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Stefan Ankirchner, Maike Klein, Thomas Kruse, Mikhail Urusov

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For a one-dimensional continuous strong Markov process Y we present an explicit construction of a convex function q such that $q(Y_t) - t$, $t \geq 0$, is a local martingale. As an application we deduce some integrability properties of Y evaluated at stopping times and present a proof of Feller's test for explosions based directly on that function q.

1 Setting

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (P_y)_{y \in I}, (Y_t)_{t \geq 0})$ be a one-dimensional continuous strong Markov process in the sense of Section VII.3 in [3]. We assume that the state space is an open, half-open or closed interval $I \subseteq \mathbb{R}$. We denote by $I^\circ = (l, r)$ the interior of I, where $-\infty \leq l < r \leq \infty$, and we set $\bar{I} = [l, r]$. Recall that by the definition we have $P_y[Y_0 = y] = 1$ for all $y \in I$.

For every $x \in \bar{I}$ we denote the first hitting time of x by $H_x = \inf\{t \geq 0 : Y_t = x\}$ (where we use the convention $\inf \emptyset = \infty$). Moreover, for $a < b$ in $\bar{I}$ we denote by $H_{a,b}$ the first exit time of Y from (a, b), i.e. $H_{a,b} = H_a \wedge H_b$. Throughout we assume that Y is regular. This means that for every $y \in I^\circ$ and $x \in I$ we have that $P_y[H_x < \infty] > 0$.

Without loss of generality we suppose that the diffusion Y is in natural scale. If Y is not in natural scale, then there exists a strictly increasing continuous function $s : I \to \mathbb{R}$, the so-called scale function, such that $s(Y_t)$, $t \geq 0$, is in natural scale. Recall that, for $a, b \in I$, $a < b$, $y \in (a, b)$, it holds

$$E_y H_{a,b} < \infty, \tag{1}$$

in particular, $H_{a,b} < \infty$, P_y-a.s., and

$$P_y(Y_{H_{a,b}} = a) = \frac{b - y}{b - a}, \qquad P_y(Y_{H_{a,b}} = b) = \frac{y - a}{b - a} \tag{2}$$

(see e.g. Prop. 3.1 and 3.2, Chapter VII in [3]).

Let m be the speed measure of the Markov process Y (see VII.3.7 and VII.3.10 in [3]). Recall that for all $a < b$ in I° we have

$$0 < m([a,b]) < \infty.$$

We say that the boundary point l or r is accessible if $l \in I$ or $r \in I$, respectively. That is, an accessible boundary point is attained in finite time with a positive probability when started in the interior of the state space (recall regularity of Y). In particular, it follows from (2) that infinite boundary points are always inaccessible. Throughout we make the assumption that if a boundary point is accessible, then it is absorbing. In other words, we do not allow for reflection or killing. As a consequence, the process Y is a P_y-local martingale for all $y \in I$.

We now introduce a function that turns out to be useful for determining integrability properties and the boundary behavior of Y. Let $q : I^\circ \times \bar{I} \to [0, \infty]$ be defined by

$$q(y,x) = \frac{1}{2} m(\{y\})|x-y| + \int_y^x m((y,u))du, \tag{3}$$

where for $u < y$ we set $m((y,u)) := -m((u,y))$.

Notice that, for $y \in I^\circ$, the function $q(y,\cdot)$ is strictly convex on $\bar{I}$, decreasing on $[l,y]$ and increasing on $[y,r]$.

Moreover, the definition immediately implies that q is finite on $I^\circ \times I^\circ$. We show below that q is even finite on $I^\circ \times I$ (see Lemma 2.3).

Finally, a straightforward calculation shows that for all $y, z \in I^\circ$ and $x \in I$ we have

$$q(z,x) = q(y,x) - q(y,z) - \frac{\partial^0 q}{\partial x}(y,z)(x-z), \tag{4}$$

where $\frac{\partial^0 q}{\partial x}(y,x) = \frac{1}{2}(\frac{\partial^+ q}{\partial x} + \frac{\partial^- q}{\partial x})(y,x)$.

2 $q(y,Y)$ has a linear compensator

In this section we show that the process $q(y,Y_t) - t$ is a local martingale up to the time where Y attains the boundary of I. More precisely:

Theorem 2.1. *Let $y \in I^\circ$. Then the process $q(y,Y_t) - (t \wedge H_{l,r})$, $t \in [0,\infty)$, is a P_y-local martingale starting in zero.*

For the proof of Theorem 2.1 we need the following auxiliary results.

Lemma 2.2. *Let $a, b \in I$ with $a < b$ and $y \in (a,b)$. Then*

$$E_y H_{a,b} = E_y q(y, Y_{H_{a,b}}). \tag{5}$$

Proof. Recall that the speed measure m satisfies

$$E_y H_{a,b} = \int_{(a,b)} \frac{(b - x \vee y)(x \wedge y - a)}{b-a} m(dx) \tag{6}$$

(see e.g. Thm 3.6, Chapter VII in [3]). This implies

$$\begin{aligned}
E_y H_{a,b} =& \frac{1}{b-a}\left[\int_{(a,y)} (b-y)(x-a)m(dx) + \int_{(y,b)} (b-x)(y-a)m(dx) + (b-y)(y-a)m(\{y\})\right] \\
=& \frac{b-y}{b-a}\left[\int_{(a,y)} (x-a)m(dx) + (y-a)\frac{m(\{y\})}{2}\right] \\
&+ \frac{y-a}{b-a}\left[\int_{(y,b)} (b-x)m(dx) + (b-y)\frac{m(\{y\})}{2}\right] \\
=& \frac{b-y}{b-a} q(y,a) + \frac{y-a}{b-a} q(y,b) \\
=& E_y q(y, Y_{H_{a,b}}),
\end{aligned}$$

where in the last equality we use (1) and (2). □

Lemma 2.3. *q is finite on $I^\circ \times I$.*

Proof. Let $a, b \in I$ such that $a < b$ and let $y \in (a,b)$. Recall that by (1) we have $E_y H_{a,b} < \infty$. Together with Formulas (5) and (2) this entails that if the boundary point l is attained in finite time with a positive probability (in particular, $l > -\infty$), then $q(y,l)$ has to be finite. The same argument applies to the right-hand boundary r. □

Notice that Lemma 2.3 garantees that $q(y, Y_t) - (t \wedge H_{l,r})$, $t \in [0,\infty)$, is a real-valued process. We can now prove the main result of this section.

Proof of Theorem 2.1. Let $a, b \in I$ with $a < y < b$. We first show that $q(y, Y_{t \wedge H_{a,b}}) - t \wedge H_{a,b}$, $t \in [0,\infty)$, is a P_y-martingale. For this purpose observe that for all $t \in [0,\infty)$ it holds

$$E_y\left[q(y, Y_{H_{a,b}}) - H_{a,b} | \mathcal{F}_t\right] = \quad (q(y, Y_{H_{a,b}}) - H_{a,b}) 1_{\{H_{a,b} \le t\}} + E_y\left[q(y, Y_{H_{a,b}}) - H_{a,b} | \mathcal{F}_t\right] 1_{\{H_{a,b} > t\}}. \tag{7}$$

On the event $\{H_{a,b} > t\}$ we have $q(y, Y_{H_{a,b}}) - H_{a,b} = q(y, Y_{H_{a,b}}) \circ \theta_t - H_{a,b} \circ \theta_t - t$, where θ_t denotes the shift operator for Y (see Chapter III in [3]). The Markov property and (5) imply that on the event $\{H_{a,b} > t\}$ we have P_y-a.s.

$$\begin{aligned}
E_y\left[q(y, Y_{H_{a,b}}) - H_{a,b} | \mathcal{F}_t\right] =& E_z[q(y, Y_{H_{a,b}}) - H_{a,b}]\Big|_{z=Y_t} - t \\
=& E_z[q(y, Y_{H_{a,b}}) - q(z, Y_{H_{a,b}})]\Big|_{z=Y_t} - t.
\end{aligned} \tag{8}$$

Formula (4) yields for all $z \in (a,b)$

$$q(y, Y_{H_{a,b}}) - q(z, Y_{H_{a,b}}) = q(y,z) + \frac{\partial^0 q}{\partial x}(y,z)(Y_{H_{a,b}} - z).$$

Since $E_z[Y_{H_{a,b}} - z] = 0$ for all $z \in (a,b)$, Equation (8) implies that on the event $\{H_{a,b} > t\}$ we have P_y-a.s.

$$E_y\left[q(y, Y_{H_{a,b}}) - H_{a,b} | \mathcal{F}_t\right] = q(y, Y_t) - t.$$

Together with (7) this yields for all $t \in [0,\infty)$

$$E_y\left[q(y, Y_{H_{a,b}}) - H_{a,b} | \mathcal{F}_t\right] = q(y, Y_{t \wedge H_{a,b}}) - t \wedge H_{a,b},$$

which shows that $q(y, Y_{t\wedge H_{a,b}}) - t \wedge H_{a,b}$, $t \in [0,\infty)$, is a P_y-martingale.

The statement of the lemma follows via a localization argument. If $l \notin I$, then choose a decreasing sequence $(l_n)_{n\in\mathbb{N}} \subseteq I$ with $l_1 < y$ and $\lim_{n\to\infty} l_n = l$. If $l \in I$, set $l_n = l$ for all $n \in \mathbb{N}$. Similarly, if $r \notin I$, then choose an increasing sequence $(r_n)_{n\in\mathbb{N}} \subseteq I$ with $r_1 > y$ and $\lim_{n\to\infty} r_n = r$, and if $r \in I$, then set $r_n = r$ for all $n \in \mathbb{N}$. The sequence of stopping times $\inf\{t \geq 0 \colon X_t \notin [l_n, r_n]\}$, $n \in \mathbb{N}$, is then a localizing sequence for the process $q(y, Y_t) - (t \wedge H_{l,r})$, $t \in [0,\infty)$. □

3 Integrability properties and Feller's test

The local martingale property of $q(y, Y_t) - (t \wedge H_{l,r})$ allows to derive some integrability properties of Y. We first show that Y is integrable at integrable stopping times.

Proposition 3.1. *Let $y \in I^\circ$. For any stopping time τ satisfying $E_y[\tau] < \infty$ we have $E_y|Y_\tau| < \infty$.*

Let us remark that Lemma 1 in [2] is a similar result for deterministic times.

Proof. Let τ be a stopping time with $E_y[\tau] < \infty$. We can assume that $\tau \leq H_{l,r}$ (else replace τ by $\tau \wedge H_{l,r}$). The stopped process $Z_t = q(y, Y_{t\wedge\tau}) - t \wedge \tau$, $t \geq 0$, is also a local martingale. Let (τ_n) be a localizing sequence for (Z_t). Then $E_y[q(y, Y_{t\wedge\tau\wedge\tau_n})] = E_y[t\wedge\tau\wedge\tau_n]$. By applying Fatou's lemma to the left-hand side and monotone convergence to the right-hand side we obtain

$$E_y[q(y, Y_\tau)] \leq E_y[\tau]. \tag{9}$$

Since $q(y,\cdot)$ is strictly convex on I, decreasing on (l,y) and increasing on (y,r), there exists a constant $C \in (0,\infty)$ such that

$$|x - y| \leq C(1 + q(y,x)), \qquad \text{for all } x \in I. \tag{10}$$

Combining (9) and (10) entails

$$E_y|Y_\tau| \leq |y| + E_y|Y_\tau - y| \leq |y| + C(1 + E_y(q(y, Y_\tau)) < \infty.$$

□

Proposition 3.2. *Let $y \in I^\circ$ and $T \in [0, \infty)$. Assume that the following two implications hold true:*

(i) if $l = -\infty$, then $m\big((-\infty, y)\big) = \infty$;

(ii) if $r = \infty$, then $m\big((y, \infty)\big) = \infty$.

Then the family of random variables

$$\{Y_\tau : \tau \text{ stopping time with } E_y[\tau] \le T\}$$

is uniformly integrable under P_y.

Proof. Assumption (i) (resp. (ii)) means that $\lim_{x\to-\infty} q(y,x)/|x| = \infty$ (resp. $\lim_{x\to\infty} q(y,x)/x = \infty$) whenever $l = -\infty$ (resp. $r = \infty$). The statement now follows from Estimate (9) and de la Vallée-Poussin's theorem. □

We next recall a result known as Feller's test for explosions.

Theorem 3.3. *For any $y \in I^\circ$ the following equivalences hold true:*

$$l \text{ is accessible} \iff q(y,l) < \infty, \tag{11}$$

$$r \text{ is accessible} \iff q(y,r) < \infty. \tag{12}$$

In contrast to the classical approach (see e.g. Thm 5.29, Chapter 5 in [1] or Thm 52.1, Chapter V in [4] for the proofs in the SDE setting), we do not construct an appropriate solution of the related Sturm-Liouville problem in the proof, but rather work directly with the function q.

Proof. Notice that the implications from the left-hand side to the right-hand side follow from Lemma 2.3.

We now prove via contradiction that $q(y,r) < \infty$ entails that r is accessible. If $r = \infty$, then $q(y,r) = \infty$. So, suppose that $r < \infty$, $q(y,r) < \infty$ and that r is not accessible. Choose $a, b \in I$ such that $l < a < y < b < r$. By Lemma 2.2 we have

$$E_y[H_{a,b}] = E_y\left[q\big(y, Y_{H_{a,b}}\big)\right] = \frac{b-y}{b-a} q(y,a) + \frac{y-a}{b-a} q(y,b). \tag{13}$$

By letting b converge to r from below, we obtain

$$\begin{aligned} E_y[H_a] = E_y[H_{a,r}] &= \lim_{b\uparrow r} E_y[H_{a,b}] = \lim_{b\uparrow r}\left\{\frac{b-y}{b-a} q(y,a) + \frac{y-a}{b-a} q(y,b)\right\} \\ &= \frac{r-y}{r-a} q(y,a) + \frac{y-a}{r-a} q(y,r) < \infty. \end{aligned}$$

In particular, $H_a < \infty$, P_y-a.s. This further entails that the stopped process Y^{H_a} is a bounded P_y-local martingale, and hence a uniformly integrable martingale converging in $L^1(P_y)$. The limit, however, is the constant a, which contradicts the $L^1(P_y)$ convergence. Thus we have shown that $q(y,r) < \infty$ implies that r is accessible.

In a similar way one can show that $q(y,l) < \infty$ entails that l is accessible. □

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