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Abstract

A new iteration method for nonlinear vibrations has been developed by decomposing the periodic solution in two parts corresponding to low and high harmonics. For a nonlinear forced oscillator, the iteration schema is proposed with different formulations for these two parts. Then, the schema is deduced by using the harmonic balance technique. This method has proven to converge to the periodic solutions provided that a convergence condition is satisfied. The convergence is also demonstrated analytically for linear oscillators. Moreover, the new method has been applied to Duffing oscillators as an example. The numerical results show that each iteration schema converges in a domain of the excitation frequency and it can converge to different solutions of the nonlinear oscillator.

Keywords: Forced nonlinear oscillator, harmonic balance method, iteration procedure, Duffing oscillator.

1. Introduction

The harmonic balance method (HBM) has many applications in nonlinear dynamics. The original principle of this method is to express the periodic solution in terms of Fourier series with limited numbers of harmonics and to substitute this expression to the dynamic equation in order to find out balance

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of all harmonics. A typical difficulty of this method is linked to the dependence of the quality of the approach on the way to carry enough terms in the solution and check the order of the coefficients for all the neglected harmonics [1, 2].

Some alternative techniques based on HBM have been developed in order to solve this difficulty. The Galerkin procedure [3–6] was used to calculate incrementally the Fourier coefficients of the classical HBM. Another technique called the iteration procedures were presented by Mickens [7, 8] and applied to different kinds of non-linearities [9, 10]. The high dimensional harmonic balance (HDHB) was developed by Hall et al. [11]. This method is based on a constant Fourier transformation matrix which is an approximation of the matrix related to the non-linearity deduced from the classical HBM. This method has some advantages in calculation in case of high dimension, but it can lead to non-physical solutions [12–14]. Recently, other new methods based on HBM were developed by Cochelin et al. [15–17] and Ju et al. [18–20].

In this paper, we propose a new method for forced nonlinear oscillators based on the iteration procedures and HBM. The existing methods propose to use the same iteration schema for all harmonics. This new method provides different schemas which adapt to the frequency of the force. By decomposing the periodical solution in low and high harmonic components, an iteration schema is presented and then developed by using HBM. The schemas are proved to converge to the periodic solution, provided that a sufficient condition is satisfied. Thus, the harmonic decomposition is a new way to build iteration schemas which could be applied to other problems using HMB (e.g. the high dimensional problems). Moreover, this method is demonstrated to converge to the analytic solution of a linear oscillator and it is applied to a forced Duffing oscillator. The numerical results show that each iteration schema converges in each range of the excitation frequency and it can converge to different solutions of the oscillator.

2. Iteration schemas

Let's consider a forced nonlinear oscillator given by

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \beta\dot{u} + \omega_0^2 u + f(u,\dot{u}) = A\cos\Omega t \tag{1}$$

where $f(u,\dot{u})$ is a nonlinear function. This equation is rewritten to read

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + \Omega^2 u = (\Omega^2 - \omega_0^2)u - \beta\dot{u} - f(u,\dot{u}) + A\cos\Omega t \equiv G(u,\dot{u}) \tag{2}$$

A classical iteration schema is often proposed as follows

$$\frac{\mathrm{d}^2 u_{k+1}}{\mathrm{d}t^2} + \Omega^2 u_{k+1} = \mathcal{L}(G(u_i,\dot{u}_j)_{i,j\le k}) \tag{3}$$

where $\mathcal{L}$ is a linear form which is chosen for each function $G(u,\dot{u})$ in order to meet the convergence. Now we will build a different iteration schema by considering the periodic solution defined by series $\{q_{nk}\}, \{p_{nk}\}$ (with $1 \le n \le N$) as follows

$$u_{Nk}(t) = \frac{q_{0k}}{2} + \sum_{n=1}^{N} q_{nk}\cos n\Omega t + p_{nk}\sin n\Omega t \tag{4}$$

For each $\mathcal{N}$ $(0 \le \mathcal{N} \le N)$, we decompose u_{Nk} into two terms which correspond to low and high harmonics as follows

$$\begin{aligned} X_k(t) &= \frac{q_{0k}}{2} + \sum_{n=1}^{\mathcal{N}} q_{nk}\cos n\Omega t + p_{nk}\sin n\Omega t && (5)\\ Y_k(t) &= \sum_{n=\mathcal{N}+1}^{N} q_{nk}\cos n\Omega t + p_{nk}\sin n\Omega t && (6)\\ u_{Nk}(t) &= X_k(t) + Y_k(t) && (7) \end{aligned}$$

Then, the proposed iteration schema is

$$\begin{aligned} \frac{\mathrm{d}^2 Y_{k+1}}{\mathrm{d}t^2} + \beta\dot{X}_{k+1} + \omega_0^2 X_{k+1} =& A\cos\Omega t - f(u_{Nk},\dot{u}_{Nk}) \\ & - \frac{\mathrm{d}^2 X_k}{\mathrm{d}t^2} - \beta\dot{Y}_k - \omega_0^2 Y_k \end{aligned} \tag{8}$$

Here, we take the initial values (i.e. $k = 1$) by $q_{n1} = p_{n1} = 0$. Now we will develop this schema by using the harmonic balance technique. By performing the Fourier series development of equation (8), we obtain the following results

$$\begin{aligned}\frac{1}{T}\int_0^T\left(\frac{\mathrm{d}^2Y_{k+1}}{\mathrm{d}t^2}+\beta\dot{X}_{k+1}+\omega_0^2X_{k+1}\right)\cos n\Omega t\ \mathrm{d}t &= A\delta_{1n}-\mathcal{C}_{nk}\\ &-\frac{1}{T}\int_0^T\left(\frac{\mathrm{d}^2X_k}{\mathrm{d}t^2}+\beta\dot{Y}_k+\omega_0^2Y_k\right)\cos n\Omega t\ \mathrm{d}t\end{aligned}\tag{9}$$

$$\begin{aligned}\frac{1}{T}\int_0^T\left(\frac{\mathrm{d}^2Y_{k+1}}{\mathrm{d}t^2}+\beta\dot{X}_{k+1}+\omega_0^2X_{k+1}\right)\sin n\Omega t\ \mathrm{d}t &= -\mathcal{S}_{nk}\\ &-\frac{1}{T}\int_0^T\left(\frac{\mathrm{d}^2X_k}{\mathrm{d}t^2}+\beta\dot{Y}_k+\omega_0^2Y_k\right)\sin n\Omega t\ \mathrm{d}t\end{aligned}\tag{10}$$

where $T=\frac{2\pi}{\Omega}$; $\delta_{1n}=1$ if $n=1$ and $\delta_{1n}=0$ if other. $\mathcal{C}_{nk},\mathcal{S}_{nk}$ are the Fourier coefficients of $f(u_{Nk},\dot{u}_{Nk})$

$$\begin{cases}\mathcal{C}_{nk}=\dfrac{2}{T}\displaystyle\int_0^T f(u_{Nk},\dot{u}_{Nk})\cos n\Omega t\mathrm{d}t\\ \mathcal{S}_{nk}=\dfrac{2}{T}\displaystyle\int_0^T f(u_{Nk},\dot{u}_{Nk})\sin n\Omega t\mathrm{d}t\end{cases}\tag{11}$$

Thereafter, by substituting equations (5) and (6) into equations (9) and (10), we obtain

For $n=0$

$$\omega_0^2q_{0(k+1)}=-\mathcal{C}_{0k}$$

For $0<n\le\mathcal{N}$

$$\begin{aligned}\beta n\Omega p_{n(k+1)}+\omega_0^2q_{n(k+1)}&=A\delta_{n1}-\mathcal{C}_{nk}+n^2\Omega^2q_{nk}\\ -\beta n\Omega q_{n(k+1)}+\omega_0^2p_{n(k+1)}&=-\mathcal{S}_{nk}+n^2\Omega^2p_{nk}\end{aligned}$$

For $\mathcal{N}<n\le N$

$$\begin{aligned}-n^2\Omega^2q_{n(k+1)}&=A\delta_{1n}-\mathcal{C}_{nk}-\omega_0^2q_{nk}-\beta n\Omega p_{nk}\\ -n^2\Omega^2p_{n(k+1)}&=-\mathcal{S}_{nk}-\omega_0^2p_{nk}+\beta n\Omega q_{nk}\end{aligned}$$

Finally, we have deduced the iteration schema (8) to the results in Table 1 where $\{p_{nk}\},\{q_{nk}\}$ are represented as recurrent sequences. By taking the initial values ($p_{n1}=q_{n1}=0$), we can compute these series for a number of iteration K to get the solution of equation (1). Here, we have one iteration schema corresponding to each value of $\mathcal{N}$. In the next section, we will prove that these schemas converge to the periodic solution of equation (1) if a sufficient condition is satisfied.

Table 1: Frequency dependent iteration schema

$n = 0$	$q_{n(k+1)} = -\dfrac{\mathcal{C}_{0k}}{\omega_0^2} \quad ; \quad p_{n(k+1)} = 0$
$0 < n \le \mathcal{N}$	$p_{n(k+1)} = \dfrac{n^2\Omega^2(\beta n\Omega q_{nk} + \omega_0^2 p_{nk})}{(\beta n\Omega)^2 + \omega_0^4} - \dfrac{\beta n\Omega\mathcal{C}_{nk} + \omega_0^2\mathcal{S}_{nk} + \beta n\Omega A\delta_{n1}}{(\beta n\Omega)^2 + \omega_0^4}$ $q_{n(k+1)} = \dfrac{n^2\Omega^2(-\beta n\Omega p_{nk} + \omega_0^2 q_{nk})}{(\beta n\Omega)^2 + \omega_0^4} + \dfrac{\beta n\Omega\mathcal{S}_{nk} - \omega_0^2\mathcal{C}_{nk} + \omega_0^2 A\delta_{n1}}{(\beta n\Omega)^2 + \omega_0^4}$
$\mathcal{N} < n \le N$	$q_{n(k+1)} = \dfrac{\omega_0^2}{n^2\Omega^2} q_{nk} + \dfrac{\beta}{n\Omega} p_{nk} + \dfrac{\mathcal{C}_{nk}}{n^2\Omega^2} - \dfrac{A\delta_{n1}}{n^2\Omega^2}$ $p_{n(k+1)} = \dfrac{\omega_0^2}{n^2\Omega^2} p_{nk} - \dfrac{\beta}{n\Omega} q_{nk} + \dfrac{\mathcal{S}_{nk}}{n^2\Omega^2}$

3. Sufficient condition of convergence

Lemma: If $f \in \mathcal{C}^1$ and if the series of functions $\{u_{Nk}\}$ defined by equation (4) and the schema in Table 1 is uniformly convergent when N, k tends to infinity, their limit is a periodic solution of equation (1).

Demonstration: Suppose that $u_{Nk}(t)$ converges uniformly to $u(t)$. We have to prove that

(i) $u(t)$ is periodic of period T

(ii) $u(t)$ is a solution of equation (1).

We prove (i) by contrary. Suppose that it exists τ such that $|u(\tau) - u(\tau + T)| = a > 0$. Because $u_{Nk}(t)$ is uniformly convergent, for $0 < 2\epsilon < a$ it exists M, K such that $\forall N > M, k > K$ we have

$$|u_{Nk}(\tau) - u(\tau)| < \epsilon$$

$$|u_{Nk}(\tau + T) - u(\tau + T)| < \epsilon$$

In addition, we have $u_{Nk}(\tau) = u_{Nk}(\tau + T)$. Thus

$$a = |u(\tau) - u(\tau + T)| \le |u_{Nk}(\tau) - u(\tau)| + |u_{Nk}(\tau) - u(\tau + T)| < 2\epsilon$$

The last inequality cannot be true because $2\epsilon < a$. By consequence, $u(t)$ is periodic.

To prove (ii), it is sufficient to show that $\forall \epsilon > 0, \exists M', K'$ such that

$$|\ddot{u}_{Nk} + \omega_0^2 u_{Nk} + f(u_{Nk}, \dot{u}_{Nk}) - K \cos \Omega t| < C\epsilon \;\forall N > M', k > K' \tag{12}$$

where C is a constant.

By using the triangular inequality, we have

$$\begin{aligned}&\left|\ddot{u}_{N(k+1)} + \omega_0^2 u_{N(k+1)} + f(u_{N(k+1)}, \dot{u}_{N(k+1)}) - \Gamma \cos \Omega t\right| \leq \omega_0^2 \left|Y_{k+1} - Y_k\right| \\ &\quad + \beta \left|\dot{Y}_{k+1} - \dot{Y}_k\right| + \left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}(X_{k+1} - X_k)\right| + \left|f\left(u_{N(k+1)}, \dot{u}_{N(k+1)}\right) - f(u_{Nk}, \dot{u}_{Nk})\right| \\ &\quad + \left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}(Y_{k+1} + X_k) + \beta(\dot{X}_{(k+1)} + \dot{Y}_k) + \omega_0^2(X_{k+1} + Y_k) + f(u_{Nk}, \dot{u}_{Nk}) - A \cos \Omega t\right|\end{aligned}$$

We note f_a, f_b, f_c, f_d and f_e the five terms of the last inequality, that means

$$\begin{aligned} f_a =& \omega_0^2 \left|Y_{k+1} - Y_k\right| \\ f_b =& \beta \left|\dot{Y}_{k+1} - \dot{Y}_k\right| \\ f_c =& \left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}(X_{k+1} - X_k)\right| \\ f_d =& \left|f\left(u_{N(k+1)}, \dot{u}_{N(k+1)}\right) - f(u_{Nk}, \dot{u}_{Nk})\right| \\ f_e =& \left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}(Y_{k+1} + X_k) + \beta(\dot{X}_{(k+1)} + \dot{Y}_k)\right. \\ & \left. + \omega_0^2(X_{k+1} + Y_k) + f(u_{Nk}, \dot{u}_{Nk}) - A \cos \Omega t\right| \end{aligned}$$

In order to prove (12), we have to prove that $f_a, f_b, ..., f_e$ are measured by ϵ.

Because u_{Nk} is uniformly convergent, it is clear that f_a is measured by ϵ. To prove that f_b and f_c are measured by ϵ, it is sufficient, for example, to take the definition of X_{mk} from equation (5) with remark that

$$\begin{aligned} f_c &= \left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}(X_{(k+1)} - X_k)\right| \\ &= \left|\sum_{n=1}^{m} n^2\Omega^2 \left[(p_{n(k+1)} - p_{nk}) \sin n\Omega t + (q_{n(k+1)} - q_{nk}) \cos n\Omega t\right]\right| \\ &\leq \sum_{n=1}^{m} n^2\Omega^2 \left(\left|p_{n(k+1)} - p_{nk}\right| + \left|q_{n(k+1)} - q_{nk}\right|\right) \end{aligned}$$

As the series $\{u_{Nk}\}$ converge uniformly, the series p_{nk}, q_{nk} are uniformly convergent when k tends to infinity. Thus, f_b and f_c are measured by ϵ.

In order to measure f_d, we use a following property: for $f \in \mathcal{C}^1$, it exists C such that $|f(u) - f(\tilde{u})| \le C\epsilon$ when $|u - \tilde{u}| < \epsilon$. As the series u_{Nk} converges uniformly, we can chose k large enough for $|u_{N(k+1)} - u_{Nk}| < \epsilon$. By consequence, f_d is measured by ϵ.

For f_e, by using the iteration schema in Table 1, we obtain

$$\begin{aligned}&\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}(Y_{(k+1)} + X_k) + \Omega^2(X_{(k+1)} - X_k) + \omega_0^2 u_{Nk} + f(u_{Nk}, \dot{u}_{Nk}) - A\cos\Omega t\right| \\ &\quad = \left| f(u_{mk}, \dot{u}_{mk}) - \left(\frac{\mathcal{C}_{0k}}{2} + \sum_{n=1}^{m} \mathcal{C}_{nk}\cos n\Omega t + \mathcal{S}_{nk}\sin n\Omega t\right)\right|\end{aligned}$$

Thus, f_e is exactly the difference between the function $f(u_{Nk}, \dot{u}_{Nk})$ and its Fourier series development. Because u_{Nk} is periodic and bounded, $f(u_{Nk}, \dot{u}_{Nk})$ is also periodic and bounded. Thus, this difference is measured by ϵ when N is large enough. Therefore, the five terms are measured by ϵ, and the lemma is proved.

4. Examples

4.1. Linear oscillator

We consider a forced linear oscillator, i.e. $f(u, \dot{u}) = 0$. The iteration schema in Table 1 becomes

$$\begin{array}{lll} n = 0 & q_{0(k+1)} &= 0 \\ 1 \le n \le \mathcal{N} & p_{n(k+1)} &= \dfrac{n^2\Omega^2(\beta n\Omega q_{nk} + \omega_0^2 p_{nk})}{(\beta n\Omega)^2 + \omega_0^4} + \dfrac{\beta n\Omega A\delta_{n1}}{(\beta n\Omega)^2 + \omega_0^4} \\ & q_{n(k+1)} &= \dfrac{n^2\Omega^2(-\beta n\Omega p_{nk} + \omega_0^2 q_{nk})}{(\beta n\Omega)^2 + \omega_0^4} + \dfrac{\omega_0^2 A\delta_{n1}}{(\beta n\Omega)^2 + \omega_0^4} \\ \mathcal{N} \le n \le N & q_{n(k+1)} &= \dfrac{\omega_0^2}{n^2\Omega^2} q_{nk} + \dfrac{\beta}{n\Omega} p_{nk} - \dfrac{A\delta_{n1}}{n^2\Omega^2} \\ & p_{n(k+1)} &= \dfrac{\omega_0^2}{n^2\Omega^2} p_{nk} - \dfrac{\beta}{n\Omega} q_{nk} \end{array}$$

We will prove that when the schema converges, its limit is the analytic solution.

Case $\mathcal{N} = 0$

For $n \geq 2$, we have

$$q_{n(k+1)} + \mathrm{i}p_{n(k+1)} = \left(\frac{\omega_0^2}{n^2\Omega^2} - \frac{\mathrm{i}\beta}{n\Omega}\right)(q_{nk} + \mathrm{i}p_{nk})$$
$$q_{n(k+1)} - \mathrm{i}p_{n(k+1)} = \left(\frac{\omega_0^2}{n^2\Omega^2} + \frac{\mathrm{i}\beta}{n\Omega}\right)(q_{nk} - \mathrm{i}p_{nk})$$

The last equations define two geometrical sequences and they converge if and only if $\left|\frac{\omega_0^2}{n^2\Omega^2} + \frac{\mathrm{i}\beta}{n\Omega}\right| < 1$; their limits are zeros (i.e., $q_{nk}, p_{nk} \to 0$ when $k \to \infty$).

For $n = 1$, we have

$$\begin{aligned} q_{1(k+1)} + \mathrm{i}p_{1(k+1)} - A_1 &= \left(\frac{\omega_0^2}{\Omega^2} - \frac{\mathrm{i}\beta}{\Omega}\right)(q_{1k} + \mathrm{i}p_{1k} - A_1) \\ q_{1(k+1)} - \mathrm{i}p_{1(k+1)} - A_2 &= \left(\frac{\omega_0^2}{\Omega^2} + \frac{\mathrm{i}\beta}{\Omega}\right)(q_{1k} - \mathrm{i}p_{1k} - A_2) \end{aligned} \tag{13}$$

where $A_1 = -A/\left(\Omega^2 - \omega_0^2 + \mathrm{i}\beta\Omega\right)$ and $A_2 = -A/\left(\Omega^2 - \omega_0^2 - \mathrm{i}\beta\Omega\right)$. The last equations define also two geometrical sequences which converge to zeros with the same condition for $n \geq 2$. By consequence, the limits of $\{q_{1k}\}$ and $\{p_{1k}\}$ are

$$\begin{cases} \mathbf{q}_1 = \dfrac{(\omega_0^2 - \Omega^2)A}{(\omega_0^2 - \Omega^2)^2 + (\beta\Omega)^2} \\ \mathbf{p}_1 = \dfrac{\beta\Omega A}{(\omega_0^2 - \Omega^2)^2 + (\beta\Omega)^2} \end{cases} \tag{14}$$

Equation (14) is exactly the analytic solution of the forced linear oscillator. Thus, the sufficient condition is thus justified in this case. In addition, equation (13) shows that the schema converges linearly and the convergence rate is calculated by

$$\mu = \left|\frac{\omega_0^2}{\Omega^2} + \frac{\mathrm{i}\beta}{\Omega}\right| \tag{15}$$

The condition of the convergence $\mu < 1$ can be rewritten as $\Omega^4 - \omega_0^4 > (\beta\Omega)^2$. It means that Ω is larger than the resonance frequency of the oscillator Ω_r.

Case $\mathcal{N} = 1$

For $n \geq 2$, we have the same results with the previous case, i.e., $q_{nk}, p_{nk} \to 0$ when $k \to \infty$ if and only if $\left|\frac{\omega_0^2}{n^2\Omega^2} + \frac{\mathrm{i}\beta}{n\Omega}\right| < 1$ for $n \geq 2$.

For $n = 1$, we have

$$q_{1(k+1)} + \mathrm{i}p_{1(k+1)} - A_1 = \frac{\Omega^2}{-\mathrm{i}\beta\Omega + \omega_0^2}(q_{1k} + \mathrm{i}p_{1k} - A_1)$$
$$q_{1(k+1)} - \mathrm{i}p_{1(k+1)} - A_2 = \frac{\Omega^2}{\mathrm{i}\beta\Omega + \omega_0^2}(q_{1k} - \mathrm{i}p_{1k} - A_2)$$

We see that the last two equations define also two geometrical sequences which converge if and only if $\left|\frac{\Omega^2}{\mathrm{i}\beta\Omega+\omega_0^2}\right| < 1$. The condition of convergence for $n = 1$ and $n \geq 2$ of the schema can be rewritten as follows

$$\begin{cases} \Omega^4 - \omega_0^4 < (\beta\Omega)^2 \\ 16\Omega^4 - \omega_0^4 > 4(\beta\Omega)^2 \end{cases} \tag{16}$$

We deduce that the iteration schema $\mathcal{N} = 1$ converges linearly when Ω is smaller than the resonance frequency and the convergence rate is

$$\mu = \left|\frac{\Omega^2}{\mathrm{i}\beta\Omega + \omega_0^2}\right| \tag{17}$$

Case $\mathcal{N} \geq 2$

Similarly, the schema converges linearly to the analytic solution if and only if

$$\begin{cases} \mathcal{N}^4\Omega^4 - \omega_0^4 < (\beta\mathcal{N}\Omega)^2 \\ (\mathcal{N}+1)^4\Omega^4 - \omega_0^4 > (\beta(\mathcal{N}+1)\Omega)^2 \end{cases} \tag{18}$$

Remark: From equations (13) and (17), when $\Omega \simeq \Omega_r$, the convergence rate μ can be approached by

$$\mu \simeq \left(1 - \frac{\Delta\Omega}{\Omega_r}\right)^2 \tag{19}$$

where $\Delta\Omega = |\Omega - \Omega_r|$. It means that $\mu^K \simeq 1 - 2K\frac{\Delta\Omega}{\Omega_r}$. Thus, the number of iteration can be estimated by $K \sim \frac{1}{2}\frac{\Omega_r}{\Delta\Omega}$.

Numerical example

The iteration method is used for a forced linear oscillator with $\omega_0 = 1, A = 1$. Figure 1 shows the amplitude $\mathcal{A} = \sqrt{q_1^2 + p_1^2}$ in function of the excitation frequency Ω with different values of the damping coefficient β. Here, we take the number of harmonics $N = 3$ and the number of iteration $K = 300$ for all frequencies Ω. The figure shows that the iteration method converges to the analytic solution. Particularly, the schema with $\mathcal{N} = 0$ is convergent for low frequency Ω and the schema with $\mathcal{N} = 1$ is convergent for high frequency Ω. This numerical result agrees well with the previous demonstration.

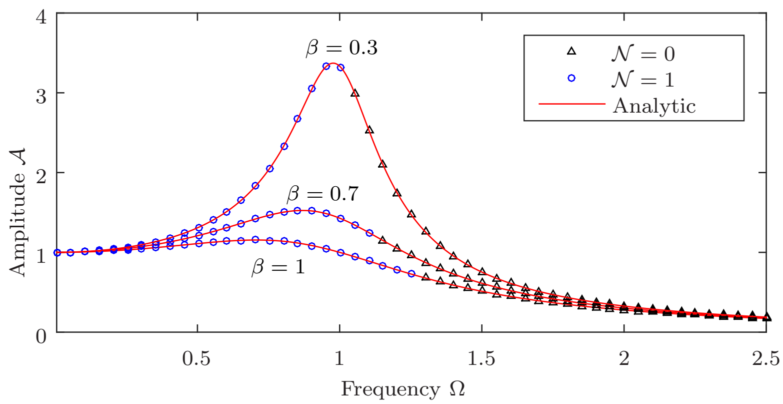

Figure 1: Forced linear oscillator by analytic and iteration methods

4.2. Duffing oscillator

Let's consider another example, a forced Duffing oscillator. This oscillator has many applications in physics and it can be described by

$$\ddot{u} + \beta\dot{u} + u + \varepsilon u^3 = A\cos\Omega t \tag{20}$$

When ε is small, the perturbation techniques [1, 21] can be applied for this oscillator. This technique demonstrates that the periodical solution depends on the frequency Ω by

$$\mathcal{A}^2\left\{\beta^2\Omega^2 + \left(1 - \Omega^2 + \frac{3}{4}\varepsilon\mathcal{A}^2\right)^2\right\} = A^2, \quad \tan\vartheta = \frac{\beta\Omega}{1 - \omega^2 + \frac{3}{4}\varepsilon\mathcal{A}^2} \tag{21}$$

where $\mathcal{A} = \sqrt{q_1^2 + p_1^2}$ is the amplitude and $\tan\vartheta = p_1/q_1$ is the phase of the first harmonic.

Figure 2 shows the numerical results with different $\mathcal{N}$ for a forced Duffing oscillator with parameters $\omega = 1,\ \beta = 0.1,\ \varepsilon = 0.01$ and $A = 1$. In this example, we take the number of harmonics $N = 7$ et the number of iteration $K = 200$ for all frequencies Ω. Here, the schema is considered to converge when the amplitude of the last iteration changes less than 1%. We see that each iteration schema converges in each frequency domain of Ω. Particularly, in the domain I, two schemas converges to two different solutions, and in domain II, different schemas converge to the same solution.

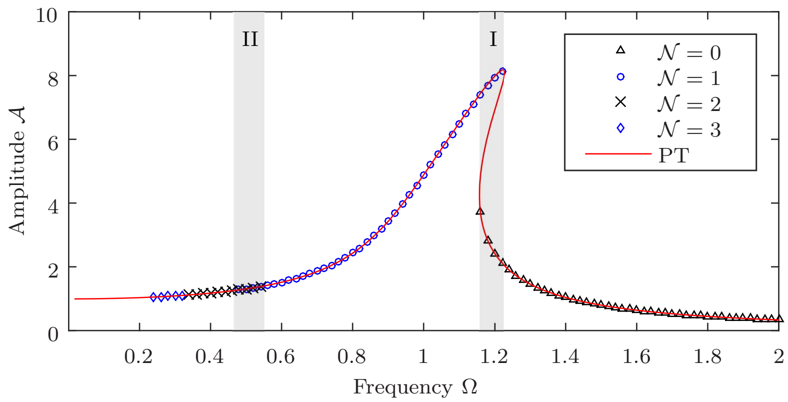

Figure 2: Forced Duffing oscillator by the perturbation technique (PT) and the iteration method. Parameter: $\omega = 1$, $\beta = 0.1$, $\varepsilon = 0.01$ and $A = 1$

Figure 3 shows the results for the Duffing oscillator with different nonlinear parameter ε. We see that the schema $\mathcal{N} = 0$ converges for high frequency Ω and the schema $\mathcal{N} = 1$ converges for low frequency Ω. There is no schema of Table 1 converge to the unstable solution of the Duffing oscillator.

In order to estimate the convergence rate of the schemas, the numerical error of equation (20) is calculated by

$$\mathcal{O}(\varepsilon) = \max |\ddot{u} + \beta\dot{u} + u + \varepsilon u^3 - A\cos\Omega t|$$

where $u(t)$ is the numerical result in each iteration. Figure 4 shows the error versus the number of iteration for the Duffing oscillator with $\varepsilon = 0.01$, $N = 10$ and $K = 1000$. We can see that the schemas converge in logarithmic scale of the number of iteration. Moreover, when the frequency Ω approaches the resonance frequency (green cycle markers), the convergence is relatively slow.

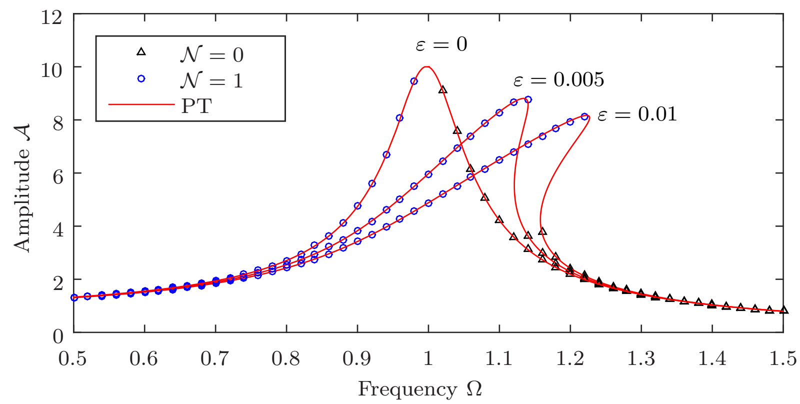

Figure 3: Forced Duffing oscillator by the perturbation technique (PT) and the iteration method

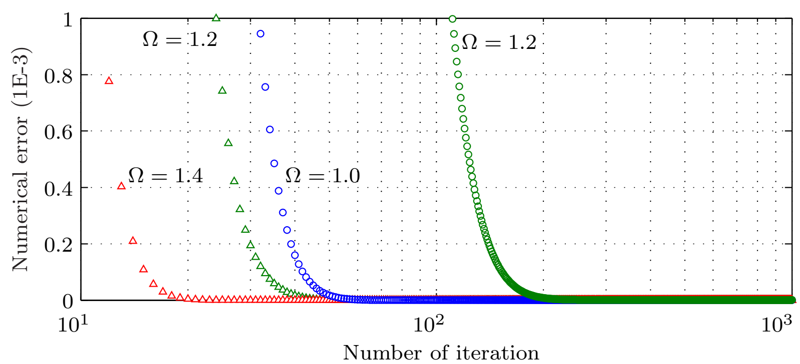

Figure 4: Numerical error in each iteration with the schema $\mathcal{N} = 0$ (triangle) and $\mathcal{N} = 1$ (cycle) for the Duffing oscillator

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5. Conclusion

When the periodic solution of a dynamical system is decomposed into two parts corresponding low and high harmonics, we can propose an iteration schema with different formulations for the two parts. This decomposition can be built to adapt the system in term of convergence. The application for forced Duffing oscillators show that each schema converges in each range of the excitation frequency and we can reach different solutions by using different schema.

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