}{2}$, $b = n+1$, $c = n+2$. We have

$$\int_{\phi=-\pi/2}^{0} \cos(\phi)^{2n+1}\ \mathrm{d}\phi = \frac{1}{2}B\left(\frac{1}{2},n+1\right). \tag{4.36}$$

Finally, using the same approach,

$$\int_{\phi=0}^{\pi/2-\phi_i} \cos(\phi)^{2n+1}\ \mathrm{d}\phi = \int_{\phi=0}^{\pi/2} \cos(\phi)^{2n+1}\ \mathrm{d}\phi + \int_{\phi=\pi/2}^{\pi/2-\phi_i} \cos(\phi)^{2n+1}\ \mathrm{d}\phi \tag{4.37}$$

$$= \int_{1}^{0} \frac{X^{2n+1}}{-\sqrt{1-X^2}}\ \mathrm{d}X + \int_{0}^{\cos(\pi/2-\phi_i)} \frac{X^{2n+1}}{-\sqrt{1-X^2}}\ \mathrm{d}X \tag{4.38}$$

$$= \frac{1}{2n+2}B\left(\frac{1}{2},n+1\right) - \frac{\sin(\phi_i)^{2n+2}}{2n+2}\ {}_2F_1\left(\frac{1}{2},n+1,n+2,\sin(\phi_i)^2\right). \tag{4.39}$$

We can add the three terms and get

$$\int_{\phi=-\pi/2-\phi_i}^{\pi/2-\phi_i} \cos(\phi)^{2n+1}\,\mathrm{d}\phi = B\left(\frac{1}{2},n+1\right) - \frac{\sin(\phi_i)^{2n+2}}{n+1}\,{}_2F_1\left(\frac{1}{2},n+1,n+2,\sin(\phi_i)^2\right). \tag{4.40}$$

We proved that

$$\boxed{\sigma_t(\omega) = \rho\frac{A(\omega)}{N_{\cos}(n)}\frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+n\right)}{\Gamma(2+n)}\left(\cos(\phi_i)\left(B\left(\frac{1}{2},n+1\right) - \frac{\sin(\phi_i)^{2n+2}}{n+1}\,{}_2F_1\left(\frac{1}{2},n+1,n+2,\sin(\phi_i)^2\right)\right) - \frac{2\sin(\phi_i)^{2n+2}}{2n+1}\right).} \tag{4.41}$$

As explained before, we don't use this expression in our implementation because more efficient expressions can be found for each particular value of n, when n is small.

4.5.2. Pre-computing simple closed-form expressions for particular values n

Using symbolic integration [MGH*05], we have found that for each particular n, $\sigma_t^{\cos}(\omega)$ can be written

$$\sigma_t^{\cos}(\omega) = \rho A(\omega)\begin{bmatrix} 1 \\ (\omega\cdot\xi_D)^2 \\ (\omega\cdot\xi_D)^4 \\ \cdots\cdot \\ (\omega\cdot\xi_D)^{2n} \end{bmatrix}\cdot C_1$$

with C_1 a $n+1$ vector of coefficients. The evaluation cost is linear in n and this expression is very efficient when n is small. In practice we pre-computed coefficient up to $n=20$ (corresponding to a $\cos^{40}$ lobe). We have found similar expressions for sine lobes. These coefficients can be found in our code (files *matrixcoslobe.h* and *matrixsinlobe.h*).

4.6. Single scattering coeff σ_{ss} for $\cos^n$ and $\sin^n$ microflake distributions

General closed-form expressions for σ_{ss} are even more complex than expressions for σ_t (Eq. 4.41) and have no practical interest for rendering. Like we did for σ_t, we pre-computed simpler and exact closed-form solutions for each n up to $n=20$. We have found using symbolic integration that

$$\int(\xi_A\cdot\omega')^2 D_{\cos}(m,\xi_D,n)\langle m\cdot\omega\rangle\,\mathrm{d}m = \begin{bmatrix} 1 \\ (\omega\cdot\xi_D)^2 \\ (\omega\cdot\xi_D)^4 \\ \cdots\cdot \\ (\omega\cdot\xi_D)^{2n} \end{bmatrix}\cdot C_2\cdot\begin{bmatrix} (\omega\cdot\xi_A)^2 \\ (\xi_A\cdot\xi_D)^2 \\ 1 \end{bmatrix} + (\xi_A\cdot\xi_D)(\xi_A\cdot\omega)\begin{bmatrix} (\omega\cdot\xi_D) \\ (\omega\cdot\xi_D)^3 \\ \cdots\cdot\cdot \\ (\omega\cdot\xi_D)^{2n-1} \end{bmatrix}\cdot C_3 \tag{4.42}$$

with $\omega' = -\omega + 2m(\omega\cdot m)$, C_2 a $(n+1)\times 3$ matrix of coefficients and C_3 a vector of coefficients of length n. Given Equation 4.1, we sum contributions of the three shadowing axes and get

$$\int A(\omega')D_{\cos}(m,\xi_D,n)\langle m\cdot\omega\rangle\,\mathrm{d}m = \begin{bmatrix} 1 \\ (\omega\cdot\xi_D)^2 \\ (\omega\cdot\xi_D)^4 \\ \cdots\cdot \\ (\omega\cdot\xi_D)^{2n} \end{bmatrix}\cdot C_2\cdot\begin{bmatrix} a_1(\omega\cdot\xi_{A1})^2 + a_2(\omega\cdot\xi_{A2})^2 + a_3(\omega\cdot\xi_{A3})^2 \\ a_1(\xi_{A1}\cdot\xi_D)^2 + a_2(\xi_{A2}\cdot\xi_D)^2 + a_3(\xi_{A3}\cdot\xi_D)^2 \\ a_1+a_2+a_3 \end{bmatrix} \tag{4.43}$$

$$+\left(a_1(\xi_{A1}\cdot\xi_D)(\xi_{A1}\cdot\omega) + a_2(\xi_{A2}\cdot\xi_D)(\xi_{A2}\cdot\omega) + a_3(\xi_{A3}\cdot\xi_D)(\xi_{A3}\cdot\omega)\right)\begin{bmatrix} (\omega\cdot\xi_D) \\ (\omega\cdot\xi_D)^3 \\ \cdots\cdot\cdot \\ (\omega\cdot\xi_D)^{2n-1} \end{bmatrix}\cdot C_3 \tag{4.44}$$

We can write

$$a_1(\omega\cdot\xi_{A1})^2 + a_2(\omega\cdot\xi_{A2})^2 + a_3(\omega\cdot\xi_{A3})^2 = \omega^T S_A\,\omega$$

$$a_1(\xi_{A1}\cdot\xi_D)^2 + a_2(\xi_{A2}\cdot\xi_D)^2 + a_3(\xi_{A3}\cdot\xi_D)^2 = \xi_D^T S_A\,\xi_D$$

$$a_1(\xi_{A1}\cdot\xi_D)(\xi_{A1}\cdot\omega) + a_2(\xi_{A2}\cdot\xi_D)(\xi_{A2}\cdot\omega) + a_3(\xi_{A3}\cdot\xi_D)(\xi_{A3}\cdot\omega) = \omega^T S_A\,\xi_D$$

$$a_1 + a_2 + a_3 = \mathrm{tr}(A)$$

Finally we get, for a cosine lobe $D_{\cos}$,

$$\sigma_{ss}(\omega) = \alpha_{ss}(\lambda)\rho A(\omega) \begin{bmatrix} 1 \\ (\omega\cdot\xi_D)^2 \\ (\omega\cdot\xi_D)^4 \\ \cdots \\ (\omega\cdot\xi_D)^{2n} \end{bmatrix} \cdot C_2 \cdot \begin{bmatrix} \omega^T S_A\ \omega \\ \xi_D^T\ S_A\ \xi_D \\ \mathrm{trace}(A) \end{bmatrix} + \alpha_{ss}(\lambda)\rho A(\omega)\left(\omega^T S_A\ \xi_D\right) \begin{bmatrix} (\omega\cdot\xi_D) \\ (\omega\cdot\xi_D)^3 \\ \cdots\cdot \\ (\omega\cdot\xi_D)^{2n-1} \end{bmatrix} \cdot C_3 \tag{4.45}$$

Expressions in the case of sine lobes $D_{\sin}$ are similar, only coefficients in C_2 and C_3 are different. All coefficients can be found in our code in files *matrixcoslobe.h* and *matrixsinlobe.h*.

4.7. Closed-form expression for $\int \sigma_t$ and $\int \sigma_{ss}$

Integrals $\int \sigma_t(\omega)\ \mathrm{d}\omega$ and $\int \sigma_{ss}(\omega)\ \mathrm{d}\omega$ are involved in our multiple scattering phase function (Eq. 2.12). We provide here derivations of their closed-from expressions.

4.7.1. Closed-form expression of $\int \sigma_t(\omega)$ for cosine lobes $D_{\cos}(m,\xi_D,n)$

We want to find a closed-form expression for a cosine distribution:

$$\int \sigma_t(\omega)\ \mathrm{d}\omega = \rho \int A(\omega) \int D_{\cos}(m,\xi_D,n)\langle m\cdot\omega\rangle\ \mathrm{d}m\ \mathrm{d}\omega. \tag{4.46}$$

We first invert integrals:

$$\int A(\omega) \int D_{\cos}(m,\xi_D,n)\langle m\cdot\omega\rangle\ \mathrm{d}m\ \mathrm{d}\omega = \int D_{\cos}(m,\xi_D,n) \int A(\omega)\langle m\cdot\omega\rangle\ \mathrm{d}\omega\ \mathrm{d}m. \tag{4.47}$$

We can decompose our shadowing function (Eq. 4.1):

$$\int A(\omega)\langle m\cdot\omega\rangle\ \mathrm{d}\omega = \sum a_i \int (\omega\cdot\xi_{A_i})^2\langle m\cdot\omega\rangle\ \mathrm{d}\omega. \tag{4.48}$$

We choose a convenient parametrization: we take $m = [1,0,0]^T$ and $\xi_{A_i} = [\sin(\theta_A), 0, \cos(\theta_A)]^T$, so that we can write

$$\int (\omega\cdot\xi_{A_i})^2\langle m\cdot\omega\rangle\ \mathrm{d}\omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} (\sin(\theta_A)\cos(\phi)\sin(\theta)+\cos(\theta_A)\cos(\theta))^2\cos(\theta)\sin(\theta)\ \mathrm{d}\phi\ \mathrm{d}\theta = \frac{\pi}{4}\left(\cos(\theta_A)^2+1\right) = \frac{\pi}{4}\left((\xi_{A_i}\cdot m)^2+1\right). \tag{4.49}$$

We can now substitute this expression in Eq. 4.47 and get

$$\int D_{\cos}(m,\xi_D,n)\frac{\pi}{4}\left((\xi_{A_i}\cdot m)^2+1\right)\ \mathrm{d}m = \int \frac{2n+1}{4\pi}(\xi_D\cdot m)^{2n}\frac{\pi}{4}\left((\xi_{A_i}\cdot m)^2+1\right)\ \mathrm{d}m. \tag{4.50}$$

Choosing a parametrization such that $\xi_D = [0,0,1]^T$ and $\xi_{A_i} = [\sin(\theta_A), 0, \cos(\theta_A)]^T$, this writes

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{2n+1}{4\pi}\cos(\theta)^{2n}\frac{\pi}{4}\left((\sin(\theta_A)\cos(\phi)\sin(\theta)+\cos(\theta_A)\cos(\theta))^2+1\right)\sin(\theta)\ \mathrm{d}m = \frac{\pi\left(n(\xi_D\cdot\xi_{A_i})^2+n+2\right)}{4n+6}. \tag{4.51}$$

Finally, we can sum and factor the formula for the three axes, and we get

$$\int \sigma_t(\omega)\ \mathrm{d}\omega = \rho a_1\frac{\pi\left(n(\xi_D\cdot\xi_{A_1})^2+n+2\right)}{4n+6} + \rho a_2\frac{\pi\left(n(\xi_D\cdot\xi_{A_2})^2+n+2\right)}{4n+6} + \rho a_3\frac{\pi\left(n(\xi_D\cdot\xi_{A_3})^2+n+2\right)}{4n+6}. \tag{4.52}$$

Using Eq. 4.2, we can write, for a distribution $D_{\cos}(m,\xi_D,n)$:

$$\boxed{\int \sigma_t(\omega)\ \mathrm{d}\omega = \rho \int A(\omega) \int D_{\cos}(m,\xi_D,n)\langle m\cdot\omega\rangle\ \mathrm{d}m\ \mathrm{d}\omega = \rho\frac{\pi\left(n(\omega^T S_A\ \omega)+\mathrm{tr}(S_A)(n+2)\right)}{4n+6}} \tag{4.53}$$

4.7.2. Closed-form expression of $\int \sigma_t(\omega)$ for sine lobe $D_{\sin}(m,\xi_D,n)$

The derivation is almost the same as for the $D_{\cos}(m,\xi_D,n)$ in the previous subsection. We need to compute

$$\int D_{\sin}(m,\xi_D,n)\frac{\pi}{4}\left((\xi_{A_i}\cdot m)^2+1\right)\,\mathrm{d}m = \int \frac{2\pi^{\frac{3}{2}}\Gamma(1+n)}{\Gamma\left(\frac{3}{2}+n\right)}\left(1-(\xi_D\cdot m)^2\right)^n\frac{\pi}{4}\left((\xi_{A_i}\cdot m)^2+1\right)\,\mathrm{d}m. \tag{4.54}$$

Choosing the same convenient frame in which $\xi_D=[0,0,1]^T$ and $\xi_{A_i}=[\sin(\theta_A),0,\cos(\theta_A)]^T$, this writes

$$2\times\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\frac{\pi}{4}\left((\sin(\theta_A)\cos(\phi)\sin(\theta)+\cos(\theta_A)\cos(\theta))^2+1\right)\sin(\theta)^{2n+1}\,\mathrm{d}m = \frac{\pi\left(3n+4-n\,(\xi_D\cdot\xi_{A_i})^2\right)}{8n+12}. \tag{4.55}$$

Again, we have to sum this for each shadowing axis. For a $D_{\sin}(m,\xi_D,n)$ distribution, we get:

$$\boxed{\int\sigma_t(\omega)\,\mathrm{d}\omega = \rho\int A(\omega)\int D_{\sin}(m,\xi_D,n)\langle m\cdot\omega\rangle\,\mathrm{d}m\,\mathrm{d}\omega = \rho\frac{\pi\left((3n+4)\,\mathrm{tr}(S_A)-n(\omega^T S_A\,\omega)\right)}{8n+12}} \tag{4.56}$$

4.7.3. Closed-form expression of $\int \sigma_{ss}(\omega)$ for cosine lobe $D_{\cos}(m,\xi_D,n)$

In the case of a distribution $D_{\cos}(m,\xi_D,n)$, we look for a closed-form expression of the integral

$$\int\sigma_{ss}(\omega)\,\mathrm{d}\omega = \rho\alpha_{ss}(\lambda)\int A(\omega)\int A(\omega')D_{\cos}(m,\xi_D,n)\langle\omega\cdot m\rangle\,\mathrm{d}m\,\mathrm{d}\omega, \tag{4.57}$$

with $\omega'=-\omega+2m(\omega\cdot m)$. We invert integrals and get

$$\int D_{\cos}(m,\xi_D,n)\int A(\omega)A(\omega')\langle\omega\cdot m\rangle\,\mathrm{d}\omega\,\mathrm{d}m. \tag{4.58}$$

We expand each shadowing term using Equation 4.1 and get 9 terms of the form:

$$\int D_{\cos}(m,\xi_D,n)\int A(\omega)A(\omega')\langle\omega\cdot m\rangle\,\mathrm{d}\omega\,\mathrm{d}m = \sum_{i,j\in\{1,2,3\}}\int D_{\cos}(m,\xi_D,n)a_ia_j\int(\omega\cdot\xi_{Ai})^2(\omega'\cdot\xi_{Aj})^2\langle\omega\cdot m\rangle\,\mathrm{d}\omega. \tag{4.59}$$

We first consider the case $i=j$, and then the case $i\neq j$.

4.7.3.1. Case $i=j$

Without loss of generality, we work in the reference frame in which $m=[1,0,0]$ and $\xi_{Ai}=[\cos(\phi_{Ai}),\sin(\phi_{Ai}),0]$. We find

$$\begin{aligned}
&\int(\omega\cdot\xi_{Ai})^2((-\omega+2m(\omega\cdot m))\cdot\xi_{Ai})^2\langle\omega\cdot m\rangle\,\mathrm{d}\omega && (4.60)\\
&=\int_{\phi=-\pi/2}^{\pi/2}\int_{\theta=0}^{\pi}(\cos(\phi_{Ai})\cos(\phi)+\sin(\phi_{Ai})\sin(\phi))^2(\cos(\phi_{Ai})\cos(\phi)-\sin(\phi_{Ai})\sin(\phi))^2\cos(\phi)\sin(\theta)^6\,\mathrm{d}\omega\,\mathrm{d}\theta && (4.61)\\
&=\frac{1}{24}\pi\left(15(m\cdot\xi_{Ai})^4-10(m\cdot\xi_{Ai})^2+3\right). && (4.62)
\end{aligned}$$

We integrate this result with $D_{\cos}(m,\xi_D,n)$ (recall that we want a closed-form expression for Equation 4.57) and find

$$\int D_{\cos}(m,\xi_D,n)\frac{1}{24}\pi\left(15(m\cdot\xi_{Ai})^4-10(m\cdot\xi_{Ai})^2+3\right)\,\mathrm{d}m = \pi\frac{15n(n-1)\,(\xi_D\cdot\xi_{Ai})^4-10n(n-2)\,(\xi_D\cdot\xi_{Ai})^2+3n^2+7n+10}{24n^2+96n+90}. \tag{4.63}$$

4.7.3.2. Case $i\neq j$

We choose a reference frame such that $m=[0,0,1]^T$, $A_i=[\sin(\theta_{Ai}),0,\cos(\theta_{Ai})]^T$ and $A_j=[\cos(\phi_{Aj})\sin(\theta_{Aj}),\sin(\phi_{Aj})\sin(\theta_{Aj}),\cos(\theta_{Aj})]^T$. We have

$$\xi_{Ai}\perp\xi_{Aj}\Rightarrow\sin(\theta_{Ai})\cos(\phi_{Aj})\sin(\theta_{Aj})+\cos(\theta_{Aj})\cos(\theta_{Ai})=0.$$

We develop and integrate as before, and find

$$\int(\omega\cdot\xi_{Ai})^2((-\omega+2m(\omega\cdot m))\cdot\xi_{Aj})^2\langle\omega\cdot m\rangle\,\mathrm{d}\omega = \frac{\pi}{24}\left(15(m\cdot\xi_{Ai})^2(m\cdot\xi_{Aj})^2+(m\cdot\xi_{Ai})^2+(m\cdot\xi_{Aj})^2+1\right). \tag{4.64}$$

Finally, we get

$$\int D_{\cos}(m,\xi_D,n)\frac{\pi}{24}\left(15(m\cdot\xi_{Ai})^2(m\cdot\xi_{Aj})^2+(m\cdot\xi_{Ai})^2+(m\cdot\xi_{Aj})^2+1\right)\,\mathrm{d}m=\pi\frac{15C_1C_2n(n-1)+(n^2+10n)(C_1+C_2)+n^2+5n+10}{24n^2+96n+90} \tag{4.65}$$

with $C_1=(\xi_{Ai}\cdot\xi_D)^2$ and $C_2=(\xi_{Aj}\cdot\xi_D)^2$.

4.7.3.3. Sum of all terms

Given Equation 4.59, 4.63 and 4.65, we can write the closed-form expression of $\int\sigma_{ss}(\omega)$ in the case of a cosine lobe $D_{\cos}(m,\xi_D,n)$. After simplification, we get:

$$\boxed{\begin{aligned}\int\sigma_{ss}(\omega)\,\mathrm{d}\omega=\frac{\pi\rho\alpha_{ss}(\lambda)}{24n^2+96n+90}\Big(&15n(n-1)\left(\xi_D^TS_A\xi_D\right)^2\\&-10n(n-2)\xi_D^TS_AS_A\xi_D+2n(n+10)\left(\xi_D^TS_A\xi_D\ \mathrm{tr}(S_A)-\xi_D^TS_AS_A\xi_D\right)\\&+(3n^2+7n+10)\,\mathrm{tr}(S_AS_A)+(n^2+5n+10)\left(\mathrm{tr}(S_A)^2-\mathrm{tr}(S_AS_A)\right)\Big)\end{aligned}} \tag{4.66}$$

4.7.4. Closed-form expression of $\int\sigma_{ss}(\omega)$ for sine lobe $D_{\sin}(m,\xi_D,n)$

The derivation is almost the same as for cosine lobes.

4.7.4.1. Case $i=j$

We have shown (Sec. 4.7.3.1) that

$$\int(\omega\cdot\xi_{A_i})^2((-\omega+2m(\omega\cdot m))\cdot\xi_{Ai})^2\langle\omega\cdot m\rangle\,\mathrm{d}\omega=\frac{\pi}{24}\left(15(m\cdot\xi_{Ai})^4-10(m\cdot\xi_{Ai})^2+3\right) \tag{4.67}$$

Then

$$\int D_{\sin}(m,\xi_D,n)\frac{\pi}{24}\left(15(m\cdot\xi_{Ai})^4-10(m\cdot\xi_{Ai})^2+3\right)\,\mathrm{d}m=\pi\frac{(45n^2-45n)(\xi_D\cdot\xi_{Ai})^4+(-50n^2+10n)(\xi_D\cdot\xi_{Ai})^2+29n^2+91n+80}{192n^2+768n+720} \tag{4.68}$$

4.7.4.2. Case $i\neq j$

We have found previously (Sec. 4.7.3.2) that

$$\int(\omega\cdot\xi_{Ai})^2((-\omega+2m(\omega\cdot m))\cdot\xi_{Aj})^2\langle\omega\cdot m\rangle\,\mathrm{d}\omega=\frac{1}{24}\pi\left(15(m\cdot\xi_{Ai})^2(m\cdot\xi_{Aj})^2+(m\cdot\xi_{Ai})^2+(m\cdot\xi_{Aj})^2+1\right) \tag{4.69}$$

We get

$$\int D_{\sin}(m,\xi_D,n)\frac{\pi}{24}\left(15(m\cdot\xi_{Ai})^2(m\cdot\xi_{Aj})^2+(m\cdot\xi_{Ai})^2+(m\cdot\xi_{Aj})^2+1\right)\mathrm{d}m=\pi\frac{45C_1C_2n(n-1)-(C_1+C_2)n(19n+25)+31n^2+105n+80}{192n^2+768n+720} \tag{4.70}$$

With $C_1=(\xi_{A1}\cdot\xi_D)^2$ and $C_2=(\xi_{A2}\cdot\xi_D)^2$

4.7.4.3. Sum of all terms

Again, we sum terms from cases $i=j$ and cases $i\neq j$, and after simplification we get the closed-form expression of $\int\sigma_{ss}$ for sine lobes $D_{\sin}(m,\xi_D,n)$:

$$\boxed{\begin{aligned}\int\sigma_{ss}(\omega)\,\mathrm{d}\omega=\frac{\pi\rho\alpha_{ss}(\lambda)}{192n^2+768n+720}\Big(&45n(n-1)\left(\xi_D^TS_A\xi_D\right)^2+10n(1-5n)\xi_D^TS_AS_A\xi_D\\&-2n(19n+25)\left(\xi_D^TS_A\xi_D\,\mathrm{tr}(S_A)-\xi_D^TS_AS_A\xi_D\right)\\&+(29n^2+91n+80)\,\mathrm{tr}(S_AS_A)+(31n^2+105n+80)\left(\mathrm{tr}(S_A)^2-\mathrm{tr}(S_AS_A)\right)\Big)\end{aligned}} \tag{4.71}$$

4.8. Sampling the distribution of visible normals for a cosine lobe $D_{\cos}(m, \xi_D, n)$

We address here the problem of sampling the distribution of visible microflake normals [HDCD15] given that the microflake distribution is a cosine lobe $D_{\cos}(m, \xi_D, n)$. This sampling procedure is useful when using trigonometric lobes in the standard microflake model or in the simplified self-shadowing model. Indeed, sampling visible normal distributions is actually the same problem as sampling the specular phase function. We want to sample the probability density function

$$D_{\omega_i}(\omega_o) = \frac{D_{\cos}(\omega_h, \xi_D, n)}{4 \int D_{\cos}(m, \xi_D, n) \langle \omega_i \cdot m \rangle \, \mathrm{d}m} \tag{4.72}$$

with $\omega_h = \dfrac{\omega_i + \omega_o}{\|\omega_i + \omega_o\|}$. It is more convenient to work in the space of microflake normals, and to sample normals instead of outgoing directions. Using the Jacobian of the transformation from microflake normals to outgoing directions [WMLT07], the *pdf* for sampling microflake normals writes

$$D_{\omega_i}(m) = \frac{D_{\cos}(m, \xi_D, n) \langle \omega_i \cdot m \rangle}{\int D_{\cos}(m, \xi_D, n) \langle \omega_i \cdot m \rangle \, \mathrm{d}m}. \tag{4.73}$$

We could not find a procedure for sampling directly this function. We propose a sampling procedure based on rejection sampling (Sec. 3).

4.8.1. Using rejection sampling for sampling visible normals of $D_{\cos}(m, \xi_D, n)$

We have found that it is possible to sample the function

$$\frac{D_{\cos}(m, \xi_D, n)(\omega_i \cdot m)^2}{\int D_{\cos}(m, \xi_D, n)(\omega_i \cdot m)^2 \, \mathrm{d}m}$$

which is relatively similar to $D_{\omega_i}(m)$ (Eq. 4.73) on the hemisphere for which $D_{\omega_i}(m)$ is not null. We propose a rejection sampling method based on this function. We introduce the normalized *pdf*

$$D^s_{\omega_i}(m) = \frac{D_{\cos}(m, \xi_D, n)((\omega_1 \cdot m)^2 + 1/4)}{\int D_{\cos}(m, \xi_D, n)((\omega_1 \cdot m)^2 + 1/4) \, \mathrm{d}m}$$

because we have

$$|\omega_1 \cdot m| \leq (\omega_1 \cdot m)^2 + 1/4$$

so that we can write

$$\frac{D_{\cos}(m, \xi_D, n)|\omega_i \cdot m|}{\int D_{\cos}(m, \xi_D, n)|\omega_i \cdot m| \, \mathrm{d}m} \leq M D^s_{\omega_i}(m)$$

with

$$M = \frac{\int D_{\cos}(m, \xi_D, n)((\omega_1 \cdot m)^2 + 1/4) \, \mathrm{d}m}{\int D_{\cos}(m, \xi_D, n)|m \cdot \omega_i| \, \mathrm{d}m}.$$

Using rejection sampling, we can generate samples from the *pdf*

$$\frac{D_{\cos}(m, \xi_D, n)|\omega_i \cdot m|}{\int D_{\cos}(m, \xi_D, n)|\omega_i \cdot m| \, \mathrm{d}m}$$

by generating samples from $D^s_{\omega_i}(m)$ and accepting samples with probability

$$\frac{D_{\cos}(m, \xi_D, n)|\omega_i \cdot m|}{\int D_{\cos}(m, \xi_D, n)|\omega_i \cdot m| \, \mathrm{d}m} \frac{1}{M D^s_{\omega_i}(m)} = \frac{|\omega_1 \cdot m|}{(\omega_1 \cdot m)^2 + 1/4}.$$

Once we get a sample m, we invert its direction if $\omega_i \cdot m < 0$ and get a sample for D_{ω_i}.

4.8.2. Sampling $D^s_{\omega_i}(m)$

We show here how to sample the *pdf*

$$D^s_{\omega_i}(m) = \frac{D_{\cos}(m, \xi_D, n)((\omega_1 \cdot m)^2 + 1/4)}{\int D_{\cos}(m, \xi_D, n)((\omega_1 \cdot m)^2 + 1/4) \, \mathrm{d}m}.$$

We choose a reference frame such that $\xi_D = [0, 0, 1]^T$ and $\omega = [\sin(t_i), 0, \cos(t_i)]^T$, and find

$$D^s_{\omega_i}(\phi, \theta) = \cos(\theta)^{2n}((\cos(t_i)\cos(\phi)\sin(\theta) + \sin(t_i)\cos(\theta))^2 + 1/4) \frac{(2n+1)(2n+3)}{\pi(8n\cos(t_i)^2 + 2n + 7)}.$$

The marginal *pdf* for θ is given by

$$f_\theta(\theta) = \int D^s_{\omega_i}(\phi,\theta)\sin(\theta)\,\mathrm{d}\phi = \frac{(2n+3)(2n+1)\cos(\theta)^{2n}\sin(\theta)}{8\cos(ti)^2 n+2n+7}\left(\cos(\theta)^2\left(3\cos(t_i)^2-1\right)-\cos(t_i)^2+\frac{3}{2}\right).$$

For sampling this function, we use inverse transform sampling, meaning that we sample a random number $U\in(0,1)$ and solve for θ the equation

$$U = \int_0^\theta f_\theta(x)\,\mathrm{d}x.$$

We find that

$$\int_0^\theta f_\theta(x)\,\mathrm{d}x = \frac{-\cos(\theta)^{2n+1}}{8\cos(t_i)^2 n+2n+7}\left((2n+1)\left(3\cos(t_i)^2-1\right)\cos(\theta)^2+(2n+3)\left(-\cos(t_i)^2+\frac{3}{2}\right)\right).$$

In our implementation, we solve this function numerically using Brent's method, although other methods could also be used. Once θ has been sampled, ϕ must be sampled from the *pdf*

$$\frac{D^s_{\omega_i}(\phi,\theta)}{f_\theta(\theta)}.$$

This leads to the following equation for sampling ϕ:

$$V = \int_0^\phi \frac{D^s_{\omega_i}(x,\theta)}{f_\theta(\theta)}\,\mathrm{d}x = \frac{2\sin(t_i)^2\sin(\theta)^2(\cos(\phi)\sin(\phi)+\phi)+8\sin(\theta)\cos(\theta)\cos(t_i)\sin(t_i)\sin(\phi)+4\cos(\theta)^2\cos(t_i)^2\phi+\phi}{2\pi(6\cos(\theta)^2\cos(t_i)^2-2\cos(\theta)^2-2\cos(t_i)^2+3))}, \tag{4.74}$$

V being another random number in $(0,1)$. Again, we invert this equation numerically with Brent's method, and we obtain a sample $m = \begin{bmatrix}\cos(\phi)\sin(\theta)\\ \sin(\phi)\sin(\theta)\\ \cos(\theta)\end{bmatrix}$.

4.9. Sampling the distribution of visible normals for a cosine lobe $D_{\sin}(m,\xi_D,n)$

For sine lobes, we use the fact that $D_{\sin}(m,\xi_D,n)$ can be written as a sum of cosine lobes $D_{\cos}(m,\xi_D,n)$ (Sec. 4.4). We first sample one of the cosine lobes proportionally to their projected area in direction ω_i, and then sample a visible normal from this lobe using the procedure derived in the previous subsection (Sec. 4.8).

4.10. Sampling single scattering phase function f_{ss} for $D_{\cos}(m,\xi_D,n)$

In subsections 4.8 and 4.9, we have derived sampling procedures for the distribution of visible normals. Now, we derive similar sampling procedures for the single scattering phase function f_{ss}. The single scattering phase function is given by

$$f_{ss}(\omega_i\to\omega_o) = \frac{\rho\alpha_{ss}(\lambda)A(\omega_o)A(\omega_i)D(\omega_h)}{4\sigma_s(\omega_i)}.$$

We use $D^s_{\omega_i}$ (Eq. 4.8.1) in the space of outgoing directions

$$D^s_{\omega_i}(\omega_o) = \frac{D(\omega_h)((\omega_h\cdot\omega_i)^2+1/4)}{4|\omega_h\cdot\omega_i|\int D(\omega_h)((\omega_h\cdot\omega_i)^2+1/4)\,\mathrm{d}m}$$

and write

$$f_{ss}(\omega_i\to\omega_o) = \frac{\rho\alpha_{ss}(\lambda)A(\omega_o)A(\omega_i)D(\omega_h)}{4\sigma_s(\omega_i)} \tag{4.75}$$

$$= D^s_{\omega_i}(\omega_o)\frac{\rho\alpha_{ss}(\lambda)A(\omega_o)A(\omega_i)}{\sigma_s(\omega_i)}\frac{|\omega_h\cdot\omega_i|\int D(\omega_h)((\omega_h\cdot\omega_i)^2+1/4)\,\mathrm{d}m}{((\omega_h\cdot\omega_i)^2+1/4)} \tag{4.76}$$

$$\le D^s_{\omega_i}(\omega_o)\frac{\rho\alpha_{ss}(\lambda)A_{max}A(\omega_i)\int D(\omega_h)((\omega_h\cdot\omega_i)^2+1/4)\,\mathrm{d}m}{\sigma_s(\omega_i)} \tag{4.77}$$

$$\le D^s_{\omega_i}(\omega_o)M_A \tag{4.78}$$

with $A_{max} = \max_\omega\ A(\omega)$ and

$$M_A = \frac{A_{max}\int D(m)((m\cdot\omega_i)^2+1/4)dm}{\int A(-\omega_i+2m(m\cdot\omega_i))D(m)|m\cdot\omega_i|dm}.$$

Using this relation between $D^s_{\omega_i}$ and f_{ss}, with can generate samples from f_{ss} by generating samples from $D^s_{\omega_i}$ and accepting them with probability

$$\frac{f_{ss}(\omega_i \to \omega_o)}{MD^s_{\omega_i}(\omega)} = \frac{A(\omega_o)|\omega_h \cdot \omega_i|}{A_{max}((\omega_h \cdot \omega_i)^2 + 1/4)}. \tag{4.79}$$

4.11. Sampling single scattering phase function f_{ss} for $D_{\sin}(m, \xi_D, n)$

In the case of a sine distribution $D_{\sin}(m, \xi_D, n)$, we rely on the sampling procedure for visible normals proposed in subsection 4.9. This method generates samples from the *pdf*

$$D_{\omega_i}(\omega_o) = \frac{D(\omega_h)}{4\int D(m)|m \cdot \omega_i|\,\mathrm{d}m}.$$

We can write

$$f_{ss}(\omega_i \to \omega_o) = \frac{\rho\alpha_{ss}(\lambda)A(\omega_i)A(\omega_o)D(\omega_h)}{4\sigma_s(\omega_i)} \tag{4.80}$$

$$\leq D_{\omega_i}(\omega_o)\frac{\rho\alpha_{ss}(\lambda)A_{max}A(\omega_i)\int D(m)|m \cdot \omega_i|\,\mathrm{d}m}{\sigma_s(\omega_i)} \tag{4.81}$$

$$\tag{4.82}$$

so we can sample f_{ss} for a sine distribution $D_{\sin}(m, \xi_D, n)$ by generating samples ω_o from $D_{\omega_i}(\omega_o)$ and accepting them with probability

$$\frac{pdf_1(\omega_o)}{Mpdf_2(\omega_o)} = \frac{\rho\alpha_{ss}(\lambda)A(\omega_i)A(\omega_o)D(\omega_h)}{4\sigma_s(\omega_i)} \frac{4\int D(m)|m \cdot \omega_i|\,\mathrm{d}m}{D(\omega_h)} \frac{\sigma_s(\omega_i)}{\rho\alpha_{ss}(\lambda)A_{max}A(\omega_i)\int D(m)|m \cdot \omega_i|\,\mathrm{d}m} \tag{4.83}$$

$$= \frac{A(\omega_o)}{A_{max}}. \tag{4.84}$$

4.12. Sampling the multiple scattering phase function f_{ms}

Our multiple scattering phase function is given by

$$f_{ms}(\omega_o) = \frac{\sigma_t(\omega_o) - \sigma_s(\omega_o)/\alpha_{ss}(\lambda)}{\int \sigma_t(\omega) - \int \sigma_s(\omega)/\alpha_{ss}(\lambda)}. \tag{4.85}$$

We observe that in the case of isotropic self-shadowing, this phase function reduces to the *pdf*

$$\frac{A(1-A)\int D(m)\langle m \cdot \omega_o\rangle dm}{\int A \int D(m)\langle m \cdot \omega_o\rangle dm d\omega - \int A \int AD(m)\langle m \cdot \omega_o\rangle dm d\omega} = \frac{\int D(m)\langle m \cdot \omega_o\rangle dm}{\int\int D(m)\langle m \cdot \omega_o\rangle dm d\omega}$$

which is proportional the projected area of microflakes in direction ω_o. Given a single microflake with normal m, we can sample a direction ω proportionally to the projection area of the microflake $|m \cdot \omega|$ by sampling a cosine lobe around direction m. Similarly, we can sample the projected area of a cloud of microflakes by sampling first a normal from the distribution of normals D, and then by sampling a cosine lobe around this normal. In subsection 4.13 and 4.14, we show how to sample normals for $D_{\cos}$ and $D_{\sin}$.

We derive a rejection sampling procedure for f_{ms} based on this *pdf*:

$$\frac{\sigma_t(\omega_o) - \sigma_s(\omega_o)/\alpha_{ss}(\lambda)}{\int \sigma_t(\omega) - \int \sigma_s(\omega)/\alpha_{ss}(\lambda)} = \frac{A(\omega_o)\int(1 - A(-\omega_o + 2m(m \cdot \omega_o)))D(m)\langle m \cdot \omega_o\rangle dm}{\int A(\omega)\int(1 - A(-\omega + 2m(m \cdot \omega)))D(m)\langle m \cdot \omega\rangle\,\mathrm{d}m\,\mathrm{d}\omega} \tag{4.86}$$

$$< \frac{\int D(m)\langle m \cdot \omega_o\rangle dm}{\int\int D(m)\langle m \cdot \omega_o\rangle dm d\omega_o} \frac{A_{max}(1 - A_{min})\int\int D(m)\langle m \cdot \omega_o\rangle dm d\omega_o}{\int A(\omega)\int(1 - A(-\omega + 2m(m \cdot \omega)))D(m)\langle m \cdot \omega\rangle\,\mathrm{d}m\,\mathrm{d}\omega}. \tag{4.87}$$

We find the following probability of acceptance:

$$\frac{A(\omega)\int(1 - A(-\omega + 2m(m \cdot \omega)))D(m)\langle m \cdot \omega\rangle\,\mathrm{d}m}{A_{max}(1 - A_{min})\int D(m)\langle m \cdot \omega_o\rangle dm}. \tag{4.88}$$

This probability tends to one when self-shadowing is almost anisotropic, and tends to be smaller when A_{min} and A_{max} are very different, meaning that this sampling procedure is less efficient is such case.

4.13. Sampling a normal from $D_{\cos}$

The distribution

$$D_{\cos}(\omega, \xi, n) = (\omega \cdot \xi)^{2n}\frac{2n+1}{4\pi}$$

can be sampled easily in the reference frame in which $\xi = [0,0,1]$, using inverse transform sampling. We first sample the azimuth ϕ uniformly in $(0,2\pi)$. Then, the marginal distribution for θ is

$$\int_0^{2\pi} \cos(\theta)^{2n} \frac{2n+1}{4\pi} \sin(\theta)\, \mathrm{d}\phi = \frac{2n+1}{2} \cos(\theta)^{2n} \sin(\theta).$$

The cumulative distribution function writes

$$\int_0^{x} \frac{2n+1}{2} \cos(\theta)^{2n} \sin(\theta) d\theta = \frac{1}{2}\left(1 - \cos(x)^{2n+1}\right).$$

This function can be inverted:

$$U = \frac{1}{2}\left(1 - \cos(x)^{2n+1}\right) \Rightarrow \cos(x)^{2n+1} = 1 - 2U,$$

If $U \geq 1/2$:

$$x = \arccos\left((1-2U)^{\frac{1}{2n+1}}\right). \tag{4.89}$$

If $U \leq 1/2$:

$$x = \pi - \arccos\left((2U-1)^{\frac{1}{2n+1}}\right). \tag{4.90}$$

Therefore, θ can be sampled by sampling U in $(0,1)$ and then using Eq. 4.89 or 4.90. Finally, the sample

$$m = \begin{bmatrix} \cos(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

must be rotated back in the original reference frame of $D_{\cos}(\omega,\xi,n)$.

4.14. Sampling a normal from $D_{\sin}$

Again, we use the fact that $D_{\sin}(m,\xi_D,n)$ can be written as a sum of cosine lobes $D_{\cos}(m,\xi_D,n)$ (Sec. 4.4). We first sample one of the cosine lobes randomly and follow the procedure derived in the previous subsection (Sec. 4.13).

4.15. Sampling a cosine lobe

Sampling simple cosine lobes $\cos(\theta)$ is very common in path tracing, we recall here the procedure. Given a lobe axis m, sampling a cosine lobe around this direction can be done easily in the frame in which $m = [0,0,1]$. In this frame, the cosine lobe on the upper hemisphere in spherical coordinates writes

$$\frac{\cos(\theta)}{\pi} \mathbb{1}_{\theta\in(0,\frac{\pi}{2})}(\theta).$$

We first sample ϕ uniformly in $(0,2\pi)$. The marginal *pdf* for θ writes

$$\int_0^{2\pi} \frac{\cos(\theta)}{\pi} \sin(\theta)\, \mathrm{d}\phi = 2\cos(\theta)\sin(\theta).$$

Then the cumulative distribution function writes

$$\int_0^{x} 2\cos(\theta)\sin(\theta)\, \mathrm{d}\theta = \sin(x)^2.$$

So θ can be sampled using a random sample U in $(0,1)$ and the inverse of the cumulative function:

$$\theta = \arcsin\left(\sqrt{U}\right).$$

4.16. Case of isotropic distribution $D(\omega) = \frac{1}{4\pi}$

We derived above closed-form expressions and sampling procedures of our self-shadowing model for sine and cosine distributions. We now consider the case of an isotropic microflake distribution

$$D(\omega) = \frac{1}{4\pi}$$

which is much simpler. The attenuation coefficient writes

$$\sigma_t(\omega_i) = \rho A(\omega_i)\frac{1}{4\pi}\int\langle\omega_i\cdot\omega\rangle d\omega = \frac{\rho A(\omega_i)}{4}$$

and the single scattering coefficient

$$\begin{aligned}
\sigma_{ss}(\omega_i) &= \rho\alpha_{ss}(\lambda)A(\omega_i)\int A(\omega_o)\frac{1}{4\pi}\langle\omega_i\cdot\omega\rangle d\omega && (4.91)\\
&= \rho\alpha_{ss}(\lambda)A(\omega_i)\frac{1}{4\pi}\frac{1}{4}\int A(\omega_o)d\omega_o && (4.92)\\
&= \rho\alpha_{ss}(\lambda)A(\omega_i)\frac{1}{4\pi}\frac{1}{4}\frac{4\pi}{3}\operatorname{tr}(S_A) && (4.93)\\
&= \rho\alpha_{ss}(\lambda)A(\omega_i)\frac{\operatorname{tr}(S_A)}{12}. && (4.94)
\end{aligned}$$

The single scattering phase function writes

$$f_{ss}(\omega_i\to\omega_o) = \frac{A(\omega_i)A(\omega_o)\frac{1}{4\pi}}{A(\omega_i)\frac{1}{4\pi}\int A(\omega)\,\mathrm{d}\omega} = \frac{3A(\omega_o)}{4\pi\operatorname{tr}(S_A)}.$$

This phase function is proportional to the self-shadowing function in the outgoing direction $A(\omega_o)$. The function A can be written as a sum of three $\cos^2$ lobes as shown in equation 4.1. Using this expression, sampling $A(\omega_o)$ can be done by sampling one of the lobe proportionally to the eigenvalues a_1, a_2 and a_3, and then sampling a $\cos^2$ lobe (almost the same as in Section 4.13) around the corresponding eigenvector. The multiple scattering phase function writes

$$f_{ms}(\omega_o) = \frac{\sigma_t(\omega_o) - \sigma_s(\omega_o)/\alpha_{ss}(\lambda)}{\int\sigma_t - \int\sigma_s/\alpha_{ss}(\lambda)}$$

with

$$\int\sigma_t(\omega)\,\mathrm{d}\omega = \rho\frac{1}{4}\int A(\omega)\,\mathrm{d}\omega = \frac{\pi\operatorname{tr}(S_A)}{3}$$

and

$$\int\sigma_{ss}(\omega)\,\mathrm{d}\omega = \rho\alpha_{ss}(\lambda)\frac{\operatorname{tr}(S_A)}{12}\int A(\omega)\,\mathrm{d}\omega = \rho\alpha_{ss}(\lambda)\frac{\operatorname{tr}(S_A)}{12}\frac{4\pi\operatorname{tr}(S_A)}{3} = \frac{\pi\operatorname{tr}(S_A)^2}{9}$$

so we have

$$f_{ms}(\omega_o) = \frac{A(\omega_o)\frac{1}{4} - A(\omega_o)\frac{\operatorname{tr}(S_A)}{12}}{\frac{\pi}{3}\operatorname{tr}(S_A) - \frac{\pi\operatorname{tr}(S_A)^2}{9}} = \frac{3A(\omega_o)}{\operatorname{tr}(S_A)4\pi} = f_{ss}(\omega_i\to\omega_o). \qquad (4.95)$$

The multiple scattering phase function f_{ms} is the same as the single scattering phase function f_{ss} in the case of an isotropic microflake normal distribution.

4.17. Combining trigonometric lobes

In previous sections, we have derived expressions for σ_t, σ_{ss}, σ_{ms}, f_{ss}, f_{ms} and sampling procedures for cosine, sine and isotropic microflake distributions. We now provide details for combining various lobes, that is when the microflake normal distribution D can be written

$$D(\omega) = \sum w_i D_i(\omega)$$

with $\sum w_i = 1$. Several expressions are linear in D, so that we get:

$$\sigma_t(\omega) = \rho A(\omega)\int\sum w_i D_i(m)\langle m\cdot\omega\rangle\,\mathrm{d}m = \sum w_i\,\sigma_t^i(\omega)$$

$$\sigma_{ss}(\omega) = \rho\alpha_{ss}(\lambda)A(\omega)\int A(-\omega+2m(m\cdot\omega))\sum w_i D_i(m)\,\mathrm{d}m = \sum w_i\ \sigma^i_{ss}(\omega)$$

$$\int \sigma_t(\omega)\,\mathrm{d}\omega = \int\sum w_i\ \sigma^i_t(\omega)\,\mathrm{d}\omega = \sum w_i\int\sigma^i_t(\omega)\,\mathrm{d}\omega$$

$$\int \sigma_{ss}(\omega)\,\mathrm{d}\omega = \int\sum w_i\ \sigma^i_{ss}(\omega)\,\mathrm{d}\omega = \sum w_i\int\sigma^i_{ss}(\omega)\,\mathrm{d}\omega$$

The single scattering phase function is not a linear combination of phase functions derived from each lobes, because of the normalization factor:

$$f_{ss}(\omega_i\to\omega_o) = \frac{\rho\alpha_{ss}(\lambda)A(\omega_i)A(\omega_o)\sum w_i D_i(\omega_h)}{4\sum w_i\ \sigma^i_{ss}(\omega)} = \sum\frac{w_i\sigma^i_{ss}(\omega)}{\sum w_i\ \sigma^i_{ss}(\omega)}\frac{\rho\alpha_{ss}(\lambda)A(\omega_i)A(\omega_o)D_i(\omega_h)}{4\sigma^i_{ss}(\omega)} = \sum\frac{w_i\sigma^i_{ss}(\omega)}{\sum w_i\ \sigma^i_{ss}(\omega)}f^i_{ss}(\omega_i\to\omega_o).$$

The weights depend on direction ω_i. Intuitively, this means that if one lobe has a small projected area in direction ω_i, or if there is more self-shadowing for this lobe because reflected rays are such that $A(\omega_o)$ is small, then this lobe contributes less to the combined phase function from direction ω_i. Similarly, the combined multiple scattering phase function writes:

$$f_{ms}(\omega_o) = \frac{\sigma_t(\omega_o)-\sigma_{ss}(\omega_o)/\alpha_{ss}(\lambda)}{\int\sigma_t(\omega)\,\mathrm{d}\omega-\int\sigma_{ss}(\omega)/\alpha_{ss}(\lambda)\,\mathrm{d}\omega} \tag{4.96}$$

$$= \frac{\sum w_i(\sigma^i_t(\omega_o)-\sigma^i_{ss}(\omega_o)/\alpha_{ss}(\lambda))}{\sum w_i(\int\sigma^i_t(\omega)\,\mathrm{d}\omega-\int\sigma^i_{ss}(\omega)/\alpha_{ss}(\lambda)\,\mathrm{d}\omega)} \tag{4.97}$$

$$= \sum\frac{w_i(\int\sigma^i_t(\omega)\,\mathrm{d}\omega-\int\sigma^i_{ss}(\omega)/\alpha_{ss}(\lambda)\,\mathrm{d}\omega)}{\sum w_i(\int\sigma^i_t(\omega)\,\mathrm{d}\omega-\int\sigma^i_{ss}(\omega)/\alpha_{ss}(\lambda)\,\mathrm{d}\omega)}\frac{\sigma^i_t(\omega_o)-\sigma^i_{ss}(\omega_o)/\alpha_{ss}(\lambda)}{\int\sigma^i_t(\omega)\,\mathrm{d}\omega-\int\sigma^i_{ss}(\omega)/\alpha_{ss}(\lambda)\,\mathrm{d}\omega} \tag{4.98}$$

$$= \sum\frac{w_i(\int\sigma^i_t(\omega_o)-\int\sigma^i_{ss}(\omega_o)/\alpha_{ss}(\lambda))}{\sum w_i(\int\sigma^i_t(\omega)\,\mathrm{d}\omega-\int\sigma^i_{ss}(\omega)/\alpha_{ss}(\lambda)\,\mathrm{d}\omega)}f^i_{ms}(\omega_o). \tag{4.99}$$

Sampling these phase functions from direction ω_i can be done by computing view dependent weights of each lobes, and then sampling one function f^i_{ss} or f^i_{ms} accordingly.

5. Implementing the simplified self-shadowing model with the SGGX distribution

In this section, we discuss the implementation of our simplified self-shadowing model (Sec. 2.5) using the SGGX distribution. The attenuation coefficient is given by

$$\sigma_t(\omega) = A\rho\int D(\omega_m)\langle\omega\cdot\omega_m\rangle\ \mathrm{d}\omega_m.$$

Using SGGX microflake distributions, this writes

$$\sigma_t(\omega) = A\rho\sqrt{\omega^T S\,\omega}$$

with S the SGGX matrix. Similarly, the single scattering coefficient writes

$$\sigma_{ss}(\omega) = \alpha_{ss}(\lambda)A\sigma_t(\omega) = \alpha_{ss}(\lambda)A^2\rho\sqrt{\omega^T S\,\omega}.$$

The single scattering phase function writes

$$f_{ss}(\omega\to\omega') = \frac{\rho\alpha_{ss}(\lambda)A^2 D(\omega_h)}{4\sigma_{ss}(\omega)} = \frac{D(\omega_h)}{4\sqrt{\omega^T S\,\omega}}$$

which is the SGGX specular phase function [HDCD15]. For the microscopic multiple scattering, we have

$$\sigma_{ms}(\omega) = \alpha_{ms}(\lambda)\sigma_t(\omega)\,(1-A) = \alpha_{ms}(\lambda)\rho A\,(1-A)\sqrt{\omega^T S\,\omega} \tag{5.1}$$

$$f_{ms}(\omega') = \frac{\sigma_{ms}(\omega')}{\int\sigma_{ms}(\omega'')\ \mathrm{d}\omega''} = \frac{\sqrt{\omega^T S\,\omega}}{\int\sqrt{\omega''^T S\,\omega''}\ \mathrm{d}\omega''}. \tag{5.2}$$

Unfortunately, there is no closed-form expression for $\int\sqrt{\omega^T S\,\omega}$. As this integral does not depend on the orientation of the SGGX distribution, but only on the eigenvalues of the SGGX matrix, we pre-compute numerically this integral for several sets of eigenvalues. Assuming that the SGGX is normalized so that its largest eigenvalue is 1, we pre-compute $\int\sqrt{\omega^T S\,\omega}$ for several pairs of smaller eigenvalues a_2 and a_3. Table 1 shows the values we used in our implementation.

The multiple scattering phase function $f_{ms}(\omega_o)$ is proportional to the projected area of microflakes in direction ω_o. As discussed in Sec. 4.12, sampling this function could be done by sampling a normal from distribution D and then sampling a cosine lobe around this normal.

$a_2 \setminus a_3$	**0.05**	**0.1**	**0.15**	**0.2**	**0.25**	**0.3**	**0.35**	**0.4**	**0.45**	**0.5**	**0.55**	**0.6**	**0.65**	**0.7**	**0.75**	**0.8**	**0.85**	**0.9**	**0.95**	**1.0**
0.05	6.98478																			
0.1	7.24191	7.48864																		
0.15	7.47046	7.70689	7.92137																	
0.2	7.67929	7.90892	8.1193	8.31181																
0.25	7.87585	8.10067	8.30395	8.49325	8.67262															
0.3	8.06218	8.28009	8.48044	8.66547	8.84188	9.0083														
0.35	8.2367	8.45133	8.64776	8.83106	9.00371	9.16961	9.32714													
0.4	8.4059	8.61678	8.80948	8.99007	9.1601	9.32419	9.47985	9.63083												
0.45	8.56783	8.77602	8.96504	9.14333	9.31145	9.47266	9.62734	9.77656	9.92073											
0.5	8.7241	8.92956	9.1162	9.29157	9.45843	9.61683	9.76971	9.91778	10.0615	10.1992										
0.55	8.87382	9.07574	9.26196	9.43575	9.60125	9.758	9.90851	10.0557	10.1965	10.3334	10.4677									
0.6	9.02212	9.22178	9.40381	9.57613	9.73718	9.89405	10.0441	10.189	10.3294	10.4648	10.5979	10.7268								
0.65	9.16639	9.36294	9.54264	9.71236	9.87291	10.0276	10.1762	10.3193	10.459	10.5934	10.7246	10.8529	10.9782							
0.7	9.30461	9.499	9.67856	9.84528	10.0051	10.1578	10.3046	10.4478	10.5851	10.7189	10.8495	10.9767	11.1013	11.2231						
0.75	9.44041	9.63153	9.80971	9.9749	10.133	10.2856	10.431	10.572	10.7088	10.8415	10.9713	11.0975	11.2208	11.3416	11.4602					
0.8	9.57287	9.76264	9.93909	10.102	10.26	10.4098	10.5546	10.6939	10.8301	10.9622	11.0911	11.2159	11.3384	11.4588	11.5763	11.6918				
0.85	9.70291	9.89037	10.0654	10.2271	10.3821	10.5327	10.675	10.8145	10.9487	11.0797	11.2076	11.3325	11.4543	11.5736	11.6905	11.805	11.9179			
0.9	9.8299	10.0156	10.1889	10.3496	10.5037	10.6512	10.7945	10.9319	11.0656	11.1956	11.3229	11.4465	11.5676	11.6861	11.8023	11.9167	12.0286	12.1387		
0.95	9.95539	10.1387	10.3096	10.4694	10.6227	10.7694	10.9105	11.0472	11.1807	11.3094	11.4362	11.5589	11.679	11.7971	11.9125	12.0262	12.1375	12.2471	12.3548	
1.0	10.0769	10.2589	10.4287	10.5875	10.7391	10.8857	11.0263	11.162	11.293	11.4217	11.5469	11.6693	11.7893	11.9066	12.0209	12.1339	12.2448	12.3537	12.4608	12.5664

Table 1: *Pre-computed values for* $\int \sqrt{\omega^T S\, \omega}$ *for eigenvalues of* S *being* 1, a_2 *and* a_3.

However, there is no efficient way of sampling a normal from SGGX distributions. Available methods are based on rejection sampling and can be inefficient in some configurations. We observed that the distribution

$$\frac{\omega^T S\, \omega}{\int \omega'^T S\, \omega' \, \mathrm{d}\omega'}$$

is rather close to $f_{ms}(\omega_o)$, and simple to sample, since it is actually the same distribution as the multiple scattering phase function in the case of anisotropic self-shadowing and isotropic microflake distribution (Sec. 4.16). Based on this function, we can derive a rejection sampling method. We know that

$$\sqrt{\omega^T S\, \omega} \;\; \leq \;\; \omega^T S\, \omega + \frac{1}{4}$$

so that

$$\frac{\sqrt{\omega^T S\, \omega}}{\int \sqrt{\omega'^T S\, \omega'}\, \mathrm{d}\omega'} \leq \frac{\omega^T S\, \omega \, + 1/4}{\int \sqrt{\omega'^T S\, \omega'}\, \mathrm{d}\omega'} \tag{5.3}$$

$$\leq \frac{\omega^T S\, \omega + 1/4}{\int (\omega'^T S\, \omega' + 1/4)\, \mathrm{d}\omega'} \, \frac{\int (\omega'^T S\, \omega' + 1/4)\, \mathrm{d}\omega'}{\int \sqrt{\omega'^T S\, \omega'}\, \mathrm{d}\omega'}. \tag{5.4}$$

We can sample

$$\frac{\omega^T S\, \omega + 1/4}{\int (\omega'^T S\, \omega' + 1/4)\, \mathrm{d}\omega'}$$

and accept samples with probability

$$\frac{\sqrt{\omega^T S\, \omega}}{\int \sqrt{\omega'^T S\, \omega'}\, \mathrm{d}\omega'} \, \frac{\int (\omega'^T S\, \omega' + 1/4)\, \mathrm{d}\omega'}{\omega^T S\, \omega + 1/4} \, \frac{\int \sqrt{\omega'^T S\, \omega'}\, \mathrm{d}\omega'}{\int (\omega'^T S\, \omega' + 1/4)\, \mathrm{d}\omega'} = \frac{\sqrt{\omega^T S\, \omega}}{\omega^T S\, \omega + 1/4}.$$

6. Evaluation of the estimation of multiple scattering albedos

In the main paper, in Sec. 7.4.4., we proposed a method for estimating multiple scattering albedos in low-resolution voxels. The multiple scattering albedos must be chosen such that the resulting *effective* albedos of low-res voxels, *i.e.* the color of the light when it leaves the voxel after one or more scattering event, matches the effective albedo of the corresponding block of input voxels. There is no closed-form expression of the effective albedo of a voxel of microflake medium, meaning that we cannot compute analytically the perceived color of a voxel given its density and its single scattering albedo. In our downsampling algorithm (Sec. 7 in the main paper), we estimate the effective albedo in the block of input voxels using ray casting, and then we rely on approximations for estimating a multiple scattering albedo for the low-resolution voxel. Thanks to our approximations, we don't need to rely on iterative optimizations for estimating multiple scattering albedos, which would be more accurate but also time consuming because effective albedos of low-resolution voxels would have to be evaluated with ray casting at each iteration.

We evaluated our method by computing the relative errors of the RGB values of the effective albedos in low-resolution voxels. For each low-resolution voxel, we compared the effective albedo (estimated with ray casting and using the multiple scattering albedo obtained with our approximation) with the effective albedo of the corresponding block of input voxels (also estimated with ray casting). In coarse LoDs (grid resolution divided by 32 in each dimension), we obtained the relative errors shown in Fig. 1. Our approximations tend to underestimate the effective albedo, meaning that our multiple scattering albedos in low-resolution voxels are slightly too dark on average. Accuracy depends on the amount of shadowing and the density in the voxels. Our approach could be improved with a better understanding of these errors.

Interestingly, cases with the highest errors in the effective albedos do not correspond to the highest visual errors at rendering (dense hair asset, Fig. 9 in the main paper). The average albedo is important, but phase functions also contribute a lot to appearance, especially when the lighting is not uniform.

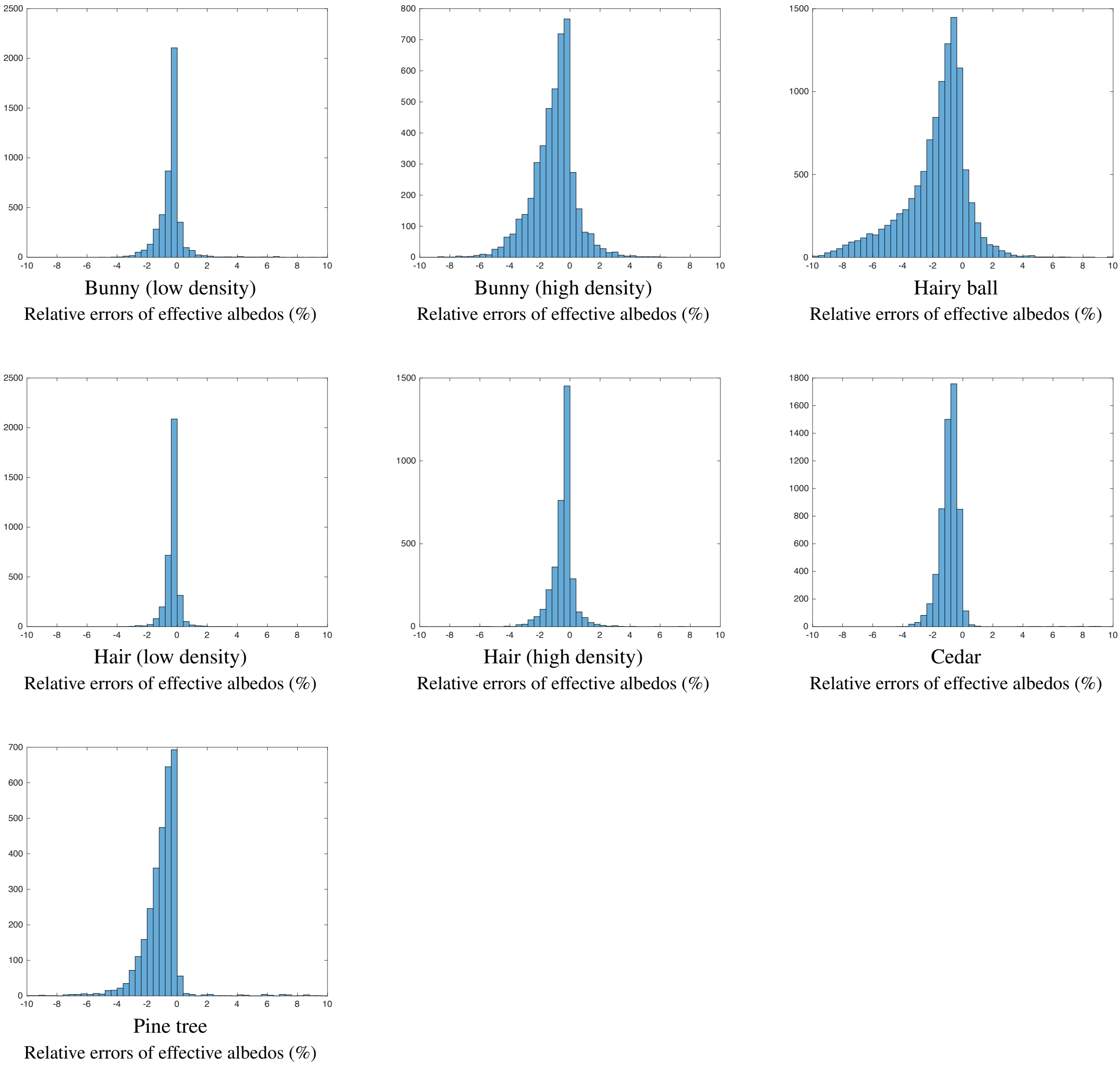


Figure 1: *Relative errors between measured effective albedos in our low-resolution voxels and reference effective albedos measured in the corresponding blocks of input voxels. Charts show relative errors for R, G and B values, in %. Each chart corresponds to one dataset used in the main paper. Vertical axes give the count of a given relative error in the low-res voxels of the LoD (empty voxels are not taken into account).*

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