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SET-MEMBERSHIP IDENTIFIABILITY OF NONLINEAR MODELS AND RELATED PARAMETER ESTIMATION PROPERTIES

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Identifiability guarantees that the mathematical model of a dynamic system is well-defined in the sense that it maps unambiguously its parameters to the output trajectories. This paper casts identifiability in a set-membership (SM) framework and relates recently introduced properties, namely SM-identifiability, μ-SM- identifiability, and ε-SM-identifiability, to the properties of parameter estimation problems. Soundness and ε-consistency are proposed to characterize these problems and the solution returned by the algorithm used to solve them. This paper also contributes by carefully motivating and comparing SM-identifiability, μ-SM- identifiability and ε-SM-identifiability to related properties found in the literature and by providing a method based on Differential Algebra to check these properties.

Keywords: Identifiability, Bounded uncertainty, Set-membership estimation, Nonlinear dynamic models.

1. Introduction

Identifiability is an important concept that decides to what extent the parameter values of a mathematical model can be *uniquely* inferred from input-output measurements, assuming that the model has the same structure as the system (Nelles, 2002).

Mathematically, this means that there exists an unambiguous mapping between the model parameters and the output trajectories. Identifiability is hence a pre-condition for safely running a parameter estimation algorithm and obtaining reliable results.

In the last years, there has been quite a lot of emphasis on bounded-error models as opposed to stochastic models for achieving several tasks, e.g. fault diagnosis and fault tolerant control (Puig, 2010; Seybold *et al.*, 2015), robust robot localization (Kieffer *et al.*, 2000), reachability analysis (Auer *et al.*, 2013; Maiga *et al.*, 2016). This has been stressed by the success of operational estimation methods aiming at computing sets guaranteed to contain the *feasible parameter/state set*, i.e. the set of all the parameter/state vectors consistent with the specified bounds. This is why bounded or also called *set-membership* estimation is qualified as *guaranteed* (Kieffer *et al.*, 2002). In this paper, we use the term *set-membership*, abbreviated as SM, having in mind that the type of sets can be of different kinds, such as ellipsoids (Kurzhanski and Valyi, 1997), boxes (Kieffer and Walter, 2011), parallelotopes (Chiscii *et al.*, 1996), zonotopes (Alamo *et al.*, 2005) or other polytopes.

Interval Analysis has brought a set of tools that indifferently apply to linear and nonlinear systems (Jaulin *et al.*, 2001) as opposed to ellipsoidal and zonotope-based estimation methods. Furthermore, its efficiency has been considerably enhanced by recent constraint propagation techniques ((Chabert and Jaulin, 2009), (Kieffer and Walter, 2011) or (Maiga *et al.*, 2013)) resulting in the most appropriate paradigm to deal with nonlinearities.

Identifiability of SM nonlinear models has been shown to give rise to three concepts: SM-identifiability, μ-SM-identifiability, and ε-SM-identifiability that the authors introduced in (Jauberthie *et al.*, 2011), (Jauberthie *et al.*, 2013). In this paper, we are interested in the way these properties impact the SM parameter estimation (SM-PE) problem. This problem is characterized by two new properties. *Soundness* guarantees that the feasible

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parameter set (FPS) is reduced to one single bounded connected set. On the other hand, *ε-consistency* is a numerical property that guarantees that the FPS and the solution set returned by a parameter estimation algorithm with precision ε are composed of an equal number of mutually disjoint connected sets. Whereas abundant literature exists about SM-PE (Jaulin *et al.*, 2001; Raïssi *et al.*, 2004; Kieffer and Walter, 2011; Milanese *et al.*, 2013; Herrero *et al.*, 2016) these problems have never been discussed in relation with SM-identifiability.

The paper is organized as follows. After the introduction, Section 2 reminds the definitions of SM-identifiability, μ-SM-identifiability, and ε-SM-identifiability and a method for checking these properties. Section 3 brings a first contribution with a thorough analysis of the links between these properties and related properties existing in the literature. Section 4 introduces the properties of SM-PE problems, namely soundness and ε-consistency, as Sections 5 and 6 derive the conditions that guaranty these properties. Finally, section 7 concludes the paper and discusses perspectives of the work.

2. Set-membership identifiability

This section resumes the framework proposed in (Jauberthie *et al.*, 2013) for SM-identifiability for the class of systems formalized below.

2.1. Class of systems. The models considered in this paper are bounded-error uncertain nonlinear models, controlled or uncontrolled, of the following form:

$$\Gamma = \begin{cases} \dot{x}(t,p) = f(x(t,p), u(t), p), \\ y(t,p) = h(x(t,p), p), \\ x(t_0, p) = x_0 \in X_0, \\ p \in P \subset \mathcal{U}_\mathcal{P},\ t_0 \leq t \leq T, \end{cases} \tag{1}$$

where :

- $x(t,p) \in \mathbb{R}^n$ and $y(t,p) \in \mathbb{R}^m$ denote the state variables and the outputs at time t respectively,
- $u(t) \in \mathbb{R}^r$ is the input vector at time t; in the case of uncontrolled models, $u(t)$ is equal to 0,
- the initial conditions x_0, if any, are assumed to belong to a bounded set X_0 and one assumes that X_0 does not contain equilibrium points of the system,
- the parameter vector p belongs to a connected set P assumed to be included in $\mathcal{U}_\mathcal{P}$, where $\mathcal{U}_\mathcal{P} \subseteq \mathbb{R}^p$ is an a priori known set of admissible parameters; the components of p are denoted p_i.
- the functions f and h are real and analytic [1] on M, where M is an open set of $\mathbb{R}^n$ such that $x(t,p) \in M$ for every $t \in [t_0,\ T]$ and $p \in P$, T is a finite or infinite time bound.

In the following, $Y(P)$ denotes the set of output trajectories, solution of Γ for any $p \in P$ and is also called the output of Γ arising from P. P^c denotes the complementary of P in $\mathcal{U}_\mathcal{P}$.

2.2. Useful concepts. Let us consider a non empty connected set Π of $\mathbb{R}^p$, $\|.\|$ a classical norm on $\mathbb{R}^p$ and d its associated distance.

The *distance*[2] between two sets Π_1 and Π_2 of $\mathbb{R}^p$ is defined by:

$$d(\Pi_1, \Pi_2) = \min_{\pi_1 \in \Pi_1, \pi_2 \in \Pi_2} d(\pi_1, \pi_2).$$

Let us define $\delta(\Pi)$ as the *diameter* of Π. $\delta(\Pi)$ is given by the least upper bound of $\{d(\pi_1,\ \pi_2),\ \pi_1, \pi_2 \in \Pi\}$. If Π is not bounded, we define $\delta(\Pi) = +\infty$ ((Bourbaki, 1989)). On the metric space (Π, d), let μ be a continuous map from Π to Π. As an extension of the definition of contraction from (Munkres, 1975), we define μ as a *set contraction* if there is a nonnegative number $k < 1$ such that for all $\Pi_1,\ \Pi_2 \subseteq \Pi$, $d(\mu(\Pi_1), \mu(\Pi_2)) < kd(\Pi_1, \Pi_2)$. In the following, $\|.\|$ denotes the Euclidean norm, $\|.\|_\infty$ the maximum norm, and $\|.\|_1$ the norm 1. These may be defined on $\mathbb{R}^\alpha$, where $\alpha \in \{n, p, m\}$, depending on the case.

2.3. Definitions. The proposed definitions are given for controlled systems but they can be formulated in a similar manner for uncontrolled systems assuming that $u(t) = 0$.

Definition 1. Given the model Γ given by (1), consider a non empty connected set $P^* \subseteq \mathcal{U}_\mathcal{P}$ and another set $\bar{P} \subseteq \mathcal{U}_\mathcal{P}$, then P^* is *globally SM-identifiable* if there exists an input u such that $Y(P^*) \neq \emptyset$ and $Y(P^*) \cap Y(\bar{P}) \neq \emptyset \Longrightarrow P^* \cap \bar{P} \neq \emptyset$.

Definition 1 states that a connected set P^* is globally SM-identifiable if the output of Γ arising from P^* does not share any trajectory with the output of Γ arising from any set $\bar{P} \subseteq P^{*c}$. As an example, consider the following nonlinear system of the form (1):

$$\dot{x} = x + t\cos(p),\ x(t_0) = x_0, \tag{2}$$

[1]The assumption that f and h are analytic on M, and hence infinitely differentiable, is needed in section 3.3 for the use of differential algebra. In particular, proving the reciprocity part of Theorem 1 requires $y(t,p)$ to be expressed as a Taylor series. This proof is provided in (Jauberthie *et al.*, 2013).

[2]To keep the concept intuitive, it is a deliberate abuse of language to call $d(\Pi_1, \Pi_2)$ a *distance* between the two sets Π_1 and Π_2 of $\mathbb{R}^p$ eventhough it does not verify all the assumptions of a distance, in particular the triangular inequality.

where p is an bounded-error parameter for which the admissible set is $\mathcal{U}_{\mathcal{P}} = [0, 2\pi]$. The solution of (2) is $x(t) = x_0 e^t + (-1 - t + e^t)\cos(p)$. It is clear that this system is not globally identifiable. It is enough to notice that any pair $(p_1 = \pi - \alpha, p_2 = \pi + \alpha)$, with $\alpha \in [0, \pi]$, results in the same trajectory, since $\cos(\pi - \alpha) = \cos(\pi + \alpha)$ for $\alpha \in [0, \pi]$. However, the trajectories arising from any set $P^* = [\pi - \alpha, \pi + \alpha]$ with $\alpha \in [0, \pi]$ are different from any trajectory arising from other regions of the parameter space. P^* is then said *globally SM-identifiable*.

The definition of μ-SM-identifiability has been proposed to ensure that the set P^* may be contracted as small as desired while still retaining the SM-identifiability property. For this purpose, a contraction μ is applied to P^* and, by the Banach fixed-point theorem, it implies that the diameter of $\mu(P^*)$ tends to zero (Munkres, 1975).

Definition 2. A non empty connected set $P^* \subseteq \mathcal{U}_{\mathcal{P}}$ is *globally μ-SM-identifiable* if $\mu(P^*)$ is globally SM-identifiable for any contraction μ from P^* to P^*.

This implies the following proposition:

Proposition 1. *If the nonempty connected set $P^* \subseteq \mathcal{U}_{\mathcal{P}}$ is globally μ-SM-identifiable then it is globally SM-identifiable. The reciprocal is not true.*

Proof. For the reciprocal, consider the system (2) and the set $P^* = [\pi - \alpha, \pi + \alpha]$ with $\alpha \in [0, \pi]$ as before. P^* has been shown to be globally SM-identifiable but it is not μ-SM-identifiable since, assuming $\alpha_1, \alpha_2 \in]0, \pi[\,, \alpha_1 \geq \alpha_2$, any set $P_1^* = [\pi - \alpha_1, \pi - \alpha_2] \subseteq P^*$ shares trajectories with its complementary set P_1^{*c} that contains $[\pi + \alpha_2, \pi + \alpha_1]$. ∎

If the diameter of $\mu(P^*)$, $\delta(\mu(P^*))$, cannot be lower than ε without loosing SM-identifiability, we refer to *ε-SM-identifiability* (Jauberthie *et al.*, 2013).

Definition 3. Consider an SM-identifiable nonempty connected set $P^* \subseteq \mathcal{U}_{\mathcal{P}}$, then P^* is *globally ε-SM-identifiable* if there exists a set contraction μ from P^* to P^* such that $\delta(\mu(P^*)) = \varepsilon$ and $\mu(P^*))$ is globally SM-identifiable and for all $\tilde{\mu}$ such that $\tilde{\mu}(P^*) \subset \mu(P^*)$, $\tilde{\mu}(P^*)$ is not globally SM-identifiable.

To summarize, interpreting identifiability in the SM framework leads to two definitions depending on whether one considers a set as a whole (SM-identifiability) or also cares about the properties of its proper subsets (μ-SM-identifiability). μ-SM-identifiability can be seen as subsuming classical identifiability in the sense that if P^* is μ-SM-identifiable, it implies that any $p \in P^*$ is identifiable in the classical sense (Ljung and Glad, 1994). ε-SM-identifiability is a kind of structural μ-SM-identifiability since subsets of delimited diameter ε that are SM-identifiable although not μ-SM-identifiable are accepted. The reader is referred to (Jauberthie *et al.*, 2011) for the extension to structural and local counterparts of these properties.

3. SM-identifiability and related concepts

The links between (μ)-SM-identifiability and classical and interval identifiability were provided in (Jauberthie *et al.*, 2011; Jauberthie *et al.*, 2013). In this section, we are interested in the links with ε-global identifiability (Braems *et al.*, 2001) and partial injectivity (Lagrange *et al.*, 2008). These links allow us to propose a method for checking (μ)-SM-identifiability.

3.1. Links with ε-global identifiability.

Global identifiability in $P^* \subset \mathcal{U}_{\mathcal{P}}$ (g.i.i.P^*) was proposed by (Braems *et al.*, 2001) as a mean to provide a stronger conclusion than structural identifiability, guaranteeing that atypical regions of non identifiability do not exist in the parameter space.

Definition 4. Given $(u, x_0) \in \mathbb{R}^r \times X_0$, the parameter p_i is globally identifiable in P^* (g.i.i.P^*) if:

$$\forall (p, \bar{p}) \in P^*,\ y(., p) \equiv y(., \bar{p}) \Rightarrow p_i = \bar{p}_i, \qquad (3)$$

and the parameter vector p is g.i.i.P^* if all its components are g.i.i.P^*.

The originality of (Braems *et al.*, 2001) is to propose a practical way to formulate the condition of Definition 4, which is to check the condition :

$$\nexists (p, \bar{p}) \in P^* \times P^* \text{such that } y(., p) \equiv y(., \bar{p}),\ \| \bar{p} - p \|_\infty > 0. \qquad (4)$$

This is a *constraint satisfaction problem* (CSP) that can be solved in a guaranteed way by *interval constraint propagation* (ICP). In practice, (Braems *et al.*, 2001) states that checking condition (4) comes back to checking :

$$\nexists (p, \bar{p}) \in P^* \times P^* \text{such that } y(., p) \equiv y(., \bar{p}),\ \| \bar{p} - p \|_\infty > \varepsilon, \qquad (5)$$

which is defined as ε-g.i.i.P^*. We have the following results:

Proposition 2. *P^* is globally μ-SM-identifiable with respect to P^* (in the sense that $\mathcal{U}_{\mathcal{P}}$ is reduced to P^*) if and only if (4) is satisfied.*

Proof. (Jauberthie *et al.*, 2013) provided the proof that if P^* is globally μ-SM-identifiable, equivalently any p in P^* is globally identifiable with respect to P^*, hence satisfying condition (3) and condition (4). ∎

Proposition 3. *If P^* is globally ε-SM-identifiable with respect to P^*, then condition (5) is satisfied.*

Proof. P^* is globally ε-SM-identifiability (cf. Definition 3) with respect to P^* if and only if there exists some subset $\tilde{P} \subset P^*$ such that $\delta(\tilde{P}) = \varepsilon$ and the interior of

$\tilde{P}$, denoted int($\tilde{P}$), as well as any $\tilde{P}' \subseteq \tilde{P}$ is not globally SM-identifiable, hence not globally identifiable. In such case, for all p, $\bar{p} \in P^* \backslash \text{int}(\tilde{P})$ satisfies condition (5). The inverse is not true because when condition (5) is satisfied, it does not provide any information about subsets $\tilde{P}' \subseteq \tilde{P}$ such that $\delta(\tilde{P}) \leq \varepsilon$. ■

From the above propositions, condition (5) does not allow one to decide between μ-SM-identifiability and ε-SM-identifiability. It can be considered to check μ-SM-identifiability accepting a numerical precision of ε.

3.2. Links with partial injectivity. The definition of partial injectivity of a function was introduced in (Lagrange *et al.*, 2008). This notion perfectly characterizes μ-SM-identifiability. A second definition named *restricted-partial injectivity* is proposed in this paper in order to characterize global SM-identifiability.

Definition 5. Consider a function $f : \mathcal{A} \to \mathcal{B}$ and any set $\mathcal{A}_1 \subseteq \mathcal{A}$. The function f is said to be a partial injection of $\mathcal{A}_1$ over $\mathcal{A}$, or $(\mathcal{A}_1, \mathcal{A})$-injective, if $\forall a_1 \in \mathcal{A}_1$, $\forall a \in \mathcal{A}$,

$$a_1 \neq a \Rightarrow f(a_1) \neq f(a).$$

f is said to be $\mathcal{A}$-injective if it is $(\mathcal{A}, \mathcal{A})$-injective.

In (Lagrange *et al.*, 2008), an algorithm based on interval analysis for testing the injectivity of a given differentiable function is presented and a solver called IAVIA (Injectivity Analysis using Interval Analysis) implemented in C++ is mentioned [3]. For a given function, the solver partitions a given box in two domains: a domain on which the function is partially injective and an indeterminate domain on which the function may or may not be injective.

In order to characterize global SM-identifiability, the notion of *restricted-partial injectivity* is introduced.

Definition 6. Consider a function $f : \mathcal{A} \to \mathcal{B}$ and any set $\mathcal{A}_1 \subseteq \mathcal{A}$. The function f is said to be a restricted-partial injection of $\mathcal{A}_1$ over $\mathcal{A}$, or $(\mathcal{A}_1, \mathcal{A})$-R-injective, if:

$$\forall a_1 \in \mathcal{A}_1, \forall a \in \mathcal{A}_1^c, \; f(a_1) \neq f(a).$$

In the following proposition, partial injectivity and restricted partial injectivity are interpreted in terms of trajectories and this formulation makes it possible the direct link with the definition of SM-identifiability and μ-SM-identifiability.

Consider the set of outputs S_u arising from $\mathcal{U}_\mathcal{P}$ for a given input u.

Proposition 4. *Given the model* Γ, P^* *is globally SM-identifiable (resp. μ-SM-identifiable) for an input u if and only if the function $\varphi : \mathcal{U}_\mathcal{P} \to S_u : p \to y(.,p)$ is $(P^*,\mathcal{U}_\mathcal{P})$-R-injective (resp. $(P^*,\mathcal{U}_\mathcal{P})$-injective).*

[3] Let us notice that the solver IAVIA has been implemented for functions $f : \mathbb{R} \to \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}^2$.

Proof. *Necessity* From the definition of global SM-identifiability, P^* and its complementary do not share trajectories, hence there do not exist common trajectories arising from these two sets which implies that φ is $(P^*,\mathcal{U}_\mathcal{P})$-R-injective.

If P^* is μ-SM-identifiable, then the property of global SM-identifiability is verified for any $\mu(P^*)$, μ being a contraction from P^* to P^*, that implies that for any $\bar{P}$ included in the complementary of $\mu(P^*)$, $Y(\mu(P^*))$ and $Y(\bar{P})$ have no common trajectories. In other words, from the Banach fixed-point theorem, the trajectory arising from $p \in P^*$ is different from any trajectory arising from $\mathcal{U}_\mathcal{P} \setminus \{p\}$ hence φ is $(P^*,\mathcal{U}_\mathcal{P})$-injective.

Sufficiency If $\bar{P}$ is such that $P^* \cap \bar{P} = \emptyset$, $\bar{P}$ is included in the complementary of P^* and since φ is $(P^*,\mathcal{U}_\mathcal{P})$-R-injective there exist no common trajectories arising from these two sets, hence P^* is globally SM-identifiable.

Assume now that φ is $(P^*,\mathcal{U}_\mathcal{P})$-injective and that for a contraction μ, $Y(\mu(P^*))$ and $Y(\bar{P})$ have common trajectories, then these trajectories arise from the same parameter. This implies that $\mu(P^*)$ and $\bar{P}$ have a non empty intersection and that P^* is globally μ-SM-identifiable. ■

Corollary 1. *The following properties are equivalent:*

- *P^* is globally μ-SM-identifiable,*
- *the function $\varphi : \mathcal{U}_\mathcal{P} \to S_u : p \to y(.,p)$ is $(P^*,\mathcal{U}_\mathcal{P})$-injective,*
- *Condition (4) is satisfied.*

Proof. The proof directly comes from propositions 2 and 4. ■

Corollary 2. *P^* is globally ε-SM-identifiable implies that φ is $(\tilde{P},\mathcal{U}_\mathcal{P})$-R-injective, with $\tilde{P} \subseteq P^*$ and $\delta(\tilde{P}) \geq \varepsilon$. The inverse is not true.*

Proof. The necessity part of proof 3.2 applies and the inverse is not true for the same reasons as in proof 3.1. ■

Testing $(P^*,\mathcal{U}_\mathcal{P})$-injectivity or $(P^*,\mathcal{U}_\mathcal{P})$-R-injectivity numerically can be done with an adaptation of IAVIA (Lagrange *et al.*, 2008) but it does not allow to decide between μ-SM-identifiability and ε-SM-identifiability or SM-identifiability and ε-SM-identifiability.

3.3. A Differential Algebra Method to perform SM-identifiability analysis. Proposition 4 points at an operational method to check SM and μ-SM-identifiability provided that the function $\varphi : \mathcal{U}_\mathcal{P} \to S_u : p \to y(.,p)$ that maps parameters and trajectories is known. Differential algebra (Kolchin, 1973) was shown to provide a way to derive an implicit form of this function (Jauberthie *et al.*, 2011) [4].

This method, whose main result is given by Theorem 1 below, is based on the use of relations linking outputs, inputs and parameters of the model. These relations are more precisely differential polynomials whose indeterminates are the variables y and u and coefficients are rational expressions in p. For obtaining such polynomials, the Rosenfeld-Groebner algorithm, which is an elimination algorithm (Boulier, 1994), implemented in the package DifferentialAlgebra of Maple is an efficient tool. The Rosenfeld-Groebner algorithm is used to eliminate state variables with the aim to obtain the relations linking only outputs, inputs and parameters. With the elimination order $\{p\} < \{y,u\} < \{x\}$ (Kolchin, 1973)(Denis-Vidal, Joly-Blanchard, Noiret and Petitot, 2001), several solutions are delivered by the algorithm. One is called the *characteristic presentation* because it corresponds to the general solution, the others being particular solutions. The characteristic presentation contains differential polynomials linking outputs, inputs and parameters of the form:

$$\begin{aligned} R_i(y,u,p) = \; & m_0^i(y,u) \\ & + \textstyle\sum_{k=1}^{n_i} \theta_k^i(p) m_k^i(y,u),\ i = 1,\dots,m, \end{aligned} \tag{6}$$

where $(\theta_k^i(p))_{1\le k\le n_i}$ are rational in p, $\theta_u^i \neq \theta_v^i$ $(u \neq v)$, $m_k^i(y,u))_{0\le k\le n_i}$ are differential polynomials with respect to y, u and $m_0^i(y,u) \neq 0$. $\{\theta_k^i(p)\}_{1\le k\le n_i}$ is called the exhaustive summary of R_i.

The size of the system is the number of outputs. For the time being, we assume that $i = 1$, that is, there is one output and $n_1 = n$, $R_1 = R$, $m_k^1(y,u) = m_k(y,u)$. The case of more outputs is considered at the end of this section.

Consider t_0^+ the right limit of t_0 [5] and l the higher order derivative of y in (6). $\Delta R(y,u)$ denotes the *functional determinant* formed from the $\{m_k(y,u)\}_{1\le k\le n}$ and given by the Wronskian (Denis-Vidal, Joly-Blanchard and Noiret, 2001)

$$\Delta R(y,u) = \left| \begin{pmatrix} m_1(y,u) & \dots & m_n(y,u) \\ m_1(y,u)^{(1)} & \dots & m_n(y,u)^{(1)} \\ & \ddots & \\ m_1(y,u)^{(n-1)} & \dots & m_n(y,u)^{(n-1)} \end{pmatrix} \right|. \tag{7}$$

Theorem 1. (from (Jauberthie *et al.*, 2011)) *Assume that the functional determinant $\Delta R(y,u)$ is not identically equal to zero*[6]. *Consider P^* a connected subset of $\mathcal{U}_\mathcal{P}$. If the function $\phi : p = (p_1,\dots,p_p) \mapsto (\theta_1(p),\dots,\theta_n(p), y(t_0^+,p),\dots,y^{(l-1)}(t_0^+,p))$ is $(P^*,\mathcal{U}_\mathcal{P})$-R-injective (resp. $(P^*,\mathcal{U}_\mathcal{P})$-injective) then P^* is globally SM-identifiable (resp. μ-SM-identifiable). Furthermore, if for a contraction μ, $\mu(P^*)$ has a diameter equal to ε and ϕ is $(\mu(P^*),\mathcal{U}_\mathcal{P})$-R-injective but not $(\mu(P^*),\mathcal{U}_\mathcal{P})$-injective then P^* is ε-SM-identifiable. In the two cases, if the coefficient of $y^{(l)}$ in (6) is not equal to 0 at t_0, then the reciprocal is valid* [7].

Remark – If $m \geq 1$, for each of the m obtained differential polynomials $R_i(y,u,p)$, the functional determinant is evaluated. If it is not identically equal to zero, the associated exhaustive summary is added to the image of the function ϕ for which (partial) injectivity has to be studied.

Theorem 1 has been used in (Ravanbod *et al.*, 2014) to provide an operational method for analyzing identifiability in an SM framework. First the μ-SM-identifiable parameter subsets are determined with IAVIA. Then, determining the maxima and minima of the function ϕ allows one to assess SM-identifiable subsets and subsets that are neither SM nor μ-SM-identifiable.

4. SM parameter estimation and properties

In this section, the SM-PE problem is presented and two important properties are introduced, namely *soundness* and *ε-consistency*. SM-identifiability is shown to play a key role in relation with this problem.

Classical parameter estimation considers a time series of noisy measured output data $y_m(t_i), i = 0,\dots,h$, where $y_m(.) \in \mathbb{R}^m$, generated by the real system on the interval $[0,T]$. The problem is formulated as finding the parameter vector p^* for which the outputs produced by the model best match the measured data according to some criterion. Minimal least squares is a common method, which formulates as:

$$p^* = \underset{p\in\mathcal{U}_\mathcal{P}}{\operatorname{argmin}} \textstyle\sum_{t=t_0}^{t_h} \|y_m(t) - y(t,p)\|^2.$$

The SM-PE problem assumes that measured outputs are corrupted by bounded-error terms that may originate from the system parameters varying within specified bounds, bounded noise, or sensor precision such that $y_m(t_i) \in Y_m(t_i), i = 0,\dots,h$, where the $Y_m(t_i)$'s are connected sets of $\mathbb{R}^m$. The SM-PE problem is formulated as finding the set of parameter vectors $\mathcal{P} \subseteq \mathbb{R}^p$ such that the arising trajectories hit all the output data sets, i.e.:

[4] Another method based on the Power Series Expansion Method inspired by (Pohjanpalo, 1978) was also proposed in (Jauberthie *et al.*, 2011).

[5] t_0^+ is considered to ensure the existence of derivatives.

[6] This assumption consists in verifying the linear independence of the $m_k(y,u)$, $k = 1,\dots,n$. For doing this, it is sufficient to find a time point at which the Wronskian is non-zero. In the framework of differential algebra, this condition consists in verifying that this functional determinant is not in the ideal obtained after eliminating state variables. In practice, it can be checked with the function Belong_To of the package DifferentialAlgebra of Maple 16.

[7] When initial conditions are not considered, the function ϕ becomes $\phi : p = (p_1,\dots,p_p) \mapsto (\theta_1(p),\dots,\theta_n(p))$ and the reciprocal of the theorem is not valid.

$$p^* \in \mathcal{P} \Leftrightarrow y(t_i, p^*) \in Y_m(t_i), \forall i = 0, \ldots, h.$$

$\mathcal{P}$ is called the *feasible parameter set* (FPS). SM-PE problems are generally solved with a branch and bound algorithm that enumerates candidate solutions thanks to a rooted tree and assumes the full parameter space as the root set. At every node, the set of trajectories arising from the considered parameter set is checked for consistency against the measurements and labelled feasible, unfeasible or undetermined. Unfeasible sets are rejected while undetermined sets are splited and checked in turn until the diameter of the candidate solution set is smaller or equal to a given threshold ε provided by the user. ε is the *precision threshold* – or the *precision* for short – of the SM-PE algorithm. The SIVIA (Set Inversion Via Interval Analysis) algorithm (Jaulin and Walter, 1993) can be cited to exemplify the above principles (branch and bound (bisection) and interval analysis). The number of bisections to be performed is generally prohibitive. Hence, recent algorithms take advantage of constraint propagation techniques to reduce the width of the boxes to be checked. In this context, the model is interpreted as the set of constraints of a Constraint Satisfaction Problem (CSP). For solving such CSP, different types of so-called *contractors* can be used (Chabert and Jaulin, 2009).

It should be noticed that such algorithms are anytime by nature, i.e. they provide a guaranteed solution independently of the stopping time, which redeems in some way their exponential complexity. The returned solution is an overestimation of the FPS given by the convex union of the candidates that have been labelled feasible and undetermined. Interestingly, the convex union may consist of one set or more (cf. (Jaulin *et al.*, 2001) for several variants). In the following, we refer to the SM-PE algorithm as to a generic SM-PE algorithm based on these principles.

When considering an SM-PE problem, one would like to know beforehand whether $\mathcal{P}$ is reduced to one single connected set or not. Like for classical parameter estimation, this property indicates whether the problem is mathematically well-posed.

Definition 7. A SM-PE problem is said to be *sound* if $\mathcal{P} \subseteq \mathcal{U}_\mathcal{P}$ is reduced to one single connected set. In this case, $\mathcal{P}$ is also said to be sound.

Given an SM-PE algorithm with precision threshold ε, we denote by $\mathcal{P}_\varepsilon$ the solution set. Then, it is important to know the properties of $\mathcal{P}_\varepsilon$ in relation to $\mathcal{P}$.

Definition 8. Assume that $\mathcal{P}$ is equal to the union of $\kappa \geq 1$ mutually disjoint connected sets, then the solution set $\mathcal{P}_\varepsilon$ is said to be *ε-consistent* if $\mathcal{P}_\varepsilon$ is equal to the union of κ_ε mutually disjoint connected sets and $\kappa_\varepsilon = \kappa$.

Overdetermination and algorithm precision result in $\mathcal{P}_\varepsilon$ overestimating $\mathcal{P}$, which may imply $\kappa_\varepsilon < \kappa$. In this latter case, at least one of the sets composing $\mathcal{P}_\varepsilon$ includes several sets composing $\mathcal{P}$. ε-consistency is analyzed in section 6.

5. Soundness

5.1. Conditions for soundness.

Proposition 5. *Consider the system Γ and assume that the set $\mathcal{P} \subseteq \mathcal{U}_\mathcal{P}$ is the FPS of an SM-PE problem for Γ, then $\mathcal{P}$ is* sound *if and only if $\mathcal{P}$ is globally SM-identifiable.*

Proof. By definition, if $\mathcal{P}$ is a globally SM-identifiable set, the trajectories of Γ arising from $\mathcal{P}$ are different from the trajectories arising from the complementary set $\mathcal{P}^c = \mathcal{U}_\mathcal{P} \setminus \mathcal{P}$. In addition, $\mathcal{P}$ is connected, hence $\mathcal{P}$ is sound. Reciprocally, if $\mathcal{P}$ is sound, by definition it is globally SM-identifiable. ■

In addition to being SM-identifiable, assume that $\mathcal{P}$ is μ-SM-identifiable for Γ. In this case, it is interesting to notice that $\mathcal{P}$ preserves soundness when the bounded error corrupting the output data is getting smaller and smaller. In this case, $\mathcal{P}$ is said to be *μ-sound* This is stated by the following result:

Proposition 6. *Given the output data sets $Y_m(t_i), i = 1, \ldots, h$, assume that $\mathcal{P}$ is sound. Then, if $\mathcal{P}$ is μ-SM-identifiable for Γ, the FPS of the same problem with contracted output data sets $\mu_i(Y_m(t_i)), i = 1, \ldots, h$, where the μ_i's are contractions, is also sound.*

Proof. This proof uses Proposition 5. The result simply comes from the fact that if $\mathcal{P}$ is μ-SM-identifiable for Γ, then $\mathcal{P}$ is obviously SM-identifiable and for all $P \subset \mathcal{P}$, P is also SM-identifiable. ■

5.2. Example. Consider the model:

$$\begin{cases} \dot{x}_1 = (p_1 + 2(1-p_2)\cos(p_1))x_1^2 + (1-p_2)x_2, \\ \dot{x}_2 = \sin(p_1)x_1, \\ y = x_1, \end{cases} \tag{8}$$

where $(p_1, p_2) \in [-1, 4] \times [0, 1/10] = \mathcal{U}_\mathcal{P}$.
By setting $c_1 = \sin(p_1)$, with the elimination order $\{c_1, p_2\} < \{y\} < \{x_1, x_2\}$, the Rosenfeld-Groebner algorithm gives the following differential polynomial:

$$R(y, u) = \ddot{y} - 2(p_1 + 2(1-p_2)\cos(p_1))\dot{y}y - (1-p_2)\sin(p_1)y. \tag{9}$$

In that case, the functional determinant is reduced to $\triangle R(y) = \det(\dot{y}y, y) = -y^2\ddot{y}$ and is not identically equal to 0.
In order to consider the initial condition, the function $\phi : (p_1, p_2) \to ((p_1 + 2(1-p_2)\cos(p_1)), (1-p_2)\sin(p_1))$ has to be studied. By using the algorithm proposed in (Ravanbod *et al.*, 2014), Figure 1 (right) is obtained. $\mathcal{U}_\mathcal{P} = [-1, 4] \times [0, 1/10]$ has been partitioned in two

domains: a domain on which the function ϕ is partially injective and hence corresponding to μ-SM-identifiable subsets in grey color on the figure (red if pdf file) and two subsets in white color in the figure, each of them producing the same image[8]. If a parameter estimation problem is formulated such that the FPS $\mathcal{P}$ is in $\mathcal{U}_{\mathcal{P}}$, we can now decide whether $\mathcal{P}$ is sound or not. Indeed, if the inverse image of the trajectories hitting the output data sets entirely lies in a μ-SM-identifiable subset, then $\mathcal{P}$ is sound. On the contrary, $\mathcal{P}$ is unsound.

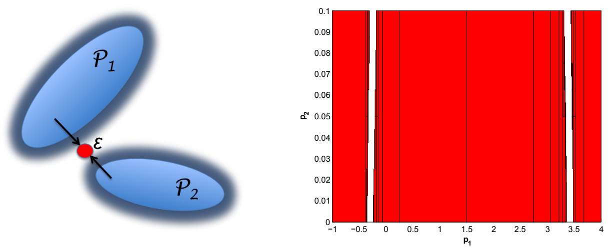

Fig. 1. A set of diameter ε interposed between $\mathcal{P}_1$, $\mathcal{P}_2$ and their undistinguishability neighborhoods (left), Partition of the parameter domain: the grey color (red if pdf) subsets are μ-SM-identifiable (right).

6. ε-consistency

6.1. Conditions for ε-consistency. ε-consistency as defined in definition 8 is a property of the solution set $\mathcal{P}_\varepsilon$ returned by the SM-PE algorithm with specified precision threshold ε. Among the problems that impact ε-consistency, two problems are analyzed in this paper:

- the SM-PE algorithm may not be able to separate the mutually disjoint connected sets composing $\mathcal{P}$ by testing topologically relevant candidate solution sets,
- trajectories arising from solution parameters may not be distinguishable from trajectories arising from non-solution parameters, given the precision of the sensors.

Proposition 7. *If the FPS $\mathcal{P}$ is sound, then the solution set $\mathcal{P}_\varepsilon$ is ε-consistent for any ε.*

Proof. If $\mathcal{P}$ is sound, it is reduced to one single connected set. Then, from the principle of branch and bound algorithms, the solution set $\mathcal{P}_\varepsilon$ is also reduced to one single connected set although it may be an overestimation of $\mathcal{P}$. ■

[8]Notice that IAVIA gives no information about these two domains: they are labelled undetermined.

Let's now assume that $\mathcal{P}$ is unsound and consists of κ mutually disjoint connected sets, say $\mathcal{P}_i, i = 1, \ldots, \kappa$. The fact that the SM-PE algorithm is able to separate the $\mathcal{P}_i$'s is a topological problem involving the distance between the $\mathcal{P}_i$'s and the diameter of the smallest candidate solution sets considered by the branch and bound SM-PE algorithm.

Proposition 8. *If $\mathcal{P}$ is unsound and consists of κ mutually disjoint connected sets $\mathcal{P}_i, i = 1, \ldots, \kappa$, then a necessary condition for the solution set $\mathcal{P}_\varepsilon$ returned by the SM-PE algorithm with precision threshold ε to be ε-consistent is that $d(\mathcal{P}_i, \mathcal{P}_j) > \varepsilon$, $\forall i, j = 1, \ldots, \kappa, i \neq j$.*

Proof. Without loss of generality, consider two mutually disjoint connected sets $\mathcal{P}_1$ and $\mathcal{P}_2$. The successive partitions of the parameter space arising from the branch and bound procedure provide candidate solution sets whose diameter is greater or equal to ε. $\mathcal{P}_1$ and $\mathcal{P}_2$ are separable if any candidate solution set of diameter ε can be *interposed* anywhere between the two sets, in particular just where the two sets are closest [9]. Only in such case, i.e. if $d(\mathcal{P}_1, \mathcal{P}_2) > \varepsilon$, the interposed candidate solution can be labelled unfeasible, hence rejected by the algorithm, and $\mathcal{P}_\varepsilon$ composed of two mutually disjoint sets. ■

Let's now consider the second problem related to the fact that the output data sets $Y_m(t_i), i = 0, \ldots, h$, rely on sensors with a given precision λ, i.e. $v = v_{mes} \pm \lambda$ where v is the true value and v_{mes} the measured value. In this case, two trajectories $y(., p)$ and $y(., \bar{p})$ must be distant by λ, i.e. be such that there exists $t \in [t_0, T]$, $\| y(t, p) - y(t, \bar{p}) \|_\infty > \lambda$, to be distinguishable. If the trajectories arising from non-solution parameters are not distinguishable from those arising from parameters of the the solution sets $\mathcal{P}_i$, then $\mathcal{P}_\varepsilon$ may not be ε-consistent.

The following proposition, whose proof is based on the Gronwall lemma, proves that, under some conditions, y is Lipschitz continuous with respect to the parameter vector. It provides the Lipschitz constant $K_{y,p}$ explicitly so that the conditions about parameters under which the output trajectories are distant by a given λ can be determined.

Recall first that, if a function g is real and analytic on M, an open set of $\mathbb{R}^n$, for every compact set $\mathcal{K} \subset M$, there exists a constant $K > 0$ such that for every x in $\mathcal{K}$, the following bound holds:

$$\| \frac{dg}{dx}(x) \|_\infty \leq K,$$

Since f and h defining Γ are assumed to be real and analytic on M, assuming that $x \in \mathcal{K}$, $\mathcal{K}$ a compact, they

[9]*Interposed just where the two sets are closest* means that the candidate set can be aligned with the segment $[p_1, p_2]$ that connects the two points $p_1 \in Fr(\mathcal{P}_1)$ and $p_2 \in Fr(\mathcal{P}_2)$ that are at minimum distance and that its intersection with either $\mathcal{P}_1$ or $\mathcal{P}_2$ is empty.

are Lipschitz continuous according to x. Their Lipschitz constants are respectively denoted $K_{f,x}$ and $K_{h,x}$.

Consider the following assumptions:

i) f and h defined on $[t_0, T] \times \mathcal{U}_{\mathcal{P}}$ are Lipschitz continuous according to p, their Lipschitz constants are respectively denoted $K_{f,p}$ and $K_{h,p}$

ii) the solution $x(t,p)$ of Γ is in the compact $\mathcal{K}$

iii) if the initial conditions depend on p, the function $p \mapsto x(t_0, p)$ is assumed to be Lipschitz continuous according to p and its Lipschitz constant is denoted $K_{x_0,p}$.

Proposition 9. *Assume that the assumptions i), ii) and iii) are verified, then y is Lipschitz continuous according to p and its Lipschitz constant $K_{y,p}$ is given by:*

$$K_{y,p} = K_{h,x}(K_{x_0,p} + K_{f,p}(T - t_0))e^{K_{f,x}(T-t_0)} + K_{h,p}.$$

If the initial conditions do not depend on p, $K_{y,p}$ is given by:

$$K_{y,p} = K_{h,x} K_{f,p}(T - t_0)e^{K_{f,x}(T-t_0)} + K_{h,p}.$$

Proof. First, integrating the equation $\dot{x}(t,p) = f(x(t,p), u(t), p)$ on $[0,t]$ and considering the difference between $x(t,p)$ and $x(t,\bar{p})$, one gets:

$$\begin{aligned}
&\| x(t,p) - x(t,\bar{p}) \| \le \| x(t_0,p) - x(t_0,\bar{p}) \| \\
&\quad + \int_{t_0}^{t} \| f(x(s,p), u(s), p) - f(x(s,\bar{p}), u(s), \bar{p}) \| \, ds \\
&\quad \le \| x(t_0,p) - x(t_0,\bar{p}) \| \\
&\quad + \int_{t_0}^{t} \| f(x(s,p), u(s), p) - f(x(s,p), u(s), \bar{p}) \| \, ds \\
&\quad + \int_{t_0}^{t} \| f(x(s,p), u(s), \bar{p}) - f(x(s,\bar{p}), u(s), \bar{p}) \| \, ds.
\end{aligned} \tag{10}$$

Using the assumption i) about lipschitz continuity of f, we deduce that:

$$\begin{aligned}
&\| x(t,p) - x(t,\bar{p}) \| \le \| x(t_0,p) - x(t_0,\bar{p}) \| \\
&\quad + K_{f,p}(T - t_0) \| p - \bar{p} \| \\
&\quad + K_{f,x} \int_{t_0}^{t} \| x(s,p) - x(s,\bar{p}) \| \, ds.
\end{aligned} \tag{11}$$

Then, the application of Gronwall lemma and assumption iii) gives:

$$\begin{aligned}
&\| x(t,p) - x(t,\bar{p}) \| \\
&\quad \le (K_{x_0,p} + K_{f,p}(T - t_0)) \| p - \bar{p} \| \, e^{K_{f,x}(T-t_0)}.
\end{aligned} \tag{12}$$

Finally, in using the hypothesis on h, the following inequalities are obtained:

$$\begin{aligned}
&\| y(t,p) - y(t,\bar{p}) \| \le \| h(x(t,p),p) - h(x(t,\bar{p}),p) \| \\
&\quad + \| h(x(t,\bar{p}),p) - h(x(t,\bar{p}),\bar{p}) \| \\
&\quad \le K_{h,x} \| x(t,p) - x(t,\bar{p}) \| + K_{h,p} \| p - \bar{p} \| \\
&\quad \le K_{h,x} (K_{x_0,p} + K_{f,p}(T - t_0)) \| p - \bar{p} \| \, e^{K_{f,x}(T-t_0)} \\
&\quad + K_{h,p} \| p - \bar{p} \| \\
&\quad \le (K_{h,x}(K_{x_0,p} + K_{f,p}(T - t_0))e^{K_{f,x}(T-t_0)} \\
&\quad + K_{h,p}) \| p - \bar{p} \|
\end{aligned} \tag{13}$$

■

If the initial conditions do not depend on the parameters then it is sufficient to set $K_{x_0,p} = 0$ in the previous proof.

From this result, one can determine the minimal distance between two parameter vectors p and $\bar{p}$ for which the trajectories $y(t,p)$ and $y(t,\bar{p})$ are ensured to be distant by λ. Applying the Lipschitz result of $y(t,p)$ with respect to p, the following corollary is deduced.

Corollary 3. *Let us consider two trajectories $y(.,p)$ and $y(.,\bar{p})$ arising from Γ, then under the conditions i), ii), and iii), we have:*

$$\text{if } \exists t \in [t_0, T], \ \| y(t,p) - y(t,\bar{p}) \|_\infty > \lambda \text{ then } \| p - \bar{p} \| > \frac{\lambda}{K_{y,p}}.$$

Proof. Since y is Lipschitz continuous according to the parameter vector p, one gets:

$$\lambda < \| y(t,p) - y(t,\bar{p}) \|_\infty < K_{y,p} \| p - \bar{p} \|, \tag{14}$$

which implies:

$$\| p - \bar{p} \| > \frac{\lambda}{K_{y,p}}.$$

■

This result means that the $\mathcal{P}_i$'s composing $\mathcal{P}$ are surrounded by a neighborhood that may generate trajectories that are not distinguishable from those arising from their inside parameters.

Putting together the results of Proposition 8 and Corollary 3, we obtain the following condition for ε-consistency.

Proposition 10. *Considering the system Γ with the assumptions i), ii), iii) and assuming that $\mathcal{P}$ is unsound and consists of κ mutually disjoint connected sets $\mathcal{P}_i, i = 1, \ldots, \kappa$, if the solution set $\mathcal{P}_\varepsilon$ returned by the SM-PE algorithm with precision threshold ε is ε-consistent, then $d(\mathcal{P}_i, \mathcal{P}_j) > \varepsilon + \frac{2\lambda}{K_{y,p}}, \forall i,j = 1, \ldots, \kappa$, where λ is the precision of the sensors.*

Proof. Without loss of generality, consider that $\mathcal{P}$ consists of two mutually disjoint connected sets $\mathcal{P}_1$ and $\mathcal{P}_2$. Let's denote by $\mathcal{N}_{\mathcal{P}_1}$ and $\mathcal{N}_{\mathcal{P}_2}$ the undistinguishability neighborhoods of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. The sets to be separated by the SM-PE algorithm are hence $\mathcal{P}_1 \cup \mathcal{N}_{\mathcal{P}_1}$ and $\mathcal{P}_2 \cup \mathcal{N}_{\mathcal{P}_2}$. Then, Proposition 8 applied to these sets implies $d(\mathcal{P}_1 \cup \mathcal{N}_{\mathcal{P}_1}, \mathcal{P}_2 \cup \mathcal{N}_{\mathcal{P}_2}) > \varepsilon$. The characterization of $\mathcal{N}_{\mathcal{P}_1}$ and $\mathcal{N}_{\mathcal{P}_2}$ provided by Corollary 3 hence implies $d(\mathcal{P}_i, \mathcal{P}_j) > \varepsilon + \frac{2\lambda}{K_{y,p}}$ as illustrated in Figure 1 (left). ■

Remark– The reciprocal is also true if the inclusion function $P \rightarrow [Y](P)$ used by the SM-PE algorithm to predict the set of trajectories arising from a given candidate parameter set P is such that $[Y](P) = Y(P)$, which is rarely the case.

6.2. Example. Consider the following example defined on $[0, T]$:

$$\begin{cases} \dot{x}_1(t,p) = \cos(p)x_2(t,p),\ x_1(0) = (\frac{1}{\sqrt{2}}+1)/2, \\ \dot{x}_2(t,p) = -x_1(t,p)\cos(p) + \big(1 - x_1(t,p)^2 \\ \quad -2x_2(t,p)^2\big)\, x_2(t,p),\ x_2(0) = 0, \\ y(t,p) = x_1(t,p). \end{cases} \tag{15}$$

The functions f and h are defined by $f(x,p) = (\cos(p)x_2, -x_1\cos(p) + (1 - x_1^2 - 2x_2^2)x_2)^T$, and $h(x,p) = x_1$ where $x = (x_1, x_2)^T$ and ^{T} denotes the transpose of the considered vector. The solution $(x_1(t), x_2(t))^T$ remains in the ring R defined by the two circles centered at $(0,0)$ with radii $\frac{1}{\sqrt{2}}$ and 1. Indeed, we have:

$$\frac{d}{dt}\left(\frac{x_1^2 + x_2^2}{2}\right) = x_1\frac{dx_1}{dt} + x_2\frac{dx_2}{dt} = (1 - x_1^2 - 2x_2^2)x_2^2.$$

Since $1 - x_1^2 - 2x_2^2$ is positive for $x_1^2 + x_2^2 < 1/2$ and negative for $x_1^2 + x_2^2 > 1$, $x_1^2 + x_2^2$ increases when $x_1^2 + x_2^2 < 1/2$ and decreases when $x_1^2 + x_2^2 > 1$. One can conclude that according to the initial condition, the solution remains in the ring R. The following step consists in finding the Lipschitz constants. Let's consider $z = (z_1, z_2)^T \in R$. Clearly, $K_{h,p} = 0$, $K_{h,x} = 1$ and $K_{f,p} = 1$ since $\| x \| = \| (x_1, x_2)^T \| < 1$. For $K_{f,x}$, by reordering the terms and by adding $x_1^2 z_2 - x_1^2 z_2$ at line 3, one gets:

$$\begin{aligned} &\| f(x,p) - f(z,p)\|_1 \leq |\cos(p)|(|x_2 - z_2| + |x_1 - z_1|) \\ &+|x_2 - z_2| + |(x_1^2 - 2x_2^2)x_2 - (z_1^2 - 2z_2^2)z_2| \\ &\leq (|x_1 - z_1| + 2|x_2 - z_2|) + |x_1^2 x_2 - x_1^2 z_2 + x_1^2 z_2 \\ &-z_1^2 z_2 - 2(x_2^3 - z_2^3)| \\ &\leq (|x_1 - z_1| + 2|x_2 - z_2|) + x_1^2|x_2 - z_2| \\ &+|z_2||x_1^2 - z_1^2| + 2|x_2^3 - z_2^3|. \end{aligned} \tag{16}$$

Since $|x_1| < 1$, $|z_1| < 1$, we deduce, on the one hand that $|x_1^2 - z_1^2| \leq |x_1 - z_1|(|x_1| + |z_1|) \leq 2|x_1 - z_1|$ and on the other hand that $|z_2^3 - x_2^3| = |x_2 - z_2||z_2^2 + 2x_2 z_2 + x_2^2| \leq 4|x_2 - z_2|$. Hence the following inequality:

$$\begin{aligned} &\| f(x,p) - f(z,p)\|_1 \leq (|x_1 - z_1| + 2|x_2 - z_2|) \\ &+|x_2 - z_2| + 2|x_1 - z_1| + 8|x_2 - z_2| \leq 11\|x - z\|_1. \end{aligned} \tag{17}$$

Using the equivalence between norm 1 and maximum norm, we get:

$$\|f(x,p) - f(z,p)\|_\infty \leq 22 \| x - z \|,$$

hence $K_{f,x} = 22$ and from Proposition 9, the Lipschitz constant $K_{y,p}$ is equal to Te^{22T}. Taking $[0,T] = [0,1]$, sensor precision $\lambda = 0,01$, and SM-PE algorithm precision threshold $\varepsilon = 0,001$, then from Proposition 10 the solution set is ε-consistent implies that $d(\mathcal{P}_i, \mathcal{P}_j) > \varepsilon + \frac{2\lambda}{K_{y,p}} \simeq 0,001 + 5,579.10^{-12} \simeq 0,001, \forall i,j = 1,\ldots,\kappa$. In this example, the SM-PE algorithm precision is dominant over sensor precision with respect to ε-consistency.

7. Discussion and conclusions

This paper casts identifiability in an SM framework and relates the properties introduced in (Jauberthie *et al.*, 2011) and (Jauberthie *et al.*, 2013), namely SM-/μ-SM-/ε-SM-identifiability, to the properties of SM-PE problems. Soundness and ε-consistency are proposed to characterize an SM-PE problem. Soundness is a theoretical property that assesses that the SM-PE problem is well-posed. ε-consistency guarantees that the structure of the FPS is well reflected in the solution returned by the SM-PE algorithm.

SM-/μ-SM-/ε-SM-identifiability are compared to related properties existing in the literature, in particular partial injectivity. The Differential Algebra based method proposed to check these properties leads to checking partial-injectivity and a newly introduced property named partial-R-injectivity. The algorithm proposed for this (Ravanbod *et al.*, 2014) remains of exponential complexity like many interval-based algorithms but it is still useful for medium-size problems.

ε-consistency is a complex property for which only necessary conditions are provided. It is impacted by several features of the SM-PE problem, including sensor precision and the overestimation involved in the computation of the image of a parameter set. Evaluating this overestimation and how it impacts ε-consistency remains an open problem.

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