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► To cite this version:

Y. S. Choi, S. T. Noah. Forced periodic vibration of unsymmetric peicewise-linear systems. Journal of Sound and Vibration, 1988, 121 (1), pp.117 - 126. 10.1016/S0022-460X(88)80064-6 . hal-01692298

HAL Id: hal-01692298

<https://hal.science/hal-01692298>

Submitted on 24 Jan 2018

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FORCED PERIODIC VIBRATION OF UNSYMMETRIC PIECEWISE-LINEAR SYSTEMS

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The forced steady response of a single degree of freedom system involving a large non-linearity, represented by unsymmetric piecewise-linear stiffness, is determined by a harmonic balance Newton-Raphson method with the application of the fast Fourier transformation (FFT) algorithm. All possible subharmonic, harmonic, and superharmonic responses are sought. The responses of two systems involving subharmonic and superharmonic dominant motions are calculated by the newly developed method and compared with previously published findings. The results obtained by using this method reveal the details of the response of the systems more efficiently than previous methods.

1. INTRODUCTION

Vibration problems involving small non-linearities lend themselves to closed form solutions obtained by using perturbation techniques. When the nonlinearities cannot be assumed small, general solution methods are usually not available. The dynamic characteristics of general non-linear systems are too complex to predict as compared to those of linear systems. For example, the response to periodic excitation may result in multiple-valued response, chaotic response or jump phenomena.

A common solution procedure for strong non-linear vibration problems is the direct numerical integration. This procedure can yield both transient and steady state response for given initial conditions and specified frequencies. The disadvantages of this approach include the following: (i) it would be too expensive and tedious to completely characterize the dynamic behavior of the system in relatively large ranges of their various parameters; (ii) in a multiple-valued response region, all possible solutions could be found only by changing the initial conditions in a trial and error process; (iii) to guarantee the convergence of the solution, the time-step integration method would have to be chosen carefully; this is one of the drawbacks to direct numerical integration.

The determination of closed form solutions for piecewise-linear systems was attempted by Maezawa [1] using Fourier series expansions to represent stable, unstable and multi-valued responses. Unfortunately, a solution method based on assumed response patterns often suffers restrictions. Unpredicted types of vibration, not included in the assumed response shapes may be excluded in these methods. Furthermore, it is almost impossible to find the exact duration of contact, θ_0 , in some ranges of θ_0 (which was treated as a parameter in Maezawa's solution procedure) because of the severe non-linear nature of the relations involving θ_0 .

Yamauchi [2] and Saito [3] used fast Fourier transformation (FFT) to determine successfully the non-linear response of a modified Jeffcott model for a rotating disk. Their solution did not include possible subharmonic response which might occur. Saito finally solved a set of iterative linear simultaneous equations in complex variable form which

were produced from the Taylor series expansion. The complex form introduced unduly complicated the solution procedure.

In this paper, the dynamic characteristics of an unsymmetric piecewise-linear spring-mass system under harmonic excitation is examined. All possible steady periodic, subharmonic, superharmonic, and harmonic responses are included. To simplify the calculation of the simultaneous complex equations only the real variables are used instead of the complex variables used in reference [3].

The calculated subharmonic response by the newly developed method is compared with the results of direct numerical integration. Also, the superharmonic response of an unsymmetric stiffness system with an offset is obtained and compared with the results given in reference [1]. The results for both cases closely agree with the previously published results, and in addition, provide information about the unstable periodic solutions which cannot be obtained by a numerical integration procedure.

2. EQUATION OF MOTION

The system considered consists of a mass, a viscous damper, and an unsymmetric piecewise-linear spring system, and is excited by a harmonic force. The equation of motion can be written as

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + g(x) = A \sin \omega t, \quad (1)$$

where m , c , t , and x denote the mass, viscous damping coefficient, time and displacement, respectively. The parameters A and ω denote the amplitude and frequency of the forcing function. The unsymmetric piecewise-linear force, $g(x)$, is depicted in Figure 1 and is expressed as follows:

$$g(x) = \begin{cases} K_1 x, & x < h \\ K_2 x + \delta, & x \geq h \end{cases}, \quad (2)$$

where K_1 and K_2 are the spring constants, h is an offset value, and δ is defined as $\delta = (K_1 - K_2)h$.

There is no restriction on the magnitude of the K_1 and K_2 values, so that $g(x)$ exists for all real x , although the derivative of $g(x)$ with respect to x is of a finite discontinuity at $x = h$. Because of this arbitrary piecewise-linear nature of $g(x)$ the equation of motion (1) describes a system involving a large non-linearity.

For the free undamped vibration of the system and no offset ($h = 0$), the period of this unsymmetric system will be composed of two half sine waves, so that the assumed period, T , can be expressed as

$$T = \pi(\sqrt{m/K_1} + \sqrt{m/K_2}). \quad (3)$$

Figure 1. Unsymmetric piecewise-linear restoring force.

The bilinear radian frequency can therefore be written as $\omega_n = 2\pi/T = \sqrt{\bar{K}/m}$, where $\bar{K}$ is an equivalent stiffness, defined as

$$\bar{K} = 4K_1K_2/(\sqrt{K_1} + \sqrt{K_2})^2. \quad (4)$$

In order to determine the subharmonic vibration of order $1/\nu$, ν being a positive integer, a dimensionless time, θ , is adopted as the new independent variable,

$$\nu\theta = \omega t, \quad (5)$$

where ν is taken as unity when the harmonic or superharmonic vibrations are sought. The period of the subharmonic vibration will be ν -times the period of the exciting force.

With the above equivalent stiffness and the consideration of subharmonic vibrations, equation (1) can be non-dimensionalized by introducing the following parameters, as in reference [4]: a stiffness ratio, $\alpha = K_2/K_1$, a damping ratio, $\zeta = c/2m\omega_n$, and a frequency ratio $\eta = \omega/\omega_n$. Equation (1) can then be written as

$$y'' + (2\zeta\nu/\eta)y' + \nu^2 g^*(y) = (\nu^2/\eta^2) \sin \nu\theta, \quad (6)$$

where

$$g^*(y) = \begin{cases} \left[\frac{(1+\sqrt{\alpha})}{2\eta} \right]^2 y, & y < \xi^* \\ \left[\frac{(1+\sqrt{\alpha})}{2\eta\sqrt{\alpha}} \right]^2 y + \delta^*, & y \geq \xi^* \end{cases},$$

$$\omega_n = \sqrt{\bar{K}/m}, \quad y = x\bar{K}/A, \quad \xi^* = h\bar{K}/A, \quad \delta^* = \delta\bar{K}/\eta^2 A \quad (7)$$

and δ^* is defined so as to make g^* continuous at ξ^* . The prime denotes differentiation with respect to the scaled time, θ .

3. STEADY STATE SOLUTION

The steady periodic solution of equation (6), including subharmonic, superharmonic and harmonic vibrations can be written as the Fourier series

$$y(\theta) = a_0 + 2 \sum_{n=1}^N (a_n \cos n\theta - b_n \sin n\theta), \quad (8)$$

where N is the number of harmonics to be taken into account under the assumption of small bandwidth in the final solution. The multiple of 2 and the minus sign in b_n are adopted to facilitate accommodating complex Fourier coefficients when FFT is used. In the same way, the non-linear restoring force, $g(\theta)$, can also be expressed as a Fourier series of the following form:

$$g^*(\theta) = c_0 + 2 \sum_{n=1}^N (c_n \cos n\theta - d_n \sin n\theta). \quad (9)$$

Since this non-linear restoring force was derived from the piecewise-linear form of the stiffness in equation (6), each of c_0 , c_n , and d_n will be functions of a_0 , a_n , and b_n . Substituting equation (8), and equation (9) into equation (6) and applying a harmonic balance procedure yields $2N+1$ non-linear simultaneous equations, involving $4N+2$

unknowns. The harmonic balance procedure gives, for the constant term,

$$\nu^2 c_0 = 0, \quad (10a)$$

for the cosine terms,

$$-n^2 a_n - n(2\zeta\nu/\eta) b_n + \nu^2 c_n = 0, \quad n = 1, \dots, N, \quad (10b)$$

and for the sine terms,

$$n^2 b_n - n \frac{2\zeta\nu}{\eta} a_n + \nu^2 d_n = \begin{cases} 0 & \text{if } n \neq \nu \\ 0.5\nu^2/\eta^2 & \text{if } n = \nu \end{cases} \quad n = 1, \dots, N. \quad (10c)$$

Since c_0 , c_n , and d_n are functions of a_0 , a_n , and b_n , then once a_0 , a_n , and b_n are determined, c_0 , c_n , and d_n can be readily calculated based on the relations given by equation (6). This means that if a_0 , a_n , and b_n are given, $y(\theta)$ can be found, and with the relation of equation (6) $g^*(\theta)$ can then be generated. The application of the FFT procedure could generate c_0 , c_n , and d_n through its efficient computational algorithm. Therefore, the relation given by equation (6) with the application of the FFT algorithm could regenerate the $2N+1$ equations which gives the relations implicitly between the Fourier coefficients of $y(\theta)$, a_0 , a_n , and b_n and those of $g^*(\theta)$, c_0 , c_n , and d_n . Consequently, equation (10) would have $2N+1$ unknowns, a_0 , a_n , and b_n .

The resulting non-linear simultaneous equations can be handled by using a Newton-Raphson iteration method. If initial values are given properly, the solution converges very rapidly. A guess of initial values for the unknowns a_0 , a_n , and b_n can be made with a linear approximation. From equation (6) the linear steady state solutions can be found with equivalent stiffness as shown in equation (4) instead of the piecewise-linear stiffness. In the case of multiple steady state solutions for a given frequency ratio, all solutions found from the equivalent stiffness calculation are pursued as possible starting points for the harmonic balance method. Small increments Δa_0 , Δa_n , and Δb_n are assumed to be close to the solution in question, so that one can write

$$a_0 = a_0^0 + \Delta a_0, \quad a_n = a_n^0 + \Delta a_n, \quad b_n = b_n^0 + \Delta b_n, \quad (11)$$

where the superscript zero indicates initial guessed values for the first iteration and old values for the next iteration.

Similarly, the nonlinear restoring force terms c_0 , c_n , and d_n can also be expressed as follows:

$$c_0 = c_0^0 + \Delta c_0, \quad c_n = c_n^0 + \Delta c_n, \quad d_n = d_n^0 + \Delta d_n. \quad (12)$$

Since c_0 , c_n , and d_n depend on a_0 , a_n , and b_n , the increments can be expanded as

$$\begin{aligned} \Delta c_0 &= \frac{\partial c_0}{\partial a_0} \Delta a_0 + \sum_{m=1}^N \left(\frac{\partial c_0}{\partial a_m} \Delta a_m + \frac{\partial c_0}{\partial b_m} \Delta b_m \right), \\ \Delta c_n &= \frac{\partial c_n}{\partial a_0} \Delta a_0 + \sum_{m=1}^N \left(\frac{\partial c_n}{\partial a_m} \Delta a_m + \frac{\partial c_n}{\partial b_m} \Delta b_m \right), \quad \Delta d_n = \frac{\partial d_n}{\partial a_0} \Delta a_0 + \sum_{m=1}^N \left(\frac{\partial d_n}{\partial a_m} \Delta a_m + \frac{\partial d_n}{\partial b_m} \Delta b_m \right), \end{aligned} \quad (13)$$

where the partial derivatives are to be calculated at the current state value. In order to

account for the large non-linearity of the problem, the increments Δa_0 , Δb_n , and Δb_n must be chosen comparatively smaller than a_0 , a_n , and b_n whenever numerical differentiation is performed.

The numerical differentiation procedure can be done by using forward differentiation as follows: (1) a_0 is changed by adding the preset small Δa_0 , but other Fourier coefficients, a_n , b_n are not changed; (2) the newly assumed discrete displacement, $y(\theta)$ is generated by using a discrete inverse Fourier transform; (3) the discrete non-linear force, $g^*(\theta)$, is calculated by using equation (7); (4) the Fourier coefficients of the discrete non-linear force, c_0 , c_n , and d_n are calculated by using the fast Fourier transformation (FFT); (5) the differential values of $\partial c_0/\partial a_0$, $\partial c_i/\partial a_0$, $\partial d_i/\partial a_0$ are calculated as

$$\partial c_0/\partial a_0 = (c_0 - c_0^0)/\Delta a_0, \quad \partial c_i/\partial a_0 = (c_i - c_i^0)/\Delta a_0, \quad \partial d_i/\partial a_0 = (d_i - d_i^0)/\Delta a_0, \quad (14)$$

where $i = 1, 2, \dots, N$; (6) the calculations in 1 to 5 are repeated for each a_i and b_i .

Substituting equation (11), equation (12), and equation (13) into equation (10) results in the following incremental equations:

$$\begin{aligned} & \frac{\partial c_0}{\partial a_0} \Delta a_0 + \sum_{m=1}^N \left(\frac{\partial c_0}{\partial a_m} \Delta a_m + \frac{\partial c_0}{\partial b_m} \Delta b_m \right) = -c_0^0, \\ & -n^2 \Delta a_n - n \frac{2\zeta\nu}{\eta} \Delta b_n + \nu^2 \left[\frac{\partial c_n}{\partial a_0} \Delta a_0 + \sum_{m=1}^N \left(\frac{\partial c_n}{\partial a_m} \Delta a_m + \frac{\partial c_n}{\partial b_m} \Delta b_m \right) \right] = n^2 a_n^0 + n \frac{2\zeta\nu}{\eta} b_n^0 - \nu^2 c_n^0, \\ & n^2 \Delta b_n - n \frac{2\zeta\nu}{\eta} \Delta a_n + \nu^2 \left[\frac{\partial d_n}{\partial a_0} \Delta a_0 + \sum_{m=1}^N \left(\frac{\partial d_n}{\partial a_m} \Delta a_m + \frac{\partial d_n}{\partial b_m} \Delta b_m \right) \right] \\ & = -n^2 b_n^0 + n \frac{2\zeta\nu}{\eta} a_n^0 - \nu^2 d_n^0 + 0.5 \frac{\nu^2}{\eta^2}, \end{aligned} \quad (15)$$

Here $0.5\nu^2/\eta^2$ is only included when n equals to ν . Equation (15) can be put in the matrix form

$$[K]\{\Delta y\} = \{R\}, \quad (16)$$

where

$$[K] = \begin{pmatrix} \frac{\partial c_0}{\partial a_0} & \frac{\partial c_0}{\partial a_1} & \frac{\partial c_0}{\partial b_1} & \dots & \frac{\partial c_0}{\partial a_N} & \frac{\partial c_0}{\partial b_N} \\ \nu^2 \frac{\partial c_1}{\partial a_0} & -1 + \nu^2 \frac{\partial c_1}{\partial a_1} & -\frac{2\zeta\nu}{\eta} + \nu^2 \frac{\partial c_1}{\partial b_1} & \dots & \nu^2 \frac{\partial c_1}{\partial a_N} & \nu^2 \frac{\partial c_1}{\partial b_N} \\ \nu^2 \frac{\partial d_1}{\partial a_0} & -\frac{2\zeta\nu}{\eta} + \nu^2 \frac{\partial d_1}{\partial a_1} & 1 + \nu^2 \frac{\partial d_1}{\partial b_1} & \dots & \nu^2 \frac{\partial d_1}{\partial a_N} & \nu^2 \frac{\partial d_1}{\partial b_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \nu^2 \frac{\partial c_N}{\partial a_0} & \nu^2 \frac{\partial c_N}{\partial a_1} & \nu^2 \frac{\partial c_N}{\partial b_1} & \dots & -N^2 + \nu^2 \frac{\partial c_N}{\partial a_N} & -N \frac{2\zeta\nu}{\eta} + \nu^2 \frac{\partial c_N}{\partial b_N} \\ \nu^2 \frac{\partial d_N}{\partial a_0} & \nu^2 \frac{\partial d_N}{\partial a_1} & \nu^2 \frac{\partial d_N}{\partial b_1} & \dots & -N \frac{2\zeta\nu}{\eta} + \nu^2 \frac{\partial d_N}{\partial a_N} & N^2 + \nu^2 \frac{\partial d_N}{\partial b_N} \end{pmatrix}$$

$$\{\Delta y\} = \{\Delta a_0, \Delta a_1, \Delta b_1, \dots, \Delta a_N, \Delta b_N\}^T$$

$$\{R\} = \begin{pmatrix} -c_0^0 \\ a_1^0 + \frac{2\zeta\nu}{\eta} b_1^0 - \nu^2 c_1^0 \\ -b_1^0 + \frac{2\zeta\nu}{\eta} a_1^0 - \nu^2 d_1^0 + \frac{\nu^2}{\eta^2} \\ \vdots \\ N^2 a_N^0 + N \frac{2\zeta\nu}{\eta} b_N^0 - \nu^2 c_N^0 \\ -N^2 b_N^0 + N \frac{2\zeta\nu}{\eta} a_N^0 - \nu^2 d_N^0 + 0.5 \frac{\nu^2}{\eta^2} \end{pmatrix}.$$

Here $[K]$ corresponds to the Jacobian matrix whose elements are calculated at every step. This iteration will continue until all the components of the correction vector $\{R\}$ become sufficiently close to zero.

The calculation procedure presented here is basically an iterative process. Initial guessed values for a new case can be obtained by choosing those previously computed values for the conditions close to the new case. To find multiple solutions, the initial values are calculated differently for the assumed linear equivalent stiffness case since the non-dimensionalized equation (6) will be different depending on the subharmonic ratio ν . Then, for a certain case the response will trace a different path. The method will yield both stable and unstable possible periodic solutions.

4. NUMERICAL RESULTS AND DISCUSSION

A system with stiffness of $\alpha > 1$ was considered as the simple unsymmetric bilinear system of Figure 1 for zero gap length. Figure 2 shows an overview of the response of this system. The numerical results obtained by the FFT method are compared with those obtained by using direct numerical integration. The average steady state amplitude, $\bar{y}$, defined as half of the peak-to-peak of the response y , is plotted as a function of the frequency ratio, η , with the damping ratio, ζ , and stiffness ratio, α , fixed.

The direct numerical integration results were obtained by the fourth-order Runge-Kutta method with initial conditions, y_0 and y'_0 , set to zero. The direct numerical integration continued until the motion reached steady state periodic motion, so that

$$y(2\pi j) = y(2\pi(j + \nu)), \quad y'(2\pi j) = y'(2\pi(j + \nu)), \quad (17)$$

where ν is the order of the subharmonic vibration and a prime denotes differentiation with respect to the dimensionless time, θ . In this numerical marching scheme, the time increment should be small in order to find exact contact time at every contact point. Any large deviation of the contact time will result in a false steady state solution. At some range of η the time increment is given by one five-hundredth of the exciting period.

Figure 2 shows good agreement between the results obtained by using the FFT method and those obtained by direct numerical integration. The only difference is that the harmonic response is shown to exist as given by the FFT approach throughout the whole range of frequency ratios where the harmonic response is stable or unstable. This is because the direct numerical integration can give only stable solutions. Also, the results of direct numerical integration depend on given initial conditions, making it difficult to find all possible solutions. The FFT method yields, on the other hand, all possible equilibrium solutions without consideration of the initial conditions.

Figure 2. Comparison between numerical and FFT results in frequency domain. O, Numerical solution; —, FFT solution.

To find an adequate number of harmonic components to be retained for the analysis, numerical experiments were carried out and their results are shown in Figure 3. The steady state second order subharmonic response at a frequency ratio of 2 was found by direct numerical integration, and by the FFT method with 2, 4, and 8 harmonic components. In order to avoid the aliasing effects, the number of discrete data were taken as four times that of the harmonic components. As shown in Figure 3, when 4 or 8 harmonic components were considered, good agreement between the results with those of direct numerical integration has occurred. The result shows that four harmonic components are sufficient for this case.

The variation of subharmonic response with respect to variation in the stiffness ratio is shown in Figure 4. The figure shows that only a harmonic response exists in the case where the stiffness ratio is unity, i.e., linear vibration, and no fourth order subharmonic response exists for a stiffness ratio of five. This behavior was also shown by Thompson and Elvey [4].

Figure 3. Effect of retained number of Fourier coefficients. —, Numerical solution; - - -, $N = 2$; O, $N = 4$; +, $N = 8$.

Figure 4. Effect of stiffness ratio as determined by FFT method. $\zeta = 0.1$; ---, $\alpha = 10$; —, $\alpha = 5$; +, $\alpha = 1$.

Figure 5. Effect of damping ratio as determined by FFT method. $\alpha = 10$.

Figure 6. Comparison between Maczawa's results and FFT results for superharmonic resonance; $A/K_1 d = 1$. O, FFT solution; +, Maczawa's solution.

Figure 7. Effect of non-dimensionalized displacement, A/K_1d , in superharmonic resonance. +, $A/K_1d = 2$; O, $A/K_1d = 1$.

The dependence of the response on the damping ratio is shown in Figure 5, where the stiffness ratio is fixed as 10 and the damping ratio takes the values of 0.1, 0.3 and 0.5. The figure shows clearly that higher damping suppresses the higher order subharmonic responses.

Cases involving superharmonic responses were also examined and compared with the results of Maezawa [1]. The system considered has an unsymmetric piecewise-linear stiffness, an offset, and no damping. As shown in Figure 6, the results obtained by using the FFT technique are very close to those obtained by Maezawa. The results show the superharmonic resonance to occur in the neighborhood of 0.35 and 0.55 frequency ratios. The shift of the main resonance near the frequency ratio of unity is due to the adoption of K_1 as a reference stiffness for the calculation of the natural frequency. As compared to the complicated derivations and lengthy calculations by Maezawa's approach, the FFT method employed in the present work proves to be simpler and more versatile. In Figure 7, a comparison between the results obtained by the two methods is made for different non-dimensionalized values of A/K_1d . The trend shows close agreement with that of Maezawa.

5. CONCLUSION

A numerical algorithm for determining the steady forced vibration of a highly non-linear system in which an FFT technique is used has been described. The algorithm is comparatively easy to formulate and results in saving of computer time as compared to that for a direct numerical integration method.

The newly developed method was applied to unsymmetric piecewise-linear systems. The approach yielded results which are in good agreement with previously published findings, and can be used effectively to determine all possible steady state solutions. For subharmonic resonances, the results show that reducing the stiffness ratio or increasing the damping ratio would lead to suppressing the higher order subharmonic responses. The results also show that the superharmonic responses of an unsymmetric piecewise-linear system occur only in some specific frequency range.

The approach developed here is being extended to the analysis of non-conservative multi-degree-of-freedom systems.

ACKNOWLEDGMENT

This work is being carried out as part of a research project supported by NASA-Marshall space flight center under contract No. NAS8-36182. This support is gratefully acknowledged. The authors are also grateful to Thomas Fox of NASA for his input as to similar methods used in non-linear control analysis and for his interest.

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