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Abstract. In molecular dynamics, several algorithms have been designed over the past few years to accelerate the sampling of the exit event from a metastable domain, that is to say the time spent and the exit point from the domain. Some of them are based on the fact that the exit event from a metastable region is well approximated by a Markov jump process. In this work, we present recent results on the exit event from a metastable region for the overdamped Langevin dynamics obtained in [22,23,57]. These results aim in particular at justifying the use of a Markov jump process parametrized by the Eyring-Kramers law to model the exit event from a metastable region.

Keywords: Exit event, metastability, Eyring-Kramers and overdamped Langevin.

The objective of this note is to give motivations (Section 1) and outlines of the proofs (Section 2) of results recently obtained in [22, 23, 57]. These results justify the use of the Eyring-Kramers formulas together with a kinetic Monte Carlo model to model the exit event from a metastable state for the overdamped Langevin dynamics. Such results are particularly useful to justify algorithms and models which use such formulas to build reduced description of the overdamped Langevin dynamics.

1 Exit event from a metastable domain and Markov jump process

1.1 Overdamped Langevin dynamics and metastability

Let $(X_t)_{t\geq 0}$ be the stochastic process solution to the overdamped Langevin dynamics in $\mathbb{R}^d$:

$$dX_t = -\nabla f(X_t)dt + \sqrt{h}\, dB_t, \tag{1}$$

where $f \in C^\infty(\mathbb{R}^d, \mathbb{R})$ is the potential function, $h > 0$ is the temperature and $(B_t)_{t\geq 0}$ is a standard d-dimensional Brownian motion. The overdamped

Langevin dynamics can be used for instance to describe the motion of the atoms of a molecule or the diffusion of impurities in a crystal (see for instance [52, Sections 2 and 3] or [10]). The term $-\nabla f(X_t)$ in (1) sends the process towards local minima of f, while thanks to the noise term $\sqrt{h}\, dB_t$, the process X_t may jump from one basin of attraction of the dynamics $\dot{x} = -\nabla f(x)$ to another one. If the temperature is small (i.e. $h \ll 1$), the process $(X_t)_{t\geq 0}$ remains during a very long period of time trapped around a neighborhood of a local minimum of f, called a metastable state, before going to another region. For that reason, the process (1) is said to be metastable. More precisely, a domain $\Omega \subset \mathbb{R}^d$ is said to be metastable for the probability measure μ supported in Ω if, when $X_0 \sim \mu$, the process (1) reaches a local equilibrium in Ω long before escaping from it. This will be made more precise below using the notion of quasi-stationary distribution (see Section 1.5). The move from one metastable region to another is typically related to a macroscopic change of configuration of the system. Metastability implies a separation of timescales which is one of the major issues when trying to have access to the macroscopic evolution of the system using simulations made at the microscopic level. Indeed, in practice, many transitions cannot be observed by integrating directly the trajectories of the process (1). To overcome this difficulty, some algorithms use the fact that the exit event from a metastable region can be well approximated by a Markov jump process with transition rates computed with the Eyring-Kramers formula, see for example the Temperature Accelerated Dynamics method [62] that will be described below.

1.2 Markov jump process and Eyring-Kramers law

Kinetic Monte Carlo methods. Let $\Omega \subset \mathbb{R}^d$ be a domain of the configuration space and let us assume that the process (1) is initially distributed according to the probability measure μ (i.e. $X_0 \sim \mu$) which is supported in Ω and for which the exit event from Ω is metastable. Let us denote by $(\Omega_i)_{i=1,\dots,n}$ the surrounding domains of Ω (see Figure 1), each of them corresponding to a macroscopic state of the system. Many reduced models and algorithms rely on the fact that the exit event from Ω, i.e. the next visited state by the process (1) among the Ω_i's as well as the time spent by the process (1) in Ω, is efficiently approximated by a Markov jump process using kinetic Monte Carlo methods [8, 25, 60, 61, 67, 68]. Kinetic Monte Carlo methods simulate a Markov jump process in a discrete state space. To use a kinetic Monte Carlo algorithm in order to sample the exit event from Ω, one needs for $i \in \{1, \dots, n\}$ the transition rate k_i to go from the state Ω to the state Ω_i. A kinetic Monte Carlo algorithm generates the next visited state Y among the Ω_i's and the time T spent in Ω for the process (1) as follows:

1. First sample T as an exponential random variable with parameter $\sum_{i=1}^n k_i$, i.e.:
$$T \sim \mathcal{E}\Big(\sum_{i=1}^n k_i\Big). \tag{2}$$
2. Then, sample the next visited state Y independently from T, i.e
$$Y \perp\!\!\!\perp T \tag{3}$$

using the following law : for all $i \in \{1, \dots, n\}$,

$$\mathbb{P}\big[Y = i\big] \;=\; \frac{k_i}{\sum_{\ell=1}^n k_\ell}. \tag{4}$$

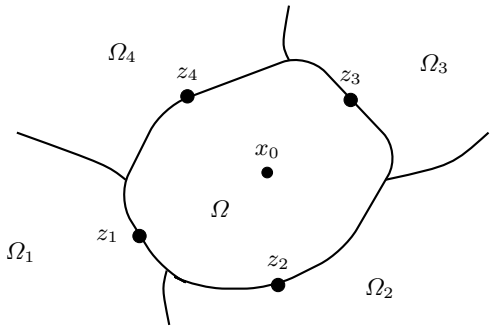

Fig. 1. Representation of the domain Ω, the surrounding domains $(\Omega_i)_{i=1,\dots,4}$ of Ω, the global minimum x_0 of f in Ω and $\{z_i\} = \mathrm{argmin}_{\partial\Omega\cap\Omega_i} f$ ($i \in \{1,2,3,4\}$).

Remark 1. Let us give an equivalent way to sample T and Y in a Monte Carlo method. Let $(\tau_i)_{i\in\{1,\dots,n\}}$ be n independent random variables such that for all $i \in \{1, \dots, n\}$, τ_i is exponentially distributed with parameter k_i. Then, the couple (T, Y) has the same law as $(\min_{j\in\{1,\dots,n\}} \tau_j, \mathrm{argmin}_{j\in\{1,\dots,n\}} \tau_j)$.

Eyring-Kramers law. In practice, the transition rates $(k_i)_{i\in\{1,\dots,n\}}$ are computed using the Eyring-Kramers formula [29, 67]:

$$k_i = A_i \, e^{-\frac{2}{\hbar}(f(z_i)-f(x_0))}, \tag{5}$$

where $x_0 \in \Omega$ is the unique global minimum of f in $\overline{\Omega}$ and $\{z_i\} = \mathrm{argmin}_{\partial\Omega\cap\partial\Omega_i} f$, see Figure 1. We here assume for simplicity that the minimum is attained at one single point z_i but the results below can be generalized to more general settings. If Ω is the basin of attraction of x_0 for the dynamics $\dot{x} = -\nabla f(x)$ so that z_i is a saddle point of f (i.e. a critical point of index 1), then, for the overdamped Langevin dynamics (1), the prefactor A_i writes:

$$A_i = \frac{|\lambda(z_i)|}{2\pi} \frac{\sqrt{\det \mathrm{Hess}\, f(x_0)}}{\sqrt{|\det \mathrm{Hess}\, f(z_i)|}}, \tag{6}$$

where $\lambda(z_i)$ is the negative eigenvalue of the Hessian matrix of f at z_i. Notice that the formula (6) requires that x_0 and z_i are non degenerate critical points of f. The formulas (5) and (6) have been first obtained in the small temperature regime by Kramers [42] (see the review of the literature [29]).

Remark 2. In the Physics literature, the approximation of the macroscopic evolution of the system with a Markov jump process with transition rates computed with the Eyring-Kramers formula (5)-(6) is sometimes called the Harmonic Transition State Theory [48, 64].

1.3 The *temperature accelerated dynamics* algorithm.

The temperature accelerated dynamics (TAD) algorithm proposed by M.R. Sørensen and A.F Voter [62] aims at efficiently approximating the exit event from a metastable domain for the dynamics (1) in order to have access to the macroscopic evolution of the system. We also refer to [1] for a mathematical analysis of this algorithm in a one-dimensional setting.

The basic idea of the TAD algorithm is the following: the exit time from the metastable domain Ω increases exponentially with the inverse of the temperature, see indeed (2)-(5). The idea is then to simulate the process at higher temperature to accelerate the simulation of the exit event. Let us assume that the process $(X_t)_{t\geq 0}$, evolving at the temperature h_{low} is at some time $t_0 \geq 0$ in the domain $\Omega \subset \mathbb{R}^d$ which is metastable for the initial condition $X_{t_0} \in \Omega$. Following [62], let us assume that the process instantaneously reaches the local equilibrium in Ω, i.e. that X_{t_0} is distributed according to this local equilibrium. The existence and the uniqueness of the local equilibrium in Ω as well as the convergence toward this local equilibrium is made more precise in Section 1.5 using the notion of quasi-stationary distribution. To ensure the convergence towards the local equilibrium in Ω, a decorrelation step may be used before running the TAD algorithm, see step (M1) in [1, Section 2.2].
As in the previous section, one denotes by $(\Omega_i)_{i=1,\ldots,n}$ the surrounding domains of Ω (see Figure 1), each of them corresponding to a macroscopic state of the system and, for $i \in \{1,\ldots,n\}$, $\{z_i\} = \mathrm{argmin}_{\partial\Omega\cap\partial\Omega_i} f$. To sample the next visited state among the Ω_i's as well as the time T spent in Ω for the process (1), the TAD algorithm proceeds as follows. Let us introduce $T_{sim} = 0$ (which is the simulation time) and $T_{stop} = +\infty$ (which is the stopping time), and iterate the following steps.

1. Let $(Y_t)_{t\geq T_{sim}}$ be the solution to the evolution equation (1) but for the temperature $h_{high} > h_{low}$, starting from the local equilibrium in Ω at temperature h_{high}. Let $(Y_t)_{t\geq T_{sim}}$ evolve until it leaves Ω and denote by

$$T_{sim} + \tau$$

the first exit time from Ω for the process $(Y_t)_{t\geq T_{sim}}$. Let $j \in \{1,\ldots,n\}$ be such that $Y_{T_{sim}+\tau} \in \partial\Omega_j \cap \partial\Omega$. Then, set $T_{sim} = T_{sim} + \tau$. If it is the first time an exit from Ω through z_j for the process $(Y_t)_{t\geq 0}$ is observed (else one goes directly to the next step), set $\tau_j(h_{high}) = T_{sim}$ and extrapolate the time to $\tau_j(h_{low})$ with the formula

$$\tau_j(h_{low}) = \tau_j(h_{high})\, e^{2\left(\frac{1}{h_{low}} - \frac{1}{h_{high}}\right)(f(z_j)-f(x_0))}, \tag{7}$$

where we recall $x_0 \in \Omega$ is the unique global minimum of f in $\overline{\Omega}$. Then, update the minimum exit time $\tau_{min}(h_{low})$ among the $\tau_j(h_{low})$'s which have been observed so far. Finally, compute a new time T_{stop} so that there is a very small probability (say $\alpha \ll 1$) to observe an exit event from Ω at the temperature h_{high} which, using (7), would change the value of $\tau_{min}(h_{low})$. We refer to [62] or [1] for the computation of T_{stop}.

2. If $T_{sim} \leq T_{stop}$ then go back to the first step starting from the local equilibrium in Ω at time T_{sim}, else go to the next step.
3. Set $T = \tau_{min}(h_{low})$ and $Y = \ell$ where ℓ is such that $\tau_\ell(h_{low}) = \tau_{min}(h_{low})$. Finally, send X_{t_0+T} to Ω_ℓ and evolve the process (1) with the new initial condition X_{t_0+T}.

Remark 3. In [62], when the process $(Y_t)_{t \geq T_{sim}}$ leaves Ω, it is reflected back in Ω and it is then assumed that it reaches instantaneously the local equilibrium in Ω at temperature h_{high}.

Remark 4. One can use a decorrelation step before running the TAD algorithm and the sampling of $Y_{T_{sim}}$ according to the local equilibrium in Ω at the beginning of the step 1 to ensure that the underlying Markov jump process is justified, see [1].

The extrapolation formula (7) which is at the heart of the TAD algorithm relies on the properties of the underlying Markov jump process used to accelerate the exit event from a metastable state and where transition times are exponentially distributed with parameters computed with the Eyring-Kramers formula, see Remark 1 and Equation (5). In the algorithm TAD, it is indeed assumed that the exit event from Ω can be modeled with a kinetic Monte Carlo method where the transition rates are computed with the Eyring-Kramers law (5)-(6). Then, at high temperature, one checks that under this assumption, each $\tau_i(h_{high})$ ($i \in \{1, \ldots, n\}$) is an exponential law of parameter $A_i \, e^{-\frac{2}{h_{high}}(f(z_i)-f(x_0))}$ (see Remark 1). The formula (7) allows to construct for all $i \in \{1, \ldots, n\}$, an exit time $\tau_i(h_{low})$ which is an exponential law of parameter $A_i \, e^{-\frac{2}{h_{low}}(f(z_i)-f(x_0))}$. By considering the couple $(\min_{i \in \{1,\ldots,n\}} \tau_i(h_{low}), \operatorname{argmin}_{i \in \{1,\ldots,n\}} \tau_i(h_{low}))$, one has access to the exit event from Ω (see Remark 1).

Remark 5. There are other algorithms which use the properties of the underlying Markov jump process to accelerate the simulation of the exit event from a metastable state, see for instance [65] and [66].

Our objective is to justify rigorously that a Markov jump process with transition rates computed with the Eyring-Kramers formula (5) can be used to model the exit event from a metastable domain Ω for the overdamped Langevin process (1). Before, let us recall mathematical contributions on the exit event from a domain and on the Eyring-Kramers formula (5).

1.4 Mathematical literature on the exit event from a domain and on the Eyring-Kramers formulas

In the mathematical literature, there are mainly two approaches to the study of the asymptotic behaviour of the exit event from a domain when $h \to 0$: the global approaches and the local approaches.

Global approaches. The global approaches study the asymptotic behaviours in the limit $h \to 0$ of the eigenvalues of the infinitesimal generator

$$L_{f,h}^{(0)} = -\frac{h}{2}\Delta + \nabla f \cdot \nabla \tag{8}$$

of the diffusion (1) on $\mathbb{R}^d$. Let us give for example a result obtained in [6, 7]. To this end, let us assume that the potential $f : \mathbb{R}^d \to \mathbb{R}$ is a Morse function, has m local minima $\{x_1, \ldots, x_m\}$ and that for h small enough $\int_{\mathbb{R}^d} e^{-\frac{2}{h}f} < +\infty$. Let us recall that $\phi : \mathbb{R}^d \to \mathbb{R}$ is a Morse function if all its critical points are non degenerate. For a Morse function $\phi : \mathbb{R}^d \to \mathbb{R}$, we say that x is a saddle point of ϕ if x is a critical point of ϕ such that the Hessian matrix of ϕ at x has exactly one negative eigenvalue (i.e. x is a critical point of ϕ of index 1). Then, from [35], the operator $L_{f,h}^{(0)}$ has exactly m exponentially small eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ when $h \to 0$ with $\lambda_1 = 0 < \lambda_2 \leq \ldots \leq \lambda_m$ (i.e., when $h \to 0$, for all $i \in \{1, \ldots, m\}$, $\lambda_i = O(e^{-\frac{c}{h}})$ for some $c > 0$ independent of h). Moreover, sharp asymptotic estimates can be derived for the eigenvalues $\{\lambda_2, \ldots, \lambda_m\}$. In [6,7], the following results are obtained. Let us assume that $\{x_1\} = \mathrm{argmin}_{\mathbb{R}^d} f$. For $k \in \{2, \ldots, m\}$ and $B_k = \{x \in \{x_1, \ldots, x_m\} \setminus \{x_k\}, f(x) \leq f(x_k)\}$ (i.e. B_k is the set of local minima of f which are lower in energy than x_k), one denotes by $\mathcal{P}(x_k, B_k)$ the set of curves $\gamma \in C^0([0,1], \mathbb{R}^d)$ such that $\gamma(0) = x_k$ and $\gamma(1) \in B_k$. Let us finally assume that:

1. For all $k \in \{2, \ldots, m\}$, there exists a unique saddle point z_k (i.e. a critical point of f of index 1) such that $f(z_k) = \inf_{\gamma \in \mathcal{P}(x_k, B_k)} \sup_{t \in [0,1]} f(\gamma(t))$.
2. The values $\big(f(z_k) - f(x_k)\big)_{k \in \{2,\ldots,m\}}$ are all distinct.

These assumptions imply that the map $x_k \in \{x_2, \ldots, x_m\} \mapsto z_k$ is injective. The set $\{x_2, \ldots, x_m\}$ is then labeled such that the sequence $\big(f(z_k) - f(x_k)\big)_{k \in \{2,\ldots,m\}}$ is strictly decreasing. The previous assumptions also imply the existence of a cascade of events, which occur with different timescales, to go from one local minimum x_k of f to the global minimum x_1 of f in $\mathbb{R}^d$, see for instance Figure 2. Then, one has for $k \in \{2, \ldots, m\}$, in the limit $h \to 0$:

$$\lambda_k = \frac{|\lambda(z_k)|}{2\pi} \frac{\sqrt{\det \mathrm{Hess} f(x_k)}}{|\sqrt{\det \mathrm{Hess} f(z_k)|}} e^{-\frac{2}{h}(f(z_k) - f(x_k))}(1 + o(1)), \tag{9}$$

where $\lambda(z_k)$ is the negative eigenvalue of the Hessian matrix of f at z_k. In the articles [6, 7], using a potential-theoretic approach, the sharp equivalent (9) is obtained and each of the eigenvalues λ_k (for $k \in \{2, \ldots, m\}$) is shown to be the inverse of the average time it takes for the process (1) to go from x_k to B_k. We

also refer to [24] for similar results. In [30], another proof of (9) is given using tools from semi-classical analysis. Let us also mention [54] for a generalization of the results obtained in [30]. Notice that the results presented above do not provide any information concerning the average time it takes for the process (1) to go from the global minimum of f to a local minimum of f when $h \to 0$. One also refers to [44] for generalization of [6,7] for a class of non reversible processes when f has two local minima, and to [11–13, 37, 55] for related results.

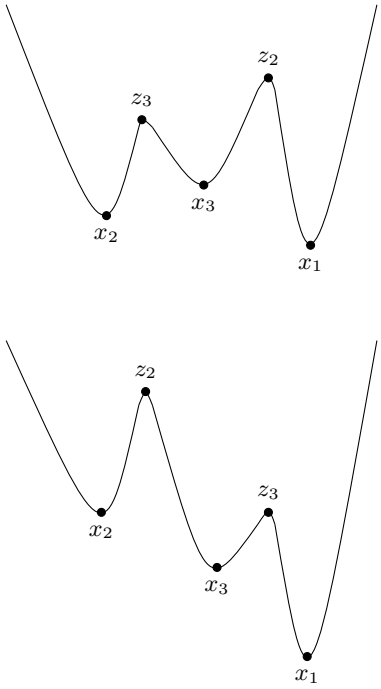

Fig. 2. Examples of two labelings of the local minima $\{x_1, x_2, x_3\}$ of f in dimension one.

Remark 6. The global approaches have been used in [60, 61] to construct a Markovian dynamics by projecting the infinitesimal generator $L^{(0)}_{f,h}$ of the diffusion (1) with a Galerkin method onto the vector space associated with the m small eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$. This projection leads to a very good approximation of $L^{(0)}_{f,h}$ in the limit $h \to 0$. The question is then how to relate the transition events (or the trajectories) of the obtained Markov process to the exit events (or the trajectories) of the original one.

Local approaches. The local approaches consist in studying the asymptotic behaviour when $h \to 0$ of the exit event $(\tau_\Omega, X_{\tau_\Omega})$ from a domain $\Omega \subset \mathbb{R}^d$, where $\tau_\Omega := \inf\{t \geq 0, X_t \notin \Omega\}$.

One of the most well-known approaches is the large deviation theory developed by Freidlin and Wentzell in the 1970s. We refer to the book [26] which summarizes their main contributions. This theory is based on the study of small pieces of the trajectories of the process defined with a suitable increasing sequence of stopping times. The rate function is fundamental in this theory: it quantifies the cost of deviating from a deterministic trajectory when $h \to 0$. The rate functional was first introduced by Schilder [59] for a Brownian motion. Some typical results from [26] (see Theorem 2.1, Theorem 4.1, and Theorem 5.1 there) are the following. Let Ω be a C^∞ open and connected bounded subset of $\mathbb{R}^d$. Let us assume that $\partial_n f > 0$ on $\partial\Omega$ (where ∂_n is the outward normal derivative to Ω) and that f has a unique non degenerate critical point x_0 in Ω such that $f(x_0) = \min_{\overline{\Omega}} f$. Then, for all $x \in \Omega$:

$$\lim_{h\to 0} h \ln \mathbb{E}_x\big[\tau_\Omega\big] = 2\big(\inf_{\partial\Omega} f - f(x_0)\big).$$

The notation $\mathbb{E}_x$ stands for the expectation given the fact that $X_0 = x$. Moreover, let $x \in \Omega$ such that $f(x) < \inf_{\partial\Omega} f$. Then, for any $\gamma > 0$ and $\delta_0 > 0$, there exist $\delta \in (0, \delta_0]$ and $h_0 > 0$ such that for all $h \in (0, h_0)$ and for all $y \in \partial\Omega$:

$$e^{-\frac{2}{h}(f(y)-\inf_{\partial\Omega} f)} e^{-\frac{\gamma}{h}} \leq \mathbb{P}_x\big[|X_{\tau_\Omega} - y| < \delta\big] \leq e^{-\frac{2}{h}(f(y)-\inf_{\partial\Omega} f)} e^{\frac{\gamma}{h}}.$$

The notation $\mathbb{P}_x$ stands for the probability given the fact that $X_0 = x$. Lastly, if the infimum of f on $\partial\Omega$ is attained at one single point $y_0 \in \partial\Omega$, then for all $\delta > 0$:

$$\lim_{h\to 0} \mathbb{P}_x\big[|X_{\tau_\Omega} - y_0| < \delta\big] = 1.$$

A result due to Day [14] (see also [49, 50]) concerning the law of τ_Ω is the following. When $h \to 0$, the exit time τ_Ω converges in law to an exponentially distributed random variable and for all $x \in \Omega$

$$\lim_{h\to 0} \lambda_h \mathbb{E}_x\big[\tau_\Omega\big] = 1,$$

where λ_h is the principal eigenvalue of the infinitesimal generator of the diffusion (1) associated with Dirichlet boundary conditions on $\partial\Omega$ (see Proposition 2 below). The interest of this approach is that it can be applied to very general dynamics. However, when it is used to prove that the Eyring-Kramers formulas (5) can be used to study the exit distribution from Ω, it only provides the exponential rates (not the prefactor A_i in (5)) and does not give error bounds when $h \to 0$.

There are also approaches which are based on techniques developed for partial differential equations. In [51, 52], using formal computations, when $\partial_n f > 0$ on $\partial\Omega$ and f has a unique non degenerate critical point x_0 in Ω such that

$f(x_0) = \min_{\overline{\Omega}} f$, the following formula is derived: for any $F \in C^\infty(\partial\Omega, \mathbb{R})$ and $x \in \Omega$, one has when $h \to 0$:

$$\mathbb{E}_x\big[F\big(X_{\tau_\Omega}\big)\big] = \frac{\int_{\partial\Omega} F(z)\partial_n f(z)\, e^{-\frac{2}{h}f(z)}\, dz}{\int_{\partial\Omega} \partial_n f\, e^{-\frac{2}{h}f}\, d\sigma} + o(1). \tag{10}$$

The formal asymptotic estimate (10) implies that the law of X_{τ_Ω} concentrates on points where f attains its minimum on $\partial\Omega$. Moreover, an asymptotic equivalent of $\mathbb{E}_x\big[\tau_\Omega\big]$ when $h \to 0$ is also formulated in [56] through formal computations. These results are obtained injecting formal asymptotic expansions in powers of h in the partial differential equations satisfied by $x \in \Omega \mapsto \mathbb{E}_x\big[F\big(X_{\tau_\Omega}\big)\big]$ and $x \in \Omega \mapsto \mathbb{E}_x\big[\tau_\Omega\big]$. We also refer to [52], where using formal computations, asymptotic formulas are obtained concerning both the concentration of the law of X_{τ_Ω} on $\operatorname{argmin}_{\partial\Omega} f$ and $\mathbb{E}_x\big[\tau_\Omega\big]$ when Ω is the union of basins of attraction of the dynamics $\frac{d}{dt}\gamma(t) = -\nabla f(\gamma(t))$. When $\partial_n f > 0$ on $\partial\Omega$ and f has a unique non degenerate critical point x_0 in Ω such that $f(x_0) = \min_{\overline{\Omega}} f$, the formula (10) is proved rigorously by Kamin in [40], and is extended to a non reversible diffusion process $(Y_t)_{t\geq 0}$ solution to $dY_t = b(Y_t)\, dt + \sqrt{h}\, dB_t$ in [15, 16, 39, 58] when Ω contains one attractor of the dynamics $\frac{d}{dt}\gamma(t) = b(\gamma(t))$ and $b(x) \cdot n < 0$ for all $x \in \partial\Omega$. However, the results [15,16,39,40,58] do not provide any information on the probability to leave Ω through a point which is not a global minimum of f on $\partial\Omega$.

Finally, let us mention [20, 21, 31, 37, 46, 49, 50] for a study of the asymptotic behaviour in the limit $h \to 0$ of λ_h and u_h (see Proposition 2 below). The reader can also refer to [19] for a review of the different techniques used to study the asymptotic behaviour of X_{τ_Ω} when $h \to 0$ and to [2] for a review of the different techniques used to study the asymptotic behaviour of τ_Ω when $h \to 0$.

Remark 7. Some authors proved the convergence to a Markov jump process in some specific geometric settings and after a rescaling in time. We refer to [41] for a one-dimensional diffusion in a double well and [27,50] for a study in higher dimension. In [63], assuming that all the saddle points of f are at the same height, it is proved that a suitable rescaling of the time leads to a convergence of the diffusion process to a Markov jump process between the global minima of f.

The results presented in this work (see [22, 23]) follow a local approach. The quasi-stationary distribution of the process (1) on Ω is the cornerstone of the analysis. They state that, under some geometric assumptions, the Eyring-Kramers formulas (with prefactors) can be used to model the exit event from a metastable state, and provide explicit error bounds.

1.5 Quasi-stationary distribution and transition rates

Local equilibrium. Let Ω be a C^∞ open bounded connected subset of $\mathbb{R}^d$ and $f \in C^\infty(\overline{\Omega}, \mathbb{R})$. Let us recall that $\tau_\Omega := \inf\{t \geq 0, X_t \notin \Omega\}$ denotes the first

exit time from Ω. The quasi-stationary distribution of the process (1) on Ω is defined as follows.

Definition 1. *A probability measure ν_h on Ω is a quasi-stationary distribution of the process* (1) *on Ω if for all $t > 0$ and any measurable set $A \subset \Omega$,*

$$\mathbb{P}_{\nu_h}\big[X_t \in A \big| t < \tau_\Omega\big] = \nu_h(A).$$

The notation $\mathbb{P}_\mu$ stands for the probability given the fact that the process (1) is initially distributed according to μ i.e. $X_0 \sim \mu$. The next proposition [9, 45] shows that the law of the process (1) at time t conditioned not to leave Ω on the interval $(0, t)$ converges to the quasi-stationary distribution.

Proposition 1. *Let Ω be a C^∞ open connected and bounded subset of $\mathbb{R}^d$ and $f \in C^\infty(\overline{\Omega}, \mathbb{R})$. Then, there exist a unique probability measure ν_h on Ω and $c > 0$ such that for any probability measure μ on Ω, there exist $C(\mu) > 0$ and $t(\mu) > 0$ such that for all $t \geq t(\mu)$ and all measurable set $A \subset \Omega$:*

$$\big|\mathbb{P}_\mu\big[X_t \in A \big| t < \tau_\Omega\big] - \nu_h(A)\big| \leq C(\mu) e^{-ct}. \tag{11}$$

Moreover, ν_h is the unique quasi-stationary distribution of the process (1) *on Ω.*

Proposition 1 indicates that the quasi-stationary distribution ν_h can be seen as a local equilibrium of the process (1) in Ω.

The quasi-stationary distribution ν_h can be expressed with the principal eigenfunction of the infinitesimal generator $L^{(0)}_{f,h}$ (see (8)) of the diffusion (1) associated with Dirichlet boundary conditions on $\partial\Omega$. To this end, let us introduce the following Hilbert spaces $L^2_w(\Omega) = \big\{u : \Omega \to \mathbb{R}, \int_\Omega u^2 e^{-\frac{2}{h}f} < \infty\big\}$ and for $q \in \{1, 2\}$,

$$H^q_w(\Omega) = \big\{u \in L^2_w(\Omega),\, \forall \alpha \in \mathbb{N}^d,\, |\alpha| \leq q,\, \partial_\alpha u \in L^2_w(\Omega)\big\}. \tag{12}$$

The subscript w in the notation $L^2_w(\Omega)$ and $H^q_w(\Omega)$ refers to the fact that the weight function $x \in \Omega \mapsto e^{-\frac{2}{h}f(x)}$ appears in the inner product. Moreover, let us denote by $H^1_{0,w}(\Omega) = \{u \in H^1_w(\Omega),\, u = 0 \text{ on } \partial\Omega\}$. Let us recall the following result [45].

Proposition 2. *Let Ω be a C^∞ open connected and bounded subset of $\mathbb{R}^d$ and $f \in C^\infty(\overline{\Omega}, \mathbb{R})$. Then, the operator $L^{(0)}_{f,h}$ with domain $H^1_{0,w}(\Omega) \cap H^2_w(\Omega)$ on $L^2_w(\Omega)$, which is denoted by $L^{D,(0)}_{f,h}$, is self-adjoint, positive and has compact resolvent. Furthermore, the smallest eigenvalue λ_h of $L^{D,(0)}_{f,h}$ is non degenerate and any eigenfunction associated with λ_h has a sign on Ω.*

In the following, one denotes by u_h an eigenfunction associated with λ_h. The smallest eigenvalue λ_h of $L^{D,(0)}_{f,h}$ is called the principal eigenvalue of $L^{D,(0)}_{f,h}$ and u_h a principal eigenfunction of $L^{D,(0)}_{f,h}$. Without loss of generality, one assumes that

$$u_h > 0 \text{ on } \Omega \quad \text{and} \quad \int_\Omega u_h^2\, e^{-\frac{2}{h}f} = 1. \tag{13}$$

Then, the quasi-stationary distribution ν_h of the process (1) in Ω is given by (see [45]):

$$\nu_h(dx) = \frac{u_h(x)\ e^{-\frac{2}{h}f(x)}}{\int_\Omega u_h\, e^{-\frac{2}{h}f}}\, dx. \tag{14}$$

Moreover, the following result shows that when $X_0 \sim \nu_h$, the law of the exit event $(\tau_\Omega, X_{\tau_\Omega})$ is explicitly known in terms of λ_h and u_h (see [45]).

Proposition 3. *Let us assume that $X_0 \sim \nu_h$, where ν_h is the quasi-stationary distribution of the process (1) in Ω. Then, τ_Ω and X_{τ_Ω} are independent. Moreover, τ_Ω is exponentially distributed with parameter λ_h and for any open set $\Sigma \subset \partial\Omega$, one has:*

$$\mathbb{P}_{\nu_h}\big[X_{\tau_\Omega} \in \Sigma\big] = -\frac{h}{2\lambda_h} \frac{\displaystyle\int_\Sigma \partial_n u_h(z) e^{-\frac{2}{h}f(z)} \sigma(dz)}{\displaystyle\int_\Omega u_h e^{-\frac{2}{h}f}}, \tag{15}$$

where $\sigma(dz)$ is the Lebesgue measure on $\partial\Omega$.

Approximation of the exit event with a Markov jump process. Let us now provide justifications to the use of a Markov jump process with transition rates computed with the Eyring-Kramers formula (5) to model the exit event from a metastable domain Ω. In view of (11), one can be more precise on the definition of the metastability of a domain Ω given in Section 1.1. For a probability measure μ supported in Ω, the domain Ω is said to be metastable if, when $X_0 \sim \mu$, the convergence to the quasi-stationary distribution ν_h in (1) is much quicker than the exit from Ω. Since the process (1) is a Markov process, it is then relevant to study the exit event from Ω starting from the quasi-stationary distribution ν_h, i.e. $X_0 \sim \nu_h$. As a consequence of Proposition 3, the exit time is exponentially distributed and is independent of the next visited state. These two properties are the fundamental features of kinetic Monte Carlo methods, see indeed (2) and (3). It thus remains to prove that the transition rates can be computed with the Eyring-Kramers formula (5). For that purpose, let us first give an expression of the transition rates. Recall that $(\Omega_i)_{i=1,\dots,n}$ denotes the surrounding domains of Ω (see Figure 1). For $i \in \{1, \dots, n\}$, we define the transition rate to go from Ω to Ω_i as follows:

$$k_i^L := \frac{1}{\mathbb{E}_{\nu_h}\big[\tau_\Omega\big]} \mathbb{P}_{\nu_h}\big[X_{\tau_\Omega} \in \partial\Omega \cap \partial\Omega_i\big], \tag{16}$$

where we recall, ν_h is the quasi-stationary distribution of the process (1) in Ω. The superscript L in (16) indicates that the microscopic evolution of the system is governed by the overdamped Langevin process (1). Notice that, using Proposition 3, it holds for all $i \in \{1, \dots, n\}$:

$$\mathbb{P}_{\nu_h}\big[X_{\tau_\Omega} \in \partial\Omega \cap \partial\Omega_i\big] \ = \ \frac{k_i^L}{\sum_{\ell=1}^n k_\ell^L}.$$

Thus, the expressions (16) are compatible with the use of a kinetic Monte Carlo algorithm, see (2) and (4). Indeed, starting from the quasi-stationary distribution ν_h, the exit event from Ω can be exactly modeled using the rates (16): the exit time is exponentially distributed with parameter $\sum_{\ell=1}^n k_\ell^L$, independent of the exit point, and the exit point is in $\partial\Omega_i \cap \partial\Omega$ with probability $k_i^L / \sum_{\ell=1}^n k_\ell^L$. The remaining question is thus following: does the transition rate (16) satisfy the Eyring-Kramers law (5) in the limit $h \to 0$?

Notice that, using Proposition 3, for $i \in \{1, \ldots, n\}$, the transition rate defined by (16) writes:

$$k_i^L = -\frac{h}{2} \frac{\displaystyle\int_{\partial\Omega \cap \partial\Omega_i} \partial_n u_h(z)\, e^{-\frac{2}{h} f(z)} \sigma(dz)}{\displaystyle\int_\Omega u_h\, e^{-\frac{2}{h} f}}, \tag{17}$$

where we recall, u_h is the eigenfunction associated with the principal eigenvalue λ_h of $L_{f,h}^{D,(0)}$.

The remainder of this work is dedicated to the presentation of recent results in [23], [22] and [57] which aim at studying the asymptotic behaviour of the exit event $(\tau_\Omega, X_{\tau_\Omega})$ from a metastable domain Ω in the limit $h \to 0$. In particular, the results give a sharp asymptotic formula of the transition rates (17) when $h \to 0$.

Remark 8. If one wants to recover the expression of the prefactor (6), one has to multiply by $\frac{1}{2}$ the expression (16). This can be explained as follows. Once the process (1) reaches $\partial\Omega \cap \partial\Omega_i$, it has, in the limit $h \to 0$, a one-half probability to come back in Ω and a one-half probability to go in Ω_i. If z_i is a non degenerate saddle point of f, this result is not difficult to prove in dimension 1. Indeed, it is proved in [57, Section A.1.2.2], that when reaching $\partial\Omega \cap \partial\Omega_i$, the probability that the process (1) goes in Ω_i is $\frac{1}{2} + O(h)$ in the limit $h \to 0$. To extend this result to higher dimensions, one can use a suitable set of coordinates around z_i.

2 Main results on the exit event

In all this section, $\Omega \subset \mathbb{R}^d$ is C^∞ open, bounded and connected, and $f \in C^\infty(\overline{\Omega}, \mathbb{R})$ [3]. The purpose of this section is to present recent results obtained in [22] and [23]. Both [22] and [23] are mainly concerned with studying the asymptotic behaviour when $h \to 0$ of the exit law of a domain Ω of the process (1). In [22], when Ω only contains one local minimum of f and $\partial_n f > 0$ on $\partial\Omega$, we obtain sharp asymptotic equivalents when $h \to 0$ of the probability that the process (1) leaves Ω through a subset Σ of $\partial\Omega$ starting from the quasi-stationary distribution or from a deterministic initial condition in Ω. Then, these asymptotic equivalents are used to compute the asymptotic behaviour of

[3] Actually, all the results presented in this section are proved in [22] and [23] in the more general setting: $\overline{\Omega} = \Omega \cup \partial\Omega$ is a C^∞ oriented compact and connected Riemannian manifold of dimension d with boundary $\partial\Omega$.

the transition rates (16). In [23], we explicit a more general setting than the one considered in [22] where we identify the most probable places of exit of Ω as well as their relative probabilities starting from the quasi-stationary distribution or deterministic initial conditions in Ω. More precisely, we consider in [23] the case when Ω contains several local minima of f and $|\nabla f| \neq 0$ on $\partial\Omega$.

2.1 Sharp asymptotic estimates on the exit event from a domain

In this section, we present the results of [22] which give sharp asymptotic estimates on the law of X_{τ_Ω} and on the expectation of τ_Ω when $h \to 0$. These results give in particular the asymptotic estimates of the transition rates $(k_j^L)_{j=1,\dots,n}$ defined in (16).

Geometric setting. Let us give the geometric setting which is considered in this section:

- **[H1]** The function $f : \overline{\Omega} \to \mathbb{R}$ and the restriction of f to Ω, denoted by $f|_{\partial\Omega}$, are Morse functions. Moreover, $|\nabla f|(x) \neq 0$ for all $x \in \partial\Omega$.
- **[H2]** The function f has a unique global minimum x_0 in $\overline{\Omega}$ and

$$\min_{\partial\Omega} f > \min_{\overline{\Omega}} f = \min_{\Omega} f = f(x_0).$$

  The point x_0 is the unique critical point of f in $\overline{\Omega}$. The function $f|_{\partial\Omega}$ has exactly $n \geq 1$ local minima which are denoted by $(z_i)_{i=1,\dots,n}$. They are ordered such that

$$f(z_1) \leq \dots \leq f(z_n).$$

- **[H3]** $\partial_n f(x) > 0$ for all $x \in \partial\Omega$.

Under the assumption **[H2]**, one denotes by $n_0 \in \{1, \dots, n\}$ the number of global minima of $f|_{\partial\Omega}$, i.e.:

$$f(z_1) = \dots = f(z_{n_0}) < f(z_{n_{n_0+1}}) \leq \dots \leq f(z_n).$$

On Figure 3, one gives a schematic representation in dimension 2 of a function f satisfying the assumptions **[H1]**, **[H2]**, and **[H3]**, and of its restriction to $\partial\Omega$, in the case $n = 4$ and $n_0 = 2$.

Remark 9. The assumption **[H1]** implies that f does not have any saddle point (i.e. critical point of index 1) on $\partial\Omega$. Actually, under **[H1]**, **[H2]**, and **[H3]**, the points $(z_i)_{i=1,\dots,n}$ play geometrically the role of saddle points and are called *generalized saddle points* of f on $\partial\Omega$, see [31, Section 5.2]. This can be explained by the fact that, under **[H1]**, **[H2]**, **[H3]** and when f is extended by $-\infty$ outside $\overline{\Omega}$, the points $(z_i)_{i=1,\dots,n}$ are geometrically saddle points of f (the extension of f by $-\infty$ is consistent with the Dirichlet boundary conditions used to define $L_{f,h}^{D,(0)}$) in the following sense. For all $i \in \{1, \dots, n\}$, z_i is a local minimum of $f|_{\partial\Omega}$ and a local maximum of $f|_{D_i}$, where D_i is the straight line passing through z_i and orthogonal to $\partial\Omega$ at z_i.

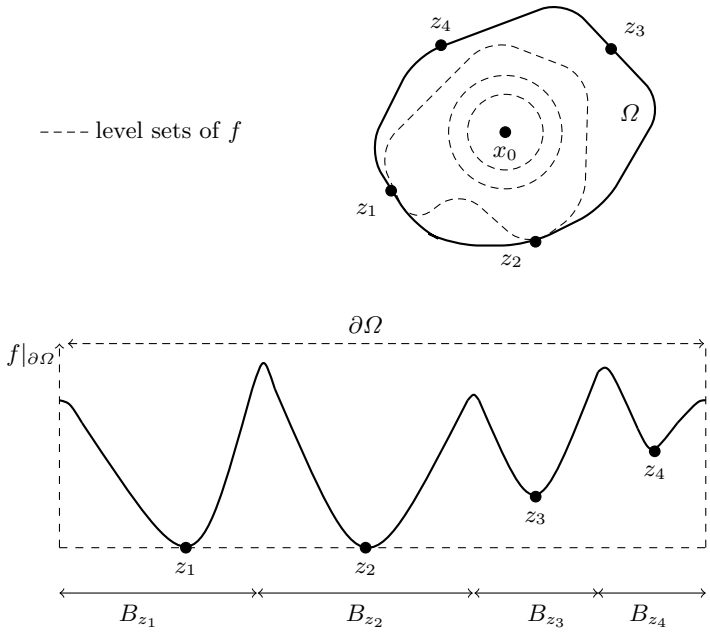

Fig. 3. Schematic representation in dimension 2 of a function f satisfying the assumptions **[H1]**, **[H2]**, and **[H3]**, and of its restriction $f|_{\partial\Omega}$ to $\partial\Omega$. On the figure, $n = 4$ and $n_0 = 2$.

Remark 10. Notice that under **[H1]**, **[H2]**, and **[H3]**, extending f by reflection outside $\overline{\Omega}$ in a neighborhood of z_i also implies that z_i is a geometric saddle point of f as defined in Remark 9. In dimension one, such a construction was considered by Kramers in [43] to derive formulas for transition rates, as explained in [53]. Moreover, as in Remark 8, it can be proved in dimension 1 (exactly as in [57, Section A.1.2.2]), that when reaching $\partial\Omega \cap \partial\Omega_i$, the probability that the process (1) goes in Ω_i is $\frac{1}{2} + O(h)$ when $h \to 0$. To extend this result to higher dimensions, one can use a suitable set of coordinates around z_i.

Let us now define $g : \overline{\Omega} \to \mathbb{R}^+$ by

$$g(x) = \left|\nabla f(x)\right| \text{ when } x \in \Omega \text{ and } g(x) = \left|\nabla_T f(x)\right| \text{ when } x \in \partial\Omega, \tag{18}$$

where $\nabla_T f$ is the tangential gradient of f in $\partial\Omega$. Let us recall that for $x \in \partial\Omega$, $\nabla_T f(x)$ is defined by $\nabla_T f(x) = \nabla f(x) - (\nabla f(x) \cdot n)\, n$, where n is the unit outward normal to $\partial\Omega$ at x. The assumptions one needs to state the results in this section depend on the Agmon distance in $\overline{\Omega}$ between the points $(z_i)_{i=1,\ldots,n}$. The Agmon distance is defined as follows: for any $x \in \overline{\Omega}$ and $y \in \overline{\Omega}$,

$$d_a(x,y) := \inf_{\gamma \in \mathrm{Lip}(x,y)} L(\gamma, (0,1)), \tag{19}$$

where $\mathrm{Lip}(x,y)$ is the set of Lipschitz curves $\gamma : [0,1] \to \overline{\Omega}$ which are such that $\gamma(0) = x$ and $\gamma(1) = y$, and where for $\gamma \in \mathrm{Lip}(x,y)$,

$$L(\gamma,(0,1)) = \int_0^1 g(\gamma(t))|\gamma'(t)|dt.$$

Remark 11. Let us give some common points and differences between the quasipotential V introduced in [26, Section 2] and the Agmon distance (19). Contrary to the quasipotential V, the Agmon distance (19) is symmetric. Moreover, let us consider $x \neq y \in \overline{\Omega}$ such that there exists a curve $\gamma : [0,1] \to \overline{\Omega}$ with $\frac{d}{dt}\gamma(t) = -\nabla f(\gamma(t))$, $\gamma(0) = x$ and $\gamma(1) = y$. Then, the Agmon distance (19) between x and y equals $f(x) - f(y) = V(y,x) > 0$ but $V(x,y) = 0 \neq d_a(x,y)$.

Finally, let us define the following sets. For $i \in \{1,\dots,n\}$, B_{z_i} is the basin of attraction of z_i for the dynamics $\frac{d}{dt}x(t) = -\nabla_T f\big(x(t)\big)$ in $\partial\Omega$, i.e. $B_{z_i} = \{y \in \partial\Omega,\ \lim_{t\to\infty} x(t) = z_i \text{ if } x(0) = y\}$ (see for instance Figure 3). Moreover, one defines for $i \in \{1,\dots,n\}$:

$$B^c_{z_i} := \partial\Omega \setminus B_{z_i}.$$

Main results. Let us now give the main results of this section.

Proposition 4. *Let u_h be the eigenfunction associated with the principal eigenvalue λ_h of $L^{D,(0)}_{f,h}$ which satisfies normalization* (13)*. Let us assume that the hypotheses **[H1]**, **[H2]**, **[H3]** are satisfied. Then, in the limit $h \to 0$, one has:*

$$\lambda_h = \frac{\sqrt{\det\ \mathrm{Hess} f(x_0)}}{\sqrt{\pi h}} \sum_{i=1}^{n_0} \frac{\partial_n f(z_i)}{\sqrt{\det\ \mathrm{Hess} f|_{\partial\Omega}(z_i)}}\ e^{-\frac{2}{h}(f(z_1)-f(x_0))}\,(1+O(h)) \tag{20}$$

and

$$\int_\Omega u_h(x)\ e^{-\frac{2}{h}f(x)}dx = \frac{\pi^{\frac{d}{4}}}{(\det\ \mathrm{Hess} f(x_0))^{1/4}}\ h^{\frac{d}{4}}\ e^{-\frac{1}{h}f(x_0)}(1+O(h)). \tag{21}$$

Furthermore, one obtains the following theorem on the asymptotic behaviour of $\partial_n u_h$, which is one of the main results of [22].

Theorem 1. *Let us assume that **[H1]**, **[H2]**, and **[H3]** are satisfied and that the following inequalities hold:*

$$f(z_1) - f(x_0) > f(z_n) - f(z_1) \tag{22}$$

and for all $i \in \{1,\dots,n\}$,

$$d_a(z_i, B^c_{z_i}) > \max[f(z_n) - f(z_i),\ f(z_i) - f(z_1)]. \tag{23}$$

Let $i \in \{1,\dots,n\}$ and $\Sigma_i \subset \partial\Omega$ be an open set containing z_i and such that $\overline{\Sigma_i} \subset B_{z_i}$. Let u_h be the eigenfunction associated with the principal eigenvalue of $L^{D,(0)}_{f,h}$ which satisfies (13)*. Then, in the limit $h \to 0$:*

$$\int_{\Sigma_i} \partial_n u_h\, e^{-\frac{2}{h}f} = C_i(h)\, e^{-\frac{2f(z_i)-f(x_0)}{h}}\,(1+O(h))\,, \tag{24}$$

where $C_i(h) = -\dfrac{(\det \mathrm{Hess} f(x_0))^{1/4} \partial_n f(z_i) 2\pi^{\frac{d-2}{4}}}{\sqrt{\det \mathrm{Hess} f|_{\partial\Omega}(z_i)}}\, h^{\frac{d-6}{4}}$.

These results have the following consequences.

Corollary 1. *Let us assume that all the assumptions of Theorem 1 are satisfied. Let* $i \in \{1, \ldots, n\}$ *and* $\Sigma_i \subset \partial\Omega$ *be an open set containing* z_i *and such that* $\overline{\Sigma_i} \subset B_{z_i}$*. Then, in the limit* $h \to 0$*:*

$$\mathbb{P}_{\nu_h}\left[X_{\tau_\Omega} \in \Sigma_i\right] = \frac{\partial_n f(z_i)}{\sqrt{\det \mathrm{Hess} f|_{\partial\Omega}(z_i)}} \left(\sum_{k=1}^{n_0} \frac{\partial_n f(z_k)}{\sqrt{\det \mathrm{Hess} f|_{\partial\Omega}(z_k)}} \right)^{-1} \times e^{-\frac{2}{h}(f(z_i) - f(z_1))}(1 + O(h)), \tag{25}$$

where ν_h *is the quasi-stationary distribution of the process* (1) *in* Ω *(see* (14)*). Moreover, if* Σ_i *is the common boundary between the state* Ω *and a state* Ω_i*, then, when* $h \to 0$

$$k_i^L = \frac{1}{\sqrt{\pi h}} \partial_n f(z_i) \frac{\sqrt{\det \mathrm{Hess} f(x_0)}}{\sqrt{\det \mathrm{Hess} f|_{\partial\Omega}(z_i)}} e^{-\frac{2}{h}(f(z_i) - f(x_0))}(1 + O(h)), \tag{26}$$

where k_i^L *is the transition rate* (16) *to go from* Ω *to* Ω_i*.*

Notice that since z_i is not a saddle point of f, the prefactor in (26) is not the prefactor $\frac{1}{2}A_i$ (see Remark 10 for the explanation of the multiplicative term $\frac{1}{2}$), where A_i is defined by (6), but it is actually the expected prefactor for a generalized saddle point of f (see Remarks 9 and 10).

The asymptotic estimate (25) is a consequence of Proposition 4, Theorem 1 together with (15), and (26) is a consequence of Proposition 4, Theorem 1 and (17). The main difficulty is to prove (24) which requires a sharp equivalent of the quantity $\int_{\Sigma_i} \partial_n u_h \, e^{-\frac{2}{h}f}$ when z_i is not a global minimum of f on $\partial\Omega$, i.e. when $i \in \{n_0 + 1, \ldots, n\}$.
In [22], numerical simulations are provided to check that (25) holds and to discuss the necessity of the assumptions (23) to obtain (25). Furthermore, in [22], the results (24) and (25) are generalized to sets $\Sigma \subset \partial\Omega$ which do not necessarily contain a point $z \in \{z_1, \ldots, z_n\}$: this is the other main results of [22] which is not presented here. Moreover, with the help of "leveling" results on the function $x \mapsto \mathbb{E}_x[F(X_{\tau_\Omega})]$, we generalized (25) to deterministic initial conditions in Ω (i.e. when $X_0 = x \in \Omega$) which are the initial conditions considered in the theory of large deviations [26].
The proofs of Proposition 4 and Theorem 1 are based on tools from semi-classical analysis and more precisely, they are based on techniques developed in [31–35,46].

Starting points of the proofs of Proposition 4 and Theorem 1. Let us recall that u_h is the eigenfunction associated with the principal eigenvalue λ_h of $L_{f,h}^{D,(0)}$ which satisfies normalization (13). In view of (15) and in order to obtain (25), one wants to study the asymptotic behaviour when $h \to 0$ of ∇u_h

on $\partial\Omega$. The starting point of the proofs of Proposition 4 and Theorem 1 is the fact that ∇u_h is solution to an eigenvalue problem for the same eigenvalue λ_h. Indeed, recall that u_h is solution to $L^{(0)}_{f,h}\, u_h = \lambda_h u_h$ in Ω and $u_h = 0$ on $\partial\Omega$. If one differentiates this relation, ∇u_h is solution to

$$\begin{cases} L^{(1)}_{f,h} \nabla u_h = \lambda_h \nabla u_h \text{ in } \Omega, \\ \nabla_T u_h = 0 \text{ on } \partial\Omega, \\ \left(-\dfrac{h}{2}\mathrm{div} + \nabla f \cdot \right) \nabla u_h = 0 \text{ on } \partial\Omega, \end{cases} \tag{27}$$

where $L^{(1)}_{f,h} = -\frac{h}{2}\Delta + \nabla f \cdot \nabla + \mathrm{Hess}\, f$ is an operator acting on 1-forms (namely on vector fields). In the following the operator $L^{(1)}_{f,h}$ with tangential boundary conditions (27) is denoted by $L^{D,(1)}_{f,h}$. From (27), ∇u_h is therefore an eigenform of $L^{D,(1)}_{f,h}$ associated with λ_h. For $p \in \{0,1\}$, let us denote, by $\pi^{(p)}_h$ the orthogonal projector of $L^{D,(p)}_{f,h}$ associated with the eigenvalues of $L^{D,(p)}_{f,h}$ smaller than $\frac{\sqrt{h}}{2}$. Another crucial ingredient for the proofs of Proposition 4 and Theorem 1 is the fact that, from [31, Chapter 3],

$$\mathrm{Ran}\, \pi^{(0)}_h = \mathrm{Span}\, u_h \text{ and } \dim \mathrm{Ran}\, \pi^{(1)}_h = n. \tag{28}$$

Therefore, from (27), it holds

$$\nabla u_h \in \mathrm{Ran}\, \pi^{(1)}_h, \tag{29}$$

and from (13) and the fact that $\langle L^{(0)}_{f,h}\, u_h, u_h \rangle_{L^2_w} = \frac{h}{2}\|\nabla u_h\|^2_{L^2_w}$, one has

$$\lambda_h = \frac{h}{2}\|\nabla u_h\|^2_{L^2_w}. \tag{30}$$

Thus, to study the asymptotic behaviour when $h \to 0$ of λ_h, u_h and ∇u_h, we construct a suitable orthonormal basis of $\mathrm{Ran}\, \pi^{(1)}_h$. This basis is constructed using so-called quasi-modes.

Sketch of the proofs of Proposition 4 and Theorem 1. Let us give the sketch of the proof of (25) which is the main result of [22]. Recall that from Proposition 2, one works in the Hilbert space $L^2_w(\Omega)$. The spaces $L^2_w(\Omega)$ and $H^1_w(\Omega)$ (see (12)) extend naturally on 1-forms as follows

$$\Lambda^1 L^2_w(\Omega) := \Big\{ u = {}^t(u_1, \dots, u_d) : \Omega \to \mathbb{R}^d,\, \forall k \in \{1, \dots, d\},\, \int_\Omega u_k^2 e^{-\frac{2}{h}f} < \infty \Big\},$$

and

$$\Lambda^1 H^1_w(\Omega) := \big\{ u = {}^t(u_1, \dots, u_d) : \Omega \to \mathbb{R}^d,\, \forall (i,k) \in \{1, \dots, d\}^2,\, \partial_i u_k \in L^2_w(\Omega) \big\}.$$

In the following, one denotes by $\|.\|_{L^2_w}$ (resp. $\|.\|_{H^1_w}$) the norm of $L^2_w(\Omega)$ and of $\Lambda^1 L^2_w(\Omega)$ (resp. $H^1_w(\Omega)$ and $\Lambda^1 H^1_w(\Omega)$). Finally, $\langle .,. \rangle_{L^2_w}$ stands for both the scalar product associated with the norm of $L^2_w(\Omega)$ and with the norm of $\Lambda^1 L^2_w(\Omega)$. In view of (29) and (28), one has for all orthonormal basis $(\psi_j)_{j\in\{1,\dots,n\}}$ of $\mathrm{Ran}\,\pi_h^{(1)}$, in $L^2_w(\Omega)$:

$$\nabla u_h = \sum_{j=1}^n \langle \nabla u_h, \psi_j \rangle_{L^2_w} \psi_j, \tag{31}$$

and from (30), it holds

$$\lambda_h = \frac{h}{2} \sum_{j=1}^n \left| \langle \nabla u_h, \psi_j \rangle_{L^2_w} \right|^2. \tag{32}$$

In particular, one has for all $k \in \{1, \dots, n\}$,

$$\int_{\Sigma_k} \partial_n u_h \, e^{-\frac{2}{h}f} = \sum_{j=1}^n \langle \nabla u_h, \psi_j \rangle_{L^2_w} \int_{\Sigma_k} \psi_j \cdot n \, e^{-\frac{2}{h}f}, \tag{33}$$

where we recall that Σ_k is an open set of $\partial\Omega$ such that $z_k \in \Sigma_k$ and $\overline{\Sigma_k} \subset B_{z_k}$.

Step 1: approximation of u_h. Under **[H1]**, **[H2]**, and **[H3]**, it is not difficult to find a good approximation of u_h. Indeed, let us consider,

$$\tilde{u} := \frac{\chi}{\|\chi\|_{L^2_w}}, \tag{34}$$

where $\chi \in C_c^\infty(\Omega, \mathbb{R}^+)$ and $\chi = 1$ on $\{x \in \Omega, d(x, \partial\Omega) \geq \varepsilon\}$ where $\varepsilon > 0$. In particular, for ε small enough, $\chi = 1$ in a neighboorhood of x_0 (which is assumed in the following). Let us explain why $\tilde{u}$ is a good approximation of u_h. Since $L^{D,(0)}_{f,h}$ is self adjoint on $L^2_w(\Omega)$, one has

$$\left\| (1 - \pi_h^{(0)}) \tilde{u} \right\|^2_{L^2_w} \leq \frac{C}{\sqrt{h}} \left\langle L^{D,(0)}_{f,h} \tilde{u}, \tilde{u} \right\rangle_{L^2_w} = \frac{Ch}{2\sqrt{h}} \frac{\int_\Omega |\nabla \chi|^2 e^{-\frac{2}{h}f}}{\int_\Omega \chi^2 e^{-\frac{2}{h}f}}.$$

Since $f(x_0) = \min_\Omega f < \min_{\partial\Omega} f$ and x_0 is the unique global minimum of f on $\overline{\Omega}$ (see **[H2]**), one has using Laplace's method (x_0 is a non degenerate critical point of f and $\chi(x_0) = 1$):

$$\int_\Omega \chi^2 e^{-\frac{2}{h}f} = \frac{(\pi h)^{\frac{d}{2}}}{\sqrt{\det \mathrm{Hess} f(x_0)}} e^{-\frac{2}{h}f(x_0)} (1 + O(h)).$$

Therefore, for any $\delta > 0$, choosing ε small enough, it holds when $h \to 0$:

$$\left\| (1 - \pi_h^{(0)}) \tilde{u} \right\|^2_{L^2_w} = O(e^{-\frac{2}{h}(f(z_1) - f(x_0) - \delta)}),$$

and thus:

$$\pi_h^{(0)} \tilde{u} = \tilde{u} + O(e^{-\frac{1}{h}(f(z_1) - f(x_0) - \delta)}) \text{ in } L^2_w(\Omega).$$

From (28) and since $\chi \geq 0$, one has for any $\delta > 0$ (choosing ε small enough), when $h \to 0$

$$u_h = \frac{\pi_h^{(0)} \tilde{u}}{\|\pi_h^{(0)} \tilde{u}\|_{L^2_w}} = \tilde{u} + O(e^{-\frac{1}{h}(f(z_1)-f(x_0)-\delta)}) \text{ in } L^2_w(\Omega). \tag{35}$$

Since $\|\tilde{u}\|_{L^2_w} = 1$, this last relation justifies that $\tilde{u}$ is a good approximation of u_h in $L^2_w(\Omega)$. Notice that (35) implies (21).

Step 2: construction of a basis of $\operatorname{Ran} \pi_h^{(1)}$ *to prove Theorem 1.* In view of (33), the idea is to construct a family of 1-forms $(\widetilde{\psi}_j)_{j\in\{1,\dots,n\}}$ which forms, when projected on $\operatorname{Ran} \pi_h^{(1)}$, a basis of $\operatorname{Ran} \pi_h^{(1)}$ and which allows to obtain sharp asymptotic estimates on $\partial_n u_h$ on all the Σ_j's when $h \to 0$. In the literature, such a 1-form $\widetilde{\psi}_j$ is called a quasi-mode (for $L^{D,(1)}_{f,h}$). A quasi-mode for $L^{D,(1)}_{f,h}$ is a smooth 1-form w such that for some norm, it holds when $h \to 0$:

$$\pi_h^{(1)} w = w + o(1), \tag{36}$$

To prove Theorem 1, one of the major issues is the construction of a basis $(\widetilde{\psi}_j)_{j\in\{1,\dots,n\}}$ so that the remainder term $o(1)$ in (36), when $w = \widetilde{\psi}_k$, is of the order (see (23))

$$\big\|(1-\pi_h^{(1)})\widetilde{\psi}_k\big\|_{H^1_w} = O\big(e^{-\frac{1}{h}\max[f(z_n)-f(z_k),\, f(z_k)-f(z_1)]}\big). \tag{37}$$

This implies that $\big(\pi_h^{(1)}\widetilde{\psi}_j\big)_{j\in\{1,\dots,n\}}$ is a basis of $\operatorname{Ran} \pi_h^{(1)}$ and above all, after a Gram-Schmidt procedure on $\big(\pi_h^{(1)}\widetilde{\psi}_j\big)_{j\in\{1,\dots,n\}}$, when $h \to 0$, that for all $k \in \{1,\dots,n\}$ (see (33)):

$$\int_{\Sigma_k} \partial_n u_h \, e^{-\frac{2}{h}f} = \sum_{j=1}^n \langle \nabla\tilde{u}, \widetilde{\psi}_j\rangle_{L^2_w} \int_{\Sigma_k} \widetilde{\psi}_j \cdot n \, e^{-\frac{2}{h}f} + O\big(e^{-\frac{2f(z_k)-f(x_0)+c}{h}}\big) \tag{38}$$

and (see (32))

$$\lambda_h = \frac{h}{2} \sum_{j=1}^n |\langle \nabla\tilde{u}, \widetilde{\psi}_j\rangle_{L^2_w}|^2 + O\big(e^{-\frac{2}{h}(f(z_1)-f(x_0)+c)}\big) \tag{39}$$

for some $c > 0$ independent of h. Here, we recall, $\tilde{u}$ (see (34)) is a good approximation of u_h (see (35)). Let us now explain how we will construct the family $\big(\widetilde{\psi}_j\big)_{j\in\{1,\dots,n\}}$ in order to obtain (38) and (39). Then, we explain how the terms $\Big(\displaystyle\int_{\Sigma_j} \widetilde{\psi}_j \cdot n \, e^{-\frac{2}{h}f}\Big)_{j\in\{1,\dots,n\}}$ and $\Big(\langle \nabla\tilde{u}, \widetilde{\psi}_j\rangle_{L^2_w}\Big)_{j\in\{1,\dots,n\}}$ appearing in (38) and (39) are computed.

Step 2a: construction of the family $(\widetilde{\psi}_j)_{j\in\{1,\dots,n\}}$. To construct each 1-form $\widetilde{\psi}_j$, the idea is to construct an operator $L^{(1)}_{f,h}$ with mixed tangential Dirichlet and

Neumann boundary conditions on a domain $\dot{\Omega}_j \subset \Omega$ which is such that $\big(\{z_1,\dots,z_n\} \cup \{x_0\}\big) \cap \dot{\Omega}_j = \{z_j\}$. For $j \in \{1,\dots,n\}$, $\widetilde{\psi}_j$ is said to be associated with the generalized saddle point z_j. The goal of the boundary conditions is to ensure that when $h \to 0$, each of these operators has only one exponentially small eigenvalue (i.e. this eigenvalue is $O\big(e^{-\frac{c}{h}}\big)$ for some $c > 0$ independent of h), the other eigenvalues being larger than $\sqrt{h}$. Then, we show that each of these small eigenvalues actually equals 0 using the Witten complex structure associated with these boundary conditions on $\partial\dot{\Omega}_j$. To construct such operators $L^{(1)}_{f,h}$ with mixed boundary conditions on $\dot{\Omega}_j$, the recent results of [38] and [28] are used. The 1-form $\widetilde{\psi}_j$ associated with z_j is then defined using an eigenform $v^{(1)}_{h,j}$ associated with the eigenvalue 0 of the operator $L^{(1)}_{f,h}$ associated with mixed boundary conditions on $\dot{\Omega}_j$:

$$\widetilde{\psi}_j := \frac{\chi_j\, v^{(1)}_{h,j}}{\|\chi_j\, v^{(1)}_{h,j}\|_{L^2_w}}, \tag{40}$$

where χ_j is a well chosen cut-off function with support in $\overline{\dot{\Omega}_j}$. Notice that for $j \in \{1,\dots,n\}$, the quasi-mode $\widetilde{\psi}_j$ is not only constructed in a neighbourhood of z_j: it has a support as large as needed in Ω. This is a difference with previous construction in the literature, such as [31]. We need such quasi-modes for the following reasons. Firstly, we compute the probability that the process (1) leaves Ω through open sets Σ_j which are arbitrarily large in B_{z_j}. Secondly, we use the fact that the quasi-mode $\widetilde{\psi}_j$ decreases very fast away from z_j to get (37). This is needed to state the hypothesis (23) in terms of Agmon distances, see next step.

Step 2b: Accuracy of the quasi-mode $\widetilde{\psi}_j$ for $j \in \{1,\dots,n\}$. To obtain a sufficiently small remainder term in (36) (to get (37) and then (38)), one needs to quantify the decrease of the quasi-mode $\widetilde{\psi}_j$ outside a neighboorhood of z_j. This decrease is obtained with Agmon estimates on $v^{(1)}_{h,j}$ which allow to localize $\widetilde{\psi}_j$ in a neighboorhood of z_j. For $j \in \{1,\dots,n\}$, we prove the following Agmon estimate on $v^{(1)}_{h,j}$:

$$\big\|\chi_j\, v^{(1)}_{h,j} e^{\frac{1}{h} d_a(.,z_j)}\big\|_{H^1_w} = O(h^{-N}), \tag{41}$$

for some $N \in \mathbb{N}$ and where d_a is the Agmon distance defined in (19). To obtain (41), we study the properties of this distance. The boundary of Ω introduces technical difficulties. The Agmon estimate (41) is obtained adapting to our case techniques developed in [31, 46]. For all $j \in \{1,\dots,n\}$, using the fact that $\big\|(1-\pi^{(1)}_h)\widetilde{\psi}_j\big\|^2_{L^2_w} \leq \frac{C}{\sqrt{h}} \big\langle L^{D,(1)}_{f,h}\widetilde{\psi}_j, \widetilde{\psi}_j\big\rangle_{L^2_w}$ and (41), one shows that

$$\big\|(1-\pi^{(1)}_h)\widetilde{\psi}_j\big\|^2_{L^2_w} \leq C\, h^{-q}\, e^{-\frac{2}{h}\inf_{\mathrm{supp}\nabla\chi_j} d_a(.,z_j)},$$

for some $q > 0$. Thus, in order to get (37), the support of $\nabla\chi_j$ has to be arbitrarily close to x_0 and $B^c_{z_j}$. This explains the assumptions (22) and (23), and

the fact that the quasi-mode $\widetilde{\psi}_j$ is not constructed in a neighboorhood of z_j but in a domain $\dot{\Omega}_j$ arbitrarily large in Ω. This is one of the main differences compared with [31]. At the end of this step, one has a family $(\widetilde{\psi}_j)_{j\in\{1,\dots,n\}}$ which satisfies (37). This allows us to obtain, in the limit $h \to 0$ (see (38)), for some $c > 0$ independent of h and for all $k \in \{1, \dots, n\}$:

$$\int_{\Sigma_k} \partial_n u_h \, e^{-\frac{2}{h}f} = \sum_{j=1}^{n} \langle \nabla \tilde{u}, \widetilde{\psi}_j \rangle_{L^2_w} \int_{\Sigma_k} \widetilde{\psi}_j \cdot n \, e^{-\frac{2}{h}f} + O\big(e^{-\frac{2f(z_k)-f(x_0)+c}{h}}\big).$$

Etape 3: computations of $\Big(\int_{\Sigma_j} \widetilde{\psi}_j \cdot n \, e^{-\frac{2}{h}f} \Big)_{j\in\{1,\dots,n\}}$ *and* $\Big(\langle \nabla \tilde{u}, \widetilde{\psi}_j \rangle_{L^2_w} \Big)_{j\in\{1,\dots,n\}}$. In view of (38) and (39), for all $j \in \{1, \dots, n\}$, one needs to compute the terms

$$\int_{\Sigma_j} \widetilde{\psi}_j \cdot n \, e^{-\frac{2}{h}f} \text{ and } \langle \nabla \tilde{u}, \widetilde{\psi}_j \rangle_{L^2_w}.$$

To do that, we use for all $j \in \{1, \dots, n\}$ a WKB approximation of $v^{(1)}_{h,j}$, denoted by $v^{(1)}_{z_j,wkb}$. In the literature we follow, $v^{(1)}_{z_j,wkb}$ is constructed in a neighboorhood of z_j (see [31, 46]). To prove Theorem 1, we extend the construction of $v^{(1)}_{z_j,wkb}$ to neighbourhoods in $\overline{\Omega}$ of arbitrarily large closed sets included in B_{z_j} (indeed, there is no restriction on the size of Σ_j in B_{z_j}). Then, the comparison between $v^{(1)}_{h,j}$ and $v^{(1)}_{z_j,wkb}$ is also extended to neighbourhoods in $\overline{\Omega}$ of arbitrarily large closed sets included in B_{z_j}. Once the terms $\Big(\int_{\Sigma_j} \widetilde{\psi}_j \cdot n \, e^{-\frac{2}{h}f} \Big)_{j\in\{1,\dots,n\}}$ and $\Big(\langle \nabla \tilde{u}, \widetilde{\psi}_j \rangle_{L^2_w} \Big)_{j\in\{1,\dots,n\}}$ are computed, one concludes the proof of (20) using (39) and the proof of (24) using (38).

2.2 Most probable exit points from a bounded domain

Setting and motivation. In this section, we present recent results from [23] on the concentration of the law of X_{τ_Ω} on a subset of $\operatorname{argmin}_{\partial\Omega} f = \{z \in \partial\Omega,\ f(z) = \min_{\partial\Omega} f\}$ when $h \to 0$ in a more general geometric setting than the one of Theorem 1. The main purpose of these results is to prove an asymptotic formula when $h \to 0$ for the concentration of the law of X_{τ_Ω} on a set of points of $\operatorname{argmin}_{\partial\Omega} f$ when Ω contains several local minima of f and when $\partial_n f$ is not necessarily positive on $\partial\Omega$.

Let $\mathcal{Y} \subset \partial\Omega$. We say that the law of X_{τ_Ω} concentrates on $\mathcal{Y}$ if for all neighborhood $\mathcal{V}_\mathcal{Y}$ of $\mathcal{Y}$ in $\partial\Omega$, one has

$$\lim_{h\to 0} \mathbb{P}\left[X_{\tau_\Omega} \in \mathcal{V}_\mathcal{Y}\right] = 1,$$

and if for all $x \in \mathcal{Y}$ and all neighborhood $\mathcal{V}_x$ of x in $\partial\Omega$, it holds:

$$\lim_{h\to 0} \mathbb{P}\left[X_{\tau_\Omega} \in \mathcal{V}_x\right] > 0.$$

In [51, 52, 56], when $\partial_n f(x) = 0$ for all $x \in \partial\Omega$ or when $\partial_n f(x) > 0$ for all $x \in \partial\Omega$ (and with additional assumptions on f), it has been shown that the law of X_{τ_Ω} concentrates on points where f attains its minimum on $\partial\Omega$ (see (10)). Later on, it has been proved in [15, 16, 39, 40, 58] when $\partial_n f > 0$ on $\partial\Omega$ and f has a unique non degenerate critical point in Ω (which is necessarily its global minimum in $\overline{\Omega}$). Tools developed in semi-classical analysis allow us to generalize this geometric setting. For instance, we consider several critical points of f in Ω and we drop the assumptions $\partial_n f > 0$ on $\partial\Omega$ (however we do not consider the case when f has saddle points on $\partial\Omega$). Assuming that f and $f|_{\partial\Omega}$ are Morse functions, and $|\nabla f| \neq 0$ on $\partial\Omega$, we raise the following questions:

- What are the geometric conditions ensuring that, when $X_0 \sim \nu_h$, the law of X_{τ_Ω} concentrates on points where f attains its minimum on $\partial\Omega$ (or a subset of these points)?
- What are the conditions which ensure that these results extend to some deterministic initial conditions in Ω ?

The results of [23] aim at answering these questions. Let us recall that when f and $f|_{\partial\Omega}$ are Morse functions and when $|\nabla f| \neq 0$ on $\partial\Omega$, the elements of the set

$$\{z \text{ is a local minimim of } f|_{\partial\Omega}\} \cap \{z \in \partial\Omega,\, \partial_n f(z) > 0\} \tag{42}$$

are the generalized saddle points of f on $\partial\Omega$ and play the role of saddle points of f on $\partial\Omega$, see Remark 9. Before stating the main results of [23], let us discuss the two questions above with one-dimensional examples.

Remark 12. The assumption that the drift term b in (1) is of the form $b = -\nabla f$ is essential here to the existence of a limiting exit distribution of Ω when $h \to 0$. If it is not the case and when for instance the boundary of Ω is a periodic orbit of the dynamics $\frac{d}{dt}x(t) = b\big(x(t)\big)$, the phenomenon of cycling discovered by Day in [17, 18] prevents the existence of a limiting exit distribution when $h \to 0$. We also refer to [3–5] for the study of this phenomenon of cycling.

One-dimensional examples. To discuss the two questions raised in the previous section, one considers two one-dimensional examples.

Example 1. The goal is here to construct a one-dimensional example for which, starting from the global minimum of f in Ω or from the quasi-stationary distribution ν_h, the law of X_{τ_Ω} does not concentrate on points where f attains its minimum on $\partial\Omega$. To this end, let us consider the function f represented in Figure 4 for which one has the following result.

Proposition 5. *Let $z_1 < z_2$ and $f \in C^\infty([z_1, z_2], \mathbb{R})$ be a Morse function. Let us assume that $f(z_1) < f(z_2)$, $\{x \in [z_1, z_2], f'(x) = 0\} = \{c, x_1\}$ with $z_1 < c < x_1 < z_2$ and $f(x_1) < f(z_1) < f(z_2) < f(c)$ (see Figure 4). Then, for all $x \in (c, z_2]$, there exists $\varepsilon > 0$ such that when $h \to 0$:*

$$\mathbb{P}_x[X_{\tau_{(z_1,z_2)}} = z_1] = O(e^{-\frac{\varepsilon}{h}}) \text{ and thus } \mathbb{P}_x[X_{\tau_{(z_1,z_2)}} = z_2] = 1 + O(e^{-\frac{\varepsilon}{h}}).$$

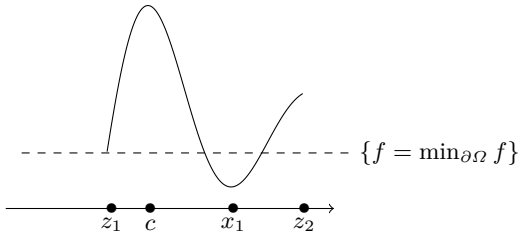

Fig. 4. Example of a function f such that, starting from the global minimum x_1 of f in Ω or from the quasi-stationary distribution ν_h, the law of X_{τ_Ω} concentrates on z_2 whereas $f(z_2) > \min_{\partial\Omega} f = f(z_1)$.

Moreover, there exists $\varepsilon > 0$ such that when $h \to 0$:

$$\mathbb{P}_{\nu_h}[X_{\tau_{(z_1,z_2)}} = z_1] = O(e^{-\frac{\varepsilon}{h}}) \quad \textit{and thus} \quad \mathbb{P}_{\nu_h}[X_{\tau_{(z_1,z_2)}} = z_2] = 1 + O(e^{-\frac{\varepsilon}{h}}),$$

where ν_h is the quasi-stationary distribution of the process (1) *in* (z_1, z_2).

The proof of Proposition 5 is based on the fact that in one dimension, explicit formulas can be written for $x \mapsto \mathbb{P}_x[X_{\tau_{(z_1,z_2)}} = z_j]$ ($j \in \{1, 2\}$), see [57, Section A.5.3.1] or [23]. According to Proposition 5, when $h \to 0$ and when $X_0 = x \in (c, z_2)$ or $X_0 \sim \nu_h$, the process (1) leaves $\Omega = (z_1, z_2)$ through z_2. However, the generalized saddle point z_2 (see (42)) is not the global minimum of f on $\partial\Omega$. This fact can be explained as follows: the potential barrier $f(c) - f(x_1)$ is larger than the potential barrier $f(z_2) - f(x_1)$. Thus, the law of X_{τ_Ω} when $X_0 = x \in (c, z_2)$ cannot concentrate on z_1 since it is less costly to leave Ω through z_2 rather than to cross the barrier $f(c) - f(x_1)$ to exit through z_1. Moreover, it can be proved that the quasi-stationary distribution ν_h concentrates in any neighborhood of x_1 in the limit $h \to 0$, which explains why the law of X_{τ_Ω} when $X_0 \sim \nu_h$ also concentrates on z_2. Concerning the two questions raised in the previous section, this example indicates that in the small temperature regime, there exist cases for which the process (1), starting from the global minimum of f in Ω or from ν_h, leaves Ω through a point which is not a global minimum of $f|_{\partial\Omega}$.

This example also suggests the following. If one wants the law of X_{τ_Ω} to concentrate when $h \to 0$ on points in $\partial\Omega$ where f attains its minimum, one should exclude cases when the largest timescales for the diffusion process in Ω are not related to energetic barriers involving points of $\partial\Omega$ where $f|_{\partial\Omega}$ attains its minimum. In order to exclude such cases, we will assume in the following that the closure of each of the connected components of $\{f < \min_{\partial\Omega} f\}$ intersects $\partial\Omega$.

Notice that if one modifies the function f in the vicinity of z_1 such that $\partial_n f(z_1) > 0$ and $\operatorname{argmin}_{\overline{\Omega}} f = \{x_1\}$, z_1 is then a generalized order one saddle point and the previous conclusions remain unchanged.

Example 2. Let us construct a one-dimensional example for which the concentration of the law of X_{τ_Ω} on $\operatorname{argmin}_{\partial\Omega} f$ is not the same starting from the global

minima of f in Ω or from the quasi-stationary distribution ν_h. For this purpose, let us consider $z_1 > 0,\ z_2 := -z_1,\ z = 0$ and $f \in C^\infty([z_1, z_2], \mathbb{R})$ such that

$$f \text{ is a Morse and even function, } \{x \in [z_1, z_2], f'(x) = 0\} = \{x_1, z, x_2\}, \quad (43)$$

where

$$z_1 < x_1 < z < x_2 < z_2,\ f(z_1) = f(z_2) > f(x_1) = f(x_2),\ f(z_1) < f(z). \quad (44)$$

A function f satisfying (43) and (44) is represented in Figure 5. One has the following result.

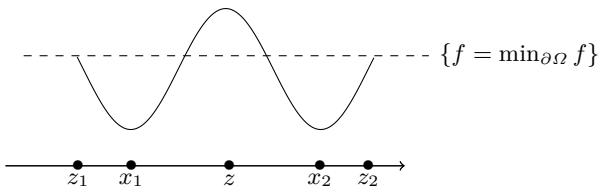

Fig. 5. One-dimensional example where (43) and (44) are satisfied.

Proposition 6. *Let $z_1 > 0$, $z_2 := -z_1$, $z = 0$ and $f \in C^\infty([z_1, z_2], \mathbb{R})$ which satisfies* (43) *and* (44)*. Then, one has for all $h > 0$,*

$$\mathbb{P}_{\nu_h}[X_{\tau_{(z_1,z_2)}} = z_1] = \frac{1}{2} \quad \text{and} \quad \mathbb{P}_{\nu_h}[X_{\tau_{(z_1,z_2)}} = z_2] = \frac{1}{2}, \quad (45)$$

where ν_h is the quasi-stationary distribution of the process (1) *in (z_1, z_2). Moreover, for all $x \in (z_1, z)$, there exists $c > 0$ such that when $h \to 0$,*

$$\mathbb{P}_x[X_{\tau_{(z_1,z_2)}} = z_1] = 1 + O(e^{-\frac{c}{h}}) \quad \text{and} \quad \mathbb{P}_x[X_{\tau_{(z_1,z_2)}} = z_2] = O(e^{-\frac{c}{h}}), \quad (46)$$

and for all $x \in (z, z_2)$, there exists $c > 0$ such that when $h \to 0$

$$\mathbb{P}_x[X_{\tau_{(z_1,z_2)}} = z_1] = O(e^{-\frac{c}{h}}) \quad \text{and} \quad \mathbb{P}_x[X_{\tau_{(z_1,z_2)}} = z_2] = 1 + O(e^{-\frac{c}{h}}). \quad (47)$$

The asymptotic estimate (45) is a consequence of the fact that f is an even function (see [23, Section 1]). The asymptotic estimates (46) and (47) are proved exactly as Proposition 5, see [23, Section 1]. Let us also mention that Proposition 6 is a consequence of the results [47]. Concerning the two questions raised in the previous section, Proposition 6 shows that, when f satisfies (43) and (44), the concentration of the law of X_{τ_Ω} on $\{z_1, z_2\}$ is not the same starting from $x \in (z_1, z_2) \setminus \{z\}$ or from ν_h. This is due to the fact that in this case the quasi-stationary distribution ν_h has an equal repartition in all disjoint neighboorhoods of x_1 and x_2, i.e. for every $(a_1, b_1) \subset (z_1, z)$ and $(a_2, b_2) \subset (z, z_2)$ such that

$a_1 < x_1 < b_1$ and $a_2 < x_2 < b_2$, it holds for any $j \in \{1,2\}$, $\lim_{h\to 0} \int_{a_j}^{b_j} \nu_h = \frac{1}{2}$ (see [47]). When $X_0 = x \in (z_1, z_2) \setminus \{z\}$, the asymptotic estimates (46) and (47) can be explained by the existence of a barrier $f(z) - f(x_1)$ which is larger than $f(z_1) - f(x_1)$. In order to exclude such cases, we will assume in the following that there exists a connected component C of $\{f < \min_{\partial\Omega} f\}$, such that $\operatorname{argmin}_{\overline{\Omega}} f \subset C$.

Main results on the exit point distribution. In this section, a simplified version of the results of [23] is presented. The aim is to exhibit a simple geometric setting for which, on the one hand, the law of X_{τ_Ω} concentrates on the same points of $\partial\Omega$ when $X_0 \sim \nu_h$ or $X_0 = x \in \Omega$ for some $x \in \{f < \min_{\partial\Omega} f\}$ and, on the other hand, this concentration occurs on generalized saddle points of f which belong to $\operatorname{argmin}_{\partial\Omega} f$. To this end, let us define the two following assumptions:

- **[H-Morse]** The function $f : \overline{\Omega} \to \mathbb{R}$ is C^∞. The functions $f : \overline{\Omega} \to \mathbb{R}$ and $f|_{\partial\Omega}$ are Morse functions. Moreover, $|\nabla f|(x) \neq 0$ for all $x \in \partial\Omega$.

- **[H-Min]** The open set $\{f < \min_{\partial\Omega} f\}$ is nonempty, contains all the local minima of f in Ω and the closure of each of the connected components of $\{f < \min_{\partial\Omega} f\}$ intersects $\partial\Omega$. Furthermore, there exists a connected component C of $\{f < \min_{\partial\Omega} f\}$ such that $\operatorname{argmin}_{\overline{\Omega}} f \subset C$.

Notice that under **[H-Morse]** and **[H-Min]**, it holds $\min_{\partial\Omega} f > \min_{\overline{\Omega}} f = \min_{\Omega} f$. Under the assumptions **[H-Morse]** and **[H-Min]**, one defines the set of points $\{z_1, \ldots, z_{k_0}\}$ by

$$\overline{C} \cap \partial\Omega = \{z_1, \ldots, z_{k_0}\}. \tag{48}$$

Remark 13. As already explained, the points $z_1, \ldots, z_{k_0}$ are generalized saddle points of f on $\partial\Omega$ (see (42)) since they satisfy

$$\{z_1, \ldots, z_{k_0}\} \subset \{z \in \partial\Omega,\, \partial_n f(z) > 0\} \cap \operatorname{argmin}_{\partial\Omega} f. \tag{49}$$

Remark 14. Under **[H-Min]**, the normal derivative of f can change sign and the function f can have saddle points in Ω higher than $\min_{\partial\Omega} f$, see for instance Figure 6.

As shown in the following theorem, the assumption **[H-Min]** ensures that the quasi-stationary distribution ν_h concentrates in neighborhoods of the global minima of f in C and, starting from $x \in C$ or from ν_h, that the concentration of the law of X_{τ_Ω} when $h \to 0$ occurs on the set of generalized saddle points $\{z_1, \ldots, z_{k_0}\}$ (see (48)). Notice that the assumption **[H-Min]** is not satisfied in the two examples given in the previous section (see Figures 4 and 5).

Theorem 2. *Let us assume that the hypotheses **[H-Morse]** and **[H-Min]** are satisfied. Let ν_h be the quasi-stationary distribution of the process* (1) *in Ω*

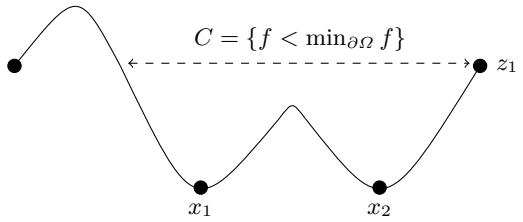

Fig. 6. A one-dimensional example where **[H-Morse]** and **[H-Min]** are satisfied, the normal derivative of f changes sign and the function f has a saddle point in Ω higher than $\min_{\partial\Omega} f$. In this example, $\{f < \min_{\partial\Omega} f\}$ is connected and thus $C = \{f < \min_{\partial\Omega} f\}$. Moreover, $\overline{C} \cap \partial\Omega = \{z_1\}$.

(see (14)*). Let* $\mathcal{V}$ *be an open subset of* Ω*. Then, if* $\mathcal{V} \cap \operatorname{argmin}_C f \neq \emptyset$*, one has in the limit* $h \to 0$*:*

$$\nu_h(\mathcal{V}) = \frac{\sum_{x \in \mathcal{V} \cap \operatorname{argmin}_C f} \left(\det \operatorname{Hess} f(x)\right)^{-\frac{1}{2}}}{\sum_{x \in \operatorname{argmin}_C f} \left(\det \operatorname{Hess} f(x)\right)^{-\frac{1}{2}}} \left(1 + O(h)\right).$$

When $\overline{\mathcal{V}} \cap \operatorname{argmin}_C f = \emptyset$*, there exists* $c > 0$ *such that when* $h \to 0$*:*

$$\nu_h(\mathcal{V}) = O\left(e^{-\frac{c}{h}}\right).$$

In addition, let $F \in C^\infty(\partial\Omega, \mathbb{R})$*. Then, when* $h \to 0$*:*

$$\mathbb{E}_{\nu_h}\left[F\left(X_{\tau_\Omega}\right)\right] = \sum_{i=1}^{k_0} F(z_i)\, a_i + O(h^{\frac{1}{4}}), \tag{50}$$

where for $i \in \{1, \ldots, k_0\}$*,*

$$a_i = \frac{\partial_n f(z_i)}{\sqrt{\det \operatorname{Hess} f\big|_{\partial\Omega}(z_i)}} \left(\sum_{j=1}^{k_0} \frac{\partial_n f(z_j)}{\sqrt{\det \operatorname{Hess} f\big|_{\partial\Omega}(z_j)}} \right)^{-1}. \tag{51}$$

Finally, (50) *holds when* $X_0 = x \in C$*.*

Remark 15. In [23], one also gives sharp asymptotic estimates of λ_h and $\partial_n u_h$ in a more general setting than the one of Theorem 2 (for instance, we study the case when f has local minima higher than $\min_{\partial\Omega} f$). However, in [23], we do not study the precise asymptotic behaviour of X_{τ_Ω} when $h \to 0$ near generalized saddle points z of f on $\partial\Omega$ which are such that $f(z) > \min_{\partial\Omega} f$ as we did in [22] (see Corollary 1). Finally, in [23], the optimality of the remainder term $O(h^{\frac{1}{4}})$ in (50) is discussed and improved in some situations.

Ideas and sketch of the proof of Theorem 2. In this section, one gives the sketch of the proof of (50) which is the main result of Theorem 2. Recall that from (15), for $F \in C^\infty(\partial\Omega, \mathbb{R})$

$$\mathbb{E}_{\nu_h}\big[F(X_{\tau_\Omega})\big] = -\frac{h}{2\lambda_h}\frac{\displaystyle\int_\Sigma F\,\partial_n u_h e^{-\frac{2}{h}f}}{\displaystyle\int_\Omega u_h e^{-\frac{2}{h}f}},$$

where u_h is the eigenfunction associated with the principal eigenvalue λ_h of $L^{D,(0)}_{f,h}$. Therefore, to prove (50), one studies the asymptotic behaviour when $h \to 0$ of the following quantities

$$\lambda_h,\ \partial_n u_h \text{ and } \int_\Omega u_h e^{-\frac{2}{h}f}. \tag{52}$$

Under the assumptions **[H-Morse]** and **[H-Min]**, one defines

$$m_0 := \text{Card}\,\Big(\{z \in \Omega, z \text{ is a local minimum of } f\}\Big)$$

and

$$\begin{aligned} m_1 :=& \text{Card}\,\Big(\{z \text{ is a local minimum of } f|_{\partial\Omega}\} \cap \{z \in \partial\Omega,\, \partial_n f(z) > 0\}\Big) \\ &+ \text{Card}\,\Big(\{z \text{ is saddle point of } f\}\Big). \end{aligned} \tag{53}$$

The integer m_1 is the number of generalized saddle points of f in $\overline{\Omega}$ (see [31, Section 5.2]). To study the asymptotic behaviour when $h \to 0$ of the quantities involved in (52), the starting point is to again observe that ∇u_h is solution to an eigenvalue problem for the same eigenvalue λ_h (as already explained at the end of Section 2.1). Indeed, ∇u_h is solution to (see (27))

$$\begin{cases} L^{(1)}_{f,h}\nabla u_h = \lambda_h \nabla u_h \text{ in } \Omega, \\ \nabla_T u_h = 0 \text{ on } \partial\Omega, \\ \left(-\dfrac{h}{2}\text{div} + \nabla f\cdot\right)\nabla u_h = 0 \text{ on } \partial\Omega, \end{cases} \tag{54}$$

where we recall that $L^{(1)}_{f,h} = -\frac{h}{2}\Delta + \nabla f \cdot \nabla + \text{Hess}\, f$ is an operator acting on 1-forms. Let us also recall that the operator $L^{(1)}_{f,h}$ with tangential boundary conditions (54) is denoted by $L^{D,(1)}_{f,h}$. From (54), ∇u_h is an eigenform of $L^{D,(1)}_{f,h}$ associated with λ_h.

The second ingredient is the following result: under the assumptions **[H-Morse]** and **[H-Min]** and when $h \to 0$, the operator $L^{D,(0)}_{f,h}$ has exactly m_0 eigenvalues smaller than $\frac{\sqrt{h}}{2}$ and $L^{D,(1)}_{f,h}$ has exactly m_1 eigenvalues smaller than $\frac{\sqrt{h}}{2}$ (see [31, Chapter 3]). Actually, all theses small eigenvalues are exponentially small when

$h \to 0$, i.e. they are all $O\big(e^{-\frac{c}{h}}\big)$ for some $c > 0$ independent of h. In particular λ_h is an exponentially small eigenvalue of $L_{f,h}^{D,(0)}$ and of $L_{f,h}^{D,(1)}$. Let us denote by $\pi_h^{(0)}$ (resp. $\pi_h^{(1)}$) the orthogonal projector in $L^2_w(\Omega)$ onto the m_0 (resp. m_1) smallest eigenvalues of $L_{f,h}^{D,(0)}$ (resp. $L_{f,h}^{D,(1)}$). Then, according to the foregoing, one has when $h \to 0$:

$$\dim \operatorname{Ran} \pi_h^{(0)} = m_0, \quad \dim \operatorname{Ran} \pi_h^{(1)} = m_1$$

and

$$\nabla u_h \in \operatorname{Ran} \pi_h^{(1)}.$$

Let us now explain how we prove Theorem 2. To this end, let us introduce the set of local minima of f in Ω,

$$\mathsf{U}_0^{\Omega} := \{x \in \Omega,\, x \text{ is a local minimum of } f\},$$

and the set of generalized saddle points of f in $\overline{\Omega}$,

$$\mathsf{U}_1^{\overline{\Omega}} = \Big(\{z \text{ is a local minimum of } f|_{\partial\Omega}\} \cap \{z \in \partial\Omega,\, \partial_n f(z) > 0\}\Big) \bigcup \{z \text{ is a saddle point of } f\}.$$

Let us recall that $m_0 = \operatorname{Card}\big(\mathsf{U}_0^{\Omega}\big)$ and, from (53), that $m_1 = \operatorname{Card}\big(\mathsf{U}_1^{\overline{\Omega}}\big)$. The first step to prove Theorem 2 consists in constructing two maps $\widetilde{\mathbf{j}}$ and $\mathbf{j}$. The goal of the map $\mathbf{j}$ is to associate each local minimum x of f with a set of generalized saddle points $\mathbf{j}(x) \subset \mathsf{U}_1^{\overline{\Omega}}$ such that

$$\forall z, y \in \mathbf{j}(x),\ f(z) = f(y),$$

and such that, in the limit $h \to 0$, there exists at least one eigenvalue of $L_{f,h}^{D,(0)}$ whose exponential rate of decay is $2\big(f(\mathbf{j}(x)) - f(x)\big)$ i.e.

$$\exists \lambda \in \sigma\big(L_{f,h}^{D,(0)}\big) \text{ such that } \lim_{h\to 0} h \ln \lambda = -2\big(f(\mathbf{j}(x)) - f(x)\big).$$

The aim of the map $\widetilde{\mathbf{j}}$ is to associate each local minimum x of f with the connected component of $\{f < f(\mathbf{j}(x))\}$ which contains x.

The second step consists in constructing bases of $\operatorname{Ran} \pi_h^{(0)}$ and $\operatorname{Ran} \pi_h^{(1)}$. To this end, one constructs two families of quasi-modes, denoted by $(\widetilde{u}_k)_{k\in\{1,\dots,m_0\}}$ and $(\widetilde{\psi}_j)_{j\in\{1,\dots,m_1\}}$, which are then respectively projected onto $\operatorname{Ran} \pi_h^{(0)}$ and $\operatorname{Ran} \pi_h^{(1)}$. To construct the family of 1-forms $(\widetilde{\psi}_j)_{j\in\{1,\dots,m_1\}}$, we proceed as follows. For each saddle point z of f in Ω, following the procedure of [30], one constructs a 1-form supported in a neighboorhood of z in Ω. For a local minimum z of $f|_{\partial\Omega}$ such that $\partial_n f(z) > 0$, one constructs a 1-form supported in a neighboorhood of z in $\overline{\Omega}$ as made in [31]. To construct the family of functions $(\widetilde{u}_k)_{k\in\{1,\dots,m_0\}}$, one constructs for each local minimum x of f a smooth function whose support is almost $\widetilde{\mathbf{j}}(x)$ (this construction is close to the one made in [30, 31, 36, 46, 54]).

The next step consists in finding a sharp asymptotic equivalent for λ_h when $h \to 0$. The quantity $\frac{2}{h}\lambda_h$ equals the square of the smallest singular values of the finite dimensional operator

$$\nabla : \mathrm{Ran}\, \pi_h^{(0)} \to \mathrm{Ran}\, \pi_h^{(1)}.$$

To study the asymptotic behaviour when $h \to 0$ of this smallest singular value, one uses the bases of $\mathrm{Ran}\, \pi_h^{(0)}$ and $\mathrm{Ran}\, \pi_h^{(1)}$ which have been constructed previously. The analysis of this finite dimensional problem is inspired by [36] and also yields the asymptotic equivalent of $\int_\Omega u_h\, e^{-\frac{2}{h}f}$ when $h \to 0$.

Then, we study the asymptotic behaviour of the normal derivative of u_h on $\partial\Omega$ when $h \to 0$ to deduce that the law of X_{τ_Ω} concentrates when $h \to 0$ on $\overline{C} \cap \partial\Omega = \{z_1, \ldots, z_{k_0}\}$ when $X_0 \sim \nu_h$.
Lastly, one proves "leveling" results on the function

$$x \mapsto \mathbb{E}_x[F(X_{\tau_\Omega})]$$

to obtain that when $X_0 = x \in C$, the law of X_{τ_Ω} also concentrates when $h \to 0$ on $\{z_1, \ldots, z_{k_0}\}$.

To conclude, the main results of [23] are the following:

1. One uses techniques from semi-classical analysis to study the asymptotic behaviours of λ_h and $\partial_n u_h$ when $h \to 0$, and then, the concentration of the law of X_{τ_Ω} on a subset of $\mathrm{argmin}_{\partial\Omega} f$ when $X_0 \sim \nu_h$.
2. One identifies the points of $\mathrm{argmin}_{\partial\Omega} f$ where the law of X_{τ_Ω} concentrates when $X_0 \sim \nu_h$: this set of points is $\{z_1, \ldots, z_{k_0}\}$. Moreover, explicit formulas for their relative probabilities are given (see indeed (51)) as well as precise remainder terms.
3. One extends the previous results on the law of X_{τ_Ω} to a deterministic initial condition in Ω: $X_0 = x$ where $x \in C$.
4. These results hold under weak assumptions on the function f and one-dimensional examples are given to explain why the geometric assumptions are needed to get them.

Conclusion. We presented recent results which justify the use of a kinetic Monte Carlo model parametrized by Eyring-Kramers formulas to model the exit event from a metastable state Ω for the overdamped Langevin dynamics (1). Our analysis is for the moment limited to situations where $|\nabla f| \neq 0$ on $\partial\Omega$, which does not allow to consider order one saddle points on $\partial\Omega$. The extensions of [22] and [23] which are currently under study are the following: the case when f has saddle points on $\partial\Omega$ and the case when the diffusion process $X_t = (q_t, p_t)$ is solution to the Langevin stochastic differential equation

$$\begin{cases} dq_t = p_t dt, \\ dp_t = -\nabla f(q_t) dt - \gamma\, p_t dt + \sqrt{h\gamma}\, dB_t, \end{cases}$$

where $(q_t, p_t) \in \Omega \times \mathbb{R}^d$, Ω being a bounded open subset of $\mathbb{R}^d$.

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