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F. Hache · I. Elishakoff · N. Challamel

Free vibration analysis of plates taking into account rotary inertia and shear deformation via three alternative theories: a Lévy-type solution

Abstract This paper deals with the exact calculation of natural frequencies of a plate with two opposite edges simply supported considering three versions of the Uflyand–Mindlin plate theory: the original Uflyand–Mindlin plate theory, the truncated version of this model as suggested by Elishakoff, and the recently proposed Uflyand–Mindlin plate theory based on slope inertia. The comparison between the frequencies using the different models and with those found in the literature using numerical methods shows the efficiency of the models and the methods presented hereinafter.

1 Introduction

Rectangular plates are widely used in engineering practice. It appears crucial to understand their free vibrational behavior. Different models exist to describe the mechanical behavior of plates. The Kirchhoff–Love (KL) plate theory [1], also referred to as the classical plate theory, is applicable to thin plates. Within this theory the effects of shear deformation and rotatory inertia are neglected resulting in the overestimation of vibration frequencies. This error becomes pronounced with the increase in the thickness of the plate. Thus, the classical plate theory produces accurate solutions until the thickness-to-length ratio reaches a value of about $1/20$ [2]. Reissner [3] proposed a static theory taking into account the shear effect. In 1948 Uflyand [4], and several years later, in 1951, Mindlin [5] developed the thick plate theory or first-order shear deformation theory (FSDT) including effects of shear deformation and rotatory inertia. Several decades later, Elishakoff [6] suggested a truncated version of this theory. A variational derivation based on slope inertia is also leading to a similar equation with an additional term in the governing differential equation [7] and in the boundary condition for a free edge. Since the last decades, the Uflyand–Mindlin plate model has been widely studied (see, for example, Irschik [8,9], Brunelle and Roberts [10], Brunelle [11], Sharma et al. [12]) and different methods were developed in order to determine the natural frequencies [13–16].

Liew et al. [13] presented the numerical p-version Ritz method and obtained the natural frequencies for different non-dimensional thickness. The Rayleigh–Ritz method is widely used in the literature. Introduced by Rayleigh [17] and further developed by Ritz [18], this method assumes that the displacement is a linear

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combination of arbitrarily chosen admissible functions. Each amplitude coefficient associated with the functions must, according to the variational principle, minimize the energy functional. Since the last decades, many investigations are focused on the determination of appropriate functions [19–22]. The accuracy depends on the number of functions used in the series representing the displacement field, the expression of these functions; scientists have to find a compromise between accuracy and computational economy, as Liew et al. [13] mention.

Leissa [23] found the exact characteristic equations for thin rectangular plates having two opposite sides simply supported by using the Lévy approach. In this method, he distinguished the symmetric and antisymmetric modes. In addition to a better understanding of the vibrational mechanics, the problem is simplified and reduced to the determination of frequencies for which the determinant of a matrix 3×3 vanishes. Later, Gorman [24] generalized this work and developed the superposition method providing the natural frequencies for any boundary conditions. Later, Gorman and Ding [25] applied this method to obtain an exact expression of the solution for point supported Uflyand–Mindlin plates. Hashemi and Arsanjani [26] distinguished the symmetric and antisymmetric modes by using the Lévy approach, extending the work of Gorman to the free vibration of a thick plate.

The aim of this paper is to obtain the natural frequencies of a thick plate through the Lévy approach for the three alternative theories, namely (a) the original Uflyand–Mindlin plate theory, (b) the truncated Uflyand–Mindlin plate theory, and (c) the Uflyand–Mindlin plate theory based on slope inertia, and to compare them with the results reported in the literature. Detailed numerical results are reported for all three versions for the first 10 natural frequencies for each of the six combinations of boundary conditions.

2 Derivation of Uflyand–Mindlin plate models

2.1 Original and truncated Uflyand–Mindlin plate models

It appears instructive to provide the variational derivation of the original Uflyand–Mindlin’s equations presented in studies by Mindlin [5] and the equations of the Uflyand–Mindlin’s theory based on slope inertia. Consider a thick rectangular plate of length a , width b , and uniform thickness h , as shown in Fig. 1. Hereinafter, plates with two opposite edges simply supported will be considered. For the purpose of description, a special notation will be adopted, the same as the one commonly used in the literature [23,26,27] for the boundary conditions. The symbolism SCSF, for example, will identify a plate with the edges $x = 0$ and $x = a$ simply supported, and the two others clamped and free [19] (see Fig. 3).

ψ_x and ψ_y are the bending rotations of a transverse normal about the x and y axis respectively, as shown in Fig. 2.

The potential energy is given in the following form:

$$V = \iint_{\Omega} \frac{1}{2} \left(D \left\{ \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right)^2 - 2(1-\nu) \left[\frac{\partial \psi_x}{\partial x} \frac{\partial \psi_y}{\partial y} - \frac{1}{4} \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right)^2 \right] \right\} + \kappa^2 G h \left[\left(\frac{\partial w}{\partial x} + \psi_x \right)^2 + \left(\frac{\partial w}{\partial y} + \psi_y \right)^2 \right] \right) dx dy, \quad (1)$$

Fig. 1 Plate of dimensions $a \times b \times h$

Fig. 2 Rotations of a transverse normal about the y axis

Fig. 3 Set of considered boundary conditions: SSSS, SCSC, SFSF, SFSS, SSSC, SFSC

where $D = Eh^3/12(1 - \nu^2)$ is the plate's flexural rigidity, ν the Poisson's ratio, κ^2 the shear coefficient, and G the shear modulus of elasticity.

The expression of the kinetic energy is the following:

$$T = \frac{1}{2} \iint_{\Omega} \rho h \left(\frac{\partial w}{\partial t} \right)^2 + \frac{\rho h^3}{12} \left[\left(\frac{\partial \psi_x}{\partial t} \right)^2 + \left(\frac{\partial \psi_y}{\partial t} \right)^2 \right] dx dy, \quad (2)$$

where Ω is the area of the mid-surface of the plate. Here it is worth noting that the term V is responsible for the shear effect, whereas T is associated with rotary inertia. Mindlin [5] and Liew et al. [13], when using the expression for kinetic energy, do not mention that de facto by using the expression $\left[\rho I (\partial \psi_x / \partial t)^2 + \rho I (\partial \psi_y / \partial t)^2 \right]$ they, in fact, correct the rotary inertia term which should be with the shear effect.

According to Hamilton's principle

$$\delta \int_{t_i}^t \Pi dt = 0, \quad (3)$$

where the Lagrangian Π is given by:

$$\Pi = T - V = \frac{1}{2} \iint_{\Omega} \left\{ \frac{\rho h^3}{12} \left[\left(\frac{\partial \psi_x}{\partial t} \right)^2 + \left(\frac{\partial \psi_y}{\partial t} \right)^2 \right] + \rho h \left(\frac{\partial w}{\partial t} \right)^2 \right\} dx dy$$

$$\begin{aligned}
& -\frac{1}{2} \iint_{\Omega} \left(D \left\{ \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right)^2 - 2(1-\nu) \left[\frac{\partial \psi_x}{\partial x} \frac{\partial \psi_y}{\partial y} - \frac{1}{4} \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right)^2 \right] \right\} \right. \\
& \left. + \kappa^2 Gh \left[\left(\frac{\partial w}{\partial x} + \psi_x \right)^2 + \left(\frac{\partial w}{\partial y} + \psi_y \right)^2 \right] \right) dx dy
\end{aligned} \tag{4}$$

In view of Eqs. (3) and (4), one obtains:

$$\begin{aligned}
& \int_{t_i}^t \iint_{\Omega} \left\{ -D \left(\frac{\partial \psi_x}{\partial x} \frac{\partial \delta \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \frac{\partial \delta \psi_y}{\partial y} + \nu \frac{\partial \psi_x}{\partial x} \frac{\partial \delta \psi_y}{\partial y} + \nu \frac{\partial \delta \psi_x}{\partial x} \frac{\partial \psi_y}{\partial y} \right) \right. \\
& - \frac{D(1-\nu)}{2} \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \left(\frac{\partial \delta \psi_x}{\partial y} + \frac{\partial \delta \psi_y}{\partial x} \right) \\
& - \kappa^2 Gh \left[\left(\frac{\partial w}{\partial x} + \psi_x \right) \left(\frac{\partial \delta w}{\partial x} + \delta \psi_x \right) + \left(\frac{\partial w}{\partial y} + \psi_y \right) \left(\frac{\partial \delta w}{\partial y} + \delta \psi_y \right) \right] \\
& \left. + \rho h \frac{\partial w}{\partial t} \frac{\partial \delta w}{\partial t} + \frac{\rho h^3}{12} \left(\frac{\partial \psi_x}{\partial t} \frac{\partial \delta \psi_x}{\partial t} + \frac{\partial \psi_y}{\partial t} \frac{\partial \delta \psi_y}{\partial t} \right) \right\} dx dy dt = 0.
\end{aligned} \tag{5}$$

Integration by parts results in:

$$\begin{aligned}
& \int_{t_i}^t \iint_{\Omega} \left\{ D \left(\frac{\partial^2 \psi_x}{\partial x^2} \delta \psi_x + \frac{\partial^2 \psi_y}{\partial y^2} \delta \psi_y + \nu \frac{\partial^2 \psi_x}{\partial x \partial y} \delta \psi_y + \nu \frac{\partial^2 \psi_y}{\partial x \partial y} \delta \psi_x \right) \right. \\
& + \frac{D(1-\nu)}{2} \left(\frac{\partial^2 \psi_x}{\partial y^2} \delta \psi_x + \frac{\partial^2 \psi_y}{\partial x^2} \delta \psi_y + \frac{\partial^2 \psi_x}{\partial x \partial y} \delta \psi_y + \frac{\partial^2 \psi_y}{\partial x \partial y} \delta \psi_x \right) \\
& - \kappa^2 Gh \left[\left(\psi_x \delta \psi_x - \frac{\partial \psi_x}{\partial x} \delta w \right) + \left(\psi_y \delta \psi_y - \frac{\partial \psi_y}{\partial y} \delta w \right) + \left(\frac{\partial w}{\partial x} \delta \psi_x - \frac{\partial^2 w}{\partial x^2} \delta w \right) \right. \\
& \left. + \left(\frac{\partial w}{\partial y} \delta \psi_y - \frac{\partial^2 w}{\partial y^2} \delta w \right) \right] - \rho h \frac{\partial^2 w}{\partial t^2} \delta w - \frac{\rho h^3}{12} \left(\frac{\partial^2 \psi_x}{\partial t^2} \delta \psi_x + \frac{\partial^2 \psi_y}{\partial t^2} \delta \psi_y \right) \Big\} dx dy dt \\
& - \int_{t_i}^t \oint_{\Gamma} \left\{ D \left(\frac{\partial \psi_x}{\partial x} \delta \psi_x dy - \frac{\partial \psi_y}{\partial y} \delta \psi_y dx - \nu \frac{\partial \psi_x}{\partial x} \delta \psi_y dx + \nu \frac{\partial \psi_y}{\partial y} \delta \psi_x dy \right) \right. \\
& + \frac{D(1-\nu)}{2} \left(-\frac{\partial \psi_x}{\partial y} \delta \psi_x dy + \frac{\partial \psi_x}{\partial y} \delta \psi_y dy - \frac{\partial \psi_y}{\partial x} \delta \psi_x dx + \frac{\partial \psi_y}{\partial x} \delta \psi_y dy \right) \\
& \left. + \kappa^2 Gh \left(\psi_x \delta w dy + \frac{\partial w}{\partial x} \delta w dy - \psi_y \delta w dx - \frac{\partial w}{\partial y} \delta w dx \right) \right\} dt = 0,
\end{aligned} \tag{6}$$

where Γ is the boundary path. Grouping the terms in the foregoing functional with respect to the variation terms yields

$$\begin{aligned}
& \int_{t_i}^t \iint_{\Omega} \left\{ \left[D \left(\frac{\partial^2 \psi_x}{\partial x^2} + \nu \frac{\partial^2 \psi_y}{\partial x \partial y} \right) + \frac{D(1-\nu)}{2} \left(\frac{\partial^2 \psi_x}{\partial y^2} + \nu \frac{\partial^2 \psi_y}{\partial x \partial y} \right) - \kappa^2 Gh \left(\psi_x + \frac{\partial w}{\partial x} \right) - \frac{\rho h^3}{12} \frac{\partial^2 \psi_x}{\partial t^2} \right] \delta \psi_x \right. \\
& + \left[D \left(\frac{\partial^2 \psi_y}{\partial y^2} + \nu \frac{\partial^2 \psi_x}{\partial x \partial y} \right) + \frac{D(1-\nu)}{2} \left(\frac{\partial^2 \psi_y}{\partial x^2} + \nu \frac{\partial^2 \psi_x}{\partial x \partial y} \right) - \kappa^2 Gh \left(\psi_y + \frac{\partial w}{\partial y} \right) - \frac{\rho h^3}{12} \frac{\partial^2 \psi_y}{\partial t^2} \right] \delta \psi_y \\
& + \left[\kappa^2 Gh \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) - \rho h \frac{\partial^2 w}{\partial t^2} \right] \delta w \Big\} dx dy dt \\
& - \int_{t_i}^t \oint_{\Gamma} \left\{ \left[D \left(\frac{\partial \psi_x}{\partial x} dy + \nu \frac{\partial \psi_y}{\partial y} dy \right) - \frac{D(1-\nu)}{2} \left(\frac{\partial \psi_x}{\partial y} dx + \frac{\partial \psi_y}{\partial x} dx \right) \right] \delta \psi_x \right.
\end{aligned}$$

$$\begin{aligned}
& + \left\{ -D \left(\frac{\partial \psi_y}{\partial y} dx + \nu \frac{\partial \psi_x}{\partial x} dx \right) + \frac{D(1-\nu)}{2} \left(\frac{\partial \psi_x}{\partial y} dy + \frac{\partial \psi_y}{\partial x} dx \right) \right\} \delta \psi_y \\
& + \kappa^2 Gh \left(\psi_x dy + \frac{\partial w}{\partial x} dy - \psi_y dx - \frac{\partial w}{\partial y} dx \right) \delta w \Big] dt = 0.
\end{aligned} \tag{7}$$

Equating the coefficients of the variation terms to zero for the functional over the plate area, the equations of motions are obtained [8, 16] as

$$\frac{D}{2} \left[(1-\nu) \nabla^2 \psi_x + (1+\nu) \left(\frac{\partial^2 \psi_x}{\partial x^2} + \frac{\partial \psi_y}{\partial x \partial y} \right) \right] - \kappa^2 Gh \left(\psi_x + \frac{\partial w}{\partial x} \right) = \frac{\rho h^3}{12} \frac{\partial^2 \psi_x}{\partial t^2}, \tag{8}$$

$$\frac{D}{2} \left[(1-\nu) \nabla^2 \psi_y + (1+\nu) \left(\frac{\partial \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial y^2} \right) \right] - \kappa^2 Gh \left(\psi_y + \frac{\partial w}{\partial y} \right) = \frac{\rho h^3}{12} \frac{\partial^2 \psi_y}{\partial t^2}, \tag{9}$$

$$\kappa^2 Gh \left(\nabla^2 w + \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right) = \rho h \frac{\partial^2 w}{\partial t^2}. \tag{10}$$

Some algebraic manipulations lead to the governing differential equation in terms of the plate's displacement:

$$D \nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} - \rho \frac{h^3}{12} \left(1 + \frac{12}{h^3} \frac{D}{\kappa^2 G} \right) \frac{\partial^2}{\partial t^2} \nabla^2 w + \frac{\rho^2 h^3}{12 \kappa^2 G} \frac{\partial^4 w}{\partial t^4} = 0. \tag{11}$$

In his paper, Elishakoff, Hache and Challamel [7], following Elishakoff [6] for the Timoshenko beams, suggested that $\partial^2 \psi_x / \partial t^2$ and $\partial^2 \psi_y / \partial t^2$ in Eqs. (8) and (9) ought to be, respectively, replaced by $\partial^3 w / \partial x \partial t^2$ and $\partial^3 w / \partial y \partial t^2$ for Mindlin plates.

After some algebraic manipulations, a truncated simpler governing differential equation is obtained:

$$D \nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} - \rho \frac{h^3}{12} \left(1 + \frac{12}{h^3} \frac{D}{\kappa^2 G} \right) \frac{\partial^2}{\partial t^2} \nabla^2 w = 0 \tag{12}$$

instead of Eq. (11). This truncated Mindlin plate model can be supported by asymptotic arguments from 3d-elasticity, as it is proven for the truncated Bresse–Timoshenko model [28].

In the case where $12D/h^3 \kappa^2 G \gg 1$, one obtains the reduced truncated Mindlin plate equation:

$$D \nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} - \rho \frac{D}{\kappa^2 G} \frac{\partial^2}{\partial t^2} \nabla^2 w = 0. \tag{13}$$

Note that this equation is also obtained by Mindlin [28] without any comment on the benefit or drawback in comparison with original Uflyand [4] and Mindlin [5] derivations of Eq. (11). For boundary conditions, the line integral of Eq. (6) is set to zero and rewritten as:

$$\begin{aligned}
& \int_{t_i}^t \oint_{\Gamma} \left[D \left(\frac{\partial \psi_x}{\partial x} + \nu \frac{\partial \psi_y}{\partial y} \right) \delta \psi_x dy - D \left(\frac{\partial \psi_y}{\partial y} + \nu \frac{\partial \psi_x}{\partial x} \right) \delta \psi_y dx + \frac{D(1-\nu)}{2} \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \delta \psi_y dy \right. \\
& \left. - \frac{D(1-\nu)}{2} \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \delta \psi_x dx + \kappa^2 Gh \left(\psi_x + \frac{\partial w}{\partial x} \right) \delta w dy - \kappa^2 Gh \left(\psi_y + \frac{\partial w}{\partial y} \right) \delta w dx \right] dt = 0.
\end{aligned} \tag{14}$$

At the boundaries of the plate, for edges parallel to the x axis,

$$\begin{aligned}
D \left(\frac{\partial \psi_x}{\partial x} + \nu \frac{\partial \psi_y}{\partial y} \right) &= 0 \quad \text{or} \quad \psi_x \\
\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} &= 0 \quad \text{or} \quad \psi_y \\
\kappa^2 Gh \left(\psi_y + \frac{\partial w}{\partial y} \right) &= 0 \quad \text{or} \quad w
\end{aligned} \tag{15}$$

are specified. For edges parallel to the y axis,

$$\begin{aligned}
D \left(\frac{\partial \psi_y}{\partial y} + \nu \frac{\partial \psi_x}{\partial x} \right) &= 0 \quad \text{or} \quad \psi_y \\
\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} &= 0 \quad \text{or} \quad \psi_x \\
\kappa^2 Gh \left(\psi_x + \frac{\partial w}{\partial x} \right) &= 0 \quad \text{or} \quad w
\end{aligned} \tag{16}$$

are specified. It is assumed in this case that the boundary conditions of the truncated Mindlin plate model and the original Mindlin plate model are the same.

2.2 Uflyand–Mindlin plate model based on slope inertia

In his paper, Mindlin [5] uses the exact expression of the kinetic energy in three dimensions given by the general linear theory of elasticity. By using Eq. (2) which contains also a correction in order to take into account the shear effect, Mindlin “overcorrected,” as it were, the kinetic energy. It is suggested [7] to replace the expression of the kinetic energy given in Eq. (2) by:

$$T = \frac{1}{2} \iint \frac{\rho h^3}{12} \left[\left(\frac{\partial^2 w}{\partial t \partial x} \right)^2 + \left(\frac{\partial^2 w}{\partial t \partial y} \right)^2 \right] + \rho h \left(\frac{\partial w}{\partial t} \right)^2 dx dy. \quad (17)$$

Substituting Eqs. (1) and (17) into Eq. (3) yields

$$\begin{aligned} \int_{t_i}^t \iint_{\Omega} \left\{ -D \left(\frac{\partial \psi_x}{\partial x} \frac{\partial \delta \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \frac{\partial \delta \psi_y}{\partial y} + \nu \frac{\partial \psi_x}{\partial x} \frac{\partial \delta \psi_y}{\partial y} + \nu \frac{\partial \delta \psi_x}{\partial x} \frac{\partial \psi_y}{\partial y} \right) \right. \\ - \frac{D(1-\nu)}{2} \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \left(\frac{\partial \delta \psi_x}{\partial y} + \frac{\partial \delta \psi_y}{\partial x} \right) \\ - \kappa^2 G h \left[\left(\frac{\partial w}{\partial x} + \psi_x \right) \left(\frac{\partial \delta w}{\partial x} + \delta \psi_x \right) + \left(\frac{\partial w}{\partial y} + \psi_y \right) \left(\frac{\partial \delta w}{\partial y} + \delta \psi_y \right) \right] \\ \left. + \rho h \frac{\partial w}{\partial t} \frac{\partial \delta w}{\partial t} + \frac{\rho h^3}{12} \left(\frac{\partial^2 w}{\partial t \partial x} \frac{\partial^2 \delta w}{\partial t \partial x} + \frac{\partial^2 w}{\partial t \partial y} \frac{\partial^2 \delta w}{\partial t \partial y} \right) \right\} dx dy dt = 0. \end{aligned} \quad (18)$$

Integrating by parts yields

$$\begin{aligned} \int_{t_i}^t \iint_{\Omega} \left\{ \left[D \left(\frac{\partial^2 \psi_x}{\partial x^2} + \nu \frac{\partial^2 \psi_y}{\partial x \partial y} \right) + \frac{D(1-\nu)}{2} \left(\frac{\partial^2 \psi_x}{\partial y^2} + \nu \frac{\partial^2 \psi_y}{\partial x \partial y} \right) - \kappa^2 G h \left(\psi_x + \frac{\partial w}{\partial x} \right) \right] \delta \psi_x \right. \\ + \left[D \left(\frac{\partial^2 \psi_y}{\partial y^2} + \nu \frac{\partial^2 \psi_x}{\partial x \partial y} \right) + \frac{D(1-\nu)}{2} \left(\frac{\partial^2 \psi_y}{\partial x^2} + \nu \frac{\partial^2 \psi_x}{\partial x \partial y} \right) - \kappa^2 G h \left(\psi_y + \frac{\partial w}{\partial y} \right) \right] \delta \psi_y \\ + \left[\kappa^2 G h \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) - \rho h \frac{\partial^2 w}{\partial t^2} + \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} \nabla^2 w \right] \delta w \Big\} dx dy dt \\ - \int_{t_i}^t \oint_{\Gamma} \left[\left\{ D \left(\frac{\partial \psi_x}{\partial x} dy + \nu \frac{\partial \psi_y}{\partial y} dx \right) - \frac{D(1-\nu)}{2} \left(\frac{\partial \psi_x}{\partial y} dx + \frac{\partial \psi_y}{\partial x} dy \right) \right\} \delta \psi_x \right. \\ + \left\{ -D \left(\frac{\partial \psi_y}{\partial y} dx + \nu \frac{\partial \psi_x}{\partial x} dy \right) + \frac{D(1-\nu)}{2} \left(\frac{\partial \psi_x}{\partial y} dy + \frac{\partial \psi_y}{\partial x} dx \right) \right\} \delta \psi_y \\ + \kappa^2 G h \left(\psi_x dy + \frac{\partial w}{\partial x} dy - \psi_y dx - \frac{\partial w}{\partial y} dx \right) \delta w \\ \left. + \frac{\rho h^3}{12} \left(\frac{\partial^2 w}{\partial t^2 \partial x} dy + \frac{\partial^2 w}{\partial t^2 \partial y} dx \right) \delta w \right] dt = 0. \end{aligned} \quad (19)$$

Equating the coefficients of the variation terms to zero for the functional over the plate area, the equations of motion are obtained as follows:

$$\frac{D}{2} \left[(1-\nu) \nabla^2 \psi_x + (1+\nu) \left(\frac{\partial^2 \psi_x}{\partial x^2} + \frac{\partial \psi_y}{\partial x \partial y} \right) \right] - \kappa^2 G h \left(\psi_x + \frac{\partial w}{\partial x} \right) = 0, \quad (20)$$

$$\frac{D}{2} \left[(1-\nu) \nabla^2 \psi_y + (1+\nu) \left(\frac{\partial \psi_x}{\partial x \partial y} + \frac{\partial^2 \psi_y}{\partial y^2} \right) \right] - \kappa^2 G h \left(\psi_y + \frac{\partial w}{\partial y} \right) = 0, \quad (21)$$

$$\kappa^2 G h \left(\nabla^2 w + \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right) = \rho h \left(1 - \frac{h^2}{12} \nabla^2 \right) \frac{\partial^2 w}{\partial t^2}. \quad (22)$$

From Eqs. (20) to (22), the governing equation is obtained:

$$D \nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} - \frac{\rho h^3}{12} \left(1 + \frac{12}{h^3} \frac{D}{\kappa^2 G} \right) \frac{\partial^2}{\partial t^2} \nabla^2 w + \frac{\rho h^2 D}{12 \kappa^2 G} \frac{\partial^2}{\partial t^2} \nabla^4 w = 0. \quad (23)$$

The difference with original Uflyand–Mindlin equations is twofold: (a) an additional, the last, term occurs, and (b) the fourth-order time derivative, that is characteristic of the original Uflyand–Mindlin plate theory, does not appear.

For boundary conditions, the line integral of Eq. (19) is set to zero and rewritten as:

$$\begin{aligned} \int_{t_i}^t \oint_{\Gamma} \left[\left\{ D \left(\frac{\partial \psi_x}{\partial x} + \nu \frac{\partial \psi_y}{\partial y} \right) dy - \frac{D(1-\mu)}{2} \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) dx \right\} \delta \psi_x \right. \\ \left. + \left\{ -D \left(\frac{\partial \psi_y}{\partial y} + \nu \frac{\partial \psi_x}{\partial x} \right) dx + \frac{D(1-\nu)}{2} \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) dy \right\} \delta \psi_y \right. \\ \left. + \left[\left(\frac{\rho h^3}{12} \frac{\partial^2 w}{\partial t^2 \partial x} + \kappa^2 G h \psi_x + \kappa^2 G h \frac{\partial w}{\partial x} \right) dy - \left(\frac{\rho h^3}{12} \frac{\partial^2 w}{\partial t^2 \partial y} + \kappa^2 G h \psi_y - \kappa^2 G h \frac{\partial w}{\partial y} \right) dx \right] \delta w \right] dt = 0. \end{aligned} \quad (24)$$

Equation (24) implies the boundaries of the plate.

For edges parallel to the x axis,

$$\begin{aligned} D \left(\frac{\partial \psi_x}{\partial x} + \nu \frac{\partial \psi_y}{\partial y} \right) &= 0 \quad \text{or} \quad \psi_x \\ \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} &= 0 \quad \text{or} \quad \psi_y \\ \kappa^2 G h \left(\psi_y + \frac{\partial w}{\partial y} \right) - \frac{\rho h^3}{12} \frac{\partial^2 w}{\partial t^2 \partial y} &= 0 \quad \text{or} \quad w \end{aligned} \quad (25)$$

are specified. For edges parallel to the y axis,

$$\begin{aligned} D \left(\frac{\partial \psi_y}{\partial y} + \nu \frac{\partial \psi_x}{\partial x} \right) &= 0 \quad \text{or} \quad \psi_y \\ \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} &= 0 \quad \text{or} \quad \psi_x \\ \kappa^2 G h \left(\psi_x + \frac{\partial w}{\partial x} \right) + \frac{\rho h^3}{12} \frac{\partial^2 w}{\partial t^2 \partial x} &= 0 \quad \text{or} \quad w \end{aligned} \quad (26)$$

are specified. When the edge is simply supported or clamped, the boundary conditions at the edge match the ones given by the original and the truncated Uflyand–Mindlin plate theory.

3 Transformation of governing differential equations

Introducing control parameters γ_1 , γ_2 , and γ_3 , with $(\gamma_1, \gamma_2, \gamma_3)$ equal to $(1, 0, 0)$, $(0, 0, 1)$ and $(0, 1, 0)$ for the original Uflyand–Mindlin plate theory, the truncated Uflyand–Mindlin theory and the Uflyand–Mindlin model based on slope inertia, also denoted UM, truncated UM, and slope inertia UM, respectively, the equations of motion are given as follows for all Uflyand–Mindlin plate models, setting $C = D(1 - \nu)/2$:

$$D \frac{\partial}{\partial x} \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right) + C \frac{\partial}{\partial y} \left(\frac{\partial \psi_x}{\partial y} - \frac{\partial \psi_y}{\partial x} \right) - \kappa^2 G h (\psi_x + w_{,x}) = \gamma_1 \frac{\rho h^3}{12} \frac{\partial^2 \psi_x}{\partial t^2} + \gamma_3 \frac{\rho h^3}{12} \frac{\partial^3 w}{\partial x \partial t^2}, \quad (27)$$

$$D \frac{\partial}{\partial y} \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right) - C \frac{\partial}{\partial x} \left(\frac{\partial \psi_x}{\partial y} - \frac{\partial \psi_y}{\partial x} \right) - \kappa^2 G h (\psi_y + w_{,y}) = \gamma_1 \frac{\rho h^3}{12} \frac{\partial^2 \psi_y}{\partial t^2} + \gamma_3 \frac{\rho h^3}{12} \frac{\partial^3 w}{\partial y \partial t^2}, \quad (28)$$

$$\kappa^2 G h \left(\frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} + \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = \left(\rho h - \gamma_2 \frac{\rho h^3}{12} \nabla^2 \right) \frac{\partial^2 w}{\partial t^2}. \quad (29)$$

In order to solve these three coupled partial differential equations, it is more convenient to deal with uncoupled equations [26,27,32]. The governing equations can be rewritten as:

$$D \zeta_{,x} + C \varphi_{,y} - \kappa^2 G h (\psi_x + w_{,x}) = \gamma_1 \frac{\rho h^3}{12} \ddot{\psi}_x + \gamma_3 \frac{\rho h^3}{12} \frac{\partial^3 w}{\partial x \partial t^2}, \quad (30)$$

$$D \zeta_{,y} - C \varphi_{,x} - \kappa^2 G h (\psi_y + w_{,y}) = \gamma_1 \frac{\rho h^3}{12} \ddot{\psi}_y + \gamma_3 \frac{\rho h^3}{12} \frac{\partial^3 w}{\partial y \partial t^2}, \quad (31)$$

$$\kappa^2 G h (\psi_{x,x} + \psi_{y,y} + w_{,xx} + w_{,yy}) = \left(\rho h - \gamma_2 \frac{\rho h^3}{12} \nabla^2 \right) \ddot{w}, \quad (32)$$

where the function φ is introduced as [32]:

$$\varphi = \psi_{x,y} - \psi_{y,x} \quad (33)$$

Differentiating Eqs. (30) and (31) with respect to y and x , respectively, and subtracting Eq. (31) from (30), one obtains:

$$C \nabla^2 (\psi_{x,y} - \psi_{y,x}) - \kappa^2 G h (\psi_{x,y} - \psi_{y,x}) = \gamma_1 \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} (\psi_{x,y} - \psi_{y,x}). \quad (34)$$

This yields

$$C \nabla^2 \varphi - \kappa^2 G h \varphi = \gamma_1 \frac{\rho h^3}{12} \frac{\partial^2}{\partial t^2} \varphi. \quad (35)$$

Moreover, Eqs. (11), (12), and (23) are written in general form as follows [7]:

$$D \left(1 + \gamma_2 \frac{\rho h^2}{12 \kappa^2 G} \frac{\partial^2}{\partial t^2} \right) \nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} - \rho \frac{h^3}{12} \left(1 + \frac{12}{h^3} \frac{D}{\kappa^2 G} \right) \frac{\partial^2}{\partial t^2} \nabla^2 w + \gamma_1 \frac{\rho^2 h^3}{12 \kappa^2 G} \frac{\partial^4 w}{\partial t^4} = 0. \quad (36)$$

Thus, the system of three coupled equations is reduced to a system of two uncoupled equations. The solution of such a system is known.

Moreover, the rotation angles ψ_x and ψ_y can be found from Eqs. (30) to (31) by the following relation:

$$\left(\kappa^2 G h - \gamma_1 \frac{\rho h^3}{12} \omega^2 \right) \psi_x = \frac{\partial}{\partial x} \left[-\rho \omega^2 \frac{D \left(1 - \gamma_2 \frac{h^2}{12} \nabla^2 \right)}{\kappa^2 G} w - D \nabla^2 w - \kappa^2 G h w \right] + C \frac{\partial \varphi}{\partial y} + \gamma_3 \frac{\rho h^3}{12} \omega^2 \frac{\partial w}{\partial x}, \quad (37)$$

$$\left(\kappa^2 G h - \gamma_1 \frac{\rho h^3}{12} \omega^2 \right) \psi_y = \frac{\partial}{\partial y} \left[-\rho \omega^2 \frac{D \left(1 - \gamma_2 \frac{h^2}{12} \nabla^2 \right)}{\kappa^2 G} w - D \nabla^2 w - \kappa^2 G h w \right] - C \frac{\partial \varphi}{\partial x} + \gamma_3 \frac{\rho h^3}{12} \omega^2 \frac{\partial w}{\partial y}. \quad (38)$$

It is assumed that

$$(w; \psi_x; \psi_y; \varphi)(x, y, t) = (\bar{w}; \bar{\psi}_x; \bar{\psi}_y; \bar{\varphi})(x, y) e^{i\omega t}. \quad (39)$$

Substituting Eq. (39) into Eqs. (35) and (36) leads to:

$$C \nabla^2 \bar{\varphi} - \left(\kappa^2 G h - \gamma_1 \frac{\rho h^3}{12} \omega^2 \right) \bar{\varphi} = 0, \quad (40)$$

$$D \left(1 - \frac{\rho h^2}{12 \kappa^2 G} \omega^2 \gamma_2 \right) \nabla^4 \bar{w} + \rho \frac{h^3}{12} \left(1 + \frac{12}{h^3} \frac{D}{\kappa^2 G} \right) \omega^2 \nabla^2 \bar{w} + \left(\gamma_1 \frac{\rho^2 h^3}{12 \kappa^2 G} \omega^2 - \rho h \right) \omega^2 \bar{w} = 0. \quad (41)$$

4 Rectangular plates with four edges simply supported

Consider a plate with all the edges simply supported (Fig. 3a). The following non-dimensional numbers are defined:

$$\lambda = \omega b^2 \sqrt{\frac{\rho h}{D}}; \beta = \frac{E}{G(1-\nu^2)}; \gamma = \eta b = \frac{\eta}{\chi} a; \bar{h} = \frac{h}{a}; \xi = \frac{x}{a}; \chi = \frac{a}{b}. \quad (42)$$

In view of the boundary conditions, the solution is given by Navier [29] as

$$\bar{w}(x, y) = a \sin(n\pi \eta) \sin(m\pi \xi), \quad (43)$$

where m and n are the numbers of half-waves in the x and y direction, respectively.

Substituting Eqs. (42) and (43) into Eq. (41),

$$\begin{aligned}
& [(\chi n)^2 + m^2]^2 \pi^4 - \chi^4 \lambda^2 \left[1 + \frac{\bar{h}^2 \pi^2}{12} \left(1 + \frac{\beta}{\kappa^2} \right) [(\chi n)^2 + m^2] + [(\chi n)^2 + m^2]^2 \frac{\beta \bar{h}^4 \pi^4}{144 \kappa^2} \gamma_2 \right] \\
& + \frac{\beta \chi^8 \bar{h}^4}{144 \kappa^2} \gamma_1 \lambda^4 = 0.
\end{aligned} \tag{44}$$

For the original Uflyand–Mindlin model, $\gamma_1 = 1$ and $\gamma_2 = 0$. The solution of Eq. (45) is expressed as:

$$\begin{aligned}
\lambda = & \frac{12}{\chi^2 \bar{h}^2} \sqrt{\frac{\kappa^2}{2\beta} \left(1 + \frac{\bar{h}^2 \pi^2}{12} \left(1 + \frac{\beta}{\kappa^2} \right) [(\chi n)^2 + m^2] \right.} \\
& \left. \pm \sqrt{\left[1 + \frac{\bar{h}^2 \pi^2}{12} \left(1 + \frac{\beta}{\kappa^2} \right) [(\chi n)^2 + m^2] \right]^2 - \frac{\beta \pi^4 \bar{h}^4}{36 \kappa^2} [(\chi n)^2 + m^2]^2} \right)^{\frac{1}{2}}.
\end{aligned} \tag{45}$$

It is seen that the natural frequency is given by two different expressions. The second expression is associated with a second branch of frequencies.

This result matches with the one of Irschik [8], Wang [34], and Wang and Wang [36]. For the truncated Uflyand–Mindlin model and the slope inertia-based Uflyand–Mindlin model, the natural frequencies are given by:

$$[(\chi n)^2 + m^2]^2 \pi^4 - \chi^4 \lambda^2 \left[1 + \frac{\bar{h}^2 \pi^2}{12} \left(1 + \frac{\beta}{\kappa^2} \right) [(\chi n)^2 + m^2] + [(\chi n)^2 + m^2]^2 \frac{\beta \bar{h}^4 \pi^4}{144 \kappa^2} \gamma_2 \right] = 0. \tag{46}$$

From Eq. (47), one obtains

$$\lambda = \frac{[(\chi n)^2 + m^2] \pi^2}{\chi^2 \sqrt{1 + \frac{\bar{h}^2 \pi^2}{12} \left(1 + \frac{\beta}{\kappa^2} \right) [(\chi n)^2 + m^2] + [(\chi n)^2 + m^2]^2 \frac{\beta \bar{h}^4 \pi^4}{144 \kappa^2} \gamma_2}}. \tag{47}$$

Since γ_2 equals zero for the truncated model and equals unity for the slope inertia-based model, the Uflyand–Mindlin model based on slope inertia provides smaller values for natural frequencies than does the truncated Uflyand–Mindlin model. The study of the natural frequencies for the plates having simply supported two opposite edges will show that this result is valid also to other sets of boundary conditions. Moreover, it is worth noting that in contrast to the original Uflyand–Mindlin plate theory, these two models provide one unique branch of frequencies. Therefore, the results provided by these two theories can be compared with the frequencies predicted by the first branch of the original Uflyand–Mindlin theory.

5 Solutions for two opposite simply supported edges: Lévy’s approach

Consider a plate with simple supports along the edges $\xi = 0$ and $\xi = 1$. Following the approach developed by Lévy (see, for instance, Chen and Liu [30], Szilard [31], and Jomehzadeh [32]):

$$\bar{w}(\xi, \eta) = \sum_{m=1}^{\infty} a W_m(\eta) \sin(m\pi\xi); \quad \bar{\psi}_x(\xi, \eta) = \sum_{m=1}^{\infty} \phi_x(\eta) \cos(m\pi\xi); \quad \bar{\psi}_y(\xi, \eta) = \sum_{m=1}^{\infty} \phi_y(\eta) \sin(m\pi\xi). \tag{48}$$

Consequently, using Eq. (33),

$$\bar{\varphi}(\xi, \eta) = \frac{1}{a} \sum_{m=1}^{\infty} \varphi_m(\eta) \cos(m\pi\xi). \tag{49}$$

Substituting Eqs. (48) and (49) into Eqs. (40) and (41) leads to the non-dimensional equations of motion:

$$\Lambda_1 \frac{d^4 W_m}{d\eta^4} + \Lambda_2 \frac{d^2 W_m}{d\eta^2} + \Lambda_3 W_m = 0, \tag{50}$$

$$\Lambda_4 \frac{d^2 \varphi_m}{d\eta^2} + \Lambda_5 \varphi_m = 0, \quad (51)$$

where the material constants Λ_i ($i = 1, \dots, 5$) are given by

$$\begin{aligned} \Lambda_1 &= \chi^4 \left(1 - \gamma_2 \frac{\beta \chi^4 \bar{h}^4}{144 \kappa^2} \lambda^2 \right), \\ \Lambda_2 &= \left[\chi^6 \lambda^2 \frac{\bar{h}^2}{12} \left(1 + \frac{\beta}{\kappa^2} \right) - 2 (\alpha m \pi)^2 \left(1 - \frac{\beta \chi^4 \bar{h}^4}{144 \kappa^2} \lambda^2 \gamma_2 \right) \right], \\ \Lambda_3 &= \left[(m \pi)^4 - \chi^4 \lambda^2 \left[(m \pi)^4 \frac{\beta \bar{h}^4}{144 \kappa^2} \gamma_2 + \frac{(m \pi \bar{h})^2}{12} \left(1 + \frac{\beta}{\kappa^2} \right) + 1 \right] + \frac{\beta \chi^8 \bar{h}^4}{144 \kappa^2} \gamma_1 \lambda^4 \right], \\ \Lambda_4 &= \beta \chi^2 \frac{\bar{h}^2}{24} (1 - \nu), \\ \Lambda_5 &= - \left[(m \pi)^2 \beta \frac{\bar{h}^2}{24} (1 - \nu) + \kappa^2 - \gamma_1 \frac{\beta \chi^4 \bar{h}^4}{144} \lambda^2 \right], \end{aligned} \quad (52)$$

and Eqs. (37) and (38) become:

$$\begin{aligned} & \left(\kappa^2 - \gamma_1 \lambda^2 \beta \frac{\chi^4 \bar{h}^4}{144} \right) \psi_x \\ &= -m \pi \left[\left[\frac{\beta^2 \chi^4 \bar{h}^4}{144} \lambda^2 \frac{1}{\kappa^2} \left(1 + (m \pi)^2 \gamma_2 \frac{\bar{h}^2}{12} \right) - (m \pi)^2 \frac{\beta \bar{h}^2}{12} + \kappa^2 - \gamma_3 \lambda^2 \beta \frac{\chi^4 \bar{h}^4}{144} \right] W \right. \\ & \quad \left. + \left[\chi^2 \frac{\beta \bar{h}^2}{12} - \frac{\beta^2 \chi^6 \bar{h}^6}{1728} \lambda^2 \frac{1}{\kappa^2} \gamma_2 \right] \frac{d^2 W}{d\eta^2} \right] + \chi \beta \frac{\bar{h}^2}{24} (1 - \nu) \frac{\partial \varphi_m}{\partial \eta}, \end{aligned} \quad (53)$$

$$\begin{aligned} & \left(\kappa^2 - \gamma_1 \lambda^2 \beta \frac{\chi^4 \bar{h}^4}{144} \right) \psi_y \\ &= -\chi \frac{\partial}{\partial \eta} \left[\left[\frac{\beta^2 \chi^4 \bar{h}^4}{144} \lambda^2 \frac{1}{\kappa^2} \left(1 + (m \pi)^2 \gamma_2 \frac{\bar{h}^2}{12} \right) - (m \pi)^2 \frac{\beta \bar{h}^2}{12} + \kappa^2 - \gamma_3 \lambda^2 \beta \frac{\chi^4 \bar{h}^4}{144} \right] W \right. \\ & \quad \left. + \left[\chi^2 \frac{\beta \bar{h}^2}{12} - \frac{\beta^2 \chi^6 \bar{h}^6}{1728} \lambda^2 \frac{1}{\kappa^2} \gamma_2 \right] \frac{d^2 W}{d\eta^2} \right] + m \pi \beta \frac{\bar{h}^2}{24} (1 - \nu) \varphi_m. \end{aligned} \quad (54)$$

The roots of the characteristic polynomial of Eq. (50) are:

$$r_{\pm}^2 = \frac{-\Lambda_2 \pm \sqrt{\Lambda_2^2 - 4 \Lambda_1 \Lambda_3}}{2 \Lambda_1}. \quad (55)$$

r_-^2 and r_+^2 are both equal to $(m \pi / \alpha)^2$, positive number, for λ equal to 0. There is a transition frequency, denoted λ_-^* , such that for λ greater than λ_-^* , r_-^2 becomes negative, and so r_- is an imaginary number. Similarly, when λ is greater than a value λ_+^* , r_+ is an imaginary number. For instance, the values of λ_-^* and λ_+^* are listed in the following table for the eight first values of m , with an aspect ratio α , a thickness ratio $\bar{h}$, a shear correction factor κ^2 , and a Poisson ratio ν equal to 1, 0.2, 0.86667, and 0.3, respectively, for the three versions of the Uflyand–Mindlin plate theory. For the particular case of the truncated Uflyand–Mindlin plate model, one can show that λ_+^* go to infinity. Thus, r_+ is always real, independent of the frequency.

The table shows that there is zero or one unique root for any model: The sign of λ_-^* and λ_+^* changes only once. Moreover, λ_-^* is smaller than λ_+^* . λ_-^* and λ_+^* are reached for a certain value of the order of the natural frequency. This value depends on the model, the boundary conditions, and the geometric parameters such as the thickness ratio. For instance, for a plate with all edges simply supported, as shown in Table 2, keeping the same set of parameters in Table 1 $(\kappa^2; \nu; \alpha; \bar{h}) = (0.86667; 0.3; 1; 0.2)$, λ is immediately greater than λ_-^* , and it is greater than λ_+^* after the 18th and the 34th natural frequency for original Uflyand–Mindlin plate theory and the Uflyand–Mindlin plate model based on slope inertia, respectively. It is worth noting that, for this particular case, considering the original Uflyand–Mindlin plate theory, the second branch of frequencies and the transition to the frequency greater than λ_+^* occur at the same moment. Thus, the second transition frequency λ_+^* is reached for big orders of frequencies, not considered usually in the literature.

Thus, we consider the different models and different cases. The solution of the differential equation depends on the value of the frequency λ .

Table 1 Eight first values of the transition frequencies λ_-^* and λ_+^* for the three versions of the Uflyand–Mindlin plate model with $(\kappa^2; \nu; \bar{h}) = (0.86667; 0.3; 1; 0.2)$

	m	1	2	3	4	5	6	7	8
Original	λ_-^*	9.25	31.94	60.76	91.91	123.79	155.82	187.74	219.487
	λ_+^*	176.27	204.24	241.53	283.89	329.32	376.76	425.61	475.50
Truncated	λ_-^*	9.24	31.55	58.93	87.44	115.88	143.99	171.77	199.28
	λ_+^*	Infinite							
Slope inertia	λ_-^*	9.21	30.46	52.61	70.00	82.27	90.73	96.61	100.79
	λ_+^*	165.23	165.23	165.23	165.23	165.23	165.23	165.23	165.23

If $\lambda \leq \lambda_-^* < \lambda_+^*$ then $r_-^2 \geq 0$; $r_+^2 > 0$. The solution of the differential equation is given by

$$W_m(\eta) = C_1 \cosh r_+ \eta + C_2 \sinh r_+ \eta + C_3 \cosh r_- \eta + C_4 \sinh r_- \eta. \quad (56)$$

Define $\tilde{r}_-$ and $\tilde{r}_+$ as follows:

$$\tilde{r}_-^2 = \frac{\Lambda_2 + \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}; \quad \tilde{r}_+^2 = \frac{\Lambda_2 - \sqrt{\Lambda_2^2 - 4\Lambda_1\Lambda_3}}{2\Lambda_1}. \quad (57)$$

If $\lambda_-^* < \lambda \leq \lambda_+^*$ then $r_-^2 < 0$; $r_+^2 \geq 0$; and $\cosh r_- \eta$ and $\sinh r_- \eta$ are replaced by $\cos \tilde{r}_- \eta$ and $\sin \tilde{r}_- \eta$, respectively. If $\lambda_-^* < \lambda$; $\lambda_+^* < \lambda$ then, in addition to the previously given transformation, $\cosh r_+ \eta$ and $\sinh r_+ \eta$ are replaced by $\cos \tilde{r}_+ \eta$ and $\sin \tilde{r}_+ \eta$, respectively.

This last case does not appear in the paper in the paper of Hashemi and Arsanjani [26]. Indeed, he determines the first values of the natural frequencies. Thus, the frequency is always smaller than the second transition value λ_+^* .

Thus, the process to determine the natural frequency is the following. First of all, for fixed values of the aspect ratio α , the thickness ratio $\bar{h}$, the mode shape number m , the shear coefficient κ^2 , and the Poisson's ratio ν , the solution λ is obtained by considering all the possible expressions of the displacements given by Eq. (56).

The roots of the characteristic equation Eq. (51) are:

$$u_{\pm} = \pm \sqrt{\frac{\Lambda_5}{\Lambda_4}}, \quad (58)$$

hence, the solution is expressed as follows:

$$\varphi_m(\eta) = C_5 \sinh u_+ \eta + C_6 \cosh u_+ \eta. \quad (59)$$

This expression coincides with the result obtained by Jomehzadeh and Saidi [32].

In the following, three different boundary conditions are studied: simply supported, free, and clamped (fixed) (see, for instance, Wang et al. [33]).

Clamped edges:

$$w = 0; \quad \psi_x = 0; \quad \psi_y = 0 \quad (60)$$

Simply supported:

$$w = 0; \quad \frac{\partial \psi_y}{\partial y} + \nu \frac{\partial \psi_x}{\partial x} = 0; \quad \psi_x = 0 \quad (61)$$

Free edges:

$$\kappa^2 G h \left(\psi_y + \frac{\partial w}{\partial y} \right) - \gamma_2 \frac{\rho h^3}{12} \omega^2 \frac{\partial w}{\partial y} = 0; \quad \frac{\partial \psi_y}{\partial y} + \nu \frac{\partial \psi_x}{\partial x} = 0; \quad \frac{1}{2} (1 - \nu) D \left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) = 0 \quad (62)$$

6 Numerical results

6.1 Two opposite edges simply supported and two other edges both clamped or free

First of all, for a thin plate, the non-dimensional natural frequencies are the solutions of characteristic equations given by Leissa [19] and Wang and Wang [36]. Following notations are introduced:

$$\lambda_1 = \frac{1}{b} \sqrt{(m\pi)^2 + \lambda^2}; \quad \lambda_2 = \frac{1}{b} \sqrt{(m\pi)^2 - \lambda^2}; \quad \tilde{\lambda}_2 = \frac{1}{b} \sqrt{\lambda^2 - (m\pi)^2}. \quad (63)$$

Thus, for an SCSC plate,

For $m\pi < \lambda$

$$2\lambda_1 \tilde{\lambda}_2 \left[1 - \cosh \frac{\lambda_1}{\chi} \cos \frac{\tilde{\lambda}_2}{\chi} \right] + (\lambda_1^2 - \tilde{\lambda}_2^2) \sinh \frac{\lambda_1}{\chi} \sin \frac{\tilde{\lambda}_2}{\chi} = 0; \quad (64)$$

If $\lambda \leq m\pi$ then $\tilde{\lambda}_2$, $\cos(\tilde{\lambda}_2/\chi)$, and $\sin(\tilde{\lambda}_2/\chi)$ are replaced by λ_2 , $\cosh(\lambda_2/\chi)$, and $\sinh(\lambda_2/\chi)$, respectively. For an SFSF plate, for $m\pi < \lambda$

$$\left\{ \lambda_1^2 [k - (1 - \nu)(m\pi)^2]^4 - \tilde{\lambda}_2^2 [k + (1 - \nu)(m\pi)^2]^4 \right\} \sinh \frac{\lambda_1}{\chi} \sin \frac{\tilde{\lambda}_2}{\chi} + 2\lambda_1 \tilde{\lambda}_2 \left[(\lambda_1 \tilde{\lambda}_2)^2 - \nu(1 - \nu)(m\pi)^4 \right] \left(1 - \cosh \frac{\lambda_1}{\chi} \cos \frac{\tilde{\lambda}_2}{\chi} \right) = 0. \quad (65)$$

If $\lambda \leq m\pi$ then $\tilde{\lambda}_2$, $\cos(\tilde{\lambda}_2/\chi)$, and $\sin(\tilde{\lambda}_2/\chi)$ are replaced by λ_2 , $\cosh(\lambda_2/\chi)$, and $\sinh(-\lambda_2/\chi)$, respectively.

For a thick plate, when two opposite edges have the same boundary conditions (SFSF and SCSC), because of the symmetry of the problem, it is suspected that all possible modes of vibration will either be symmetric or antisymmetric with respect to the central axis and, consequently, the ξ axis is arbitrarily located at the center of the plate. Two cases are considered: symmetric and antisymmetric modes. In contrast to the simply supported case, the coefficients C_1 , C_3 , and C_4 are not both equal to 0 and the wave forms are only approximately sinusoidal in the y direction. Moreover, defining the node lines as the lines across which the displacements vanish [26], the node lines in the y direction will be parallel to the y axis. For two opposite edges both clamped or free, the node lines will be also parallel to the x axis. The edges $\xi = 0$ and $\xi = 1$ are simply supported. As explained by Leissa [19], for the particular case and in the y direction, the wave forms are found to be sine function exactly.

Symmetric modes Restricting to this family of modes, the antisymmetric terms are deleted from the previous equations. The boundary conditions are applied at the edges $\eta = -1/2$ and $\eta = 1/2$. Because $\partial w / \partial y$ is a function of ψ_y , when the symmetric modes for w are retained, only the antisymmetric modes are kept from the expression of ψ_y . According to Eqs. (37) and (38), φ_m can be expressed with only ψ_y and the derivative following x of w , both being antisymmetric. Consequently, only the antisymmetric term of φ_m is retained. The displacement is expressed as:

For $\lambda \leq \lambda_-^* < \lambda_+^*$ then $r_-^2 \geq 0$; $r_+^2 > 0$. The solution of the differential equation is given by

$$W_m(\eta) = C_1 \cosh r_+ \eta + C_3 \cosh r_- \eta. \quad (66)$$

If $\lambda_-^* < \lambda \leq \lambda_+^*$ then $r_-^2 < 0$; $r_+^2 \geq 0$ and $\cosh r_- \eta$ is replaced by $\cos \tilde{r}_- \eta$. If $\lambda_-^* < \lambda$; $\lambda_+^* < \lambda$ then $\cosh r_- \eta$ and $\cosh r_+ \eta$ are replaced by $\cos \tilde{r}_- \eta$ and $\cos \tilde{r}_+ \eta$, respectively. Moreover, φ_m is given by

$$\varphi_m(y) = C_5 \sinh u_+ \eta. \quad (67)$$

Applying the boundary conditions, one obtains a system of three equations with three unknowns, which can be written in matrix form:

$$Z_S \begin{pmatrix} C_1 \\ C_3 \\ C_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (68)$$

where Z_S is a matrix 3×3 . In order to have a non-trivial solution, the determinant of Z_S must vanish and it provides the natural frequencies of the plate.

The symmetric modes have an axis of symmetry with respect to the y coordinate. Thus, $(m, n) = (1, 1)$, $(m, n) = (2, 1)$; $(m, n) = (3, 1)$ or $(m, n) = (3, 1)$ are symmetric [26].

Antisymmetric modes By the same way, the symmetric terms are deleted from the previous equations. The boundary conditions are applied at the edges $\eta = -1/2$ and $\eta = 1/2$. Only the symmetric term of φ_m is kept. The displacement is expressed as:

- $\lambda \leq \lambda_-^* < \lambda_+^*$ then $r_-^2 \geq 0$; $r_+^2 > 0$.

$$W_m(\eta) = C_2 \sinh r_+ \eta + C_4 \sinh r_- \eta \quad (69)$$

• If $\lambda_-^* < \lambda \leq \lambda_+^*$ then $\sinh r_- \eta$ is replaced by $\sin \tilde{r}_- \eta$. If $\lambda_-^* < \lambda$ then $\sinh r_- \eta$ and $\sinh r_+ \eta$ are replaced by $\sin \tilde{r}_- \eta$ and $\sin \tilde{r}_+ \eta$, respectively. Moreover, φ_m is given by

$$\varphi_m(y) = C_6 \cosh u_+ \eta \quad (70)$$

Applying the boundary conditions, one obtains a system of three equations with three unknowns, which is written in matrix form:

$$Z_{AS} \begin{pmatrix} C_2 \\ C_4 \\ C_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (71)$$

where Z_{AS} is a matrix 3×3 . In order to have a non-trivial solution, the determinant of Z_{AS} must vanish and it provides the natural frequencies of the plate.

The antisymmetric modes have an axis of symmetry with respect to the y coordinate. Thus, $(m, n) = (1, 2)$, $(m, n) = (2, 2)$; $(m, n) = (3, 2)$ or $(m, n) = (4, 2)$ are antisymmetric [26].

By considering separately the symmetric and the antisymmetric modes for the case of a plate with two simply supported opposite edges and the two other opposite edges having the same boundary conditions, Leissa [23] and Gorman [24,25] reduce the problem to the calculation of values of frequency for which the determinant vanishes. Consequently, the time to calculate the determinant is considerably reduced. Indeed, it is much easier and efficient to calculate two determinants of matrices 3×3 (total of 18 coefficients) than one determinant of a matrix 6×6 (total of 36 coefficients).

First of all, three functions H_1 , H_2 , and H_3 are defined such as

$$\begin{aligned} H_1(x) &= \frac{\beta^2 \chi^4 \bar{h}^4 \lambda^2}{144 \kappa^2} - [\chi^2 x + (m\pi)^2] \frac{\beta \bar{h}^2}{12} \left(1 - \gamma_2 \lambda^2 \bar{h}^4 \chi^4 \frac{\beta}{144 \kappa^2} \right) + \kappa^2 - \gamma_3 \lambda^2 \beta \frac{\chi^4 \bar{h}^4}{144}, \\ H_2(x) &= \chi \left(\kappa^2 - [\gamma_1 + \gamma_2] \frac{\beta \bar{h}^4 \chi^4}{144} \lambda^2 - H_1(x) \right), \\ H_3(x) &= [\chi^2 x + v(m\pi)^2]. \end{aligned} \quad (72)$$

The following notations are also considered:

$$\begin{aligned} H_{i+} &= H_i(-r_+^2); \quad H_{i-} = H_i(\tilde{r}_+^2); \quad H_{i+} = H_i(-r_-^2); \quad H_{i-} = H_i(\tilde{r}_-^2) \quad (i = 1, 2, 3), \\ C_x &= \cosh x; \quad S_x = \sinh x; \quad \tilde{C}_x = \cos \tilde{x}; \quad \tilde{S}_x = \sin \tilde{x}; \quad \bar{\beta} = (1 - \nu) \frac{\beta \bar{h}^2}{24}. \end{aligned} \quad (73)$$

For the particular case of a plate with two opposite edges that are clamped, the matrices Z_S and Z_{AS} are expressed as follows:

$$Z_S = \begin{pmatrix} C_{r_+} & C_{r_-} & 0 \\ H_{1+} C_{r_+} & H_{1-} C_{r_-} & -\frac{\chi \bar{\beta}}{m\pi} u_+ C_{u_+} \\ H_{1+} r_+ S_{r_+} & H_{1-} r_- S_{r_-} & -\frac{\bar{\beta} m\pi}{\chi} S_{u_+} \end{pmatrix}; \quad Z_{AS} = \begin{pmatrix} S_{r_+} & S_{r_-} & 0 \\ H_{1+} S_{r_+} & H_{1-} S_{r_-} & -\frac{\bar{\beta} \chi}{m\pi} u_+ S_{u_+} \\ H_{1+} r_+ C_{r_+} & H_{1-} r_- C_{r_-} & -\frac{\bar{\beta} m\pi}{\chi} C_{u_+} \end{pmatrix}. \quad (74)$$

For the particular case of a plate with two opposite edges that are free, the matrices Z_S and Z_{AS} are expressed as follows:

$$\begin{aligned} Z_S &= \begin{pmatrix} H_{3+}H_{1+}C_{\frac{r_+}{2}} & H_{3-}H_{1-}C_{\frac{r_-}{2}} & \bar{\beta}(1-\nu)\chi m\pi u_+C_{\frac{u_+}{2}} \\ H_{2+}r_+S_{\frac{r_+}{2}} & H_{2-}r_-S_{\frac{r_-}{2}} & m\pi\bar{\beta}S_{\frac{u_+}{2}} \\ m\pi\chi H_{1+}r_+S_{\frac{r_+}{2}} & m\pi\chi H_{1-}r_-S_{\frac{r_-}{2}} & -\frac{\bar{\beta}}{2}[\chi^2u_+^2 + (m\pi)^2]S_{\frac{u_+}{2}} \end{pmatrix}, \\ Z_{AS} &= \begin{pmatrix} H_{3+}H_{1+}S_{\frac{r_+}{2}} & H_{3-}H_{1-}S_{\frac{r_-}{2}} & \bar{\beta}(1-\nu)\chi m\pi u_+S_{\frac{u_+}{2}} \\ H_{2+}r_+C_{\frac{r_+}{2}} & H_{2-}r_-C_{\frac{r_-}{2}} & m\pi\bar{\beta}C_{\frac{u_+}{2}} \\ m\pi\chi H_{1+}r_+C_{\frac{r_+}{2}} & m\pi\chi H_{1-}r_-C_{\frac{r_-}{2}} & -\frac{\bar{\beta}}{2}(1-\nu)[\chi^2u_+^2 + (m\pi)^2]C_{\frac{u_+}{2}} \end{pmatrix}. \end{aligned} \quad (75)$$

In both cases, if $\lambda_-^* < \lambda \leq \lambda_+^*$ then $r_-^2 < 0$; $r_+^2 \geq 0$, $-r_-^2$, $C_{\frac{r_-}{2}}$; $r_-S_{\frac{r_-}{2}}$; $S_{\frac{r_-}{2}}$ and $r_-C_{\frac{r_-}{2}}$ are replaced by $\tilde{r}_-^2$, $\tilde{C}_{\frac{r_-}{2}}$, $-\tilde{r}_-\tilde{S}_{\frac{r_-}{2}}$, $\tilde{S}_{\frac{r_-}{2}}$ and $\tilde{r}_-\tilde{C}_{\frac{r_-}{2}}$, respectively. If $\lambda_-^* < \lambda$; $\lambda_+^* < \lambda$ then $r_-^2 < 0$; $r_+^2 < 0$, $-r_+^2$, $C_{\frac{r_+}{2}}$; $r_+S_{\frac{r_+}{2}}$; $S_{\frac{r_+}{2}}$ and $r_+C_{\frac{r_+}{2}}$ are replaced by $\tilde{r}_-^2$, $\tilde{C}_{\frac{r_+}{2}}$, $-\tilde{r}_+\tilde{S}_{\frac{r_+}{2}}$, $\tilde{S}_{\frac{r_+}{2}}$ and $\tilde{r}_+\tilde{C}_{\frac{r_+}{2}}$, respectively.

6.2 Two other opposite edges with differing boundary conditions

When two opposite edges have differing boundary conditions (SSSF, SSSC, and SCSF), the problem cannot be separated in symmetric and antisymmetric modes. The ξ axis is arbitrarily located at one of the edges of the plate. Thus, the boundary conditions are applied at the edges $\eta = 0$ and $\eta = 1$. In contrast to the case of two opposite edges having the same boundary conditions, the node lines are not straight or parallel to the x axis. Applying the boundary conditions, one obtains a system of six equations with six unknowns, which is in matrix form:

$$Z \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (76)$$

where Z is a matrix 6×6 . In order to have a non-trivial solution, the determinant of Z must vanish and it provides the natural frequencies of the plate.

For the particular case of a plate with the edge $\eta = 0$ free and the edge $\eta = 1$ simply supported (SSSF plate), the coefficients Z_{ij} ($i = 1, 2, 3, 4, 5, 6$) for the matrix Z are

- $\lambda \leq \lambda_-^* < \lambda_+^*$ then $r_-^2 \geq 0$; $r_+^2 > 0$.

$$\begin{aligned} Z_{12} &= H_{3+}H_{1+}; Z_{14} = H_{3-}H_{1-}; Z_{16} = \chi\bar{\beta}(1-\nu)m\pi u_+; Z_{21} = H_{2+}r_+; Z_{23} = H_{2-}r_-; Z_{25} = \bar{\beta}m\pi; \\ Z_{31} &= -2\chi m\pi r_+H_{1+}; Z_{33} = -2\chi m\pi r_-H_{1-}; Z_{35} = \bar{\beta}[\chi^2u_+^2 + (m\pi)^2]; Z_{41} = S_{r_+}; Z_{42} = C_{r_+}; \\ Z_{43} &= S_{r_-}; Z_{44} = C_{r_-}; Z_{51} = H_{3+}H_{1+}S_{r_+}; Z_{52} = H_{3+}H_{1+}C_{r_+}; Z_{53} = H_{3-}H_{1-}S_{r_-}; Z_{54} = H_{3-}H_{1-}C_{r_-}; \\ Z_{55} &= \bar{\beta}(1-\nu)\chi m\pi u_+S_{u_+}; Z_{56} = \bar{\beta}(1-\nu)\chi m\pi u_+C_{u_+}; Z_{61} = H_{1+}S_{r_+}; Z_{62} = H_{1+}C_{r_+}; \\ Z_{63} &= H_{1-}S_{r_-}; Z_{64} = H_{1-}C_{r_-}; Z_{65} = -\frac{\bar{\beta}\chi}{m\pi}S_{u_+}; Z_{66} = \frac{\bar{\beta}\chi}{m\pi}C_{u_+}; \\ Z_{11} &= Z_{13} = Z_{15} = Z_{22} = Z_{24} = Z_{26} = Z_{32} = Z_{34} = Z_{36} = Z_{45} = Z_{46} = 0 \end{aligned} \quad (77)$$

- $\lambda_-^* < \lambda \leq \lambda_+^*$ then $r_-^2 < 0$; $r_+^2 \geq 0$ and the coefficients $\tilde{Z}_{14}$, $\tilde{Z}_{23}$, $\tilde{Z}_{33}$, $\tilde{Z}_{43}$, $\tilde{Z}_{44}$, $\tilde{Z}_{53}$, $\tilde{Z}_{54}$, $\tilde{Z}_{63}$ and $\tilde{Z}_{64}$ are replaced by

$$\begin{aligned} Z_{14} &= \tilde{H}_{3-}\tilde{H}_{1-}; Z_{23} = \tilde{H}_{2-}\tilde{r}_-; Z_{33} = -2m\pi\chi\tilde{H}_{1-}\tilde{r}_-; Z_{43} = \tilde{S}_{r_-}; Z_{44} = \tilde{C}_{r_-}; \\ Z_{53} &= \tilde{H}_{3-}\tilde{H}_{1-}\tilde{S}_{r_-}; Z_{54} = \tilde{H}_{3-}\tilde{H}_{1-}\tilde{C}_{r_-}; Z_{63} = \tilde{H}_{1-}\tilde{S}_{r_-}; Z_{64} = \tilde{H}_{1-}\tilde{C}_{r_-} \end{aligned} \quad (78)$$

• $\lambda_-^* < \lambda$; $\lambda_+^* < \lambda$ then $r_-^2 < 0$; $r_+^2 < 0$ and matrix Z is given by the coefficients $\tilde{Z}_{ij}$ with $\tilde{Z}_{14}$, $\tilde{Z}_{23}$, $\tilde{Z}_{33}$, $\tilde{Z}_{43}$, $\tilde{Z}_{44}$, $\tilde{Z}_{53}$, $\tilde{Z}_{54}$, $\tilde{Z}_{63}$ and $\tilde{Z}_{64}$ given in the previous equations and the other coefficients given by

$$\begin{aligned} \tilde{Z}_{16} &= Z_{16}; \tilde{Z}_{25} = Z_{25}; \tilde{Z}_{35} = Z_{35}; \tilde{Z}_{55} = Z_{55}; Z_{56} = \tilde{Z}_{56}; \tilde{Z}_{65} = Z_{65}; \tilde{Z}_{66} = Z_{66} \\ Z_{12} &= \tilde{H}_{3+}\tilde{H}_{1+}; Z_{21} = \tilde{H}_{2+}\tilde{r}_+; Z_{31} = -2m\pi\chi\tilde{H}_{1+}\tilde{r}_+; Z_{41} = \tilde{S}_{r+}; \\ Z_{42} &= \tilde{C}_{r+}; Z_{51} = \tilde{H}_{3+}\tilde{H}_{1+}\tilde{S}_{r+}; Z_{52} = \tilde{H}_{3+}\tilde{H}_{1+}\tilde{C}_{r+}; Z_{61} = \tilde{r}_+\tilde{H}_{1+}\tilde{S}_{r+}; Z_{62} = \tilde{r}_+\tilde{H}_{1+}\tilde{C}_{r+}; \\ Z_{11} &= Z_{13} = Z_{15} = Z_{22} = Z_{24} = Z_{26} = Z_{32} = Z_{34} = Z_{36} = Z_{45} = Z_{46} = 0 \end{aligned} \quad (79)$$

Moreover, for a thin plate, the frequencies are obtained through the following characteristic equation [19],

• $m\pi < \lambda$

$$\tilde{\lambda}_2[\lambda + (1 - \nu)(m\pi)^2]^2 \sinh \frac{\lambda_1}{\chi} \cos \frac{\tilde{\lambda}_2}{\chi} - \lambda_1[k - (1 - \nu)(m\pi)^2]^2 \cosh \frac{\lambda_1}{\chi} \sin \frac{\tilde{\lambda}_2}{\chi} = 0 \quad (80)$$

For $\lambda \leq m\pi$, $\tilde{\lambda}_2$, $\sin(\tilde{\lambda}_2/\chi)$, and $\cos(\tilde{\lambda}_2/\chi)$ are replaced by λ_2 , $\sin(\lambda_2/\chi)$, and $\cos(\lambda_2/\chi)$, respectively.

For the particular case of a plate with the edge $\eta = 0$ simply supported and the edge $\eta = 1$ clamped (SSSC plate), the coefficients Z_{ij} ($i = 1, 2, 3, 4, 5, 6$) for the matrix Z are

• $\lambda \leq \lambda_-^* < \lambda_+^*$ then $r_-^2 \geq 0$; $r_+^2 > 0$.

$$\begin{aligned} Z_{12} &= Z_{14} = 1; Z_{16} = Z_{11} = Z_{13} = Z_{15} = Z_{21} = Z_{23} = Z_{25} = Z_{32} = Z_{34} = Z_{36} = Z_{45} = Z_{46} = 0 \\ Z_{22} &= H_{3+}H_{1+}; Z_{24} = H_{3-}H_{1-}; Z_{26} = \bar{\beta}(1 - \nu)\chi m\pi u_+; Z_{32} = H_{1+}; Z_{34} = H_{1-}; Z_{36} = -\bar{\beta}\frac{m\pi}{\chi}; \\ Z_{41} &= S_{r+}; Z_{42} = C_{r+}; Z_{43} = S_{r-}; Z_{44} = C_{r-}; Z_{51} = H_{1+}S_{r+}; Z_{52} = H_{1+}C_{r+}; Z_{53} = H_{1-}S_{r-}; \\ Z_{54} &= H_{1-}C_{r-}; Z_{55} = -\frac{\bar{\beta}\chi}{m\pi}u_+S_{u+}; Z_{56} = -\frac{\bar{\beta}\chi}{m\pi}u_+C_{u+}; Z_{61} = H_{1+}r_+C_{r+}; Z_{62} = H_{1+}r_+S_{r+}; \\ Z_{63} &= H_{1-}r_-C_{r-}; Z_{64} = H_{1-}r_-S_{r-}; Z_{65} = -\bar{\beta}\frac{m\pi}{\chi}C_{u+}; Z_{66} = -\bar{\beta}\frac{m\pi}{\chi}S_{u+} \end{aligned} \quad (81)$$

• $\lambda_-^* < \lambda \leq \lambda_+^*$ then $r_-^2 < 0$; $r_+^2 \geq 0$ and the coefficients $\tilde{Z}_{24}$, $\tilde{Z}_{34}$, $\tilde{Z}_{43}$, $\tilde{Z}_{44}$, $\tilde{Z}_{53}$, $\tilde{Z}_{54}$, $\tilde{Z}_{63}$ and $\tilde{Z}_{64}$ are replaced by

$$\begin{aligned} Z_{24} &= \tilde{H}_{3-}\tilde{H}_{1-}; Z_{34} = \tilde{H}_{1-}; Z_{43} = \tilde{S}_{r-}; Z_{44} = \tilde{C}_{r-}; \\ Z_{53} &= \tilde{H}_{1-}\tilde{S}_{r-}; Z_{54} = \tilde{H}_{1-}\tilde{C}_{r-}; Z_{63} = \tilde{H}_{1-}\tilde{r}_-\tilde{C}_{r-}; Z_{64} = -\tilde{H}_{1-}\tilde{r}_-\tilde{S}_{r-} \end{aligned} \quad (82)$$

• $\lambda_-^* < \lambda$; $\lambda_+^* < \lambda$ then $r_-^2 < 0$; $r_+^2 < 0$, and matrix Z is given by the coefficients $\tilde{Z}_{ij}$ with $\tilde{Z}_{24}$, $\tilde{Z}_{34}$, $\tilde{Z}_{43}$, $\tilde{Z}_{44}$, $\tilde{Z}_{53}$, $\tilde{Z}_{54}$, $\tilde{Z}_{63}$ and $\tilde{Z}_{64}$ given in the previous equations and the other coefficients given by

$$\begin{aligned} \tilde{Z}_{26} &= Z_{26}; \tilde{Z}_{36} = Z_{36}; \tilde{Z}_{55} = Z_{55}; \tilde{Z}_{56} = Z_{56}; \tilde{Z}_{65} = Z_{65}; \tilde{Z}_{66} = Z_{66} \\ Z_{22} &= \tilde{H}_{3+}\tilde{H}_{1+}; Z_{32} = \tilde{H}_{1+}; \\ Z_{41} &= \tilde{S}_{r+}; Z_{42} = \tilde{C}_{r+}; Z_{51} = \tilde{H}_{1+}\tilde{S}_{r+}; Z_{52} = \tilde{H}_{1+}\tilde{C}_{r+}; Z_{61} = \tilde{H}_{1+}\tilde{r}_+\tilde{C}_{r+}; Z_{62} = -\tilde{H}_{1+}\tilde{r}_+\tilde{S}_{r+}; \\ Z_{12} &= Z_{14} = 1; Z_{16} = Z_{11} = Z_{13} = Z_{15} = Z_{21} = Z_{23} = Z_{25} = Z_{32} = Z_{34} = Z_{36} = Z_{45} = Z_{46} = 0 \end{aligned} \quad (83)$$

Moreover, for a thin plate, the frequencies are obtained through the following characteristic equation [19],

• $m\pi < \lambda$

$$\lambda_1 \cosh \frac{\lambda_1}{\chi} \sin \frac{\tilde{\lambda}_2}{\chi} - \tilde{\lambda}_2 \sinh \frac{\lambda_1}{\chi} \cos \frac{\tilde{\lambda}_2}{\chi} = 0. \quad (84)$$

For $\lambda \leq m\pi$, $\tilde{\lambda}_2$, $\sin(\tilde{\lambda}_2/\chi)$, and $\cos(\tilde{\lambda}_2/\chi)$ are replaced by λ_2 , $\sin(\lambda_2/\chi)$, and $\cos(\lambda_2/\chi)$, respectively.

For the particular case of a plate with the edge $\eta = 0$ free and the edge $\eta = 1$ clamped (SCSF plate), the coefficients Z_{ij} ($i = 1, 2, 3, 4, 5, 6$) for the matrix Z are

• $\lambda \leq \lambda_-^* < \lambda_+^*$ then $r_-^2 \geq 0$; $r_+^2 > 0$

$$\begin{aligned} Z_{12} &= H_{3+}H_{1+}; Z_{14} = H_{3-}H_{1-}; Z_{16} = \chi\bar{\beta}(1-\nu)m\pi u_+; Z_{21} = H_{2+r+}; Z_{23} = H_{2-r-}; Z_{25} = m\pi\bar{\beta}; \\ Z_{31} &= -2\chi m\pi H_{1+r+}; Z_{33} = -2\chi m\pi H_{1-r-}; Z_{35} = \bar{\beta}[\chi^2 u_+^2 + (m\pi)^2]; Z_{41} = S_{r+}; Z_{42} = C_{r+}; \\ Z_{43} &= S_{r-}; Z_{44} = C_{r-}; Z_{51} = H_{1+}S_{r+}; Z_{52} = H_{1+}C_{r+}; Z_{53} = H_{1-}S_{r-}; Z_{54} = H_{1-}C_{r-}; \\ Z_{55} &= -\frac{\bar{\beta}\chi}{m\pi}u_+S_{u+}; Z_{56} = -\frac{\bar{\beta}\chi}{m\pi}u_+C_{u+}; Z_{61} = H_{1+r+}C_{r+}; Z_{62} = H_{1+r+}S_{r+}; \\ Z_{63} &= H_{1-r-}C_{r-}; Z_{64} = H_{1-r-}S_{r-}; Z_{65} = -\bar{\beta}\frac{m\pi}{\chi}C_{u+}; Z_{66} = -\bar{\beta}\frac{m\pi}{\chi}S_{u+} \\ Z_{11} &= Z_{13} = Z_{15} = Z_{22} = Z_{24} = Z_{26} = Z_{32} = Z_{34} = Z_{36} = Z_{45} = Z_{46} = 0 \end{aligned} \quad (85)$$

• $\lambda_-^* < \lambda \leq \lambda_+^*$ then $r_-^2 < 0$; $r_+^2 \geq 0$ and the coefficients $\tilde{Z}_{14}, \tilde{Z}_{23}, \tilde{Z}_{33}, \tilde{Z}_{43}, \tilde{Z}_{44}, \tilde{Z}_{53}, \tilde{Z}_{54}, \tilde{Z}_{63}$ and $\tilde{Z}_{64}$ are replaced by

$$\begin{aligned} Z_{14} &= \tilde{H}_{3-}\tilde{H}_{1-}; Z_{23} = \tilde{H}_{2-}\tilde{r}_-; Z_{33} = -2m\pi\chi\tilde{H}_{1-}\tilde{r}_-; Z_{43} = \tilde{S}_{r-}; Z_{44} = \tilde{C}_{r-}; \\ Z_{53} &= \tilde{H}_{1-}\tilde{S}_{r-}; Z_{54} = \tilde{H}_{1-}\tilde{C}_{r-}; Z_{63} = \tilde{H}_{1-}\tilde{r}_-\tilde{C}_{r-}; Z_{64} = -\tilde{H}_{1-}\tilde{r}_-\tilde{S}_{r-} \end{aligned} \quad (86)$$

• $\lambda_-^* < \lambda$; $\lambda_+^* < \lambda$ then $r_-^2 < 0$; $r_+^2 < 0$ and matrix Z is given by the coefficients $\tilde{Z}_{ij}$ with $\tilde{Z}_{24}, \tilde{Z}_{34}, \tilde{Z}_{43}, \tilde{Z}_{44}, \tilde{Z}_{53}, \tilde{Z}_{54}, \tilde{Z}_{63}$ and $\tilde{Z}_{64}$ given in the previous equations and the other coefficients given by

$$\begin{aligned} \tilde{Z}_{16} &= Z_{16}; \tilde{Z}_{25} = Z_{25}; \tilde{Z}_{35} = Z_{35}; \tilde{Z}_{55} = Z_{55}; \tilde{Z}_{56} = Z_{56}; \tilde{Z}_{65} = Z_{65}; \tilde{Z}_{66} = Z_{66} \\ Z_{12} &= \tilde{H}_{3+}\tilde{H}_{1+}; Z_{21} = \tilde{H}_{2+}\tilde{r}_+; Z_{31} = -2m\pi\chi\tilde{H}_{1+}\tilde{r}_+; Z_{41} = \tilde{S}_{r+}; Z_{42} = \tilde{C}_{r+}; \\ Z_{51} &= \tilde{H}_{1+}\tilde{S}_{r+}; Z_{52} = \tilde{H}_{1+}\tilde{C}_{r+}; Z_{61} = \tilde{H}_{1+}\tilde{r}_+\tilde{C}_{r+}; Z_{62} = -\tilde{H}_{1+}\tilde{r}_+\tilde{S}_{r+} \\ Z_{11} &= Z_{13} = Z_{15} = Z_{22} = Z_{24} = Z_{26} = Z_{32} = Z_{34} = Z_{36} = Z_{45} = Z_{46} = 0. \end{aligned} \quad (87)$$

Moreover, for a thin plate, the frequencies are obtained through the following characteristic equation [19], for $m\pi < \lambda$

$$\begin{aligned} \lambda_1\tilde{\lambda}_2[\lambda^2 - (1-\nu)^2(m\pi)^4] + (m\pi)^2[(1-2\nu)\lambda^2 - (1-\nu)^2(m\pi)^4]\sinh\frac{\lambda_1}{\chi}\sin\frac{\tilde{\lambda}_2}{\chi} \\ + \lambda_1\tilde{\lambda}_2[\lambda^2 + (1-\nu)^2(m\pi)^4]\cosh\frac{\lambda_1}{\chi}\cos\frac{\tilde{\lambda}_2}{\chi} = 0. \end{aligned} \quad (88)$$

For $\lambda \leq m\pi$, $\tilde{\lambda}_2$, $\sin(\tilde{\lambda}_2/\chi)$, and $\cos(\tilde{\lambda}_2/\chi)$ are replaced by λ_2 , $\sin(\lambda_2/\chi)$, and $\cos(\lambda_2/\chi)$, respectively.

7 Numerical results: discussion

Numerical results for the non-dimensional frequency λ were obtained for each of the six cases considered, an aspect ratio equal to 1 (square plate) and 2 and a Poisson ratio 0.3, following the closed-form solution obtained through the Navier's approach for an all edges simply supported plate (Table 2) and the exact solutions for the other cases given by the Lévy approach (see Tables 3, 4, 5).

For each case previously presented (SCSC, SFSF, SSSC, SSSF, SCSF), the first natural frequencies have been compared with those of the classical plate theory in the monograph of Leissa [19], those of Wang and Wang [36] and those obtained by Hashemi [26] (see Table 5). Different shear correction factors κ^2 are used in the literature. Hashemi and Arsanjani [26] or Wang and Wang [36] took 8.667. Hereinafter, this value is retained in order to compare our results with the results of the literature. However, other values could be used: $\kappa^2 = \pi^2/12$ (used by Mindlin [5,35]), $\kappa^2 = 0.822$, $\kappa^2 = 0.88$ [37,38]. Three different thickness ratios are considered: $\bar{h} = 0.01$; $\bar{h} = 0.1$ and $\bar{h} = 0.2$. In each table, the five lowest frequencies are displayed.

All the results presented in this paper are obtained from the exact solution and so are extremely accurate, much more than any numerical method such as the Rayleigh-Ritz method that includes naturally some approximations. Thus, considering the original Uflyand-Mindlin theory, the results match with those obtained by Hashemi and Arsanjani for the original Uflyand-Mindlin plate theory. An error smaller than 1% can occur due to the numerical calculations. The same calculations have been performed for $\kappa^2 = 0.822$ and coincide

Table 2 First ten non-dimensional natural frequencies of a SSSS considering three versions of the Uflyand–Mindlin plate theory for different aspect and thickness ratios ($\kappa^2 = 0.86667$; $\nu = 0.3$)

χ	$\bar{h} = 0$	$\bar{h} = 0.01$			$\bar{h} = 0.1$			$\bar{h} = 0.2$			Hashemi Arsanjani [26]		
		KL theory	Original UM	Truncated UM	Slope inertia UM	Hashemi Arsanjani [26]	Original UM	Truncated UM	Slope inertia UM	Original UM		Truncated UM	Slope inertia UM
1	19.739	19.732	19.732	19.732	19.732	19.084	19.077	19.069	19.084	17.505	17.429	17.333	17.506
	49.348	49.303	49.304	49.304	49.304	45.585	45.492	45.385	45.585	38.385	37.773	36.823	38.385
	78.957	78.839	78.845	78.845	78.846	70.022	69.715	69.33	70.022	55.586	54.089	51.405	55.586
	98.696	98.518	98.522	98.522	98.522	83.365	84.838	84.147	83.365	65.719	63.529	59.297	65.719
	128.305	128.01	128.011	128.011	128.011	107.177	106.207	104.862	107.178	79.476	76.167	59.297	79.476
	167.783	167.283	167.282	167.281	167.282	134.359	132.613	130.021	134.359	95.809	90.953	69.171	95.809
	177.653	177.088	177.091	177.089	–	140.857	138.889	135.92	–	99.608	94.363	69.171	–
	197.392	196.703	196.698	196.697	–	153.539	151.092	147.292	–	106.919	100.901	79.678	–
	246.74	245.659	245.657	245.656	–	183.588	179.788	173.483	–	123.796	115.879	86.113	–
2	256.609	255.444	255.439	255.437	–	189.343	185.249	178.375	–	144.836	118.678	96.39	–
	12.337	12.326	12.326	12.326	12.326	11.396	11.373	11.346	11.396	9.596	9.443	9.206	9.596
	19.7393	19.709	19.711	19.711	19.711	17.505	17.429	17.333	17.505	13.896	13.522	12.851	13.897
	32.0763	32.003	32.003	32.003	32.003	26.794	26.552	26.216	26.794	19.869	19.042	17.293	19.869
	41.9458	41.821	41.820	41.820	41.821	33.590	33.153	32.505	33.590	23.952	22.738	19.919	23.952
	49.348	49.176	49.175	49.174	49.175	38.476	37.773	36.822	38.385	26.729	25.225	21.528	26.730
	61.685	61.415	61.414	61.413	61.415	45.884	44.947	43.371	45.897	30.947	28.969	23.718	30.949
	71.554	71.192	71.191	71.190	71.191	51.539	50.285	48.107	51.539	34.033	31.687	25.142	34.033
	78.956	78.515	78.514	78.513	78.515	55.586	54.089	51.405	55.586	36.209	33.596	26.064	36.209
91.293	90.704	90.703	90.701	–	62.022	60.098	56.478	–	39.617	36.575	27.384	–	
98.696	98.007	98.006	98.004	–	65.719	63.529	59.297	–	41.549	38.261	28.069	–	

Table 3 First five non-dimensional natural frequencies of a SCSC plate considering three versions of the Uflyand–Mindlin plate theory for different aspect and thickness ratios ($\kappa^2 = 0.86667$; $\nu = 0.3$)

χ	$\bar{h} = 0$	$\bar{h} = 0.01$					$\bar{h} = 0.1$					$\bar{h} = 0.2$					
		KL theory	Original UM	Truncated UM	Slope inertia UM	Wang Wang [36]	Hashemi Arsanjani [26]	Original UM	Truncated UM	Slope inertia UM	Wang Wang [36]	Hashemi Arsanjani [26]	Original UM	Truncated UM	Slope inertia UM	Wang Wang [36]	Hashemi Arsanjani [26]
SCSC																	
1	28.946	28.925	28.925	28.925	28.925	–	28.925	26.735	26.710	26.680	–	26.737	22.500	22.330	22.170	–	22.510
	54.743	54.670	54.670	54.670	54.670	–	54.674	49.260	49.140	49.010	–	49.261	40.140	39.410	38.410	–	40.138
	69.32	69.200	69.200	69.200	69.200	–	69.199	59.470	59.280	59.060	–	59.480	45.060	43.960	42.820	–	45.057
	94.584	94.370	94.370	94.370	94.370	–	94.369	79.190	78.740	78.240	–	79.195	59.120	57.120	54.400	–	59.123
2	102.21	102.010	102.010	102.010	–	102.011	87.210	86.640	85.920	–	87.207	66.370	64.050	59.830	–	66.371	–
	23.816	23.740	23.727	23.720	23.741	23.741	18.790	18.690	18.590	18.799	18.799	13.030	12.551	12.258	13.032	13.032	13.032
	28.951	28.840	28.840	28.840	28.848	28.848	22.510	22.340	22.180	22.510	22.510	15.740	15.070	14.410	15.743	15.743	
	39.089	38.920	38.920	38.920	38.929	38.929	29.790	29.440	29.040	29.788	29.787	20.740	19.607	17.969	20.738	20.738	
3	54.743	54.470	54.470	54.450	54.471	54.471	40.140	39.410	38.410	40.139	40.138	25.685	23.476	21.315	25.684	25.684	25.684
	63.535	63.090	63.090	63.090	63.099	63.099	41.690	40.770	39.910	41.695	41.695	27.140	25.302	21.823	27.144	27.144	27.144

Table 4 First five non-dimensional natural frequencies of a SFSF and a SSSF plate considering three versions of the Uflyand–Mindlin plate theory for different aspect and thickness ratios ($\kappa^2 = 0.86667$; $\nu = 0.3$)

χ	$\bar{h} = 0$	$\bar{h} = 0.1$					$\bar{h} = 0.2$									
		KL theory	Original UM	Truncated UM	Slope inertia UM	Wang Wang [36]	Hashemi Arsanjani [26]	Original UM	Truncated UM	Slope inertia UM	Wang Wang [36]	Hashemi Arsanjani [26]				
SFSF																
1	9.631	9.625	9.621	9.622	–	9.627	9.441	9.448	9.439	–	9.446	8.996	9.013	8.972	–	8.999
	16.135	16.096	16.085	16.084	–	16.097	15.391	15.472	15.384	–	15.405	14.128	14.357	14.037	–	14.134
	36.726	36.610	36.614	36.609	–	36.611	33.910	34.329	33.827	–	33.916	29.251	30.305	28.518	–	29.256
	38.945	38.903	38.904	38.903	–	38.904	36.424	36.441	36.321	–	36.425	31.436	31.238	30.534	–	31.434
2	46.737	46.635	46.633	46.630	–	46.639	42.884	43.054	42.719	–	42.887	36.162	36.276	34.838	–	36.165
	2.378	2.367	2.367	2.367	2.377	2.377	2.325	2.328	2.326	2.33265	2.333	5.22	2.229	2.217	2.225	2.225
	6.881	6.835	6.836	6.835	6.839	6.840	6.238	6.358	6.234	6.243	6.243	5.329	5.684	5.259	5.332	5.332
	9.632	9.617	9.618	9.618	9.619	9.619	8.999	9.014	8.974	8.999	9.000	7.774	7.751	7.555	7.774	7.774
1	16.135	16.050	16.053	16.050	16.051	16.051	14.132	14.361	14.041	14.134	14.134	11.335	11.812	10.735	11.335	11.335
	21.821	21.772	21.774	21.772	21.773	21.773	19.054	19.042	18.834	19.054	19.054	14.905	14.622	13.654	14.905	14.905
	SSSF															
	11.685	11.653	11.653	11.653	–	11.675	11.355	11.372	11.351	–	11.381	10.698	10.752	10.652	–	10.722
1	27.756	27.699	27.700	27.699	–	27.704	26.188	26.310	26.152	–	26.191	23.233	23.519	22.847	–	23.243
	41.197	41.134	41.135	41.134	–	41.147	38.345	38.377	38.236	–	38.361	32.873	32.697	31.846	–	32.892
	59.066	58.938	58.941	58.938	–	58.943	53.382	53.490	53.063	–	53.385	43.856	43.623	43.107	–	43.868
	61.861	61.727	61.734	61.725	–	61.731	55.759	56.149	55.399	–	55.762	45.582	46.138	45.875	–	46.686
2	4.034	4.022	3.993	3.993	4.024	4.024	3.849	3.853	3.833	3.851	3.851	3.532	3.583	3.509	3.534	3.534
	11.685	11.659	11.656	11.656	11.659	11.660	10.703	10.762	10.673	10.722	10.722	9.041	9.065	8.705	9.0412	9.041
	18.821	18.760	18.760	18.755	18.764	18.764	16.587	16.885	16.440	16.593	16.593	13.198	13.871	12.292	13.200	13.200
	24.010	23.931	23.942	23.941	23.944	23.944	20.670	20.690	20.398	20.679	20.680	15.941	15.692	14.461	15.949	15.949
1	27.756	27.618	27.627	27.620	27.626	27.626	23.242	23.525	22.858	23.243	23.243	17.493	18.013	15.36	17.494	17.494

Table 5 First five non-dimensional natural frequencies of a SSSC and a SCSF plate considering three versions of the Uflyand–Mindlin plate theory for different aspect and thickness ratios ($\chi = 1; \kappa^2 = 0.86667; \nu = 0.3$)

χ	$\bar{h} = 0$	$\bar{h} = 0.01$					$\bar{h} = 0.1$					$\bar{h} = 0.2$					
		KL theory	Original UM	Truncated UM	Slope inertia UM	Wang Wang [36]	Hashemi Arsanjani [26]	Original UM	Truncated UM	Slope inertia UM	Wang Wang [36]	Hashemi Arsanjani [26]	Original UM	Truncated UM	Slope inertia UM	Wang Wang [36]	Hashemi Arsanjani [26]
SSSC																	
1	23.646	23.639	23.639	23.639	23.639	–	23.633	22.419	22.407	22.394	–	22.426	19.792	19.682	19.557	–	19.799
	51.674	51.618	51.619	51.618	51.618	–	51.621	47.224	47.119	47.001	–	47.225	39.203	38.541	37.563	–	39.203
	58.646	58.563	58.563	58.563	58.563	–	58.569	52.322	52.183	52.027	–	52.325	41.781	40.944	38.888	–	41.781
	86.131	85.978	85.976	85.978	85.978	–	86.979	74.401	74.029	73.587	–	74.402	57.338	55.606	52.89	–	57.338
	100.267	100.080	100.085	100.083	100.083	–	100.083	86.221	85.676	84.971	–	86.219	66.045	63.782	59.551	–	66.032
2	17.332	17.288	17.288	17.288	17.299	17.299	17.299	14.861	14.816	14.755	14.870	14.870	11.257	10.989	10.706	11.264	11.264
	23.645	23.592	23.592	23.592	23.592	23.592	23.592	19.792	19.682	19.557	19.799	19.549	14.782	14.282	13.612	14.781	14.781
	35.051	34.941	34.946	34.942	34.946	34.946	34.946	28.169	27.878	27.511	28.169	28.169	20.293	19.322	17.616	20.287	20.287
	51.675	51.462	51.468	51.461	51.463	51.463	51.463	37.793	37.195	36.362	37.796	37.796	24.945	23.161	20.684	24.931	24.931
	52.102	51.846	51.846	51.845	51.850	51.850	51.849	39.221	38.545	37.565	39.203	39.203	26.935	25.264	21.670	26.931	26.932
SCSF																	
1	12.687	12.671	12.671	12.672	12.672	–	12.673	12.236	12.261	12.232	–	12.261	11.373	11.441	11.321	–	11.393
	33.064	32.992	32.993	32.993	32.993	–	32.993	30.465	30.648	30.402	–	30.474	25.887	26.219	25.382	–	25.898
	41.702	41.644	41.647	41.647	41.647	–	41.647	38.708	38.747	38.578	–	38.713	33.054	32.882	32.016	–	33.075
	63.014	62.856	62.859	62.859	62.859	–	62.859	55.969	56.107	55.609	–	55.974	45.041	44.793	42.682	–	45.045
	72.395	72.215	72.216	72.217	72.217	–	72.217	62.948	63.382	62.45	–	62.953	48.888	49.253	46.016	–	48.891
2	5.704	5.686	5.686	5.688	5.688	5.688	5.688	5.293	5.329	5.289	5.297	5.297	4.622	4.709	4.569	4.623	4.623
	12.687	12.649	12.649	12.659	12.651	12.651	12.651	11.387	11.449	11.34	11.393	11.393	9.384	9.425	9.025	9.387	9.387
	24.695	24.612	24.613	24.616	24.616	24.616	24.616	20.256	20.621	20.006	20.259	20.259	14.469	15.088	14.561	14.469	14.469
	24.944	24.846	24.84558	24.853	24.846	24.846	24.846	21.015	21.046	20.725	21.019	21.019	16.073	15.809	16.282	16.074	16.074
	33.064	32.865	32.868	32.87	32.871	32.871	32.871	25.896	26.221	25.393	25.898	25.898	18.272	18.671	19.678	18.272	18.272

with those obtained by Dawe [38]. This validates the accuracy of the calculations and the solutions developed in this paper.

In their paper, Liew et al. [13] calculate the natural frequencies by using the numerical Rayleigh–Ritz method. This method is extremely fast to implement and can be used in any situation. However, this gives an upper bound, but our results allow to question the accuracy of such methods.

The objective of the present paper is to present the analytical determination of the natural frequencies of thick plates considering three versions of the Uflyand–Mindlin theory by using the Navier approach (all edges simply supported) or, more generally, the Lévy method (at least two opposite edges simply supported). Whatever the boundary conditions, the thickness, or the aspect ratio, the three models provide extremely close results. The natural frequencies are bigger for the truncated Uflyand–Mindlin theory than for the two other models, whereas the Uflyand–Mindlin theory based on slope inertia provides lower frequencies. The analytical difference between the truncated Uflyand–Mindlin theory and the Uflyand–Mindlin theory based on slope inertia is clearly established in the case of a plate with all edges simply supported (see Eq. (47))

Considering a square plate ($\alpha = 1$), the modes (1,2) and (2,1) provide the same natural frequency in the case of SSSS, SCSC, and SFSF due to the symmetries. This result confirms the determination of the analytical solutions in which the geometrical symmetries are used to simplify the calculations.

As explained above, according to Eq. (48), for any case considered in this paper, the wave forms are, of course, sine functions in the x direction. When the two other edges are both clamped or free, the node lines will be also parallel to the x axis. For the other cases the forms are not straight and only approximately sinusoidal.

The influence of thickness ratio on the non-dimensional natural frequency considering different boundary conditions is established by taking the aspect ratio α constant and varying $\bar{h}$ from 0.01 to 0.2. For a thickness ratio equal to 0.01, the results are close to those obtained from the thin plate theory (classical plate theory). When the thickness ratio is small, the rotary inertia and the shear effect should have no influence. Thus, all the results (equations and solutions) for a thin plate, developed by Leissa [19], are obtainable by taking κ^2 going to infinity. It can be observed that, as the thickness ratio increases, the frequency parameter decreases with the thickness ratio increasing. Indeed, when $\bar{h}$ is large, the rotary inertia and the shear effect cannot be ignored. This influence becomes more important when the mode of the frequency increases.

To study the effect of the boundary conditions, the thickness and aspect ratios are taken equal to 0.2 and 1, respectively, and the five first non-dimensional frequencies are listed in Tables 2, 3, 4 and 5 for the six different boundary conditions. Irrespective of the model, the difference between the models increases with the order of the natural frequency. For higher orders, the Uflyand–Mindlin plate theory provides lower non-dimensional natural frequencies than the two other Uflyand–Mindlin models. This difference increases with the mode number. From this table, it is established that $\lambda_{SFSF} < \lambda_{SSSF} < \lambda_{SCSF} < \lambda_{SSSS} < \lambda_{SSSC} < \lambda_{SCSC}$. Thus, the lowest frequencies are obtained when one of the edges is free. In contrast to the SFSF, the SCSC provides the highest frequencies. For instance, considering the original Uflyand–Mindlin theory, the fundamental natural frequency is 60% more important for the SCSC case (22.509) than for the SFSF case (8.997). Thus, higher constraints at the edges increase the overall rigidity of the plate, resulting in a higher frequency response as expected [26]. It is worth noting that there is no apparent correlation between the boundary conditions and the difference between the Uflyand–Mindlin theories and the Kirchhoff–Love model.

Furthermore, the comparison of the non-dimensional natural frequency for an aspect ratio equal to 1 (square plate) and 2 shows that it decreases with increasing plate aspect ratio if the relative thickness ratio and boundary conditions are kept constant.

Ultimately, it is worth noting that for the original Uflyand–Mindlin plate theory, two cases have to be considered, following the values of the natural frequency. Thus, the truncated Uflyand–Mindlin plate theory and the Uflyand–Mindlin plate theory based on slope inertia provide very accurate results and are much simpler compared to the original one.

8 Conclusion

The aim of this study was to analyze the different models describing the mechanical behavior of a plate under free vibrations. Three models have been studied: original Uflyand–Mindlin, truncated, and Uflyand–Mindlin based on slope inertia. For each of these models, the first five natural frequencies of a plate have been calculated, considering six different boundary conditions. The difference between the models depends on the aspect ratio, the order of the frequency, and the boundary conditions. When the flexural rigidity of the plate increases (for instance, by considering one of the edges free), the natural frequencies decrease, whereas the differences between the models increase.

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