STABILITY PARAMETERS FOR QUIVER SHEAVES

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ABSTRACT. In this paper, we will begin the systematic study of the influence of the choice of a faithful representation on the notion of (semi)stability for decorated principal bundles. We will prove boundedness for slope semistable quiver sheaves.

INTRODUCTION

Hitchin used the technique of dimensional reduction of self-duality equations in four dimensions to derive interesting equations for a vector bundle over a compact riemann surface endowed with a holomorphic structure and an endomorphism ([13], [14]). This led to the theory of higgs bundles on compact riemann surfaces. Likewise, García-Prada applied it to the hermite–einstein equations for a bundle on the product of a compact riemann surface and the projective line and found coupled vortex equations on the compact riemann surface whose solutions may be interpreted as holomorphic triples. In a purely algebraic context, homogeneous bundles on a homogeneous variety G/P , G a complex reductive algebraic group, $P \subset G$ a parabolic subgroup, may be described in terms of representations of $\mathfrak{a}(\mathfrak{n})$ (infinite) quiver ([7], [12], [3], [18]). (This is a dimensional reduction from G/P to a point.) Álvarez-Cónsul and García-Prada generalized this to identify homogeneous vector bundles on the product of a compact kähler manifold and G/P with quiver bundles on the kähler manifold [3]. In that work, the concept of a quiver sheaf was developed. Furthermore, the papers [3] and [2] contain the parameter dependent notion of slope semistability for these objects. The homological algebra of quiver sheaves was investigated in [10]. Gieseker type notions of semistability for quiver sheaves were discussed in [22], [24], [1], [26] and moduli spaces constructed. Quiver sheaves have discrete invariants of a topological flavor, namely the hilbert polynomials of the participating sheaves. For fixed values of these discrete invariants, we now have a family of moduli spaces, depending on a countable set of stability parameters. The present article is motivated by the study of the variation of the moduli spaces with the stability parameter. The basic property we will establish here is that there are only finitely many distinct semistability concepts and, so, only finitely many distinct moduli spaces. This continues our work in [4], [16], [27]. In [5], we discussed the chamber structure on the set of stability parameters which explains how moduli spaces associated with different stability parameters interact. The result of the present paper show that there are only finitely many chambers. In this way, we get a basic qualitative picture of the family of moduli spaces.

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We will use the set-up of [5]. Introductions to quiver sheaves are also contained in the references mentioned in the previous paragraph. We will work over a fixed polarized complex projective manifold $(X, \mathcal{O}_X(1))$. Fix a quiver $Q = (V, A, t, h)$ and a tuple $\underline{M} = (M_a, a \in A)$ of locally free $\mathcal{O}_X$ -modules of finite rank. An $\underline{M}$ -twisted Q -sheaf is a tuple $(\mathcal{E}_v, v \in V, \varphi_a, a \in A)$ in which $\mathcal{E}_v$ is a torsion free coherent $\mathcal{O}_X$ -module, $v \in V$, and $\varphi_a: M_a \otimes \mathcal{E}_{t(a)} \rightarrow \mathcal{E}_{h(a)}$ is a (twisted) homomorphism, $a \in A$. We will refer to the tuple $(\text{rk}(\mathcal{E}_v), v \in V, \deg(\mathcal{E}_v), v \in V)$ as the *type* of $(\mathcal{E}_v, v \in V, \varphi_a, a \in A)$. The notion of slope semistability for $\underline{M}$ -twisted Q -sheaves depends on a tuple $\underline{\kappa} \in (\mathbb{R}_{>0})^{\times \#V}$ and a tuple $\underline{\chi} \in \mathbb{R}^{\#V}$. For such parameters $\underline{\kappa} = (\kappa_v, v \in V)$, $\underline{\chi} = (\chi_v, v \in V)$ and a tuple $(\mathcal{F}_v, v \in V)$ of coherent $\mathcal{O}_X$ -modules, we define

- the $\underline{\kappa}$ -rank as $\text{rk}_{\underline{\kappa}}(\mathcal{F}_v, v \in V) := \sum_{v \in V} \kappa_v \cdot \text{rk}(\mathcal{F}_v)$,
- the $(\underline{\kappa}, \underline{\chi})$ -degree as $\deg_{\underline{\kappa}, \underline{\chi}}(\mathcal{F}_v, v \in V) := \sum_{v \in V} (\kappa_v \cdot \deg(\mathcal{F}_v) + \chi_v \cdot \text{rk}(\mathcal{F}_v))$,
- and, if the $\underline{\kappa}$ -rank is positive, the $(\underline{\kappa}, \underline{\chi})$ -slope as $\mu_{\underline{\kappa}, \underline{\chi}}(\mathcal{F}_v, v \in V) := \frac{\deg_{\underline{\kappa}, \underline{\chi}}(\mathcal{F}_v, v \in V)}{\text{rk}_{\underline{\kappa}}(\mathcal{F}_v, v \in V)}$.

An $\underline{M}$ -twisted Q -sheaf $(\mathcal{E}_v, v \in V, \varphi_a, a \in A)$ is $(\underline{\kappa}, \underline{\chi})$ -slope semistable, if the inequality

$$\mu_{\underline{\kappa}, \underline{\chi}}(\mathcal{F}_v, v \in V) \leq \mu_{\underline{\kappa}, \underline{\chi}}(\mathcal{E}_v, v \in V)$$

is satisfied for every non-trivial proper Q -subsheaf $(\mathcal{F}_v, v \in V)$ of $(\mathcal{E}_v, v \in V, \varphi_a, a \in A)$. If one fixes the type $(\underline{n}, \underline{d})$ and the stability parameter $(\underline{\kappa}, \underline{\chi})$, then it is easy to show that there exists a constant C with

$$\mu_{\max}(\mathcal{E}_{v_0}) \leq C,$$

for every vertex $v_0 \in V$ and every $(\underline{\kappa}, \underline{\chi})$ -slope semistable Q -sheaf $(\mathcal{E}_v, v \in V, \varphi_a, a \in A)$ of type $(\underline{n}, \underline{d})$ ([25], Exercise 2.5.4.1, [26], Proposition 2.1). In [27], we proved that the result still holds, if one fixes just $(\underline{n}, \underline{d})$ and $\underline{\kappa}$ and lets $\underline{\chi}$ vary freely. In this paper, we will complete the picture and show:

Main Theorem. *Let $(X, \mathcal{O}_X(1))$ be a polarized complex projective manifold, $Q = (V, A, t, h)$ a quiver, $\underline{M} = (M_a, a \in A)$ a tuple of twisting vector bundles, and $(\underline{n}, \underline{d})$ a type. Then, there exists a constant C , such that, for every stability parameter $(\underline{\kappa}, \underline{\chi}) \in (\mathbb{R}_{>0})^{\times \#V} \times \mathbb{R}^{\#V}$, every $(\underline{\kappa}, \underline{\chi})$ -slope semistable Q -sheaf $(\mathcal{E}_v, v \in V, \varphi_a, a \in A)$ of type $(\underline{n}, \underline{d})$, and every vertex $v_0 \in V$, the inequality*

$$\mu_{\max}(\mathcal{E}_{v_0}) \leq C$$

is satisfied.

This result is crucial for exploring moduli spaces. First of all, it immediately implies:

Corollary. *Let $(X, \mathcal{O}_X(1))$ be a polarized complex projective manifold, $Q = (V, A, t, h)$ a quiver, $\underline{M} = (M_a, a \in A)$ a tuple of twisting vector bundles, and $\underline{P} = (P_v, v \in V)$ a tuple of Hilbert polynomials. Then, the family of torsion free coherent $\mathcal{O}_X$ -modules $\mathcal{F}$ for which there exist a stability parameter $(\underline{\kappa}, \underline{\chi})$, a $(\underline{\kappa}, \underline{\chi})$ -slope semistable $\underline{M}$ -twisted Q -sheaf $(\mathcal{E}_v, v \in V, \varphi_a, a \in A)$ with $P(\mathcal{E}_v) = P_v$, $v \in V$, and a vertex $v_0 \in V$ with $\mathcal{F} \cong \mathcal{E}_{v_0}$ is bounded.*

To study slope semistability for quiver sheaves further, let us point out that several normalizations are possible. Obviously, for $(\underline{\kappa}, \underline{\chi}) \in (\mathbb{R}_{>0})^{\times \#V} \times \mathbb{R}^{\#V}$ and $\lambda \in \mathbb{R}_{>0}$, the notions of $(\underline{\kappa}, \underline{\chi})$ -slope semistability and of $(\lambda \cdot \underline{\kappa}, \lambda \cdot \underline{\chi})$ -slope semistability are equivalent. This means that we may require

$$(1) \quad \underline{\kappa} \in \Xi := \left\{ \underline{\nu} \in (\mathbb{R}_{>0})^{\times \#V} \mid \|\underline{\nu}\| = 1 \right\}.$$

Here, $\|\cdot\|$ stands for the maximum norm on $\mathbb{R}^{\#V}$. Next, one may also check that one may choose

$$(\underline{\kappa}, \underline{\chi}) \in \Pi := \left\{ (\underline{\nu}, \underline{\psi}) \in \Xi \times \mathbb{R}^{\#V} \mid \sum_{v \in V} (\nu_v \cdot d_v + \psi_v \cdot r_v) = 0 \right\}.$$

Using this normalization, we defined in [5], Section 2.5, a locally finite subdivision of $\Xi \times \mathbb{R}^{\#V}$ into locally closed chambers, such that the notion of $(\underline{\kappa}, \underline{\chi})$ -slope semistability remains constant within each chamber. If we combine [5], Proposition 2.6.1, with the main result of the present article, we obtain:

Theorem. *In the situation of the corollary, there is a finite set $\{(\underline{\kappa}_1, \underline{\chi}_1), \dots, (\underline{\kappa}_s, \underline{\chi}_s)\} \subset (\mathbb{R}_{>0})^{\times \#V} \times \mathbb{R}^{\#V}$ of stability parameters, such that, for any $(\underline{\kappa}, \underline{\chi}) \in (\mathbb{R}_{>0})^{\times \#V} \times \mathbb{R}^{\#V}$, there is an index $j \in \{1, \dots, s\}$, such that the notions of $(\underline{\kappa}, \underline{\chi})$ -slope (semi)stability and $(\underline{\kappa}_j, \underline{\chi}_j)$ -slope (semi)stability for $\underline{M}$ -twisted Q -sheaves $(\mathcal{E}_v, v \in V, \varphi_a, a \in A)$ with $P(\mathcal{E}_v) = P_v, v \in V$, are equivalent.*

As usual, the chamber decomposition gives an indication how the moduli associated with different stability parameters interact (see, e.g., [4], Section 6, [16], Section 3.5).

Let us comment a bit on the techniques in this paper. Quiver sheaves are examples of decorated principal bundles in the sense of [25]. Recall that a *rational principal bundle* on X is a pair $(U, \mathcal{P})$ in which $U \subset X$ is a big open subset¹ and $\mathcal{P}$ is a principal G -bundle on U . For example, if $\mathcal{E}$ is a torsion free coherent $\mathcal{O}_X$ -module of rank $n \geq 1$, then the locus U where $\mathcal{E}$ is locally free is a big open subset, and the frame bundle of $\mathcal{E}|_U$ is a principal $\mathrm{GL}_n(\mathbb{C})$ -bundle, so that $\mathcal{E}$ gives rise to a rational principal $\mathrm{GL}_n(\mathbb{C})$ -bundle. We fix a representation $\sigma: G \rightarrow \mathrm{GL}(H)$ and a line bundle $\mathcal{L}$ on X . Then, an $\mathcal{L}$ -twisted affine σ -bump is a pair $((U, \mathcal{P}), \varphi)$ which consists of a rational principal G -bundle $(U, \mathcal{P})$ on X and a homomorphism $\varphi: \mathcal{P}_\sigma \rightarrow \mathcal{L}|_U$. Here, $\mathcal{P}_\sigma$ is the vector bundle on U with typical fiber H that is associated with $\mathcal{P}$ and the G -action on H induced by σ . Let us remind you of the significance of the three different types of stability parameters that we have used above. The first one is the choice of a polarization $\mathcal{O}_X(1)$ on X . This is irrelevant, if X is a curve, and is quite tricky to understand, if $\dim(X) \geq 3$, even if one deals “just” with vector bundles or torsion free coherent sheaves ([19], [21], [11]). We will not talk about it in this paper. The second parameter is the choice of a character χ of G . This appeared already in the first examples of decorated vector bundles [8] and was treated in a general manner in [27]. The last parameter is the choice of a faithful representation $\kappa: G \rightarrow \mathrm{GL}(K)$. More generally, one may fix a maximal torus on $T \subset G$ and a Weyl invariant pairing on $X_\star(T) \otimes_{\mathbb{Z}} \mathbb{C}$ (compare [17]). To my knowledge, this parameter has not been systematically studied in the algebro geometric context. In this paper, we will make a first general contribution. So

¹This means that the closed subset $Z := X \setminus U$ has codimension at least two in X .

far, two methods have been used in study boundedness questions for (decorated) principal bundles. The first one, appearing in [23], [27], is based on the formalism of the instability flag, in particular, on its application to the study of semistable rational principal bundles by Ramanan and Ramanathan [20]. The second method was introduced in [9] and works for direct sums of tensor powers of the standard representation of $\mathrm{GL}_n(\mathbb{C})$. It is technically much easier. Note that the second method cannot be directly applied in the present situation, because the estimates one gets with this approach involve terms of the form χ_v/κ_v (see [5], Proposition 2.4.2, or [24], Remark 3.3.2), for some vertices v of V , and these terms might get arbitrarily large, if $\underline{\kappa}$ approaches the boundary of Ξ .² An important observation we make here is that this approach still works well, if $\underline{\chi} = 0$ (see the proof of Theorem 4.2). In order to control $\underline{\chi}$, we recur again to the mechanism of the instability flag. The latter depends on the choice of a Weyl invariant norm on the space of real characters of a maximal torus of G . In our set-up, G is a direct product of general linear groups, and the norm corresponds to the stability parameter $\underline{\kappa}$. We need to control the instability flag when $\underline{\kappa}$ approaches the boundary of Ξ . This requires some refined estimates. Maybe this is the first time that the dependence of the instability on the chosen Weyl invariant norm has an impact.

Conventions. We will use freely the notation and the results from the papers [5], [27]. In addition, we will adopt the following terminology: Given a tuple $\underline{k} = (k_1, \dots, k_w)$ of integers, we set $|\underline{k}| := k_1 + \dots + k_w$. For a natural number $n \in \mathbb{N}$, the symbol $[n]$ is a short-hand for the set $\{1, \dots, n\}$. We will assume that the vertex set V of the quiver Q is of the form $[w]$ for some natural number $w \geq 1$. The letter $\mathbb{K}$ denotes the function field $\mathbb{C}(X)$ of X . For $w \in \mathbb{N}$, the symbol $\mathbb{R}_+^w$ stands for $(\mathbb{R}_{>0})^{\times w}$, the space of w -tuples of positive real numbers.

If we have fixed $Q, \underline{M}, \mathcal{L}$, as well as the type $(\underline{n}, \underline{d})$, and if we speak of “constants”, “bounds” or similar concepts, it is understood that they do only depend on these background data.

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1. PRELIMINARIES: CHARACTERS AND COCHARACTERS

First, we fix a natural number $n \geq 1$ and let $\mathrm{GL}_n(\mathbb{C})$ be the linear algebraic group consisting of invertible $(n \times n)$ -matrices and $D \subset \mathrm{GL}_n(\mathbb{C})$ the maximal torus formed by the invertible diagonal $(n \times n)$ -matrices. For $i \in [n]$, we define the cocharacter $e_i: \mathbb{C}^* \rightarrow D$, $z \mapsto \mathrm{diag}(0, \dots, 0, z, 0, \dots, 0)$, z occupying the i -th slot, and the character $e^i: D \rightarrow \mathbb{C}^*$, $\mathrm{diag}(m_1, \dots, m_n) \mapsto m_i$. Then, $\{e_1, \dots, e_n\}$ and $\{e^1, \dots, e^n\}$ are bases for the free abelian groups $X_*(D) := \mathrm{Hom}_{\mathrm{alg. gp.}}(\mathbb{C}^*, D)$ and $X(D) := X^*(D) := \mathrm{Hom}_{\mathrm{alg. gp.}}(D, \mathbb{C}^*)$ of cocharacters and characters of D , respectively. These bases are dual to each other with respect to the natural pairing $\langle \cdot, \cdot \rangle: X(D) \times X_*(D) \rightarrow \mathrm{Hom}_{\mathrm{alg. gp.}}(\mathbb{C}^*, \mathbb{C}^*) \cong \mathbb{Z}$, $(\chi, \lambda) \mapsto \chi \circ \lambda$. Using the basis

²It could, however, be applied when $\underline{\kappa}$ is fixed in order to give a simpler proof of the results [27], at least for $G = \mathrm{GL}_n(\mathbb{C})$.

$\{e_1, \dots, e_n\}$ of $X_\star(D)$, we introduce the pairing

$$\begin{aligned} (\cdot, \cdot)_\star: X_\star(D) \times X_\star(D) &\longrightarrow \mathbb{Z} \\ \left(\sum_{i=1}^n \chi_i \cdot e_i, \sum_{i=1}^n \psi_i \cdot e_i \right) &\longmapsto \sum_{i=1}^n \chi_i \cdot \psi_i. \end{aligned}$$

There is a similar pairing $(\cdot, \cdot): X(D) \times X(D) \longrightarrow \mathbb{Z}$.

Next, we introduce the real vector spaces

$$X_{\star, \mathbb{R}}(D) := X_\star(D) \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad X_{\mathbb{R}}(D) := X(D) \otimes_{\mathbb{Z}} \mathbb{R}.$$

The pairings $(\cdot, \cdot)_{\star, \mathbb{R}}: X_{\star, \mathbb{R}}(D) \times X_{\star, \mathbb{R}}(D) \longrightarrow \mathbb{R}$, $\langle \cdot, \cdot \rangle_{\mathbb{R}}: X_{\mathbb{R}}(D) \times X_{\star, \mathbb{R}}(D) \longrightarrow \mathbb{R}$, and $(\cdot, \cdot)_{\mathbb{R}}: X_{\mathbb{R}}(D) \times X_{\mathbb{R}}(D) \longrightarrow \mathbb{R}$ are the scalar extensions of the pairings discussed before. The norm on $X_{\star, \mathbb{R}}(D)$ and $X_{\mathbb{R}}(D)$ induced by $(\cdot, \cdot)_{\star, \mathbb{R}}$ and $(\cdot, \cdot)_{\mathbb{R}}$ will be denoted by $\|\cdot\|_\star$ and $\|\cdot\|$, respectively. For a real cocharacter $l = \sum_{i=1}^n \chi_i \cdot e_i \in$

$X_{\star, \mathbb{R}}(D)$, we define the *dual real character* $l^\vee := \sum_{i=1}^n \chi_i \cdot e^i \in X_{\mathbb{R}}(D)$. In terms of the above pairings, $l^\vee$ is characterized by the property

$$\forall l' \in X_{\star, \mathbb{R}}(D): \quad (l, l')_{\star, \mathbb{R}} = \langle l^\vee, l' \rangle_{\mathbb{R}}.$$

Now, let $w \geq 1$ be a natural number, $\underline{n} = (n_1, \dots, n_w)$ a tuple of positive natural numbers, and $\underline{\kappa} = (\kappa_1, \dots, \kappa_w)$ a tuple of **non-negative** real numbers. We set

$$\mathrm{GL}_{\underline{n}}(\mathbb{C}) := \bigtimes_{i=1}^w \mathrm{GL}_{n_i}(\mathbb{C}),$$

let $D_i \subset \mathrm{GL}_{n_i}(\mathbb{C})$ be the maximal torus consisting of diagonal matrices and $D = \bigtimes_{i=1}^w D_i \subset \mathrm{GL}_{\underline{n}}(\mathbb{C})$. Then,

$$X_{\star, \mathbb{R}}(D) = \bigoplus_{i=1}^w X_{\star, \mathbb{R}}(D_i) \quad \text{and} \quad X_{\mathbb{R}}(D) = \bigoplus_{i=1}^w X_{\mathbb{R}}(D_i).$$

So, we write a real cocharacter $l \in X_{\star, \mathbb{R}}(D)$ as a tuple $l = (l_1, \dots, l_w)$ with $l_i \in X_{\star, \mathbb{R}}(D_i)$, $i \in [w]$. We use an analogous notation for real characters of D .

Next, introduce the pairing

$$\begin{aligned} (\cdot, \cdot)_{\star, \underline{\kappa}}: X_{\star, \mathbb{R}}(D) \times X_{\star, \mathbb{R}}(D) &\longrightarrow \mathbb{R} \\ ((l_1, \dots, l_w), (l'_1, \dots, l'_w)) &\longmapsto \sum_{i=1}^w \kappa_i \cdot (l_i, l'_i)_{\star, \mathbb{R}}, \end{aligned}$$

and, similarly, $(\cdot, \cdot)_{\underline{\kappa}}: X_{\mathbb{R}}(D) \times X_{\mathbb{R}}(D) \longrightarrow \mathbb{R}$. Note that these pairings become degenerate, if not all the entries of $\underline{\kappa}$ are positive. The induced (semi-)norms will be denoted by

$$\|\cdot\|_{\star, \underline{\kappa}}: X_{\star, \mathbb{R}}(D) \longrightarrow \mathbb{R}, \quad \text{and} \quad \|\cdot\|_{\underline{\kappa}}: X_{\mathbb{R}}(D) \longrightarrow \mathbb{R}.$$

The natural pairing between characters and cocharacters is

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\mathbb{R}}: X_{\mathbb{R}}(D) \times X_{\star, \mathbb{R}}(D) &\longrightarrow \mathbb{Z} \\ ((x_1, \dots, x_w), (l_1, \dots, l_w)) &\longmapsto \sum_{i=1}^w \langle x_i, l_i \rangle_{\mathbb{R}}. \end{aligned}$$

For a real cocharacter $l = (l_1, \dots, l_w) \in X_{\star, \mathbb{R}}(D)$, we define the *dual real character* $l_{\underline{\kappa}}^\vee := (\kappa_1 \cdot l_1^\vee, \dots, \kappa_w \cdot l_w^\vee) \in X_{\mathbb{R}}(D)$. The formula

$$\forall l' \in X_{\star, \mathbb{R}}(D) : \quad (l, l')_{\star, \underline{\kappa}} = \langle l_{\underline{\kappa}}^\vee, l' \rangle_{\mathbb{R}}$$

still holds true.

Remark 1.1. Suppose that $\underline{\kappa} = (\kappa_1, \dots, \kappa_w)$ is a tuple of **positive integers**. Then, we have the embedding

$$\iota_{\underline{\kappa}} : \mathrm{GL}_{\underline{n}}(\mathbb{C}) \longrightarrow \mathrm{GL}_N(\mathbb{C}), \quad N := \kappa_1 \cdot n_1 + \dots + \kappa_w \cdot n_w,$$

that maps a tuple $(m_1, \dots, m_w)$ of matrices to the block diagonal $(N \times N)$ -matrix in which the block m_1 is first repeated κ_1 times, then the block m_2 κ_2 times, and so on. In this case, the pairing $(\cdot, \cdot)_{\star, \underline{\kappa}}$ and the norm $\|\cdot\|_{\star, \underline{\kappa}}$ on $X_{\star, \mathbb{R}}(D)$ are simply those induced by the corresponding pairing and norm on $X_{\star, \mathbb{R}}(T)$, $T \subset \mathrm{GL}_N(\mathbb{C})$ the maximal torus of diagonal matrices that were described at the beginning.

2. A REFINED ANALYSIS OF THE INSTABILITY FLAG

In this section, we will adopt the notation of Section 2 of [27]. Abbreviate $G := \mathrm{GL}_{\underline{n}}(\mathbb{C})$. Fix a Borel subgroup $D \subset B \subset G$. We start with a finite dimensional complex vector space H and a representation $\sigma : G \longrightarrow \mathrm{GL}(H)$. Then, we form $R := H \oplus \mathbb{C}$ and $\rho := \sigma \oplus \mathbb{1} : G \longrightarrow \mathrm{GL}(R)$. We need to investigate the instability one parameter subgroups of points of the shape $r = (h, 1)$, $h \in H$. Let $\chi \in X_{\mathbb{R}}(G) := X(G) \otimes_{\mathbb{Z}} \mathbb{R}$ be a real character of G . The point r is χ -(semi)stable, if

$$\forall g \in G \forall l \in \mathcal{C}(B, D) : \quad \mu_{\mathrm{St}_D(\sigma(g)(h))}(l) \leq 0 \implies \langle \chi, l \rangle_{\mathbb{R}} (\geq) 0.$$

For the following, fix a norm $\|\cdot\|$ on $X_{\mathbb{R}}(G)$. Note that, for $\eta \in \mathbb{R}_{>0}$, the condition of χ -(semi)stability is equivalent to the one of $(\eta \cdot \chi)$ -(semi)stability.

Proposition 2.1. *There is a constant $K > 0$, such that, for a character $\chi \in X_{\mathbb{R}}(G)$ with*

$$\|\chi\| < \frac{1}{K},$$

the following conditions are equivalent:

- *The point r is χ -(semi)stable.*
- *The point $[r] = [h, 1] \in P(R)$ is GIT-(semi)stable with respect to the natural linearization of the G -action in $\mathcal{O}_{P(R)}(1)$ twisted by the character χ .*

This is Corollary 2.11 in [27]. For subsequent arguments, we will have to review the proof of this result.

Proof. First note that, for any character $\chi \in X_{\mathbb{R}}(G)$, the second condition implies the first one. Let

$$(2) \quad \mathcal{C}(B, D) = \bigcup_{i=1}^h B_i$$

be the decomposition of the Weyl chamber into cones described in [27], p. 457, and

$$\mathcal{T} = \{ \lambda_1, \dots, \lambda_m \}$$

the set of minimal integral generators of edges of these cones. Introduce

$$L_i := \max \left\{ -\langle \psi, \lambda_i \rangle_{\mathbb{R}} \mid \psi \in X_{\mathbb{R}}(G) \wedge \|\psi\| = 1 \right\}, \quad i \in [m],$$

$$L := \max\{L_1, \dots, L_m\}.$$

Next, let $\mathcal{S}$ be the set of states of the representation σ . Then, we pick a positive natural number M with the property

$$\forall \omega \in \mathcal{S}, i \in [m] : \quad \langle \omega, \lambda_i \rangle \neq 0 \quad \implies \quad |\langle \omega, \lambda_i \rangle| > \frac{1}{M}.$$

The proposition is true for $K := M \cdot L$. $\square$

We need some more data. Let $\|\cdot\|$ be a norm on $X_{\star, \mathbb{R}}(D)$ and set

$$\mathcal{T} := \left\{ l \in X_{\star, \mathbb{R}}(D) \mid \|l\| = 1 \right\}.$$

Furthermore, we fix a **finite** subset $\Gamma \subset G$ with the property

$$(3) \quad \left\{ \text{St}_D(\sigma(\gamma)(h)) \mid \gamma \in \Gamma \right\} = \left\{ \text{St}_D(\sigma(g)(h)) \mid g \in G \right\}.$$

Pick a stability parameter $\underline{\kappa} \in \Xi$,³ let $\chi \in X_{\mathbb{R}}(D)$, and assume that the point $r = (h, 1)$ is χ -unstable. We call a point $(\gamma_0, l_0) \in \Gamma \times \mathcal{T}$ at which the function

$$\begin{aligned} \nu_{\chi, \underline{\kappa}} : \Gamma \times \mathcal{T} &\longrightarrow \mathbb{R} \\ (\gamma, l) &\longmapsto \frac{\mu(\rho(\gamma)(r), l) + \langle \chi, l \rangle_{\mathbb{R}}}{\|l\|_{\star, \underline{\kappa}}} \end{aligned}$$

attains its minimum an *instability point* of r and $l_r := \gamma_0^{-1} \cdot l_0 \cdot \gamma_0$ an *instability one parameter subgroup* of r . In [23], Theorem 2.10, and [27], Theorem 3.9, the following result was obtained:

Proposition 2.2. *There is a constant $K_{\underline{\kappa}}$, such that, for a stability parameter $\chi \in X_{\mathbb{R}}(G)$ with $\|\chi\| < 1/K_{\underline{\kappa}}$ and a χ -unstable point $r = (h, 1) \in R$, an instability point (γ_0, l_0) and the instability one parameter subgroup l_r satisfy*

$$\mu(r, l_r) = \mu(\rho(\gamma_0)(r), l_0) = 0.$$

As indicated by the notation, the constant $K_{\underline{\kappa}}$ **depends on $\underline{\kappa}$** . Unfortunately, there is no uniform version of this result. For this reason, we need more precise information about $K_{\underline{\kappa}}$. First, we note that we can determine (γ_0, l_0) without any restriction on χ . To this end, we look at the closed subset

$$(\Gamma \times \mathcal{T})_r := \left\{ (\gamma, l) \in \Gamma \times \mathcal{T} \mid \mu(\rho(\gamma)(r), l) = 0 \right\}$$

of $\Gamma \times \mathcal{T}$. Let $(\gamma_0, l_0) \in (\Gamma \times \mathcal{T})_r$ be a point at which the function $\nu_{\chi, \underline{\kappa}}$ takes its minimal value. By Proposition 2.2, (γ_0, l_0) is an instability point of r with respect to $(1/N) \cdot \chi$, for all $N \gg 0$. In particular, $\gamma_0^{-1} \cdot l_0 \cdot \gamma_0$ is uniquely defined, and χ and $\eta \cdot \chi$ yield the same point, $\eta \in \mathbb{R}_{>0}$.

Next, we remind the reader of the salient feature of an instability point (γ_0, l_0) . First, l_0 defines a filtration

$$\{0\} = R_0 \subsetneq R_1 \subsetneq \dots \subsetneq R_{s-1} \subsetneq R_s = R.$$

Let $Q_G(l_0) \subset G$ be the closed subgroup that stabilizes this filtration and $L_G(l_0) \subset Q_G(l_0)$ the centralizer of l_0 . It is a Levi subgroup of $Q_G(l_0)$.

³see Equation (1)

Remark 2.3. Note that

$$L_G(l_0) \cong \bigtimes_{j=1}^s \mathrm{GL}(R_j/R_{j-1}).$$

In particular, there are only finitely many possibilities for $L_G(l_0)$ as the parameters vary, call them $H_1, \dots, H_t$.

Let $j(r)$ be maximal among the indices $j \in [s]$ for which the image of $\rho(\gamma_0)(r)$ in R_j/R_{j-1} is non-zero and $\bar{r}$ the image of $\rho(\gamma_0)(r)$ in $R_{j(r)}/R_{j(r)-1}$.

Remark 2.4. i) The vector space $R_{j(r)}/R_{j(r)-1}$ is in a natural way an $L_G(l_0)$ -module. Again, there are only finitely many options for this module. In the notation of Remark 2.3, these modules give rise to finitely many representations denoted by $\rho_j^i: H_i \rightarrow \mathrm{GL}(S_j^i)$, $j \in [u_i]$, $i \in [t]$.

ii) Since we have $\mu(\rho(\gamma_0)(r), l_0) = 0$, the representation $R_{j(r)}/R_{j(r)-1}$ contains the trivial representation $\mathbb{1}$ as a direct summand.

Let $\tilde{\chi} := l_{0, \underline{\kappa}}^\vee \in X_{\mathbb{R}}(D)$ be the dual real character of l_0 (see Page 5), i.e., $\tilde{\chi}$ satisfies

$$(4) \quad \forall l \in X_{\star, \mathbb{R}}(D) : \quad \langle \tilde{\chi}, l \rangle_{\mathbb{R}} = (l_0, l)_{\star, \underline{\kappa}}.$$

Finally, we set

$$(5) \quad \chi_r := \langle \tilde{\chi}, l_0 \rangle_{\mathbb{R}} \cdot \tilde{\chi}.$$

The following is a crucial result of Ramanan and Ramanathan ([20], Proposition 1.12):

Proposition 2.5. *Assume $\chi \in X_{\mathbb{R}}(D)$ is a character with $\|\chi\| < K_{\underline{\kappa}}$. The point $[\bar{r}] \in P(R_{j(r)}/R_{j(r)-1})$ is GIT-semistable with respect to the natural linearization of the $L_G(l_0)$ -action in $\mathcal{O}_{P(R_{j(r)}/R_{j(r)-1})}(1)$ twisted by the character*

$$\chi + \frac{\chi_r}{\|l_0\|_{\star, \underline{\kappa}}^2}.$$

Remark 2.6. If $\eta \in \mathbb{R}_{>0}$ is a positive constant and χ_r^η is the character from (5) associated with $\eta \cdot \chi$, then

$$\eta \cdot \chi + \frac{\chi_r^\eta}{\|l_0\|_{\star, \underline{\kappa}}^2} = \eta \cdot \left(\chi + \frac{\chi_r}{\|l_0\|_{\star, \underline{\kappa}}^2} \right).$$

As we explained before, we need not assume $\|\chi\| < 1/K_{\underline{\kappa}}$ for defining (γ_0, l_0) and $\bar{r}$. Moreover, if $\chi \in X_{\mathbb{R}}(G)$ is any character and $(\gamma_0, l_0) \in (\Gamma \times \mathcal{Y})_r$ is a point where $\nu_{\underline{\kappa}, \chi}$ takes on its minimal value, then the associated point $\bar{r} \in R_{j(r)}/R_{j(r)-1}$ verifies the condition of semistability with respect to the character

$$\chi + \frac{\chi_r}{\|l_0\|_{\star, \underline{\kappa}}^2},$$

for all points $(h, l) \in H_i \times X_{\star, \mathbb{R}}(D)$ with $\mu(\rho_j^i(h)(\bar{r}), l) = 0$. Here, $i \in [t]$, $j \in [u_i]$ are the indices with $H_i = L_G(l)$ and $\rho_j^i = R_{j(r)}/R_{j(r)-1}$. Remark 2.4, ii), shows that $\mu(\rho_j^i(h)(\bar{r}), l) \geq 0$ holds for any $(h, l) \in H_i \times X_{\star, \mathbb{R}}(D)$.

It remains to look at those $(h, l) \in H_i \times X_{\star, \mathbb{R}}(D)$ with $\mu(\rho_j^i(h)(\bar{r}), l) > 0$. As usual, one may restrict to a finite number of such (h, l) 's: Observe that D is a

maximal torus of H_i , $i \in [t]$. We fix a Borel subgroup $D \subset B_i \subset H_i$, $i \in [t]$. Then, we may find a decomposition

$$\mathcal{C}(B_i, D) = \bigcup_{k=1}^{m_i} B_k^i$$

as in (2), working for H_i and $\rho_1^i, \dots, \rho_{u_i}^i$, $i \in [t]$. We add the one parameter subgroups of D which occur as minimal integral generators of one of the above cones to $\mathcal{T}$ and obtain the larger set

$$\mathcal{T} \subset \mathcal{T}' = \{ \lambda_1, \dots, \lambda_v \}.$$

We have to find out how negative $\langle \chi_r, \lambda_i \rangle_{\mathbb{R}} / \|l_0\|_{\star, \underline{\kappa}}$ can get, $i \in [v]$. To this end, note that, for $i \in [v]$, (4) and the Cauchy–Schwarz inequality give

$$\langle \chi_r, \lambda_i \rangle_{\mathbb{R}} = \langle \chi, l_0 \rangle_{\mathbb{R}} \cdot (l_0, \lambda_i)_{\star, \underline{\kappa}} \geq \langle \chi, l_0 \rangle_{\mathbb{R}} \cdot \|l_0\|_{\star, \underline{\kappa}} \cdot \|\lambda_i\|_{\star, \underline{\kappa}}.$$

We set⁴

$$E_i := \sup \{ \|\lambda_i\|_{\star, \underline{\nu}} \mid \underline{\nu} \in \Xi \}, \quad E := \max \{ E_1, \dots, E_v \},$$

and

$$(6) \quad F_i := \min \{ \langle \psi, \lambda_i \rangle_{\mathbb{R}} \mid \psi \in X_{\mathbb{R}}(G) : \|\psi\| = 1 \}, \quad F := -\min \{ F_1, \dots, F_v \}.$$

Altogether, we find

$$\langle \chi, \lambda_i \rangle_{\mathbb{R}} + \frac{\langle \chi_r, \lambda_i \rangle_{\mathbb{R}}}{\|l_0\|_{\star, \underline{\kappa}}^2} \geq -\|\chi\| \cdot \left(F - \frac{\langle \chi, l_0 \rangle_{\mathbb{R}}}{\|\chi\| \cdot \|l_0\|_{\star, \underline{\kappa}}} \cdot E \right), \quad i \in [v].$$

For each $i \in [t]$ and each $j \in [u_i]$, the set of states of ρ_j^i is contained in $\mathcal{S} \cup \{0\}$. Recall that $\mathcal{S}$ stands for the set of states of σ . As before, we let M' be a positive natural number with the property

$$\forall \omega \in \mathcal{S}, i \in [v] : \quad \langle \omega, \lambda_i \rangle \neq 0 \implies |\langle \omega, \lambda_i \rangle| > \frac{1}{M'}.$$

Invoking Remark 2.6, we obtain the following variant of Proposition 2.5.

Proposition 2.7. *Assume $\chi \in X_{\mathbb{R}}(D)$ is a character with $\|\chi\| = 1$, $\underline{\kappa} \in \Xi$, and $r = (h, 1) \in R$ is a χ -unstable point. Let $(\gamma_0, l_0) \in (\Gamma \times \Upsilon)_r$ be a point at which $v_{\chi, \underline{\kappa}}|_{(\Gamma \times \Upsilon)_r}$ attains its minimal value and*

$$0 < \eta < \frac{1}{M \cdot \left(F - E \cdot \frac{\langle \chi, l_0 \rangle_{\mathbb{R}}}{\|l_0\|_{\star, \underline{\kappa}}} \right)}$$

a positive real number. Then, the point $[\bar{\tau}] \in P(R_{j(r)}/R_{j(r)-1})$ is GIT-semistable with respect to the natural linearization of the $L_G(l_0)$ -action in $\mathcal{O}_{P(R_{j(r)}/R_{j(r)-1})}(1)$ twisted by the character

$$\eta \cdot \left(\chi + \frac{\chi_r}{\|l_0\|_{\star, \underline{\kappa}}^2} \right).$$

⁴The supremum exists as a real number, because the function $\underline{\nu} \mapsto \|\lambda_i\|_{\star, \underline{\nu}}$ can be extended to the closure $\bar{\Xi}$ (see Section 1).

Remark 2.8. Let $A \subset D$ be the subgroup of scalar matrices and assume that A acts trivially on H . This is, for example, the case, if H is the space $\text{Rep}(Q, \underline{n}) := \bigoplus_{a \in A} \text{Hom}(\mathbb{C}^{n_{t(a)}}, \mathbb{C}^{n_{h(a)}})$ of representations of a dimension vector $\underline{n}$ of a quiver Q .

Next, let $\underline{\kappa} = (\kappa_1, \dots, \kappa_w)$ be a tuple of positive integers and $\iota_{\underline{\kappa}}: \text{GL}_{\underline{n}}(\mathbb{C}) \rightarrow \text{GL}_N(\mathbb{C})$ the embedding described in Remark 1.1. Set

$$D' := D \cap \text{SL}_N(\mathbb{C}).$$

Then,

$$X_{\star, \mathbb{R}}(D) = X_{\star, \mathbb{R}}(D') \oplus X_{\star, \mathbb{R}}(A) \cong X_{\star, \mathbb{R}}(D') \oplus \mathbb{R}.$$

Note that $X_{\star, \mathbb{R}}(D')$ and $X_{\star, \mathbb{R}}(A)$ are orthogonal with respect to the pairing $(\cdot, \cdot)_{\star, \underline{\kappa}}$ introduced in Section 1.

Next, let $\underline{\chi} = (\chi_1, \dots, \chi_w)$ be a tuple of rational numbers subject to the condition

$$(7) \quad \sum_{i=1}^w \chi_i \cdot n_i = 0.$$

This means exactly that the rational character χ of $\text{GL}_{\underline{n}}(\mathbb{C})$ associated with $\underline{\chi}$ is trivial on A . Now, let $r = (h, 1) \in R$ be a χ -unstable point and $l_r \in X_{\star, \mathbb{Q}}(D)$ an instability one parameter subgroup with respect to the character χ . We suppose $\mu(r, l_r) = 0$. We decompose $l_r = l' + l''$ with $l' \in X_{\star, \mathbb{Q}}(D')$ and $l'' \in X_{\star, \mathbb{Q}}(A)$. Then,

$$\langle \chi, l_r \rangle_{\mathbb{R}} = \langle \chi, l' \rangle_{\mathbb{R}} \quad \text{and} \quad \|l_r\|_{\star, \underline{\kappa}}^2 = \|l'\|_{\star, \underline{\kappa}}^2 + \|l''\|_{\star, \underline{\kappa}}^2.$$

This shows that $l'' = 0$, i.e., $l_r \in X_{\star, \mathbb{R}}(D')$.

3. BOUNDEDNESS OF SOME OF THE STABILITY PARAMETERS

As in [27], we will use the ideas of Ramanan and Ramanathan [20] in order to obtain bounds on the stability parameters. The theory of semistable decorated principal bundles on curves has been described in [25]. The extension of the basic notions to rational principal bundles on higher dimensional projective manifolds has been outlined in [25], Section 2.9.2, and [27], Section 1. We continue to look at the structure group $G = \text{GL}_{\underline{n}}(\mathbb{C})$ and a representation $\sigma: G \rightarrow \text{GL}(H)$ which is homogeneous of degree zero in the sense of [24], Section 3.1. As before, we fix $m > 0$ and the line bundle $\mathcal{O}_X(m)$. Recall that an $\mathcal{O}_X(m)$ -twisted affine σ -bump is a pair $((U, \mathcal{P}), \varphi)$ which consists of a rational principal G -bundle $(U, \mathcal{P})$ on X and a homomorphism $\varphi: \mathcal{P}_{\sigma} \rightarrow \mathcal{O}_X(m)|_U$. Next, fix an injective homomorphism $\varepsilon: \mathcal{O}_X \rightarrow \mathcal{O}_X(m)$. With an $\mathcal{O}_X(m)$ -twisted affine σ -bump $((U, \mathcal{P}), \varphi)$, we associate the $\mathcal{O}_X(m)$ -twisted affine ρ -bump $((U, \mathcal{P}), \psi)$ with $\psi := \varphi \oplus \varepsilon|_U: \mathcal{P}_{\rho} = \mathcal{P}_{\sigma} \oplus \mathcal{O}_U \rightarrow \mathcal{O}_X(m)|_U$.

In the introduction, we spoke about the normalization $\sum_{i=1}^w (\kappa_i \cdot d_i + \chi_i \cdot n_i) = 0$. Here, we will need another one. Set

$$(8) \quad N := \left\{ \underline{\psi} = (\psi_1, \dots, \psi_w) \in \mathbb{R}^w \mid \sum_{i=1}^w \psi_i \cdot n_i = 0 \right\}.$$

As recalled in the introduction, we may restrict to stability parameters $(\underline{\kappa}, \underline{\chi})$ with $\|\underline{\kappa}\| = 1$ and $\underline{\chi} \in N$. The same holds in general ([25], Remark 2.5.3.5, ii). Note that an element $\underline{\chi} \in N$ may be viewed as a real character of G which is trivial on $\mathbb{G}_m$ embedded diagonally.

Let us briefly review the definition of (semi)stability. Fix $\underline{\kappa} \in \mathbb{R}_+^w$ and $\underline{\chi} \in N$. Write $\delta_i: G \rightarrow \mathbb{G}_m$ for the character $(g_i, i \in [w]) \mapsto \det(g_i), i \in [w]$. We set

$$X_{\star, \mathbb{R}}(D)^{\underline{\kappa}\text{-SL}} := \left\{ (l_1, \dots, l_w) \in X_{\star, \mathbb{R}}(D) \mid \sum_{i=1}^w \kappa_i \cdot \langle \delta_i, l_i \rangle_{\mathbb{R}} = 0 \right\}.$$

An element $l \in X_{\star, \mathbb{R}}(D)^{\underline{\kappa}\text{-SL}}$ defines a weighted flag $(\underline{W}^\bullet, \underline{\gamma})$ (compare [24], Section 3.2) in the $[w]$ -split vector space $(\mathbb{C}^{n_i}, i \in [w])$. Note that the entries of the tuple $\underline{\gamma}$ may be real numbers in our case. The flag $\underline{W}^\bullet$ defines a parabolic subgroup of $\bar{G}$ that will be denoted by $Q_G(l)$.

Now, let $(U, \mathcal{P})$ be a rational principal G -bundle on X and $(E_i, i \in [w])$ the corresponding $[w]$ -split vector bundle on U . Suppose $l \in X_{\star, \mathbb{R}}(D)^{\underline{\kappa}\text{-SL}}$. A *reduction* of $(U, \mathcal{P})$ to l is a section

$$\beta: U' \rightarrow \mathcal{P}|_{U'}/Q_G(l)$$

over a big open subset $U' \subset U$. It corresponds to a filtration

$$\underline{E}^\bullet: 0 \subsetneq (E_i^1, i \in [w]) \subsetneq \dots \subsetneq (E_i^s, i \in [w]) \subsetneq (E_i|_{U'}, i \in [w])$$

of the $[w]$ -split vector bundle $(E_i|_{U'}, i \in [w])$ on U' where the rank of E_i^j agrees with the dimension of W_i^j from the filtration $\underline{W}^\bullet$, $i \in [w]$, $j \in [s]$. Let $\underline{\gamma} := (\gamma_1, \dots, \gamma_{s+1})$ be the tuple of real numbers from the weighted flag $(\underline{W}^\bullet, \underline{\gamma})$ associated with l and set

$$\alpha_i := \frac{\gamma_{i+1} - \gamma_i}{\sum_{i=1}^w \kappa_i \cdot n_i}, \quad i \in [s], \quad \underline{\alpha} := (\alpha_1, \dots, \alpha_s).$$

Next, we define

$$L_{\underline{\kappa}, \underline{\chi}}(\underline{E}^\bullet, \underline{\alpha}) :=$$

$$\sum_{j=1}^s \alpha_j \cdot \left(\deg_{\underline{\kappa}, \underline{\chi}}(E_i, i \in [w]) \cdot \text{rk}_{\underline{\kappa}}(E_i^j, i \in [w]) - \deg_{\underline{\kappa}, \underline{\chi}}(E_i^j, i \in [w]) \cdot \text{rk}_{\underline{\kappa}}(E_i, i \in [w]) \right).$$

Remark 3.1. Suppose that $\underline{\kappa}$ is a tuple of positive integers.

i) As described in Remark 1.1, there is a corresponding faithful representation $\iota_{\underline{\kappa}}: G \rightarrow \text{GL}_N(\mathbb{C})$. Given a genuine one parameter subgroup $\lambda \in X_\star(D) \cap \text{SL}_N(\mathbb{C})$, the pairing $(\cdot, \cdot)_\star$ may be used to define a character χ_λ on $Q_G(\lambda)$. A reduction β of $(U, \mathcal{P})$ to λ gives a principal $Q_G(\lambda)$ -bundle on U' . This bundle and χ_λ define a line bundle $\mathcal{L}_\beta$ on U' . The number $L_{\underline{\kappa}, 0}(\underline{E}^\bullet, \underline{\alpha})$ does compute the degree of that line bundle (see [25], Exercise 2.4.9.2),⁵ and there is the following identity

$$L_{\underline{\kappa}, \underline{\chi}}(\underline{E}^\bullet, \underline{\alpha}) = L_{\underline{\kappa}, 0}(\underline{E}^\bullet, \underline{\alpha}) + \langle \underline{\chi}, \lambda \rangle = \deg(\mathcal{L}_\beta) + \langle \underline{\chi}, \lambda \rangle.$$

For this, we refer to [25], Exercise 2.5.2.4 and Remark 2.5.3.5, i).

ii) As in Remark 2.8, the decomposition $X_{\star, \mathbb{R}}(D) = X_{\star, \mathbb{R}}^{\underline{\kappa}\text{-SL}}(D) \oplus X_{\star, \mathbb{R}}(\Lambda)$ is orthogonal with respect to the pairing $(\cdot, \cdot)_{\star, \underline{\kappa}}$. Suppose $l \in X_{\star, \mathbb{R}}(\Lambda)$. Then, $Q_G(l) = G$ and $\beta := \text{id}_U: U \rightarrow U = \mathcal{P}/G$ is a reduction with $\mu_\sigma(\beta, \varphi) = 0$.⁶ because σ is homogeneous of degree zero. Furthermore, $\underline{\chi} \in N$ gives $\langle \underline{\chi}, l \rangle_{\mathbb{R}} = 0$. If l is

⁵The exercise does not depend on the fact that G is semisimple, but rather on the fact that the image of κ , corresponding to ι , here, is contained in $\text{SL}(W)$.

⁶For the definition of $\mu_\sigma(\beta, \varphi)$, we refer to [25], Page 347, or [27], Page 449.

the standard generator of $l \in X_{\star, \mathbb{R}}(\Lambda)$, we get the condition $\sum_{i=1}^w \kappa_i \cdot \deg(E_i) \geq 0$. In the same manner, $-l$ and β give $\sum_{i=1}^w \kappa_i \cdot \deg(E_i) \leq 0$. So, we obtain the topological restriction $\sum_{i=1}^w \kappa_i \cdot \deg(E_i) = 0$. Omitting $X_{\star, \mathbb{R}}(\Lambda)$ just means omitting this topological restriction.

We continue with the above notation. Assume that $\varphi: \mathcal{P}_\sigma \rightarrow \mathcal{O}_X(m)|_U$ is an $\mathcal{O}_X(m)|_U$ -twisted homomorphism. The definition of $\mu_\sigma(\beta, \varphi)$ given in [27], p. 449, makes sense in the more general set-up we are currently discussing.

An $\mathcal{O}_X(m)$ -twisted affine σ -bump $((U, \mathcal{P}), \varphi)$ is $(\underline{\kappa}, \underline{\chi})$ -(semi)stable, if the inequality

$$L_{\underline{\kappa}, \underline{\chi}}(\underline{E}^\bullet, \underline{\alpha})(\geq) 0$$

holds for every $\lambda \in X_{\star, \mathbb{R}}(D)^{\underline{\kappa} \cdot \text{SL}}$ and every reduction $\beta: U' \rightarrow \mathcal{P}_{|U'}/Q_G(\lambda)$ with

$$\mu_\sigma(\beta, \varphi) \leq 0.$$

Remark 3.2. i) Let $((U, \mathcal{P}), \varphi)$ be an $\mathcal{O}_X(m)$ -twisted affine σ -bump and $((U, \mathcal{P}), \psi)$ the associated $\mathcal{O}_X(m)$ -twisted affine ρ -bump. Then, for λ and β as before,

$$\mu_\rho(\beta, \psi) = \max\{\mu_\sigma(\beta, \varphi), 0\}.$$

This means that $((U, \mathcal{P}), \varphi)$ is $(\underline{\kappa}, \underline{\chi})$ -(semi)stable if and only if $((U, \mathcal{P}), \psi)$ is.

ii) Let $\underline{\kappa} \in \mathbb{R}_+^w$, $\underline{\chi} \in N$ and $\eta > 0$. It is obvious that the notions of $(\underline{\kappa}, \underline{\chi})$ -(semi)stability and $(\eta \cdot \underline{\kappa}, \eta \cdot \underline{\chi})$ -(semi)stability are equivalent. In particular, if $\underline{\kappa}$ and $\underline{\chi}$ are rational, they may be replaced by integral parameters. Then, we are in the usual set-up.

iii) Assume that $\underline{\kappa}$ is integral and $\iota_{\underline{\kappa}}: G \rightarrow \text{GL}_N(\mathbb{C})$ is the corresponding faithful representation. The arguments of [25], Example 1.5.1.18, show that the notion of (semi)stability has to be checked only for finitely many conjugacy classes of (integral) one parameter subgroups. In particular, the notion of $(\underline{\kappa}, \underline{\chi})$ -(semi)stability has to be checked only for (integral) one parameter subgroups $\lambda \in X_\star(D) \cap \text{SL}_N(\mathbb{C})$.

iv) Let Q be a quiver and $H = \text{Rep}(Q, \underline{n})$ the space of representations of Q with dimension vector $\underline{n}$. Then, the above concept of (semi)stability is the same as the one presented in the introduction. This results from [25], Proposition 1.5.1.22. In [5], Remark 2.5.4, we checked that it is sufficient to look at rational stability parameters. A similar result holds for arbitrary representations, but the arguments are more tedious.

The idea is to apply the theory of the instability flag at the generic point of X . Let $((U, \mathcal{P}), \varphi)$ be an $\mathcal{O}_X(m)$ -twisted affine σ -bump. Since G is a special group,⁷ we may choose a trivialization $\mathcal{P}_{|\text{Spec}(\mathbb{K})} \cong G_{\mathbb{K}} := G \times_{\text{Spec}(\mathbb{K})} \text{Spec}(\mathbb{K})$. This induces a trivialization $\mathcal{P}_{\sigma|\text{Spec}(\mathbb{K})} \cong H_{\mathbb{K}} := H \otimes_{\mathbb{C}} \mathbb{K}$. The decoration φ yields a point $\varphi_{\mathbb{K}} \in H_{\mathbb{K}}$ and $\varphi \oplus \varepsilon|_U$ a point $\psi_{\mathbb{K}} \in R_{\mathbb{K}}$, $R_{\mathbb{K}} := R \otimes_{\mathbb{C}} \mathbb{K} = H_{\mathbb{K}} \oplus \mathbb{K}$. For a tuple $\underline{\chi} \in N$, we may speak about $\underline{\chi}$ -(semi)stability of $\psi_{\mathbb{K}}$ with respect to the $G_{\mathbb{K}}$ -action on R . We say that $((U, \mathcal{P}), \varphi)$ is *generically* $\underline{\chi}$ -(semi)stable, if $\psi_{\mathbb{K}}$ is $\underline{\chi}$ -(semi)stable in the sense

⁷This point is not essential for our argument. In general, we may pass to a finite extension of the function field of X .

of Section 2. If $((U, \mathcal{P}), \varphi)$ is not generically $\underline{\chi}$ -semistable, then it is said to be *generically $\underline{\chi}$ -unstable*. We define

$$(9) \quad \Omega := \left\{ \underline{\psi} \in N \mid \|\underline{\psi}\| = 1 \right\}.$$

Suppose $\Omega' \subset \Omega$ is a closed subset. We call $((U, \mathcal{P}), \varphi)$ *generically Ω' -unstable*, if it is generically $\underline{\chi}$ -unstable, for all $\underline{\chi} \in \Omega'$. If $((U, \mathcal{P}), \varphi)$ is generically Ω -unstable, we also say that it is *generically totally unstable*. For later purposes, we let

$$\Omega'_Q := \Omega' \cap (\mathbb{R}_{>0} \cdot \mathbb{Q}^w) \quad \text{and} \quad \Xi_Q := \Xi \cap (\mathbb{R}_{>0} \cdot \mathbb{Q}^w)$$

consist of those elements that generate rays containing rational points.

Last but not least, we need a topological invariant. We assign to a principal G -bundle $\mathcal{P}$ on X the tuple $(n_i, i \in [w], d_i, i \in [w])$ consisting of the ranks and degrees of the vector bundles in the corresponding $[w]$ -split vector bundle $(E_i, i \in [w])$. We define the function

$$\begin{aligned} d_{\mathcal{P}}: \mathbb{R}^w &\longrightarrow \mathbb{R} \\ \underline{\psi} = (\psi_1, \dots, \psi_w) &\longmapsto \sum_{i=1}^w \psi_i \cdot d_i. \end{aligned}$$

Theorem 3.3. *Fix the type $(\underline{n}, \underline{d})$, and let $\Omega' \subset \Omega$ be a closed subset. Then, there is a constant C_1 , depending only on $(\underline{n}, \underline{d})$, σ , and Ω' , such that, for rational stability parameters $\underline{\kappa} \in \Xi_Q$, $\underline{\chi} \in \mathbb{R}^w \setminus \{0\}$ with $\underline{\chi}/\|\underline{\chi}\| \in \Omega'_Q$, the existence of a $(\underline{\kappa}, \underline{\chi})$ -slope semistable $\mathcal{O}_X(m)$ -twisted affine σ -bump in which $(U, \mathcal{P})$ has type $(\underline{n}, \underline{d})$ and which is generically Ω' -unstable implies*

$$\|\underline{\chi}\| \leq C_1.$$

Proof. To ease notation and to use the same notation as in Section 1 and 2, we write χ instead of $\underline{\chi}$. We use elements of the proof of Theorem 3.1 in [23] and Theorem 4.2 in [27]. Define $\overline{\chi} := \chi/\|\chi\|$. Since the point $\psi_{\mathbb{K}}$ is $\overline{\chi}$ -unstable, it defines an instability point $(\gamma_0, l_0) \in (\tilde{\Gamma}_{\overline{\mathbb{K}}} \times \mathcal{Y})_r$. Here, $\overline{\mathbb{K}}$ is the algebraic closure of $\mathbb{K}$ and $\tilde{\Gamma}_{\overline{\mathbb{K}}} \subset G_{\overline{\mathbb{K}}} := G_{\text{Spec}(\mathbb{C})} \times \overline{\mathbb{K}}$ is an appropriate finite subset (compare (3)).

The problem in the proof is that the decisive quantity $\langle \overline{\chi}, l_0 \rangle_{\mathbb{R}} / \|l_0\|_{\star, \underline{\kappa}}$ could get arbitrarily negative when $\underline{\kappa}$ approaches the boundary of Ξ . So, we may assume without loss of generality⁸ that

$$-\frac{\langle \overline{\chi}, l_0 \rangle_{\mathbb{R}}}{\|l_0\|_{\star, \underline{\kappa}}} > F,$$

F as defined in (6), and apply Proposition 2.7 with

$$\eta := \frac{-1}{2 \cdot M \cdot E} \cdot \frac{\|l_0\|_{\star, \underline{\kappa}}}{\langle \overline{\chi}, l_0 \rangle_{\mathbb{R}}}.$$

Now, we pick a natural number $T > 0$, such that

$$T \cdot \overline{\chi}, \quad T \cdot \underline{\kappa}, \quad T \cdot l_0, \quad N := \frac{T}{\eta}$$

are all integral.

⁸In Equation (11), we will see that there is always a lower bound.

Note. Let $\tilde{\chi}_r$ be defined with respect to $T \cdot l_0$ and $T \cdot \underline{\kappa}$. Then,

$$\frac{\langle \tilde{\chi}_r, T \cdot l_0 \rangle}{\|T \cdot l_0\|_{\star, T \cdot \underline{\kappa}}^2} = \frac{T^4 \cdot \langle \chi_r, l_0 \rangle_{\mathbb{R}}}{T^3 \cdot \|l_0\|_{\star, \underline{\kappa}}^2} = T \cdot \frac{\langle \chi_r, l_0 \rangle_{\mathbb{R}}}{\|l_0\|_{\star, \underline{\kappa}}^2}.$$

We infer from Proposition 2.7 that $[\psi_{\mathbb{K}}] \in P(\overline{R}_{\mathbb{K}})$, $\overline{R}_{\mathbb{K}} := (R_{j(r)}/R_{j(r)-1}) \otimes_{\mathbb{C}} \mathbb{K}$, is semistable with respect to the linearization in $\mathcal{O}_{P(R_{\mathbb{K}})}(N)$ twisted by the character

$$T \cdot \left(\overline{\chi} + \frac{\chi_r}{\|l_0\|_{\star, \underline{\kappa}}^2} \right).$$

As we pointed out after Proposition 2.2, $l := \gamma_0^{-1} \cdot l_0 \cdot \gamma_0$ is a genuine instability one parameter subgroup for $\psi_{\mathbb{K}}$. Thus, $T \cdot l \in G_{\mathbb{K}}$. Note also that $l_0 \in X_{\star, \mathbb{R}}(D') = X_{\star, \mathbb{R}}(D)^{\kappa\text{-SL}}$, by Remark 2.8. Using the trivialization of $\mathcal{P}_{|\text{Spec}(\mathbb{K})}$, we get a point

$$\beta_{\mathbb{K}}: \text{Spec}(\mathbb{K}) \longrightarrow \mathcal{P}_{|\text{Spec}(\mathbb{K})}/Q_G(T \cdot l_0).$$

This can be extended to a reduction

$$\beta: U' \longrightarrow \mathcal{P}_{|U'}/Q_G(T \cdot l_0)$$

over a big open subset $U' \subset U \subset X$. The argument of [23], (3.21), and [27], p. 479f, implies that

$$-\frac{1}{T} \cdot \frac{\langle \overline{\chi}, l_0 \rangle_{\mathbb{R}}}{\|l_0\|_{\star, \underline{\kappa}}^2} \cdot \deg(\mathcal{L}_{\beta}) \leq T \cdot \left(d_{\mathcal{P}}(-\chi) + \frac{1}{\eta} \cdot m \cdot \deg(\mathcal{O}_X(1)) \right).$$

The continuous function $\psi \mapsto d_{\mathcal{P}}(-\psi)$ admits a maximum K_1 on the set Ω' , and

$$\frac{1}{\eta} \cdot \frac{\|l_0\|_{\star, \underline{\kappa}}^2}{\langle \overline{\chi}, l_0 \rangle_{\mathbb{R}}} = -2 \cdot M \cdot E \cdot \|l_0\|_{\star, \underline{\kappa}}.$$

We find

$$\deg(\mathcal{L}_{\beta}) \leq T^2 \cdot K_2 \cdot \|l_0\|_{\star, \underline{\kappa}}, \quad K_2 := -\frac{K_1}{F} - 2 \cdot M \cdot E \cdot m \cdot \deg(\mathcal{O}_X(1)).$$

Set

$$K_3 := \max \left\{ \|\lambda_i\|_{\star, \underline{\nu}} \mid i \in [m], \underline{\nu} \in \Omega \right\}.$$

Let $R_{\overline{\mathbb{K}}} := R \otimes_{\mathbb{C}} \mathbb{K}$, $G_{\overline{\mathbb{K}}} := G \otimes_{\text{Spec}(\mathbb{C})} \text{Spec}(\overline{\mathbb{K}})$, and $\rho_{\overline{\mathbb{K}}}$ the representation of $G_{\overline{\mathbb{K}}}$ on $R_{\overline{\mathbb{K}}}$. We view $\psi_{\mathbb{K}}$ as an element of $R_{\overline{\mathbb{K}}}$ and set

$$(10) \quad \mathfrak{S} := \left\{ \text{St}_D(\rho_{\overline{\mathbb{K}}}(\tilde{\gamma})(\psi_{\mathbb{K}})) \mid \tilde{\gamma} \in \tilde{\Gamma}_{\overline{\mathbb{K}}} \right\}.$$

For every $S \in \mathfrak{S}$, let $I(S) := \{i \in [m] \mid \mu_S(\lambda_i) = 0\}$, and

$$I := \bigcup_{S \in \mathfrak{S}} I(S).$$

Then,

$$\begin{aligned} \Omega' &\longrightarrow \mathbb{R} \\ \psi &\longmapsto \min \left\{ \langle \psi, \lambda_i \rangle_{\mathbb{R}} \mid i \in I \right\} \end{aligned}$$

is a continuous function, and the assumption implies that it has a negative maximum. Since there are only finitely many possibilities for $\mathfrak{S}$, we may find a constant $K_4 > 0$ which depends only on σ , such that this maximum is smaller than $-K_4$.

For $i \in I$, we have

$$\frac{\langle \bar{\chi}, l_0 \rangle_{\mathbb{R}}}{\|l_0\|_{\star, \underline{\kappa}}} \leq \frac{\langle \bar{\chi}, \lambda_i \rangle_{\mathbb{R}}}{\|\lambda_i\|_{\star, \underline{\kappa}}}.$$

It follows readily that

$$(11) \quad \frac{\langle \bar{\chi}, l_0 \rangle_{\mathbb{R}}}{\|l_0\|_{\star, \underline{\kappa}}} \leq -\frac{K_4}{K_3}.$$

Recall (Remark 3.2, ii) that a ρ -bump which is $(\underline{\kappa}, \chi)$ -slope semistable is also $(T \cdot \underline{\kappa}, T \cdot \chi)$ -slope semistable. In particular, we find

$$0 \leq \deg(\mathcal{L}_\beta) + \langle T \cdot \chi, T \cdot l_0 \rangle \leq T^2 \cdot \|l_0\|_{\star, \underline{\kappa}} \cdot \left(K_2 - \|\chi\| \cdot \frac{K_4}{K_3} \right),$$

and this gives

$$\|\chi\| \leq \frac{K_2 \cdot K_3}{K_4}.$$

This concludes the proof of the theorem. $\square$

Remark 3.4. i) The above arguments provide an alternative proof for Theorem 4.2 in [27] for the structure group $\mathrm{GL}_n(\mathbb{C})$.

ii) Let us use the conventions of the proof of Theorem 3.3. Suppose $((U, \mathcal{P}), \varphi)$ is an $\mathcal{O}_X(m)$ -twisted affine σ -bump. The set $\mathfrak{S}$ associated with it in (10) is called the *generic set of sets of states* of the affine bump. Since there are only finitely many possibilities for $\mathfrak{S}$, we may fix the generic set of sets of states for our discussion. The notion of χ -(semi)stability of a point in R or $R_{\mathbb{K}}$ depends only on its set of sets of states. So, it makes sense to define

$$\mathcal{K}^{\mathrm{ss}}(\mathfrak{S}) := \{ \psi \in N \mid \mathfrak{S} \text{ is } \psi\text{-semistable} \}.$$

In [27], p. 481ff, we explained how to deal with the case that $\mathcal{K}^{\mathrm{ss}}(\mathfrak{S}) \neq \{0\}$. The main point is that there is a decomposition of $\mathcal{K}^{\mathrm{ss}}(\mathfrak{S})$ into finitely many polyhedral subcones, such that the conditions of ψ -stability and -semistability for the set of sets of states $\mathfrak{S}$ are constant within the relative interior of each of these cones. In each of these relative interiors, we pick an integral element. Let $\chi_1, \dots, \chi_t$ be the resulting characters of G . In addition, we need to fix for χ_j a positive integer N_j , satisfying certain requirements, $j \in [t]$. Now, set $\tilde{\rho}_j := S^{N_j}(\rho) \otimes \chi_j$, $j \in [t]$. It is possible to reduce the problem to Theorem 3.3 for bumps attached to the representations $\tilde{\rho}_1, \dots, \tilde{\rho}_t$. This will enable us to go from the case of generically totally unstable quiver sheaves to arbitrary quiver sheaves.

4. CONCLUSION OF THE PROOF OF THE MAIN THEOREM

We need to introduce some more notation. Let $s \geq 1$ be a natural number and

$$\mathcal{A} = ((n_i^j, i \in [w]), j \in [s])$$

a collection of tuples of integers, such that

- $0 \leq n_i^j \leq n_i^{j+1} \leq n_i$, $j \in [s-1]$, $i \in [w]$,
- $0 < \sum_{i=1}^w n_i^j < \sum_{i=1}^w n_i^{j+1} < \sum_{i=1}^w n_i$, $j \in [s-1]$.

Pick a tuple $\underline{\kappa} = (\kappa_i, i \in [w]) \in (\mathbb{R}_{\geq 0})^{\times w}$ of non-negative real numbers and define further

- $n_i^0 := 0$, $n_i^{s+1} := n_i$, $i \in [w]$,

- $\bar{n}_{\underline{\kappa}}^j := \sum_{i=1}^w \kappa_i \cdot (n_i^j - n_i^{j-1}), j \in [s+1],$
- $\bar{n}_{\underline{\kappa}}^0 := 0.$

Next, set

$$\begin{aligned} \mathbb{R}_{\text{sasc}}^{s+1} &:= \left\{ \underline{\delta} = (\delta_1, \dots, \delta_{s+1}) \in \mathbb{R}^{s+1} \mid \delta_1 < \dots < \delta_{s+1} \right\}, \\ \mathbb{R}_{\text{asc}}^{s+1} &:= \left\{ \underline{\delta} = (\delta_1, \dots, \delta_{s+1}) \in \mathbb{R}^{s+1} \mid \delta_1 \leq \dots \leq \delta_{s+1} \right\}, \end{aligned}$$

and

$$\begin{aligned} \text{OP}_{\mathcal{H}, \underline{\kappa}} &:= \left\{ (\delta_1, \dots, \delta_{s+1}) \in \mathbb{R}_{\text{sasc}}^{s+1} \mid \sum_{j=1}^{s+1} (\bar{n}_{\underline{\kappa}}^j - \bar{n}_{\underline{\kappa}}^{j-1}) \cdot \delta_j = 0 \right\}, \\ \overline{\text{OP}}_{\mathcal{H}, \underline{\kappa}} &:= \left\{ (\delta_1, \dots, \delta_{s+1}) \in \mathbb{R}_{\text{asc}}^{s+1} \mid \sum_{j=1}^{s+1} (\bar{n}_{\underline{\kappa}}^j - \bar{n}_{\underline{\kappa}}^{j-1}) \cdot \delta_j = 0 \right\}. \end{aligned}$$

Finally, we introduce

$$\begin{aligned} \Omega\Pi_{\mathcal{H}} &:= \left\{ (\underline{\nu}, \underline{\delta}) \in \mathbb{R}_+^w \times \mathbb{R}_{\text{sasc}}^{s+1} \mid \underline{\delta} \in \text{OP}_{\mathcal{H}, \underline{\nu}} \right\}, \\ \overline{\Omega\Pi}_{\mathcal{H}} &:= \left\{ (\underline{\nu}, \underline{\delta}) \in (\mathbb{R}_{\geq 0})^w \times \mathbb{R}_{\text{asc}}^{s+1} \mid \underline{\delta} \in \overline{\text{OP}}_{\mathcal{H}, \underline{\nu}} \right\}. \end{aligned}$$

In our arguments, the map^{9,10}

$$\begin{aligned} \text{ev}_{\mathcal{H}}: \overline{\Omega\Pi}_{\mathcal{H}} \times N &\longrightarrow \mathbb{R} \\ (\underline{\nu}, \underline{\delta}, \underline{\psi}) &\longmapsto \sum_{i=1}^w \sum_{j=1}^s (\psi_i \cdot (\delta_{j+1} - \delta_j) \cdot n_i^j) \end{aligned}$$

will play an important rôle. Suppose $B \subset N$ is a compact subset. For $(\underline{\nu}, \underline{\delta}) \in \overline{\Omega\Pi}_{\mathcal{H}}$, set

$$\mathbb{M}(\underline{\nu}, \underline{\delta}) := \max \left\{ \text{ev}(\underline{\nu}, \underline{\delta}, \underline{\psi}) \mid \underline{\psi} \in B \right\}.$$

Then,

$$\begin{aligned} \mathbb{M}: \overline{\Omega\Pi}_{\mathcal{H}} &\longrightarrow \mathbb{R} \\ (\underline{\nu}, \underline{\delta}) &\longmapsto \mathbb{M}(\underline{\nu}, \underline{\delta}) \end{aligned}$$

is a continuous function ([6], p. 116). Denoting the maximum norm on $\mathbb{R}^{s+1}$ and $\mathbb{R}^w$ by $\|\cdot\|$, we set

$$\begin{aligned} \Omega\Pi\Xi_{\mathcal{H}} &:= \left\{ (\underline{\nu}, \underline{\delta}) \in \Omega\Pi_{\mathcal{H}} \mid \|\underline{\nu}\| = 1, \|\underline{\delta}\| = 1 \right\}, \\ \overline{\Omega\Pi\Xi}_{\mathcal{H}} &:= \left\{ (\underline{\nu}, \underline{\delta}) \in \overline{\Omega\Pi}_{\mathcal{H}} \mid \|\underline{\nu}\| = 1, \|\underline{\delta}\| = 1 \right\}. \end{aligned}$$

The function $\mathbb{M}$ admits a maximum on $\overline{\Omega\Pi\Xi}_{\mathcal{H}}$, call it C_2 .

Recall that we are assuming that ρ is a homogeneous representation of degree zero. Set $W := \mathbb{C}^{[\underline{n}]}$. According to [24], Proposition 3.1.2, [25], Proposition 2.5.1.2,

⁹The vector space N has been defined in (8).

¹⁰See [24], Remark 3.4.3, [26], Remark 2.1, ii).

we may find positive integers a, b, c , subject to the condition $a = |\underline{n}| \cdot c$, such that ρ is a direct summand of the natural representation $\rho_{a,b,c}$ of $\mathrm{GL}_{\underline{n}}(\mathbb{C})$ on

$$W_{a,b,c} := (W^{\otimes a})^{\oplus b} \otimes \left(\left(\bigwedge W \right)^{\otimes c} \right)^{\vee}.$$

For this reason, we will assume $\rho = \rho_{a,b,c}$ for the rest of this section.

Next, let us look at a $[w]$ -split vector bundle $(E_i, i \in [w])$ on U and a filtration

$$\underline{E}^{\bullet} : 0 \subsetneq (E_i^1, i \in [w]) \subsetneq \cdots \subsetneq (E_i^s, i \in [w]) \subsetneq (E_i, i \in [w])$$

of $(E_i, i \in [w])$.¹¹ This defines the tuple

$$\mathcal{H}(\underline{E}^{\bullet}) = ((\mathrm{rk}(E_i^j), i \in [w]), j \in [s]),$$

satisfying the conditions stated above. Suppose that

$$E := \bigoplus_{i=1}^w E_i$$

is endowed with a non-zero tensor field

$$\psi : E_{a,b} = (E^{\otimes a})^{\oplus b} \longrightarrow \det(E)^{\otimes c} \otimes \mathcal{O}_X(m)|_U.$$

We write

$$\psi = \psi^1 + \cdots + \psi^b$$

with

$$\psi^{\beta} : E^{\otimes a} \longrightarrow \det(E)^{\otimes c} \otimes (\mathcal{O}_X(m))|_U, \quad \beta \in [b].$$

For $i_0 \in [w]$, we let

$$\iota_{i_0} : E_{i_0} \longrightarrow E$$

be the obvious inclusion map, and, for $\beta \in [b]$ and a tuple $\underline{i} = (i_1, \dots, i_a)$ with $i_{\alpha} \in [w]$, $\alpha \in [a]$, we define

$$\psi_{\underline{i}}^{\beta} := \psi^{\beta} \circ (\iota_{i_1} \otimes \cdots \otimes \iota_{i_a}).$$

Next, suppose $\underline{\gamma} = (\gamma_1, \dots, \gamma_{s+1}) \in \overline{\mathrm{OP}}_{\mathcal{H}(\underline{E}^{\bullet})}$. Then, setting $E^j := \bigoplus_{i \in [w]} E_i^j$, $j \in [s+1]$,

$$\begin{aligned} & \mu(\underline{E}^{\bullet}, \underline{\gamma}, \psi) \\ &:= -\min \left\{ \gamma_{j_1} + \cdots + \gamma_{j_a} \mid \underline{j} = (j_1, \dots, j_a) \in [s+1]^{\times a} : \psi|_{(E^{j_1} \otimes \cdots \otimes E^{j_a})^{\oplus b}} \neq 0 \right\} \\ &= -\min \left\{ \gamma_{j_1} + \cdots + \gamma_{j_a} \mid \underline{j} = (j_1, \dots, j_a) \in [s+1]^{\times a} : \exists \beta \in [b] : \right. \\ (12) \quad & \left. \psi_{|E^{j_1} \otimes \cdots \otimes E^{j_a}}^{\beta} \neq 0 \right\} \\ &= -\min \left\{ \gamma_{j_1} + \cdots + \gamma_{j_a} \mid \underline{j} = (j_1, \dots, j_a) \in [s+1]^{\times a} : \exists \beta \in [b] : \right. \\ & \left. \exists \underline{i} = (i_1, \dots, i_a) \in [w]^{\times a} : \psi_{\underline{i}|E_{i_1}^{j_1} \otimes \cdots \otimes E_{i_a}^{j_a}}^{\beta} \neq 0 \right\}. \end{aligned}$$

¹¹This means that, on a big open subset $U' \subset U$, $E_{i|U'}^j$ will be a subbundle of $E_{i|U'}^{j+1}$, $E_i^{s+1} := E_i$, $i \in [w]$, $j \in [s]$.

Remark 4.1. The function $\overline{\Omega\Pi\Xi}_{\mathcal{H}(\underline{E}^\bullet)} \longrightarrow \mathbb{R}, (\underline{\nu}, \underline{\delta}) \longmapsto \mu(\underline{E}^\bullet, \underline{\delta}, \psi)$, is piecewise linear and, therefore, continuous. It depends only on the set

$$\left\{ (\beta, \underline{i}, \underline{j}) \in [b] \times [w]^{\times a} \times [s+1]^{\times a} \mid \psi_{\underline{i}|E_{i_1}^{j_1} \otimes \dots \otimes E_{i_a}^{j_a}}^\beta \neq 0 \right\}.$$

Having fixed, w , a , b , $\underline{n}$, and $\mathcal{H}$, there are only finitely many possibilities for this set. Also, given $\underline{n}$, the discrete invariant $\mathcal{H}$ can admit only finitely many values. So, the function $\underline{\delta} \longmapsto \mu(\underline{E}^\bullet, \underline{\delta}, \psi)$ belongs to a finite set of continuous functions. Let $\{\Phi_1, \dots, \Phi_t\}$ be this set of functions.

To conclude the preparations, we need to discuss the Harder–Narasimhan filtration. Let $\underline{\kappa} \in \mathbb{R}_+^w$ be a tuple of positive real numbers. In the all the notions related to slope semistability, $\underline{\kappa}$ will stand for $(\underline{\kappa}, 0)$. As explained in [24], Remark 3.3.2, [5], Proposition 2.4.2, a $[w]$ -split sheaf $(E_i, i \in [w])$ is $\underline{\kappa}$ -slope semistable¹² if and only if E_j , $j \in [w]$, is a slope semistable sheaf and

$$\mu(E_j) = \mu_{\underline{\kappa}}(E_i, i \in [w]), \quad j \in [w].$$

Now, any $[w]$ -split sheaf $(E_i, i \in [w])$ possesses a Harder–Narasimhan filtration

$$\underline{E}^\bullet : \quad 0 \subsetneq (E_i^1, i \in [w]) \subsetneq \dots \subsetneq (E_i^s, i \in [w]) \subsetneq (E_i, i \in [w])$$

with respect to the stability parameter $\underline{\kappa}$. For $i \in [w]$, it induces the filtration

$$E_i^\bullet : \quad 0 \subseteq E_i^1 \subseteq \dots \subseteq E_i^s \subseteq E_i.$$

After removing improper inclusions, we get

$$\widetilde{E}_i^\bullet : \quad 0 \subsetneq \widetilde{E}_i^1 \subsetneq \dots \subsetneq \widetilde{E}_i^{s_i} \subsetneq E_i.$$

By our previous remark, this is the Harder–Narasimhan filtration of E_i , $i \in [w]$. Conversely, it is now clear how the Harder–Narasimhan filtration of $(E_i, i \in [w])$ is built from the Harder–Narasimhan filtrations of $E_1, \dots, E_w$. In particular,

$$\begin{aligned} & \left\{ \mu_{\underline{\kappa}}(E_i^1, i \in [w]), \dots, \mu_{\underline{\kappa}}(E_i/E_i^s, i \in [w]) \right\} \\ &= \left\{ \mu(\widetilde{E}_1^1), \dots, \mu(E_1/\widetilde{E}_1^{s_1}), \dots, \mu(\widetilde{E}_w^1), \dots, \mu(E_w/\widetilde{E}_w^{s_w}) \right\}. \end{aligned}$$

We see that the Harder–Narasimhan filtration of $(E_i, i \in [w])$ **does not depend** on $\underline{\kappa}$.

4.1. The case of totally unstable quiver sheaves. Note that, by construction, ψ is generically semistable. This is equivalent to the fact that, for every filtration $\underline{E}^\bullet$ of $(E_i, i \in [w])$ and every tuple $\underline{\gamma} \in \text{OP}_{\mathcal{H}(\underline{E}^\bullet), \underline{\kappa}}$,¹³

$$\mu(\underline{E}^\bullet, \underline{\gamma}, \psi) \geq 0.$$

For $\underline{\chi} \in N$, $(E_i, i \in [w], \psi)$ will be $(\underline{\kappa}, \underline{\chi})$ -slope semistable, if, for every filtration $\underline{E}^\bullet$ of $(E_i, i \in [w])$ and every tuple $\underline{\gamma} \in \text{OP}_{\mathcal{H}(\underline{E}^\bullet), \underline{\kappa}}$, satisfying

$$\mu(\underline{E}^\bullet, \underline{\gamma}, \psi) = 0,$$

the inequality

$$L(\underline{E}^\bullet, \underline{\gamma}) + \text{ev}_{\mathcal{H}}(\underline{\kappa}, \underline{\gamma}, \underline{\chi}) \geq 0,$$

¹²Observe that slope semistability is defined for vector bundles on big open subsets of X .

¹³see Remark 3.2, i)

with

$$L(\underline{E}^\bullet, \underline{\gamma}) := \sum_{j=1}^s \left(\frac{\gamma_{j+1} - \gamma_j}{\text{rk}_{\underline{\kappa}}(E_i, i \in [w])} \right) \cdot \left(\deg_{\underline{\kappa}}(E_i, i \in [w]) \cdot \text{rk}_{\underline{\kappa}}(E_i^j, i \in [w]) - \deg_{\underline{\kappa}}(E_i^j, i \in [w]) \cdot \text{rk}_{\underline{\kappa}}(E_i, i \in [w]) \right),$$

holds true.

Theorem 4.2. *Let the situation be as in Theorem 3.3. Then, there is a constant C_3 , such that, for every $\underline{\kappa} \in \Xi$, every $\underline{\chi} \in N$ with $\underline{\chi} = 0$ or $\underline{\chi}/\|\underline{\chi}\| \in \Omega'$, and every $(\underline{\kappa}, \underline{\chi})$ -slope semistable totally Ω' -unstable $\mathcal{O}_X(m)$ -twisted affine σ -bump $((U, \mathcal{P}), \psi)$ in which $(U, \mathcal{P})$ has type $(\underline{n}, \underline{d})$, the associated $[w]$ -split vector bundle $(E_i, i \in [w])$ satisfies*

$$\mu_{\max}(E_i) \leq C_3, \quad i \in [w].$$

Proof. We introduce one more piece of notation which we will use in the proof. For $\underline{\kappa} = (\kappa_1, \dots, \kappa_w) \in (\mathbb{R}_{\geq 0})^{\times w} \setminus \{0\}$, we set

$$\begin{aligned} n(\underline{\kappa}) &:= \kappa_1 \cdot n_1 + \dots + \kappa_w \cdot n_w, \\ d(\underline{\kappa}) &:= \kappa_1 \cdot d_1 + \dots + \kappa_w \cdot d_w, \\ \mu(\underline{\kappa}) &:= \frac{d(\underline{\kappa})}{n(\underline{\kappa})}. \end{aligned}$$

Let $\underline{\kappa}_0 \in \mathbb{R}_+^w$, $\underline{\chi}_0 \in N$ with $\underline{\chi}_0 = 0$ or $\underline{\chi}_0/\|\underline{\chi}_0\| \in \Omega'$ be stability parameters, $((U, \mathcal{P}), \psi)$ a $(\underline{\kappa}_0, \underline{\chi}_0)$ -slope semistable totally Ω' -unstable $\mathcal{O}_X(m)$ -twisted affine σ -bump in which $(U, \mathcal{P})$ has type $(\underline{n}, \underline{d})$, and $(E_i, i \in [w])$ the associated $[w]$ -split vector bundle on U . Using the notation from above, we assume that $\underline{E}^\bullet$ is the Harder–Narasimhan filtration of $(E_i, i \in [w])$ with respect to the stability parameter $\underline{\kappa}_0$. Then, by [5], claim in the proof of Proposition 2.5.2,

$$\forall \underline{\gamma} \in \overline{\text{OP}}_{\mathcal{H}(\underline{E}^\bullet), \underline{\kappa}_0} \setminus \{0\} : L(\underline{E}^\bullet, \underline{\gamma}) < 0.$$

It follows that, for every $\underline{\gamma} \in \text{OP}_{\mathcal{H}(\underline{E}^\bullet), \underline{\kappa}_0}$ with $\mu(\underline{E}^\bullet, \underline{\gamma}, \psi) = 0$,

$$\text{ev}(\underline{\kappa}_0, \underline{\gamma}, \underline{\chi}_0) > 0.$$

Using the constant C_1 from Theorem 3.3, we introduce the compact set

$$B := \left\{ \underline{\psi} \in N \mid \underline{\psi} = 0 \vee \frac{\underline{\psi}}{\|\underline{\psi}\|} \in \Omega', \|\underline{\psi}\| \leq C_1 \right\}.$$

Then, it follows that

$$\mathbb{M}(\underline{\kappa}_0, \underline{\gamma}) > 0.$$

Let $\tau_0 \in [t]$ be such that $(\underline{\nu}, \underline{\delta}) \mapsto \mu(\underline{E}^\bullet, \underline{\delta}, \psi)$ agrees with the function Φ_{τ_0} . We form the continuous function ([6], p. 116)

$$\begin{aligned} f_{\tau_0} : \overline{\text{OP}}_{\mathcal{H}(\underline{E}^\bullet)} &\longrightarrow \mathbb{R} \\ (\underline{\nu}, \underline{\delta}) &\longmapsto \mathbb{M}(\underline{\nu}, \underline{\delta}) + \Phi_{\tau_0}(\underline{\nu}, \underline{\delta}). \end{aligned}$$

Note that f_{τ_0} has the property

$$\forall \underline{\kappa} \in (\mathbb{R}_{\geq 0})^{\times s} \forall \underline{\gamma} \in \mathbb{R}_{\text{asc}} \setminus \{0\} : f_{\tau_0}(\underline{\kappa}, \underline{\gamma}) = \|\underline{\gamma}\| \cdot f_{\tau_0}\left(\underline{\kappa}, \frac{\underline{\gamma}}{\|\underline{\gamma}\|}\right).$$

By definition of $\underline{E}^\bullet$,

$$\min \left\{ f_{\tau_0}(\underline{\kappa}_0, \underline{\delta}) \mid \underline{\delta} \in \overline{\text{OP}}_{\mathcal{H}(\underline{E}^\bullet), \underline{\kappa}_0} : \|\underline{\delta}\| = 1 \right\} > 0.$$

We introduce the continuous function ([6], p. 115)

$$\begin{aligned} F_{\tau_0} : (\mathbb{R}_{\geq 0})^{\times w} &\longrightarrow \mathbb{R} \\ \underline{\nu} &\longmapsto \min \left\{ f_{\tau_0}(\underline{\nu}, \underline{\delta}) \mid \underline{\delta} \in \overline{\text{OP}}_{\mathcal{H}(\underline{E}^\bullet), \underline{\nu}} : \|\underline{\delta}\| = 1 \right\}. \end{aligned}$$

The image of $\overline{\Xi} := \{ \underline{\nu} \in (\mathbb{R}_{\geq 0})^{\times w} \mid \|\underline{\nu}\| = 1 \}$ under F is a compact set in $\mathbb{R}$. By what we have just observed, it admits a positive maximum M_{τ_0} . This constant depends only on the input data. The preimage of the set $(M_{\tau_0}/2, \infty)$ under $F|_{\overline{\Xi}}$ is a non-empty open subset of $\overline{\Xi}$. So, it contains an element $\underline{\kappa}'_0 \in \mathbb{R}_+^w$.

For $\underline{\kappa} \in (\mathbb{R}_{\geq 0})^{\times w}$, set

$$\mu_{\underline{\kappa}}^j := \mu_{\underline{\kappa}}(E_i^j, i \in [w]), \quad j \in [s-1].$$

Note that

$$\underline{\gamma}_0 := (\mu(\underline{\kappa}'_0) - \mu_{\underline{\kappa}'_0}^1, \dots, \mu(\underline{\kappa}'_0) - \mu_{\underline{\kappa}'_0}^{s+1}) \in \text{OP}_{\mathcal{H}(\underline{E}^\bullet), \underline{\kappa}'_0}.$$

We need an upper bound for $f_{\tau_0}(\underline{\kappa}'_0, \underline{\gamma}_0)$. We use Formula (12). Let $\beta \in [b]$, $\underline{i} = (i_1, \dots, i_a) \in [w]^{\times a}$, $\underline{j} = (j_1, \dots, j_a) \in [s+1]^{\times a}$ be data which compute $\mu(\underline{E}^\bullet, \underline{\gamma}_0, \psi)$. Then, we have a non-zero map

$$E_{i_1}^{j_1} \otimes \dots \otimes E_{i_a}^{j_a} \longrightarrow \det(E)^{\otimes c} \otimes \mathcal{O}_X(m)|_U.$$

It follows that

$$\mu_{\min}(E_{i_1}^{j_1}) + \dots + \mu_{\min}(E_{i_a}^{j_a}) = \mu_{\min}(E_{i_1}^{j_1} \otimes \dots \otimes E_{i_a}^{j_a}) \leq c \cdot |d| + \deg(\mathcal{O}_X(m)).$$

For $l \in [a]$, let $j'_l \in [s+1]$ be the first index with $E_{i_l}^{j'_l} = E_{i_l}^{j_l}$. The comments on the Harder–Narasimhan filtration made just before Section 4.1 show

$$(13) \quad \mu_{\min}(E_{i_l}^{j'_l}) = \mu_{\underline{\kappa}'_0}^{j'_l} \geq \mu_{\underline{\kappa}'_0}^{j_l}, \quad l \in [a].$$

We find

$$f_{\tau_0}(\underline{\kappa}'_0, \underline{\gamma}_0) \leq \max \left\{ C_2, c \cdot |d| + \deg(\mathcal{O}_X(m)) - a \cdot \mu(\underline{\kappa}'_0) \right\}.$$

Note that $\underline{\nu} \longmapsto \mu(\underline{\nu})$ is a continuous function on $(\mathbb{R}_{\geq 0})^{\times w} \setminus \{0\}$. It admits a minimum C_4 on the set $\overline{\Xi}$. So,

$$f_{\tau_0}(\underline{\kappa}'_0, \underline{\gamma}_0) \leq C_5 := \max \left\{ C_2, c \cdot |d| + \deg(\mathcal{O}_X(m)) - a \cdot C_4 \right\}.$$

The constant C_5 depends only on the input data.

Altogether we find

$$C_5 \geq f_{\tau_0}(\underline{\kappa}'_0, \underline{\gamma}_0) = \|\underline{\gamma}_0\| \cdot f_{\tau_0} \left(\underline{\kappa}'_0, \frac{\underline{\gamma}_0}{\|\underline{\gamma}_0\|} \right) \geq \|\underline{\gamma}_0\| \cdot \frac{M_{\tau_0}}{2},$$

i.e.,

$$\|\underline{\gamma}_0\| \leq \frac{2 \cdot C_5}{M_{\tau_0}}.$$

This means

$$\mu_{\underline{\kappa}'_0}^1 - \mu(\underline{\kappa}'_0) \leq \frac{2 \cdot C_5}{M_{\tau_0}}$$

or

$$\mu(\underline{\kappa}'_0) - \mu_{\underline{\nu}'_0}^{s+1} \leq \frac{2 \cdot C_5}{M_{\tau_0}}.$$

Recall that $\underline{\nu} \mapsto \mu(\underline{\nu})$ admits the minimal value C_4 on the set $\overline{\Xi}$. It also attains a maximal value C_6 on this set. The first inequality and the remarks on the Harder–Narasimhan filtration before Section 4.1 give

$$\mu_{\max}(E) = \mu_{\underline{\kappa}'_0}^1 \leq \frac{2 \cdot C_5}{M_{\tau_0}} + C_6$$

and the second one

$$\mu_{\min}(E) = \mu_{\underline{\kappa}'_0}^{s+1} \geq -\frac{2 \cdot C_5}{M_{\tau_0}} + C_4.$$

Both inequalities show that the maximal slope of E is bounded from above. $\square$

4.2. The remaining case. We return to the setting of Remark 3.4, ii). The notation and basic constructions are explained in [27], p. 481ff. Pick a potential generic set of sets of states $\mathfrak{S}$. There are three situations to consider: 1) The stability parameter χ satisfies $\chi/\|\chi\| \in \Omega'$; ¹⁴ 2) $\chi \in \mathcal{K}^{\text{ss}}(\mathfrak{S})$; 3) $\chi \notin \mathcal{K}^{\text{ss}}(\mathfrak{S})$ and $\vartheta(\chi) > 0$.

In the first case, we look at totally Ω' -unstable $\mathcal{O}_X(m)$ -twisted affine σ -bumps which are $(\underline{\kappa}, \chi)$ -semistable with respect to some parameters $\underline{\kappa} \in \Xi$, $\chi \in \mathbb{R}_{>0} \cdot \Omega'$. Here, we may directly apply Theorem 3.3 and 4.2.

Let us speak about the second case. Fix $\underline{\kappa} \in \Xi$, $\chi \in \mathcal{K}^{\text{ss}}(\mathfrak{S})$, and let $(U, \mathcal{P}, \psi)$ be a $(\underline{\kappa}, \chi)$ -semistable $\mathcal{O}_X(m)$ -twisted affine ρ -bump. It satisfies the condition

$$L_{\underline{\kappa}}(\underline{E}^\bullet, \underline{\alpha})(\geq)0,$$

for every $\lambda \in X_{\star, \mathbb{R}}(D)^{\underline{\kappa}\text{-SL}}$ and every reduction $\beta: U' \rightarrow \mathcal{P}_{|U'}/Q_G(\lambda)$ with

$$\mu_\rho(\beta, \psi) = 0 \quad \text{and} \quad \langle \chi, \lambda \rangle = 0.$$

This condition does not change, if we replace χ by $\ell \cdot \chi$, for some positive real number ℓ . So, we may fix any constant C_7 and assume $\|\chi\| < C_7$. We see that the arguments in the proof of Theorem 4.2 may be adapted to cover this case.

For the third case, pick a face F of the cone $\mathcal{K}^{\text{ss}}(\mathfrak{S})$ and an integral character χ_F in the relative interior of that face. Then, there are a new twisting line bundle $\mathcal{L}_F := \mathcal{O}_X(N_F \cdot m) \otimes \mathcal{L}_{\chi_F}^\vee$, a closed subset $\Omega_F \subset \Omega$, ¹⁵ and a certain representation $\tilde{\rho}_F$. The latter is homogeneous of degree zero. We need to study totally Ω_F -unstable $\mathcal{L}_F$ -twisted affine $\tilde{\rho}_F$ -bumps which are $(\underline{\kappa}, \chi)$ -semistable with respect to parameters $\underline{\kappa} \in \Xi$ and $\chi \in \mathbb{R}_{>0} \cdot \Omega_F$. Theorem 3.3 and 4.2 apply to that situation. $\square$

Example 4.3. Let $w \geq 2$ be a natural number and $\tilde{A}_w = ([w], A, t, h)$ the quiver whose arrow set is $A = \{a_1, \dots, a_w\}$, $t(a_i) = i$, and $h(a_i) = i+1$, $i \in [w]$ (Figure 1). Here $\bar{i} = i$, $i \in [w-1]$, $w+1 = 1$.

¹⁴defined in loc. cit.

¹⁵Let us take the opportunity to correct some notation in that article. On Page 481, second line from below, we should set $\gamma := \alpha - \alpha_0$, and, on Page 483, line 11,

$$\Omega_F := \left\{ \beta \in N_F \mid \|\beta\| = 1 \right\}.$$

FIGURE 1. A circular quiver

Koike computed the semi-invariants for this quiver [15]. His result implies that, for a given dimension vector $\underline{n} = (n_i, i \in [w])$, a representation $(f_{a_i}, i \in [w]) \in \text{Rep}(\tilde{A}_w, \underline{n})$ is totally unstable if and only if, for all $i, j \in [w]$, the homomorphism $f: \mathbb{C}^{n_i} \rightarrow \mathbb{C}^{n_j}$ associated with the shortest non-constant path from i to j is not an isomorphism. Note that this example includes the quiver A_w depicted in Figure 2.

$$A_w : 1 \xrightarrow{a_2} 2 \xrightarrow{a_3} \dots \xrightarrow{a_{w-1}} w-1 \xrightarrow{a_w} w.$$

FIGURE 2. A linear quiver

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