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WHEN DOES A PERTURBED MOSER-TRUDINGER INEQUALITY ADMIT AN EXTREMAL?

PIERRE-DAMIEN THIZY

Abstract. In this paper, we are interested in several questions raised mainly in [17] (see also [18, 20]). We consider the perturbed Moser-Trudinger inequality \( I^\alpha_g(\Omega) \) below, at the critical level \( \alpha = 4\pi \), where \( g \), satisfying \( g(t) \to 0 \) as \( t \to +\infty \), can be seen as a perturbation with respect to the original case \( g \equiv 0 \). Under some additional assumptions, ensuring basically that \( g \) does not oscillates too fast as \( t \to +\infty \), we identify a new condition on \( g \) for this inequality to have an extremal. This condition covers the case \( g \equiv 0 \) studied in [3, 12, 23]. We prove also that this condition is sharp in the sense that, if it is not satisfied, \( I^\alpha_g(\Omega) \) may have no extremal.

1. Introduction

Let \( \Omega \) be a smooth, bounded domain of \( \mathbb{R}^2 \) and let \( H^1_0 = H^1_0(\Omega) \) be the standard Sobolev space, obtained as the completion of the set of smooth functions with compact support in \( \Omega \), with respect to the norm \( \| \cdot \|_{H^1_0} \) given by
\[
\| u \|_{H^1_0}^2 = \int_{\Omega} |\nabla u(x)|^2 dx.
\]
Throughout the paper, \( \Omega \) is assumed to be connected. Let \( g \) be such that
\[
g \in C^1(\mathbb{R}), \quad \lim_{s \to +\infty} g(s) = 0, \quad g(t) > -1 \text{ and } g(t) = g(-t) \text{ for all } t. \tag{1.1}
\]
Then, we have that
\[
C_{g,\alpha}(\Omega) := \sup_{u \in H^1_0: \|u\|_{H^1_0}^2 \leq \alpha} \int_{\Omega} (1 + g(u)) \exp(u^2) dx \quad (I^\alpha_g(\Omega))
\]
is finite for \( 0 < \alpha < 4\pi \) and equals \( +\infty \) for \( \alpha > 4\pi \). This result was first obtained by Moser [19] in the unperturbed case \( g \equiv 0 \). Still by [19], we easily extend the \( g \equiv 0 \) case to the case of \( g \) as in (1.1). At last, [19] gives also the existence of an extremal for \( (I^\alpha_0(\Omega)) \) if \( 0 < \alpha < 4\pi \) (see Lemma 3.1). If now \( \alpha = 4\pi \), getting the existence of an extremal is more challenging; however Carleson-Chang [3], Struwe [23] and Flucher [12] were also able to prove that \( (I^\alpha_0(\Omega)) \) admits an extremal in the unperturbed case \( g \equiv 0 \). Yet, surprisingly, McLeod and Peletier [18] conjectured that there should exist a \( g \) as in (1.1) such that \( (I^\alpha_{4\pi}(\Omega)) \) does not admit any extremal function. Through a nice but very implicit procedure, Pruss [20] was able prove that such a \( g \) does exist. Observe that, since \( g(u) \to 0 \) as \( u \to +\infty \) in (1.1), \( (1 + g(u)) \exp(u^2) \) in \( (I^\alpha_0(\Omega)) \) sounds like a very mild perturbation of \( \exp(u^2) \) as \( u \to +\infty \) and then, this naturally raises the following question:

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Question 1. To what extent does the existence of an extremal for the critical Moser-Trudinger inequality \((I^4_{0\gamma}(\Omega))\) really depend on asymptotic properties of the function \(t \mapsto \exp(t^2)\) as \(t \to +\infty\)?

To investigate this question, we may rephrase it as follows: for what \(g\) satisfying (1.1) does \((I^4_{0\gamma}(\Omega))\) admit an extremal? This is Open problem 2 in Mancini and Martinazzi [17], stated in this paper for \(\Omega = \mathbb{D}^2\), the unit disk of \(\mathbb{R}^2\). In order to state our main general result, we introduce now some notations. For a first reading, one can go directly to Corollary 1.1, which aims to give a less general but more readable statement. We let \(H : (0, +\infty) \to \mathbb{R}\) be given by

\[
H(t) = 1 + g(t) + \frac{g'(t)}{2t},
\]

so that we have

\[
[(1 + g(t)) \exp(t^2)]' = 2tH(t) \exp(t^2).
\]

We set \(tH(t) = 0\) for \(t = 0\), so that \(t \mapsto tH(t)\) is continuous at \(0\) by (1.1). This function \(H\) comes into play, since the Euler-Lagrange associated to \((I^4_{0\gamma}(\Omega))\) reads as

\[
\begin{cases}
\Delta u = \lambda uH(u) \exp(u^2) \text{ in } \Omega, \\
u = 0 \text{ in } \partial \Omega,
\end{cases}
\]

where \(\lambda \in \mathbb{R}\) is a Lagrange multiplier and \(\Delta = -\partial_{xx} - \partial_{yy}\) (see also Lemma 3.1 below). Now, we make some further assumptions on the behavior of \(g\) at \(+\infty\) and at \(0\). First, we assume that there exist \(\delta_0 \in (0,1)\) and a sequence of real numbers \(A = (A(\gamma))_\gamma\) such that

a) \(H\left(\gamma - \frac{t}{\gamma}\right) = H(\gamma) \left(1 + A(\gamma) t + o(|A(\gamma)| + \gamma^{-4})\right)\), in \(C^0_{loc}(0, +\infty)\), as \(\gamma \to +\infty\),

b) \(\exists C > 0, \left|H\left(\gamma - \frac{t}{\gamma}\right) - H(\gamma)\right| \leq C|H(\gamma)||A(\gamma)| + \gamma^{-4}\exp(\delta_0 t),\)

for all \(\gamma \gg 1\) and all \(0 \leq t \leq \gamma^2\),

c) \(\lim_{\gamma \to +\infty} A(\gamma) = 0\).

We also assume that there exist \(\delta'_0 \in (0,1), \kappa \geq 0, \varepsilon_0 \in \{-1, +1\}\), \(F\) given by \(F(t) := \varepsilon_0 t^\kappa\), and a sequence \(B = (B(\gamma))_\gamma\) of positive real numbers such that

a) \(\frac{t}{\gamma} H\left(\frac{t}{\gamma}\right) = B(\gamma) F(t) + o(|B(\gamma)| + \gamma^{-1})\), in \(C^0_{loc}(0, +\infty)\), as \(\gamma \to +\infty\),

b) \(\exists C > 0, \left|\frac{t}{\gamma} H\left(\frac{t}{\gamma}\right)\right| \leq C(|B(\gamma)| + \gamma^{-1}) \exp(\delta'_0 t),\)

for all \(\gamma \gg 1\) and all \(0 \leq t \leq \gamma^2\).

Observe that we may have \(B(\gamma) = o(\gamma^{-1})\) as \(\gamma \to +\infty\), in which case the precise formula for \(F\) is not really significant. Since \(t \mapsto (1 + g(t)) \exp(t^2)\) is an even \(C^1\) function, we have that

\[
\lim_{\gamma \to +\infty} B(\gamma) = 0,
\]
in view of (1.3) and (1.6). Following rather standard notations, we may split the Green's function $G$ of $\Delta$, with zero Dirichlet boundary conditions in $\Omega$, according to

$$G_x(y) = \frac{1}{4\pi} \left( \log \frac{1}{|x-y|^2} + \mathcal{H}_x(y) \right),$$

for all $x \neq y$ in $\Omega$, where $\mathcal{H}_x$ is harmonic in $\Omega$ and coincides with $-\log \frac{1}{|x-y|^2}$ in $\partial \Omega$. Then the Robin function $x \mapsto \mathcal{H}_x(x)$ is smooth in $\Omega$, and goes to $-\infty$ as $x \to \partial \Omega$, so that we may set

$$M = \max_{x \in \Omega} \mathcal{H}_x(x),$$

$$K_\Omega = \{ y \in \Omega \text{ s.t. } \mathcal{H}_0(y) = M \}$$

and

$$S = \max_{x \in K_\Omega} \int_\Omega G_z(y) F(4\pi G_z(y)) dy,$$

where $F$ is as in (1.6). For $N \geq 1$, we let $g_N$ be given by

$$(1 + g_N(t)) \exp(t^2) = (1 + g(t))(1 + t^2) + (1 + g(t)) \left( \sum_{k=N+1}^{+\infty} \frac{t^{2k}}{k!} \right),$$

so that $g_N \leq g$, $g_N(0) = g(0)$ for all $N \geq 1$, while $g = g_N$ for $N = 1$. We also set

$$\Lambda_g(\Omega) := \max_{\omega \in H^1_0(\Omega) \cap \mathcal{S}} \int_\Omega \left( (1 + g(u))(1 + u^2) - (1 + g(0)) \right) dx$$

(1.10)

We are now in position to state our main result, giving a new, very general and basically sharp picture about the existence of extremals for the perturbed Moser-Trudinger ($I^g_{4\pi}(\Omega)$).

**Theorem 1.1** (Existence and non-existence of an extremal). Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^2$. Let $g$ be such that (1.1) and (1.5)-(1.6) hold true for $H$ as in (1.2), and let $A$, $B$ and $F$ be thus given. Assume that

$$l = \lim_{\gamma \to +\infty} \frac{\gamma^{-4} + A(\gamma)/2 + 4\gamma^{-3} \exp(-1 - M)B(\gamma)S}{\gamma^{-4} + |A(\gamma)| + \gamma^{-3}|B(\gamma)|}$$

(1.12)

exists, where $M$ and $S$ are given by (1.9). Then

(1) if $l > 0$ or $\Lambda_g(\Omega) \geq \pi \exp(1 + M)$, $(I^g_{4\pi}(\Omega))$ admits an extremal, where $\Lambda_g(\Omega)$ is as in (1.11);

(2) if $l < 0$ and $\Lambda_g(\Omega) < \pi \exp(1 + M)$, there exists $N_0 \geq 1$ such that $(I^{gN}_{4\pi}(\Omega))$ admits no extremal for all $N \geq N_0$, where $g_N$ is given by (1.10).

Observe that, for all given $N \geq 1$, $g_N$ satisfies (1.1) and (1.5)-(1.6), with the same $A$, $B$ and $F$ as the original $g$. Moreover it is clear that $\Lambda_{gN}(\Omega) \leq \Lambda_g(\Omega)$. Then, this second assertion in Theorem 1.1 proves that the assumptions on $g$ in the first assertion are basically sharp to get the existence of an extremal for $(I^g_{4\pi}(\Omega))$. As a remark, Pruss concludes in [20] that the existence of an extremal for the critical Moser-Trudinger inequality is in some sense accidental and relies on non-asymptotic properties of $\exp(u^2)$. Theorem 1.1 clarifies this tricky situation: the existence or nonexistence of an extremal for $(I^g_{4\pi}(\Omega))$ may really depend on a balance of the asymptotic properties of $g$ both at infinity (given by $A(\gamma)$) and at zero (given by $B(\gamma)$). Yet, it may also depend on the non-asymptotic quantity $\Lambda_g(\Omega)$ (see Corollary 1.2). Observe that $\Lambda_0(\Omega) = (4\pi)/\lambda_1(\Omega)$ in the unperturbed case $g \equiv 0$, where $\lambda_1(\Omega)$ is the first eigenvalue of the Neumann problem on $\Omega$. In particular, if $\Lambda_0(\Omega) > \pi \exp(1 + M)$, then $(I^g_{4\pi}(\Omega))$ admits no extremal; see Corollary 1.4.
where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of $\Delta$ in $\Omega$.

From now on, we illustrate Theorem 1.1 by two corollaries dealing with less general but more explicit situations. Let $c, c' \in \mathbb{R}$, $(a, b), (a', b') \in \mathcal{E}$, where

$$\mathcal{E} = \{(a, b) \in [0, +\infty) \times \mathbb{R} \mid b > 0 \text{ if } a = 0\}.$$  

(1.13)

Let $R' > 0$ be a large positive constant. If one picks $g$ such that

$$g(t) = \begin{cases}  
g_0(t) := g(0) + ct^{a+1} \log(1/t)^{-b} \text{ in } (0, 1/R'], \\
g_\infty(t) := ct^{-a'}(\log t)^{-b'} \text{ in } [R', +\infty),
\end{cases}$$  

(1.14)

$l$ in (1.12) of Theorem 1.1 can be made more explicit. Indeed, we can then set

$$B(\gamma) = \frac{1 + g(0)}{\gamma} + \frac{c(a + 1)}{2} \gamma^{-a}(\log \gamma)^{-b},$$

$$A(\gamma) = c' \times \begin{cases}  
a' \gamma^{-(a'+2)}(\log \gamma)^{-b'} \text{ if } a' > 0, \\
b' \gamma^{-2}(\log \gamma)^{-(b'+1)} \text{ if } a' = 0,
\end{cases}$$  

(1.15)

(see also Lemma 3.2). Theorem 1.1 is even more explicit in the particular case $\Omega = \mathbb{D}^2$. Indeed, in this case we have that $K_\mathbb{D}^2 = \{0\}$ in (1.9) and $G_0(x) = \frac{\pi}{2\pi} \log \frac{1}{|x|}$. Still on the unit disk $\mathbb{D}^2$, it is known that

$$\Lambda_0(\mathbb{D}^2) = \frac{4\pi}{\lambda_1(\mathbb{D}^2)} < \pi e,$$  

(1.16)

($\lambda_1(\mathbb{D}^2) \simeq 5.78$). Property (1.16) shows in particular that the second assertion $\Lambda_0(\mathbb{D}^2) \geq \pi e$ of Theorem 1.1, Part (1), is not satisfied. In some sense, this is an additional motivation for the nice approach of [3], proving the existence of an extremal for $(I^\alpha_\pi(\mathbb{D}^2))$ via asymptotic analysis. As an illustration and a very particular case of Theorem 1.1, we get the following corollary.

**Corollary 1.1** (Case $\Omega = \mathbb{D}^2$). Assume that $\Omega = \mathbb{D}^2$. Let $c' \neq 0$ and $(a', b') \in \mathcal{E}$ be given, where $\mathcal{E}$ is as in (1.13). Let $g_\infty$ be as in (1.14).

1. If we assume $a' > 2$ or $c' > 0$, then for all even function $g \in C^2(\mathbb{R})$, zero in a neighborhood of 0, such that $g > -1$ and

$$g^{(i)}(t) = g_\infty^{(i)}(t)(1 + o(1))$$  

(1.17)

as $t \to +\infty$ for all $i \in \{0, 1, 2\}$, $(I^\alpha_\pi(\mathbb{D}^2))$ admits an extremal.

2. If we assume $a' < 2$ and $c' < 0$, there exists an even function $g \in C^2(\mathbb{R})$, zero in a neighborhood of 0, such that $g > -1$ and such that (1.17) holds true, while $(I^\alpha_\pi(\mathbb{D}^2))$ admits no extremal.

Our main concern in Corollary 1.1 is to write a readable statement. In this result, the existence of an extremal in the unperturbed case $g \equiv 0$ is recovered for quickly decaying $g$’s, namely if $a' > 2$ (see [17]). But a threshold phenomenon appears (only if $c' < 0$) and there are no more extremal for slowlier decaying $g$’s, namely for $a' < 2$. Note that Theorem 1.1 also allows to point out the existence of a threshold $c' < 0$ in the border case $a' = 2, b' = 0$. Indeed, proving Corollary 1.1 basically reduces to give an explicit formula for $l$ in (1.12), which only depends on $\Omega$ and on the asymptotics of $g$ at $+\infty$ and at 0. On the contrary, we do not care about the precise asymptotics of $g$ in the following corollary, thus illustrating the role of $\Lambda_0(\Omega)$ in Theorem 1.1.
Corollary 1.2 (Extremal for $\Lambda_g(\Omega)$ large). Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^2$. Let $\lambda_1(\Omega) > 0$ be the first Dirichlet eigenvalue of $\Delta$ in $\Omega$ and $M$ be given as in (1.9). Let $\tilde{A}$ be such that $4(1 + \tilde{A}) > \lambda_1(\Omega) \exp(1 + M)$ and let $C > \tilde{A}$ be given. Then there exists $R \gg 1$ such that $(I_{4\pi}^g(\Omega))$ admits an extremal for all $g$ satisfying (1.1) and
\[ g(0) = \tilde{A}, \quad g \geq g(0) \text{ in } [1/R, R] \quad \text{and} \quad |g| \leq C \text{ in } \mathbb{R}. \quad (1.18) \]

As a remark, in the process of the proof below (see Remark 2.1), we answer the very interesting Open problem 6 of [17].

This paper is organized as follows. Theorem 1.1, and Corollaries 1.1 and 1.2 are proved in Section 2. Theorem 1.1 follows from Propositions 2.1 and 2.2, proved in Section 4. Both Propositions 2.1 and 2.2 are consequences of key Lemma 3.3, which is proved in Section 3, using some radial analysis results obtained in Appendix A.

2. Proof of the main results

We begin by proving Corollary 1.1, assuming that Theorem 1.1 holds true.

Proof of Corollary 1.1. The first part of Corollary 1.1 is a straightforward consequence of the first part of Theorem 1.1: plugging the formulas of (1.15) in (1.12), we get that $l > 0$ for $g$ as in Case (1) of Corollary 1.1. In order to prove the second part of Corollary 1.1, we apply the second part of Theorem 1.1. Let $\chi$ be a smooth nonnegative function in $\mathbb{R}$ such that $\chi(t) = 0$ for all $t \leq 1/2$ and $\chi(t) = 1$ for all $t \geq 1$. By the Sobolev inequality and standard integration theory, we can check that $g_R := g_{\infty} \times \chi(\cdot/R)$ satisfies $\Lambda_{g_R}(\mathbb{D}^2) \to \Lambda_0(\mathbb{D}^2)$ as $R \to +\infty$. Then, by (1.15), (1.16), assuming $a' < 2, c' < 0$, the second part of Theorem 1.1 applies, starting from $g = g_R$, for $R \gg 1$ fixed sufficiently large. Observe that, for all given $N \gg 1$, $(g_R)_N$ (given by (1.10) for $g = g_R$) satisfies (1.17). Corollary 1.1 is proved.

Proof of Corollary 1.2. Let $\Omega, \tilde{A}, \lambda_1(\Omega), C$ be as in the statement of the corollary. By Theorem 1.2, it is sufficient to prove that there exists $R \gg 1$ such that for all $g$ satisfying (1.1) and (1.18), we have that $\Lambda_g(\Omega) \geq \pi \exp(1 + M)$, where $\Lambda_g(\Omega)$ is as in (1.11). Let $v > 0$ in $\Omega$ be the first eigenvalue of $\Delta$ normalized according to $\|v\|_{H^1_0}^2 = 4\pi$. For all $g$ satisfying (1.18), we have that
\[
\Lambda_g(\Omega) \geq \int_{\Omega} ((1 + g(0))v^2 + (g(v) - g(0))(1 + v^2)) \, dx \\
\geq (1 + \tilde{A}) \frac{4\pi}{\lambda_1(\Omega)} + \int_{\{v \in [1/R, R]\}} (g(v) - g(0))(1 + v^2) \, dx,
\]
and, since we have
\[
\left| \int_{\{v \in [1/R, R]\}} (g(v) - g(0))(1 + v^2) \, dx \right| \leq (|\tilde{A}| + C)(1 + \|v\|_{L^\infty}^2) |\{v \not\in [1/R, R]\}| \to 0
\]
as $R \to +\infty$, we get the result using that $4(1 + \tilde{A}) > \lambda_1(\Omega) \exp(1 + M)$. 

The following proposition is the core of the argument to get the existence of an extremal in Theorem 1.1, Part (1). Its proof is postponed in Section 4. It uses the tools developed in Druet-Thizy [9] that allow us to push the asymptotic analysis
of a concentrating sequence of extremals \((u_\varepsilon)_\varepsilon\) further than in previous works. In the process of the proof of Proposition 2.1 (see Lemma 4.1), we show first that a concentration point \(\bar{x}\) of such \(u_\varepsilon\)'s realizes \(M\) in (1.9). But in the case where \(|B(\gamma)|\) matters in (1.12) or, in other words, where \(\gamma^3|A(\gamma)| + \gamma^{-1} \lesssim |B(\gamma)|\) as \(\gamma \to +\infty\), we also show that \(S\) in (1.9) has to be attained at \(\bar{x}\).

**Proposition 2.1.** Let \(\Omega\) be a smooth bounded domain of \(\mathbb{R}^2\). Let \(g\) be such that (1.1) and (1.5)-(1.6) hold true, for \(H\) as in (1.2), and let \(A, B\) and \(F\) be thus given. Let \((u_\varepsilon)_\varepsilon\) be a sequence of nonnegative functions such that \(u_\varepsilon\) is a maximizer for \(I^{g,(1-\varepsilon)}(\Omega)\), for all \(0 < \varepsilon \ll 1\). Assume that

\[ u_\varepsilon \to 0 \quad \text{in} \quad H^1_0, \tag{2.1} \]

as \(\varepsilon \to 0\). Then, \(|u_\varepsilon|_{H^1_0}^2 = 4\pi (1 - \varepsilon)\), there exists a sequence \((\lambda_\varepsilon)_\varepsilon\) of real numbers such that \(u_\varepsilon\) solves in \(H^1_0\)

\[
\begin{align*}
\Delta u_\varepsilon &= \lambda_\varepsilon u_\varepsilon H(u_\varepsilon) \exp(u_\varepsilon^2), & u_\varepsilon > 0 \quad \text{in} \quad \Omega, \\
\quad & \quad u_\varepsilon = 0 \quad \text{on} \quad \partial \Omega, \\
\end{align*}
\tag{2.2}
\]

\(u_\varepsilon \in C^{1,\theta}(\overline{\Omega}) \ (0 < \theta < 1)\) and we have that

\[ \gamma_\varepsilon := \max_{y \in \Omega} u_\varepsilon \rightarrow +\infty. \tag{2.3} \]

Moreover, we have that

\[
\lim_{\varepsilon \to 0} \int_\Omega (1 + g(u_\varepsilon)) \exp(u_\varepsilon^2) dx = |\Omega|(1 + g(0)) + \pi \exp(1 + M) \tag{2.4}
\]

and that

\[
||u_\varepsilon||_{H^1_0}^2 = 4\pi \left( 1 + I(\gamma_\varepsilon) + o(\gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)| + \gamma_\varepsilon^{-3}|B(\gamma_\varepsilon)|) \right) \tag{2.5}
\]

as \(\varepsilon \to 0\), where

\[
I(\gamma_\varepsilon) := \gamma_\varepsilon^{-4} + A(\gamma_\varepsilon)/2 + 4\gamma_\varepsilon^{-3} \exp(-1 - M)B(\gamma_\varepsilon)S, \tag{2.6}
\]

where \(|\Omega|\) stands for the volume of the domain \(\Omega\) and where \(M\) and \(S\) are as in (1.9).

**Remark 2.1.** Let \(g, H\) be such that (1.1), (1.2), (1.5)-(1.7) hold true. Let \(u_\varepsilon\) be a maximizer for \((I^{g,(1-\varepsilon)}(\Omega))\) such that (2.1) holds true, as in Proposition 2.1. Then, for such a sequence \((u_\varepsilon)_\varepsilon\) satisfying in particular (2.2) and (2.3), we get in the process of the proof (see (3.16) below) that the term \(I(\gamma_\varepsilon)\) in (2.5) is necessarily smaller than \(o(\gamma_\varepsilon^{-2})\) as \(\varepsilon \to 0\). Moreover this threshold \(o(\gamma_\varepsilon^{-2})\) is sharp, in the sense that this term may be for instance of size \(\gamma_\varepsilon^{-(2+a')}\), for all given \(a' \in (0,2]\). This can be seen by picking an appropriate \(g\) such that \(I^{g,(1-\varepsilon)}(\Omega)\) has no extremal, as in Corollary 1.1, and by using Proposition 2.1. Observe that, for such a \(g\), assumption (2.1) is indeed automatically true. This gives an answer to Open Problem 6 in [17].

**Proof of Theorem 1.1, Part (1): existence of an extremal for \((I^{g,(1-\varepsilon)}(\Omega))\).** We first prove the existence of an extremal stated in Part (1) of Theorem 1.1. Let \(g\) be such that (1.1) and (1.5)-(1.6) hold true, for \(H\) as in (1.2), and let \(A, B\) and \(F\) be thus given. Assume either that \(l > 0\) in (1.12) or that \(A_g(\Omega) \geq \pi \exp(1 + M)\). Using Lemma 3.1, let \((u_\varepsilon)_\varepsilon\) be a sequence of nonnegative functions such that \(u_\varepsilon\) is a maximizer
for \((I^{g}_{\pi(1-\varepsilon)}(\Omega))\), for all \(0 < \varepsilon \ll 1\). Then, up to a subsequence, \((u_{\varepsilon})_{\varepsilon}\) converges a.e. and weakly in \(H^1_0\) to some \(u_0\). Independently, we check that

\[
\lim_{\varepsilon \to 0} C_{g,4\pi(1-\varepsilon)}(\Omega) = C_{g,4\pi}(\Omega),
\]

(2.7)

where \(C_{g,\alpha}(\Omega)\) is as in \((I^{g}_{\pi}(\Omega))\). Indeed, if one assumes by contradiction that the \(C_{g,4\pi(1-\varepsilon)}(\Omega)\)'s increase to some \(\bar{l} < C_{g,4\pi}(\Omega)\) as \(\varepsilon \to 0\), then we may choose some nonnegative \(u\) such that \(\|u\|_{H^1_0}^2 \leq 4\pi\) and \(\int_{\Omega}(1 + g(u))\exp(u^2)dx > \bar{l}\). But, picking \(v_\varepsilon = u\sqrt{1 - \varepsilon}\), we have that \(\|v_\varepsilon\|_{H^1_0}^2 < 4\pi\), and

\[
\lim_{\varepsilon \to 0} \int_{\Omega} (1 + g(v_\varepsilon))\exp(v_\varepsilon^2)dx = \int_{\Omega} (1 + g(u))\exp(u^2)dx,
\]

by the dominated convergence theorem, using (1.1), \(v_\varepsilon^2 \leq u^2\) and \(\exp(u^2) \in L^1(\Omega)\). But this contradicts the definition of \(\bar{l}\) and concludes the proof of (2.7). Now, by (2.7) and since \(\|u_0\|_{H^1_0}^2 \leq 4\pi\), in order to get that \(u_0\) is the extremal for \((I^{g}_{\frac{\pi}{4}}(\Omega))\) we look for, it is sufficient to prove that

\[
\lim_{\varepsilon \to 0} \int_{\Omega} (1 + g(u_\varepsilon))\exp(u_\varepsilon^2)dx = \int_{\Omega} (1 + g(u_0))\exp(u_0^2)dx.
\]

(2.8)

If \(u_0 = 0\), then Proposition 2.1 gives a contradiction: either by (2.4) and (2.7) if \(\Lambda_\beta(\Omega) \geq \pi \exp(1 + M)\), since it is clear that

\[
C_{g,4\pi}(\Omega) > \Lambda_\beta(\Omega) + (1 + g(0))|\Omega|,
\]

or by (2.5)-(2.6) if \(l > 0\), since \(\|u_\varepsilon\|_{H^1_0}^2 \leq 4\pi\). Thus, we necessarily have that \(u_0 \neq 0\). Then, noting that \(\|u_\varepsilon - u_0\|_{H^1_0}^2 \leq 4\pi - \|u_0\|_{H^1_0}^2 + o(1)\), the standard Moser-Trudinger inequality \((I^{g}_{\frac{\pi}{4}}(\Omega))\) and some integration theory give that (2.8) still holds true, and Part (1) of Theorem 1.1 is proved in any case.

The following proposition is the core of the argument to get the non-existence of an extremal in Theorem 1.1, Part (2). Its proof is postponed in Section 4.

**Proposition 2.2.** Let \(\Omega\) be a smooth bounded domain of \(\mathbb{R}^2\). Let \(g\) be such that

(1.1) and (1.5)-(1.6) hold true, for \(H\) as in (1.2), and let \(A\), \(B\) and \(F\) be thus given. Assume that \(\Lambda_\beta(\Omega) < \pi \exp(1 + M)\), where \(M\) is as in (1.9) and \(\Lambda_\beta(\Omega)\) as in (1.11).

Assume that there exists a sequence of positive integers \((N_\varepsilon)_{\varepsilon}\) such that

\[
\lim_{\varepsilon \to 0} N_\varepsilon = +\infty
\]

(2.9)

and such that \((I^{g_{N_\varepsilon}}_{\frac{\pi}{4}}(\Omega))\) admits a nonnegative extremal \(u_\varepsilon\) for all \(\varepsilon > 0\), where \(g_{N_\varepsilon}\) is as in (1.10). Then we have (2.1) and that \(\|u_\varepsilon\|_{H^1_0}^2 = 4\pi\) for all \(0 < \varepsilon \ll 1\). Moreover, we have \(u_\varepsilon \in C^{1,\theta}(\Omega)\) \((0 < \theta < 1)\), (2.3) and that

\[
\|u_\varepsilon\|_{H^1_0}^2 \leq 4\pi \left(1 + I(\gamma_\varepsilon) + o(\gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)| + |B(\gamma_\varepsilon)|)\right)
\]

(2.10)

as \(\varepsilon \to 0\), where \(I(\gamma_\varepsilon)\) is given by (2.6).

**Proof of Theorem 1.1, Part (2): non-existence of an extremal for \((I^{g_{N_\varepsilon}}_{\frac{\pi}{4}}(\Omega))\), \(N \geq N_0\).**

Let \(g\) be such that (1.1) and (1.5)-(1.6) hold true, for \(H\) as in (1.2), and let \(A\), \(B\) and \(F\) be thus given. Assume \(l < 0\) and \(\Lambda_\beta(\Omega) < \pi \exp(1 + M)\), where \(l\) is as in (1.12), \(\Lambda_\beta\) as in (1.11) and \(M\) as in (1.9). In order to prove Part (2) of Theorem 1.1, we assume by contradiction that there exists a sequence \((N_\varepsilon)_{\varepsilon}\) of positive integers satisfying (2.9) and such that \((I^{g_{N_\varepsilon}}_{\frac{\pi}{4}}(\Omega))\) admits an extremal, for \(g_{N_\varepsilon}\) as in (1.10).
We let \((u_\varepsilon)_\varepsilon\) be a sequence of nonnegative functions such that \(u_\varepsilon\) is a maximizer for \((I_{g_\varepsilon}^N(\Omega))\), for all \(\varepsilon > 0\). But this is not possible by Proposition 2.2, since \(\|u_\varepsilon\|^2_{H_0^1} = 4\pi\) contradicts (2.10), since we also assume now \(l < 0\). This concludes the proof of Part (2) of Theorem 1.1.

\[
\square
\]

3. Blow-up analysis in the strongly perturbed Moser-Trudinger regime

In this section, we aim to prove the main blow-up analysis results that we need to get both Propositions 2.1 and 2.2. The following preliminary lemma deals with the existence of an extremal for the perturbed Moser-Trudinger inequality \((I_\alpha^g(\Omega))\) in the subcritical case \(0 < \alpha < 4\pi\). Its proof relies on integration theory combined with \((I_0^g(\Omega))\), and on standard variational techniques. It is omitted here and the interested reader may find more details in the proof of Proposition 6 of [17].

Lemma 3.1. Let \(\Omega\) be a smooth bounded domain of \(\mathbb{R}^2\). Let \(g\) be such that (1.1) holds true. Then, \((I_\alpha^g(\Omega))\) admits a nonnegative extremal \(u_\alpha\) for all \(0 < \alpha < 4\pi\). Moreover, we have the following alternative

1) either \(\|u_\alpha\|^2_{H_0^1} < \alpha\) and \(u_\alpha H(u_\alpha) = 0\) a.e.,

2) or \(\|u_\alpha\|^2_{H_0^1} = \alpha\) and there exists \(\lambda \in \mathbb{R}\) such that \(u_\alpha\) solves in \(H_0^1\) the Euler-Lagrange equation (1.4).

Remark 3.1. The first alternative in Lemma 3.1 may occur in general, but does not if \(t \mapsto (1 + g(t)) \exp(t^2)\) increases in \((0, +\infty)\).

The following lemma investigates more precisely the behavior of \(g\) and \(H\), when we assume (1.1) together with (1.5).

Lemma 3.2. Let \(\Omega\) be a smooth bounded domain of \(\mathbb{R}^2\). Let \(g\) be such that (1.1), (1.5) and (1.6) hold true, for \(H\) as in (1.2), and let \(A, B\) and \(\delta_0, \delta'_0, F, \kappa\) be thus given. Then we have that

\[
a) \left(1 + g \left(\frac{t}{\gamma}\right)\right) \exp\left(\frac{t^2}{\gamma^2}\right) = \left(1 + g(0)\right) + \frac{2B(\gamma)F(t)t}{\gamma(\kappa + 1)} + o\left(\frac{|B(\gamma)|}{\gamma} + \frac{1}{\gamma^2}\right),
\]

in \(C_0^0((0, +\infty)\), as \(\gamma \to +\infty\),

\[
b) \exists C > 0,
\]

\[
\left|\left(1 + g \left(\frac{t}{\gamma}\right)\right) \exp\left(\frac{t^2}{\gamma^2}\right) - \left(1 + g(0)\right)\right| \leq C \left(\frac{|B(\gamma)|}{\gamma} + \frac{1}{\gamma^2}\right) t \exp(\delta'_0 t),
\]

for all \(\gamma \gg 1\) and all \(0 \leq t \leq \gamma\),

\[
c) \|g\|_{L^\infty(\mathbb{R})} < +\infty,
\]
and that

\[ a) \ 1 + g \left( \gamma - \frac{t}{\gamma} \right) = H(\gamma) \left( 1 + A(\gamma) \left( t + \frac{1}{2} \right) + o\left( |A(\gamma)| + \gamma^{-4} \right) \right), \]

in \( C^0_{\text{loc}}((0, +\infty)_t) \), as \( \gamma \to +\infty \),

\[ b) \ \exists C > 0, \left| 1 + g \left( \gamma - \frac{t}{\gamma} \right) - H(\gamma) \right| \leq C |H(\gamma)| |A(\gamma)| + \gamma^{-4} \exp(\delta_0 t), \]

for all \( \gamma \gg 1 \) and all \( 0 \leq t \leq \gamma \).

In particular, we have that

\[ H(\gamma) \to 1 \text{ as } \gamma \to +\infty. \tag{3.3} \]

Proof of Lemma 3.2. We first prove (3.3). Using (1.3), we write

\[ (1 + g(r)) \exp(r^2) - (1 + g(0)) = 2 \int_0^r sH(s) \exp(s^2) ds, \tag{3.4} \]

for all \( r \geq 0 \). Then, as \( \gamma \to +\infty \), setting \( r = \gamma \), we can write

\[ 1 + g(\gamma) = \exp(-\gamma^2) (1 + g(0)) + 2 \int_0^{\gamma^2} \left( 1 - \frac{u}{\gamma^2} \right) H \left( \gamma - \frac{u}{\gamma} \right) \exp \left( -2u + \frac{u^2}{\gamma^2} \right) du, \]

\[ = O \left( \exp(-\gamma^2) \right) + 2H(\gamma) \int_0^{\gamma^2} \left( 1 - \frac{u}{\gamma^2} \right) \exp \left( -2u + \frac{u^2}{\gamma^2} \right) du, \]

\[ + O \left( |H(\gamma)| |A(\gamma)| + \gamma^{-4} \right) \int_0^{\gamma^2} \exp(-1 - \delta_0) \exp \left( -u \left( 1 - \frac{u}{\gamma^2} \right) \right) du, \]

\[ = O \left( \exp(-\gamma^2) \right) + H(\gamma) \left( 1 + \exp(-\gamma^2) \right) + o(H(\gamma)), \]

using (1.5). This proves (3.3) since \( g \) satisfies (1.1). Observe that parts \( a \) and \( b \) of (3.1) follow from (1.6) and (3.4) with \( r = t/\gamma \), while part \( c \) of (3.1) is a straightforward consequence of (1.1). We prove now part \( b \) of (3.2). As \( \gamma \to +\infty \), we write for all \( 0 \leq t \leq \gamma \)

\[ \left( 1 + g \left( \gamma - \frac{t}{\gamma} \right) \right) \exp \left( \left( \gamma - \frac{t}{\gamma} \right)^2 \right) - (1 + g(\gamma - 1)) \exp((\gamma - 1)^2), \]

\[ = 2 \int_{\gamma}^{\gamma - \frac{t}{\gamma}} rH(r) \exp(r^2) dr, \]

\[ = 2 \int_{\gamma}^{\gamma} \left( 1 - \frac{u}{\gamma^2} \right) H \left( \gamma - \frac{u}{\gamma} \right) \exp \left( \gamma^2 - 2u + \frac{u^2}{\gamma^2} \right) du, \]

\[ = H(\gamma) \left( \exp \left( \left( \gamma - \frac{t}{\gamma} \right)^2 \right) - \exp((\gamma - 1)^2) \right) \]

\[ + O \left( |H(\gamma)| |A(\gamma)| + \gamma^{-4} \right) \int_{\gamma}^{\gamma} \exp \left( \gamma^2 - (2 - \delta_0)u \right) du, \]

using \( b \) in (1.5). Multiplying the above identity by \( \exp(-(\gamma - (t/\gamma))^2) \), using \( t \leq \gamma \), (1.1) and (3.3), part \( b \) of (3.2) easily follows. Using now \( a \) of (1.5) in the above before last inequality, we also get part \( a \) of (3.2). \( \square \)
In the sequel, for all integer $N \geq 1$, we let $\varphi_N$ be given by (see also (3.38) below)

$$\varphi_N(t) = \sum_{k=N+1}^{+\infty} \frac{t^k}{k!}.$$  \hfill (3.5)

The main results of this section are stated in the following lemma.

**Lemma 3.3.** Let $\Omega$ be a smooth bounded domain of $\mathbb{R}^2$. Let $g$ be such that (1.1) and (1.5)-(1.6) hold true, for $H$ as in (1.2), and let $A$, $B$ and $F$ be thus given. Let $(\alpha_\varepsilon)$ be a sequence of numbers in $(0, 4\pi]$. Let $(N_\varepsilon)$ be a sequence of positive integers. Assume that

$$\lim_{\varepsilon \to 0} \alpha_\varepsilon = 4\pi \text{ and that } u_\varepsilon \text{ is an extremal for } (I_{gN_\varepsilon(\alpha_\varepsilon)}),$$  \hfill (3.6)

for all $0 < \varepsilon \ll 1$, where $g_{N_\varepsilon}$ is as in (1.10). Assume in addition that we are in one of the following two cases:

- **(Case 1)** $\lim_{\varepsilon \to 0} N_\varepsilon = +\infty$, $\alpha_\varepsilon = 4\pi$ for all $\varepsilon$, and $\Lambda_g(\Omega) < \pi \exp(1 + M)$, \hfill (3.7)

where $\Lambda_g(\Omega)$ is as in (1.11) and $M$ as in (1.9), or

- **(Case 2)** $N_\varepsilon = 1$ for all $\varepsilon$ and (2.1) holds true.

Then, up to a subsequence,

$$\|u_\varepsilon\|_{H^1_0}^2 = \alpha_\varepsilon,$$  \hfill (3.8)

$u_\varepsilon \in C^{1,\theta}(\Omega)$ ($0 < \theta < 1$) solves

$$\begin{cases}
\Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon H_{N_\varepsilon}(u_\varepsilon) \exp(u_\varepsilon^2), & u_\varepsilon > 0 \text{ in } \Omega, \\
u_\varepsilon = 0 & \text{on } \partial \Omega,
\end{cases}$$  \hfill (3.9)

where $H_N(t) = 1 + g_N(t) + \frac{g_N''(t)}{2}$. Moreover, we have (2.4), that

$$\lambda_\varepsilon = \frac{4 + o(1)}{\gamma_\varepsilon^2 \exp(1 + M)},$$  \hfill (3.10)

that

$$A(\gamma_\varepsilon) - 2\xi_\varepsilon = o\left(\frac{1}{\gamma_\varepsilon}\right),$$  \hfill (3.11)

and that

$$x_\varepsilon \to \bar{x}, \quad (\bar{x} \in K)$$  \hfill (3.12)

as $\varepsilon \to 0$, where $x_\varepsilon, \gamma_\varepsilon$ satisfy

$$u_\varepsilon(x_\varepsilon) = \max_{\Omega} u_\varepsilon = \gamma_\varepsilon \to +\infty,$$  \hfill (3.13)

as $\varepsilon \to 0$, where $\xi_\varepsilon$ is given by (3.14)

$$\xi_\varepsilon = \frac{\gamma_\varepsilon^{2(N_\varepsilon - 1)}}{\varphi_{N_\varepsilon - 1}(\gamma_\varepsilon^2)(N_\varepsilon - 1)!},$$  \hfill (3.14)

and where $\tilde{\xi}_\varepsilon$ is given by

$$\tilde{\xi}_\varepsilon = \max \left(\frac{1}{\gamma_\varepsilon}, |A(\gamma_\varepsilon)|, \xi_\varepsilon\right).$$  \hfill (3.15)

At last, (3.137)-(3.139) below hold true, for $\mu_\varepsilon$ as in (3.42) and $t_\varepsilon$ as in (3.43).
Observe that \( N_\varepsilon = 1 \) in (Case 2) reduces to say that \( g_{N_\varepsilon} = g \). From (3.31) obtained in the process of the proof below, we get that \( \xi_\varepsilon = o(1/\gamma_\varepsilon^2) \) in (Case 2), so that (3.11) is then equivalent to

\[
A(\gamma_\varepsilon) = o\left(\frac{1}{\gamma_\varepsilon^2}\right),
\]

as discussed in Remark 2.1.

**Proof of Lemma 3.3.** We start by several basic steps. First, a test function computation gives the following result.

**Step 3.1.** For all \( g \) such that (1.1) holds true, we have that

\[
C_{g,4\pi}(\Omega) \geq |\Omega|(1 + g(0)) + \pi \exp(1 + M),
\]

where \( C_{g,4\pi}(\Omega) \) is as in (I\( g \)) (\( \alpha = 4\pi \)) and where \( M \) is as in (1.9).

**Proof of Step 3.1.** In order to get (3.17), it is sufficient to prove that there exists functions \( f_\varepsilon \in H_{0}^1 \) such that \( \| f_\varepsilon \|_{H_{0}^1} = 4\pi \) and such that

\[
\int_{\Omega} (1 + g(f_\varepsilon)) \exp(f_\varepsilon^2) \; dy \geq |\Omega|(1 + g(0)) + \pi \exp(1 + M) + o(1),
\]

as \( \varepsilon \to 0 \). In order to reuse these computations later, we fix any sequence \( (z_\varepsilon)_\varepsilon \) of points in \( \Omega \) such that

\[
\varepsilon^2 \frac{d(z_\varepsilon, \partial \Omega)^2}{d(z_\varepsilon, \partial \Omega)^2} = o\left(\frac{1}{\varepsilon}\right)^{-1}.
\]

For \( 0 < \varepsilon < 1 \), we let \( v_\varepsilon \) be given by \( v_\varepsilon(y) = \log \frac{1}{\varepsilon^2 + |y - z_\varepsilon|^2} + \mathcal{H}_{z_\varepsilon, \varepsilon} \), where \( \mathcal{H}_{z_\varepsilon, \varepsilon} \) is harmonic in \( \Omega \) and such that \( v_\varepsilon \) is zero on \( \partial \Omega \). Then, by the maximum principle and (1.8), we have that

\[
\mathcal{H}_{z_\varepsilon, \varepsilon}(y) = \mathcal{H}_{z_\varepsilon}(y) + O\left(\frac{\varepsilon^2}{d(z_\varepsilon, \partial \Omega)^2}\right) \quad \text{for all } y \in \Omega,
\]

where \( \mathcal{H}_{z_\varepsilon} \) is as in (1.8). Then, integrating by parts, we compute

\[
\| v_\varepsilon \|_{H_{0}^1}^2 = \int_{\Omega} v_\varepsilon \Delta v_\varepsilon \; dy,
\]

\[
= \int_{\Omega} \varepsilon^2 \left(1 + \frac{|z_\varepsilon - y|^2}{\varepsilon^2}\right) \left( \log 1 + \frac{1}{1 + \frac{|y - z_\varepsilon|^2}{\varepsilon^2}} + \mathcal{H}_{z_\varepsilon, \varepsilon}(y) \right) \; dy,
\]

\[
= 4\pi \left( \log \frac{1}{\varepsilon^2} + o(1) \right) - 4\pi (1 + o(1))
\]

\[
+ 4\pi \left( \mathcal{H}_{z_\varepsilon}(z_\varepsilon) + o(1) \right),
\]

\[
= 4\pi \left( \log \frac{1}{\varepsilon^2} - 1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon) \right) + o(1),
\]

where the change of variable \( z = |y - z_\varepsilon|/\varepsilon \), (3.19), (3.20) and

\[
\mathcal{H}_{z_\varepsilon}(z_\varepsilon + \varepsilon z) = \mathcal{H}_{z_\varepsilon}(z_\varepsilon) + O\left(\frac{\varepsilon |z|}{d(z_\varepsilon, \partial \Omega)}\right),
\]

(3.22)
We can write \( r \) as \( \epsilon \) (see for instance Appendix B in [9]) are used. Let \( f \) be given by \( 4\pi v^2 = f^2_{\epsilon} \| v \|_{R_0}^2 \). We can write
\[
 f_{\epsilon}(y)^2 = \left( \frac{\log \frac{1}{|z_{\epsilon} \cdot y|^2 + \epsilon^2}}{\log \frac{1}{\epsilon^2}} \right)^2 + 2H_{z_{\epsilon}, \epsilon}(y) \log \frac{1}{|z_{\epsilon} \cdot y|^2 + \epsilon^2} + H_{z_{\epsilon}, \epsilon}(y) \]
using (3.21). Then, writing \( \log \frac{1}{|z_{\epsilon} \cdot y|^2 + \epsilon^2} = \log \frac{1}{\epsilon^2} + \log \frac{1}{1 + \frac{|z_{\epsilon} \cdot y|^2}{\epsilon^2}} \), we get
\[
\int_{B_{z_{\epsilon}}(\hat{r}_{\epsilon}) \cap \Omega} (1 + g(f_{\epsilon})) \exp(f^2_{\epsilon}) \, dy
= \int_{B_{z_{\epsilon}}(\hat{r}_{\epsilon}) \cap \Omega} (1 + o(1)) \exp \left( -\frac{2\hat{t}_{\epsilon}(y)}{\epsilon} + 2H_{z_{\epsilon}, \epsilon}(y) - H_{z_{\epsilon}}(z_{\epsilon}) + 1 \right) \times
\exp \left( \frac{\hat{t}^2_{\epsilon}}{\log \frac{1}{\epsilon^2}} + O \left( \frac{1 + \hat{t}_{\epsilon}}{\log \frac{1}{\epsilon^2}} + \frac{1 + \hat{t}^2_{\epsilon}}{\log \frac{1}{\epsilon^2}} \right) \right) \, dy
= \pi \exp(H_{z_{\epsilon}}(z_{\epsilon}) + 1) (1 + o(1))
\]
as \( \epsilon \to 0 \), using (1.1), (3.20) and (3.22), where \( \hat{t}_{\epsilon}(y) = \log \left( 1 + \frac{|z_{\epsilon} \cdot y|^2}{\epsilon^2} \right) \) and where \( \hat{r}_{\epsilon} \) is given by \( \hat{r}_{\epsilon}(\hat{r}_{\epsilon}) = \frac{1}{2} \log \frac{1}{\epsilon^2} \). Now, we can check that
\[
f_{\epsilon}(y)^2 \leq \left( \frac{\log \frac{1}{\epsilon^2} + O(1)}{\log \frac{1}{\epsilon^2}} \right)^{-1} \left( \frac{\log \frac{1}{|z_{\epsilon} \cdot y|^2} + O(1)}{\log \frac{1}{|z_{\epsilon} \cdot y|^2}} \right)^2,
\]
\[
\leq \left( \frac{\log \frac{1}{|z_{\epsilon} \cdot y|^2} + O(1)}{\log \frac{1}{|z_{\epsilon} \cdot y|^2}} \right) \times \left( \frac{1}{2} + o(1) \right)
\]
for all \( y \in \Omega \setminus B_{z_{\epsilon}}(\hat{r}_{\epsilon}) \), using (1.8), (3.20) and our definition of \( \hat{r}_{\epsilon} \), so that we also get
\[
\int_{B_{z_{\epsilon}}(\Omega \setminus \hat{r}_{\epsilon})} (1 + g(f_{\epsilon})) \exp(f^2_{\epsilon}) \, dy \to (1 + g(0))|\Omega|
\]
as \( \epsilon \to 0 \), by the dominated convergence theorem, using (1.1). Property (3.18) and then Step 3.1 follow from (3.23) and (3.24), choosing \( z_{\epsilon} \in K_{\Omega} \) as in (1.9). \( \square \)

From now on, we make the assumptions of Lemma 3.3. In particular, we assume that either (Case 1), or (Case 2) holds true. Given an integer \( N \geq 1 \), observe that Step 3.1 applies to \( g_N \), since \( g_N \) satisfies (1.1), if \( g \) does. Then, using \( \alpha_{\epsilon} = 4\pi \) in (Case 1), or (2.7) and \( g_{N\epsilon} = g \) in (Case 2), we get that
\[
|\Omega|(1 + g(0)) + \pi \exp(1 + M) \leq \left\{ \begin{array}{ll}
C_{g_{N\epsilon}, 4\pi} \quad \text{in (Case 1)}, \\
C_{g_{N\epsilon}, \alpha_{\epsilon} + o(1)} \quad \text{in (Case 2)}.
\end{array} \right.
\]
as \( \epsilon \to 0^+ \), where \( C_{g, \alpha}(\Omega) \) is as in formula (\( P_0^\alpha(\Omega) \)) and where \( M \) is as in (1.9). Let us rewrite now (3.9) in a more convenient way. Let \( \Psi_N \) be given by
\[
\Psi_N(t) = (1 + g_N(t)) \exp(t^2).
\]
Observe in particular that
\[
(1 + g(t))(1 + t^2) \leq \Psi_N(t) \leq (1 + g(t)) \exp(t^2),
\]
for all $t$ and all $N$, by (1.1). Using (1.2), (1.3) and (1.10), we may rewrite (3.9) as

\[
\begin{aligned}
\Delta u_{\varepsilon} &= \frac{\lambda_{\varepsilon}}{2} \Psi'_{N_{\varepsilon}}(u_{\varepsilon}), \quad u_{\varepsilon} > 0 \text{ in } \Omega, \\
u_{\varepsilon} &= 0 \text{ on } \partial \Omega,
\end{aligned}
\]

(3.27)

with

\[
\Psi'_{N}(t) = 2t H(t)(1 + t^2 + \varphi_{N}(t^2)) + 2t(1 + g(t))\left(\frac{t^{2N}}{N!} - t^2\right)
\]

\[
= 2t H(t)\varphi_{N}(t^2) + 2t\left(1 + \frac{t^{2N}}{N!}\right)(1 + g(t)) + g'(t)(1 + t^2).
\]

(3.28)

Indeed, in (3.9), it turns out that

\[
H_{N}(t) = \Psi'_{N}(t) \exp(-t^2).
\]

(3.29)

Observe that by (1.1) and (3.3), using the first line of (3.28), we clearly have that there exists $C > 0$ such that

\[
|\Psi'_{N_{\varepsilon}}(t)| \leq C t \exp(t^2),
\]

(3.30)

for all $t \geq 0$ and all $\varepsilon$. In (Case 2), (2.1) is assumed to be true. We prove now that (2.1) also holds true in (Case 1).

**Step 3.2.** Assume that we are in (Case 1). Then (2.1) holds true. Moreover, we have that

\[
\liminf_{\varepsilon \to 0} \frac{\varphi_{N_{\varepsilon}}(\gamma_{\varepsilon}^2)}{\exp(\gamma_{\varepsilon}^2)} > 0,
\]

(3.31)

and, in other words, that

\[
\liminf_{\varepsilon \to 0} \frac{\gamma_{\varepsilon}^2 - N_{\varepsilon}}{\sqrt{N_{\varepsilon}}} > -\infty,
\]

(3.32)

where $\gamma_{\varepsilon} = \text{ess sup } u_{\varepsilon}$ and $\varphi_{N}$ is as in (3.5).

**Proof of Step 3.2.** By (3.6) and (3.25), we get that

\[
\int_{\Omega} \Psi_{N}(u_{\varepsilon}) dy \geq (1 + g(0))|\Omega| + \pi \exp(1 + M).
\]

(3.33)

Writing now

\[
\Psi_{N}(t) = (1 + g(0)) + ((1 + g(t))(1 + t^2) - (1 + g(0))) + (1 + g(t))\varphi_{N}(t^2)
\]

and using (1.1), we also get

\[
\int_{\Omega} \Psi_{N_{\varepsilon}}(u_{\varepsilon}) dy \leq (1 + g(0))|\Omega| + \Lambda_{g}(\Omega) + \int_{\Omega} (1 + g(u_{\varepsilon}))\varphi_{N_{\varepsilon}}(u_{\varepsilon}^2) dy
\]

(3.34)

where $\Lambda_{g}$ is as in (1.11). Then by (1.1) and (3.7), we get from (3.33) and (3.34) that

\[
\liminf_{\varepsilon \to 0} \int_{\Omega} \varphi_{N_{\varepsilon}}(u_{\varepsilon}^2) dy > 0.
\]

(3.35)

Up to a subsequence, $u_{\varepsilon} \rightharpoonup u_0$ in $H^1_0$, for some $u_0 \in H^1_0$ such that $\|u_0\|^2_{H^1_0} \leq 4\pi$. Let $0 < \beta \ll 1$ be given. We have that

\[
u_{\varepsilon}^2 \leq (1 + \beta)(u_{\varepsilon} - u_0)^2 + \left(1 + \frac{1}{\beta}\right) u_0^2.
\]

(3.36)
Independently, by Moser-Trudinger’s inequality, we have that
\[ u \in H_0^1 \implies \forall p \in [1, +\infty), \quad \exp(u^2) \in L^p. \tag{3.37} \]
If \( u_0 \not= 0 \), \( \lim_{\varepsilon \to 0} \|u_\varepsilon - u_0\|_{H_0^1}^2 < 4\pi \) and, by (3.36), (3.37), Moser’s and Hölder’s inequalities, there exists \( p_0 > 1 \) such that \( (\exp(u_\varepsilon^2))_\varepsilon \) is bounded in \( L^{p_0} \). Then, by standard integration theory, since \( \varphi_{N_\varepsilon} \leq \exp \) in \([0, +\infty)\) and since \( N_\varepsilon \to +\infty \) in (Case 1), we get
\[ u_0 \not= 0 \implies \int_{\Omega} \varphi_{N_\varepsilon}(u_\varepsilon^2) dy = o(1) \]
as \( \varepsilon \to 0 \), which proves (2.1), in view of (3.35). Noting that the function \( t \mapsto \varphi_N(t) \exp(-t) \) increases in \([0, +\infty)\), we can write
\[ \int_{\Omega} \varphi_{N_\varepsilon}(u_\varepsilon^2) dy \leq \frac{\varphi_N(\gamma_\varepsilon^2)}{\exp(\gamma_\varepsilon^2)} \int_{\Omega} \exp(u_\varepsilon^2) dy, \]
and conclude that (3.31) holds true by (3.35) and Moser’s inequality. Observe that
\[ \varphi_N(\Gamma) = \exp(\Gamma) \int_0^\Gamma \exp(-s) \frac{s^N}{N!} ds. \tag{3.38} \]
Setting \( \Gamma = \gamma_\varepsilon^2, N = N_\varepsilon \) and \( s = N_\varepsilon + u_\sqrt{N_\varepsilon} \), we clearly get (3.32) from (3.31). \( \square \)

The next steps applies in both (Case 1) and (Case 2).

**Step 3.3.** We have that (3.8), (3.9) hold true, and that \( u_\varepsilon \) is in \( C^{1, \theta}(\Omega) \).

**Proof of Step 3.3.** Since \( u_\varepsilon \in L^1 \), note that \( \bar{\mu}_\varepsilon \) given by
\[ \bar{\mu}_\varepsilon(t) := \{|x \in \Omega \text{ s.t. } u_\varepsilon(x) > t| \} \]
is continuous in \([0, \gamma_\varepsilon]\). By (3.6) and the considerations as in Lemma 3.1, either (3.8) and (3.9) hold true, or \( \Psi_{N_\varepsilon}'(u_\varepsilon) \equiv 0 \) almost everywhere in \( \Omega \). Then, if we assume by contradiction that this second alternative holds true, since \( \Psi_{N_\varepsilon}' \) is continuous, we get that \( \Psi_{N_\varepsilon}' = 0 \) in \([0, \gamma_\varepsilon]\). Then, since \( \Psi_{N_\varepsilon}(0) = 1 \), there must be the case that
\[ (1 + g(t)) = \frac{1}{1 + t^2 + \varphi_{N_\varepsilon}(t^2)} \]for all \( t \in [0, \gamma_\varepsilon] \). Now we prove that
\[ \gamma_\varepsilon \to +\infty. \tag{3.40} \]
as \( \varepsilon \to 0 \). This is merely a consequence of Step 3.2 in (Case 1). In (Case 2), (2.1) is assumed. Thus, up to a subsequence, \( u_\varepsilon \to 0 \) a.e. and if we assume by contradiction that \( \gamma_\varepsilon = O(1) \), we contradict (3.6) and (3.25) by the dominated convergence theorem. This concludes the proof of (3.40). Then (3.39) contradicts that \( g(t) \to 0 \) as \( t \to +\infty \) in (1.1), which proves (3.8) and (3.9). By (3.37), the regularity of \( u_\varepsilon \) comes from (3.9) under its form (3.27), by (1.1), (3.3), (3.28) and standard elliptic theory (see for instance Gilbarg-Trudinger [14]). \( \square \)

The previous steps give in particular that (3.13) makes sense and holds true.

**Step 3.4.** There holds that \( \lambda_\varepsilon > 0 \) for all \( 0 < \varepsilon \ll 1 \). Moreover
\[ \lambda_\varepsilon \to 0, \tag{3.41} \]
as \( \varepsilon \to 0 \), where \( \lambda_\varepsilon \) is as in (3.9).
Proof of Step 3.4. By (3.6) and (3.25), we have that
\[ \liminf_{\varepsilon \to 0} \int_{\Omega} \Psi_{N_\varepsilon}(u_\varepsilon) \, dx > 0, \]
so that, by (1.1), (2.1), (3.3), (3.26), (3.28) and integration theory
\[ \liminf_{\varepsilon \to 0} \int_{\Omega} (\Psi_{N_\varepsilon}'(u_\varepsilon) + 2(1 + g(u_\varepsilon))u_\varepsilon^3) \, u_\varepsilon \, dx = +\infty. \]
But by (1.1), (2.1) and Rellich-Kondrachov’s theorem, we get that
\[ \int_{\Omega} (1 + g(u_\varepsilon))u_\varepsilon^4 \, dx = o(1). \]
Then, multiplying (3.27) by \( u_\varepsilon \) and integrating by parts, we get that \( \lambda_\varepsilon > 0 \) and
\[ 4\pi + o(1) = \int_{\Omega} |\nabla u_\varepsilon|^2 \, dx \gg \lambda_\varepsilon, \]
which proves (3.41).

Remark 3.2. Note that (Case 1) is particularly delicate to handle, since the nonlinearities \( (\Psi_{N_\varepsilon})_\varepsilon \) are not of uniform critical growth, even in the very general framework of [8, Definition 1]. A more intuitive way to see this is the following: if \( (\tilde{\gamma}_\varepsilon)_\varepsilon \) is a sequence of positive real numbers such that \( \tilde{\gamma}_\varepsilon \to +\infty \), but not too fast, in the sense that \( \tilde{\gamma}_\varepsilon^2 \ll N_\varepsilon \), then it can be checked with (1.1) and (3.3) that
\[ \frac{\lambda_\varepsilon}{2} \Psi_{N_\varepsilon}'(\tilde{\gamma}_\varepsilon) = \tilde{\lambda}_\varepsilon (1 + o(1)) \tilde{\gamma}_\varepsilon^{2N_\varepsilon+1}, \]
as \( \varepsilon \to 0 \), where \( \tilde{\lambda}_\varepsilon = \lambda_\varepsilon/(N_\varepsilon! \varepsilon). \) Then, in the regime \( 0 \leq u_\varepsilon \leq \tilde{\gamma}_\varepsilon \), at least formally, (3.27) looks at first order like the Lane-Emden problem, namely
\[
\begin{aligned}
\Delta u_\varepsilon &= \tilde{\lambda}_\varepsilon u_\varepsilon^{2N_\varepsilon+1}, \quad u_\varepsilon > 0 \text{ in } \Omega, \\
u_\varepsilon &= 0 \text{ on } \partial \Omega, \\
N_\varepsilon &\to +\infty,
\end{aligned}
\]
(Lane-Emden problem)
for which very interesting, but very different concentration phenomena were pointed out (see for instance [2, 6, 7, 11, 21, 22]). A real difficulty to conclude the subsequent proofs is to extend the analysis developed in [1, 8, 9] for the Moser-Trudinger "purely critical" regime, in order to deal also with such other intermediate regimes. As a last remark, a much simpler version of the techniques developed here permits also to answer some open questions about the Lane-Emden problem, as performed in [10].

We let \( t_\varepsilon \) be given by
\[ t_\varepsilon(x) = \log \left( 1 + \frac{|x - x_\varepsilon|^2}{\mu_\varepsilon^2} \right). \quad (3.43) \]
Here and in the sequel, for a radially symmetric function \( f \) around of \( x_\varepsilon \) (resp. around 0), we will often write \( f(r) \) instead of \( f(x) \) for \( |x - x_\varepsilon| = r \) (resp. \( |x| = r \).
Step 3.5. We have that
\[
\gamma_\varepsilon (\gamma_\varepsilon - u_\varepsilon (x_\varepsilon - \mu_\varepsilon)) \to T_0 := \log \left(1 + |\cdot|^2\right) \text{ in } C^{1,\theta}_{\text{loc}}(\mathbb{R}^2),
\] (3.44)
where \(\gamma_\varepsilon, x_\varepsilon\) are as in (3.13) and \(\mu_\varepsilon\) is as in (3.42). Moreover, we have that
\[
\liminf_{\varepsilon \to 0} \lambda_\varepsilon \gamma_\varepsilon^2 > 0.
\] (3.45)

At this stage, we can observe that
\[
\log \frac{1}{\mu_\varepsilon^2} = \gamma_\varepsilon^2 (1 + o(1)),
\] (3.46)
as \(\varepsilon \to 0\), by (3.3), (3.31), (3.41), (3.42), (3.45).

Proof of Step 3.5. We first sketch the proof of (3.44). In (Case 2), (3.44) follows closely Step 1 of the proof of [8, Proposition 1]. Thus, we focus now on the the proof of (3.44) in (Case 1). Observe that
\[
\sup_{t \in \mathbb{R}} t^N N! \exp(-t^2) = \frac{N!}{\sqrt{2\pi N}} \exp(-N) = \frac{1 + o(1)}{\sqrt{2\pi N}}
\] (3.47)
by Stirling’s formula. Then, by (1.1), (3.3), (3.13), (3.28), (3.31) and (3.40), we have that
\[
\frac{\Psi'_{N_\varepsilon}(u_\varepsilon)}{2} = u_\varepsilon H(u_\varepsilon)\varphi_{N_\varepsilon}(u_\varepsilon^2) + u_\varepsilon(1 + g(u_\varepsilon))\frac{u_\varepsilon^{2N_\varepsilon}}{N_\varepsilon!} + O(\gamma_\varepsilon^2)
\] (3.48)
\[
\leq (1 + o(1))\gamma_\varepsilon \varphi_{N_\varepsilon - \gamma_\varepsilon^2}.
\]
Let \(\tau_\varepsilon\) be given in \((\Omega - x_\varepsilon)/\mu_\varepsilon\) by
\[
u_\varepsilon(x_\varepsilon + \mu_\varepsilon) = \gamma_\varepsilon - \frac{\tau_\varepsilon}{\gamma_\varepsilon}.
\] (3.49)
Then, since \(\Delta \tau_\varepsilon = -\mu_\varepsilon^2 \gamma_\varepsilon (\Delta u_\varepsilon)(x_\varepsilon + \mu_\varepsilon)\), we get from (3.27), (3.42) and (3.48), that there exists \(C > 0\) such that \(|\Delta \tau_\varepsilon| \leq C\), while \(\tau_\varepsilon \geq 0, \tau_\varepsilon(0) = 0\). As in [8, p.231], we have that \(\mu_\varepsilon = o(d(x_\varepsilon, \partial \Omega))\). Then, by standard elliptic theory, there exists \(\tau_0\) such that
\[
\tau_\varepsilon \to \tau_0 \text{ in } C^{1,\theta}_{\text{loc}}(\mathbb{R}^2),
\] (3.50)
as \(\varepsilon \to 0\). Note that for all \(\Gamma, T > 0\) and all \(N\), we have that
\[
\varphi_N(T) = \varphi_N(\Gamma) \exp(-((\Gamma - T)) - \exp(T) \int_T^\Gamma \exp(-s) \frac{s^N}{N!} ds.
\] (3.51)
Writing the previous identity for \(N = N_\varepsilon - 1, \Gamma = \gamma_\varepsilon^2 \) and \(T = u_\varepsilon^2 = \gamma_\varepsilon^2 - 2\tau_\varepsilon + \frac{\tau_\varepsilon^2}{2}\varepsilon\^2\), noting from (3.47) and (3.50) that
\[
\int_{u_\varepsilon^2}^{\gamma_\varepsilon^2} \exp(-s) \frac{s^{N_\varepsilon - 1}}{(N_\varepsilon - 1)!} ds = O\left(\frac{1}{\sqrt{N_\varepsilon}}\right)
\]
in \(\mathbb{R}^2_{\text{loc}}\) and resuming the arguments to get (3.48), we get that
\[
\Delta (-\tau_0) = 4 \exp(-\tau_0)
\] (3.52)
using also (3.27), (3.31) and (3.42). Now, choosing \(R \gg 1\) such that \(|g(t)| < 1\) and \(H(t) > 0\) for all \(t \geq R\), we easily see that
\[
u_\varepsilon \left[\Psi'_{N_\varepsilon}(u_\varepsilon)\right]^{-} \leq 2\|t \mapsto tH(t)\|_{L^\infty(0,R)} \exp(R^2) u_\varepsilon + 4u_\varepsilon^4
\] (3.53)
by (1.1), (3.3) and (3.28), where \( t^- = -\min(t, 0) \). Then, we have that

\[
\frac{\lambda_\varepsilon}{2} \int_{\Omega} u_\varepsilon \left[ \Psi'_{N_\varepsilon}(u_\varepsilon) \right]^+ dy = 4\pi + o(1) ,
\]

(3.54)

by (3.6), (3.27), (3.41) and (3.53), where \( t^+ = \max(t, 0) \). Then, for all \( A \gg 1 \), we get that

\[
4 \int_{B_0(A)} \exp(-\tau_0) dy \leq \liminf_{\varepsilon \to 0} \lambda_\varepsilon \int_{\Omega} u_\varepsilon \left[ \Psi'_{N_\varepsilon}(u_\varepsilon) \right]^+ dy ,
\]

by (3.50) and, since \( A \) is arbitrary, we get from (3.54) that \( \int_{\mathbb{R}^2} \exp(-\tau_0) dy < +\infty \).

Then, by the classification result Chen-Li \([4]\), since \( \tau_0 \geq 0 \) and \( \tau_0(0) = 0 \), we get that \( \tau_0(y) = \log(1 + |y|^2) \). Thus (3.44) is proved by (3.50). Similarly, we may also choose some \( A_\varepsilon \)'s, such that \( A_\varepsilon \to +\infty \) and such that

\[
\frac{\lambda_\varepsilon}{2} \int_{B_{A_\varepsilon}(\gamma_\varepsilon)} \Psi'_{N_\varepsilon}(u_\varepsilon) dy = \frac{4\pi + o(1)}{\gamma_\varepsilon^q} .
\]

(3.55)

Since \( 0 < \Psi_{N_\varepsilon}(t) \leq (1 + g(t)) \exp(t^2) \) for all \( t \geq 0 \), and since \( C_{g, 4\pi}(\Omega) < +\infty \), we get (3.45) from (1.1) and (3.55). This concludes the proof of Step 3.5. \( \square \)

By Step 3.5 and estimates in its proof, since we assume \( \|u_\varepsilon\|_{H^1_0}^2 \leq 4\pi \), we get that

\[
\lim_{R \to +\infty} \lim_{\varepsilon \to 0} \int_{\Omega \backslash B_{A_\varepsilon}(R\gamma_\varepsilon)} (\Delta u_\varepsilon(y))^+ u_\varepsilon dy = 0 .
\]

(3.56)

We let \( \Omega_\varepsilon \) be given by

\[
\Omega_\varepsilon = \begin{cases} 
\{ y \in \Omega \ s.t. \ \varphi_{N_\varepsilon-1}(u_\varepsilon(y)^2) \geq u_\varepsilon(y)^2 + 1 \} & \text{in (Case 1),} \\
\Omega & \text{in (Case 2).}
\end{cases}
\]

(3.57)

Now, despite the difficulty pointed out in Remark 3.2, we are able to get the following weak, but global pointwise estimates.

**Step 3.6.** There exists \( C > 0 \) such that

\[
|\cdot|_{-x_\varepsilon}^2 |\Delta u_\varepsilon| u_\varepsilon \leq C \quad \text{in } \Omega_\varepsilon
\]

(3.58)

and such that

\[
|\cdot|_{-x_\varepsilon} |\nabla u_\varepsilon| u_\varepsilon \leq C \quad \text{in } \Omega_\varepsilon
\]

(3.59)

for all \( \varepsilon \), where \( \Omega_\varepsilon \) is as in (3.57).

In (Case 2), it is not so difficult to adapt the arguments of [8, §3.4] to get Step 3.6. Thus, in the proof of Step 3.6 just below, we assume that we are in (Case 1).

Then observe that \( \Omega_\varepsilon \neq \emptyset \) by Step 3.2. Given \( \eta_0 \in (0, 1) \), writing

\[
\varphi_{N_\varepsilon-1}(tN_\varepsilon) = \frac{t^{N_\varepsilon}N_\varepsilon^{N_\varepsilon}(1 + o(1))}{N_\varepsilon!} \leq \min u_\varepsilon^2
\]

for all \( 0 < \varepsilon \ll 1 \), uniformly in \( |t| \leq \eta_0 \), the unique positive solution \( \Gamma_\varepsilon \) of

\[
\varphi_{N_\varepsilon-1}(\Gamma_\varepsilon) = \Gamma_\varepsilon + 1 \quad \text{satisfies} \quad \Gamma_\varepsilon = (1 + o(1)) \frac{N_\varepsilon}{\varepsilon} .
\]

Then, since \( \varphi_{N_\varepsilon-1}/(1 + \cdot) \) increases in \((0, +\infty)\), we clearly get that

\[
(1 + o(1)) \frac{N_\varepsilon}{\varepsilon} \leq \min u_\varepsilon^2 .
\]

(3.60)
Proof of Step 3.6, Formula (3.58). As aforementioned, we still assume that we are in (Case 1). Thus, in particular, we assume that \( N_{\varepsilon} \to +\infty \) as \( \varepsilon \to 0 \). Assume now by contradiction that

\[
\max_{y \in \Omega_{\varepsilon}} |y - x_{\varepsilon}| |\Delta u_{\varepsilon}(y)||u_{\varepsilon}(y)| = |y_{\varepsilon} - x_{\varepsilon}| |\Delta u_{\varepsilon}(y_{\varepsilon})||u_{\varepsilon}(y_{\varepsilon})| \to +\infty
\]  
(3.61)

as \( \varepsilon \to 0 \), for some \( y_{\varepsilon} \)'s such that \( y_{\varepsilon} \in \Omega_{\varepsilon} \). First for all sequence \((\tilde{z}_{\varepsilon})_{\varepsilon}\) such that \( \tilde{z}_{\varepsilon} \in \Omega_{\varepsilon} \), we have that \( \Delta u_{\varepsilon}(\tilde{z}_{\varepsilon}) > 0 \), that \( g'(u_{\varepsilon}(\tilde{z}_{\varepsilon})) = o(u_{\varepsilon}(\tilde{z}_{\varepsilon})) \) and that

\[
\Psi'_{N_{\varepsilon}}(u_{\varepsilon}(\tilde{z}_{\varepsilon})) = (1 + o(1))2u_{\varepsilon}(\tilde{z}_{\varepsilon})\varphi_{N_{\varepsilon}-1}(u_{\varepsilon}(\tilde{z}_{\varepsilon})^2),
\]  
(3.62)

as \( \varepsilon \to 0 \), using (1.1), (1.5), (3.3), (3.28) and (3.60). Besides, we have that

\[
u_{\varepsilon}(y_{\varepsilon}) \to +\infty,
\]  
(3.63)

as \( \varepsilon \to 0 \). Let \( \nu_{\varepsilon} > 0 \) be given by

\[
\nu_{\varepsilon}^2 |\Delta u_{\varepsilon}(y_{\varepsilon})||u_{\varepsilon}(y_{\varepsilon})| = 1.
\]  
(3.64)

Then, in view of (3.61) and (3.64), we have that

\[
\lim_{\varepsilon \to 0} \frac{|y_{\varepsilon} - x_{\varepsilon}|}{\nu_{\varepsilon}} = +\infty,
\]  
(3.65)

and, in view of Step 3.5, that

\[
\lim_{\varepsilon \to 0} \frac{|y_{\varepsilon} - x_{\varepsilon}|}{\mu_{\varepsilon}} = +\infty.
\]  
(3.66)

For \( R > 0 \), we set \( \Omega_{R,\varepsilon} = B_{y_{\varepsilon}}(R\nu_{\varepsilon}) \cap \Omega \) and \( \tilde{\Omega}_{R,\varepsilon} = (\Omega_{R,\varepsilon} - y_{\varepsilon})/\nu_{\varepsilon} \). Up to harmless rotations and since \( \Omega \) is smooth, we may assume that there exists \( B \in [0, +\infty) \) such that \( \tilde{\Omega}_{0,R} \to (-\infty, B) \times \mathbb{R} \) as \( R \to +\infty \), where \( \tilde{\Omega}_{R,\varepsilon} \to \tilde{\Omega}_{0,R} \) as \( \varepsilon \to 0 \). In this proof, for \( z \in \tilde{\Omega}_{R,\varepsilon} \), we write \( z_{\varepsilon} = y_{\varepsilon} + \nu_{\varepsilon}z \in \Omega_{R,\varepsilon} \). Let \( \tilde{u}_{\varepsilon} \) be given by

\[
\tilde{u}_{\varepsilon}(z) = u_{\varepsilon}(y_{\varepsilon})(u_{\varepsilon}(z_{\varepsilon}) - u_{\varepsilon}(y_{\varepsilon})),
\]  
(3.67)

so that we get

\[
(\Delta \tilde{u}_{\varepsilon})(z) = (\Delta u_{\varepsilon})(z_{\varepsilon}) = \frac{\Psi'_{N_{\varepsilon}}(z_{\varepsilon})}{\Psi'_{N_{\varepsilon}}(y_{\varepsilon})}.
\]  
(3.68)

First, we prove that for all \( R > 0 \), there exists \( C_R > 0 \) such that

\[
|\Delta \tilde{u}_{\varepsilon}| \leq C_R \text{ in } \tilde{\Omega}_{R,\varepsilon},
\]  
(3.69)

for all \( 0 < \varepsilon \ll 1 \). Otherwise, by (3.68), assume by contradiction that there exists \( z_{\varepsilon} \in \Omega_{R,\varepsilon} \) such that

\[
|\Psi'_{N_{\varepsilon}}(z_{\varepsilon})| \gg \Psi'_{N_{\varepsilon}}(y_{\varepsilon})
\]  
(3.70)

as \( \varepsilon \to 0 \). If, still by contradiction, \( z_{\varepsilon} \not\in \Omega_{\varepsilon} \), we have that \( u_{\varepsilon}(z_{\varepsilon}) < u_{\varepsilon}(y_{\varepsilon}) \), that

\[
\varphi_{N_{\varepsilon}-1}(u_{\varepsilon}(z_{\varepsilon})^2) < \varphi_{N_{\varepsilon}-1}(u_{\varepsilon}(y_{\varepsilon})^2),
\]

by definition of \( \Omega_{\varepsilon} \) and since \( \varphi_{N}/(1 + \cdot) \) increases in \([0, +\infty)\), and then that

\[
|\Psi'_{N_{\varepsilon}}(u_{\varepsilon}(z_{\varepsilon}))| \lesssim u_{\varepsilon}(z_{\varepsilon})(1 + u_{\varepsilon}(z_{\varepsilon})^2 + \varphi_{N_{\varepsilon}-1}(u_{\varepsilon}(z_{\varepsilon})^2)) \lesssim \Psi'_{N_{\varepsilon}}(u_{\varepsilon}(y_{\varepsilon})),
\]

using (1.1), (3.3), (3.28), (3.62) and \( y_{\varepsilon} \in \Omega_{\varepsilon} \) again. This contradicts (3.70) and then it must be the case that \( z_{\varepsilon} \in \Omega_{\varepsilon} \). Thus, since \( y_{\varepsilon} \) is a maximizer on \( \Omega_{\varepsilon} \) in \( (3.61) \), we get from (3.65) and (3.70) that \( u_{\varepsilon}(z_{\varepsilon}) \ll u_{\varepsilon}(y_{\varepsilon}) \). But this is not possible by (3.62) and (3.70), which proves (3.69). Now we prove that, for all \( R > 0 \),

\[
\limsup_{\varepsilon \to 0} \sup_{z \in \tilde{\Omega}_{R,\varepsilon}} \tilde{u}_{\varepsilon}(z) \leq 0.
\]  
(3.71)
Until the end of this proof, we set $\tilde{\gamma}_\varepsilon := u_\varepsilon(y_\varepsilon)$. If (3.71) does not hold true, since $\tilde{u}_\varepsilon(0) = 0$ and by continuity, we may assume that there exist $z_\varepsilon \in \Omega_{R,\varepsilon}$ such that

$$
\beta_\varepsilon := [\tilde{\gamma}_\varepsilon (u_\varepsilon(z_\varepsilon) - \tilde{\gamma}_\varepsilon)] \to \beta_0 \in (0, +\infty),
$$

(3.72)
as $\varepsilon \to 0$. Since $u_\varepsilon(z_\varepsilon) > u_\varepsilon(y_\varepsilon)$ for $0 < \varepsilon \ll 1$ by (3.72), we have that $z_\varepsilon \in \Omega_\varepsilon$. Moreover, since $y_\varepsilon$ is maximizing in (3.61), we then get from (3.62), (3.63) and (3.65) that

$$
\varphi_{N,-1}(u_\varepsilon(z_\varepsilon))^2 \leq (1 + o(1)) \varphi_{N,-1}(\tilde{\gamma}_\varepsilon^2).
$$

(3.73)

Independently, since $\varphi_N$ is convex, we get that

$$
\varphi_{N,-1}(u_\varepsilon(z_\varepsilon))^2 \geq \varphi_{N,-1}(\tilde{\gamma}_\varepsilon^2) + \varphi'_{N,-1}(\tilde{\gamma}_\varepsilon^2) (u_\varepsilon(z_\varepsilon)^2 - \tilde{\gamma}_\varepsilon^2),
$$

(3.74)
using (3.72) and $\varphi'_N(t) \geq \varphi_N(t)$ for $t \geq 0$. But (3.72)-(3.74) cannot hold true simultaneously, which proves (3.71). As in [8, p.231], $\tilde{u}_\varepsilon(0) = 0$, $u_\varepsilon = 0$ on $\partial \Omega$, (3.69) and (3.71) imply that

$$
\lim_{\varepsilon \to 0} \frac{d(y_\varepsilon, \partial \Omega)}{\nu_\varepsilon} = +\infty.
$$

(3.75)

Moreover, by standard elliptic theory, $\tilde{u}_\varepsilon(0) = 0$, (3.69), (3.71) and (3.75) give that

$$
\tilde{u}_\varepsilon \to u_0 \text{ in } C^1_{loc}(\mathbb{R}^2),
$$

(3.76)
as $\varepsilon \to 0$, for some $u_0 \in C^1(\mathbb{R}^2)$. Given $R > 0$, we prove now that

$$
\liminf_{\varepsilon \to 0} \inf_{z \in \tilde{\Omega}_{R,\varepsilon}} (\Delta \tilde{u}_\varepsilon)(z) > 0.
$$

(3.77)

Using (3.28), (3.63) and (3.76), we have that

$$
\Psi'_{N,-1}(u_\varepsilon) = 2\tilde{\gamma}_\varepsilon \varphi_{N,-1}(u_\varepsilon^2)(1 + o(1)) + o(\tilde{\gamma}_\varepsilon^3),
$$

uniformly in $\Omega_{R,\varepsilon}$. Then, coming back to (3.68), using (3.62) and $y_\varepsilon \in \Omega_\varepsilon$, we get that

$$
(\Delta \tilde{u}_\varepsilon)(z) = (1 + o(1)) \frac{\varphi_{N,-1}(u_\varepsilon(z_\varepsilon)^2)}{\varphi_{N,-1}(\tilde{\gamma}_\varepsilon^2)} + o(1),
$$

uniformly in $z \in \tilde{\Omega}_{R,\varepsilon}$. Now, we write (3.51) with $\Gamma = \tilde{\gamma}_\varepsilon^2$ and $T = u_\varepsilon^2$. Then, in order to conclude the proof of (3.77), using also (3.38), it is sufficient to check that there exists $\eta_R < 1$ such that

$$
I_\varepsilon := \exp(u_\varepsilon^2) \frac{\varphi'_{N,-1}(\tilde{\gamma}_\varepsilon^2) \exp(- (\tilde{\gamma}_\varepsilon^2 - u_\varepsilon^2))}{\varphi_{N,-1}(\tilde{\gamma}_\varepsilon^2)} \int_{u_\varepsilon^2}^{\tilde{\gamma}_\varepsilon^2} \exp(-s) \frac{N_s}{\varphi_{N,-1}(\tilde{\gamma}_\varepsilon^2)} ds = \int_{u_\varepsilon^2}^{\tilde{\gamma}_\varepsilon^2} \exp(-s) \frac{N_s}{\varphi_{N,-1}(\tilde{\gamma}_\varepsilon^2)} ds \
\leq \eta_R,
$$

(3.78)
for all $0 < \varepsilon < 1$, uniformly in $\Omega_{R,\varepsilon}$, where $\tilde{N}_\varepsilon = N_\varepsilon - 1$. If $u_\varepsilon \geq \tilde{\gamma}_\varepsilon$, the last inequality in (3.78) is obvious. If now $u_\varepsilon < \tilde{\gamma}_\varepsilon$, we write

$$I_\varepsilon \leq \frac{\int_{u_\varepsilon^2 - \frac{3}{2}}^{0} \exp(-t) \left(1 + \frac{t}{\varepsilon^2} \right)^{N_\varepsilon} dt}{\int_{2(u_\varepsilon^2 - \frac{3}{2})}^{0} \exp(-t) \left(1 + \frac{t}{\varepsilon^2} \right)^{N_\varepsilon} dt} \leq \frac{\int_{u_\varepsilon^2 - \frac{3}{2}}^{0} \exp \left(t \left( \frac{S_\varepsilon^2}{\varepsilon^2} - 1 + O \left( \frac{S_\varepsilon^2}{\varepsilon^2} \right) \right) \right) dt}{\int_{2(u_\varepsilon^2 - \frac{3}{2})}^{0} \exp \left(t \left( \frac{S_\varepsilon^2}{\varepsilon^2} - 1 + O \left( \frac{S_\varepsilon^2}{\varepsilon^2} \right) \right) \right) dt} \quad (3.79)$$

using (3.76), where $I_\varepsilon$ is as in (3.78). We get the last inequality using (3.60) and $y_\varepsilon \in \Omega_\varepsilon$: (3.78) and then (3.77) are proved in any case. But (3.63), (3.65), (3.66), (3.76) and (3.77) clearly contradict (3.56), which concludes the proof of (3.58). □

**Proof of Step 3.6, Formula (3.59).** Remember that we assume that (Case 1) holds true. Assume then by contradiction that there exists $(y_\varepsilon)_{\varepsilon}$ such that $y_\varepsilon \in \Omega_\varepsilon$ and

$$\max_{y \in \Omega_\varepsilon} |y - x_\varepsilon||\nabla u_\varepsilon(y)||u_\varepsilon(y)| = |y_\varepsilon - x_\varepsilon||\nabla u_\varepsilon(y_\varepsilon)||u_\varepsilon(y_\varepsilon)| := C_\varepsilon \to +\infty \quad (3.80)$$

as $\varepsilon \to 0$. Then, by (3.60), (3.63) holds true. Let $\nu_\varepsilon > 0$ be given by

$$\nu_\varepsilon = \min \{|x_\varepsilon - y_\varepsilon|, d(y_\varepsilon, \partial \Omega)|. \quad (3.81)$$

For all $R > 1$ and all $\varepsilon$, let $\Omega_{R,\varepsilon}$ and $\tilde{\Omega}_{R,\varepsilon}$ be given by the formulas above (3.67). Let $w_\varepsilon$ be given by

$$w_\varepsilon(z) = u_\varepsilon(y_\varepsilon + \nu_\varepsilon z). \quad (3.82)$$

Since $\|u_\varepsilon\|_{H_0^1} \leq 4\pi$, we get from Moser’s inequality that $\int_{\tilde{\Omega}} \exp(u_\varepsilon^2) dy = O(1)$ and then that, for all given $p \geq 1$,

$$\|\nu_\varepsilon^{2/p} w_\varepsilon\|_{L^p(\tilde{\Omega}_{R,\varepsilon})} = O(1), \quad (3.83)$$

for all $\varepsilon$. Now, for any given $R > 1$ and all sequence $(z_\varepsilon)_{\varepsilon}$ such that $z_\varepsilon \in \Omega_{R,\varepsilon} \setminus \{x_\varepsilon\}$ (i.e. $\tilde{z}_\varepsilon := (z_\varepsilon - y_\varepsilon)/\nu_\varepsilon \in \Omega_{R,\varepsilon} \setminus \{\tilde{x}_\varepsilon\}$), we get that

$$|\Delta w_\varepsilon(\tilde{z}_\varepsilon)| = \nu_\varepsilon^2 |\Delta u_\varepsilon(z_\varepsilon)| \lesssim \begin{cases} \frac{1}{\nu_\varepsilon^2(z_\varepsilon) \|\nabla u_\varepsilon(z_\varepsilon)\|^2} & \text{if } z_\varepsilon \in \Omega_\varepsilon, \\ \lambda_\varepsilon \nu_\varepsilon^2 |\Psi_{N_\varepsilon}(u_\varepsilon(z_\varepsilon))| & \text{if } z_\varepsilon \in \tilde{\Omega}_\varepsilon \end{cases} \quad (3.84)$$

using (3.58) for the first line, and (3.28) for the second one. Then, using either (3.60) or (3.41) with (3.83), we get that

$$\|\Delta w_\varepsilon\|_{L^p(\tilde{\Omega}_{R,\varepsilon} \setminus B_{\varepsilon}(1/R))} \to 0 \quad (3.84)$$

as $\varepsilon \to 0$. Independently, since $\|u_\varepsilon\|_{H_0^1} = O(1)$, we easy get that

$$\int_{\tilde{\Omega}_{R,\varepsilon}} |\nabla w_\varepsilon|^2 dz = O(1). \quad (3.85)$$

Set $\tilde{x}_\varepsilon = \frac{x_\varepsilon - y_\varepsilon}{\nu_\varepsilon}$. Observe that $|\tilde{x}_\varepsilon| \geq 1$. Now, we claim that up to a subsequence,

$$\nu_\varepsilon \to 0 \text{ and } \frac{d(y_\varepsilon, \partial \Omega)}{|x_\varepsilon - y_\varepsilon|} \to +\infty, \quad (3.86)$$

as $\varepsilon \to 0$. In particular, by (3.81), this implies that $\nu_\varepsilon = |x_\varepsilon - y_\varepsilon|$. Now we prove (3.86). Indeed, if we assume by contradiction that (3.86) does not hold, for
all $R \gg 1$ sufficiently large, we get that the $(w_{z}/u_{z}(y_{z}))$’s converge locally out of $B_{\delta z}(1/2)$ to some $C^{1}$ function which is 1 at 0 and 0 on the non-empty and smooth boundary of $\lim_{R \to +\infty} \lim_{\varepsilon \to 0} \Omega_{R,\varepsilon}$ (maybe after a harmless rotation). We use here the Harnack inequality and elliptic theory with (3.63), (3.84) (with $p > 2$) and (3.85), since $u_{z} = 0$ in $\partial \Omega$. This clearly contradicts (3.85) and (3.86) is proved. Up to a subsequence, we may now assume that

$$\tilde{x}_{\varepsilon} \to \tilde{x}, \quad |\tilde{x}| = 1,$$

(3.87)
as $\varepsilon \to 0$. By (3.63), (3.84), (3.85), and similar arguments including again Harnack’s principle, we get that

$$\frac{w_{z}(y_{z})}{u_{z}(y_{z})} \to 1 \text{ in } C^{1}_{loc}(\mathbb{R}^{2}\backslash\{\tilde{x}\}),$$

(3.88)using also (3.86). By (3.83) and (3.88), we get that for all $p \geq 1$

$$u_{\varepsilon}^{2/p}u_{z}(y_{z}) = O(1),$$

(3.89)as $\varepsilon \to 0$. Let now $\tilde{w}_{\varepsilon}$ be given by $\tilde{w}_{\varepsilon} = \frac{w_{z} - w_{z}(0)}{\varepsilon |\nabla w_{z}(y_{z})|}$, so that $|\nabla \tilde{w}_{\varepsilon}(0)| = 1$. For any given $R > 1$ and all sequence $(z_{\varepsilon})$ such that $\tilde{z}_{\varepsilon} := (z_{\varepsilon} - y_{z})/\varepsilon_{\varepsilon} \in \tilde{\Omega}_{R,\varepsilon}\backslash B_{\varepsilon}(1/R)$, we get that

$$|\Delta \tilde{w}_{\varepsilon}(\tilde{z}_{\varepsilon})| = \frac{u_{z}(y_{z})}{C_{\varepsilon}}|\Delta w_{z}(\tilde{z}_{\varepsilon})| \lesssim \begin{cases} \frac{1}{(\varepsilon_{\varepsilon} |\nabla w_{z}(y_{z})|)^{2}} \text{ if } z_{\varepsilon} \in \Omega_{\varepsilon}, \\ \frac{1}{C_{\varepsilon}^{2}u_{z}^{2}(y_{z})^{4}} \text{ if } z_{\varepsilon} \notin \Omega_{\varepsilon}, \end{cases}$$

for all $\varepsilon$, using (3.58), (3.80) and (3.88). Then, by (3.41), (3.80), (3.86) and (3.89) (with $p \geq 4$), we get that

$$\Delta \tilde{w}_{\varepsilon} \to 0 \text{ in } L_{loc}^{\infty}(\mathbb{R}^{2}\backslash\{\tilde{x}\}),$$

(3.90)as $\varepsilon \to 0$. By (3.80), (3.87) and (3.88), given $R > 1$ and $\tilde{z}_{\varepsilon} \in \tilde{\Omega}_{R,\varepsilon}\backslash B_{\varepsilon}(1/R)$, we get that

$$|\nabla \tilde{w}_{\varepsilon}(\tilde{z}_{\varepsilon})| = \frac{|\nabla u_{z}(z_{\varepsilon})|}{|\nabla w_{z}(y_{z})|} \leq \frac{u_{z}(y_{z})}{u_{z}(z_{\varepsilon})} \frac{1}{|x_{\varepsilon} - z_{\varepsilon}|} \leq \frac{1 + o(1)}{|x_{\varepsilon} - z_{\varepsilon}|},$$

(3.91)for all $0 < \varepsilon \ll 1$. Then, by (3.90), (3.91) and since $\tilde{w}_{\varepsilon}(0) = 0$, there exists a harmonic function $H$ in $\mathbb{R}^{2}\backslash\{\tilde{x}\}$ such that $\lim_{\varepsilon \to 0} \tilde{w}_{\varepsilon} = H$ in $C^{1}_{loc}(\mathbb{R}^{2}\backslash\{\tilde{x}\})$. Now, for all given $\beta > 0$, integrating by parts, we get that

$$\int_{\partial B_{\delta z}(\beta \nu_{z})} u_{z} \partial_{\nu_{z}} u_{z} d\sigma = O \left( \int_{\Omega} |\nabla u_{z}|^{2} \, dy \right) + O \left( \int_{\Omega} u_{z} \Delta u_{z} \, dy \right) = O(1),$$

(3.92)

$$= C_{\varepsilon} \left( \int_{\partial B_{\delta z}(\beta \nu_{z})} \partial_{\nu_{z}} H d\sigma + o(1) \right),$$

using (3.80) and (3.88), as $\varepsilon \to 0$. Since $C_{\varepsilon} \to +\infty$, we get from (3.92) that $\int_{\partial B_{\delta z}(\beta \nu_{z})} \partial_{\nu_{z}} H d\sigma = 0$. Then, also by (3.91), $\beta$ being arbitrary, $H$ is bounded around $\tilde{x}$ and then the singularity at $\tilde{x}$ is removable. By the Liouville theorem, $H$ is constant in $\mathbb{R}^{2}$, which is not possible since $|\nabla \tilde{w}_{\varepsilon}(0)| = |\nabla H(0)| = 1$. This concludes the proof of (3.59).
Remark 3.3. Note that we do not assume that the continuous function $\Psi'_N$ is positive and increasing in $[0, +\infty)$. Then, standard moving plane techniques \cite{1, 5, 13, 15} do not apply. We use in the proof below the variational characterization (3.6) of the $u_\varepsilon$’s to get that $\bar{x} \in K_\Omega$, $K_\Omega$ as in (1.9), and that, in particular, $\bar{x} \notin \partial \Omega$ in (3.12).

Let $B_\varepsilon$ be the radial solution around $x_\varepsilon$ of
\[
\begin{cases}
\Delta B_\varepsilon = \frac{\lambda_\varepsilon}{2} \Psi'_N(B_\varepsilon), \\
B_\varepsilon(x_\varepsilon) = \gamma_\varepsilon,
\end{cases}
\] (3.93)
where $\gamma_\varepsilon$ is still given by (3.13). Let $\bar{u}_\varepsilon$ be given by
\[
\bar{u}_\varepsilon(z) = \frac{1}{2\pi |x_\varepsilon - z|} \int_{\partial B_{x_\varepsilon}(|x_\varepsilon - z|)} u_\varepsilon d\sigma,
\] (3.94)
for all $z \neq x_\varepsilon$ and $\bar{u}_\varepsilon(x_\varepsilon) = u_\varepsilon(x_\varepsilon) = \gamma_\varepsilon$. Let $\varepsilon_0 \in (\sqrt{1/\varepsilon}, 1)$ be given. Let $\rho_\varepsilon > 0$ be given by
\[
t_\varepsilon(\rho_\varepsilon) = (1 - \varepsilon_0)\gamma_\varepsilon^2.
\] (3.95)
By (3.3), (3.41), (3.42) and (3.45), we have that
\[
\rho_\varepsilon^2 = \exp(-(\varepsilon_0 + o(1))\gamma_\varepsilon^2).
\] (3.96)
Let $r_\varepsilon$ be given by
\[
r_\varepsilon = \sup \left\{ r \in (0, \rho_\varepsilon] \text{ s.t. } |\bar{u}_\varepsilon - B_\varepsilon| \leq \frac{1}{\gamma_\varepsilon} \text{ in } B_{x_\varepsilon}(r) \right\}.
\] (3.97)
Observe that $r_\varepsilon \gg \mu_\varepsilon$ by Step 3.5 and Appendix A. Then, we state the following key result.

Step 3.7. We have that
\[
\bar{u}_\varepsilon(r_\varepsilon) = B_\varepsilon(r_\varepsilon) + o \left( \frac{1}{\gamma_\varepsilon} \right),
\] (3.98)
and then that $r_\varepsilon = \rho_\varepsilon$ for all $0 < \varepsilon \ll 1$. Moreover, there exists $C > 0$ such that
\[
|\nabla(B_\varepsilon - u_\varepsilon)| \leq \frac{C}{\rho_\varepsilon \gamma_\varepsilon} \text{ in } B_{x_\varepsilon}(\rho_\varepsilon),
\] (3.99)
for all $0 < \varepsilon \ll 1$, where $(x_\varepsilon)_\varepsilon$ is as in (3.13), $B_\varepsilon$ as in (3.93), $\bar{u}_\varepsilon$ as in (3.94), $\rho_\varepsilon$ as in (3.95) and $r_\varepsilon$ as in (3.97).

Since $B_\varepsilon(x_\varepsilon) = u_\varepsilon(x_\varepsilon) = \gamma_\varepsilon$, (3.99) obviously implies that
\[
|B_\varepsilon - u_\varepsilon| \leq \frac{C}{\rho_\varepsilon \gamma_\varepsilon} \text{ in } B_{x_\varepsilon}(\rho_\varepsilon),
\] (3.100)
for all $0 < \varepsilon \ll 1$. Then, combined with Appendix A, Step 3.7 provides pointwise estimates of the $u_\varepsilon$’s in $B_{x_\varepsilon}(\rho_\varepsilon)$.

Proof of Step 3.7. The proof of Lemma 3.7 follows the lines of \cite[Section 3]{9}. We only recall here the argument in the more delicate (Case 1). Let $v_\varepsilon$ be given by
\[
u_\varepsilon = B_\varepsilon + v_\varepsilon.
\] (3.101)
By Appendix A, we have that $B_\varepsilon$ is well defined, radially decreasing in $B_{x_\varepsilon}(\rho_\varepsilon)$, and that
\[
B_\varepsilon = \gamma_\varepsilon - \frac{t_\varepsilon}{\gamma_\varepsilon} + o \left( \frac{t_\varepsilon}{\gamma_\varepsilon} \right)
\] (3.102)
uniformly in $B_{x_{e}}(\rho_{e})$ as $\varepsilon \to 0$. Then, we get from (3.96) and (3.97) that
\[
\min_{B_{x_{e}}(r_{e})} u_{\varepsilon} \geq \gamma_{\varepsilon}(\varepsilon_{0} + o(1)).
\] (3.103)
First, (3.103) combined with (3.32), with (3.60) and with our assumption $\varepsilon_{0}^{2} > 1/e$ implies that $B_{x_{e}}(r_{e}) \subset \Omega_{\varepsilon}$. Then, we can use (3.59) to get also from (3.103) that
\[
||x_{e} - |\nabla u_{\varepsilon}||_{L^{\infty}(B_{x_{e}}(r_{e}))} = O\left(\frac{1}{\gamma_{\varepsilon}}\right),
\] (3.104)
which implies by (3.97) that
\[
\|v_{\varepsilon}\|_{L^{\infty}(B_{x_{e}}(r_{e}))} = O\left(\frac{1}{\gamma_{\varepsilon}}\right),
\] (3.105)
by the mean value property. Therefore, since
\[
B_{\varepsilon} \leq \gamma_{\varepsilon}
\] (3.106)
in $B_{x_{e}}(r_{e})$ and by (1.1), (1.5), Lemma 3.2, (3.27), (3.28), (3.93), (3.97), (3.102) and (3.103), we get that there exists $C, C' > 0$ such that
\[
|\Delta v_{\varepsilon}| \leq C\lambda_{\varepsilon}\gamma_{\varepsilon}^{2} \varphi_{N_{\varepsilon}}^{-2}\left(\gamma_{\varepsilon}^{2} - 2t_{\varepsilon}(1 + o(1)) + \frac{t_{\varepsilon}^{2}}{\gamma_{\varepsilon}^{2}}\right)|v_{\varepsilon}|\mbox{ in } B_{x_{e}}(r_{e}),
\] an then that
\[
|\Delta v_{\varepsilon}| \leq C'\frac{\exp\left(-2t_{\varepsilon}(1 + o(1)) + \frac{t_{\varepsilon}^{2}}{\gamma_{\varepsilon}^{2}}\right)}{\rho_{e}^{2}}|v_{\varepsilon}|\mbox{ in } B_{x_{e}}(r_{e})
\] (3.107)
by (3.31), (3.42) and (3.47). Observe that, for all $\Gamma, \delta > 0$,
\[
\varphi_{N}(\Gamma) = \delta \exp(\Gamma) \implies \forall T \in [0, \Gamma], \quad \varphi_{N}(T) \leq \delta \exp(T),
\] (3.108)
since $\varphi_{N}' \geq \varphi_{N}$ in $[0, +\infty]$. Starting now from (3.104)-(3.107), we can compute and argue as in [9, Section 3] in order to get (3.98)-(3.99).

\[\square\]

Conclusion of the proof of Lemma 3.3. In order to conclude the proof of Lemma 3.3, by Steps 3.1-3.7, it remains to prove (2.4), (3.10)-(3.12), and (3.137)-(3.139) below. Let $\varepsilon_{0} \in (\varepsilon_{0}, 1)$ be fixed and let $\rho_{e}' > 0$ be given by
\[
t_{\varepsilon}(\rho_{e}') = (1 - \varepsilon_{0})\gamma_{\varepsilon}^{2},
\] (3.109)
so that, by (3.46),
\[
(\rho_{e}')^{2} = \exp(-\varepsilon_{0}(1 + o(1))\gamma_{\varepsilon}^{2}.
\] (3.110)

\[\bullet\]

1. In this first point, we aim to get pointwise estimates of the $u_{\varepsilon}$'s out of $B_{x_{e}}(\rho_{e}')$. Let $G$ be the Green's function in (1.8). It is known that (see for instance [9, Appendix B]) there exists $C > 0$ such that
\[
|\nabla y G_{x}(y)| \leq \frac{C}{|x - y|}, \quad \mbox{and} \quad 0 < G_{x}(y) \leq \frac{1}{2\pi} \log\frac{C}{|x - y|},
\] (3.111)
for all $x, y \in \Omega$, $x \neq y$. By (3.99) and since $\|u_{\varepsilon}\|_{H^{2}}^{2} \leq 4\pi$, it is possible to prove (see for instance the proof of [9, Claim 4.6]) that, given $p < 1/\varepsilon_{0}'$,\[
\|\exp(u_{\varepsilon}^{2})\|_{L^{p}(B_{x_{e}}(\rho_{e}'/2))} = O(1).
\] (3.112)
for all \( \varepsilon \), where \( B_{x_\varepsilon}(\rho'_\varepsilon/2) = \Omega \setminus B_{x_\varepsilon}(\rho'_\varepsilon/2) \). In the sequel, \( p' > 1 \) is chosen such that
\[
\frac{1}{p} + \frac{1}{p'} < 1. \tag{3.113}
\]
Let now \((z_\varepsilon)_\varepsilon\) be any sequence of points in \( B_{x_\varepsilon}(\rho'_\varepsilon) \). By the Green’s representation formula and (3.27), we can write that
\[
u_\varepsilon(z_\varepsilon) = \frac{\lambda_\varepsilon}{2} \int_{\Omega} G_{x_\varepsilon}(y) \Psi'_{N_\varepsilon}(u_\varepsilon(y)) \, dy. \tag{3.114}
\]
By (3.111), we have that there exists \( C > \) 0 such that
\[
|G_{x_\varepsilon}(x_\varepsilon) - G_{x_\varepsilon}| \leq C |x_\varepsilon - | \rho'_\varepsilon| \tag{3.115}
\]
in \( B_{x_\varepsilon}(\rho'_\varepsilon/2) \), for all \( \varepsilon \). By (3.46) and (3.96), we have that
\[
\frac{| - x_\varepsilon |}{\gamma_\varepsilon \rho_\varepsilon} = o \left( \frac{t_\varepsilon}{\gamma_\varepsilon} \right) \text{ in } \tilde{\Omega}_\varepsilon := \{ y \text{ s.t. } t_\varepsilon(y) \leq \gamma_\varepsilon \}, \tag{3.116}
\]
as \( \varepsilon \to 0 \), and then, by (3.100), (A.9) holds true for \( u_\varepsilon \) as in (3.101). Independently, using (3.30), (3.42), (3.100) and (A.3) with (A.7), we clearly get that there exists \( C > 0 \) such that
\[
\lambda_\varepsilon |\Psi'_{N_\varepsilon}(u_\varepsilon)| \leq C \frac{\exp \left( -2t_\varepsilon + \frac{t_\varepsilon^2}{\gamma_\varepsilon} \right)}{\rho'_\varepsilon} \text{ in } B_{x_\varepsilon}(\rho'_\varepsilon/2) \setminus \tilde{\Omega}_\varepsilon, \tag{3.117}
\]
for all \( \varepsilon \). Then, we get that
\[
u_\varepsilon(z_\varepsilon) = G_{x_\varepsilon}(x_\varepsilon) \int_{B_{x_\varepsilon}(\rho'_\varepsilon/2)} \frac{\lambda_\varepsilon \Psi'_{N_\varepsilon}(u_\varepsilon)}{2} \, dy + O \left( \int_{B_{x_\varepsilon}(\rho'_\varepsilon/2)} \frac{\exp \left( -2t_\varepsilon + \frac{t_\varepsilon^2}{\gamma_\varepsilon} \right) \, | - x_\varepsilon |}{\rho'_\varepsilon} \, dy \right) + O \left( \lambda_\varepsilon \| u_\varepsilon \|_{L^{p'}} \right) \tag{3.118}
\]
\[
= G_{x_\varepsilon}(x_\varepsilon) \frac{4\pi}{\gamma_\varepsilon} \left( 1 + \frac{1}{\gamma_\varepsilon} \frac{A(\gamma_\varepsilon) - 2\xi_\varepsilon}{2} + o(\zeta_\varepsilon) \right) + o \left( \frac{1}{\gamma_\varepsilon} + \| u_\varepsilon \|_{L^{p'}} \right),
\]
where \( p' \) is fixed in (3.113), \( \zeta_\varepsilon \) is given by (3.15) and \( x_\varepsilon \) by (3.14). Concerning the first estimate of (3.118), (3.115), (3.117) and a rough version of (A.9) are used to get the first two terms, while (3.30), (3.111), (3.112) and Hölder’s inequality are used to get the last one. Concerning the second estimate of (3.118), (3.41), (3.46), (3.96), (A.2)-(A.4), \( \varepsilon_0 > 1/2 \), the dominated convergence theorem, (A.9) and (3.117) are used. Using first that \( u_\varepsilon \leq \gamma_\varepsilon \) and (3.96) in \( B_{x_\varepsilon}(\rho_\varepsilon) \), and then (3.118) with (3.111) in \( \Omega \setminus B_{x_\varepsilon}(\rho_\varepsilon) \), we get that
\[
\| u_\varepsilon \|_{L^{p'}} = o \left( \frac{1}{\gamma_\varepsilon} + \| u_\varepsilon \|_{L^{p'}} \right) + O \left( \frac{1}{\gamma_\varepsilon} \right). \tag{3.119}
\]
Summarizing, we get from (3.118) and (3.119) that
\[
u_\varepsilon(z_\varepsilon) = \frac{4\pi G_{x_\varepsilon}(x_\varepsilon)}{\gamma_\varepsilon} \left( 1 + \frac{1}{\gamma_\varepsilon} \frac{A(\gamma_\varepsilon) - 2\xi_\varepsilon}{2} + o(\zeta_\varepsilon) \right) + o \left( \frac{1}{\gamma_\varepsilon} \right). \tag{3.120}
\]
• (2) In this second point, we prove that

\[
\lambda_\varepsilon \leq \frac{4 + o(1)}{\gamma_\varepsilon^2 \exp(1 + M)},
\]

as \(\varepsilon \to 0\), for \(M\) as in (1.9). Observe that (3.120) implies that

\[
u_\varepsilon = (1 + o(1)) \frac{4\pi G_{x_\varepsilon} + o(1)}{\gamma_\varepsilon}
\]

in \(\Omega \setminus B_{x_\varepsilon}(\rho_\varepsilon)\). Then, by (1.1), (3.111) and (3.122), our definition of \(\rho_\varepsilon\) and the dominated convergence theorem, we get that

\[
\lim_{\varepsilon \to 0} \int_{\Omega \setminus B_{x_\varepsilon}(\rho_\varepsilon)} \Psi_{N_\varepsilon}(\nu_\varepsilon) \, dy = |\Omega|(1 + g(0)).
\]

Independently, (A.7) and (3.100) give that

\[
u_\varepsilon = \gamma_\varepsilon - \frac{(1 + o(1)) t_\varepsilon}{\gamma_\varepsilon}
\]

in \(B_{x_\varepsilon}(\rho_\varepsilon)\), since \(\mu_\varepsilon \ll \rho_\varepsilon\). Then, using (3.31), (3.47), \(\varepsilon_0^2 > 1/e\) and resuming the arguments to get (3.62), we have that

\[
\Psi_{N_\varepsilon}(\nu_\varepsilon) = (1 + o(1)) \phi_{N_\varepsilon} - \frac{1}{4\pi \phi_{N_\varepsilon}^2}(u_\varepsilon^2)
\]

in \(B_{x_\varepsilon}(\rho_\varepsilon)\). Independently, we get that

\[
\int_{B_{x_\varepsilon}(\rho_\varepsilon)} \Psi_{N_\varepsilon}(\nu_\varepsilon) \, dy = 4\pi \frac{(1 + o(1))}{\gamma_\varepsilon^2 \lambda_\varepsilon}
\]

as \(\varepsilon \to 0\), by (3.31), (3.42), (3.124), (3.125), with (3.51) for \(|y - x_\varepsilon| \lesssim \mu_\varepsilon\), or with (3.108) and the dominated convergence theorem for \(|y - x_\varepsilon| \gg \mu_\varepsilon\). Then, because of (3.6), we get that (3.121) holds true, by combining (3.123), (3.126) with (3.25).

• (3) In this point, we conclude the proof of (3.10), and prove (2.4) and (3.12). For \(R > 1\), let \(\chi_{x_\varepsilon,R}\) be given in \(\Omega_{x_\varepsilon,R} := \Omega \setminus B_{x_\varepsilon}(R\mu_\varepsilon)\) by

\[
\chi_{x_\varepsilon,R} = 4\pi \Lambda_{x_\varepsilon,R} G_{x_\varepsilon},
\]

for \(\Lambda_{x_\varepsilon,R} > 0\) to be chosen later such that

\[
\nabla \chi_{x_\varepsilon,R} \leq \nu_\varepsilon \text{ on } \partial B_{x_\varepsilon}(R\mu_\varepsilon).
\]

Integrating by parts, we can write that

\[
\int_{\Omega_{x_\varepsilon,R}} \nabla \nu_\varepsilon^2 \, dy = \int_{\Omega_{x_\varepsilon,R}} \nabla \chi_{x_\varepsilon,R}^2 \, dy - 2 \int_{\partial B_{x_\varepsilon}(R\mu_\varepsilon)} (\partial_\nu \chi_{x_\varepsilon,R})(u_\varepsilon - \chi_{x_\varepsilon,R}) \, d\sigma
\]

\[+ \int_{\Omega_{x_\varepsilon,R}} \nabla (u_\varepsilon - \chi_{x_\varepsilon,R})^2 \, dy,
\]

\[
\geq \int_{\Omega_{x_\varepsilon,R}} \nabla \chi_{x_\varepsilon,R}^2 \, dy,
\]

where \(\nu\) is the unit outward normal to the boundary of \(B_{x_\varepsilon}(R\mu_\varepsilon)\), using (3.128). Indeed, by [9, Appendix B] for instance, since \(d(x_\varepsilon, \partial \Omega) > \mu_\varepsilon\) by Step 3.5, we have that

\[
\partial_\nu G_{x_\varepsilon} = -\frac{1}{2\pi R\mu_\varepsilon} + O\left(\frac{1}{d(x_\varepsilon, \partial \Omega)}\right) \text{ on } \partial B_{x_\varepsilon}(R\mu_\varepsilon).
\]
Now, by (3.3), (3.42), (3.44), (3.47), (3.96), in order to have (3.128), we can choose \( \Lambda_{\varepsilon, R} \) such that

\[
\Lambda_{\varepsilon, R} = \frac{1}{\gamma_{\varepsilon}} \left( 1 - \frac{\log(1 + R^2) + o(1)}{\gamma_{\varepsilon}^2} \right) \times \left( 1 + \frac{\log \frac{\delta_{\varepsilon, \lambda_{\varepsilon}}^2}{4R^2} + \mathcal{H}_{x_{\varepsilon}}(x_{\varepsilon}) + O \left( \frac{\gamma_{\varepsilon}^2}{R^2} \right) }{\gamma_{\varepsilon}^2} \right)^{-1},
\]

with \( \delta_{\varepsilon} \in (0, 1] \) as in (3.31). In (3.131), the term

\[
\frac{\mu_{\varepsilon}^2}{\rho_{\varepsilon}^2} = o(1)
\]

by (3.95), arguing as in (3.22), since \( d(x_{\varepsilon}, \partial\Omega) > \rho_{\varepsilon} \) by Step 3.7. Now, by (1.8), (3.130), (3.46) and (3.96) again, we compute and get that

\[
\int_{\Omega_{\varepsilon, R}} |\nabla \chi_{\varepsilon, R}|^2 dy \geq - \int_{\partial B_{x_{\varepsilon}}(R_{\rho_{\varepsilon}})} (\partial_{\nu} \chi_{\varepsilon, R}) \chi_{\varepsilon, R} d\sigma,
\]

\[
\geq 4\pi \left( 1 - \frac{2\log(1 + R^2) + o(1)}{\gamma_{\varepsilon}^2} \right) \left( 1 + \frac{\log \frac{\delta_{\varepsilon, \lambda_{\varepsilon}}^2}{4R^2} + \mathcal{H}_{x_{\varepsilon}}(x_{\varepsilon}) + o(1)}{\gamma_{\varepsilon}^2} \right)^{-1}
\]

using also (3.131). Independently, we compute

\[
\int_{B_{x_{\varepsilon}}(R_{\rho_{\varepsilon}})} |\nabla u_{\varepsilon}|^2 dy = \frac{4\pi}{\gamma_{\varepsilon}^2} \left( \log(1 + R^2) - \frac{R^2}{1 + R^2} + o(1) \right),
\]

by (3.44). Then, since \( \|u_{\varepsilon}\|_{H^1_0} \leq 4\pi \) by (3.6), by (3.129), (3.132) and (3.133), we get that

\[
\log \frac{\delta_{\varepsilon, \lambda_{\varepsilon}} + \mathcal{H}_{x_{\varepsilon}}(x_{\varepsilon})}{\gamma_{\varepsilon}^2} \geq o(1).
\]

Moreover, using also the definition (1.9) of \( M \), (3.121), \( \delta_{\varepsilon} \leq 1 \) and that \( R > 0 \) may be arbitrarily large, we get together that

\[
\delta_{\varepsilon} \to 1,
\]

and that (3.10) and (3.12) hold true. Observe that (3.134) can be obtained directly (Case 2). Then, (2.4) follows from (3.10), (3.123) and (3.126).

• (4) Now we prove (3.11). Since \( \varepsilon_0' > \varepsilon_0 \), we get from (3.96), (3.100), (3.110) and (A.7) that

\[
u_{\varepsilon} = \gamma_{\varepsilon} - \frac{t_{\varepsilon}}{\gamma_{\varepsilon}} - \frac{t_{\varepsilon}}{\gamma_{\varepsilon}^2} - (A(\gamma_{\varepsilon}) - 2\xi_{\varepsilon}) \frac{t_{\varepsilon}}{2\gamma_{\varepsilon}} + o \left( \frac{t_{\varepsilon} \xi_{\varepsilon}}{\gamma_{\varepsilon}} \right)
\]

uniformly in \( \{ y \in B_{x_{\varepsilon}}(\rho_{\varepsilon}') \text{ s.t. } t_{\varepsilon} \geq \gamma_{\varepsilon}/4 \} \), using also (A.3). Then, noting that the averages of (3.120) and (3.135) have to match on \( \partial B_{x_{\varepsilon}}(\rho_{\varepsilon}') \), we compute and get that

\[
\lambda_{\varepsilon} = \frac{4}{\gamma_{\varepsilon}^2} \exp \left( 1 + M + \frac{\gamma_{\varepsilon}^2(A(\gamma_{\varepsilon}) - 2\xi_{\varepsilon})}{2} + o(\xi_{\varepsilon}^2) \right).
\]
by (3.12), (3.134) and (3.42) with (3.3) and (3.47), observing that
\[ 1 \lesssim \gamma^2 G_{x_x} \lesssim 1, \quad 1 \lesssim \gamma^2 \xi \lesssim 1 \]
on \partial B_{x_x}(\rho'_{\varepsilon}), by (3.109) and (3.110) with (1.8) and (3.12). By (3.10) and (3.136), (3.11) is proved.

- (5) Here, we conclude the proof of Lemma 3.3. As an immediate consequence of (3.120), we get that
\[ \left| u_{\varepsilon}(y) - \frac{4\pi G_{x_x}(y)}{\gamma_{\varepsilon}} \right| = o \left( \frac{G_{x_x}(y)}{\gamma_{\varepsilon}} \right) \]as \( \varepsilon \to 0 \), uniformly in \( B_{x_x}(\rho'_{\varepsilon}) \). Pushing now one step further the above computations with very similar arguments, we easily get that
\[ u_{\varepsilon} = \gamma_{\varepsilon} - \frac{t_{\varepsilon}}{\gamma_{\varepsilon}} + \frac{S_0_{x_x}}{\gamma_{\varepsilon}} + \frac{S_1_{x_x}}{\gamma_{\varepsilon}} + (A(\gamma_{\varepsilon}) - 2\xi_{\varepsilon}) \frac{S_{2_{x_x}}}{\gamma_{\varepsilon}} + o \left( \frac{t_{\varepsilon}}{\gamma_{\varepsilon}} \right) \],
in \( B_{x_x}(\rho'_{\varepsilon}) \), where the \( S_{i_{x_x}} \)’s are as in (A.5). At last, using in particular (3.10) with (1.6) to improve the estimates in Point (1) of this proof, we get that
\[ u_{\varepsilon}(y) = G_{x_x}(y) \left( \frac{4\pi}{\gamma_{\varepsilon}} + \sum_{i=0}^{1} A_i \frac{1}{\gamma_{\varepsilon}^{i+2}} + \frac{A_2(A(\gamma_{\varepsilon}) - 2\xi_{\varepsilon})}{\gamma_{\varepsilon}} \right) \]
\[ + \frac{4B(\gamma_{\varepsilon})}{\gamma_{\varepsilon}^2 \exp(1 + H_{x_x}(x_{\varepsilon}))} \int_{\Omega} G_{y_x}(x)F(4\pi G_{x_x}(x)) \, dx \]
\[ + o \left( \frac{\zeta_{\varepsilon}}{\gamma_{\varepsilon}} G_{x_x}(y) + \frac{|B(\gamma_{\varepsilon})|}{\gamma_{\varepsilon}^2} \right) \],
in \( B_{x_x}(\rho'_{\varepsilon}) \), where \( F \) and \( B(\gamma_{\varepsilon}) \) are given in (1.6), where the \( A_i \)’s are as in (A.3), and where \( \zeta_{\varepsilon} \) is given in (A.8).

\( \square \)

Lemma 3.3 is proved.

\( \square \)

4. PROOF OF PROPOSITION 2.1

Proof of Proposition 2.1. Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^2 \). Let \( g \) be such that (1.1) and (1.5)-(1.6) hold true, for \( H \) as in (1.2), and let \( A(\gamma), B(\gamma) \) and \( F \) be thus given. Let \( (u_{\varepsilon})_{\varepsilon} \) be a sequence of nonnegative functions such that \( u_{\varepsilon} \) is a maximizer for \( (I_{g_{\varepsilon}}^0(1-\varepsilon))_{\varepsilon} \), for all \( 0 < \varepsilon \ll 1 \). Assume that (2.1) holds true. Then, we apply Lemma 3.3. (Case 2): (3.8) holds true for \( \alpha_{\varepsilon} = 4\pi(1-\varepsilon) \); there exists a sequence \( \{\lambda_{i_{x_x}}\}_{i} \) of real numbers such that \( u_{\varepsilon} \) is \( C^{1,\theta} \) and solves (2.2) in \( H_{\varepsilon}^0 \), using (3.9); (2.3) holds true by (3.13), (2.4) is also true. Moreover, \( (3.10)-(3.13), (3.137)-(3.139) \) and (A.9) (\( v_{\varepsilon} \) as in (3.101)) hold true still by Lemma 3.3. In order to conclude the proof of Proposition (2.1), it remains to prove (2.5)-(2.6). At last, we let \( \mu_{\varepsilon} \) be given by (3.42), for \( N_{x_x} = 1 \), since we consider here (Case 2)
In view of (3.139), for $z \in \Omega$, we let now $U_{\varepsilon,z}$ be given by
\[
U_{\varepsilon,z}(x) = \frac{1}{\gamma} \left( \log \frac{1}{|x-z|^2 + \mu_\varepsilon^2} + \tilde{H}_{-1,\varepsilon,z}(x) \right)_{(\ast)}
\]
\[+ \frac{1}{\gamma} \sum_{i=0}^{\gamma} \left( S_i \left( \frac{x-z}{\mu_\varepsilon} \right) + A_i \left( \log \frac{1}{\mu_\varepsilon^2} + \tilde{H}_{i,\varepsilon,z}(x) - B_i \right) \right)_{(\ast\ast)}
\]
\[+ \frac{A(\varepsilon)}{\gamma} \left( S_2 \left( \frac{x-z}{\mu_\varepsilon} \right) + A_2 \left( \log \frac{1}{\mu_\varepsilon^2} + \tilde{H}_{2,\varepsilon,z}(x) - B_2 \right) \right)_{(\ast\ast\ast)}
\]
\[+ \frac{4B(\varepsilon)}{\gamma^2} \exp(1 + \tilde{H}_z(z)) \int_\Omega G_z(y) F(4\pi G_z(y)) dy \]
where the $A_i, B_i$ are as in (A.3), where $H$ is as in (1.8), where the $\tilde{H}_{i,\varepsilon}$ are the unique harmonic functions in $\Omega$ such that the expressions involved in brackets $(\ast), (\ast\ast), (\ast\ast\ast)$ of (4.1) were zero at the boundary, and where $\mu_\varepsilon$ is given by
\[
U_{\varepsilon,z}(x) = \gamma_\varepsilon.
\]

The following result concludes the proof of Proposition 2.1.

**Lemma 4.1.** We have that
\[
S = \int_\Omega G_z(y) F(4\pi G_z(y)) dy, \quad \text{if} \quad \frac{\gamma_\varepsilon^{-3}}{\gamma_\varepsilon + |A(\gamma_\varepsilon)|} \not\rightarrow 0,
\]
as $\varepsilon \rightarrow 0$, where $S$ is as in (1.9) and $\varepsilon$ as in (3.12). Moreover, (2.5) holds true in any case.

**Proof of Lemma 4.1.** Let $K$ be a compact subset of $\Omega$ and $(z_\varepsilon)_{\varepsilon}$ be a given sequence of points of $K$. For simplicity, we let in the proof below $\hat{\gamma}_\varepsilon$ be given by
\[
\hat{\gamma}_\varepsilon = \max \left( \frac{1}{\gamma_\varepsilon^4} |A(\gamma_\varepsilon)|, \frac{|B(\gamma_\varepsilon)|}{\gamma_\varepsilon^3} \right).
\]
Observe also that we get from (3.16), (3.138) and (A.3) that
\[
\left| u_{\varepsilon}(y) - \left( \gamma_\varepsilon - \frac{t_{\varepsilon}(y)}{\gamma_\varepsilon} \right) \right| \leq C \gamma_\varepsilon,
\]
in $\{ y \text{ s.t. } \gamma_\varepsilon/2 \leq t_{\varepsilon}(y) \leq \gamma_\varepsilon(\gamma_\varepsilon - 1/2) \}$, as $\varepsilon \rightarrow 0$.

1. We first derive the following more explicit expression of the $\mu_\varepsilon$ from (4.2):
\[
\frac{4}{\mu_\varepsilon^2} \exp(\gamma_\varepsilon^2) \gamma_\varepsilon^2 \left( 1 + O \left( \hat{\gamma}_\varepsilon + \gamma_\varepsilon^4 |A(\gamma_\varepsilon)|^2 \right) \right) \times
\]
\[\left( 1 - \frac{\gamma_\varepsilon^2 A(\gamma_\varepsilon)}{2} - \frac{4B(\varepsilon)}{\gamma_\varepsilon^2} \exp(1 + \tilde{H}_z(z)) \int_\Omega G_z(y) F(4\pi G_z(y)) dy \right) \]
as $\varepsilon \rightarrow 0$. By the maximum principle and (A.3), we get that there exists $C_K > 0$ such that $|\hat{H}_{j,\varepsilon,z_{\varepsilon}}| \leq C_K$ in $\Omega$, so that, by elliptic theory, the $\hat{H}_{j,\varepsilon,z_{\varepsilon}}$’s are also bounded in $C^1_{\text{loc}}(\Omega)$ for all $\varepsilon$ and $j$. We get from (4.2) that $|\log \frac{1}{\mu_\varepsilon^2} - \gamma_\varepsilon^2| \leq C_K'$, and then that
\[
|\hat{H}_{j,\varepsilon,z_{\varepsilon}} - \hat{H}_z| \leq C_K' \gamma_\varepsilon^2 \exp (-2\gamma_\varepsilon^2) \text{ in } \Omega,
\]
for all $0 < \varepsilon \ll 1$ and $j \in \{-1, \ldots, 2\}$, by the maximum principle, (1.8) and (A.3). Rewriting then (4.2) as
\[
\gamma_{\varepsilon}^2 = \log \frac{1}{\mu_{\varepsilon}^2} \left(1 + \frac{A_0}{4a_{\gamma_{\varepsilon}}^2} + \frac{A_1}{4a_{\gamma_{\varepsilon}}^4} + \frac{A(\gamma_{\varepsilon})A_2}{4\pi} + \mathcal{H}_{\varepsilon}\left(1 + \frac{A_0}{4a_{\gamma_{\varepsilon}}^2}\right)\right) + \frac{B_0}{\gamma_{\varepsilon}^2} + \frac{4B(\gamma_{\varepsilon})}{\gamma_{\varepsilon}\exp(1 + \mathcal{H}_{\varepsilon} \circ \gamma_{\varepsilon})} \int_{\Omega} G_{\varepsilon}(y) F(4\pi G_{\varepsilon}(y)) \, dy\] (4.8)
we easily get (4.6), using (3.16) and (A.3) with $\frac{A_2}{4\pi} - \frac{A^2}{16\pi^2} - B_0 = 0$.

\begin{itemize}
  \item (2) We prove now that
  \[
  \int_{\Omega} |\nabla U_{\varepsilon,z_{\varepsilon}}|^2 \, dx = 4\pi \left(1 + I_{z_{\varepsilon}}(\gamma_{\varepsilon}) + o\left(\tilde{\zeta}_{\varepsilon}\right)\right),
  \tag{4.9}
  \]
as $\varepsilon \to 0$, where $I_{z_{\varepsilon}}(\gamma_{\varepsilon})$ is given by
\[
I_{z_{\varepsilon}}(\gamma_{\varepsilon}) = \gamma_{\varepsilon}^{-4} + \frac{A(\gamma_{\varepsilon})}{2} + \frac{4B(\gamma_{\varepsilon})}{\gamma_{\varepsilon}^2 \exp(1 + \mathcal{H}_{\varepsilon} \circ \gamma_{\varepsilon})} \int_{\Omega} G_{\varepsilon}(y) F(4\pi G_{\varepsilon}(y)) \, dy,
\tag{4.10}
\]
and where $U_{\varepsilon,z_{\varepsilon}}$ is given by (4.1)-(4.2). By (1.6) and elliptic theory,
\[
\left(x \mapsto \int_{\Omega} G_{\varepsilon}(y) F(4\pi G_{\varepsilon}(y)) \, dy\right)_{\varepsilon}
\]
is a bounded sequence in $C^1(\Omega)$. (4.11)

By construction of the $\tilde{\mathcal{H}}_{\varepsilon}$, we can write that
\[
\int_{\Omega} |\nabla U_{\varepsilon,z_{\varepsilon}}|^2 \, dy = \int_{\Omega} \Delta U_{\varepsilon,z_{\varepsilon}}(y) \, dy,
\]
where
\[
\left\{ \begin{array}{l}
\{y; \tilde{t}_{\varepsilon}(y) \leq \gamma_{\varepsilon}\} \\
\{y; \tilde{t}_{\varepsilon}(y) \geq \gamma_{\varepsilon}(\gamma_{\varepsilon} - 1)\}
\end{array}\right.
\]
\[
\int_{\{y; \tilde{t}_{\varepsilon}(y) \leq \gamma_{\varepsilon}\}} \left(\frac{\Delta (\tilde{t}_{\varepsilon})}{\gamma_{\varepsilon}} + \frac{\tilde{S}_{0,\varepsilon}}{\gamma_{\varepsilon}^3} + \frac{\tilde{S}_{1,\varepsilon}}{\gamma_{\varepsilon}^5} + \frac{\mathcal{A}(\gamma_{\varepsilon})\Delta \tilde{S}_{2,\varepsilon}}{\gamma_{\varepsilon}}\right) \times
\left(\gamma_{\varepsilon} - \tilde{t}_{\varepsilon} \gamma_{\varepsilon}^3 + \tilde{S}_{0,\varepsilon} \gamma_{\varepsilon}^3 + O\left(\left(\left|A(\gamma_{\varepsilon})\right| + \frac{1}{\gamma_{\varepsilon}}\right)(1 + \tilde{t}_{\varepsilon}) + \frac{|y - z_{\varepsilon}|}{\gamma_{\varepsilon}}\right)\right) \, dy
\tag{4.12}
\]
\[
+ \int_{\{y; \tilde{t}_{\varepsilon}(y) \geq \gamma_{\varepsilon}(\gamma_{\varepsilon} - 1)\}} \left(\frac{O \left(\tilde{\mu}_{\varepsilon}^2 \gamma_{\varepsilon}^4 + \frac{4\pi G_{\varepsilon}(y)}{\gamma_{\varepsilon}} \exp(1 + \mathcal{H}_{\varepsilon} \circ \gamma_{\varepsilon}) \right) F(4\pi G_{\varepsilon}(y))}{\gamma_{\varepsilon}^2} + O\left(\frac{G_{\varepsilon}(y)}{\gamma_{\varepsilon}^3} + \frac{|B(\gamma_{\varepsilon})|}{\gamma_{\varepsilon}^2}\right)\right) \, dy,
\]
where $\tilde{t}_{\varepsilon}(y) = \log \left(1 + |y - z_{\varepsilon}|^2 / \tilde{\mu}_{\varepsilon}^2\right)$ and $\tilde{S}_{i,\varepsilon} = S_i(|y - z_{\varepsilon}| / \tilde{\mu}_{\varepsilon})$. We use also here (1.8) with (3.16), and the estimates of Point (1) of this proof, including (4.6)-(4.7). The integral on $\{\tilde{t}_{\varepsilon} \in (\gamma_{\varepsilon}, \gamma_{\varepsilon}(\gamma_{\varepsilon} - 1))\}$ gives a $o(\gamma_{\varepsilon}^{-4})$ term. Estimate (4.9) follows from (4.12), Appendix A and some computations that we do not develop here again (see also [17], §5).

\begin{itemize}
  \item (3) We prove now that
  \[
  \int_{\Omega} |\nabla u_{\varepsilon}|^2 \, dx = 4\pi \left(1 + I_{x_{\varepsilon}}(\gamma_{\varepsilon}) + o\left(\tilde{\zeta}_{\varepsilon}\right)\right),
  \tag{4.13}
  \]
as $\varepsilon \to 0$, where $I_{x_{\varepsilon}}(\gamma_{\varepsilon})$ is given by (4.10), for $(x_{\varepsilon})_{\varepsilon}$ as in (3.13). Now, we can push one step further the argument involving (3.136), writing now that both formulas
(3.138) and (3.139) must also coincide on \( \partial B_x (\rho'_\varepsilon) \), where \( \rho'_\varepsilon > 0 \) is as in (3.109).

We compute and then get for \( \mu_\varepsilon \) in (3.42) the analogue of (4.6) for \( \tilde{\mu}_\varepsilon \)

\[
\lambda_\varepsilon H(\gamma_\varepsilon) = \frac{4}{\mu_\varepsilon^2 \exp(\gamma_\varepsilon^2)} \left( 1 + o \left( \frac{1}{\gamma_\varepsilon^2} \right) \right)
\]

\[
= \frac{4}{\gamma_\varepsilon^2 \exp(1 + \mathcal{H}_{x,\varepsilon}(x_\varepsilon))} \left( 1 + o \left( \gamma_\varepsilon^2 \xi_\varepsilon \right) \right) \times \left( 1 - \frac{\gamma_\varepsilon^2 A(\gamma_\varepsilon)}{2} - \frac{4B(\gamma_\varepsilon)}{\gamma_\varepsilon} \right) \exp \left( \mathcal{H}_{x,\varepsilon}(x_\varepsilon) \right) \int_{\Omega} G_{x,\varepsilon}(y) F \left( 4\pi G_{x,\varepsilon}(y) \right) dy,
\]

using (1.8), (3.16), (A.3)-(A.7). Independently, integrating by parts, resuming some computations in Appendix A and using (2.2), (3.12), (3.46), Point (1), and (3.137)-(3.139) (see also (3.101) and (A.9)), we get that

\[
\int_{\Omega} |\nabla u_\varepsilon|^2 dx = \int_{\Omega} u_\varepsilon \left( \lambda_\varepsilon H(u_\varepsilon)u_\varepsilon \exp(u_\varepsilon^2) \right) dx,
\]

\[
= \int_{\Omega} U_{\varepsilon,x_\varepsilon} \Delta U_{\varepsilon,x_\varepsilon} dx + o \left( \xi_\varepsilon \right).
\]

In order to get the second equality and to apply the dominated convergence theorem, it may be useful to split \( \Omega \) according

\[
\Omega = \{ y \text{ s.t. } t_\varepsilon(y) \leq \gamma_\varepsilon \} \cup \{ y \text{ s.t. } t_\varepsilon(y) > \gamma_\varepsilon \text{ and } \log \frac{1}{|x_\varepsilon - y|^2} \geq \frac{1 - \delta_0}{2} \gamma_\varepsilon^2 \}
\]

\[
\cup \{ y \text{ s.t. } \log \frac{1}{|x_\varepsilon - y|^2} < \frac{1 - \delta_0}{2} \gamma_\varepsilon^2 \},
\]

where \( \delta_0 \) is as in (1.6), and to use the first line of (4.14) with (1.5) (resp. with (3.30)) in the first region (resp. in the second region), or (1.6)-(1.7) in the last region. Observe that the argument here is to show that \( U_{\varepsilon,x_\varepsilon} \) (resp. \( \Delta U_{\varepsilon,x_\varepsilon} \)) is in some sense the main part of the expansion of \( u_\varepsilon \) (resp. \( \Delta u_\varepsilon \)). Thus we get (4.13) from (4.9) and (4.15).

\bullet (4) We prove now that, for any fixed sequence \( (\eta_\varepsilon)_\varepsilon \) of real numbers such that \( \eta_\varepsilon = o \gamma_\varepsilon^{-2} \), we have that

\[
\int_{\Omega} (1 + g(V_{\varepsilon,z_\varepsilon})) \exp \left( V_{\varepsilon,z_\varepsilon}^2 \right) dy
\]

\[
= |\Omega| (1 + g(0)) + \pi \exp(1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon)) (1 - \eta_\varepsilon \gamma_\varepsilon^2) \times
\]

\[
H(\gamma_\varepsilon) \left( 1 + \gamma_\varepsilon^2 I_{z_\varepsilon}(\gamma_\varepsilon) + \frac{1}{\gamma_\varepsilon^2} + o \left( \gamma_\varepsilon^2 (\xi + |\eta_\varepsilon|) \right) \right) \times
\]

\[
\left( 1 + \frac{8B(\gamma_\varepsilon)}{\gamma_\varepsilon (\kappa + 1)} \exp(1 + \mathcal{H}_{z_\varepsilon}(z_\varepsilon)) \int_{\Omega} G_{z_\varepsilon}(y) F \left( 4\pi G_{z_\varepsilon}(y) \right) dy \right),
\]

where \( \kappa \) is as in (1.6) and where \( V_{\varepsilon,z_\varepsilon} \geq 0 \) is given by

\[
V_{\varepsilon,z_\varepsilon}^2 = (1 - \eta_\varepsilon) U_{\varepsilon,z_\varepsilon}^2,
\]

where \( U_{\varepsilon,z_\varepsilon} \) is given in (4.1). Computations in the spirit of the proof of (4.15) give that

\[
\int_{\Omega} (1 + g(U_{\varepsilon,x_\varepsilon})) \exp \left( U_{\varepsilon,x_\varepsilon}^2 \right) dy = \int_{\Omega} (1 + g(u_\varepsilon)) \exp \left( u_\varepsilon^2 \right) dy + o \left( \gamma_\varepsilon^2 \xi_\varepsilon \right),
\]
Independently, we get from (1.6), (3.1) (part computing explicitly 
that for all 
It remains to prove (4.16). We compute and get that 
for all y such that \( \bar{t}_e(y) \leq \gamma_e \), using (1.7), (4.1)-(4.2), (4.6), (4.11) and (A.3). Then we get 
using (3.2) and (4.17) with (4.20). Then combining \( \eta_e = o(\gamma_e^{-2}) \), (3.16), (4.6), computing explicitly \( \int_{\mathbb{R}^2} \exp(-2T_0)S_0dy = 0 \) and \( \int_{\mathbb{R}^2} \exp(-2T_0)T_0^2 dy = 2\pi \), we get that 
and the dominated convergence theorem that
Combining (4.21) and (4.22), we conclude that (4.16) holds true, using (3.3) and (4.6).

• (5) We are now in position to conclude the proof of Lemma 4.1. Let \( \bar{x}_0 \) be a point in the compact \( K_\Omega \subset \Omega \) where \( S \) is attained in the third equation of (1.9). Let \( \eta_\varepsilon \) be given by

\[
(1 - \eta_\varepsilon) = \frac{4\pi(1 - \varepsilon)}{\|U_{\varepsilon, \bar{x}_0}\|_{H^1_0}^2}.
\]

(4.23)

First, we can check that

\[
\eta_\varepsilon = I_{\bar{x}_0}(\gamma_\varepsilon) - I_{x_\varepsilon}(\gamma_\varepsilon) + o(\varepsilon),
\]

so that the condition \( \eta_\varepsilon = o(\gamma_\varepsilon^{-2}) \) above (4.16) is satisfied, using (1.7), (3.6), (3.16), (4.9) and (4.13). Besides, we have that \( \|V_{\varepsilon, \bar{x}_0}\|_{H^1_0}^2 = 4\pi(1 - \varepsilon) \), by our choice (4.23) of \( \eta_\varepsilon \), and then, by (3.6), that

\[
\int_{\Omega} (1 + g(u_\varepsilon)) \exp(u_\varepsilon^2) \, dy \geq \int_{\Omega} (1 + g(V_{\varepsilon, \bar{x}_0})) \exp(V_{\varepsilon, \bar{x}_0}^2) \, dy;
\]

this implies, in view of (4.16), (4.19), (4.24) and of our choice of \( \bar{x}_0 \), that (4.3) is true and then, by (4.13) again, that (2.5)-(2.6) are true as well. This concludes the proof of Lemma 4.1. \( \square \)

Proposition 2.1 is proved.

\( \square \)

**Proof of Proposition 2.2.** Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^2 \). Let \( g \) be such that (1.1) and (1.5)-(1.6) hold true, for \( H \) as in (1.2), and let \( A, B \) and \( F \) be thus given. Assume that \( \Lambda_g(\Omega) < \pi \exp(1 + M) \), where \( M \) is as in (1.9) and \( \Lambda_g(\Omega) \) as in (1.11). Assume that there exists a sequence of positive integers \( (N_{\varepsilon})_\varepsilon \) such that (2.9) holds true and such that \( (P^\varepsilon_{\Lambda_g}(\Omega)) \) admits a nonnegative extremal \( u_\varepsilon \) for all \( \varepsilon > 0 \), where \( g_{N_{\varepsilon}} \) is as in (1.10). Then, by Lemma 3.3 in **Case 1**, we have (2.1) and that (3.8) holds true for \( \alpha_\varepsilon = 4\pi \), for all \( 0 < \varepsilon \ll 1 \). Moreover, we have \( u_\varepsilon \in C^{1,\theta}(\bar{\Omega}) \) \( (0 < \theta < 1) \) and (2.3) by (3.13). In order to conclude the proof of Proposition 2.2, it remains to prove (2.10). Still by Lemma 3.3 in **Case 1**, (3.137)-(3.139) and (A.9) \((v_\varepsilon) \) hold true. Concerning (3.137)-(3.139) and (A.9), observe that, contrary to **Case 2**, the term \( \xi_\varepsilon \) cannot be neglected in **Case 1** we are facing here. Indeed, using also now (3.31), (3.42), (3.108) and (A.9), we can resume computations of (4.12), (4.15) and Appendix A to get that

\[
\|u_\varepsilon\|_{H^1_0}^2 = 4\pi \left( 1 + \tilde{I}(\gamma_\varepsilon) + o\left( \gamma_\varepsilon^{-4} + |A(\gamma_\varepsilon)| + \gamma_\varepsilon^{-3}|B(\gamma_\varepsilon)| + \xi_\varepsilon \right) \right)
\]

as \( \varepsilon \rightarrow 0 \), where

\[
\tilde{I}(\gamma_\varepsilon) := \gamma_\varepsilon^{-4} + (A(\gamma_\varepsilon) - 2\xi_\varepsilon)/2 + 4\gamma_\varepsilon^{-3}\exp(-1 - M)B(\gamma_\varepsilon)S,
\]

so that (2.10) holds true, which concludes. \( \square \)

**Appendix A. Radial analysis**

Let \((x_\varepsilon)_{\varepsilon} \) be a sequence of points in \( \mathbb{R}^2 \) and \( (\gamma_\varepsilon)_{\varepsilon} \) be a sequence of positive real numbers such that (3.40) holds true. Let \( g \) be such that (1.1) and (1.5) holds true for \( H \) as in (1.2), and let \( A \) be thus given. Let \((N_{\varepsilon})_{\varepsilon} \) be a sequence of integers. We assume that we are in one of the following two cases:

\[
N_{\varepsilon} \rightarrow +\infty \text{ as } \varepsilon \rightarrow 0, \text{ and } (3.31)-(3.32) \text{ hold true}, \quad \text{ (Case 1)}
\]

\[
N_{\varepsilon} = 1 \text{ for all } \varepsilon. \quad \text{ (Case 2)}
\]
Let $B_\varepsilon$ be the radial solution around $x_\varepsilon$ in $\mathbb{R}^2$ of (3.93), for $\Psi_N$ as in (3.26), where $(\lambda_\varepsilon)_\varepsilon$ is any given sequence of positive real numbers. Let $T_0$ be given in $\mathbb{R}^2$ by
\[ T_0(x) = \log \left( 1 + |x|^2 \right). \] (A.1)

Let $S_i$, $i = 0, 1, 2$, be the radially symmetric solutions around 0 in $\mathbb{R}^2$ of
\[
\begin{align*}
\Delta S_0 - 8 \exp(-2T_0)S_0 &= 4 \exp(-2T_0) \left( T_0^2 - T_0 \right), \\
\Delta S_1 - 8 \exp(-2T_0)S_1 &= 4 \exp(-2T_0) \left( S_0 + 2S_0^2 - 4T_0S_0 + 2S_0T_0^2 - T_0^3 + \frac{T_0^3}{2} \right), \\
\Delta S_2 - 8 \exp(-2T_0)S_2 &= 4 \exp(-2T_0)T_0,
\end{align*}
\] (A.2)
such that $S_i(0) = 0$. In the sequel, we will use the asymptotic expansions of the $S_i$’s given by
\[
\begin{align*}
S_0(r) &= \frac{A_0}{4\pi} \log \frac{1}{r^2} + B_0 + O \left( \log(r)^2 r^{-2} \right) \text{ where } A_0 = 4\pi, \quad B_0 = \frac{\pi^2}{6} + 2, \\
S_1(r) &= \frac{A_1}{4\pi} \log \frac{1}{r^2} + B_1 + O \left( \log(r)^4 r^{-2} \right) \text{ where } A_1 = 4\pi \left( 3 + \frac{\pi^2}{6} \right), \quad B_1 \in \mathbb{R}, \\
S_2(r) &= \frac{A_2}{4\pi} \log \frac{1}{r^2} + B_2 + O \left( \log(r)^2 r^{-2} \right) \text{ where } A_2 = 2\pi, \quad B_2 \in \mathbb{R},
\end{align*}
\] (A.3)
as $r = |x| \to +\infty$. Note that in particular
\[
A_i = \int_{\mathbb{R}^2} \Delta S_i dx.
\] (A.4)
The explicit formula for $S_0$
\[
S_0(r) = -T_0(r) + \frac{2r^2}{1 + r^2} - \frac{1}{2} T_0(r)^2 + \frac{1 - r^2}{1 + r^2} \int_1^{1 + r^2} \frac{\log t}{1 - t} dt,
\]
and the expansions in (A.3) are derived in [16, 17]. Let $\varepsilon_0 \in (\sqrt{1/\gamma}, 1)$ be given. Let $\mu_\varepsilon$ be given by (3.42) and $t_\varepsilon$ by (3.43). Let $\rho_\varepsilon > 0$ be given by (3.95) and satisfying (3.96). Let $S_{i, \varepsilon}$ be then given by
\[
S_{i, \varepsilon}(x) = S_i \left( \frac{|x - x_\varepsilon|}{\mu_\varepsilon} \right),
\] (A.5)
for $i = 0, 1, 2$. Let $\xi_\varepsilon > 0$ be given by (3.14). In (Case 1) where $N_\varepsilon \to +\infty$ as $\varepsilon \to 0$, we get that $\xi_\varepsilon = O(N_\varepsilon^{-1/2})$ by (3.31) and (3.47). Then, in any case, we clearly have that
\[
\xi_\varepsilon \to 0
\] (A.6)
as $\varepsilon \to 0$. Then we are in position to state the main result of this section.

**Proposition A.1.** We have that
\[
B_\varepsilon = \gamma_\varepsilon \frac{t_\varepsilon}{\gamma_\varepsilon} + \frac{S_0_{\varepsilon}}{\gamma_\varepsilon} + \frac{S_{1, \varepsilon}}{\gamma_\varepsilon} + (A(\gamma_\varepsilon) - 2\xi_\varepsilon) \frac{S_{2, \varepsilon}}{\gamma_\varepsilon} + o \left( t_\varepsilon \left( \frac{1}{\gamma_\varepsilon} + \frac{|A(\gamma_\varepsilon)| + \xi_\varepsilon}{\gamma_\varepsilon} \right) \right),
\] (A.7)
uniformly in $[0, \rho_\varepsilon]$, as $\varepsilon \to 0$. 

In particular, using also (1.1) and (3.3), it can be checked that $B_{\varepsilon}$ is positive and radially decreasing in $[0, \rho_{\varepsilon}]$. Observe also that $\xi_{\varepsilon} \ll \gamma_{\varepsilon}^{-1}$ can be seen as a remainder term in (Case 2). Let $\zeta_{\varepsilon} > 0$ be given by

$$\zeta_{\varepsilon} = \max \left( \frac{1}{\gamma_{\varepsilon}^2}, |A(\gamma_{\varepsilon})|, \xi_{\varepsilon} \right).$$  \hspace{1cm} (A.8)

Resuming the computations below, we get as a by product of Proposition A.1 that, $v_{\varepsilon} = o \left( \frac{1}{\gamma_{\varepsilon}} \right)$ implies that

$$\frac{\lambda_{\varepsilon} \Psi_{\varepsilon}'(B_{\varepsilon} + v_{\varepsilon})}{2} = \frac{4 \exp(-2 t_{\varepsilon})}{\mu_{\varepsilon}^2 \gamma_{\varepsilon}} \left[ 1 + \frac{(\Delta S_0) \left( \frac{-x_{\varepsilon}}{\mu_{\varepsilon}} \right)}{\gamma_{\varepsilon}^2} + \frac{(\Delta S_1) \left( \frac{-x_{\varepsilon}}{\mu_{\varepsilon}} \right)}{\gamma_{\varepsilon}^4} \right]
+ (A(\gamma_{\varepsilon}) - 2 \xi_{\varepsilon}) (\Delta S_2) \left( \frac{\cdot - x_{\varepsilon}}{\mu_{\varepsilon}} \right) + o \left( \varepsilon \exp(\delta_0 t_{\varepsilon}) \right),$$  \hspace{1cm} (A.9)

uniformly in $\{ y \ s.t. \ t_{\varepsilon}(y) \leq \gamma_{\varepsilon} \}$, for some given $\delta_0 \in (0, 1)$, for $\delta_0$ as in (1.5).

**Proof of Proposition A.1.** Since both arguments are very similar to prove (Case 1) and (Case 2), for the sake of readability, we only write the proof of Claim A.1 in the more delicate (Case 1). Then, assume that we are in (Case 1). We let $\tau_{\varepsilon}$ be given by

$$B_{\varepsilon} = \gamma_{\varepsilon} - \frac{\tau_{\varepsilon}}{\gamma_{\varepsilon}}.$$  \hspace{1cm} (A.10)

Let $w_{\varepsilon}$ be given by

$$B_{\varepsilon} = \gamma_{\varepsilon} - \frac{t_{\varepsilon}}{\gamma_{\varepsilon}} + \frac{S_{0, \varepsilon}}{\gamma_{\varepsilon}^2} + \frac{S_{1, \varepsilon}}{\gamma_{\varepsilon}^5} + (A(\gamma_{\varepsilon}) - 2 \xi_{\varepsilon}) \frac{S_{2, \varepsilon}}{\gamma_{\varepsilon}} + \frac{\zeta_{\varepsilon} w_{\varepsilon}}{\gamma_{\varepsilon}}.$$  \hspace{1cm} (A.11)

Let $\tilde{\delta} > 0$ be fixed and let $\bar{r}_{\varepsilon} \geq 0$ be given by

$$\bar{r}_{\varepsilon} = \sup \{ r > 0 \ s.t. \ |\bar{w}_{\varepsilon}| \leq \tilde{\delta} t_{\varepsilon} \ \text{in} \ [0, r] \}.$$  \hspace{1cm} (A.12)

Now, since $\tilde{\delta} > 0$ may be arbitrarily small, in order to get Claim A.1, it is sufficient to prove that $\bar{r}_{\varepsilon} = \rho_{\varepsilon}$, for all $0 < \varepsilon \ll 1$. Using (A.12), we perform computations in $[0, \bar{r}_{\varepsilon}]$ and the subsequent $o(1)$ are uniformly small in this set as $\varepsilon \rightarrow 0$. First, by (1.5), (A.3), (A.6) and (A.12), we have that

$$\tau_{\varepsilon} = t_{\varepsilon}(1 + o(1)).$$  \hspace{1cm} (A.13)

Observe that, as soon as we have $\Delta B_{\varepsilon} > 0$ in $[0, \bar{r}_{\varepsilon}]$, then $B_{\varepsilon}$ is radially decreasing and (3.106) holds true in $[0, \bar{r}_{\varepsilon}]$. Let $L^H_{\varepsilon}$ and $L^g_{\varepsilon}$ be given by

$$H(B_{\varepsilon}) = H(\gamma_{\varepsilon}) \left( 1 + L^H_{\varepsilon} \right) \ \text{and then,} \ (1 + g(B_{\varepsilon})) = H(\gamma_{\varepsilon}) \left( 1 + L^H_{\varepsilon} + L^g_{\varepsilon} \right).$$  \hspace{1cm} (A.14)

In view of (A.10) and (A.13), estimates of $L^H_{\varepsilon}$, $L^g_{\varepsilon}$ are given by (1.5) and (3.2), respectively. We are now in position to expand the right-hand side of (3.93). From now on, it is convenient to denote

$$\tilde{N}_{\varepsilon} = N_{\varepsilon} - 1.$$  \hspace{1cm} (A.15)

Going back to (3.28), we have that

$$\frac{\Psi_{\varepsilon}(B_{\varepsilon})}{2} = B_{\varepsilon} H(\gamma_{\varepsilon}) \left[ (1 + L^H_{\varepsilon}) (1 + \varphi_{\tilde{N}_{\varepsilon}}(B^2_{\varepsilon})) + L^g_{\varepsilon} \left( \frac{B^{2N_{\varepsilon}}}{N_{\varepsilon}!} - B^2_{\varepsilon} \right) \right]$$  \hspace{1cm} (A.16)
By (3.95), (A.10) and (A.13) and since $\bar{r}_\varepsilon \leq \rho_\varepsilon$, we have that
\[
\min_{[0,\bar{r}_\varepsilon]} B_\varepsilon \geq (\varepsilon_0 + o(1))\gamma_\varepsilon \to +\infty \quad (A.17)
\]
as $\varepsilon \to 0$. Thus, by Stirling’s formula, we get that
\[
B^{2N_\varepsilon}/(N_\varepsilon!) \geq \exp \left( N_\varepsilon \left( \frac{\gamma^2_\varepsilon}{2N_\varepsilon} + (\log \varepsilon^2_0 + 1) + o(1) \right) \right)
\]
and then, for all given integer $k \geq 0$, that
\[
B^k_\varepsilon = o(1) \times \frac{B^{2N_\varepsilon}}{N_\varepsilon!} \quad (A.18)
\]
in $[0, \bar{r}_\varepsilon]$, as $\varepsilon \to 0$, using $\varepsilon^2_0 > 1/e$ with (3.32). Similarly, for all given integer $k \geq 0$, we have that
\[
\frac{B^k_\varepsilon}{\varphi_{N_\varepsilon}(B^2_\varepsilon)} = o(1) \quad (A.19)
\]
in $[0, \bar{r}_\varepsilon]$, as $\varepsilon \to 0$. Then, by (3.42), (A.10), (A.19) and (A.18), we may rewrite (A.16) as
\[
\frac{\lambda_\varepsilon \Psi'_{N_\varepsilon}(B_\varepsilon)}{2} = \frac{4}{B^2_\varepsilon \gamma^\varepsilon_\gamma} \left( 1 - \frac{\tau_\varepsilon}{\gamma^\varepsilon} \right) \left[ O(\exp(-\gamma^2_\varepsilon)) + \frac{\varphi_{N_\varepsilon}(B^2_\varepsilon)}{\varphi_{N_\varepsilon}(\gamma^2_\varepsilon)} \times \right.
\]
\[
\left. \left( 1 + L^H_\varepsilon + O \left( \frac{B^{2N_\varepsilon}}{N_\varepsilon! \varphi_{N_\varepsilon}(B^2_\varepsilon)} L^2_\varepsilon \right) \right) \right] \quad (A.20)
\]
in $[0, \bar{r}_\varepsilon]$, as $\varepsilon \to 0$. Indeed, by (A.17), we have that
\[
L^H_\varepsilon = o(1) \quad \text{and} \quad L^2_\varepsilon = o(1) \quad (A.21)
\]
in $[0, \bar{r}_\varepsilon]$ as $\varepsilon \to 0$, using (1.1), (3.3) and (A.14). In (A.20), the term $O(\exp(-\gamma^2_\varepsilon))$ equals $(1 + L^H_\varepsilon)/\varphi_{N_\varepsilon}(\gamma^2_\varepsilon)$ and we thus get this control by (3.31) and (A.21). In the following lines, we expand the terms of (A.20). By (3.51) with $\Gamma = \gamma^2_\varepsilon$ and $T = B^2_\varepsilon$, we get that
\[
\frac{\varphi_{N_\varepsilon}(B^2_\varepsilon)}{\varphi_{N_\varepsilon}(\gamma^2_\varepsilon)} = \exp(B^2_\varepsilon - \gamma^2_\varepsilon) - F_\varepsilon, \quad (A.22)
\]
where $F_\varepsilon$ satisfies in $[0, \bar{r}_\varepsilon]$
\[
F_\varepsilon = \frac{B^{2N_\varepsilon}}{N_\varepsilon! \varphi_{N_\varepsilon}(\gamma^2_\varepsilon)} \int^{\gamma^2_\varepsilon - B^2_\varepsilon}_0 \exp(-u) \left( 1 + \frac{u}{B^2_\varepsilon} \right)^{N_\varepsilon} du, \quad (A.23)
\]
\[
= \frac{\exp(B^2_\varepsilon)}{\varphi_{N_\varepsilon}(\gamma^2_\varepsilon)} \int_{B^2_\varepsilon}^{\gamma^2_\varepsilon} \exp(-s) s^{N_\varepsilon}/N_\varepsilon! ds, \quad (A.23)
\]
\[
= \xi_\varepsilon \exp(B^2_\varepsilon - \gamma^2_\varepsilon) \int^0_{B^2_\varepsilon - \gamma^2_\varepsilon} \exp(-y) \left( 1 + \frac{y}{\gamma^2_\varepsilon} \right)^{N_\varepsilon} dy. \quad (A.23)
\]
By (A.10) and (A.11), we may write
\[
\tau_\varepsilon = t_\varepsilon - \frac{S_{0_\varepsilon}}{\gamma^2_\varepsilon} - \frac{S_{1_\varepsilon}}{\gamma^2_\varepsilon} - (A(\gamma_\varepsilon) - 2\xi_\varepsilon) S_{2_\varepsilon} - \zeta_\varepsilon \bar{w}_\varepsilon.
\]
We set \( \bar{t}_\varepsilon = 1 + t_\varepsilon \). Then, keeping in mind (A.3), (A.6), (A.12), (A.13) and \( t_\varepsilon \leq \gamma_\varepsilon^2 \), we may compute

\[
\exp(B_\varepsilon^2 - \gamma_\varepsilon^2) = \exp\left(-2\tau_\varepsilon + \frac{\tau_\varepsilon^2}{\gamma_\varepsilon^2}\right) = \exp\left[-2\tau_\varepsilon + \frac{1}{\gamma_\varepsilon^2} \left(t_\varepsilon^2 - \frac{2t_\varepsilon S_{0,\varepsilon}}{\gamma_\varepsilon^2} + O(\zeta_\varepsilon t_\varepsilon^2)\right)\right] \tag{A.24}
\]

in \([0, \bar{r}_\varepsilon]\), as \( \varepsilon \to 0 \). Observe that

\[
\left|\exp(y) - \sum_{j=0}^{N} \frac{y^j}{j!}\right| \leq \frac{|y|^{N+1}}{(N+1)!} \exp(|y|), \tag{A.25}
\]

for all \( y \in \mathbb{R} \) and all integer \( N \geq 0 \). Then we draw from (A.24) that

\[
\left(1 - \frac{\tau_\varepsilon}{\gamma_\varepsilon^2}\right) \exp(B_\varepsilon^2 - \gamma_\varepsilon^2) = \exp(-2t_\varepsilon) \left[1 + \frac{1}{\gamma_\varepsilon^2} \left(2S_{0,\varepsilon} + t_\varepsilon^2 - t_\varepsilon\right) + \frac{1}{\gamma_\varepsilon^2} \left(2S_{1,\varepsilon} + 2S_{0,\varepsilon} + \frac{t_\varepsilon^4}{2} + 2S_{0,\varepsilon}t_\varepsilon^2 - 4S_{0,\varepsilon}t_\varepsilon - t_\varepsilon^3 + S_{0,\varepsilon}\right) + 2(A(\gamma_\varepsilon) - 2\xi_\varepsilon) S_{2,\varepsilon} + 2\zeta_\varepsilon \bar{w}_\varepsilon + O\left(\left(\frac{t_\varepsilon^5}{\gamma_\varepsilon^6} + \frac{\zeta_\varepsilon^2 t_\varepsilon^3}{\gamma_\varepsilon^2} + \zeta_\varepsilon^2 t_\varepsilon^3\right) \exp\left(o(t_\varepsilon) + \frac{t_\varepsilon^2}{\gamma_\varepsilon^2}\right)\right)\right] \tag{A.26}
\]

in \([0, \bar{r}_\varepsilon]\), as \( \varepsilon \to 0 \). Independently, by (3.31), (3.47), (A.10), (A.12), (A.13) and since \( B_\varepsilon(x_\varepsilon) = \gamma_\varepsilon^2 \), for all given \( R > 0 \), we have that

\[
\left|\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon! \varphi_{N_\varepsilon}(B_\varepsilon^2)} + \frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon! \varphi_{N_\varepsilon}(B_\varepsilon^2)}\right|_{L^\infty([0, \min\{R\mu_\varepsilon, \bar{r}_\varepsilon\}])} \leq O\left(\frac{1}{\sqrt{N_\varepsilon}}\right) \tag{A.27}
\]

and

\[
\frac{B_\varepsilon^{2N_\varepsilon}}{N_\varepsilon! \varphi_{N_\varepsilon}(B_\varepsilon^2)} \leq 1,
\]

in \([0, \bar{r}_\varepsilon]\), the second inequality being obvious by (3.5) and (A.15). In the sequel, by (3.32), we may assume that

\[
\beta_\varepsilon := \frac{\bar{N}_\varepsilon}{\gamma_\varepsilon^2} \text{ satisfies } \lim_{\varepsilon \to 0} \beta_\varepsilon = \beta_0 \in [0, 1], \tag{A.28}
\]

up to a subsequence. Now, we give estimates for \( F_\varepsilon \) given in (A.23). Up to a subsequence, we can split our results according to the following two cases

Case A: \(\lim_{\varepsilon \to 0} \frac{\gamma_\varepsilon^2 - \bar{N}_\varepsilon}{\sqrt{N_\varepsilon}} = +\infty\),

Case B: \(\frac{\gamma_\varepsilon^2 - \bar{N}_\varepsilon}{\sqrt{N_\varepsilon}} = O(1)\). \tag{A.29}
Observe that, since we assume \((3.32)\), all the possible situations are considered in (A.29). Let \((r_\varepsilon)\) be any sequence such that

\[
 r_\varepsilon \in [0, \bar{r}_\varepsilon] \tag{A.30}
\]

for all \(\varepsilon\). We prove that, in (Case A):

\[
 F_\varepsilon(r_\varepsilon) = \begin{cases} 
 O(\xi_\varepsilon \gamma_\varepsilon \exp(-2t_\varepsilon(r_\varepsilon)(\beta_0 + o(1)))) & \text{if } B_\varepsilon(r_\varepsilon)^2 \geq \hat{N}_\varepsilon + \sqrt{N_\varepsilon}, \\
 O(\exp(-1 + \varepsilon_0 + o(1))t_\varepsilon(r_\varepsilon)) & \text{if } B_\varepsilon(r_\varepsilon)^2 < \hat{N}_\varepsilon + \sqrt{N_\varepsilon}, 
\end{cases} \tag{A.31}
\]

while we get in (Case B):

\[
 F_\varepsilon(r_\varepsilon) = \begin{cases} 
 2t_\varepsilon(r_\varepsilon)\xi_\varepsilon \exp(-2t_\varepsilon(r_\varepsilon)(1 + o(1))), & \text{if } t_\varepsilon(r_\varepsilon) = o(\gamma_\varepsilon), \\
 O(t_\varepsilon(r_\varepsilon)\xi_\varepsilon \exp(-1 + \varepsilon_0 + o(1))t_\varepsilon(r_\varepsilon)) & \text{if } \gamma_\varepsilon = O(t_\varepsilon(r_\varepsilon)). 
\end{cases} \tag{A.32}
\]

Now we prove (A.31). We start with the first estimate of (A.31). Then, we assume that \(B_\varepsilon(r_\varepsilon)^2 \geq \hat{N}_\varepsilon + \sqrt{N_\varepsilon}\), and thus in particular that

\[
 1 - \frac{\hat{N}_\varepsilon}{B_\varepsilon(r_\varepsilon)^2} \geq 1 + o(1) \frac{1}{\sqrt{N_\varepsilon}}. \tag{A.33}
\]

Writing now \(F_\varepsilon\) according to the first formula of (A.23), using (3.106), (17) and \(\log(1 + t) \leq t\) for all \(t > -1\), we get first that

\[
 F_\varepsilon(r_\varepsilon) \leq \xi_\varepsilon \exp(-2r_\varepsilon(\varepsilon_0)\beta_\varepsilon) \int_0^{\gamma_\varepsilon^2-B_\varepsilon^2} \exp\left( -y \left( 1 - \frac{\hat{N}_\varepsilon}{B_\varepsilon(r_\varepsilon)^2} \right) \right) dy, \tag{A.35}
\]

and conclude the proof of the first estimate of (A.31), by (3.32), (13) and (3.33). In order to prove the second estimate of (A.31), it is sufficient to write \(F_\varepsilon\) according to the second formula of (A.23), to check that

\[
 \int_{-\infty}^\infty \exp(-s) \frac{s^{\hat{N}_\varepsilon}}{N_\varepsilon!} ds = 1,
\]

that \(r_\varepsilon \leq \bar{r}_\varepsilon \leq \rho_\varepsilon\) imply

\[
 t_\varepsilon(\bar{r}_\varepsilon) \leq (1 - \varepsilon_0)\gamma_\varepsilon^2, \tag{A.36}
\]

and to use (A.10), (13) and (3.31). Now we turn to the proof of (A.32). Then, we assume that (Case B) in (A.29) holds true and in particular that

\[
 1 - \beta_\varepsilon = O\left(\frac{1}{\gamma_\varepsilon}\right) \text{ in (Case B).} \tag{A.37}
\]

Writing \(F_\varepsilon\) according to the third estimate of (A.23), we get that

\[
 F_\varepsilon = \xi_\varepsilon \exp\left( -\tau_\varepsilon \left( 2 - \frac{\tau_\varepsilon}{\gamma_\varepsilon^2} \right) \right) \times \\
 \int_0^1 \exp\left( (\gamma_\varepsilon^2 - B_\varepsilon^2)y + \hat{N}_\varepsilon \log\left( 1 - \frac{(\gamma_\varepsilon^2 - B_\varepsilon^2)y}{\gamma_\varepsilon^2} \right) \right) dy \tag{A.38}
\]

at \(r_\varepsilon\). Expanding the log, we easily get the first estimate of (A.32) from (13), (37), (38) and the assumption \(t_\varepsilon(r_\varepsilon) = o(\gamma_\varepsilon)\). The second estimate of (A.32) can also be obtained from (A.38) by (13), (34), (36) and (37). This
concludes the proof of (A.32). Now, we prove that, in (Case A) of (A.29), we have that
\[
\int_0^{r_\varepsilon} F_\varepsilon(r) r dr = o \left( \frac{\mu^2}{\gamma_\varepsilon^2} \right).
\] (A.39)

Since \( r_\varepsilon \leq \rho_\varepsilon \), we get from (3.14), (3.31), (3.32), (A.31) and by Stirling’s formula that
\[
\int_{\{r \in [0, \bar{r}_\varepsilon], B_\varepsilon(r)^2 \geq \tilde{N}_\varepsilon + \sqrt{\tilde{N}_\varepsilon}\}} F_\varepsilon(r) r dr \\
\lesssim \exp (\gamma_\varepsilon^2[f(\beta_\varepsilon) + O((\log \gamma_\varepsilon)/\gamma_\varepsilon^2)]) \times \\
\begin{cases} 
\mu_\varepsilon^2 & \text{if } \beta_0 > 1/2, \\
\mu_\varepsilon^2 \exp(\gamma_\varepsilon^2(1 - \varepsilon_0)(1 - 2\beta_0 + o(1))) & \text{if } \beta_0 \leq 1/2,
\end{cases}
\] (A.40)

where \( f \) is the continuous function in \([0, 1]\) given for \( \beta \in (0, 1]\) by
\[
f(\beta) = \beta \log \frac{1}{\beta} + \beta - 1.
\]

Independently, since \( \bar{r}_\varepsilon \leq \rho_\varepsilon \), if
\[
r_\varepsilon \in J_\varepsilon := \left\{ r \in [0, \bar{r}_\varepsilon], B_\varepsilon(r)^2 < \tilde{N}_\varepsilon + \sqrt{\tilde{N}_\varepsilon}\right\},
\]
then \( J_\varepsilon \neq \emptyset \) and \( \gamma_\varepsilon^2 \lesssim \tilde{N}_\varepsilon \), by (A.10), (A.13) and (A.36). Thus we have that
\[
\gamma_\varepsilon \lesssim \sqrt{\tilde{N}_\varepsilon} \ll t_\varepsilon(r_\varepsilon),
\]

using that we are in (Case A) for the last estimate. Then, we get from (A.31) that
\[
\int_{J_\varepsilon} F_\varepsilon(r) r dr \lesssim \int_{\{r \leq \rho_\varepsilon, t_\varepsilon(r) \geq \gamma_\varepsilon\}} \exp (- (1 + \varepsilon_0 + o(1)) t_\varepsilon(r)) r dr = o \left( \frac{\mu^2}{\gamma_\varepsilon^2} \right).
\] (A.41)

Observe that \( f \) and \( \beta \mapsto f(\beta) + (1 - 2\beta)/2 \) are negative in \([0, 1]\) and \([0, 1/2]\) respectively. Moreover, because of (Case A) and by (3.32), we can check that
\[
\beta_\varepsilon = \frac{\tilde{N}_\varepsilon}{\gamma_\varepsilon^2} \leq \frac{1}{1 + 1/\sqrt{\tilde{N}_\varepsilon}} \leq 1 - \frac{1 + o(1)}{\sqrt{\tilde{N}_\varepsilon}} \leq 1 - \frac{1 + o(1)}{\gamma_\varepsilon},
\]

since \( \gamma_\varepsilon^2 \geq \tilde{N}_\varepsilon + \sqrt{\tilde{N}_\varepsilon} \), and then that
\[
0 < - f(\beta_\varepsilon) \lesssim 1/\gamma_\varepsilon.
\] (A.42)

Thus, we get (A.39) from the first estimate of (A.40) with (A.42), from the second estimate of (A.40) with \( 1 - \varepsilon_0 < 1 - \sqrt{1/e} < 1/2 \) and from (A.41). Computing as in (A.40), we get also that
\[
\xi_\varepsilon = o \left( \frac{1}{\gamma_\varepsilon^4} \right).
\] (A.43)
If (A.49) does not hold true, then, by (A.48), there exists \( s \) from (A.46) and (A.47) that, in (Case A) and in (Case B),

\[
\int \overline{r}^2 \exp(-\gamma^2) r dr = o \left( \frac{\mu^2}{\gamma^2} \right).
\]

Integrating by parts, observe that \( \overline{r}^2 \exp(-\gamma^2) r dr \) is dominated by \( |F_\varepsilon| + \exp(-\gamma^2) \) and \( \exp(B^2_\varepsilon - \gamma^2) \) in (Case A) (see (A.42)). By (A.13) and the second part of (A.27), using that \( \gamma \) is given by (A.8) of \( \zeta_\varepsilon \), we may rewrite (A.20) as

\[
\frac{\lambda_\varepsilon \Psi_{N_\varepsilon}(B_\varepsilon)}{2} = \frac{4}{\mu^2 \gamma^2} \left[ (1 - \frac{\tau_\varepsilon}{\gamma^2} + \frac{L^H_\varepsilon}{\gamma^2}) \exp(B^2_\varepsilon - \gamma^2) - F_\varepsilon \right. \\
+ O \left( \frac{t_\varepsilon}{\gamma^2} |F_\varepsilon| + \exp(-\gamma^2) \right) \\
+ O \left( \left( \frac{t_\varepsilon}{\gamma^2} \exp(B^2_\varepsilon - \gamma^2) + |F_\varepsilon| \right) \left( |L^H_\varepsilon| + |L^\varepsilon| \right) \right) \\
+ O \left( |L^\varepsilon| \exp(B^2_\varepsilon - \gamma^2) \frac{B_{2N_\varepsilon}^2}{N_\varepsilon! |\varphi_{N_\varepsilon}^\prime(B^2_\varepsilon)} \right).
\]

By (3.96), we clearly have that

\[
\int_0^{r_\varepsilon} \exp(-\gamma^2) r dr = o \left( \frac{\mu^2}{\gamma^2} \right).
\]

Integrating by parts, observe that \( \bar{w}_\varepsilon \) given by (A.11) satisfies

\[
\bar{w}_\varepsilon(0) = 0 \quad \text{and} \quad -r_\varepsilon \bar{w}_\varepsilon'(r_\varepsilon) = \int_0^{r_\varepsilon} (\Delta \bar{w}_\varepsilon) r dr,
\]

where, still using radial notations, \( \bar{w}_\varepsilon'(r) = \frac{d\bar{w}_\varepsilon}{dr}(r) \). Now we estimate \( \bar{w}_\varepsilon \) in \([0, \bar{r}_\varepsilon] \), by using (A.46). By (3.93), (A.11) and (A.44), we are in position to estimate the RHS of (A.46), for \( r_\varepsilon \) still as in (A.30). Assume first that we are in (Case A) of (A.29). By plugging (1.5), (3.2), (A.2), (A.3), (A.14), (A.21), (A.26), (A.27), (A.31), (A.39), (A.43), (A.45) in (A.44), by using the dominated convergence theorem and by coming back to the definition (A.8) of \( \zeta_\varepsilon \), we get that

\[
\int_0^{r_\varepsilon} |(\Delta \bar{w}_\varepsilon)| r dr \leq o \left( \frac{\mu^2}{\gamma^2} \right).
\]

The first term in the right hand side of (A.47) uses that, for all \( r \in [0, r_\varepsilon] \),

\[
|\bar{w}_\varepsilon(r)| \leq r \|\bar{w}_\varepsilon\|_{L^\infty([0, r_\varepsilon])}.
\]

Observe now that (A.47) still holds true in (Case B) of (A.29), replacing (A.31), (A.39) and (A.43) by (A.32) in the above argument. Since \( \varepsilon_0 > 1/2 \), we clearly get from (A.46) and (A.47) that, in (Case A) and in (Case B),

\[
r_\varepsilon |\bar{w}_\varepsilon'(r_\varepsilon)| = O \left( \|\bar{w}_\varepsilon\|_{L^\infty([0, r_\varepsilon])} \frac{\mu \varepsilon (r_\varepsilon / \mu) \varepsilon}{1 + (r_\varepsilon / \mu_\varepsilon)^2} \right) + o \left( \frac{(r_\varepsilon / \mu_\varepsilon)^2}{1 + (r_\varepsilon / \mu_\varepsilon)^2} \right).
\]

Now we prove that

\[
\mu \varepsilon \|\bar{w}_\varepsilon\|_{L^\infty([0, \bar{r}_\varepsilon])} = o(1) \quad \text{(A.49)}.
\]

If (A.49) does not hold true, then, by (A.48), there exists \( s_\varepsilon \in [0, \bar{r}_\varepsilon] \) such that \( s_\varepsilon = O(\mu_\varepsilon) \), \( \mu_\varepsilon = O(s_\varepsilon) \),

\[
|\bar{w}_\varepsilon'(s_\varepsilon)| = \|\bar{w}_\varepsilon\|_{L^\infty([0, \bar{r}_\varepsilon])} \quad \text{and} \quad \limsup_{\varepsilon \to 0} \mu_\varepsilon |\bar{w}_\varepsilon'(s_\varepsilon)| > 0 \quad \text{(A.50)}.
\]
In particular, up to a subsequence, we may assume that there exists \( \alpha_0 \in (0, +\infty) \) such that \( \tilde{r}_\varepsilon / \mu_\varepsilon \to \alpha_0 \) as \( \varepsilon \to 0 \). Let \( \tilde{w}_\varepsilon \) be given by

\[
\tilde{w}_\varepsilon(y) = \tilde{w}_\varepsilon(\mu_\varepsilon y)/(\mu_\varepsilon \| \tilde{w}_\varepsilon \|_{L^\infty([0, \tilde{r}_\varepsilon])}).
\]

By (A.48) and (A.50), we get that \( (\| (1 + \cdot) \tilde{w}_\varepsilon \|_{L^\infty([0, \tilde{r}_\varepsilon/\mu_\varepsilon]))} \) is a bounded sequence. Then, computing as in (A.47) and by radial elliptic theory with (3.93), we get that \( \tilde{w}_\varepsilon \to \tilde{w}_0 \) in \( C^2([0, \alpha_0]) \) if \( \alpha_0 < +\infty \) or in \( C^1_{loc}([0, \alpha_0]) \) if \( \alpha_0 = +\infty \), where \( \tilde{w}_0 \) solves

\[
\begin{aligned}
\Delta \tilde{w}_0 &= 8 \exp(-2T_0) \tilde{w}_0 \text{ in } B_0(\alpha_0), \\
\tilde{w}_0(0) &= 0, \\
\tilde{w}_0 &\text{ is radial around } 0 \in \mathbb{R}^2,
\end{aligned}
\]

still making usual radial identifications, and where \( T_0 \) is given in (A.1). By standard theory of radial elliptic equation, this implies \( \tilde{w}_0 \equiv 0 \), which contradicts (A.50) and proves (A.49). Then, since \( \tilde{w}_\varepsilon(0) \equiv 0 \) and by the fundamental theorem of calculus, we get from (A.48) with (A.49) that \( \tilde{r}_\varepsilon = \rho_\varepsilon \) in (A.12). By the discussion just above (A.13), this concludes the proof of Proposition A.1.

\[ \square \]

References


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