Positive invariance over the Callier-Desoer class

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Extended Abstract

1 Introduction

An input-output linear system given in the form of a convolution,

$$y = h \star u \ , \tag{1}$$

is BIBO-stable if its kernel belongs to the class $\mathcal{A}$ of measures of the form

$$h(t) = h_a(t) + \sum_{i \in \mathbb{N}} h_i \delta(t - t_i) \ ,$$

where h_a is in L_1, $h_i \in \mathbb{R}$, $t_i \in \mathbb{R}_+$, $t_i < t_{i+1}$ for $i \geq 0$, and $\sum_{i \in \mathbb{N}} |h_i| < \infty$. The set $\mathcal{A}$ endowed with the convolution product forms a Banach commutative algebra for the norm

$$||h||_{\mathcal{A}} = \int_0^{+\infty} |h_a(t)| dt + \sum_{i \in \mathbb{N}} |h_i| \ .$$

This norm was shown to be the induced norm when h is seen as an operator over L_∞. We indeed have

$$\sup_{u \neq 0} \frac{||h \star u||_\infty}{||u||_\infty} = ||h||_{\mathcal{A}} \ , \tag{2}$$

for every h in $\mathcal{A}$. Here, as usually, $||\cdot||_\infty$ denotes the sup-norm on L_∞, say $||u||_\infty = \operatorname{ess\,sup}_{t \geq 0} |u(t)|$, $||y||_\infty = \operatorname{ess\,sup}_{t \geq 0} |y(t)|$. This shows that every bounded input leads to a bounded output, and that $||h||_{\mathcal{A}}$ gives an exact bound on the output $y(t)$.

Of course, this result is very well-known. Many properties of the set $\mathcal{A}$ are exposed in [5], and its use in control theory was gradually introduced by various authors, among them C. A. Desoer, F. Callier [3, 2] and M. Vidyasagar [4]. The Callier-Desoer class, properly speaking, is the set of fractions of elements of $\mathcal{A}(\sigma) = \mathrm{e}^{-\sigma t} \mathcal{A}$, which is the key concept to describe robust stabilization methods for a large class of distributed systems. The matter continues to generate interesting results, see for instance P. Lakkonen [7] for a recent survey.

This result can also be interpreted in terms of reachability or invariance in an input-output setting. It shows indeed that the output $y(t)$ of system (1) remains in the interval $[-\alpha ||h||_{\mathcal{A}}, +\alpha ||h||_{\mathcal{A}}]$ if the input $u(t)$ evolves in the interval $[-\alpha, \alpha]$, for every positive real α. The aim of this communication is to develop this idea and some of its applications.

2 Polyhedral bound of the reachable set

We now consider a multivariable convolution system, defined by a kernel $H \in \mathcal{A}^{p\times m}$, say

$$y = H \star u \ , \tag{3}$$

where $u(t) \in \mathcal{U} \subset \mathbb{R}^m$. Recall that the convolution product $\star$ is defined as

$$y_i(t) = \int_0^t \sum_j H_{ij}(t-\tau)u_j(\tau)d\tau \ .$$

A basic question consists in determining the range of $y(t)$, and more precisely the reachable set of the considered system.

Definition 2.1 *System (2) and a subset $\mathcal{U}$ of $\mathbb{R}^m$ being given, we define the reachable set $\mathcal{R}(\mathcal{U})$ as the set of vectors $x \in \mathbb{R}^p$ for which there exists a control u, with $u(t) \in \mathcal{U}$, for $t \geq 0$, and an instant t such that $y(t) = x$.*

We are interested into the computation of polyhedral approximations of the reachable set of system (2). We first introduce the following definitions.

Definition 2.2 *A matrix $M \in \mathbb{R}^{m\times n}$ being given, the convex polytope of $\mathbb{R}^m$ generated by the columns of M is the set denoted $\mathcal{C}(M)$, and defined by*

$$\mathcal{C}(M) = \{x \in \mathbb{R}^m \mid \exists v \in \Gamma,\ x = Mv\} \ ,$$

with $\Gamma = \{v \in \mathbb{R}^n \mid v \geq 0, \sum_{i=0}^n v_i \leq 1\}$. The open convex polytope generated by M is the set

$$\mathcal{C}_o(N) = \{x \in \mathbb{R}^m \mid \exists v \in \Omega,\ x = Nv\} \ ,$$

with $\Omega = \{v \in \mathbb{R}^n \mid v \geq 0, \sum_{i=0}^n v_i < 1\}$.

Definition 2.3 *A matrix $P \in \mathbb{R}^{q\times p}$ and a vector $\pi \in \mathbb{R}^q$ being given, the polyhedron denoted $\mathcal{P}(p,\pi)$ is the set defined as*

$$\mathcal{P}(p,\pi) = \{z \in \mathbb{R}^p \mid Pz \leq \pi\} \ .$$

The open polyhedron $\mathcal{P}_o(P,\pi)$ is the set

$$\mathcal{P}_o(P,\pi) = \{y \in \mathbb{R}^p \mid Py < \pi\} \ .$$

For the monovariable system (1), if $\alpha < u(t) < \beta$, we remark that

$$h(t-\tau)u(\tau)d\tau < \max\{h(t-\tau)\alpha, h(t-\tau)b\} \ ,$$

in addition, this bound is exact. For the multivariable system (2), and a matrix $M \in \mathbb{R}^{m\times n}$ being given, we remark on the same way that, if $u(t) \in \mathcal{C}_o M$, for $t \leq 0$, then the following inequality holds true

$$\sum_j H_{ij}(t-\tau)u(\tau) < \max_j\{H(t-\tau)M)_{ij}\} \ ,$$

for $i = 1$ to p. From this preliminary remark, we can deduce the following.

Theorem 2.1 *System (1) being given, together with a convex polytope $\mathcal{C}_o(M)$, and a polyhedron $\mathcal{P}_o(P,\pi)$, then the inclusion $\mathcal{R}(\mathcal{C}_o(M)) \subset \mathcal{P}_o(p,\pi)$is satisfied if and only if the following condition holds true for $i = 1$ to q:*

$$\sup_{t\geq 0} \int_0^t \max_j \{(PH(\tau)M)_{ij}\}\, d\tau \leq \pi$$

In other words, the output of system (1) with input constrained in the convex polytope $\mathcal{C}_o(M)$ remains into the polyhedron $\mathcal{P}_o(P,\pi)$, for $t \geq 0$. A preliminary version of this result was obtained in [8]

3 Exact polyhedral bounds

Remark that the difference between the left and right members of the condition of Theorem 2.1 is the distance between the reachable set and the plan $\{y \in \mathbb{R}^p \mid \sum_j P_{ij} y_j = \pi_i\}$. The left member of the condition, say

$$\lambda_i = \sup_{t\geq 0} \int_0^t \max_j \{(PH(\tau)M)_{ij}\}\, d\tau \ , \tag{4}$$

is therefore so that the plan $\{y \in \mathbb{R}^p \mid \sum_j P_{ij} y_j = \lambda_i\}$ is tangent to the reachable space. If the matrix P is given, the polyhedron $\mathcal{P}_o(P,\lambda)$ is the least polyhedron whose faces are oriented according to P, and that contains the reachable set. One can also compute a point of the intersection between the face and the reachable set. We first define

$$j_k(t) = \arg\max_j \{(PH(\tau)M)_{kj}\} \ ,$$

and

$$\nu_i(k) = \sup_{t\geq 0} \int_0^t (PH(\tau)M)_{ij_k} d\tau \ ,$$

for $k = 1$ to q. Then, the matrix N which columns are the vectors $\nu(k)$, say

$$N_{ij} = \nu_i(j) \ ,$$

for $i = 1$ to p, $j = 1$ to q.

Definition 3.1 *An open set $\mathcal{R}$ being given, together with a polyhedron $\mathcal{P}_o(P,\nu)$ and a polytope $\mathcal{C}_o(N)$, we say that $\mathcal{P}_o(P,\nu)$ is an exact upper approximation of $\mathcal{R}$ if its faces are tangent to $\mathcal{S}$, and that $\mathcal{C}_o(N)$ is an exact lower approximation of $\mathcal{R}$, if its vertices are on the boundary of $\mathcal{R}$.*

Theorem 3.1 *The system (1) being given, together with the matrices P and M, and taking N and ν defined as above, the convex polytope $\mathcal{C}_o(N)$ is an exact lower approximation, and the polyhedron $\mathcal{P}_o(P,\nu)$ is an exact upper approximation, of $\mathcal{R}(\mathcal{C}_o(M))$.*

This formulation is well fitted for numerical computations. The integrals can be easily approximated using Matlab or Scilab, provided that the kernel $H(t)$ is explicitly known, or can be computed. We also remark that this formula gives the way to calculate a control law $u^{\max}$ that maximizes the output.

We shall complete this study by some remarks and examples.

4 Remarks and examples

4.1 Link with BIBO stability

The link between the bound (2) and Theorem 2.1 is more visible when H is a positive measure.

Definition 4.1 *The measure $h \in \mathcal{A}$ is said to be positive if $h_i \geq 0$, for $i \in \mathbb{N}$, and $h_a(t) \geq 0$, for $t \geq 0$. In the multivariable case, H is positive if all its entries are positive measures.*

In this case, if $u(t)$ lies in $]\alpha, \beta[$, we have for $t \geq 0$ the following inequalities

$$\alpha \int_0^t h(\tau) d\tau < y(t) < \beta \int_0^t h(\tau) d\tau \ .$$

In addition, these bounds are exact. From Theorem 2.1, we then obtain the following.

Corollary 4.1 *Let system (1) have a positive kernel h and be subject to inputs constrained by $\alpha < u(t) < \beta$, for $t \geq 0$, where α and β are real numbers such that $\alpha < \beta$. Then the reachable set of system (1) is equal the interval $]\,\alpha||h||_{\mathcal{A}}, \beta||h||_{\mathcal{A}}\,[$*

This first result can be generalized to kernels that are not necessarily positive. Every measure h in $\mathcal{A}$ can be uniquely decomposed into a difference $h = h^+ - h^-$, where h^+ and h^- are two positive measures in $\mathcal{A}$ with disjoint supports. Then if $\alpha \leq u(t) \leq \beta$, then the range of $y(t)$ is given by

$$\alpha \int_0^t h^+(\tau) d\tau - \beta \int_0^y h^-(\tau) d\tau \leq y(t) \leq \beta \int_0^t h^+(\tau) d\tau - \alpha \int_0^t h^-(\tau) d\tau \ ,$$

for any positive t, that can be rewritten as

$$\int_0^\infty \min\left\{\alpha h(\tau), \beta h(\tau)\right\} d\tau \leq y(t) \leq \int_0^t \max\left\{\alpha h(\tau), \beta h(\tau)\right\} d\tau; .$$

The latter formulation is well fitted for numerical computations, since it avoids the computation of h^+ and h^-. Indeed the infinite integral can be easily approximated using Matlab or Scilab, provided that $h(t)$ is explicitly known, or can be computed. We also remark that this formula gives the way to calculate a control law $u^{\max}$ that maximizes the output. This control law is given by

$$u^{\max}(t) = \begin{cases} \alpha & , \text{ if } \max\left\{\alpha h(\tau), \beta h(\tau)\right\} = \alpha h(\tau) \ , \\ \beta & , \text{ else } , \end{cases} \tag{5}$$

for any positive t. In the same way, the control given by

$$u^{\min}(t) = \begin{cases} \alpha & , \text{ if } \min\left\{\alpha h(\tau), \beta h(\tau)\right\} = \alpha h(\tau) \ , \\ \beta & , \text{ else } , \end{cases}$$

4.2 Closed input sets

The bounds found in Theorem 2.1 may be useful to study constrained systems. In this case, the input space $\mathcal{U}$ is generally closed, but in general, the reachable space is not closed, nor open. We have the following in the case of a monovariable positive system.

Corollary 4.2 *Let system (1) have a positive kernel h and be subject to inputs constrained by $\alpha \leq u(t) \leq \beta$, for $t \geq 0$, where α and β are real numbers such that $\alpha < \beta$. Then the reachable set of system (1) is equal the interval $[\alpha||h||_{\mathcal{A}}, \beta||h||_{\mathcal{A}}]$, if h has a finite support, and, if h has an unbounded support, $]\alpha||h||_{\mathcal{A}}, \beta||h||_{\mathcal{A}}[$, if α, β are nonzero, $[\alpha||h||_{\mathcal{A}}, \beta||h||_{\mathcal{A}}[$ if α equals 0, and $]\alpha||h||_{\mathcal{A}}, \beta||h||_{\mathcal{A}}]$ if β equals 0*

In general, the bounds (5) are reached or not. The singular part of the kernel can indeed cause discontinuities of the integral that makes the reachable space not closed. When the integral is a monotone function of the time, its behaviour at infinity, if the kernel is not of finite support, can also create the same result.

4.3 Constrained control and $\mathcal{D}$-invariance

We would finally like to develop that these techniques may be useful to design control laws for constrained systems. We consider the following system

$$\dot{y}(t) = -\mu y(t) + u(t-\theta) - w(t) \; , \tag{6}$$

where u is an input control with delay $\theta \geq 0$, w is a disturbance, μ is a constant coefficient. Such a model is used in the control of communication networks [6] and of logistic systems [1]. In the latter case, $w(t)$ corresponds to an instantaneous demand, that is assumed to be unknown, but lies into $[d_{\min}, d_{\max}]$. The variable $u(t)$ is the instantaneous production order, $y(t)$ is the inventory level, and $\mu \in [0, 1[$ is a loss factor. One wants to design of u, subject to $u(t) \in [u_{\min}, u_{\max}]$, so that $y(t) \in [y_{\min}, y_{\max}]$, for $t \geq 0$, whatever be the demand in the required range. Equation (6) can be rewritten in the form

$$y = h \star u - g \star w \; ,$$

where

$$h(t) = \begin{cases} 0 & \text{if } t < \theta \; , \\ \mathrm{e}^{-\mu t - \theta} & \text{if } t \geq \theta \end{cases} \quad ,$$

and

$$g(t) = \mathrm{e}^{-\mu t} \; , \text{ for } t \geq 0.$$

We assume that the system is well-sized, so that constant demand can be satisfied. We then have $u_{\max} \geq \mu y_{\min} + w_{\max}$, and $u_{\min} \leq \mu y_{\max} + w_{\min}$. Applying Theorem 2.1 to this case, we obtain that

$$y_{\max} - y_{\min} \geq \frac{1}{\mu}(1 - \mathrm{e}^{-\mu\theta}(w_{\max} - w_{\min}) \; .$$

Reversely, we can generalize the control laws proposed by [1, 6], that are respectively a bang-bang control and a sliding mode control, to obtain a production strategy that permit the constraint on the inventory level to be met, whatever be the demand in the specified range, if

$$y_{\max} - y_{\min} > \frac{1}{\mu}(1 - \mathrm{e}^{-\mu\theta}(w_{\max} - w_{\min}) \; .$$

These last comments will be further developed in the final version of this work, together with an example of the computation of approximation of the reachable space.

References

[1] F. Blanchini, S. Miani, R. Pesenti, F. Rinaldi and W. Ukovich. Robust control of production-distribution systems. Chap. 2 in Perspectives in robust control. Lecture Notes in Control and Information Sciences 268:13-28. Springer, New York, 2007. control : foundations & applications. Springer, 2008.

[2] F. M. Callier and C. A. Desoer. An algebra of transfer functions for distributed linear time-invariant systems. IEEE Transactions on Circuits and Systems, 25:651662, 1978. *The disturbance decoupling problem for systems over ring*, SIAM J. Control Optim., Vol. 33, 1995, pp. 750-764.

[3] C. A. Desoer and F. M. Callier. Convolution feedback systems. SIAM J. Control, 10:737-746, 1972.

[4] C. A. Desoer and M. Vidyasagar. Feedback systems: input-output properties. Academic, New York, 1975.

[5] E. Hille and R. S. Phillips. Functional analysis and semi-groups. Amer. Math. Soc. Collec. Publ., 31, 1957.

[6] P. Ignaciuk and A. Bartoszevicz. Congestion control in data transmission networks. Sliding modes and other designs. Series Communications and Control Eng. Springer, New York, 2013.

[7] P. Lakkonen, Robust regulation for infinite-dimensional systems and signals in the frequency domain, Ph. D. Thesis, Tampere University of Technology, Finland, 2013.

[8] C. Moussaoui, R. Abbou and J. J. Loiseau. On bounds of input-output systems. Reachability set determination and polyhedral constraints verification. In Proc. 19th IFAC World Congress, Cape Town, South Africa, 2014.