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Abstract The influence of a nonlinear foundation on the dynamics of a periodically supported beam has been investigated by a novel model. By using Fourier transforms and Dirac comb properties, a relation between the displacement of the beam and the reaction forces of its supports in steady-state has been established from the Euler-Bernoulli beam's equation. This relation holds for any foundation behaviours. Therefore, the dynamic equation of a support has been built by combining this relation and the constitutive law of the foundation and the supports. This equation describes a forced nonlinear oscillator provided that the moving loads are a periodical series. Then, an iteration procedure has been developed to compute the periodic solution. This procedure has been demonstrated converging to the analytic solution for linear foundations. The applications to bilinear and cubic nonlinear foundations have been performed as examples. Moreover, the influences of non-linearity on the dynamic responses have been investigated by parametric studies.

Keywords Nonlinear foundation · Periodically supported beam · Harmonic balance technique · Iteration procedure

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1 Introduction

The dynamics of a railway track on a nonlinear foundation is the subject of numerous investigations. The model of beams subjected to moving loads has been often used for this analysis. For elastic foundations, the researches on this analytic model have been summarized by Fryba [1]. Analytic and numerical methods have been developed for the visco-elastic foundations [2–5]. For nonlinear foundations, the perturbation techniques have been used in analytic models [6–8] provided that the nonlinear force of the foundation follows a cubic law. The cubic nonlinear foundation has been also studied by using the Galerkin method [9], while the effects of tensionless foundations have been also investigated with numerical methods [10, 11].

In order to take into account the discrete distribution of supports, Mead [12, 13] developed the model of a periodically supported beam subjected to moving loads. This model has been also developed for visco-elastic foundations by Metrikine et al. [14, 15] and Belotserkovskiy [16]. Although these analytical methods are well developed for linear foundations, they may not be easily applied to nonlinear foundations. Moreover, the models for a continuously supported beam may be difficult to extend to discrete supports.

This article presents a novel model for a periodically supported beam on a nonlinear foundation subjected to moving loads. Based on an analytic model [17], a dynamic equation of a support is developed which holds for any foundation behavior. This equation leads to a forced nonlinear oscillation provided that the moving loads are a periodic series. Then, a numerical method is developed by using the harmonic balance techniques and the iteration procedures for nonlinear oscillators [18, 19]. This method is proven to converges to the ana-

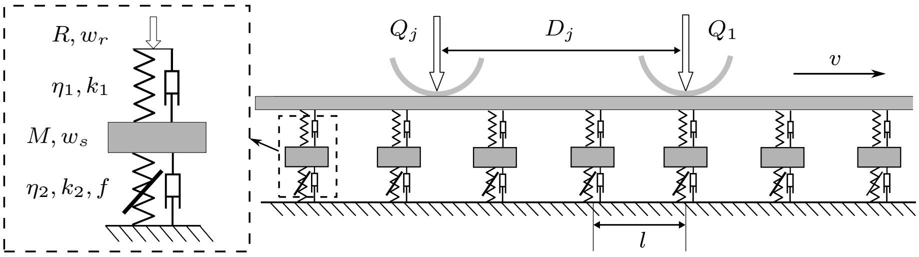

Fig. 1 Periodically supported beam on a nonlinear foundation subjected to moving forces

lytic solution when the foundation behaviour is linear. Thereafter, the bilinear and cubic laws are considered as examples of nonlinear foundations. These applications show that this method is a fast and simple way to compute the responses. Moreover, the influences of the nonlinear parameters of the foundations are investigated by parametric studies.

2 Formulations

2.1 Periodically supported beam in steady-state

Let's consider a periodically supported beam on a nonlinear viscoelastic foundation as shown in Figure 1. The beam is subjected to moving forces Q_j characterized by the distance to the first moving force D_j ($1 \leq j \leq K$ where K is the number of moving forces). It is remarkable that the mutual distance D_j of the moving loads is not restricted to be constant.

In steady-state, we suppose that all supports are equivalent and their reaction forces are described by a same function, but with a delay equal to the time for a load moving from a support to another. The total force applied on the beam can be written with the help of the Dirac function as follows

$$F(x,t) = \sum_{n=-\infty}^{\infty} R\left(t - \frac{x}{v}\right)\delta(x-nl) - \sum_{j=1}^{K} Q_j \delta(x+D_j-vt) \tag{1}$$

where l is the distance between two successive supports and $R(t)$ is the reaction force of the support at the reference origin ($x = 0$).

The response of the beam is governed by the dynamical equation of the Euler-Bernoulli beam

$$EI\frac{\partial^4 w_r(x,t)}{\partial x^4} + \rho S\frac{\partial^2 w_r(x,t)}{\partial t^2} - F(x,t) = 0 \tag{2}$$

where ρ, E are the density, the Young's modulus and S, I are the cross-section area of the beam and the area moment of inertia.

From equations (1) and (2), by performing a double Fourier transform, one temporal and one spatial, and by using the Dirac comb properties, we can deduce a relation between the Fourier transforms of the beam displacement $\hat{w}_r(0,\omega)$ and of the reaction force $\hat{R}(\omega)$ as follows (see Appendix)

$$\hat{R}(\omega) = \mathcal{K}_e \hat{w}_r(0,\omega) + \mathcal{Q}_e \tag{3}$$

where $\mathcal{K}_e$ and $\mathcal{Q}_e$ are calculated by

$$\mathcal{K}_e = 4\lambda_b^3 EI\left[\frac{\sin l\lambda_e}{\cos l\lambda_e - \cos\frac{\omega l}{v}} - \frac{\sinh l\lambda_e}{\cosh l\lambda_e - \cos\frac{\omega l}{v}}\right]^{-1} \tag{4}$$

$$\mathcal{Q}_e = \sum_{j=1}^{K} \frac{\mathcal{K}_e Q_j \mathrm{e}^{-\mathrm{i}\omega\frac{D_j}{v}}}{vEI\left[\left(\frac{\omega}{v}\right)^4 - \lambda_e^4\right]} \tag{5}$$

with $\lambda_e = \sqrt[4]{\frac{\rho S\omega^2}{EI}}$.

Equation (3) does not depend on the behaviour of the support nor on the foundations. Next, this equation will be combined with the constitutive law of the supports to get a dynamical equation of the system.

2.2 Dynamical equation of supports

Consider a system of support on a nonlinear foundation as shown in Figure 1. This system contains a viscoelastic pad under the beam with stiffness k_1 and damping coefficient η_1. The foundation under the block has a linear behaviour with stiffness k_2, damping coefficient η_2 and a nonlinear behaviour characterized by a function $f(w_s, w_s')$, where w_s, w_s' are the displacement and the velocity of the block. The reaction force of the block to the beam is

$$R(t) = -\eta_1\frac{\mathrm{d}}{\mathrm{d}t}\left[w_r(0,t) - w_s(t)\right] - k_1(w_r(0,t) - w_s(t)) \tag{6}$$

By performing the Fourier transform of the last equation with regard to time t, we obtain

$$\hat{R}(\omega) = -\kappa_p\left[\hat{w}_r(0,\omega) - \hat{w}_s(\omega)\right] \tag{7}$$

where $\kappa_p = \mathrm{i}\omega\eta_1 + k_1$. Combining the last equation and equation (3) leads to the following result

$$\begin{cases} \hat{w}_r(0,\omega) = \dfrac{\kappa_p \hat{w}_s(\omega) - \mathcal{Q}_e}{\kappa_p + \mathcal{K}_e} \\ \hat{R}(\omega) = \dfrac{\kappa_p}{\kappa_p + \mathcal{K}_e}\left(\mathcal{K}_e \hat{w}_s(\omega) + \mathcal{Q}_e\right) \end{cases} \tag{8}$$

The dynamical equation of the block on the foundation is

$$M\frac{\mathrm{d}^2 w_s(t)}{\mathrm{d}t^2} + \eta_2 \frac{\mathrm{d}w_s}{\mathrm{d}t} + k_2 w_s + f(w_s, w_s') = -R(t) \tag{9}$$

where M is the mass of the block, $f(w_s, w_s')$ is the nonlinear part of the reaction force of the foundation. By substituting equation (6) into the last equation, we have

$$Mw_s'' + \eta_s w_s' + k_s w_s + f = \eta_1 w_r'(0,t) + k_1 w_r(0,t) \tag{10}$$

where $\eta_s = \eta_1 + \eta_2$, $k_s = k_1 + k_2$ and the prime stands for the derivation in time t. Performing the Fourier transform and then the inverse Fourier transform of the right term of equation (10) leads to the following result

$$\eta_1 w_r'(0,t) + k_1 w_r(0,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \kappa_p \hat{w}_r(0,\omega) e^{\mathrm{i}\omega t}\mathrm{d}\omega \tag{11}$$

By substituting equation (11) into equation (10), we can write

$$Mw_s'' + \eta_s w_s' + k_s w_s + f(w_s, w_s') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\kappa_p^2 \hat{w}_s(\omega) - \kappa_p \mathcal{Q}_e}{\kappa_p + \mathcal{K}_e} e^{\mathrm{i}\omega t}\mathrm{d}\omega \tag{12}$$

Equation (12) is the dynamical equation of the block. This equation is similar to that of nonlinear oscillators, but it contains terms corresponding to the interaction between the beam and the block. In order to simplify these terms, we consider only the periodical solution which exists when the moving forces are a periodic series (see [17]).

2.3 Periodic series of moving loads

When a train contains many identical wagons, the loads of wheels are equal ($Q_j = Q$) and the distances D_j are given by

$$D_j = \begin{cases} jH & \text{for front wheels} \\ jH + D & \text{for back wheels} \end{cases} \tag{13}$$

where D is the distance between the front and the back wheels of a boogie, H is the distance between two front wheels of two boogies (see Figure 2). This series of moving loads may be used to represent the limit charges for a railway track. By considering the moving loads as an infinite periodic series ($j \in \mathbb{Z}$), we will use the periodicity of this series to reduce the terms on the right side of equation (12).

By substituting equation (13) into equation (5), we obtain

$$\mathcal{Q}_e = \frac{Q\mathcal{K}_e}{vEI} \frac{\left(1 + e^{-\mathrm{i}\omega\frac{D}{v}}\right)}{\left(\frac{\omega}{v}\right)^4 - \lambda_e^4} \sum_{j=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega\frac{H}{v}j} \tag{14}$$

In addition, we have the propriety of Dirac comb [20]

$$\sum_{j=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega\frac{H}{v}j} = 2\pi\frac{v}{H}\sum_{j=-\infty}^{\infty}\delta\left(\omega + \frac{2\pi v}{H}j\right) \tag{15}$$

Thus, equation (14) becomes

$$\mathcal{Q}_e = 2\pi\frac{Q\mathcal{K}_e}{EIH}\frac{1 + e^{-\mathrm{i}\omega\frac{D}{v}}}{\left(\frac{\omega}{v}\right)^4 - \lambda_e^4}\sum_{j=-\infty}^{\infty}\delta\left(\omega + \frac{2\pi v}{H}j\right) \tag{16}$$

By substituting the last equation into the last term of equation (12), we have

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\kappa_p \mathcal{Q}_e \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega}{\kappa_p + \mathcal{K}_e} &= \sum_{\substack{j=-\infty \\ \omega=\omega_j}}^{\infty} \frac{Q\mathrm{e}^{\mathrm{i}\omega_j t}}{EIH}\left[\frac{1 + \mathrm{e}^{-\mathrm{i}\omega\frac{D}{v}}}{\left(\frac{\omega}{v}\right)^4 - \lambda_e^4}\frac{\kappa_p\mathcal{K}_e}{\kappa_p + \mathcal{K}_e}\right] \\ &= \sum_{j=-\infty}^{\infty} F_j \mathrm{e}^{\mathrm{i}\omega_j t} \end{aligned} \tag{17}$$

where $\omega_j = 2\pi j\frac{v}{H}$ and F_j is calculated by

$$F_j = \frac{Q}{EIH}\left[\frac{1 + \mathrm{e}^{-\mathrm{i}\omega\frac{D}{v}}}{\left(\frac{\omega}{v}\right)^4 - \lambda_e^4}\frac{\kappa_p\mathcal{K}_e}{\kappa_p + \mathcal{K}_e}\right]_{\omega=\omega_j} \tag{18}$$

Therefore, by substituting equation (17) into equation (12), we can write

$$Mw_s'' + \eta_s w_s' + k_s w_s + f(w_s, w_s') = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\kappa_p^2 \hat{w}_s(\omega)\mathrm{e}^{\mathrm{i}\omega t}}{\kappa_p + \mathcal{K}_e}\mathrm{d}\omega - \sum_{j=-\infty}^{\infty} F_j \mathrm{e}^{\mathrm{i}\omega_j t} \tag{19}$$

Equation (19) describes a forced oscillation with the exciting force $\sum F_j \mathrm{e}^{\mathrm{i}\omega_j t}$. This force is periodical with frequency $f_0 = H/v$. Therefore, it exists a periodical solution of $w_s(t)$ which can be represented by its Fourier series

$$w_s(t) = \sum_{j=-\infty}^{\infty} c_j \mathrm{e}^{\mathrm{i}\omega_j t} \tag{20}$$

This expression can be also written as follows:

$$\hat{w}_s(\omega) = 2\pi\sum_{j=-\infty}^{\infty} c_j \delta(\omega - \omega_j)$$

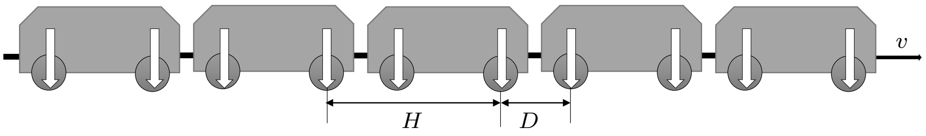

Fig. 2 Periodical series of moving loads

By substituting the last equation into the right term of equation (12), we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\kappa_p^2\hat{w}_s(\omega)}{\kappa_p+\mathcal{K}_e}\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega = \sum_{\substack{j=-\infty\\ \omega=\omega_j}}^{\infty} c_j\mathrm{e}^{\mathrm{i}\omega_j t}\left[\frac{\kappa_p^2}{\kappa_p+\mathcal{K}_e}\right] = \sum_{j=-\infty}^{\infty} c_j P_j \mathrm{e}^{\mathrm{i}\omega_j t} \quad (21)$$

where P_j is calculated by

$$P_j = \left[\frac{\kappa_p^2}{\kappa_p+\mathcal{K}_e}\right]_{\omega=\omega_j} \quad (22)$$

Thus, equation (12) becomes

$$Mw_s''+\eta_s w_s'+k_s w_s+f = \sum_j c_j P_j \mathrm{e}^{\mathrm{i}\omega_j t}-\sum_j F_j\mathrm{e}^{\mathrm{i}\omega_j t} \quad (23)$$

where F_j, P_j are computed by equations (18) and (22). Particularly, we have $P_0 = k_1$ and $F_0 = 2Q\frac{L}{H}$.

In equation (23), F_j corresponds to the moving loads and P_j corresponds to the coupling of the support system and the beam. These quantities depend on the parameters of the beam and the pad (κ_p) but they do not depend on the parameters of the foundation. Therefore, by using the periodicity of the moving loads, we have reduced the dynamic equation (12) to a forced oscillation equation. Thus, we can use analytic or numerical techniques for forced nonlinear oscillators to find out the periodic solution of this equation. In the next sections, we present a numerical method and its applications to nonlinear foundations.

3 Iteration procedure

We will use the harmonic balance method [19] and the iteration procedures [21,22] for nonlinear oscillators to develop a numerical method for the dynamical equation of the block (23). by performing the Fourier series development of equation (23), we have

$$\frac{1}{T}\int_{-T/2}^{T/2}(Mw_s''+\eta_s w'+k_s w_s+f)\mathrm{e}^{-\mathrm{i}\omega_j t}dt = c_j P_j - F_j$$

By substituting equation (20) into the last equation and after rearrangement, we can write

$$(k_s+\mathrm{i}\omega_j\eta_s)c_j+\frac{1}{T}\int_{-T/2}^{T/2} f(w_s,w_s')\mathrm{e}^{-\mathrm{i}\omega_j t}dt = (M\omega_j^2+P_j)c_j - F_j \quad (24)$$

where w_s, w_s' in the integral of the last equation are computed by equation (20)

$$w_s(t) = \sum_{j=-\infty}^{\infty} c_j\mathrm{e}^{\mathrm{i}\omega_j t}, \quad w_s'(t) = \sum_{j=-\infty}^{\infty}\mathrm{i}\omega_j c_j\mathrm{e}^{\mathrm{i}\omega_j t} \quad (25)$$

Equation (24) is the harmonic balance of equation (23). The set of this equation for all $j \in \mathbb{Z}$ establishes a system of equations with regard to variables $\{c_j\}$ which are to be determined. When the foundation behavior is linear, i.e. $f(w_s, w_s') = 0$, we can obtain easily the analytic solution calculated by

$$c_j = \frac{F_j}{P_j + M\omega_j^2 - \mathrm{i}\omega_j\eta_s - k_s} \quad (\forall j \in \mathbb{Z}) \quad (26)$$

For a general nonlinear force $f(w_s, w_s')$, we can approximate the periodic solution by using techniques for nonlinear oscillators. Here, we use iteration procedures by considering the n first harmonics of the periodic solution

$$w_{nm}(t) = \sum_{j=-n}^{n} c_j^m e^{\mathrm{i}\omega_j t} \quad \forall m \geq 1 \quad (27)$$

Here we take the initial value $c_j^1 = 0\ \forall j$. We built series $\{c_j^m\}$ such that $c_j^m \to c_j$ when $m, n \to \infty$ by inserting an index m in equation (24). Such a series $\{c_j^m\}$ is given by

$$\begin{aligned}
&\text{For } |j| \leq n_0 : \\
&\qquad (\mathrm{i}\eta_s\omega_j + k_s)c_j^{m+1} + \mathcal{F}_j^m = (M\omega_j^2 + P_j)c_j^m - F_j \\
&\text{For } n_0 < |j| \leq n : \\
&\qquad (\mathrm{i}\eta_s\omega_j + k_s)c_j^{m} + \mathcal{F}_j^m = (M\omega_j^2 + P_j)c_j^{m+1} - F_j
\end{aligned}$$

where $0 \leq n_0 \leq n$ and n_0 is chosen for the convergence of the series $\{c_j^m\}$ and $\mathcal{F}_j^m$ is calculated by

$$\mathcal{F}_j^m = \frac{1}{T}\int_{-T/2}^{T/2} f(w_{nm}, w_{nm}')e^{-\mathrm{i}\omega_j t}\mathrm{d}t \quad (28)$$

We can rewrite the series $\{c_j^m\}$ as follows

$$c_j^{m+1} = \begin{cases} \dfrac{M\omega_j^2 + P_j}{k_s + \mathrm{i}\eta_s\omega_j} c_j^m - \dfrac{\mathcal{F}_j^m + F_j}{k_s + \mathrm{i}\eta_s\omega_j} & \forall |j| \le n_0 \\ \dfrac{k_s + \mathrm{i}\eta_s\omega_j}{M\omega_j^2 + P_j} c_j^m + \dfrac{\mathcal{F}_j^m + F_j}{M\omega_j^2 + P_j} & \forall |j| > n_0 \end{cases} \quad (29)$$

The last equation defines recurrent sequences $\{c_j^m\}$ with regard to m. If these sequences $\{c_j^m\}$ converge for all $j \in \mathbb{Z}$ when $n, m \to \infty$, by replacing c_j^m, c_j^{m+1} by their limit, we find once again equation (24). Hence, these sequences converge to the solution of (24). By consequence, we have the approximation of the periodic solution from (27) by using the sequence $\{c_j^m\}$ when m is large enough. In the next section, we will use these sequences to compute the response of the block on different foundations.

4 Examples

4.1 Linear foundation

Let's consider a linear foundation, i.e. $f(w_s, w_s') = 0$. The analytic solution is given by equation (26). Now we calculate the response by the numerical method. By substituting $f(w_s, w_s') = 0$ into equation (28), we have $\mathcal{F}_j^m = 0$ and equation (29) becomes

$$c_j^{m+1} = \begin{cases} \dfrac{\left(M\omega_j^2 + P_j\right) c_j^m - F_j}{k_s + \mathrm{i}\eta_s\omega_j} & \text{if } 0 \le |j| \le n_0 \\ \dfrac{\left(k_s + \mathrm{i}\eta_s\omega_j\right) c_j^m + F_j}{M\omega_j^2 + P_j} & \text{if } n_0 < |j| \le n \end{cases}$$

By combining the last equation and equation (26), we obtain

$$c_j^{m+1} - c_j = \begin{cases} \dfrac{M\omega_j^2 + P_j}{k_s + \mathrm{i}\eta_s\omega_j} \left(c_j^m - c_j\right) & \text{if } 0 \le |j| \le n_0 \\ \dfrac{k_s + \mathrm{i}\eta_s\omega_j}{M\omega_j^2 + P_j} \left(c_j^m - c_j\right) & \text{if } n_0 < |j| \le n \end{cases}$$

The last equation describes geometric sequences which converge to zeros if and only if

$$\begin{cases} \left|\dfrac{M\omega_j^2 + P_j}{k_s + \mathrm{i}\eta_s\omega_j}\right| < 1 & \text{if } 0 \le |j| \le n_0 \\ \left|\dfrac{k_s + \mathrm{i}\eta_s\omega_j}{M\omega_j^2 + P_j}\right| < 1 & \text{if } n_0 < |j| \le n \end{cases}$$

Therefore, if n_0 is chosen so that the last inequalities are satisfied, the iteration procedure converges to the analytic solution. Otherwise, this procedure is not convergent.

Figure 3 shows an example for a linear foundation by using the numerical method with different numbers of iterations. The parameters of the railway track are given in Table 1. Here we plot the vertical displacement of a block in one period of the moving loads which corresponds to the time for the train move a distance of a wagon H. The reference time $t = 0$ corresponds to the moment when the front wheel moves over the support. The real-time response of the sleeper is the periodical series with a period as shown in the figure.

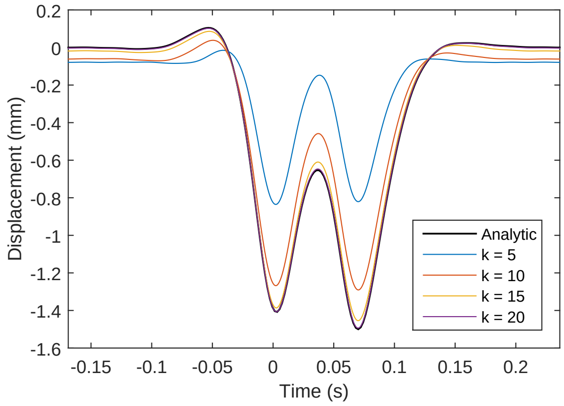

Fig. 3 Analytic solution and numerical results with different numbers of iterations for the linear foundation

The calculations have been performed with the number of harmonics $n = 15$ and the parameter $n_0 = 0$. We see that the numerical results agree well with the analytic results when the number of iterations is bigger than 15.

4.2 Cubic-nonlinear foundation

Consider a nonlinear foundation with the nonlinear part following a cubic law

$$f(w_s, w_s') = ek_3 w_s^3 \quad (30)$$

where $e = 0.8$ and $k_3 = 20$ kNmm^{-3}. Other parameters are given in Table 1. As the previous example, the displacement of a block is calculated with a number of harmonics $n = 15$ and represented in a time interval corresponding to one period of the moving loads. Figure 4 shows the results for different numbers of iterations m. When $m \ge 15$, the response becomes almost unchanged and the iteration procedure converges.

The influence of the nonlinear parameter e on the response of the block has been investigated as shown in Figure 5. When $e = 0$, the foundation is linear. The

Table 1 Parameters of railway track

Content	Notation	Unit	Value
Rail mass	ρS	kgm^{-1}	60
Rail stiffness	EI	MNm^2	6.3
Train speed	v	km/h	160
Charge per wheel	Q	kN	100
Block mass	M	kg	90
Sleeper distance	l	m	0.6
Length of boogie	D	m	3
Length of wagon	H	m	18
Stiffness of rail pad	k_1	MNm^{-1}	200
Damping coefficient of rail pad	η_1	MNsm^{-1}	1.0
Linear stiffness of foundation	k_2	MNm^{-1}	20
Linear damping coefficient of foundation	η_2	MNsm^{-1}	0.2

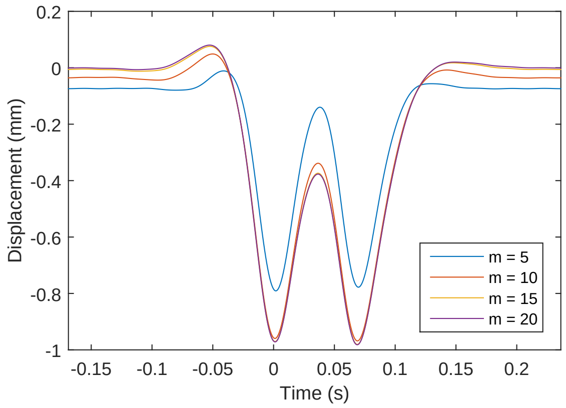

Fig. 4 Numerical results with different numbers of iterations for the cubic-nonlinear foundation

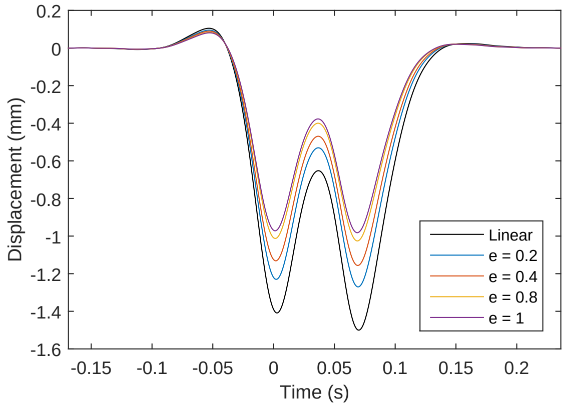

Fig. 5 Effect of the nonlinear parameter of the cubic-nonlinear foundation on the block displacement

amplitude of the displacement decreases when the nonlinear parameter increases. Indeed, the bigger e is, the harder the foundation is.

4.3 Bilinear foundation

Consider a foundation with different linear behaviours in compression and tension. Such a constitutive law can be described by a damping coefficient (η_2), stiffness in compression (k^+) and in traction (k^-). Because the linear and nonlinear parts of the behaviour are separated in the model, the nonlinear part of this foundation is given by

$$f(w_s, w_s') = \begin{cases} (k^+ - k_2)w_s & \text{if } w < 0 \\ (k^- - k_2)w_s & \text{if } w \geq 0 \end{cases} \tag{31}$$

We will calculate the displacement of a block in one period with different numbers of harmonics and iterations in order to study the convergence of the numerical method for a railway track in Table 1 and $k^+ =$ 20MKm^{-1}, $k^- =$ 10MKm^{-1}. Figure 6 shows the numerical results with different numbers of iterations while the number of harmonics is $n = 15$. When the number of iterations is bigger than 50, the numerical method converges well. We find out again the convergence with different numbers of harmonics for $m = 100$ when $n \geq 10$ as shown in Figure 7. By consequence, the result shows that the lower harmonics are more important and the high order harmonics ($n > 10$) can be negligible in this case.

The influence of the nonlinear parameter $r = k^-/k^+$ is studied by using the numerical method with $n = 15$ and $k = 100$. Figure 8 shows the displacement of the block for different parameters r. When $r = 1$, the foundation is linear and for $r = 0$, the foundation is tensionless. We see that a small traction stiffness of the foundation can make a great influence on the response of the dynamical system. It is remarkable that the computational time is almost instantaneous for these three examples, and this is an advantage of this semi-analytical method.

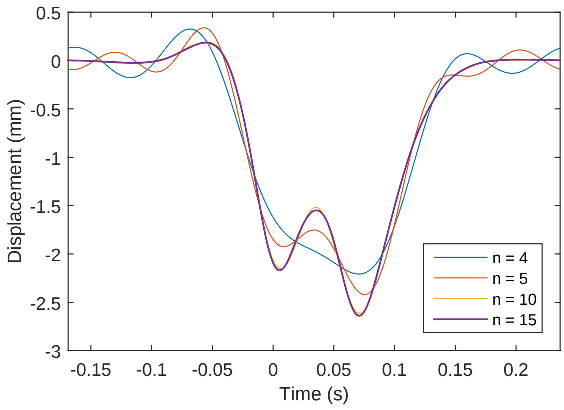

Fig. 6 Numerical results with different numbers of iterations for the bilinear foundation

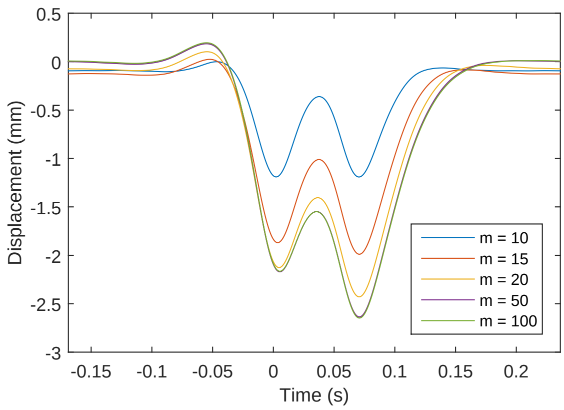

Fig. 7 Numerical results with different numbers of harmonics for the bilinear foundation

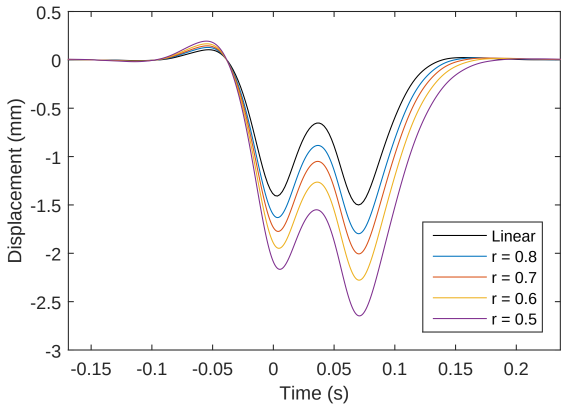

Fig. 8 Effect of the nonlinear parameter of the bilinear foundation on the block displacement

5 Conclusion

A semi-analytic model for periodically supported beams on nonlinear foundations has been developed. When the moving loads is a periodic series, the dynamic equation of the support system has been reduced to a forced nonlinear oscillation. Then, the harmonic balance method and the iteration procedure have been used to compute the periodic response. In linear cases, this method is proven to converge to the analytic solution. For other cases, this method is demonstrated to converges well with a small number of iterations of harmonics. This model is simple, fast and it is applicable for different nonlinear foundations.

Appendix

By performing the Fourier transform of equations (1) and (2) with regard to time t, we have

$$\hat{F}(x,\omega) = \hat{R}(\omega)\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\frac{\omega}{v}x}\delta(x-nl) - \sum_{j=1}^{K}\frac{Q_j}{v}\mathrm{e}^{-\mathrm{i}\frac{\omega}{v}(x+D_j)}$$

$$EI\frac{\partial^4 \hat{w}_r(x,\omega)}{\partial x^4} - \rho S\omega^2\hat{w}_r(x,\omega) - \hat{F}(x,\omega) = 0$$

where $\hat{w}_r(x,\omega)$ and $\hat{R}(\omega)$ are the Fourier transforms of $w_r(x,t)$ and $R(t)$ respectively. Then, performing the spatial Fourier transform of the last results with regard to x leads to the following result

$$(EI\lambda^4 - \rho S\omega^2)\Pi(\lambda,\omega) + 2\pi\delta\left(\lambda+\tfrac{\omega}{v}\right)\sum_{j=1}^{K}\frac{Q_j}{v}\mathrm{e}^{-\mathrm{i}\frac{\omega}{v}D_j} - \hat{R}(\omega)\sum_{n=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}(\lambda+\frac{\omega}{v})nl} = 0 \quad \text{(A.1)}$$

where $\Pi(\lambda,\omega)$ is the Fourier transform of $\hat{w}_r(x,\omega)$ with regard to x. The last term in equation (A.1) is a Dirac comb [20] which has a following propriety

$$\sum_{n=-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\left(\lambda+\frac{\omega}{v}\right)nl} = \frac{2\pi}{l}\sum_{n=-\infty}^{\infty}\delta\left(\lambda+\frac{\omega}{v}+\frac{2\pi}{l}n\right) \quad \text{(A.2)}$$

Then, $\Pi(\lambda,\omega)$ can be obtained from equation (A.1):

$$\Pi(\lambda,\omega) = \frac{2\pi}{EI(\lambda^4-\lambda_e^4)}\left[\frac{\hat{R}(\omega)}{l}\sum_{n}\delta\left(\lambda+\frac{\omega}{v}+\frac{2\pi}{l}n\right) - \delta\left(\lambda+\frac{\omega}{v}\right)\sum_{j}\frac{Q_j}{v}\mathrm{e}^{-\mathrm{i}\frac{\omega}{v}D_j}\right] \quad \text{(A.3)}$$

where $\lambda_e = \sqrt[4]{\frac{\rho S\omega^2}{EI}}$. Thereafter, the expression of $\hat{w}_r(x,\omega)$ is deduced by performing the inverse Fourier transform of $\Pi(\lambda,\omega)$

$$\hat{w}_r(x,\omega) = \frac{\hat{R}(\omega)}{lEI}\sum_{n=-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}\left(\frac{\omega}{v}+\frac{2\pi n}{l}\right)x}}{\left(\frac{\omega}{v}+\frac{2\pi n}{l}\right)^4-\lambda_e^4} - \sum_{j=1}^{K}\frac{Q_j\mathrm{e}^{-\mathrm{i}\frac{\omega}{v}(x+D_j)}}{vEI\left[\left(\frac{\omega}{v}\right)^4-\lambda_e^4\right]} \quad \text{(A.4)}$$

By substituting $x=0$ into the last equation, we obtain the vertical displacement of the beam at the support position

$$\hat{w}_r(0,\omega) = \hat{R}(\omega)\eta_e(\omega) - \sum_{j=1}^{K} \frac{Q_j \mathrm{e}^{-\mathrm{i}\omega\frac{D_j}{v}}}{vEI\left[\left(\frac{\omega}{v}\right)^4 - \lambda_e^4\right]} \quad \text{(A.5)}$$

where $\eta_e(\omega)$ is calculated by

$$\eta_e(\omega) = \frac{1}{lEI}\sum_{n=-\infty}^{\infty}\frac{1}{\left(\frac{\omega}{v}+\frac{2\pi n}{l}\right)^4 - \lambda_e^4} \quad \text{(A.6)}$$

The last expression is a sum of an infinite series which can be deduced to the function in equation (4) by a symbolic computation (see [17]) . Finally, we get equation (3) from equation (A.5).

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