Universality Results for Kinetically Constrained Spin Models in Two Dimensions

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ON THE INFECTION TIME OF KINETICALLY CONSTRAINED MODELS: UNIVERSALITY IN TWO DIMENSIONS

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ABSTRACT. Kinetically constrained models (KCM) are reversible interacting particle systems on $\mathbb{Z}^d$ with continuous time Markov dynamics of Glauber type, which represent a natural stochastic (and non-monotone) counterpart of the family of cellular automata known as $\mathcal{U}$-bootstrap percolation. KCM aslo display some of the peculiar features of the so-called "glassy dynamics", and as such they are extensively used in the physics literature to model the liquid-glass transition, a major and longstanding open problem in condensed matter physics.

We consider two-dimensional KCM with update rule $\mathcal{U}$, and focus on proving universality results for the mean infection time of the origin, in the same spirit as those recently established in the setting of $\mathcal{U}$-bootstrap percolation. We first identify what we believe are the correct universality classes, which turn out to be different from those of $\mathcal{U}$-bootstrap percolation. Then we prove universal upper bounds on the mean infection time within each class, which we conjecture to be sharp up to logarithmic corrections. In certain cases, including the well-known Duarte model, our conjecture has recently been confirmed in [31]. It turns out that for certain classes of update rules $\mathcal{U}$, the infection time for the KCM diverges much faster then for the corresponding $\mathcal{U}$-bootstrap process when the equilibrium density of infected sites goes to zero. This is due to the occurrence of energy barriers which determine the dominant behaviour for KCM, but which do not matter at all for the monotone bootstrap dynamics.

1. INTRODUCTION

Kinetically constrained models (KCM) are interacting particle systems on the integer lattice $\mathbb{Z}^d$, which were introduced in the physics literature in the 1980s in order to model the liquid-glass transition (see e.g. [23, 34] for reviews), a major and still largely open problem in condensed matter physics. The main motivation for the ongoing (and extremely active) research on KCM is that, despite their simplicity, they feature some of the main signatures of a super-cooled liquid near the glass transition point.

A generic KCM is a continuous time Markov process of Glauber type defined as follows. A configuration ω is defined by assigning to each site $x \in \mathbb{Z}^d$ an occupation variable $\omega_x \in \{0, 1\}$, corresponding to an empty or occupied site respectively. Each site

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waits an independent, mean one, exponential time and then, iff a certain local constraint is satisfied by the current configuration ω, its occupation variable is updated to be occupied with rate p and to empty with rate $q = 1 - p$. All the constraints that have been considered in the physics literature belong to the following general class [12].

Fix an *update family* $\mathcal{U} = \{X_1, \dots, X_m\}$, that is, a finite collection of finite subsets of $\mathbb{Z}^d \setminus \{\mathbf{0}\}$. Then ω satisfies the constraint at site x if there exists $X \in \mathcal{U}$ such that $\omega_y = 0$ for all $y \in X + x$. Since each update set belongs to $\mathbb{Z}^d \setminus \{\mathbf{0}\}$, the constraints never depend on the state of the to-be-updated site. As a consequence, the product Bernoulli(p) measure μ is a reversible invariant measure, and the process started at μ is stationary. Despite this trivial equilibrium measure, however, KCM display an extremely rich behaviour which is qualitatively different from that of interacting particle systems with non-degenerate birth/death rates (e.g. the stochastic Ising model). Empirically, this behaviour includes the key dynamical features of real glassy materials: anomalously long mixing times [1, 12, 30], aging [22], dynamical heterogeneities, and ergodicity breaking transitions corresponding to percolation of blocked structures [23]. However, proving any of the above rigorously turns out to be a very challenging mathematical task, one of the main reasons being that a number of the classical tools typically used to analyse reversible interacting particle systems (e.g., coupling, censoring, logarithmic Sobolev inequalities) fail for KCM.

In order to give the reader a feel for the variety of behaviours encountered when studying KCM, let us briefly discuss one of the most studied models: the so-called *East model*. This model has update family $\mathcal{U} = \big\{\{-\vec{e}_1, \}, \dots, \{-\vec{e}_d\}\big\}$, so in the one-dimensional setting $d = 1$ a site can update iff it is the neighbour "to the east" of an empty site. For $d = 1$, it was first proved [1] that the relaxation time $T_{\rm rel}(q)$ (see Definition 2.8) is finite for any $q \in (0, 1]$, and it was later shown (see [1, 12, 14]) that it diverges as

$$\exp\left(\big(1 + o(1)\big)\frac{\log(1/q)^2}{2\log 2}\right)$$

as $q \downarrow 0$. A similar scaling was later proved in any dimension $d \geqslant 1$, see [15].

Another well-studied KCM is the Friedrickson-Andersen k-facilitated model (FA-kf), whose update family consists of the k-sets of nearest neighbours of the origin: a site can be updated iff it has at least k empty nearest neighbours. For example, it is known (see [12]) that μ has exponentially decaying time auto-correlations for any $q \in (0, 1]$ and $1 \leqslant k \leqslant d$, while μ is never ergodic for $k > d$. Moreover, the relaxation time $T_{\rm rel}(q)$ diverges as $1/q^{\Theta(1)}$ when $k = 1$ [12, 38], and as a $(k-1)$-times iterated exponential of q^{-d+k-1} when $2 \leqslant k \leqslant d$ [30]. We remark that the above scalings also hold for the so-called *persistence time* (or mean infection time) $\mathbb{E}_\mu(\tau_0)$, defined as the mean over the stationary KCM process of the first time at which the origin becomes empty.

The above model-dependent results (which are, in fact, the only ones that have been proved so far) show a very large diversity of possible scalings of the persistence time, together with a strong sensitivity to the details of the update family $\mathcal{U}$. Therefore, a very natural "universality" question emerges:

Question. *Is it possible to group all possible update families $\mathcal{U}$ into distinct classes, in such a way that all members of the same class induce the same divergence of the persistence time as q approaches from above a certain critical value $q_c(\mathcal{U})$?*

Such a general question has not been addressed so far, even in the physics literature: physicists lack a general criterion to predict the different scalings. This fact is particularly unfortunate since, due to the anomalous and sharp divergence of times, numerical simulations often cannot give clear cut and reliable answers. Indeed, some of the rigorous results recalled above corrected some false conjectures that were based on numerical simulations.

The universality question stated above has, however, being addressed and successfully solved for two-dimensional $\mathcal{U}$-bootstrap percolation (see [4, 6, 9], or [33] for a recent review), a discrete cellular automaton that evolves as follows. Given a configuration of "infected" sites A_t at time t, infected sites remain infected, and a site v becomes infected at time $t+1$ if the translate by v of one of the sets in $\mathcal{U}$ belongs to A_t. One then defines the final infection set $[A]_{\mathcal{U}} := \bigcup_{t=1}^{\infty} A_t$ and the percolation threshold

$$q_c(\mathcal{U}) := \inf\Big\{q > 0 :\ \mathbb{P}_q\big([A]_{\mathcal{U}} = \mathbb{Z}^2\big) = 1\Big\},$$

where we write $\mathbb{P}_q$ to indicate that every site is included in A with probability q, independently from all other sites. The update families $\mathcal{U}$ were classified [9] into three universality classes: *supercritical*, *critical* and *subcritical* (see Definition 2.2), according to a simple geometric criterion, and it was proved [4] that $q_c(\mathcal{U}) = 0$ if $\mathcal{U}$ is supercritical or critical, while $q_c(\mathcal{U}) > 0$ is $\mathcal{U}$ is subcritical. For critical update families $\mathcal{U}$, the scaling (as $q \downarrow 0$) of the typical infection time of the origin starting from $\mathbb{P}_q$ was pinned down very precisely [6] (improving bounds obtained in [9]), and universal properties of the dynamics were determined.

In this paper we make a first important step towards establishing a similar universality picture for two dimensional KCM with supercritical or critical update family $\mathcal{U}$. Using a geometric criterion, our first contribution is to propose a classification of the two-dimensional update families into universality classes, which is inspired by, but at the same time quite different from, that established for bootstrap percolation. More precisely, we classify a supercritical update family $\mathcal{U}$ as being *supercritical unrooted* or *supercritical rooted* and a critical $\mathcal{U}$ as being *α-rooted* or *β-unrooted*, where $\alpha \leqslant \beta$ are two integers called the difficulty and the bilateral difficulty of $\mathcal{U}$ respectively (see Definitions 2.10 and 2.11). We then prove the following two main universality results (see Theorems 1 and 2 in Section 2.3) on the mean infection time $\mathbb{E}_\mu(\tau_0)$. Let us denote by $T_{\mathcal{U}}$ the median of the infection time of the origin for the $\mathcal{U}$-bootstrap process.

Supercritical KCM. *Let $\mathcal{U}$ be a supercritical two-dimensional update family. Then, as $q \to 0$,*

(a) *if $\mathcal{U}$ is unrooted*
$$\mathbb{E}_\mu(\tau_0) = e^{O(\log T_{\mathcal{U}})} = 1/q^{O(1)},$$

(b) *if $\mathcal{U}$ is rooted,*
$$\mathbb{E}_\mu(\tau_0) = e^{O((\log T_{\mathcal{U}})^2)} = e^{O((\log q)^2)}.$$

Critical KCM. *Let $\mathcal{U}$ be a critical two-dimensional update family with finite difficulty α and bilateral difficulty $\beta \leqslant +\infty$. Then, as $q \to 0$,*

(a) if $\mathcal{U}$ is α-rooted
$$\mathbb{E}_\mu(\tau_0) = e^{\tilde{O}((\log T_{\mathcal{U}})^2)} = e^{\widetilde{O}(1/q^{2\alpha})};$$

(b) if $\mathcal{U}$ is β-unrooted
$$\mathbb{E}_\mu(\tau_0) = e^{\tilde{O}((\log T_{\mathcal{U}})^{\beta/\alpha})} = e^{\widetilde{O}(1/q^{\beta})},$$

Notice the sharp difference between the behaviour of rooted and unrooted models. Even though our result only establishes universal *upper bounds* on $\mathbb{E}_\mu(\tau_0)$, we conjecture that our bounds provide the correct scaling up to logarithmic corrections. This has been recently proved for supercritical models in [31]. For critical update families, there is a matching lower bound (see Remark 2.13) for all β-unrooted models with $\alpha = \beta$ (for example, the FA-2f model). In particular, these recent advances combined with the above theorems prove two conjectures that we put forward in [33]. Among the α-rooted models, those which have been considered most extensively in the literature are the Duarte and modified Duarte model [7, 17], for which $\alpha = 1$ and $\beta = \infty$. In [31], using very different tools and ideas from those in this paper, a lower bound on $\mathbb{E}_\mu(\tau_0)$ was recently obtained for both models that matches our upper bound, including the logarithmic corrections, yielding $\mathbb{E}_\mu(\tau_0) \asymp \exp\big(O(\log(q)^4/q^2\big)$.

Providing an insight into the heuristics and/or the key steps of the proofs at this stage, before providing a clear definition of the geometrical quantities involved, would inevitably be rather vague. We therefore defer these explanations to Section 2.4.

2. Universality classes for KCM and main results

In this section we will begin by recalling the main universality results for bootstrap cellular automata. We will then define the KCM process associated to a bootstrap update family, introduce its universality classes, and state our main results about its scaling near criticality. To finish, we will provide an outline of the heuristics behind our main theorems, and a sketch of their proofs.

2.1. **The bootstrap monotone cellular automata and its universality properties.** Let us begin by defining a large class of two dimensional monotone cellular automata, which were recently introduced by Bollobás, Smith and Uzzell [9].

Definition 2.1. Let $\mathcal{U} = \{X_1, \ldots, X_m\}$ be an arbitrary finite collection of finite subsets of $\mathbb{Z}^2 \setminus \{\mathbf{0}\}$. The *$\mathcal{U}$-bootstrap process* on $\mathbb{Z}^2$ is defined as follows: given a set $A \subset \mathbb{Z}^2$ of initially *infected* sites, set $A_0 = A$, and define for each $t \geqslant 0$,

$$A_{t+1} = A_t \cup \big\{ v \in \mathbb{Z}^2 \,:\, v + X \subset A_t \text{ for some } X \in \mathcal{U} \big\}.$$

We write $[A]_{\mathcal{U}} = \bigcup_{t \geqslant 0} A_t$ for the *closure* of A under the $\mathcal{U}$-bootstrap process.

Thus, a vertex v becomes infected at time $t+1$ if the translate by v of one of the sets in $\mathcal{U}$ (which we refer to as the *update family*) is already entirely infected at time t, and infected vertices remain infected forever. For example, if we take $\mathcal{U}$ to be the family of 2-subsets of the set of nearest neighbours of the origin, we obtain the classical 2-neighbour bootstrap process, which was first introduced in 1979 by Chalupa, Leath and Reich [13]. One of the key insights of Bollobás, Smith and Uzzel [9] was that, at least in two dimensions, the typical global behaviour of the $\mathcal{U}$-bootstrap process acting on random initial sets should be determined by the action of the process on discrete half-planes.

For each unit vector $u \in S^1$, let $\mathbb{H}_u := \{x \in \mathbb{Z}^2 \,:\, \langle x, u \rangle < 0\}$ denote the discrete half-plane whose boundary is perpendicular to u.

Definition 2.2. The set of *stable directions* is

$$\mathcal{S} = \mathcal{S}(\mathcal{U}) = \big\{ u \in S^1 \,:\, [\mathbb{H}_u]_{\mathcal{U}} = \mathbb{H}_u \big\}.$$

The update family $\mathcal{U}$ is:

- *supercritical* if there exists an open semicircle in S^1 that is disjoint from $\mathcal{S}$,
- *critical* if there exists a semicircle in S^1 that has finite intersection with $\mathcal{S}$, and if every open semicircle in S^1 has non-empty intersection with $\mathcal{S}$,
- *subcritical* if every semicircle in S^1 has infinite intersection with $\mathcal{S}$.

To justify this trichotomy, we need a couple more simple definitions. Let us say that a set $A \subset \mathbb{Z}^2$ is *q-random* if each of the vertices of $\mathbb{Z}^2$ is included in A independently with probability q, and define the *critical probability* of the $\mathcal{U}$-bootstrap process on $\mathbb{Z}^2$ to be

$$q_c\big(\mathbb{Z}^2, \mathcal{U}\big) := \inf \Big\{ q \,:\, \mathbb{P}_q\big([A]_{\mathcal{U}} = \mathbb{Z}^2\big) = 1 \Big\},$$

where $\mathbb{P}_q$ denotes the product probability measure on $\mathbb{Z}^2$ with density q of infected sites. The first universality result proved in [4, 9] is as follows:

Theorem 2.3. *For any supercritical and critical update family $q_c\big(\mathbb{Z}^2, \mathcal{U}\big) = 0$, while for a subcritical update family $q_c\big(\mathbb{Z}^2, \mathcal{U}\big) > 0$.*

Given the above, for a supercritical or critical update family, the main question consists in finding the scaling as $q \to 0$ of the typical time it takes to infect the origin.

Definition 2.4. Given $A \subset \mathbb{Z}^2$, let $\tau_{BP}(A) = \min\{t \geqslant 0 : \ 0 \in A_t\}$ be the infection time of the origin. Then we set

$$T_{q,\mathcal{U}} = \inf\left\{t \geqslant 0 : \ \mathbb{P}_q\big(\tau_{BP}(A) \geqslant t\big) \leqslant \frac{1}{2}\right\}.$$

Below, if no confusion arises, we will drop the suffix q from the notation $T_{q,\mathcal{U}}$.

In order to state the main result we need some more definition. Let $\mathcal{Q}_1 \subset S^1$ denote the set of rational directions on the circle, and for each $u \in \mathcal{Q}_1$, let ℓ_u^+ be the (infinite) subset of the line $\ell_u := \{x \in \mathbb{Z}^2 : \langle x, u\rangle = 0\}$ consisting of the origin and the sites to the right of the origin as one looks in the direction of u. Similarly, let $\ell_u^- := (\ell_u \setminus \ell_u^+) \cup \{\mathbf{0}\}$ consist of the origin and the sites to the left of the origin. Given a two-dimensional bootstrap percolation update family $\mathcal{U}$, let $\alpha_{\mathcal{U}}^+(u)$ be the minimum (possibly infinite) cardinality of a set $Z \subset \mathbb{Z}^2$ such that $[\mathbb{H}_u \cup Z]_{\mathcal{U}}$ contains infinitely many sites of ℓ_u^+, and define $\alpha_{\mathcal{U}}^-(u)$ similarly (using ℓ_u^- in place of ℓ_u^+).

Definition 2.5. Given $u \in \mathcal{Q}_1$, the *difficulty* of u (with respect to $\mathcal{U}$) is[1]

$$\alpha(u) := \begin{cases} \min\left\{\alpha_{\mathcal{U}}^+(u), \alpha_{\mathcal{U}}^-(u)\right\} & \text{if } \alpha_{\mathcal{U}}^+(u) < \infty \text{ and } \alpha_{\mathcal{U}}^-(u) < \infty \\ \infty & \text{otherwise.} \end{cases}$$

Let $\mathcal{C}$ denote the collection of open semicircles of S^1. The *difficulty* of $\mathcal{U}$ is given by

$$\alpha \;:=\; \min_{C \in \mathcal{C}} \max_{u \in C} \alpha(u), \tag{2.1}$$

and the *bilateral difficulty* by

$$\beta \;:=\; \min_{C \in \mathcal{C}} \max_{u \in C} \max(\alpha(u), \alpha(-u)). \tag{2.2}$$

A critical update family $\mathcal{U}$ is *balanced* if there exists a closed semicircle C such that $\alpha(u) \leqslant \alpha$ for all $u \in C$. It is said to be *unbalanced* otherwise.

Roughly speaking, the above definition says that a direction u has finite difficulty if there exists a finite set of sites that, together with the half-plane $\mathbb{H}_u$, infect the entire line ℓ_u. Moreover, the difficulty of u is at least k if it is necessary (in order to infect ℓ_u) to find at least k infected sites that are 'close' to one another. If the open semicircle C with u as midpoint contains no direction of difficulty greater than k, then it is possible for a 'critical droplet' of infected sites to grow in the direction of u without ever finding more than k infected sites close together. As a consequence, if the bilateral difficulty is not greater than k, than there exists a direction u (the midpoint of the optimal semicircle

[1] In order to slightly simplify the notation, and since the update family $\mathcal{U}$ will always be clear from the context, we will not emphasise the dependence of the difficulty on $\mathcal{U}$.

in (2.2)) such that a suitable critical droplet is able to grow in *both directions* u *and* $-u$, without ever finding more than k infected sites close together.

We are now in a position to state the main results on the scaling of $T_{\mathcal{U}}$ for supercritical and critical update families.

Theorem 2.6. *Let $\mathcal{U}$ be a two-dimensional update family.*

(a) If $\mathcal{U}$ is supercritical then [9] $T_{\mathcal{U}} = 1/q^{\Theta(1)}$.

(b) If $\mathcal{U}$ is critical with difficulty α, then [6]
- $T_{\mathcal{U}} = \exp(\Theta(1)/q^{\alpha})$ *if $\mathcal{U}$ is balanced,*
- $T_{\mathcal{U}} = \exp(\Theta(1)\log(q)^2/q^{\alpha})$ *if $\mathcal{U}$ is unbalanced.*

Remark 2.7. Notice that in the above result the bilateral difficulty β plays no role. In some sense this is not surprising, as in bootstrap percolation whether a droplet of empty sites is able to move back and forth or only in one direction is not important. We will see that for the KCM version of the problem this feature will be instead rather crucial.

2.2. **General finite range KCM.** In this section we define a special class of two dimensional interacting particle systems known as *kinetically constrained models*. As it will appear clearly in what follows KCM are intimately connected with bootstrap cellular automata.

We will work on the probability space (Ω, μ), where $\Omega = \{0,1\}^{\mathbb{Z}^2}$ and μ is the product Bernoulli(p) measure, and we will be interested in the asymptotic regime $q \downarrow 0$, where $q = 1-p$. Given $\omega \in \Omega$ and $x \in \mathbb{Z}^2$ we will say that x is empty or infected if $\omega_x = 0$. We will say that $f : \Omega \mapsto \mathbb{R}$ is a local function if it depends on finitely many variables ω_x's.

Given a two dimensional bootstrap model with update family $\mathcal{U} = \{X_1, \dots, X_m\}$, the corresponding KCM is the Markov process on Ω associated to the Markov generator

$$(\mathcal{L}f)(\omega) = \sum_{x \in \mathbb{Z}^2} c_x(\omega)\big(\mu_x(f) - f\big)(\omega), \tag{2.3}$$

where $f : \Omega \mapsto \mathbb{R}$ is a local function, $\mu_x(f)$ denotes its average w.r.t. to the variable ω_x and c_x is the indicator function of the event that there exists an update rule $X \in \mathcal{U}$ such that $\omega_y = 0\ \forall y \in X + x$.

Informally, such a process can be described as follows. Each vertex $x \in \mathbb{Z}^2$, with rate one and independently across $\mathbb{Z}^2$, is resampled from $(\{0,1\}, \mathrm{B}(p))$ iff the update rule of the bootstrap process at x was fulfilled by the current configuration of the empty, *i.e.,* infected, sites. In what follows, we will sometimes call an update as above a *legal update* or *legal spin flip*. It follows [12] that $\mathcal{L}$ becomes the generator of a reversible Markov process on Ω, with reversible measure μ.

We now define the two main quantities characterising the dynamics of the KCM process. The first one is the relaxation time.

Definition 2.8. We say that $C > 0$ is a Poincaré constant for the KCM if for all local functions f

$$\operatorname{Var}(f) \leqslant C\mathcal{D}(f), \tag{2.4}$$

where $\mathcal{D}(f) = \sum_x \mu\big(c_x \operatorname{Var}_x(f)\big)$ is the KCM Dirichlet form of f associated to $\mathcal{L}$. If there exists a finite Poincaré constant we then define

$$T_{\mathrm{rel}}(q,\mathcal{U}) := \inf\{C :\ C \text{ is a Poincaré constant}\}.$$

Otherwise we say that the relaxation time is infinite.

A finite relaxation time implies that the reversible measure μ is mixing for the semigroup P_t with exponentially decaying time auto-correlations [29],

$$\operatorname{Var}\left(e^{t\mathcal{L}}f\right) \leqslant e^{-t/T_{\mathrm{rel}}} \operatorname{Var}(f), \qquad \forall\ f \in L^2(\mu).$$

One of the main results of [12] says that $T_{\mathrm{rel}}(q,\mathcal{U}) < +\infty$ for all super-critical and critical models.

The second (random) quantity is the hitting time

$$\tau_0 = \inf\{t \geqslant 0 :\ \omega_0(t) = 0\}.$$

In the physics literature the hitting time τ_0 is usually referred to as the *persistence time*, while in the bootstrap percolation framework it would be more conveniently dubbed *infection time*. For our purposes, the most important connection between the mean infection time $\mathbb{E}_\mu(\tau_0)$ for the stationary KCM process (*i.e.,* with μ as initial distribution) and $T_{\mathrm{rel}}(q,\mathcal{U})$ is as follows (see [11, Theorem 4.7]):

$$\mathbb{E}_\mu(\tau_0) \leqslant T_{\mathrm{rel}}(q,\mathcal{U})/q, \qquad \forall q \in (0,1). \tag{2.5}$$

The proof is quite simple. By definition τ_0 is the hitting time of $A = \{\omega :\ \omega_0 = 0\}$ and it is a standard result (see e.g. [3, Theorem 2]) that $\mathbb{P}_\mu(\tau_0 > t) \leqslant e^{-t\lambda_A}$, with

$$\lambda_A = \inf\{\mathcal{D}(f) :\ \mu(f^2) = 1,\ f\restriction_A = 0\}.$$

If we observe that $\operatorname{Var}(f) \geqslant \mu(A) = q$ for any function f satisfying $\mu(f^2) = 1$ and $f\restriction_A = 0$ we conclude that $\lambda_A \geqslant q/T_{\mathrm{rel}}(q,\mathcal{U})$ and (2.5) follows.

Remark 2.9. In general, if the initial distribution ν of the KCM process is different from μ, then it is not known that $\mathbb{E}_\nu(\tau_0)$ is finite, even under the restrictive assumption that ν is a product Bernoulli(p') measure with $p' \neq p$ and $\mathcal{U}$ is a super-critical or critical update family.

A matching lower bound on $\mathbb{E}_\mu(\tau_0)$ in terms of $T_{\mathrm{rel}}(q;\mathcal{U})$ is not known. Instead, in [30, Lemma 4.3] it is proved that

$$\mathbb{E}_\mu(\tau_0) = \Omega(T_{\mathcal{U}}). \tag{2.6}$$

2.3. **Universality results.** In this section we define precisely the universality classes for KCM with a supercritical or critical update family $\mathcal{U}$ and, for convenience, we restate our main results and conjectures on the scaling of $\mathbb{E}_\mu(\tau_0)$ as $q \to 0$. We begin with the supercritical case.

Definition 2.10. A supercritical two-dimensional update family $\mathcal{U}$ is said to be *supercritical rooted* if there exist two non-opposite stable directions in S^1. Otherwise it is called *supercritical unrooted*.

Theorem 1 (Supercritical KCM). *Let $\mathcal{U}$ be a supercritical two-dimensional update family. Then, as $q \to 0$,*

(a) *if $\mathcal{U}$ is unrooted*
$$\mathbb{E}_\mu(\tau_0) = e^{O(\log T_\mathcal{U})} = 1/q^{O(1)},$$
(b) *if $\mathcal{U}$ is rooted,*
$$\mathbb{E}_\mu(\tau_0) = e^{O((\log T_\mathcal{U})^2)} = e^{O((\log q)^2)}.$$

We now turn to the critical case. Here the distinction between critical unrooted and critical rooted is more subtle and both the difficulty α and the bilateral difficulty β (see Definition 2.5) play a role.

Definition 2.11. A critical update family $\mathcal{U}$ with finite difficulty α and bilateral difficulty β is said to be *α-rooted* if $\beta \geqslant 2\alpha$. Otherwise it is said to be *β-unrooted*[2].

Theorem 2 (Critical KCM). *Let $\mathcal{U}$ be a critical two-dimensional update family with finite difficulty α and bilateral difficulty $\beta \leqslant +\infty$. Then, as $q \to 0$,*

(a) if $\mathcal{U}$ is α-rooted
$$\mathbb{E}_\mu(\tau_0) = e^{\tilde{O}((\log T_\mathcal{U})^2)} = e^{\widetilde{O}(1/q^{2\alpha})};$$
(b) if $\mathcal{U}$ is β-unrooted
$$\mathbb{E}_\mu(\tau_0) = e^{\tilde{O}((\log T_\mathcal{U})^{\beta/\alpha})} = e^{\widetilde{O}(1/q^{\beta})},$$

Remark 2.12. As it will be clear from the proof the above bounds hold as is also for the relaxation time $T_{\text{rel}}(q;\mathcal{U})$.

Remark 2.13. If one follows carefully the derivation of the bounds of Theorem 2 (see Section 6), one finds that the dominant term in case (a) is of the form $e^{O(\log(\rho)^2)}$, with $\rho = e^{-O(\log(q)^2/q^\alpha)}$, while for case (b) is $1/\rho^{O(1)}$ with $\rho = e^{-O(\log(q)^2/q^\beta)}$. That translates into the explicit upper bounds $e^{O(\log(q)^4/q^{2\alpha})}$ and $e^{\widetilde{O}(1/q^\beta)}$ respectively. Next we briefly discuss lower bounds. The simple one, $\mathbb{E}_\mu(\tau_0) = \Omega(T_\mathcal{U})$ (2.6), matches the scalings given in Theorems 1, 2 (apart from logarithmic corrections) only for super-critical unrooted models and critical β-unrooted models with $\beta = \alpha$. For super-critical rooted models and critical models with $\beta > \alpha$, our upper bounds and the known scaling

[2] We warn the attentive reader that when $\alpha < \beta < 2\alpha$ the model is called β-unrooted while in [33] it was called α-rooted.

of $T_{\mathcal{U}}$ are very far. We conjecture that the above theorems capture the right scaling of $\mathbb{E}_\mu(\tau_0)$ apart from logarithmic corrections. The conjecture has been recently proved in [31] for all super-critical rooted models and for the Duarte model (see [7, 17, 32]), a critical model with $\alpha = 1$ and $\beta = \infty$. For the latter, the proven lower bound on $\mathbb{E}_\mu(\tau_0)$ matches the upper bound of Theorem 2, including the $\log(q)^4$ term.

Summarising, in the critical case we have

$$\mathbb{E}_\mu(\tau_0) = \Omega(T_{\mathcal{U}}), \quad \mathbb{E}_\mu(\tau_0) = e^{\tilde{O}((\log T_{\mathcal{U}})^{\min(2,\beta/\alpha)})}.$$

2.4. **Heuristics and roadmap.** We conclude this section with a high-level description of our intuition behind the proofs of Theorems 1 and 2, together with a roadmap of the actual proof.

The first key point to be stressed out is that we actually never follow the dynamics of the stationary KCM process itself, but we rather appeal to (2.5) and concentrate our efforts in proving the existence of a Poincaré constant with the correct scaling as $q \to 0$. We emphasise that this approach only works for the stationary KCM process. The second point is that, given that the Dirichlet form of the KCM is a sum of local variances ($\Leftrightarrow$ spin flips) computed with suitable infection nearby ($\Leftrightarrow$ the constraints c_x), all our reasonings will be guided by the need of having some infection next to where we want to compute the variance. Therefore, much of our intuition and all the technical tools, have been developed trying to figure out how to *effectively* move infection where we need it.

A configuration sampled from μ will always have "mesoscopic" droplets (*i.e.,* patches) of infection, typically very far from the origin. Bootstrap percolation results quantify very precisely the critical size of those droplets which allows infection to grow from them and invade the system. However, and this is a fundamental difference between bootstrap percolation and KCM, it is extremely unlikely for the stationary KCM to create around a given vertex and at any given time a very large cluster of infection. Thus, it is essential to envisage an *infection/healing* mechanism that is able to *move* infection over long distances without creating a too large excess of it[3].

At the root of our approach lies the notion of a *critical droplet*. A critical droplet is a certain finite set D whose geometry depends on the update family $\mathcal{U}$, and whose characteristic size may depend on q. For super-critical models we can take any sufficiently large (*not* depending on q) rectangle oriented along the mid-point u of a semicircle C free of stable directions. For critical models the droplet D is a more complicated object called a *quasi-stable half ring* (see Definition 4.9 and Figure 4) oriented along the midpoint u of an open semicircle with largest difficulty either α or β. The long sides of D will have length either $\Theta(q^{-\alpha}\log(1/q))$ or $\Theta(q^{-\beta}\log(1/q))$ for the α-rooted or β-unrooted case respectively, while the short sides will always have length $\Theta(1)$. The key feature of a critical droplet for super-critical models (see Section 4.2) is that,

[3] In physical terms an excess of infection is equivalent to an "energy barrier".

if it is empty, it is able to infect a suitable translate of itself in the u-direction. For unrooted supercritical models the semicircle C can be chosen in such a way that both C *and* $-C$ are free of stable directions. As a consequence, the empty critical droplet will be able to infect a suitable translate of itself in *both* directions $\pm u$. For critical models the situation changes drastically. An empty critical droplet will not be able to infect freely another critical droplet next to it in the u-direction because of the stable directions which are present in every open semicircle. However, it will be able to do so, only in the u-direction if the model is α-rooted and in the $\pm u$-directions if β-unrooted, provided it gets some help from a finite number (related to α or β) of extra empty sites nearby. If the sizes of the critical droplet are chosen as above, then these extra helping empty sites will be present with high probability (see Sections 6.1).

Having clarified what a critical droplet is, and under which circumstances it is able to infect nearby sites, we next explain what we mean by "moving a critical droplet". For simplicity we explain the heuristics only for the super-critical case. Imagine to have a sequence $D_0, D_1, \dots, D_n$ of contiguous, non-overlapping and identical critical droplets such that $D_{i+1} = D_i + d_i u$ for suitable $d_i > 0$. Suppose now that the model is unrooted and that D_0 is completely infected and call ω_i the spin configuration in D_i. Using the infection in D_0 it possible to first infect D_1, then D_2 and then, using reversibility, restore (*i.e.,* heal) the original configuration ω_1 in D_1. Using the infection in D_2 we can next infect D_3 and then, using the infection in D_3, restore ω_2 in D_2 (see the schematic diagram below where $\emptyset$ stand for infected droplets)

$$\begin{aligned}\emptyset\ \omega_1\ \omega_2\ \omega_3 \dots &\mapsto \emptyset\ \emptyset\ \omega_2\ \omega_3 \cdots \mapsto \emptyset\ \emptyset\ \emptyset\ \omega_3 \cdots \mapsto \emptyset\ \omega_1\ \emptyset\ \omega_3 \dots \\ &\mapsto \emptyset\ \omega_1\ \emptyset\ \emptyset \cdots \mapsto \emptyset\ \omega_1\ \omega_2\ \emptyset \dots\end{aligned}$$

If we continue that way, we end up moving the original infection in D_0 to the last droplet D_n without having ever created more than two extra infected critical droplets simultaneously. The above is reminiscent of how infection moves in the one dimensional 1-neighbour KCM. In the rooted super-critical case restoring ω_2 in D_2 cannot be done using the infection in D_3. In the unrooted case that was possible because infection can propagate in the u and $-u$ directions. Now we have to follow a more complicate pattern:

$$\begin{aligned}\emptyset\ \omega_1\ \omega_2\ \omega_3 \dots &\mapsto \emptyset\ \emptyset\ \omega_2\ \omega_3 \cdots \mapsto \emptyset\ \emptyset\ \emptyset\ \omega_3 \cdots \mapsto \emptyset\ \emptyset\ \emptyset\ \emptyset \dots \\ &\mapsto \emptyset\ \ \emptyset\ \omega_2\ \emptyset \cdots \mapsto \emptyset\ \omega_1\ \omega_2\ \emptyset \dots,\end{aligned}$$

in which healing is always induced by infection present in the next droplet in the $-u$ direction. This latter case is reminiscent of the one dimensional East model. In this case, a combinatorial result [16] implies that in order to move the infection to D_n one has to necessarily create $\asymp \log n$ *simultaneous* extra infected critical droplets. This logarithmic energy barrier is at the root of the different scaling of $\mathbb{E}_\mu(\tau_0)$ in rooted and unrooted supercritical models (see Theorem 1).

We can now detail a bit more our approach. We partition $\mathbb{Z}^2$ with suitable rectangular blocks $\{V_i\}_{i\in\mathbb{Z}^2}$ with shortest side orthogonal to the direction u (see Section 4.1). For super-critical models these blocks have sides of constant length, while for critical models they will have length $\approx q^{-\kappa},\ \kappa \gg \alpha$, and height either $q^{-\alpha} \times \mathrm{polylog}(q)$ or $q^{-\beta} \times \mathrm{polylog}(q)$, depending on the nature of the model. Then, given a configuration $\omega \in \Omega$, we declare a block to be *good* or *super-good* according to the following rules. For super-critical models *any* block is good, while for critical models good blocks are those which contain "enough" helping empty sites to allow an adjacent empty critical droplet to advance in the u (or $\pm u$) direction (see Definition 6.4). In both cases, a block is said to be super-good if it is good and contains an empty critical droplet fitting in the block. Good blocks turn out to be very likely w.r.t. μ (a triviality in the super-critical case), and they form a rather dense infinite cluster by standard percolation arguments. Super-good blocks are instead quite rare, with density $\rho = q^{O(1)}$ in the super-critical case, and $\rho = \exp(-\widetilde{O}(1/q^{\alpha}))$ in the critical α-rooted case or $\rho = \exp(-\widetilde{O}(1/q^{\beta}))$ in the critical β-unrooted one.

We will then prove the existence of a good Poincaré constant in three steps, each step being associated to a natural kinetically constrained *block dynamics*[4] on a certain length scale. Here the configuration in the blocks is resampled with rate one and independently among the blocks if a certain constraint is satisfied.

Our first block dynamics forces one of the blocks neighbouring V_i to be at the beginning of an oriented "thick" path γ of good blocks, with length $\approx 1/\rho$, whose last block is super-good. Using the fact that the imposed constraint is very likely, it is possible to prove (see Section 2 in [30]) that the relaxation time of this process is $O(1)$, so that the corresponding Poincaré inequality (see (3.8)) is

$$\mathrm{Var}(f) \leqslant C_1 \sum_i \mu\big(\mathbb{1}_{\Gamma_i} \,\mathrm{Var}_i(f)\big), \tag{2.7}$$

where C_1 is a constant not depending on q, and $\mathbb{1}_{\Gamma_i}$ is the indicator of the event that a good path exists for V_i.

The next idea is to convert the *long-range* constrained Poincaré inequality (2.7) into a *short-range* one of the form

$$\mathrm{Var}(f) \leqslant C_2(q) \sum_i \mu\big(\mathbb{1}_{SG_i} \,\mathrm{Var}_i(f)\big), \tag{2.8}$$

in which $\mathbb{1}_{SG_i}$ is the indicator of the event that a suitable collection of blocks *near* V_i are good and one of them is super-good. Which collections of blocks are "suitable", and which one should be super-good, depends on whether the model is rooted or unrooted; we refer the reader to Theorem 3.1 for the details. The main content of Theorem 3.1, which we present in a slightly more general setting for later convenience, is that $C_2(q)$ can be taken equal to the best Poincaré constant (*i.e.,* the relaxation time)

[4] See, e.g., Chapter 15.5 of [28] for a introduction to the technique of block dynamics in reversible Markov chains

of a one dimensional generalised 1-neighbour or East process at the effective density ρ. Section 3 is entirely dedicated to the task of formalising and proving the above claim.

The final step of the proof is to convert the Poincaré inequality (2.8) into the true Poincaré inequality for our KCM

$$\mathrm{Var}(f) \leqslant C_3(q) \sum_x \mu(c_x \,\mathrm{Var}_x(f)),$$

with a Poincaré constant $C_3(q)$ which scales with q as required by Theorems 1 and 2. In turn that requires proving that a full resampling of a block in the presence of nearby super-good and good blocks can be simulated (or reproduced) by several single site legal updates in V_i of the *original* KCM, with a global cost in the Poincaré constant compatible with Theorems 1 and 2. It is here that the full technology and results for the bootstrap percolation update family $\mathcal{U}$ come into play. While for supercritical models the above task is relatively simple (see Section 5), for critical models the problem is more complicate (a full sketch of the proof can be found in Section 6.1 and in particular in the first part of the proof of Proposition 6.6 and in Remark 6.7).

2.5. **Notation.** In our results the Bernoulli product measure $\mu = \otimes_{x\in\mathbb{Z}^2}\mathrm{B}(p)$ will play a crucial role. The parameter $q = 1 - p$ will represents the scaling parameter of our results and it always be assumed to be sufficiently small relative to all other quantities. We will write $[n] = \{1, \dots, n\}$.

If f and g are positive real-valued functions of q, then we will write $f = O(g)$ if there exists a constant $C > 0$ such that $f(q) \leqslant Cg(q)$ for every sufficiently small $q > 0$. Our asymptotic notation is mostly standard, although we just remark that if f and g are positive real-valued functions of q and diverging to $+\infty$ as $q \to 0$, then we write $f(q) = \Omega(g(q))$ if $g(q) = O(f(q))$ and $f(q) = \Theta(g(q))$ if both $f(q) = O(g(q))$ and $g(q) = O(f(q))$. It will be also quite convenient to add the less standard notation $f(q) = \widetilde{O}(g(q))$ for $f(q) = O(g(q)\log(g(q))^c)$ for some constant $c > 0$. Finally, if c_1 and c_2 are constants, then $c_1 \gg c_2 \gg 1$ means that c_2 is sufficiently large and c_1 is sufficiently large depending on c_2. Similarly, $1 \gg c_1 \gg c_2 > 0$ means that c_1 is sufficiently small and c_2 is sufficiently small depending on c_1. All constants, including those implied by the notation $O(\cdot)$, $\Omega(\cdot)$ and $\Theta(\cdot)$, are quantities that may depend on the update family $\mathcal{U}$ (and other quantities where explicitly stated) but not on the parameter q.

3. Constrained Poincaré inequalities

The aim of this section is to prove a constrained Poincaré inequality for a product measure on $S^{\mathbb{Z}^2}$, where S is a finite set. This general inequality will play an instrumental role in the proof of our main theorems, giving us precise control of the infection time for both supercritical and critical KCM.

In order to state our general constrained Poincaré inequality, we will need some notation. Let $(S, \hat{\mu})$ be a finite positive probability space, and set $\Omega = \big(S^{\mathbb{Z}^2}, \mu\big)$, where

$\mu = \otimes_{i\in\mathbb{Z}^2}\hat{\mu}$. A generic element Ω will be denoted by $\omega = \{\omega_i\}_{i\in\mathbb{Z}^2}$. For any local function f we will write $\mathrm{Var}(f)$ for its variance w.r.t. μ and $\mathrm{Var}_i(f)$ for the variance w.r.t. to the variable $\omega_i \in S$ conditioned on all the other variables $\{\omega_j\}_{j\neq i}$. For any $i \in \mathbb{Z}^2$ we set

$$\mathbb{L}^+(i) = i + \big\{\vec{e}_1, \vec{e}_2 - \vec{e}_1\big\} \qquad \text{and} \qquad \mathbb{L}^-(i) = i - \big\{\vec{e}_1, \vec{e}_2 - \vec{e}_1\big\}.$$

Finally, let $G_2 \subseteq G_1 \subseteq S$ be two events, and set $p_1 := \hat{\mu}(G_1)$ and $p_2 := \hat{\mu}(G_2)$. The main result of this section is the following theorem.

Theorem 3.1. *For any $t \in (0,1)$ there exist $\vec{T}(t), T(t)$ satisfying $\vec{T}(t) = \exp\big(O(\log 1/t)^2\big)$ and $T(t) = 1/t^{O(1)}$ as $t \to 0$, such that the following oriented and unoriented constrained Poincaré inequalities hold.*

(A) *Suppose that $G_1 = S$ and $G_2 \subseteq S$. Then, for all local functions f:*

$$\mathrm{Var}(f) \leqslant \vec{T}(p_2) \sum_{i\in\mathbb{Z}^2} \mu\left(\mathbb{1}_{\{\omega_{i+\vec{e}_1}\in G_2\}} \mathrm{Var}_i(f)\right) \tag{3.1}$$

$$\mathrm{Var}(f) \leqslant T(p_2) \sum_{i\in\mathbb{Z}^2} \mu\left(\mathbb{1}_{\{\{\omega_{i+\vec{e}_1}\in G_2\}\cup\{\omega_{i-\vec{e}_1}\in G_2\}\}} \mathrm{Var}_i(f)\right). \tag{3.2}$$

(B) *Suppose that $G_2 \subseteq G_1 \subseteq S$. Then there exists $\delta > 0$ such that, for all p_1, p_2 satisfying $\max\big\{p_2, (1-p_1)(\log p_2)^2\big\} \leqslant \delta$, and all local functions f:*

$$\begin{aligned}\mathrm{Var}(f) \leqslant \vec{T}(p_2)\bigg(&\sum_{i\in\mathbb{Z}^2} \mu\left(\mathbb{1}_{\{\omega_{i+\vec{e}_2}\in G_2\}}\mathbb{1}_{\{\omega_j\in G_1\,\forall j\in\mathbb{L}^+(i)\}} \mathrm{Var}_i(f)\right) \\ &+ \sum_{i\in\mathbb{Z}^2} \mu\left(\mathbb{1}_{\{\omega_{i+\vec{e}_1}\in G_2\}}\mathbb{1}_{\{\omega_{i-\vec{e}_1}\in G_1\}} \mathrm{Var}_i\big(f\,|\,G_1\big)\right)\bigg),\end{aligned} \tag{3.3}$$

$$\begin{aligned}\mathrm{Var}(f) \leqslant T(p_2)\bigg(&\sum_{\varepsilon=\pm1}\sum_{i\in\mathbb{Z}^2} \mu\left(\mathbb{1}_{\{\omega_{i+\varepsilon\vec{e}_2}\in G_2\}}\mathbb{1}_{\{\omega_j\in G_1\,\forall j\in\mathbb{L}^\varepsilon(i)\}} \mathrm{Var}_i(f)\right) \\ &+ \sum_{\varepsilon=\pm1}\sum_{i\in\mathbb{Z}^2} \mu\left(\mathbb{1}_{\{\omega_{i+\varepsilon\vec{e}_1}\in G_2\}}\mathbb{1}_{\{\omega_{i-\varepsilon\vec{e}_1}\in G_1\}} \mathrm{Var}_i\big(f\,|\,G_1\big)\right)\bigg).\end{aligned} \tag{3.4}$$

Remark 3.2. When proving Theorem 1 the starting point will be (3.1)/(3.2) depending on whether the model is rooted/unrooted. Similarly, for critical models we will start the proof of Theorem 2 from (3.3)/(3.4) depending on whether the model is α-rooted/β-unrooted. This choice, which will become more clear later on, is dictated by the bootstrap process according to the following rule: choose that inequality for which $V_i \subset [A]_{\mathcal{U}}$, where A is *any* initial infection such that the indicator function in front of $\mathrm{Var}_i(f)$ is equal to one.

An important role in the proof of the theorem is played by the one-dimensional East and 1-neighbour processes (see, e.g., [12]), and a certain generalization of these processes. For the reader's convenience, we begin by recalling these generalized models.

3.1. **The generalised East and 1-neighbour models.** The standard versions of these two models are ergodic interacting particle systems on $\{0,1\}^n$ with kinetic constraints, which will mean that jumps in the dynamics are facilitated by certain configurations of vertices in state 0. They are both reversible w.r.t. the product measure $\pi = B(\alpha_1) \otimes \cdots \otimes B(\alpha_n)$, where $B(\alpha)$ is the α-Bernoulli measure and $\alpha_1, \ldots, \alpha_n \in (0,1)$.

In the first process, known as the *non-homogeneous East model* (see [21, 27] and references therein), the state ω_x of each point $x \in [n]$ is resampled at rate one (independently across $[n]$) from the distribution $B(\alpha_x)$, provided that $c_x(\omega) = 1$, where

$$c_x(\omega) = \mathbb{1}_{\{\omega_{x+1}=0\}} \qquad \text{and} \qquad \omega_{n+1} := 0.$$

In the second model, known as the *non-homogeneous 1-neighbour model* (and also as the FA-1f model [2]), the resampling occurs in the same way, except in this case

$$c_x(\omega) = \max\left\{\mathbb{1}_{\{\omega_{x-1}=0\}}, \mathbb{1}_{\{\omega_{x+1}=0\}}\right\} \qquad \text{where} \qquad \omega_0 := 1 \qquad \text{and} \qquad \omega_{n+1} := 0.$$

It is known [1, 12, 14] that the corresponding relaxation times $T_{\text{East}}(n, \bar{\alpha})$ and $T_{\text{FA}}(n, \bar{\alpha})$ (where $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$) are finite *uniformly* in n. Moreover, it is shown in [12, 14] that, for small values of $q := \min\left\{1 - \alpha_x : x \in [n]\right\}$, they satisfy the bounds

$$T_{\text{East}}\big(n, \bar{\alpha}\big) = q^{-O(\min\{\log n, \log(1/q)\})} \qquad \text{and} \qquad T_{\text{FA}}\big(n, \bar{\alpha}\big) = q^{-O(1)}. \tag{3.5}$$

Remark 3.3. Strictly speaking (3.5) has only been proved in the homogeneous case $\alpha_i = \alpha$ for all $i \in [n]$, see [12]. However, it is easy to check that the proof in [12] generalizes to the non-homogeneous setting.

We will need to work in the following more general setting. Consider a finite product probability space of the form $\Omega = \otimes_{x \in [n]} (S_x, \nu_x)$, where S_x is a finite set and ν_x a positive probability measure on S_x. Given $\{\omega_x\}_{x \in [n]} \in \Omega$, we will refer to ω_x as the *the state of the vertex x*. Moreover, for each $x \in [n]$, let us fix a constraining event $S_x^g \subseteq S_x$ with $q_x := \nu_x(S_x^g) > 0$. We consider the following generalisations of the East and FA-1f processes on the space Ω.

Definition 3.4. In the *generalised East chain*, the state ω_x of each vertex $x \in [n]$ is resampled at rate one (independently across $[n]$) from the distribution ν_x, provided that $\vec{c}_x(\omega) = 1$, where

$$\vec{c}_x(\omega) = \mathbb{1}_{\{\omega_{x+1} \in S_{x+1}^g\}}$$

if $x \in \{1, \ldots, n-1\}$, and $c_n(\omega) \equiv 1$.

In the *generalised FA-1f chain*, the resampling occurs in the same way, except in this case $c_1(\omega) = \mathbb{1}_{\{\omega_2 \in G_2\}}$,

$$c_x(\omega) = \max\left\{\mathbb{1}_{\{\omega_{x-1} \in S_{x-1}^g\}}, \mathbb{1}_{\{\omega_{x+1} \in S_{x+1}^g\}}\right\}$$

if $x \in \{2, \ldots, n-1\}$, and $c_n(\omega) \equiv 1$.

In both cases, set $q := \min_x q_x = \min_x \nu_x(S_x^g)$, and set $\alpha_x := 1 - q_x$ for each $x \in [n]$.

Note that the projection variables $\eta_x = \mathbb{1}_{\{S_x^g\}}$ evolve as a standard East or FA-1f chain, and it is therefore natural to ask whether the relaxation times of these generalised constrained chains can be bounded from above in terms of the relaxation times $T_{\text{East}}(n,\bar{\alpha})$ and $T_{\text{FA}}(n,\bar{\alpha})$ respectively. The answer is affirmative, and it is the content of the following proposition (cf. [14, Proposition 3.4]), which provides us with Poincaré inequalities for the generalised East and FA-1f chains.

Proposition 3.5. *Let f be a local function. For the generalised East chain, we have*

$$\operatorname{Var}(f) \leqslant \frac{O(1)}{q} \cdot T_{\mathit{East}}(n,\bar{\alpha}) \cdot \sum_{x=1}^{n} \nu\big(\vec{c}_x \operatorname{Var}_x(f)\big), \tag{3.6}$$

and for the generalised FA-1f chain, we have

$$\operatorname{Var}(f) \leqslant \frac{O(1)}{q} \cdot T_{\mathit{FA}}(n,\bar{\alpha}) \cdot \sum_{x=1}^{n} \nu\big(c_x \operatorname{Var}_x(f)\big), \tag{3.7}$$

where $\operatorname{Var}_x(\cdot)$ denotes the conditional variance w.r.t. ν_x, given all the other variables.

The proof of this proposition, which is similar to that of [14, Proposition 3.4], is deferred to Appendix A.

3.2. **Proof of Theorem 3.1.** We begin with the proof of part (A), which is a relatively straightforward consequence of Proposition 3.5 and (3.5). The proof of part (B) is significantly more difficult, and we will require a technical result from [30] (see Proposition 3.6, below) and a careful application of Proposition 3.5 (and of convexity) after conditioning on various events.

3.2.1. *Proof of part (A).* Recall that in this setting $G_1 = S$ and $G_2 \subset S$, where $(S,\hat{\mu})$ is an arbitrary finite positive probability space. Let f be a local function and let $M > 0$ be sufficiently large so that f does not depend on the variables at vertices (m,n) with $|m| \geqslant M/2$. For each $n \in \mathbb{Z}$, let μ_n denote the product measure $\otimes_{m\in\mathbb{Z}}\, \hat{\mu}$ on $S^{\mathbb{Z}\times\{n\}}$, and note that $\mu = \otimes_{n\in\mathbb{Z}}\, \mu_n$. By construction, $\operatorname{Var}_{\mu_n}(f)$ coincides with the same conditional variance computed w.r.t. $\mu_n^M := \otimes_{m\in\mathbb{Z}\cap[-M,M]}\, \hat{\mu}$.

We apply Proposition 3.5 to the homogeneous product measure μ_n^M with the event G_2 as event S_x^g for all $x \in \{-M,\dots,M\}$. Note that $q_x = \hat{\mu}(G_2) = p_2$ for every x, and that $\operatorname{Var}_{(n,M)}(f) = \operatorname{Var}_{(n,-M)}(f) = 0$. It follows that

$$\operatorname{Var}_{\mu_n}(f) \leqslant \vec{T}(p_2) \sum_{m\in\mathbb{Z}} \mu_n \left(\mathbb{1}_{\{\omega_{(m+1,n)}\in G_2\}} \operatorname{Var}_{(m,n)}(f) \right),$$

where $\vec{T}(p_2) = \exp\Big(O\big(\log(p_2)\big)^2 \Big)$ (see (3.5)), and

$$\operatorname{Var}_{\mu_n}(f) \leqslant T(p_2) \sum_{n\in\mathbb{Z}} \mu_n \left(\mathbb{1}_{\{\omega_{(m+1,n)}\in G_2\}\cup\{\omega_{(m-1,n)}\in G_2\}} \operatorname{Var}_{(m,n)}(f) \right),$$

where $T(p_2) = p_2^{-O(1)}$ (see (3.5)). Using the standard inequality $\mathrm{Var}_\mu(f) \leqslant \sum_{n\in\mathbb{Z}} \mu\big(\mathrm{Var}_{\mu_n}(f)\big)$, the Poincaré inequalities (3.1) and (3.2) follow.

3.2.2. *Proof of part (B).* We next turn to the significantly more challenging task of proving the constrained Poincaré inequalities (3.3) and (3.4). As noted above, in addition to Proposition 3.5 we will require a technical result from [30], stated below as Proposition 3.6. In order to state this result we need some additional notation.

Recall that an *oriented path of length* n in $\mathbb{Z}^2$ is a sequence $\gamma = (i^{(1)}, \dots, i^{(n)})$ of n vertices of $\mathbb{Z}^2$ with the property that $i^{(k+1)} - i^{(k)} \in \{\vec{e}_1, \vec{e}_2\}$ for each $k \in [n-1]$. We will say that γ starts at $i^{(1)}$, ends at $i^{(n)}$, and that $i \in \gamma$ if $i = i^{(k)}$ for some $k \in [n]$. Moreover, given $\omega \in \Omega$, we will say that γ is

- ω*-good* if $\omega_i \in G_1$ for all $i \in \bigcup_{j\in\gamma} \big\{j, j+\vec{e}_1, j-\vec{e}_1\big\}$, and
- ω*-super-good* if it is good and there exists $i \in \gamma$ such that $\omega_i \in G_2$,

where $G_2 \subseteq G_1 \subseteq S$ are the events in the statement of Theorem 3.1.

In what follows it will be convenient to order the oriented paths of length n starting from a given point according to the alphabetical order of the associated strings of n unit vectors from the finite alphabet $\mathcal{X} = \{\vec{e}_1, \vec{e}_2\}$. Next, for each $i \in \mathbb{Z}^2$ we define the key event $\Gamma_i \subset \Omega$, as follows:

(i) there exists an oriented ω-good path γ, of length $\lceil 1/p_2^2 \rceil$, starting at $i + \vec{e}_2$;
(ii) the smallest such path is ω-super-good;
(iii) $\omega_j \in G_1$ for each $j \in \mathbb{L}^+(i) = i + \big\{\vec{e}_1, \vec{e}_2 - \vec{e}_1\big\}$.

In what follows, and if no confusion arises, we will abbreviate ω-good and ω-super-good into good and super-good respectively. The following starting upper bound is essentially [30, Proposition 3.4], and we therefore defer the proof to Appendix A.

Proposition 3.6. *There exists $\delta > 0$ such that, if $\max\big\{p_2, (1-p_1)(\log p_2)^2\big\} \leqslant \delta$, then*

$$\mathrm{Var}(f) \leqslant 4 \sum_{i\in\mathbb{Z}^2} \mu\big(\mathbb{1}_{\Gamma_i} \mathrm{Var}_i(f)\big)$$

for every local function f.

Observe that, while Proposition 3.5 provides us with an upper bound on the full variance in terms of the sum of local variances, the quantity $\mu\big(\mathbb{1}_{\Gamma_{(0,0)}} \mathrm{Var}_{(0,0)}(f)\big)$ is more like the average of a local variance. We will therefore need to use convexity to bound from above the average of a local variance by a full variance. In order to reduce as much as possible the potential loss of such an operation, we first perform a series of conditionings on the measure μ and use convexity only on the final conditional measure.

Roughly speaking, on the event Γ_i we first reveal, for each $j \neq i$ within distance $2/p_2^2$ of the origin, whether or not the event $\{\omega_j \in G_1\}$ holds. Given this information we know which paths of length $\lceil 1/p_2^2 \rceil$ and starting at $i + \vec{e}_2$ are good, and we define γ^* as the smallest one in the order defined above. Next, we reveal the *last* $j^* \in \gamma^*$ such that

$\{\omega_{j^*} \in G_2\}$. Note that in doing so we do not need to observe whether or not the event $\{\omega_j \in G_2\}$ holds for any earlier j (i.e., before j^* in γ^*). Finally, defining $\gamma \subset \gamma^*$ to be the part of γ^* before j^*, we reveal ω_j for all $j \in \mathbb{Z}^2$, except for $j = i$ and $j \in \gamma$.

At the end of this process we are left with a (conditional) probability measure ν on $S^{\gamma \cup \{i\}}$. We will then apply convexity and Proposition 3.5 to this measure. We now detail the above procedure.

Proof of part (B) of Theorem 3.1. Let $\delta > 0$ be given by Proposition 3.6, and assume that the events $G_2 \subseteq G_1 \subseteq S$ satisfy $\max\big\{p_2, (1-p_1)(\log p_2)^2\big\} \leqslant \delta$. By Proposition 3.6, we have

$$\mathrm{Var}(f) \leqslant 4 \sum_{i \in \mathbb{Z}^2} \mu\big(\mathbb{1}_{\Gamma_i} \mathrm{Var}_i(f)\big) \tag{3.8}$$

for every local function f. We will bound each term of the sum in (3.8). Using translation invariance, it will suffice to consider the term $i = (0,0)$.

For each $\omega \in \Gamma_{(0,0)}$, let $\gamma^* = \gamma^*(\omega)$ denote the smallest ω-good oriented path of length $\lceil 1/p_2^2 \rceil$ starting from $\vec{e}_2$, and note that γ^* is ω-super-good, since $\omega \in \Gamma_{(0,0)}$. Let $\xi = \xi(\omega) \in \gamma^*$ be the first super-good vertex encountered while travelling along γ^* backwards, *i.e.,* from its last point to its starting point $\vec{e}_2$. Finally, let γ be the portion of γ^* starting at $\vec{e}_2$ and ending at the vertex preceding ξ in γ^*.

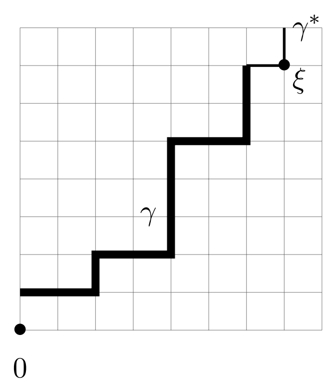


FIGURE 1. The minimal good path γ^*, the position of the first super-good vertex ξ encountered while traveling backward along γ^*, and the subpath $\gamma \subset \gamma^*$ (thick black) connecting $\vec{e}_2$ to a neighbour of ξ.

We next perform the series of conditionings on the measure μ described informally above. Let Λ be the box of side-length $4/p_2^2$ centred at the origin. We first condition on the event $\Gamma_{(0,0)}$ and on the σ-algebra generated by the events

$$\big\{\{\omega_j \in G_1\} : j \in \Lambda \setminus \{(0,0)\}\big\}.$$

Note that, since we are conditioning on the event $\Gamma_{(0,0)}$, these events determine γ^*. Next we condition on the position of ξ on γ^*; this determines the path $\gamma = (i^{(1)}, \dots, i^{(n)})$. Finally we condition on all of the variables ω_j with $j \notin \gamma \cup \{(0,0)\}$.

Let ν be the resulting conditional measure and observe that $(S^{\gamma \cup \{(0,0)\}}, \nu)$ is a product probability space of the form $\otimes_{j \in \gamma \cup \{(0,0)\}} (S_j, \nu_j)$, with $(S_{(0,0)}, \nu_{(0,0)}) = (S, \hat{\mu})$ and $(S_j, \nu_j) = \big(G_1, \hat{\mu}(\cdot \,|\, G_1)\big)$ for each $j \in \gamma$. Notice that

$$\mu\big(\mathbb{1}_{\Gamma_{(0,0)}} \operatorname{Var}_{(0,0)}(f)\big) = \mu\Big(\mathbb{1}_{\Gamma_{(0,0)}} \nu\big(\operatorname{Var}_{\nu_{(0,0)}}(f)\big)\Big) \leqslant \mu\big(\mathbb{1}_{\Gamma_{(0,0)}} \operatorname{Var}_\nu(f)\big), \tag{3.9}$$

because $\nu\big(\operatorname{Var}_{\nu_{(0,0)}}(f)\big) \leqslant \operatorname{Var}_\nu(f)$, by convexity.

We can now bound $\operatorname{Var}_\nu(f)$ from above by applying Proposition 3.5 to the measure $\nu = \otimes_{j \in \gamma \cup \{(0,0)\}} (S_j, \nu_j)$, with the super-good event G_2 as the constraining event S_j^g. Observe that $\nu\big(S_{(0,0)}^g\big) = \hat{\mu}(G_2) = p_2$ and $\nu\big(S_j^g\big) = \hat{\mu}\big(G_2 \,|\, G_1\big) = p_2/p_1$ for each $j \in \gamma$. The first Poincaré inequality (3.6) in Proposition 3.5 therefore gives

$$\mu\big(\mathbb{1}_{\Gamma_{(0,0)}} \operatorname{Var}_\nu(f)\big) \leqslant \vec{T}(p_2) \cdot \mu\bigg(\mathbb{1}_{\Gamma_{(0,0)}} \sum_{i \in \gamma \cup (0,0)} \nu\Big(\mathbb{1}_{\{\omega_{m(i)} \in G_2\}} \operatorname{Var}_{\nu_i}(f)\Big)\bigg), \tag{3.10}$$

where $m(i)$ is the next point on the path γ^* after i (*i.e.,* $m(i)$ is either $m(i) = i + \vec{e}_1$ or $m(i) = i + \vec{e}_2$) and

$$\vec{T}(p_2) = \frac{O(1)}{p_2} \sup_{n \leqslant 1/p_2^2} T_{\text{East}}(n, \bar{\alpha}) = p_2^{-\Theta(\log(1/p_2))},$$

by (3.5). Recall that in Definition 3.4 the constraint for the last point is identically equal to one (this is in order to guarantee irreducibility of the chain), and observe that this condition holds in the above setting because, by construction, $\omega_\xi \in G_2$.

Finally, we claim that (3.10) implies that

$$\begin{aligned}\mu\big(\mathbb{1}_{\Gamma_{(0,0)}} \operatorname{Var}_\nu(f)\big) \leqslant \vec{T}(p_2) \sum_{i \in \Lambda} \bigg(&\mu\Big(\mathbb{1}_{\{\omega_{i+\vec{e}_1} \in G_2\}} \mathbb{1}_{\{\omega_{i-\vec{e}_1} \in G_1\}} \operatorname{Var}_i(f \,|\, G_1)\Big) \\ &+ \mu\Big(\mathbb{1}_{\{\omega_{i+\vec{e}_2} \in G_2\}} \mathbb{1}_{\{\omega_j \in G_1 \,\forall j \in \mathbb{L}^+(i)\}} \big(\operatorname{Var}_i(f) + \operatorname{Var}_i(f \,|\, G_1)\big)\Big)\bigg). \end{aligned} \tag{3.11}$$

Indeed, note that $\operatorname{Var}_{\nu_{(0,0)}}(f) = \operatorname{Var}_{(0,0)}(f)$ and that $\operatorname{Var}_{\nu_i}(f) = \operatorname{Var}_i(f \,|\, G_1)$ for each $i \in \gamma$, and recall that, by construction, $\omega_{i+\vec{e}_1}, \omega_{i-\vec{e}_1} \in G_1$ for every $i \in \gamma$. Therefore, for each $i \in \gamma$, if $m(i) = i + \vec{e}_1$ then $\omega_{i-\vec{e}_1} \in G_1$, and if $m(i) = i + \vec{e}_2$ then $\omega_j \in G_1$ for each $j \in \mathbb{L}^+(i) = i + \big\{\vec{e}_1, \vec{e}_2 - \vec{e}_1\big\}$. Moreover, the event $\Gamma_{(0,0)}$ implies that $\omega_j \in G_1$ for each $j \in \mathbb{L}^+((0,0))$. Therefore every term of the right-hand side of (3.10) is included in the right-hand side of (3.11), and hence (3.10) implies (3.11), as claimed.

Now, combining (3.11) with (3.8) and (3.9), and noting that $\mathrm{Var}_i(f) \geqslant p_1 \mathrm{Var}_i(f \mid G_1)$ and that $|\Lambda| \leqslant p_2^{-O(1)}$, we obtain

$$\mathrm{Var}(f) \leqslant p_1^{-1} p_2^{-O(1)} \vec{T}(p_2) \sum_{i \in \mathbb{Z}^2} \bigg(\mu\Big(\mathbb{1}_{\{\omega_{i+\vec{e}_1} \in G_2\}} \mathbb{1}_{\{\omega_{i-\vec{e}_1} \in G_1\}} \mathrm{Var}_i(f \mid G_1) \Big) + \mu\Big(\mathbb{1}_{\{\omega_{i+\vec{e}_2} \in G_2\}} \mathbb{1}_{\{\omega_j \in G_1 \, \forall j \in \mathbb{L}^+(i)\}} \mathrm{Var}_i(f) \Big) \bigg),$$

which implies the oriented Poincaré inequality (3.3), as required.

The proof of the unoriented inequality (3.4) is almost the same, excapt we will use the second Poincaré inequality (3.7) in Proposition 3.5, instead of (3.6). To spell out the details, we obtain

$$\mu\big(\mathbb{1}_{\Gamma_{(0,0)}} \mathrm{Var}_\nu(f) \big) \leqslant T(p_2) \cdot \mu\bigg(\mathbb{1}_{\Gamma_{(0,0)}} \sum_{i \in \gamma \cup (0,0)} \nu\Big(c_i \, \mathrm{Var}_{\nu_i}(f) \Big) \bigg), \tag{3.12}$$

where c_i is the indicator of the event that G_2 holds for at least one of the neighbours of i on the path γ^*, and

$$T(p_2) = \frac{O(1)}{p_2} \sup_{n \leqslant 1/p_2^2} T_{\mathrm{FA}}(n, \bar{\alpha}) = p_2^{-O(1)},$$

by (3.5). Note that the constraint for the last point is again identically equal to one since $\omega_\xi \in G_2$. It follows (cf. (3.11)) that

$$\mu\big(\mathbb{1}_{\Gamma_{(0,0)}} \mathrm{Var}_\nu(f) \big) \leqslant T(p_2) \sum_{i \in \Lambda} \sum_{\varepsilon = \pm 1} \bigg(\mu\Big(\mathbb{1}_{\{\omega_{i+\varepsilon\vec{e}_1} \in G_2\}} \mathbb{1}_{\{\omega_{i-\varepsilon\vec{e}_1} \in G_1\}} \mathrm{Var}_i(f \mid G_1) \Big) + \mu\Big(\mathbb{1}_{\{\omega_{i+\varepsilon\vec{e}_2} \in G_2\}} \mathbb{1}_{\{\omega_j \in G_1 \, \forall j \in \mathbb{L}^\varepsilon(i)\}} \big(\mathrm{Var}_i(f) + \mathrm{Var}_i(f \mid G_1) \big) \Big) \bigg), \tag{3.13}$$

since $\omega_{i+\vec{e}_1}, \omega_{i-\vec{e}_1} \in G_1$ for every $i \in \gamma$, and the event $\Gamma_{(0,0)}$ implies that $\omega_j \in G_1$ for each $j \in \mathbb{L}^+((0,0))$. In particular, note that if $i \in \gamma$ and $i + \vec{e}_2 \in \gamma$, then $\omega_j \in G_1$ for each $j \in \mathbb{L}^+(i) = \mathbb{L}^-(i + \vec{e}_2) = i + \big\{ \vec{e}_1, \vec{e}_2 - \vec{e}_1 \big\}$. Therefore, as before, every term of the right-hand side of (3.12) is included in the right-hand side of (3.13).

Finally, combining (3.13) with (3.8) and (3.9), and since $\mathrm{Var}_i(f) \geqslant p_1 \mathrm{Var}_i(f \mid G_1)$ and $|\Lambda| \leqslant p_2^{-O(1)}$, we obtain

$$\mathrm{Var}(f) \leqslant p_1^{-1} p_2^{-O(1)} T(p_2) \sum_{i \in \mathbb{Z}^2} \sum_{\varepsilon = \pm 1} \bigg(\mu\Big(\mathbb{1}_{\{\omega_{i+\varepsilon\vec{e}_1} \in G_2\}} \mathbb{1}_{\{\omega_{i-\varepsilon\vec{e}_1} \in G_1\}} \mathrm{Var}_i \big(f \mid G_1 \big) \Big) + \mu\Big(\mathbb{1}_{\{\omega_{i+\varepsilon\vec{e}_2} \in G_2\}} \mathbb{1}_{\{\omega_j \in G_1 \, \forall j \in \mathbb{L}^\varepsilon(i)\}} \mathrm{Var}_i(f) \Big) \bigg),$$

which gives the unoriented Poincaré inequality (3.4), as claimed, and hence completes the proof of Theorem 3.1. $\square$

4. Towards a proof of Theorems 1 and 2

In this section we shall define the setting to which we shall apply Theorem 3.1 in order to bound from above the relaxation time, and hence the mean infection time, of supercritical and critical KCM. We will begin with a very brief informal description, before giving (in Section 4.1) the precise definition. We will then, in Sections 4.2 and 4.3, state two results from the theory of bootstrap percolation that will play an instrumental role in the proofs of Theorems 1 and 2.

Our basic strategy is to partition the lattice $\mathbb{Z}^2$ into disjoint rectangular "blocks" $\{V_i\}_{i\in\mathbb{Z}^2}$, whose size is adapted to the bootstrap update family $\mathcal{U}$. To each block V_i we associate a block random variable ω_i, which is just the collection of the $0/1$ i.i.d Bernoulli(p) variables $\{\omega_x\}_{x\in V_i}$ attached to each vertex of the block. In order to avoid confusion we will always use the letters $i, j, \dots$ for the labels of quantities associated to blocks, and the letters $x, y, \dots$ for the labels of the quantities associated to vertices of $\mathbb{Z}^2$. We will apply Theorem 3.1 to the block variables $\{\omega_i\}_{i\in\mathbb{Z}^2}$.

4.1. A concrete general setting.

Let v and $v^\perp$ be orthogonal rational directions in the first and second quadrant of $\mathbb{R}^2$ respectively. Let $\vec{v}$ be the vector joining the origin to the first site of $\mathbb{Z}^2$ in direction v, and similarly for $\vec{v}^\perp$. Let $n_1 \geqslant n_2$ be (sufficiently large) even integers, and set

$$R := \big\{x \in \mathbb{R}^2 \,:\, x = \alpha n_1 \vec{v} + \beta n_2 \vec{v}^\perp,\, \alpha, \beta \in [0,1)\big\}. \tag{4.1}$$

The finite probability space $(S, \hat{\mu})$ appearing in Section 3 will always be of the form $S = \{0,1\}^V$, where $V = R \cap \mathbb{Z}^2$, and $\hat{\mu}$ is the Bernoulli(p) product measure. Observe that the probability space $(S^{\mathbb{Z}^2}, \mu)$ is isomorphic to $\Omega = \{0,1\}^{\mathbb{Z}^2}$ equipped with the Bernoulli(p) product measure which, with a slight abuse of notation, we will continue to denote by μ. For our purposes, a convenient isomorphism between the two probability spaces is given by a kind of tilted "brick-wall" partition of $\mathbb{Z}^2$ into disjoint copies of the basic block V (see Figure 2). To be precise, for each $i = (i_1, i_2) \in \mathbb{Z}^2$, set $V_i := R_i \cap \mathbb{Z}^2$, where $R_i := R + (i_1 + i_2/2)n_1\vec{v} + i_2 n_2 \vec{v}^\perp$.

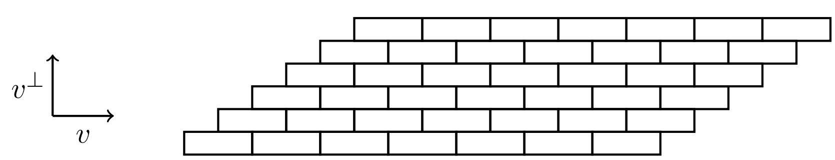


Figure 2. The partition into blocks $V_i,\ i \in \mathbb{Z}^2$

In this partition the "northern" and "southern" neighbouring blocks of V_i (*i.e.,* the blocks corresponding to $(i_1, i_2 \pm 1)$) are shifted in the direction $\vec{v}$ by $\pm n_1/2$ w.r.t. V_i. With this notation, and given $\omega \in S^{\mathbb{Z}^2}$, it is then convenient to think of the variable $\omega_i \in S$ as being the collection $\{\omega_x\}_{x\in V_i} \in \{0,1\}^{V_i}$. The local variance term $\mathrm{Var}_i(f)$ (*i.e.,* the variance of f w.r.t. the variable ω_i given all the other variables $\{\omega_j\}_{j\neq i}$), which appears in the various constrained Poincaré inequalities in the statement of Theorem 3.1, is then equal to the variance $\mathrm{Var}_{V_i}(f)$ w.r.t. the i.i.d Bernoulli(p) variables $\{\omega_x\}_{x\in V_i}$, given all the other variables $\{\omega_y\}_{y\in\mathbb{Z}^2\setminus V_i}$.

From now on, ω will always denote an element of $\{0,1\}^{\mathbb{Z}^2}$ and, given $\Lambda\subset\mathbb{R}^2$, we will write ω_Λ for the collection of i.i.d. random variables $\{\omega_x\}_{x\in\Lambda\cap\mathbb{Z}^2}$, and μ_Λ for their joint product Bernoulli(p) law. We will say that Λ is *empty* (or *empty in* ω) if ω is identically equal to 0 on $\Lambda\cap\mathbb{Z}^2$, and similarly that Λ is *filled* (or *completely occupied*) if ω is identically equal to 1 on $\Lambda\cap\mathbb{Z}^2$.

We now turn to the definitions of the good and super-good events $G_2\subset G_1\subseteq S$. The good event G_1 will depend on the update family $\mathcal{U}$, and will (roughly speaking) approximate the event that the block V_i can be "crossed" in the $\mathcal{U}$-bootstrap process with the help of a constant-width strip connecting the top and bottom of V_i. For supercritical models this event is trivial, and therefore G_1 is the entire space S; for critical models, on the other hand, G_1 will require the presence of empty vertices inside V obeying certain model-dependent geometric constraints (see Definition 6.4, below). The super-good event G_2 for supercritical models will simply require that V is empty. For critical models it will require that G_1 holds, and additionally that there exists an empty subset $\mathcal{R}$ of V, called the *quasi-stable half-ring* (see Definitions 4.9, 6.4 and Figure 4) of (large) constant width, and height equal to that of V. We reemphasize that the parameters n_1,n_2 will be chosen (depending on the model) so that the probabilities p_1 and p_2 of the events G_1 and G_2 (respectively) satisfy the key condition

$$\lim_{p\to 1}\max(p_2,\big(1-p_1\big)\big(\log p_2\big)^2)=0$$

that appears in part (B) of Theorem 3.1.

4.2. **Spreading of infection: the supercritical case.** We are now almost ready to state the property of $\mathcal{U}$-bootstrap percolation (proved by Bollobás, Smith and Uzzell [9]) that we will need when $\mathcal{U}$ is supercritical, i.e., when there exists an open semicircle $C\subset S^1$ that is free of stable directions. If $\mathcal{U}$ is rooted, then we may choose u to be the midpoint of any such semicircle; if $\mathcal{U}$ is unrooted, on the other hand, then C can be chosen in such a way that $-C$ also has no stable directions, and we choose u to be the midpoint of any such semicircle. We then apply the construction of the rectangle R and of the partition $\{V_i\}_{i\in\mathbb{Z}^2}$ described in Section 4.1 with $v=-u$.

Recall that $[V_i]_{\mathcal{U}}$ denotes the closure of $V_i=R_i\cap\mathbb{Z}^2$ under the $\mathcal{U}$-bootstrap process. The following result, proved in [9], states that a large enough rectangle can infect the rectangle to its "left" (i.e., in direction $-v$) under the $\mathcal{U}$-bootstrap process, and if $\mathcal{U}$ is unrooted then it can also infect the rectangle to its "right" (i.e., in direction v).

Proposition 4.1. *Let $\mathcal{U}$ be a supercritical two-dimensional update family. If n_1 and n_2 are sufficiently large, then the following hold:*

(i) *If $\mathcal{U}$ is unrooted, then $V_{(-1,0)} \cup V_{(1,0)} \subset [V_{(0,0)}]_{\mathcal{U}}$.*

(ii) *If $\mathcal{U}$ is rooted, then $V_{(-1,0)} \subset [V_{(0,0)}]_{\mathcal{U}}$.*

Remark 4.2. By definition, in the rooted case the semicircle $-C$ contains some stable directions. Thus, $V_{(1,0)} \not\subset [V_{(0,0)}]_{\mathcal{U}}$.

The proof of Proposition 4.1 in [9] is non-trivial, and required some important innovations, most notably the notion of "quasi-stable directions" (see Definition 4.5, below). We will therefore give here only a brief sketch, explaining how one can read the claimed inclusions out of the results of [9]

Sketch proof of Proposition 4.1. Both parts of the proposition are essentially immediate consequences of the following claim: if R is a sufficiently large rectangle, and the semicircle centred at w is entirely unstable, then $[R]_{\mathcal{U}}$ contains *every* element of $\mathbb{Z}^2$ that can be reached from R by travelling in direction w. This claim follows from [9, Lemma 5.5], since in this setting all of the quasi-stable directions in $\mathcal{S}'_U$ (see [9, Section 5.3]) are unstable, and if u is unstable then the empty set is a u-block (see [9, Definition 5.1]). We refer the reader to [9, Sections 5 and 7] for more details. $\square$

4.3. **Spreading of infection: the critical case.** We next turn to the more complicated task of precisely defining the good and super-good events for critical update families. Throughout this subsection, we will assume that $\mathcal{U}$ is a critical update family with difficulty $\alpha \in [1, \infty)$ and bilateral difficulty $\beta \in [\alpha, +\infty]$ (see Definition 2.5). Recall that we say that $\mathcal{U}$ is α-rooted if $\beta \geqslant 2\alpha$, and that $\mathcal{U}$ is β-unrooted otherwise.

We begin by noting an important property of the set of stable directions $\mathcal{S}(\mathcal{U})$.

Lemma 4.3. *If $\beta < \infty$ then $\mathcal{S}(\mathcal{U})$ consists of a finite number of isolated, rational directions. Moreover, if $\mathcal{U}$ is β-unrooted and $\alpha(u^*) = \max\big\{\alpha(u) : u \in \mathcal{S}(\mathcal{U})\big\}$, then $\alpha(u) \leqslant \beta$ for every $u \in \mathcal{S}(\mathcal{U}) \setminus \{u^*, -u^*\}$.*

Proof. By [9, Theorem 1.10], $\mathcal{S}(\mathcal{U})$ is a finite union of rational closed intervals of S^1, and if $u \in \mathcal{S}(\mathcal{U})$ then $\alpha(u) < \infty$ if and only u is an isolated point of $\mathcal{S}(\mathcal{U})$. Thus, if one of the intervals in $\mathcal{S}(\mathcal{U})$ is not an isolated point, then there exist two non-opposite stable directions in S^1, each with infinite difficulty, and so $\beta = \infty$.

Now, suppose that $\mathcal{U}$ is β-unrooted, and that $u \in \mathcal{S}(\mathcal{U})$ satisfies $\alpha(u) > \beta$ and $u \notin \{u^*, -u^*\}$. Then u and u^* are non-opposite stable directions in S^1, each with difficulty strictly greater than β, which contradicts the definition of β. $\square$

In particular, if $\mathcal{U}$ is β-unrooted then Lemma 4.3 guarantees the existence of an open semicircle C such that $(C \cup -C) \cap \mathcal{S}(\mathcal{U})$ consists of finitely many directions, each with difficulty at most β. The next lemma provides a corresponding (but slighter weaker) property for α-rooted models.

Lemma 4.4. *If $\mathcal{U}$ is α-rooted, then there exists an open semicircle C such that $C \cap \mathcal{S}(\mathcal{U})$ consists of finitely many directions, each with difficulty at most α.*

Proof. By the definition of α, there exists an open semicircle C such that each $u \in C$ has difficulty at most α. Since $\mathcal{U}$ is critical (and hence α is finite), it follows from [9, Theorem 1.10] (cf. the proof of Lemma 4.3) that each $u \in C$ is either unstable, or an isolated element of $\mathcal{S}(\mathcal{U})$, and hence $C \cap \mathcal{S}(\mathcal{U})$ is finite, as claimed. □

Let us fix (for the rest of the subsection) an open semicircle C, containing finitely many stable directions, and such that the following holds:

- if $\mathcal{U}$ is α-rooted then $\alpha(v) \leqslant \alpha$ for each $v \in C$;
- if $\mathcal{U}$ is β-unrooted then $\alpha(v) \leqslant \beta$ for each $v \in C \cup -C$.

Let us also choose C such that its mid-point u belongs to $\mathbb{Q}_1$, and denote by $\pm u^\perp$ the boundary points of C. When drawing pictures we will always think of C as the semicircle $(-\pi/2, \pi/2)$, though we emphasize that we do not assume that u is parallel to one of the axes of $\mathbb{Z}^2$. We remark that the values of $\alpha(u^\perp)$ and $\alpha(-u^\perp)$ will not be important: we will only need to use the fact that they are both finite.

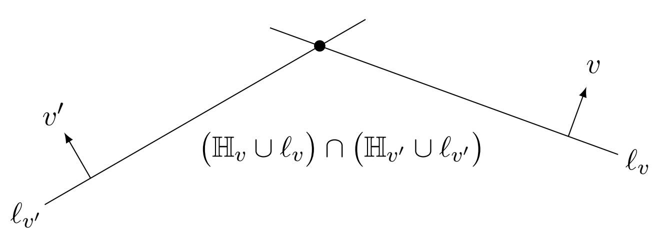
We are now ready to define one of the key notions from [9], the set of quasi-stable directions. These are directions that are not (necessarily) stable, but which nevertheless it is useful to treat as if they were.

Definition 4.5 (Quasi-stable directions)**.** We say that a direction $v \in \mathbb{Q}_1$ is quasi-stable if either $v = u$ or

$$v \in \mathcal{S}(\mathcal{U}) \cup \left(\bigcup_{X \in \mathcal{U}} \bigcup_{x \in X} \{ v \in S^1 : \langle v, x \rangle = 0 \} \right).$$

The key property of the family of quasi-stable directions is given by the following lemma, which allows us to empty the sites near the corners of "quasi-stable half rings" (see Definition 4.9, below).

Lemma 4.6 ([9, Lemma 5.3])**.** *For every pair v, v' of consecutive quasi-stable directions there exists an update rule X such that $X \subset \big(\mathbb{H}_v \cup \ell_v\big) \cap \big(\mathbb{H}_{v'} \cup \ell_{v'}\big)$.*



In order to define quasi-stable half rings, we first need to introduce some additional notation:

Definition 4.7. Let $v \in \mathbb{Q}_1$ with $\alpha(v) \leqslant \alpha$. A *$v$-strip* S is any closed parallelogram in $\mathbb{R}^2$ with long sides perpendicular to v and short sides perpendicular to $u^\perp$.

- The $+$-boundary and $-$-boundary of S, denoted $\partial_+ S$ and $\partial_- S$ respectively, are the sides of S with outer normal v and $-v$.
- The external boundary $\partial^{\text{ext}} S$ is defined as that translate of $\partial_+ S$ in the v-direction which captures for the first time a new lattice point not already present in S.
- Given $\lambda > 0$, we define $\partial^{\text{ext}}_\lambda S$ as the portion of $\partial^{\text{ext}} S$ at distance λ from its endpoints (see Figure 3).

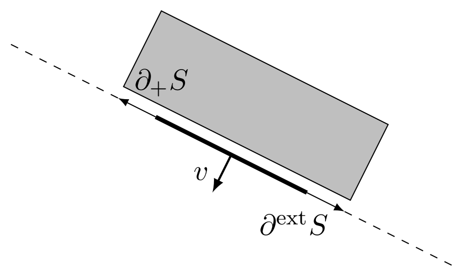


FIGURE 3. The $+$-boundary of S, the external boundary (solid segment) and its subset $\partial^{\text{ext}}_\lambda S$ (thick solid segment)

If v is a stable direction, then a v-strip needs some "help" from other infected sites in order to infect its external boundary (in the $\mathcal{U}$-bootstrap process). Our next ingredient (also first proved in [9]) provides us with a set that suffices for this purpose.

Let v be a quasi-stable direction with difficulty $\alpha(v) \leqslant \alpha$, and let $Z_v \subset \mathbb{Z}^2$ be a set of cardinality α such that $[\mathbb{H}_v \cup Z_v]_{\mathcal{U}} \cap \ell_v$ is infinite. (In the language of [6], Z_v is called a *voracious set*.) The following lemma (see [9, Lemma 5.5] and [6, Lemma 3.4]) states that if S is a sufficiently large v-strip, then a bounded number of translates of Z_v, together with $S \cap \mathbb{Z}^2$, are sufficient to infect $\partial^{\text{ext}}_\lambda S$ for some $\lambda = O(1)$. Recall that $R_{\mathcal{U}} = \max_{X \in \mathcal{U}} \max_{x \in X} |x|$ denotes the *range* of $\mathcal{U}$.

Lemma 4.8. *There exist $T_v = \{a_1, \ldots, a_r\} \subset \ell_v \cap \mathbb{Z}^2$, $b \in \ell_v \cap \mathbb{Z}^2$ and $\lambda_v > 0$ such that the following holds. If S is a sufficiently large v-strip such that $\partial^{\text{ext}} S \cap \mathbb{Z}^2 \subset \ell_v$, then*

$$\partial^{\text{ext}}_{\lambda_v} S \cap \mathbb{Z}^2 \subset \big[(S \cap \mathbb{Z}^2) \cup (Z_v + a_1 + k_1 b) \cup \cdots \cup (Z_v + a_r + k_r b)\big]_{\mathcal{U}} \tag{4.2}$$

for every $k_1, \ldots, k_r \in \mathbb{Z}$ such that $a_i + k_i b \in \partial^{\text{ext}}_{\lambda_v} S$ for every $i \in [r]$.

Let us fix for each quasi-stable direction v a set $T_v = \{a_1, \ldots, a_r\} \subset \ell_v \cap \mathbb{Z}^2$, and site $b \in \ell_v \cap \mathbb{Z}^2$, and a constant $\lambda_v > 0$ given by Lemma 4.8. If S is a sufficiently large v-strip such that $\partial^{\text{ext}} S \cap \mathbb{Z}^2 \subset \ell_v + x$ for some $x \in \mathbb{Z}^2$, then we will refer to any set of the form

$$\big((Z_v + a_1 + k_1 b) \cup \cdots \cup (Z_v + a_r + k_r b)\big) + x,$$

with $a_i + k_i b + x \in \partial^{\text{ext}}_{\lambda_v} S$ for every $i \in [r]$, as a *helping set* for S.

We are finally ready to define the key objects we will use to control the movement of empty sites in a critical KCM, the *quasi-stable half-rings*. These are non self-intersecting polygons, obtained by patching together suitable v-strips corresponding to quasi-stable directions (see Figure 4).

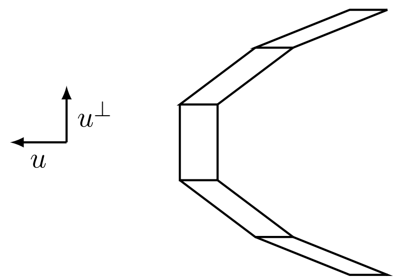


FIGURE 4. A quasi-stable half-ring.

Definition 4.9 (Quasi-stable half-rings)**.** Let $(v_1, \dots, v_m)$ be the quasi-stable directions in C, ordered in such a way that v_i and v_{i+1} are consecutive directions for any $i \in [m-1]$, and v_{i-1} comes before v_i in clockwise order. Let S_{v_i} be a v_i-strip with length ℓ_i and width w_i. We say that $\mathcal{R} := \bigcup_{i=1}^m S_{v_i}$ is a *quasi-stable half-ring* of width w and length $(\ell_1, \dots, \ell_m)$ if the following holds:

(i) $w_i = w$ for each $i \in [m]$;
(ii) $S_{v_i} \cap S_{v_j} = \emptyset$, unless v_i and v_j are consecutive directions and in that case the two strips share exactly one of their short side and no other point;
(iii) $\ell_i = \ell_j$ if v_i and v_j are symmetric w.r.t. u.

We can finally formulate the "spreading of infection" result that we will need later. Given a quasi-stable half ring $\mathcal{R}$, we will write $\mathcal{R}^*$ for the quasi-stable half ring $\mathcal{R} + su$, where $s > 0$ is minimal such that $\big(\mathcal{R}^* \setminus \mathcal{R}\big) \cap \mathbb{Z}^2$ is non-empty.

Proposition 4.10. *There exist a constant $\lambda = \lambda(\mathcal{U}) > 0$ such that following holds. Let $\mathcal{R}$ be a quasi-stable half ring of width w and length $(\ell_1, \dots, \ell_m)$, where $w, \ell_1, \dots, \ell_m \geqslant \lambda$. Let V be the set of points of $\mathbb{Z}^2$ within distance λ of $\mathcal{R} \cup \mathcal{R}^*$, and for each $i \in [m]$ let Z_i be a helping set for S_{v_i}. Then*

$$\mathcal{R}^* \subset \big[\mathcal{R} \cup Z_1 \cup \dots \cup Z_m\big]_{\mathcal{U}}^V.$$

Proof. This is a straightforward consequence of Lemmas 4.6 and 4.8. To see this, note first that, by Lemma 4.8, the closure of $\mathcal{R} \cup Z_1 \cup \dots \cup Z_m$ under the $\mathcal{U}$-bootstrap process contains all points of $\mathcal{R}^*$ except possibly those that lie within distance $O(1)$ from a corner of $\mathcal{R}$. Moreover, the path of infection described in the proof of Lemma 4.8 in [6, 9] only uses sites within distance $O(1)$ of the v-strip S. Thus, if λ is chosen large enough, we have $\partial^{\mathrm{ext}}_{\lambda/4}(S_{v_i}) \cap \mathbb{Z}^2 \subset \big[\mathcal{R} \cup Z_i\big]_{\mathcal{U}}^V$ for each $i \in [m]$.

Now, by Lemma 4.6, it follows that the set $\left[\mathcal{R} \cup Z_i \cup Z_{i+1}\right]_{\mathcal{U}}^V$ contains the remaining sites of $\partial^{\text{ext}}(S_{v_i}) \cap \mathbb{Z}^2$ and $\partial^{\text{ext}}(S_{v_{i+1}}) \cap \mathbb{Z}^2$ that lie within distance $\lambda/4$ of the intersection of S_{v_i} and $S_{v_{i+1}}$. Indeed, these sites can be infected one by one, working towards each corner, using sites in $\mathcal{R} \cup \partial^{\text{ext}}_{\lambda/4}(S_{v_i}) \cup \partial^{\text{ext}}_{\lambda/4}(S_{v_{i+1}})$. Since this holds for each $i \in [m-1]$, it follows that the whole of $\mathcal{R}^*$ is infected, as claimed. □

Given a quasi-stable half ring $\mathcal{R}$, we will write $\mathcal{R}'$ for the quasi-stable half ring $\mathcal{R}+su$, where $s > 0$ is minimal such that $\mathcal{R} \cap \mathbb{Z}^2$ and $\mathcal{R}' \cap \mathbb{Z}^2$ are disjoint.

Corollary 4.11. *There exist a constant $\lambda = \lambda(\mathcal{U}) > 0$ such that following holds. Let $\mathcal{R}$ be a quasi-stable half ring of width w and length $(\ell_1, \dots, \ell_m)$, and suppose that $w \geqslant \lambda$ and $\ell := \min\left\{\ell_1, \dots, \ell_m\right\} \geqslant \lambda$. Let V be the set of points of $\mathbb{Z}^2$ within distance λ of $\mathcal{R} \cup \mathcal{R}'$, and let $A \subset V$ be such that for any quasi-stable direction v, and any v-strip S_v such that $\partial^{\text{ext}}(S_v) \cap \mathcal{R}'$ has length at least ℓ, there exists a helping set for S_v in A. Then*

$$\mathcal{R}' \subset \left[\mathcal{R} \cup A\right]_{\mathcal{U}}^V.$$

Proof. By construction each v_i-strip of $\mathcal{R}$ has its helping set in $\mathcal{R}'$. Using Proposition 4.10, the $\mathcal{U}$-bootstrap process restricted to V is able to infect the quasi-stable half-ring $\mathcal{R}^*$. We then repeat with $\mathcal{R}$ replaced by $\mathcal{R}^*$, and so on, until the entire quasi-stable half ring $\mathcal{R}'$ has been infected. □

Observe that, under the additional assumption that each quasi-stable direction v has a helping set contained in ℓ_v, we may choose A to be a subset of $\mathcal{R}'$, but that in general we may (at some stage) need a helping set not contained in $\mathcal{R}'$ in order to advance in the u-direction.

Remark 4.12. Later on, we will also need the above results in the slightly different setting in which the first v_1-strip entering in the definition of $\mathcal{R}$ is longer than the last one while all the others v_j-strips, $j \neq 1, m$, fullfil the symmetry requirement (iii) of Definition 4.9. In this case we will refer to $\mathcal{R}$ as an *elongated* quasi-stable half-ring. For simplicity we preferred to state Proposition 4.10 in the slightly less general context, but exactly the same proof applies if $\mathcal{R}$ is an elongated quasi-stable half-ring.

5. Scaling of the infection time for supercritical KCM

In this section we shall prove Theorem 1, which gives a sharp (up to a constant factor in the exponent) upper bound on the mean infection time for a supercritical KCM. We will first (in Section 5.1) give a detailed proof in the case that $\mathcal{U}$ is unrooted, and then (in Section 5.2) explain briefly how the proof can be modified to prove the claimed bound for rooted models.

5.1. The unrooted case.

Let $\mathcal{U}$ be a supercritical, unrooted, two-dimensional update family; we are required to show that there exists a constant $\lambda = \lambda(\mathcal{U})$ such that

$$\mathbb{E}_\mu(\tau_0) \leqslant q^{-\lambda}$$

for all sufficiently small $q > 0$. To do so, recall first from (2.5) that $\mathbb{E}_\mu(\tau_0) \leqslant T_{\mathrm{rel}}(q, \mathcal{U})/q$, and therefore, by Definition 2.8, it will suffice to prove that

$$\mathrm{Var}(f) \leqslant q^{-\lambda} \sum_x \mu\big(c_x \, \mathrm{Var}_x(f)\big) \tag{5.1}$$

for some $\lambda = \lambda(\mathcal{U}) > 0$ and all local functions f, where c_x denotes the kinetic constraint for the KCM, i.e., c_x is the indicator function of the event that there exists an update rule $X \in \mathcal{U}$ such that $\omega_y = 0$ for each $y \in X + x$. We will deduce a bound of the form (5.1) from Theorem 3.1 and Proposition 4.1.

Recall the construction and notation described in Sections 4.1 and 4.2; in particular, recall the definitions of the blocks V_i, of the parameters n_1 and n_2 (which determine the side lengths of the basic rectangle R), and the choice of u as the midpoint of an open semicircle $C \subset S^1$ such that the set $C \cup -C$ contains no stable directions. As anticipated in Section 4.1, the choice of the good and super-good events $G_2 \subset G_1 \subseteq S$ entering in Theorem 3.1, is, in this case, extremely simple.

Definition 5.1. If $\mathcal{U}$ is a supercritical two-dimensional update family, then:
(a) every block V_i satisfies the *good event* G_1 for $\mathcal{U}$ (*i.e.,* $G_1 = S$);
(b) a block V_i satisfies the *super-good event* G_2 for $\mathcal{U}$ if and only if it is empty.

Let us fix the parameters n_1 and n_2 to be $O(1)$, but sufficiently large so such that Proposition 4.1 holds. It follows that if $V_{(0,0)}$ is super-good, then the blocks $V_{(-1,0)}$ and $V_{(1,0)}$ (its nearest neighbours to the left and right respectively) lie in the closure under the $\mathcal{U}$-bootstrap process of the empty sites in $V_{(0,0)}$. In particular,

$$t^{\pm} = \min\big\{t > 0 \,:\, A_t \supseteq V_{(\pm 1,0)}\big\},$$

are both finite, where A_t is the set of sites infected after t steps of the $\mathcal{U}$-bootstrap process, starting from $A_0 = V_{(0,0)}$ (see Definition 2.1). With foresight, define

$$\Lambda := \big(A_{t^-} \setminus V_{(0,0)}\big) + n_1 \vec{v}, \tag{5.2}$$

and note that $\Lambda \cap V_{\vec{e}_1} = \emptyset$ and $V_{(0,0)} \subset \Lambda$.

Proof of part (a) *of Theorem 1.* The first step is to apply Theorem 3.1 to the probability space $(S^{\mathbb{Z}^2}, \mu)$ described in Section 4.1, in which each 'block' variable $\omega_i \in S$ is given by the collection of i.i.d Bernoulli(p) variables $\{\omega_x\}_{x \in V_i} \in \{0,1\}^{V_i}$. Recall that $p_1 = \hat{\mu}(G_1)$ and $p_2 = \hat{\mu}(G_2)$ are the probabilities of the good and super-good events, respectively, and note that, in our setting, $p_1 = 1$ and $p_2 = q^{n_1 n_2} = q^{O(1)}$. It follows, using (3.2), that

$$\mathrm{Var}(f) \leqslant \frac{1}{q^{O(1)}} \sum_{i \in \mathbb{Z}^2} \mu\left(\mathbb{1}_{\{\text{either } V_{i+\vec{e}_1} \text{ or } V_{i-\vec{e}_1} \text{ is empty}\}} \mathrm{Var}_{V_i}(f)\right) \tag{5.3}$$

for all local functions f, where $\mathrm{Var}_{V_i}(f)$ denotes the variance with respect to the variables $\{\omega_x\}_{x \in V_i}$, given all the other variables $\{\omega_y\}_{y \in \mathbb{Z}^2 \setminus V_i}$.

To deduce (5.1), it will suffice (by symmetry) to prove an upper bound on the right-hand side of (5.3) of the form

$$\mu\left(\mathbb{1}_{\{V_{\vec{e}_1}\text{ is empty}\}}\operatorname{Var}_{V_{(0,0)}}(f)\right)\leqslant\frac{1}{q^{O(1)}}\sum_{x\in\Lambda}\mu\big(c_x\operatorname{Var}_x(f)\big)\tag{5.4}$$

for the set Λ defined in (5.2), since the elements of Λ are all within distance $O(1)$ from the origin, and so we may then simply sum over all $i\in\mathbb{Z}^2$.

To prove (5.4), the first step is to observe that, by the convexity of the variance, and recalling that $\Lambda\cap V_{\vec{e}_1}=\emptyset$ and $V_{(0,0)}\subset\Lambda$, we have

$$\mu\Big(\mathbb{1}_{\{V_{\vec{e}_1}\text{ is empty}\}}\operatorname{Var}_{V_{(0,0)}}(f)\Big)\leqslant\mu\left(\mathbb{1}_{\{V_{\vec{e}_1}\text{ is empty}\}}\operatorname{Var}_{\Lambda}(f)\right).$$

To conclude we appeal to the following result which, for later purposes, we formulate in a slightly more general setting that what is need here. In what follows, for any $\omega\in\Omega$ and $V\subset\mathbb{Z}^2$, we shall write $[\omega]_{\mathcal{U}}^V$ for the closure w.r.t. the bootstrap process restricted to V of the set $\{x\in\mathbb{Z}^2:\ \omega_x=0\}$.

Lemma 5.2. *Let $A,B\subset\mathbb{Z}^2$ be disjoint sets, and let $\mathcal{E}$ be an event depending only on ω_B. Suppose that there exists a set $V\supset A\cup B$ such that $B\subset[\omega]_{\mathcal{U}}^V$ for any $\omega\in\{0,1\}^V$ for which A is empty and $\omega_B\in\mathcal{E}$. Then*

$$\mu\Big(\mathbb{1}_{\{A\text{ is empty}\}}\operatorname{Var}_B\left(f\mid\mathcal{E}\right)\Big)\leqslant|V|q^{-|V|}\frac{1}{pq\mu(\mathcal{E})}\sum_{x\in V}\mu\big(c_x\operatorname{Var}_x(f)\big).\tag{5.5}$$

Before proving the lemma we conclude the proof of part (a) of Theorem 1. We apply the lemma with $A=V_{\vec{e}_1},B=\Lambda$ and $\mathcal{E}$ the trivial event, i.e., $\mathcal{E}=\Omega_B$. Indeed, by construction (see (5.2)), $B\subset\Lambda\subset[A]_{\mathcal{U}}$. Thus

$$\mu\left(\mathbb{1}_{\{V_{\vec{e}_1}\text{ is empty}\}}\operatorname{Var}_{\Lambda}(f)\right)\leqslant|\Lambda|q^{-|\Lambda|}\frac{1}{pq}\sum_{x\in\Lambda}\mu\big(c_x\operatorname{Var}_x(f)\big).\tag{5.6}$$

Since $|\Lambda|=O(1)$, we conclude that for all $i\in\mathbb{Z}^2$,

$$\mu\left(\mathbb{1}_{\{V_{i+\vec{e}_1}\text{ is empty}\}}\operatorname{Var}_{V_i}(f)\right)\leqslant\frac{1}{q^{O(1)}}\sum_{x\in\Lambda_i}\mu\big(c_x\operatorname{Var}_x(f)\big),$$

where Λ_i is the analog of Λ for the block V_i.

As noted above, summing over $i\in\mathbb{Z}^2$ and using (5.3), we obtain the Poincaré inequality (5.1) with constant $q^{-O(1)}$, and by (2.5) and Definition 2.8 it follows that there exists a constant $\lambda=\lambda(\mathcal{U})$ such that

$$\mathbb{E}_\mu(\tau_0)\leqslant\frac{T_{\mathrm{rel}}(q,\mathcal{U})}{q}\leqslant q^{-\lambda},$$

for all sufficiently small $q>0$, as required. Since the bootstrap infection time $T_{\mathcal{U}}$ of a supercritical update family satisfies $T_{\mathcal{U}}=q^{-\Theta(1)}$, it also follows that $\mathbb{E}_\mu(\tau_0)\leqslant T_{\mathcal{U}}^{O(1)}$. $\square$

Proof of Lemma 5.2. Fix $\hat\omega \in \Omega$ such that A is empty. We begin by writing

$$\operatorname{Var}_B\left(f \mid \mathcal{E}\right) \leqslant \frac{1}{\mu(\mathcal{E})} \sum_{\omega_B \in \mathcal{E}} \mu(\omega_B) \Big(f\big(\omega_B, \hat\omega_{\mathbb{Z}^2\setminus B}\big) - f\big(\omega_B \equiv 0, \hat\omega_{\mathbb{Z}^2\setminus B}\big)\Big)^2 \tag{5.7}$$

since $\mathbb{E}\big[(X-a)^2\big]$ is minimized by taking $a = \mathbb{E}[X]$. We will break each term in the sum into single spin-flips using the $\mathcal{U}$-bootstrap process as follows. Using the assumption of the lemma, there exists an ordering $\big(x^{(1)}, \dots, x^{(|V|)}\big)$ of the vertices of V such that, if $\omega^{(k)}$ denotes the configuration obtained from $\omega^{(0)} \equiv \omega$ by emptying the first k vertices, then

- V is empty in the final configuration $\omega^{(|V|)}$;
- for each $k = 1, \dots, |V|$, the constraint $c_{x^{(k)}}\big(\omega^{(k-1)}\big) = 1$.

It follows, using Cauchy-Schwarz, that for any ω in which A is empty

$$\Big(f\big(\omega_B, \omega_{\mathbb{Z}^2\setminus B}\big) - f\big(\omega_B \equiv 0, \omega_{\mathbb{Z}^2\setminus B}\big)\Big)^2 \leqslant |V| \sum_{k=1}^{|V|} c_{x^{(k)}}\big(\omega^{(k-1)}\big)\Big(f\big(\omega^{(k)}\big) - f\big(\omega^{(k-1)}\big)\Big)^2$$
$$\leqslant |V| \frac{1}{\mu_*} \frac{1}{pq} \sum_{x \in V} \sum_{\eta \in \{0,1\}^V} \mu_V(\eta) c_x(\eta, \omega_{\mathbb{Z}^2\setminus V})\, pq \Big(f\big(\eta^{(x)}, \omega_{\mathbb{Z}^2\setminus V}\big) - f\big(\eta, \omega_{\mathbb{Z}^2\setminus V}\big)\Big)^2,$$

where $\mu_* = \min_{\eta \in \{0,1\}^V} \mu_V(\eta) = q^{|V|}$ and $\eta^{(x)}$ denotes the configuration obtained from η by flipping the spin at x. Notice that the right hand side does not depend on ω_B and that $pq\big(f\big(\eta^{(x)}, \omega_{\mathbb{Z}^2\setminus V}\big) - f\big(\eta, \omega_{\mathbb{Z}^2\setminus V}\big)\big)^2$ is the local variance $\operatorname{Var}_x(f)$ computed for the configuration $(\eta, \omega_{\mathbb{Z}^2\setminus V})$. Hence,

$$\mathbb{1}_{\{A \text{ is empty}\}} \operatorname{Var}_B\left(f \mid \mathcal{E}\right) \leqslant \frac{|V| q^{-|V|}}{pq\mu(\mathcal{E})} \sum_{x \in V} \mu_V\big(c_x \operatorname{Var}_x(f)\big)(\omega_{\mathbb{Z}^2\setminus V}),$$

and the inequality (5.5) follows by averaging over ω using the measure μ. □

5.2. **The rooted case.** Let $\mathcal{U}$ be a supercritical, rooted, two-dimensional update family, let $C \subset S^1$ be a semicircle with no stable directions and recall that, thanks to (2.5), it will suffice to prove a Poincaré inequality (cf. (5.1)) with constant $1/q^{O(\log(1/q))}$. To prove this we will follow almost exactly the same route of the unrooted case, with the same definition of the blocks V_i and of the good and super-good events. We will therefore only give a very brief sketch of the proof in this new setting.

The main crucial difference w.r.t. the unrooted case is that now the opposite semicircle $-C$ will necessarily contain some stable directions. That will force us to use the oriented Poincaré inequality (3.1) from Theorem 3.1 instead of the unoriented one (3.2), because in this case (see Proposition 4.1 and Remark 4.2) a super-good block is able to infect the block to its left but not the block to its right, *i.e.,* $V_{(-1,0)} \subset [V_{(0,0)}]_{\mathcal{U}}$ but $V_{(1,0)} \not\subset [V_{(0,0)}]_{\mathcal{U}}$.

Proof of part (b) *of Theorem 1.* We again apply Theorem 3.1 to the probability space $(S^{\mathbb{Z}^2},\mu)$ described in Section 4.1, but we use (3.1) instead of (3.2). Recalling that $p_1 = 1$ and $p_2 = q^{O(1)}$, we obtain

$$\operatorname{Var}(f) \leqslant \frac{1}{q^{O(\log(1/q))}} \sum_{i\in\mathbb{Z}^2} \mu\left(\mathbb{1}_{\{V_{i+\vec{e}_1} \text{ is empty}\}} \operatorname{Var}_{V_i}(f)\right) \tag{5.8}$$

for all local functions f. As before, using translation invariance, we only examine the $i = 0$ term in the above sum. We claim that

$$\mu\left(\mathbb{1}_{\{V_{\vec{e}_1} \text{ is empty}\}} \operatorname{Var}_{V_{(0,0)}}(f)\right) \leqslant \frac{1}{q^{O(1)}} \sum_{x\in\Lambda} \mu\big(c_x \operatorname{Var}_x(f)\big) \tag{5.9}$$

where Λ is the set defined in (5.2). However, the proof of (5.9) is identical to that of (5.4), since Proposition 4.1 implies that $V_{(0,0)}$ can be entirely infected by $V_{\vec{e}_1}$. We therefore obtain the Poincaré inequality

$$\operatorname{Var}(f) \leqslant \frac{1}{q^{O(\log(1/q))}} \sum_{x} \mu\big(c_x \operatorname{Var}_x(f)\big) \tag{5.10}$$

for all local functions f. Thus $T_{\mathrm{rel}}(q,\mathcal{U}) = 1/q^{O(\log(1/q))}$ and hence

$$\mathbb{E}_\mu(\tau_0) \leqslant \frac{T_{\mathrm{rel}}(q,\mathcal{U})}{q} = 1/q^{O(\log(1/q))} = T_{\mathcal{U}}^{O(\log T_{\mathcal{U}})},$$

as required, because $T_{\mathcal{U}} = 1/q^{O(1)}$. □

6. Scaling of the infection time for critical KCM: proof of Theorem 2 under a simplifying assumption

In this section we shall prove Theorem 2 under the following additional assumption (see below): every stable direction v with finite difficulty has a voracious set that is a subset of the line ℓ_v. By doing so, we avoid some technical complications (mostly related to the geometry of the quasi-stabel half-ring) that might obscure the main ideas behind the proof. The changes necessary to treat the general case are spelled out in detail in Section 7.

Assumption 6.1. *For any stable direction $u \in \mathcal{S}$ with finite difficulty $\alpha(u)$, there exists $Z_u \subset \ell_u$ of cardinality $\alpha(u)$ such that $\big[\mathbb{H}_u \cup Z_u\big]_{\mathcal{U}} \cap \ell_u$ is infinite.*

As in Section 5, our main task will be to establish a suitable upper bound on the relaxation time $T_{\mathrm{rel}}(\mathcal{U};q)$. In Section 6.1 we will first analyse the α-rooted case and the starting point will be the constrained Poincaré inequality (3.3); the proof the β-unrooted case (see Section 6.2) will be essentially the same, the main difference being that (3.3) will be replaced by (3.4).

6.1. **α-rooted update families.** Let $\mathcal{U}$ be a critical, α-rooted, two-dimensional update family, and recall from Definition 2.11 that $\mathcal{U}$ has difficulty α, and bilateral difficulty at least 2α. The properties of $\mathcal{U}$ that we will need below have already been proved in Section 4.3; these all follow from the fact (see Lemma 4.4) that there exists an open semicircle C such that $C \cap \mathcal{S}(\mathcal{U})$ consists of finitely many directions, each with difficulty at most α. In particular, we will make crucial use of Corollary 4.11.

We will prove that, if Assumption 6.1 holds, then there exists a constant $\lambda = \lambda(\mathcal{U})$ such that

$$\mathbb{E}_\mu(\tau_0) \leqslant \frac{T_{\rm rel}(q,\mathcal{U})}{q} \leqslant \exp\Big(\lambda \cdot q^{-2\alpha}\big(\log(1/q)\big)^4\Big)$$

for all sufficiently small $q > 0$. Note that the first inequality follows from (2.5), and so, by Definition 2.8, it will suffice to prove that

$$\operatorname{Var}(f) \leqslant \exp\Big(\lambda \cdot q^{-2\alpha}\big(\log(1/q)\big)^4\Big) \sum_x \mu\big(c_x \operatorname{Var}_x(f)\big) \tag{6.1}$$

for some $\lambda = \lambda(\mathcal{U})$ and all local functions f. We will deduce a bound of the form (6.1) starting from (3.3) and using Corollary 4.11.

Remark 6.2. The choice of (3.3) instead of (3.4) as starting point of our proof is obliged. It is easy to construct α-rooted models with $\beta = +\infty$ such that, for any choice of the side lengths n_1, n_2 of the blocks V_i and of the good and super-good events G_1, G_2, fulfilling the condition $\lim_{q\to 0}(1-p_1)(\log(1/p_2))^2 = 0$, the bootstrap process is not able to infect the block V_i knowing that the block $V_{i-\vec{e}_1}$ is infected and that $V_i, V_{i+\vec{e}_1}$ are good. The well known Duarte model [17] is one such example. On the other hand, we will prove in the sequel that, by carefully choosing n_1, n_2, G_1, G_2, the block V_i is always infected starting from an arbitrary configuration satisfying the constraints appearing in (3.3) (see Figure 5). If $\beta < +\infty$ but still $\beta > 2\alpha$, it is possible to tune n_1, n_2, G_1, G_2, in such a way that (3.4) is applicable. However, as shown in Section 6.2, the best Poincaré constant obtained in this way is $e^{\widetilde{O}(1/q^\beta)}$, which is much larger than the one we get here starting from (3.3).

The geometric setting and the good and super-good events. Recall the construction and notation described in Sections 4.1 and 4.3; in particular, recall that the blocks V_i consists of the lattice points inside a $n_1 \times n_2$ rectangle in the rotated coordinates $(v, v^\perp)$, where $u = -v$ is the midpoint of an open semicircle $C \subset S^1$ such that each stable direction in C has difficulty at most α. W.l.o.g. we will always draw all our figures as if u belongs to the third quadrant. We will choose n_1, n_2 depending on q, and of the form:

$$n_1 = \big\lfloor q^{-2\kappa}\big\rfloor, \qquad n_2 = \big\lfloor \kappa^4 q^{-\alpha}\log(1/q)\big\rfloor,$$

where $\kappa = \kappa(\mathcal{U})$ is a sufficiently large constant.

In order to define the good and super-good events G_1 and G_2, we need an additional definition. Recall that $V := R \cap \mathbb{Z}^2$. A row of V is the maximal set of points of

V lying on the same line parallel to u and similarly for a column of V. We order the rows from bottom to top and the columns from left to right and observe that, by construction, the first row and the first column belong to the bottom and left side of R respectively. Let a, b be the number of rows and columns respectively and let a_i, b_j be the cardinality of the i^{th}-row and the j^{th}-column respectively. Since v is a rational direction $a = \Theta(n_2), b = \Theta(n_1), a_i = \Theta(n_1), b_j = \Theta(n_2)$ with constants depending only on the update family $\mathcal{U}$.

Definition 6.3. Given $\kappa > 0$ we say that a collection $\mathcal{M} = \{M^{(i)}\}_{i=1}^a$ of disjoint intervals of R of length κ forms an *upward* κ-stair with steps $M^{(1)}, M^{(2)}, ..$ if:

(i) for each $i \in [a]$, the i^{th}-step belongs to the line containing the i^{th}-row;

(ii) the i^{th}-step is "on the left" of the j^{th}-step if $i < j$. More precisely, let $(M_\ell^{(i)}, M_r^{(i)})$ be the abscissa (in the $(v, v^\perp)$-frame) of the left and right boundary of the i^{th}-step respectively. Then $M_r^{(i)} < M_\ell^{(j)}$ whenever $i < j$.

We refer the reader to Figure 6 for a quick visualisation of the above defintion. For later purposes it is convenient to give a partial order to the set of upward κ-stairs by saying that $\hat{\mathcal{M}} \prec \mathcal{M}$ if $\hat{M}_\ell^{(i)} \leqslant M_\ell^{(i)}$ for all $i \in [a]$.

An elementary computation proves that, for $q \ll 1$, the probability that R contains an empty upward κ-stair $\mathcal{M}$ is at least

$$\left(1 - (1 - q^\kappa)^{c\frac{n_1}{\kappa n_2}}\right)^{c' n_2}, \tag{6.2}$$

where c, c' are constants depending on $\mathcal{U}$.

We are now ready to define the good and super-good events. Recall the definitions of the quasi-stable directions $v_1, \dots, v_m$ (see Definition 4.5), of a v-strip S_v (see Definition 4.7) and that of a helping set Z for S_v given right after Lemma 4.8. The simplifying assumption made at the beginning of the section implies that the helping sets of any v_i-strip S_{v_i} belong to the line containing $\partial^{\mathrm{ext}} S_{v_i}$.

Definition 6.4 (Good and super-good events)**.**

(1) A block $V_i := R_i \cap \mathbb{Z}^2$ satisfies the good event G_1 iff:
   (a) for any quasi-stable direction v and any v-strip S such that $\partial^{\mathrm{ext}} S \cap R_i$ has length at least $n_2/10$ there exists an empty helping set $Z \subset R_i$ for S;
   (b) there exists an empty upward κ-stair within the first quarter of R_i.

(2) A block $V_i := R_i \cap \mathbb{Z}^2$ satisfies the super-good event G_2 iff, in addition to satisfying the good event G_1, there exists an empty quasi-stable half-ring $\mathcal{R}$ of parameters $(\kappa, \{\ell_j\}_{j=1}^m)$, height n_2 and entirely contained in the last quarter of R_i, such that $\min_i \ell_i \geqslant n_2/(10m)$.

Next we prove that the hypothesis for the part (B) of Theorem 3.1 holds in the above setting if κ is large enough.

Lemma 6.5. *Let $p_1 := \hat\mu(G_1)$ and $p_2 := \hat\mu(G_2)$. There exists $\kappa_0 > 0$ such that, for any $\kappa > \kappa_0$,*

$$\lim_{q\to 0}(1-p_1)(\log(1/p_2))^2 = 0$$

Proof. Using the FKG inequality, p_1 is bounded from below by the product of the probabilities of the two events characterising G_1. Using a union bound over the possible v_i-strips inside V together with the definition of a helping set, one easily verifies that the first event has probability greater than $1 - c_1 n_1 n_2 (1-q^\alpha)^{c_2\delta n_2}$, for suitable constants c_1, c_2. Using (6.2) the probability of the second event is greater than $\left(1-(1-q^\kappa)^{c\frac{n_1}{Cn_2}}\right)^{c' n_2}$. Recalling that $n_1 = \lfloor 1/q^{2\kappa}\rfloor$ and $n_2 = \frac{\kappa^4}{q^\alpha}\log(1/q)$ with $\kappa > \alpha$, we immediately get that $1-p_1 = O(q^{\kappa^2})$ for κ large enough. On the other hand, using again the FKG inequality, we have that $p_2 \geqslant q^{\kappa c'' n_2} p_1$ for some constant c'' depending on $\mathcal{U}$. Therefore $\lim_{q\to 0}(1-p_1)(\log(1/p_2))^2 = 0$ for κ large enough. □

From now on, κ will always be sufficiently large so that Lemma 6.5 applies. In particular the constrained Poincaré inequality (3.3) holds

$$\begin{aligned}\operatorname{Var}(f) \leqslant \vec{T}(p_2)\Bigg(&\sum_{i\in\mathbb{Z}^2} \mu\left(\mathbb{1}_{\{\omega_{i+\vec e_2}\in G_2\}} \mathbb{1}_{\{\omega_j\in G_1\,\forall j\in\mathbb{L}^+(i)\}} \operatorname{Var}_i(f)\right) \\ &+ \sum_{i\in\mathbb{Z}^2} \mu\left(\mathbb{1}_{\{\omega_{i+\vec e_1}\in G_2\}} \mathbb{1}_{\{\omega_{i-\vec e_1}\in G_1\}} \operatorname{Var}_i\left(f \mid G_1\right)\right)\Bigg),\end{aligned} \tag{6.3}$$

with

$$\vec{T}(p_2) = e^{O(\log(p_2)^2)} = e^{O(\log(q)^4/q^{2\alpha})}.$$

Similarly to the strategy adopted for the supercritical cases, our main goal is then to bound from above the two sums in the r.h.s. of (6.3) in terms of the Dirichlet form $\mathcal{D}(f)$ of our KCM. In turn, that requires bounding from above the two generic terms:

$$I_1(i) := \mu\left(\mathbb{1}_{\{\omega_{i+\vec e_1}\in G_2\}} \mathbb{1}_{\{\omega_{i-\vec e_1}\in G_1\}} \operatorname{Var}_{V_i}(f \mid G_1)\right),$$

and

$$I_2(i) := \mu\left(\mathbb{1}_{\{\omega_{i+\vec e_2}\in G_2\}} \mathbb{1}_{\{\omega_j\in G_1\,\forall j\in\mathbb{L}^+(i)\}} \operatorname{Var}_{V_i}(f)\right),$$

(see also Figure 5). Using translation invariance it suffices to consider only the case $i = (0,0)$.

Proposition 6.6. *Let $I_1 \equiv I_1((0,0)$ and $I_2 \equiv I_2((0,0))$. Let also $W_1 = V_{(0,0)} \cup V_{(-1,0)} \cup V_{(1,0)}$ and $W_2 = V_{(0,0)} \cup V_{(-1,0)} \cup V_{(1,0)} \cup V_{(-1,1)} \cup V_{(0,1)}$. Then there exist a $O(1)$-neighborhood $\hat W_i$ of W_i, $i = 1,2$, such that*

$$I_i \leqslant e^{O(\log(q)^2/q^\alpha)} \sum_{x\in\hat W_i} \mu\big(c_x \operatorname{Var}_x(f)\big), \quad i = 1,2.$$

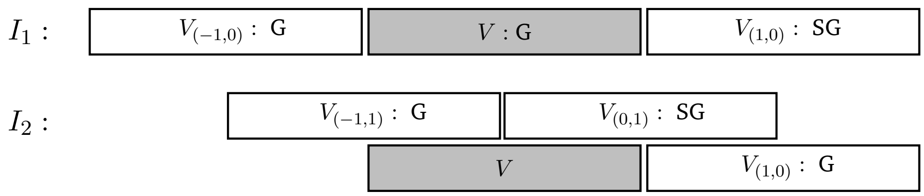


FIGURE 5. For simplicity we fix $i = (0,0)$. In I_1 the box $V \equiv V_{(0,0)}$ is conditioned to be good (G) while the boxes $V_{(-1,0)}$ and $V_{(1,0)}$ are good and super-good (SG) respectively. The set $\mathbb{L}^+((0,0))$ appearing in I_2 consists of the two boxes $V_{(1,0)}$ and $V_{(-1,1)}$. These boxes are good, the box $V_{(0,1)}$ is super-good while V is unconditioned.

By combining the proposition with (6.3), we immediately obtain a final Poincaré inequality of the form

$$\mathrm{Var}(f) \leqslant e^{O(\log(q)^4/q^{2\alpha})} \sum_x \mu\big(c_x \,\mathrm{Var}_x(f)\big)$$

as required.

Proof of Proposition 6.6. Before giving the full technical details of the proof, we first explain the high-level idea that we want to exploit.

The core of the proof. In what follows, W will denote either W_1 or W_2 according to whether we analyse I_1 or I_2. We fix $\omega \in \Omega$ such that $\omega \restriction_W$ satisfies the requirement of the good and super-good environment of the blocks (see Figure 6) and we cover the block $V = V_{(0,0)}$ with a collection of $N+1$ mutually disjoint "fibers" depending on ω. Clearly $N \leqslant |V|$. Each fiber F_i is a subset of cardinality $O(n_2)$ of W and we assume the following key properties:

(a) the fiber F_{N+1} is empty;
(b) in each fiber $F_i, i \in [N]$, a certain "helping" event H_i depending only on $\omega \restriction F_i$ occurs;
(c) the helping events H_i are such that the bootstrap process is able to infect F_i under the only assumption that F_{i+1} is empty and H_i occurs.

To be concrete, let us consider the I_1 term. In this case we will take as the $(N+1)^{th}$-fiber the right-most empty quasi-stable half-ring $\mathcal{R}$, which we know is present in the super-good block $V_{(1,0)}$. The other fibers $\{F_i\}_{i=1}^N$ will be a suitable disjoint translates of F_{N+1} in the u-direction such that $V \subset \cup_{i=1}^N F_i$. The helping event H_i will require the presence in F_i of suitable helping sets for each quasi-stable direction and the key requirement that H_i depends only on $\omega \restriction F_i$ will be guaranteed by the simplifying assumption 6.1. Finally, the third condition (c) above will hold because of Corollary 4.11. We refer the reader to Figure 6.

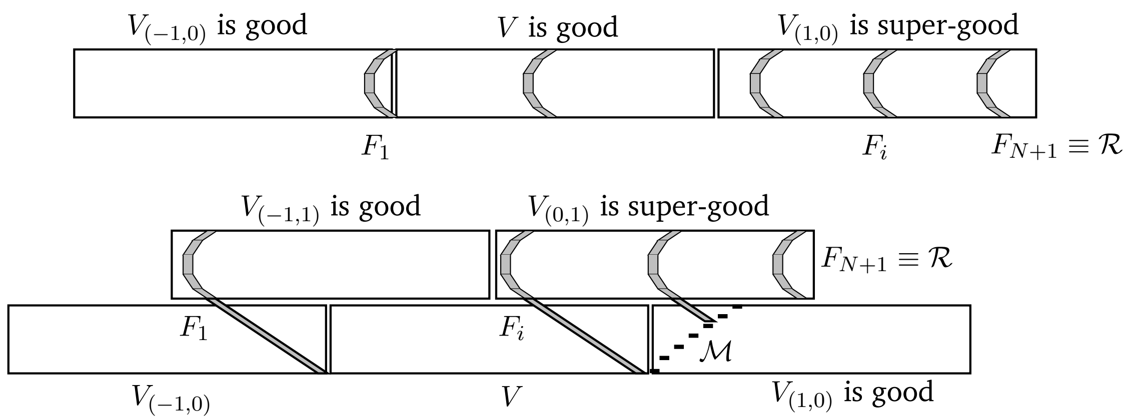


FIGURE 6. A sketchy picture illustrating the local neighborhood W_1 (above) and W_2 (below) of the block $V = V_{(0,0)}$, together with their structure of the good and super-good blocks. For I_1 (above) the fibers are simply the disjoint translates of the rightmost empty half-ring $\mathcal{R}$ in the last quarter of $V_{(1,0)}$. For I_2 (below) the fibers are not all equal and they change at the steps of the increasing stair $\mathcal{M}$ (the little horizontal intervals). Each fiber then becomes an elongated version of the rightmost empty half-ring $\mathcal{R}$.

A similar construction holds for the term I_2, but in this case, as described below, the fibers are slightly more complicate (see Figure 6).

Back to the general scheme, we will prove that both I_1 and I_2 are bounded from above by

$$(1/p_1)\mu\big(\mathbb{1}_{\{F_{N+1} \text{ is empty}\}} \operatorname{Var}_\nu(f)\big), \tag{6.4}$$

where $\operatorname{Var}_\nu(\cdot)$ is the variance computed w.r.t. the product measure $\nu = \otimes_{i=1}^N \nu_i$, where each factor ν_i is the Bernoulli(p) product measure on $S_i = \{0,1\}^{F_i}$ conditioned on the event H_i.

At this stage one realizes that one possibility to bound from above (6.4) is offered by Proposition 3.5. More precisely, one considers the generalized East process on $\otimes_{i=1}^N (S_i, \nu_i)$ with constraining event $S_i^g = \{F_i \text{ is empty}\}$ (see Definition 3.4). As required, the East constraint for the last fiber F_N is always satisfied because F_{N+1} is empty. By construction, the parameters $\{q_i\}_{i=1}^N$ of the generalized East process satisfy

$$q_i = \nu_i(S_i^g) = q^{O(n_2)} = e^{-O(\log(1/q)/q^\alpha)}.$$

By applying Proposition 3.5 to $\mathrm{Var}_\nu(f)$ and recalling the expression for T_{East}, we get

$$\begin{aligned}&\mu\big(\mathbb{1}_{\{F_{N+1}\text{ is empty}\}}\operatorname{Var}_\nu(f)\big)\\ &\leqslant e^{O(\log(q)^2/q^\alpha)}\mu\big(\mathbb{1}_{\{F_{N+1}\text{ is empty}\}}\sum_{i=1}^N \nu(\mathbb{1}_{\{F_{i+1}\text{ is empty}\}}\operatorname{Var}_{\nu_i}(f)\big)\\ &\leqslant e^{O(\log(q)^2/q^\alpha)}\frac{1}{\mu(\cap_{i=1}^N H_i)}\mu\big(\mathbb{1}_{\{F_{N+1}\text{ is empty}\}}\sum_{i=1}^N \mu_{\cup_{i=1}^N F_i}(\mathbb{1}_{\{F_{i+1}\text{ is empty}\}}\operatorname{Var}_{\nu_i}(f)\big).\end{aligned}\tag{6.5}$$

Recall that, by construction, $\mu(\cap_{i=1}^N H_i)\geqslant\mu(V_{\pm1,0}$ and $V_{(0,0)}$ are good$)=p_1^3=1-o(1)$.

Thanks to property (c) of the fibers, we can apply Lemma 5.2 with $A:=F_{i+1}, B:=F_i$ and $\mathcal{E}:=H_i$ to obtain that

$$\mu_{\cup_{i=1}^N F_i}\big(\mathbb{1}_{\{F_{i+1}\text{ is empty}\}}\operatorname{Var}_{\nu_i}(f)\big)\leqslant O(n_2)q^{-O(n_2)}\sum_{x\in\hat F_i}\mu_{\cup_{i=1}^N F_i}\big(c_x\operatorname{Var}_x(f)\big),\tag{6.6}$$

where $\hat F_i$ is a suitable $O(1)$-neighborhood of $F_i\cup F_{i+1}$. After inserting (6.6) into (6.5) we finally get that

$$I_i=e^{O(\log(q)^2/q^\alpha)}\sum_{x\in\hat W_i}\mu\big(c_x\operatorname{Var}_x(f)\big),\quad i=1,2,$$

with $\hat W_i$ a $O(1)$-neighborhood of W_i as required.

Remark 6.7. There are two reasons behind our choice of the generalized East chain rather than the generalised FA-1f chain as the fiber chain. The first one is that Proposition 4.10 is an oriented result telling us how infection moves in the u-direction only. The second one is more analytic. If at the beginning of the proof we had used (3.4) instead of (3.3) to reach (6.3), which is what we will do when treating the β-unrooted models, we would have got a version of (6.3) with $\vec T(p_2)$ replaced by $T(p_2)$. The latter is readily seen to be $e^{\widetilde O(1/q^\alpha)}$, namely the same scaling (apart from logarithmic corrections) of the relaxation time of the fiber chain above. In other words, the possible improvement offered by the smaller relaxation time of the FA-1f chain would be irrelevant.

In order to conclude the proof of the proposition, it remains to construct in detail the fibers for each case and to prove the basic inequality (6.4).

Construction of the fibers and proof of (6.4). In the sequel, for simplicity we take the large constant κ, appearing in the definition of n_1,n_2 and in the characterisation of the events G_1,G_2, such that the vector κu has integer components.

We begin by examining the term

$$I_1:=\mu\left(\mathbb{1}_{\{\omega_{\vec e_1}\in G_2\}}\mathbb{1}_{\{\omega_{-\vec e_1}\in G_1\}}\operatorname{Var}_V(f\,|\,G_1)\right).$$

Recall Definition 4.9 and let $\mathcal{R}_*$ be a quasi-stable half-ring in the last quarter of $V_{(1,0)}$ of parameters $(\kappa, \{\ell_i\}_{i=1}^m$ satisfying the requirements of the super-good event (see Definition 6.4) and such that $\partial_\pm \mathcal{R}_* \cap \mathbb{Z}^2 = \emptyset$. Let $N := \min\{j : \mathcal{R}_* + j\kappa u \subset V_{(-1,0)}\}$. With this notation we set

$$F_j = \mathcal{R}_* + (N+1-j)\kappa u,\ j \leqslant N+1$$

and call it the j-th fiber. Because of our choice of κ, $\partial_\pm F_j \cap \mathbb{Z}^2 = \emptyset\ \forall j \in [N+1]$ and $V_{(0,0)} \subset \cup_{j=1}^N F_j$.

Definition 6.8. We say that $F_j, j \in [N]$, is *helping* if for all quasi-stable directions v_i and all v_i-strip S_{v_i} such that $\partial_+ S_{v_i} \cap F_j$ has length at least ℓ_i, there exists an empty helping set for S_{v_i} in F_j. Notice that we do not require S_{v_i} to be contained in F_j.

Denote by $H_j,\ j \in [N]$, the event in the above definition and let $H_{\mathcal{R}_*}$ be the event that the lattice points contained in the rightmost empty quasi-stable half-ring in the last quarter of $V_{(1,0)}$ coincide with the lattice points in $\mathcal{R}_*$. By construction, given $H_{\mathcal{R}_*}$, the events $\{H_j\}_{j=1}^N$ are independent and knowing that the three blocks $V_{(-1,0)}, V_{(0,0)}, V_{(1,0)}$ are good implies $\cap_{j=1}^N H_j$. It follows from Corollary 4.11 that if F_j is helping and F_{j+1} is empty, then the bootstrap map is able to infect F_j. Moreover the same holds for the bootstrap process restricted to a $O(1)$-neighbourhood of $F_j \cup F_{j+1}$. Thus, the fibers $\{F_j\}_{j=1}^{N+1}$ verify the conditions (a),..,(c) of Section 6.1.

Claim 6.9. *We now claim that*

$$I_1 \leqslant (1/p_1) \sum_{\mathcal{R}_*} \mu \left(\mathbb{1}_{\{H_{\mathcal{R}_*}\}} \operatorname{Var}_\Lambda(f \mid \cap_{j=1}^N H_j) \right), \tag{6.7}$$

where $\Lambda = \cup_{j=1}^N F_j$. *Notice that* (6.7) *is exactly* (6.4) *for* I_1.

Proof of the claim. To prove the claim we set $\omega_0 \equiv \omega_{V_{(0,0)}}$ and, for any $\omega \in H_{\mathcal{R}_*}$, we let

$$a = a(\omega_{\mathbb{Z}^2 \setminus \Lambda}) = \mu_\Lambda(f \mid \cap_{j=1}^N H_j).$$

Using the standard inequality $\operatorname{Var}(X) \leqslant \mathbb{E}\big((X-a)^2\big)$ for any $a \in \mathbb{R}$ and any random variable X, the fact that

$$\{V_{(1,0)} \text{ is super-good}\} \subset \cup_{\mathcal{R}_*} H_{\mathcal{R}_*} \cap \{V_{(1,0)} \text{ is good}\}$$
$$\{V_{(\pm 1,0)} \text{ is good}\} \cap H_{\mathcal{R}_*} \subset H_{\mathcal{R}_*} \cap (\cap_{j=1}^N H_j)$$

and that on the event $H_{\mathcal{R}_*}$ the set Λ and the fibers become deterministic, we write

$$\begin{aligned} I_1 &= \mu\left(\mathbb{1}_{\{\omega_{\vec{e}_1}\in G_2\}}\mathbb{1}_{\{\omega_{-\vec{e}_1}\in G_1\}}\operatorname{Var}_V(f\,|\,G_1)\right) \\ &\leqslant (1/p_1)\sum_{\mathcal{R}_*}\mu\left(\mathbb{1}_{\{H_{\mathcal{R}_*}\}}\mathbb{1}_{\{\omega_{\pm\vec{e}_1}\in G_1\}}\mu_V\big((f-a)^2\mathbb{1}_{\{\omega_0\in G_1\}}\big)\right) \\ &\leqslant (1/p_1)\sum_{\mathcal{R}_*}\mu\left(\mathbb{1}_{\{H_{\mathcal{R}_*}\}}\mu_\Lambda\big((f-a)^2\mathbb{1}_{\{\cap_{j=1}^N H_j\}}\big)\right) \\ &\leqslant (1/p_1)\sum_{\mathcal{R}_*}\mu\left(\mathbb{1}_{\{H_{\mathcal{R}_*}\}}\operatorname{Var}_\Lambda\left(f\,|\,\mathbb{1}_{\{\cap_{j=1}^N H_j\}}\right)\right), \end{aligned}$$

where the last inequality follows from our choice of a and the trivial inequality

$$\mu_\Lambda\big((f-a)^2\mathbb{1}_{\{\cap_{j=1}^N H_j\}}\big)\leqslant\mu_\Lambda\big((f-a)^2\,|\,\mathbb{1}_{\{\cap_{j=1}^N H_j\}}\big).$$

□

We now turn to analyse the term $I_2 := \mu\left(\mathbb{1}_{\{\omega_{\vec{e}_2}\in G_2\}}\mathbb{1}_{\{\omega_j\in G_1\,\forall j\in\mathbb{L}^+\}}\operatorname{Var}_V(f)\right)$. In this case we need to slightly modify the definition of the fibers F_j in order to take into account the different local neighborhood W_2 of $V_{(0,0)}$ and the different good and super-good environment in W_2 (see Figure 5).

Recall that $a = \Theta(n_2)$ is the number of rows in a generic block and recall Definition 6.3. Let $\mathcal{M}$ be an upward κ-stair with steps $\{M^{(i)}\}_{i=1}^a$ contained in the first quarter of $V_{(1,0)}$ and assume, w.l.o.g., that the endpoints of the steps of $\mathcal{M}$ do not belong to $\mathbb{Z}^2$. Let $\mathbb{1}_{\{\mathcal{M}\}}$ be the indicator of the event that the set of lattice points contained in the rightmost empty of such stairs coincide with $\mathcal{M}\cap\mathbb{Z}^2$. Let also $\mathcal{R}_*$ and $H_{\mathcal{R}_*}$ be as before with $V_{(1,0)}$ replaced by $V_{(0,1)}$, and let S_{v_1} be the v_1-strip of $\mathcal{R}_*$. Let x_* be the abscissa of leftmost lowermost corner of the v_1-strip (*i.e.,* the first one) of $\mathcal{R}_*$ and assume, w.l.o.g., that $x_* - M_\ell^{(a)} = j_a\kappa$ with $j_a\in\mathbb{N}$. Clearly $j_a = \Theta(n_2)$. We are now ready to define the fibers $\{F_j\}_{j=1}^{N+1}$ in this new setting.

The integer N is chosen as the first integer such that the line through $\partial_- S_{v_1}$ shifted by $N\kappa$ in the u-direction crosses $V_{(-1,0)}$ but not $V_{(0,0)}$. Then we begin to define

$$F_{N+1} = \mathcal{R}_* \quad\text{and}\quad F_{N+1-j} = \mathcal{R}_* + j\kappa u \quad \text{for } j\leqslant j_a.$$

The next fiber F_{N-j_a} has a slightly different shape and it is equal to

$$F_{N-j_a} = F_{N+1-j_a}\cup M^{(a)} + \kappa u.$$

We then repeat the above procedure with $\mathcal{R}_*, S_{v_1}, M^{(a)}, j_a$ replaced by $F_{N-j_a}, S_{v_1}^{(a)}$, $M^{(a-1)}, j_{a-1}$ respectively, where j_{a-1} is defined by $M_\ell^{(a)} - M_r^{(a-1)} = j_{a-1}\kappa$. We continue the same iteration until we have swept all the steps of $\mathcal{M}$. At this stage the fiber $F_{N+1-\sum_{i=1}^a j_i}$ has the shape of an *elongated* quasi-stable half-ring $\hat{\mathcal{R}}$ whose v_1-strip has the lowest short side on the *bottom* side of $V_{(1,0)}$ and the topmost short side inside

$V_{(-1,1)} \cup V_{(0,1)}$ (see Figure 6). Finally we set

$$F_{N+1-\sum_{i=1}^{a} j_i - m} = F_{N+1-\sum_{i=1}^{a} j_i} + m\kappa u, \quad m \in [N - \sum_{i=1}^{a} j_i].$$

It is easy to check that $\partial_\pm F_j \cap \mathbb{Z}^2 = \emptyset$ for all $j \in [N]$ and that

$$F_j \cap (V_{(-1,1)} \cup V_{(0,1)}) = \mathcal{R}_* + (N+1-j)\kappa u.$$

Next we say that F_j is helping if $F_j \cap (V_{(-1,1)} \cup V_{(0,1)})$ is helping (see Definition 6.8). To finish, we need to verify property (c) of Section 6.1 for the fibers we just defined.

For j different from the special values $N - j_a, N - j_{a-1}, \ldots$ that follows again from Proposition 4.10 together with Remark 4.12. For $j = N - j_k, k \in [a]$, in order to apply Proposition 4.10 and Remark 4.12 we need F_j which is helping and $F_j - \kappa u$ which is empty. By construction, $F_j - \kappa u = F_{j+1} \cup M^{(k)}$. Thus, an empty F_{j+1} implies that $F_j - \kappa u$ is empty because, by definition, the steps of the stair $\mathcal{M}$ are empty.

□

6.2. **The β-unrooted case.** In this section we assume that the bilateral difficulty β of the updating rule $\mathcal{U}$ is smaller than 2α. We will prove that, if Assumption 6.1 holds, then there exists a constant $\lambda = \lambda(\mathcal{U})$ such that

$$\mathbb{E}_\mu(\tau_0) \leqslant \frac{T_{\mathrm{rel}}(q,\mathcal{U})}{q} \leqslant \exp\Big(\lambda \cdot \widetilde{O}(q^{-\beta})\Big)$$

for all sufficiently small $q > 0$. Note that the first inequality follows from (2.5), and so, by Definition 2.8, it will suffice to prove that

$$\mathrm{Var}(f) \leqslant \exp\Big(\lambda \cdot \widetilde{O}(q^{-\beta})\Big) \sum_x \mu\big(c_x \,\mathrm{Var}_x(f)\big) \tag{6.8}$$

for some $\lambda = \lambda(\mathcal{U})$ and all local functions f. We will deduce a bound of the form (6.8) starting from the *unoriented* constrained Poincaré inequality (3.4) and using again Corollary 4.11.

By definition there exists an open semicircle $C \subset S^1$ such that $\max(\alpha(v), \alpha(-v)) \leqslant \beta$ for all $v \in C$. As before, let u be the mid-point of C, which we assume w.l.o.g. to belong to the third quadrant, and let the blocks R_i and their side lengths n_1, n_2 be as in Section 6.1, with α replaced by β.

We need to slightly modify the definition of the good and super-good events G_1, G_2 as follows. Let $\mathcal{Q}(C), \mathcal{Q}(-C)$ be the quasi-stable directions for $C, -C$, respectively, let $m^\pm$ be their cardinality and recall the definition of an upward κ-stair.

Definition 6.10.

(1) A block $V_i := R_i \cap \mathbb{Z}^2$ satisfies the good event G_1 iff:

(a) for any quasi-stable direction $v \in \mathcal{Q}(C) \cup \mathcal{Q}(-C)$ and any v-strip S such that $\partial^{\mathrm{ext}} S \cap R_i$ has length at least $n_2/10$ there exists an empty helping set $Z \subset R_i$ for S;

(b) there exist two empty upward κ-stairs, one in the first and one in the last quarter of V.

(2) A block $V_i := R_i \cap \mathbb{Z}^2$ satisfies the super-good event G_2 iff, in addition to satisfying the good event G_1, there exist two empty quasi-stable half-rings, $\mathcal{R}^{\pm}$, the first one relative to $\mathcal{Q}(C)$ (*i.e.,* as before) and the second one relative to $\mathcal{Q}(-C)$, contained in the last and first quarter of V respectively, with height n_2 and parameters $(\kappa, \{\ell_i^{\pm}\}_{i=1}^{m^{\pm}})$ such that $\min_i \ell_i^{\pm} \geqslant n_2/(10m^{\pm})$.

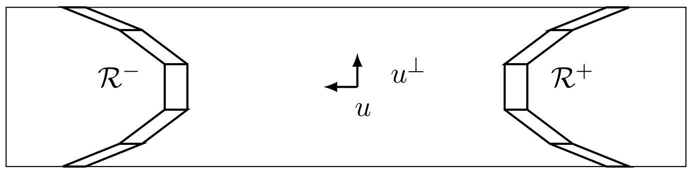


FIGURE 7. The two quasi-stable half-rings $\mathcal{R}^{\pm}$. For simplicity they have been drawn as specular to each other although in general $\mathcal{Q}(-C) \neq -\mathcal{Q}(C)$.

It is easy to check that, with the new definition of the good and super-good events, Lemma 6.5 still holds. Recall now that the unoriented constrained Poincaré inequality 3.4 has the form

$$\operatorname{Var}(f) \leqslant T(p_2) \sum_{i\in\mathbb{Z}^2} \Bigg(\sum_{\varepsilon=\pm1} \mu\left(\mathbb{1}_{\{\omega_{i+\varepsilon\vec{e}_1}\in G_2\}} \mathbb{1}_{\{\omega_{i-\varepsilon\vec{e}_1}\in G_1\}} \operatorname{Var}_i\left(f \,|\, G_1\right)\right)$$
$$+ \sum_{\varepsilon=\pm1} \mu\left(\mathbb{1}_{\{\omega_{i+\varepsilon\vec{e}_2}\in G_2\}} \mathbb{1}_{\{\omega_j\in G_1\,\forall j\in\mathbb{L}^{\varepsilon}(i)\}} \operatorname{Var}_i(f)\right) \Bigg), \qquad (6.9)$$

where $T(p_2) = 1/p_2^{O(1)} = e^{\widetilde{O}(1/q^{\beta})}$ and $\mathbb{L}^+(i) = i + \{\vec{e}_1, \vec{e}_2 - \vec{e}_1\}$, $\mathbb{L}^-(i) = i + \{-\vec{e}_1, \vec{e}_1 - \vec{e}_2\}$. Thus, for any given block V_i, we have now four terms to bound from above:

$$I_1^{\pm} := \mu\left(\mathbb{1}_{\{\omega_{i\pm\vec{e}_1}\in G_2\}} \mathbb{1}_{\{\omega_{i\mp\vec{e}_1}\in G_1\}} \operatorname{Var}_{V_i}(f \,|\, G_1)\right),$$

and

$$I_2^{\pm} := \mu\left(\mathbb{1}_{\{\omega_{V_{i\pm\vec{e}_2}}\in G_2\}} \mathbb{1}_{\{\omega_{V_j}\in G_1\,\forall j\in\mathbb{L}^{\pm}(i)\}} \operatorname{Var}_{V_i}(f)\right).$$

The terms I_1^+, I_2^+ can be treated exactly as the terms I_1, I_2 analysed in the previous section, because the new good and super-good events imply the good and super-good events for the α-rooted case. Therefore I_1^+, I_2^+ satisfy the bound given in Proposition 6.6 with α replaced by β.

The new terms, I_1^-, I_2^-, are illustrated in Figure 8. Since a good block now contains suitable empty helping sets for the quasi-stable directions in C and in $-C$, as well as an empty upward κ-stairs in the first and last quarter of the block, and since a super-good block contains an empty quasi-stable half-ring relative to C in the last quarter and one relative to $-C$ in the first quarter of the block, I_1^-, I_2^- become equal to I_1^+, I_2^+ after a rotation of π of the coordinate axes. Hence they also satisfy the bound given in Proposition 6.6 with α replaced by β.

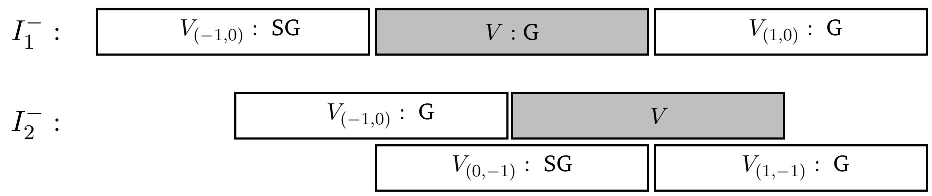


FIGURE 8

In conclusion

$$\sum_{\varepsilon\in\{0,1\}}\sum_{k\in\{1,2\}} I_k^\varepsilon(i) = e^{\widetilde{O}(1/q^\beta)} \sum_{x\in\hat{W}_i} \mu\big(c_x \operatorname{Var}_x(f)\big),$$

with $\hat{V}_i$ a $O(1)$-neighborhood of the block V_i. By summing over $i\in\mathbb{Z}^2$ the above bound and by recalling the scaling of $T(p_2)$ we finally get

$$\operatorname{Var}(f) \leqslant e^{\widetilde{O}(1/q^\beta)}\mathcal{D}(f)$$

as required.

□

7. REMOVING THE SIMPLIFYING ASSUMPTION 6.1

In this section we describe a way to avoid making the simplifying assumption 6.1. For simplicity we only treat the α-rooted case. Our solution requires a slight change in the geometry of the quasi-stable half-ring. As before, in what follows we will always work in the tilted frame $(-u, u^\perp)$, where u is the midpoint of the semicircle C (see e.g. Section 6.1).

Definition 7.1 (Generalised quasi-stable half-ring $\mathcal{R}^g$)**.** Let $\mathcal{R}$ be a quasi-stable half-ring of parameters $(w, \{\ell_i\}_{i=1}^n)$ and let $\{S_{v_i}\}_{i=1}^n$ be the minimal decomposition of $\mathcal{R}$ into v_i-strips (see Definition 4.9). The generalised version of $\mathcal{R}$, denoted $\mathcal{R}^g$, is constructed as follows. For each quasi-stable direction $v_i\in C$ let $\hat{S}_{v_i}$ be a v_i-strip of width $w/3$ and length $\ell_i/3$. Then we set

$$S_{v_i}^g = (S_{v_i}\setminus \hat{S}_{v_i}^r)\cup \hat{S}_{v_i}^l,$$

where $\hat{S}_{v_i}^l, \hat{S}_{v_i}^r$ are two copies of $\hat{S}_{v_i}$ such that (i) they share exactly one corner with S_{v_i}, (ii) $\partial_-\hat{S}_{v_i}^l \subset \partial_+ S_{v_i}$ and $\partial_-\hat{S}_{v_i}^r\subset\partial_- S_{v_i}$, (iii) they stay on the right hand-side while looking in the direction v_i from the midpoint of $\partial_+ S_{v_i}$. Finally we set (see Figure 9a)

$$\mathcal{R}^g = \cup_{i=1}^n S_{v_i}^g,$$

and we call the "core" of $\mathcal{R}^g$ the set $\mathcal{R}^g\cap\mathcal{R}$.

Recall now two key ingredients for the proof given in the previous section (see the third paragraph in Section 6.1) under the simplifying assumption 6.1:

(i) an empty quasi-stable half-ring $\mathcal{R}$ of large enough parameters $(\kappa, \{\ell_i\}_{i=1}^n)$ is able to completely infect its translate $\mathcal{R} + \kappa u$ provided that the latter is helping;

(ii) the event that a quasi-stable half-ring $\mathcal{R}$ is helping is a local event, *i.e.,* it depends only on the spin variables in $\mathcal{R}$.

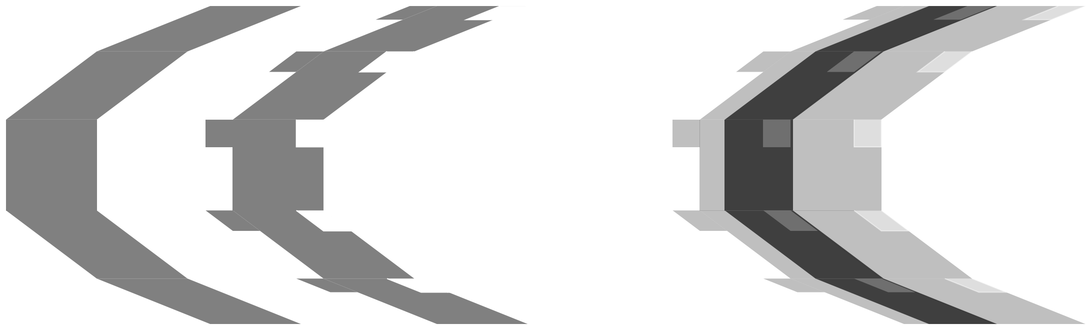
Here we prove a similar result for the generalised quasi-stable half-rings *without* the simplifying assumption.

Definition 7.2. Given a pair $(\mathcal{R}, \mathcal{R}^g)$ of parameters $(\kappa, \{\ell_i\}_{i=1}^n)$, we say that $\mathcal{R}^g$ is helping if, for all quasi-stable directions v_i and all v_i-strips S'_{v_i} of length ℓ_i and such that $\partial_+ S'_{v_i} \subset \mathcal{R}$, there exists an empty helping set for S'_{v_i} in $\mathcal{R}^g$.

Lemma 7.3. *Let $\mathcal{R}^g$ be a generalised quasi-stable half-ring of large enough parameters $(\kappa, \{\ell_i\}_{i=1}^n)$. Assume that the core of $\mathcal{R}^g$ is empty and that $\mathcal{R}^g$, $\mathcal{R}^g + \kappa u$ are helping. Then the bootstrap map is able to infect the core of $\mathcal{R}^g + \kappa u$. Similarly for the bootstrap map restricted to a $O(1)$-neighbourhood of $\mathcal{R}^g \cup (\mathcal{R}^g + \kappa u)$.*

Proof. The proof follows at once from the geometry of the generalised quasi-stable half-rings and Proposition 4.10. Fix $\mathcal{R}^g$ as in the lemma and let $\mathcal{R}$ be its standard version (*i.e.,* remove the sets $\hat{S}^r_{v_i}$ from $\mathcal{R}^g$ and add the sets $\hat{S}^l_{v_i}$). Let also $\mathcal{R}'$ be a quasi-stable half-ring with parameters $(\frac{1}{3}\kappa, \{\ell_i\}_{i=1}^n)$ such that: (a) $\partial_+\mathcal{R}' = \partial_+\mathcal{R} + \lambda u,\ \lambda \geqslant 0$, and (b) $\mathcal{R}' \cup \partial_+\mathcal{R}' \subset \mathcal{R}^g \cup (\mathcal{R}^g + \kappa u)$ (see Figure 9b). Notice that, since $\mathcal{R}'$ is a standard quasi-stable half-ring, $\mathcal{R}' \cup \partial_+\mathcal{R}'$ is necessarily a subset of the union of the core of $\mathcal{R}^g + \kappa u$ and $\mathcal{R}^g$. By construction, either $\partial_+ S'_{v_i} \subset \mathcal{R}$ or $\partial_+ S'_{v_i} \subset \mathcal{R} + \kappa u$ for every v_i-strip S'_{v_i} forming $\mathcal{R}'$. Hence S'_{v_i} has an helping empty set in $\mathcal{R}^g \cup (\mathcal{R}^g + \kappa u)$. Proposition 4.10 implies that if $\mathcal{R}_{int}$ is empty then it can infect $\partial_+\mathcal{R}_{int}$ and advance in the u-direction. Using the assumption that the core of $\mathcal{R}^g$ is empty the proof is finished. $\square$

Given the above lemma, the proof of theorem 2 proceeds exactly as the one given in Section 6 with the provision that in the estimate of the terms I_1, I_2 (or I'_1, I'_2) we define the East like process not for the quasi-stable half-rings but rather for the core of the generalised quasi-stable half-rings.



(A) A quasi-stable half-ring $\mathcal{R}$ (left) and its generalised version $\mathcal{R}^g$ (right). The core of $\mathcal{R}^g$ is $\mathcal{R}^g$ stripped off all the left protuberances (*i.e.,* the sets $\{\hat{S}^l_{v_i}\}_{i=1}^n$) .

(B) The union of $\mathcal{R}^g$ and its translate by κ in the u-direction (in lightgray) together with the half-stable ring $\mathcal{R}'$ with smaller width (dark gray).

Appendix A. Proof of proposition 3.5

We will follow closely the proof of a very similar result proved in [14, Proposition 3.4]. Let $\{P_t\}_{t\geqslant 0}$ be the Markov semigroup associated to either the generalised East chain or the generalised 1-neighbour model and let $\tau_x(\omega)$ be the first legal ring at x when the starting configuration is ω in the graphical construction of the process (see e.g. [14, Section 3.2]). Then, for any function $f : \otimes_{x\in[n]}S_x \mapsto \mathbb{R}$ with $\nu(f) = 0$, we write

$$\|P_t f\|_\infty \leqslant \max_\omega |\mathbb{E}(f(\omega_t)\mathbb{1}_{\{\tau_x(\omega)<t\ \forall x\}})| + \|f\|_\infty n \max_{x\in[n]} \mathbb{P}(\tau_x(\omega) > t), \tag{A.1}$$

where $\mathbb{P}(\cdot)$ and $\mathbb{E}(\cdot)$ denote the probability measure and associated expectation provided by the graphical construction. If $\eta(\omega) = \{\eta_x(\omega)\}_{x\in[n]}$ denotes the collection of the 0-1 variables $\eta_x = \mathbb{1}_{\{\omega_x\in S^g_x\}}$ and $\hat{\tau}_x(\eta)$ is the hitting time of the set $\{\eta' :\ \eta'_x = -\eta_x\}$ for the standard 0-1 East or 1-neighbour process, then clearly $\{\tau_x(\omega) > t\} \subset \{\hat{\tau}_x(\eta(\omega)) > t\}$. Thus

$$\mathbb{P}(\tau_x(\omega) > t) \leqslant \mathbb{P}_\eta(\hat{\tau}_x > t),$$

It follows from [11, Theorem 4.7] that

$$\mathbb{P}_\eta(\hat{\tau}_x(\eta) > t) \leqslant q^{-n}\mathbb{P}_\nu(\hat{\tau}_x > t) \leqslant \begin{cases} q^{-n}e^{-tq/T_{East}(n,\bar{\alpha})} & \text{for the East process,} \\ q^{-n}e^{-tq/T_{FA}(n,\bar{\alpha})} & \text{for the 1-neighbour process,} \end{cases}$$

where the factor q^{-n} comes from $\nu(\eta(\omega) = \eta) \geqslant q^n$. In particular, the inverse of the exponential rate of decay (in t) of the second term in the r.h.s. of (A.1) is smaller than $T_{East}(n,\bar{\alpha})/q$ or $T_{FA}(n,\bar{\alpha})/q$, depending on which of the two models we are considering.

We now analyse the first term in the r.h.s. of (A.1). Conditionally on $\cap_x\{\tau_x < t\}$ and on $\eta(\omega(t))$, the variables $\omega_x(t)$, $x \in [n]$, are independent with law $\nu_x(\cdot \mid \eta_x)$. Thus, if $g(\eta) := \nu(f \mid \eta)$ then

$$\begin{aligned}\mathbb{E}(f(\omega_t)\mathbb{1}_{\{\tau_x(\omega)<t\ \forall x\}}) &= \mathbb{E}\left(g(\eta(t))\mathbb{1}_{\{\tau_x(\omega)<t\ \forall x\}}\right) \\ &= \mathbb{E}\left(g(\eta(t))\right) - \mathbb{E}\left(g(\eta(t))\mathbb{1}_{\{\max_x \tau_x(\omega)>t\}}\right).\end{aligned}$$

The second term in the r.h.s. above can be analysed exactly as the second term in the r.h.s. of (A.1). The first term instead is just the Markov semigroup of the 0-1 East chain or 1-neighbour chain and, as such, its rate of exponential decay in t is either $T_{East}(n,\bar{\alpha})^{-1}$ or $T_{FA}(n,\bar{\alpha})^{-1}$, depending on the chosen model.

In conclusion we have proved that $\|P_t f\|_\infty \leqslant C(f,n,q)e^{-t/t^*}$, with $t^* \leqslant T_{East}(n,\bar{\alpha})/q$ or $t^* \leqslant T_{FA}(n,\bar{\alpha})/q$, depending on which of the two models we are considering. That clearly proves the proposition. $\square$

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