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ALCOVE RANDOM WALKS, k-SCHUR FUNCTIONS AND THE MINIMAL BOUNDARY OF THE k-BOUNDED PARTITION POSET

CÉDRIC LECOUVEY AND PIERRE TARRAGO

Abstract. We use k-Schur functions to get the minimal boundary of the k-bounded partition poset. This permits to describe the central random walks on affine Grassmannian elements of type A and yields a polynomial expression for their drift. We also recover Rietsch's parametrization of totally nonnegative unitriangular Toeplitz matrices without using quantum cohomology of flag varieties. All the homeomorphisms we define can moreover be made explicit by using the combinatorics of k-Schur functions and elementary computations based on Perron-Frobenius theorem.

1. Introduction

A function on the Young graph is harmonic when its value on any Young diagram λ is equal to the sum of its values on the Young diagrams obtained by adding one box to λ. The set of extremal nonnegative such functions (i.e. those that cannot be written as a convex combination) is called the minimal boundary of the Young graph. It is homeomorphic to the Thoma simplex. Kerov and Vershik proved that the extremal nonnegative harmonic functions give the asymptotic characters of the symmetric group. O'Connell's results [15] also show that they control the law of some conditioned random walks. In another but equivalent direction, Kerov-Vershik approach of these harmonic functions yields both a simple parametrization of the set of infinite totally nonnegative unitriangular Toeplitz matrices (see [4]) and a characterization of the morphisms from the algebra Λ of symmetric functions to $\mathbb{R}$ which are nonnegative on the Schur functions. These results were generalized in [11] and [12]. A crucial observation here is the connection between the Pieri rule on Schur functions and the structure of the Young graph (which is then said multiplicative in Kerov-Vershik terminology).

In [16], Rietsch obtained a parametrization for the variety $T_{\geq 0}$ of finite unitriangular $(k+1)\times(k+1)$ totally nonnegative Toeplitz matrices by $\mathbb{R}^k_{\geq 0}$ from the quantum cohomology of partial flag varieties. More precisely, such a matrix is proved to be completely determined by the datum of its k initial minors obtained by considering its south-west corners. On the combinatorial side, there is also an interesting k-analogue $\mathcal{B}_k$ of the Young lattice of partitions whose vertices are the k-bounded partitions (i.e. those with no parts greater than k). Its oriented graph structure is isomorphic to to the Hasse poset on the affine Grassmannian permutations of type A which are minimal length coset representatives in $\widetilde{W}/W$, where $\widetilde{W}$ is the affine type $A_k^{(1)}$ group and W the symmetric group of type A_k. The graph $\mathcal{B}_k$ is also multiplicative but we have then to replace the ordinary Schur functions by the k-Schur functions (see [9] and the references therein) and the algebra Λ by $\Lambda_{(k)}=\mathbb{R}[h_1,\ldots,h_k]$. The k-Schur functions were introduced by Lascoux, Lapointe and Morse [10] as a basis of $\Lambda_{(k)}$. It was established by Lam [6] that their corresponding constant structures (called k-Littlewood-Richardson coefficients) are nonnegative. This was done by interpreting $\Lambda_{(k)}$ in terms of the homology ring of the affine Grassmannian which, by works of Lam and Shimozono, can be conveniently identified with the quantum cohomology ring of

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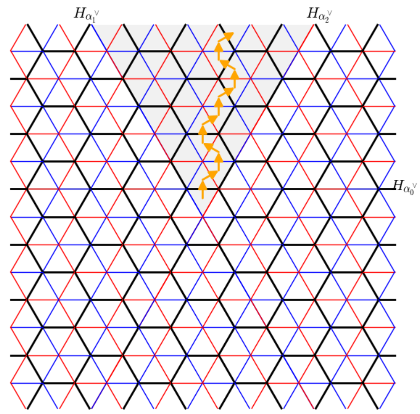

FIGURE 1. A reduced alcove walk on Grassmannian elements for $k=2$

partial flag varieties studied by Rietsch. By merging these two geometric approaches one can theoretically deduce that the set of morphisms from $\Lambda_{(k)}$ to $\mathbb{R}$, nonnegative on the k-Schur functions, are also parametrized by $\mathbb{R}_{\geq 0}^{k}$.

In this paper, we shall use another approach to avoid sophisticated geometric notions and make our construction as effective as possible. Our starting point is the combinatorics of k-Schur functions. We prove they permit to get an explicit parametrization of the morphisms φ nonnegative on the k-Schur functions, or equivalently of all the minimal t-harmonic functions with $t \geq 0$ on $\mathcal{B}_k$. Both notions are related by the simple equality $t = \varphi(s_{(1)})$. Each such morphism is in fact completely determined by its values $\vec{r} = (r_1, \ldots, r_k) \in \mathbb{R}_{\geq 0}^{k}$ on the Schur functions indexed by the rectangle partitions $R_a = (k-a+1)^a$. We get a bi-continuous (homeomorphism) parametrization which is moreover effective in the sense one can compute from $\vec{r}$ the values of φ on any k-Schur function from the Perron Frobenius vector of a matrix Φ encoding the multiplication by $s_{(1)}$ in $\Lambda_{(k)}$. Also, the primitive element theorem permits to prove that for any fixed $t \geq 0$ each $\varphi(s_{\lambda}^{(k)})$ is a rational functions on $\mathbb{R}_{\geq 0}^{k}$. It becomes then quite easy to rederive Rietsch's parametrization. So, the only place where geometry is needed in this paper is in Lam's proof of the nonnegativity of the k-Schur coefficients. As far as we are aware a complete combinatorial k-Littlewood-Richardson rule is not yet available (see nevertheless [14]).

Random walks on reduced alcoves paths have been considered by Lam in [8]. They are random walks on a particular tesselation of $\mathbb{R}^k$ by alcoves supported by hyperplanes, where each hyperplane can be crossed only once. The random walks considered in this paper are central and thus differ from those of [8]. Two trajectories with the same ends will have the same probability. We characterize all the possible laws of these alcove random walks and also get a simple algebraic expression of their drift as a rational function on $\mathbb{R}_{\geq 0}^{k}$. Our results are more precisely summarized in the following Theorem.

Theorem 1.1.

(1) *To each* $\vec{r} \in \mathbb{R}_{\geq 0}^{k}$ *corresponds a unique morphism* $\varphi : \Lambda_{(k)} \to \mathbb{R}$ *nonnegative on the* k*-Schur functions and such that* $\varphi(s_{R_a}) = r_a$ *for any* $a = 1, \ldots, k$.

(2) *To each* $\vec{r} \in \mathbb{R}_{\geq 0}^{k}$ *corresponds a unique matrix* M *in* $T_{\geq 0}$ *whose* k *southwest initial minors are exactly* $r_1, \ldots, r_k$.

(3) *Both previous one-to-one correspondences are homeomophisms, moreover* φ *and* M *can be explicitly computed from* $\vec{r}$ *by using Perron Frobenius theorem.*

(4) *The minimal boundary of $\mathcal{B}_k$ is homeomorphic to a simplex $\mathcal{S}_k$ of $\mathbb{R}^k_{\geq 0}$.*
(5) *To each $\vec{r} \in \mathcal{S}_k$ corresponds a central random walk $(v_n)_{n\geq 0}$ on affine Grassmannian elements which verifies a law of large numbers. The coordinates of its drift are the image by φ of rational fractions in the k-Schur functions. They are moreover rational on $\mathcal{S}_k$.*

The paper is organized as follows. In Section 2, we recall some background on alcoves, partitions and k-Schur functions. In Section 3, we introduce the matrix Φ and study its irreducibility. Section 4 uses classical tools of field theory to derive an expression of any k-Schur function in terms of $s_{(1)}$ and the s_{R_a} $a = 1, \dots, k$. We get the parametrization of all the minimal t-harmonic functions defined on $\mathcal{B}_k$ by $\mathbb{R}^k_{\geq 0}$ in Section 5. In Section 6, we give the law of central random walks on alcoves and compute their drift by exploiting a symmetry property of the matrix Φ. Finally, Section 7 presents consequences of our results, notably we rederive Rietsch's on finite Toeplitz matrices, establish rational expressions for the $\varphi(s^{(k)}_\lambda)$, characterize the simplex $\mathcal{S}_k$ and show the inverse limit of the minimal boundaries of the graphs $\mathcal{B}_k, k \geq 2$ is the Thomas simplex.

2. Harmonic functions on the lattice of k-bounded partitions

2.1. **The lattices $\mathcal{C}_l$ and $\mathcal{B}_k$.** In this section, we refer to [9] and [13] for the material which is not defined. Fix $l > 1$ a nonnegative integer and set $k = l - 1$. Let $\widetilde{W}$ be the affine Weyl group of type $A^{(1)}_k$. As a Coxeter group, $\widetilde{W}$ is generated by the reflections $s_0, s_1, \dots, s_k$ so that its subgroup generated by $s_1, \dots, s_k$ is isomorphic to the symmetric group S_l. Write ℓ for the length function on $\widetilde{W}$. The group $\widetilde{W}$ determines a Coxeter arrangement by considering the hyperplanes orthogonal to the roots of type $A^{(1)}_k$. The connected components of this hyperplane arrangement yield a tessellation of $\mathbb{R}^k$ by alcoves on which the action of $\widetilde{W}$ is regular. We denote by $A^{(0)}$ the fundamental alcove. Write $\widetilde{R}$ for the set of affine roots of type $A^{(1)}_k$ and R for its subset of classical roots of type A_k. The simple roots are denoted by $\alpha_0, \dots, \alpha_k$ and P is the weight lattice of type A_k with fundamental weights $\Lambda_1, \dots, \Lambda_k$.

A reduced alcove path is a sequence of alcoves $(A_1, \dots, A_m)$ such that $A_1 = A^{(0)}$ and for any $i = 1, \dots, m-1$, the alcoves A_{i+1} and A_i share a common face contained in a hyperplane H_i so that the sequence $H_1, \dots, H_{m-1}$ is without repetition (each hyperplane can be crossed only once). In the sequel, all the alcove paths we shall consider will be reduced. For any $i = 1, \dots, m-1$, let w_i be the unique element of $\widetilde{W}$ such that $A_i = w_i(A^{(0)})$. Write $\lhd$ for the weak Bruhat order on $\widetilde{W}$ and $\to$ for the covering relation $w \to w'$ if and only if $w \lhd w'$ and $\ell(w') = \ell(w) + 1$. We then have $w_1 \to w_2 \to \cdots \to w_m$.

We shall identify a partition and its Young diagram. Recall that a l-core can be seen as a partition where no box has hook length equal to l. Given a l-core λ, we denote by $\ell(\lambda)$ its length which is equal to the number of boxes of λ with hook length less that l. Recall that the residue of a box in a Young diagram is the difference modulo l between its row and column indices. We can define an arrow $\lambda \xrightarrow{i} \mu$ between the two l-cores λ and μ when $\lambda \subset \mu$ and all the boxes in μ/λ have the same residue i. By forgetting the label arrows, we get the structure of a graded rooted graph $\mathcal{C}_l$ on the l-cores. For any two vertices $\lambda \to \mu$ we have $\ell(\mu) = \ell(\lambda) + 1$. Nevertheless, the difference between the rank of the partitions λ and μ is not immediate to get in general.

The affine Grassmannian elements are the elements $w \in \widetilde{W}$ whose associated alcoves are exactly those located in the fundamental Weyl chamber (that is, in the Weyl chamber containing the fundamental alcove $A^{(0)}$). The l-cores are known to parametrize the affine Grassmannian elements. More precisely, given two l-cores such that $\lambda \xrightarrow{i} \mu$ and w the affine Grassmannian element associated to λ, $w' = ws_i$ is the affine Grassmannian element associated to μ. In particular, we get $\ell(\lambda) = \ell(w)$. So reduced alcove paths in the fundamental Weyl chamber, saturated chains of affine Grassmannian elements and paths in $\mathcal{C}_l$ naturally correspond.

A k-bounded partition is a partition λ such that $\lambda_1 \leq k$. There is a simple bijection between the l-cores and the k-bounded partitions. Start with a l core λ and delete all the boxes in the diagram of λ having a hook length greater than l (recall there is no box with hook length equal to l since λ is a l-core). This gives a skew shape and to obtain a partition, move each row so obtained on the left. The result is a k-bounded partition denoted $\mathfrak{p}(\lambda)$. For some examples and the converse bijection $\mathfrak{c}$, see [9] pages 18 and 19. This bijection permits to define an analogue of conjugation for the k-bounded partitions. Given a k-bounded partition κ set

$$\kappa^{\omega_k} = \mathfrak{p}(\mathfrak{c}(\kappa)').$$

The graph $\mathcal{B}_k$ is the image of the graph $\mathcal{C}_l$ under the bijection $\mathfrak{p}$. This means that $\mathcal{B}_k$ is the graph obtained from $\mathcal{C}_l$ by deleting all the boxes with hook length greater that l and next by aligning the rows obtained on the left. In particular, reduced alcove paths in the fundamental Weyl chamber correspond to k-bounded partitions paths in $\mathcal{B}_k$. We have the following lemma:

Lemma 2.1. *We have an arrow $\kappa \to \delta$ in $\mathcal{B}_k$ if and only if $|\delta| = |\kappa| + 1$, $\kappa \subset \delta$ and $\kappa^{\omega_k} \subset \delta^{\omega_k}$.*[1]

Let Λ be the algebra of symmetric functions in infinitely many variables over $\mathbb{R}$. It is endowed with a scalar product $\langle \cdot, \cdot \rangle$ such that $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$ for any partitions λ and μ. Let $\Lambda_{(k)}$ be the subalgebra of Λ generated by the complete homogeneous functions $h_1, \ldots, h_k$. In particular, $\{h_\lambda \mid \lambda \text{ is } k\text{-bounded}\}$ is a basis of $\Lambda_{(k)}$.

2.2. **The k-Schur functions.** We now define a distinguished basis of $\Lambda_{(k)}$ related to the graph structures of $\mathcal{C}_l$ and $\mathcal{B}_k$. Consider λ and μ two k-bounded partitions with $\lambda \subset \mu$ and $r \leq k$ a positive integer.

Definition 2.2. *We will say that μ/λ is a weak horizontal strip of size r when*
1. *μ/λ is an horizontal strip with r boxes (i.e. the boxes in μ/λ belong to different columns),*
2. *$\mu^{\omega_k}/\lambda^{\omega_k}$ is a vertical strip with r boxes (i.e. the boxes in $\mu^{\omega_k}/\lambda^{\omega_k}$ belong to different rows).*

Let us now define the notion of k-bounded semistandard tableau of shape λ a k-bounded partition and weight $\alpha = (\alpha_1, \ldots, \alpha_d)$ a composition of $|\lambda|$ *with no part larger than k.*

Definition 2.3. *A k-bounded semistandard tableau of shape λ is a semistandard filling of λ with integers in $\{1, \ldots, d\}$ such that for any $i = 1, \ldots, d$ the boxes containing i define a horizontal strip of size α_i.*

One can prove that for any k-bounded partitions λ and α the number $K^{(k)}_{\lambda,\alpha}$ of k-bounded semistandard tableaux of shape λ and weight α verifies

$$K^{(k)}_{\lambda,\lambda} = 1 \text{ and } K^{(k)}_{\lambda,\alpha} \neq 0 \Longrightarrow \alpha \leq \lambda$$

where $\leq$ is the dominant order on partitions.

Definition 2.4. *The k-Schur functions $s^{(k)}_\kappa, \kappa \in \mathcal{B}_k$ are the unique functions in $\Lambda_{(k)}$ such that*

$$h_\delta = \sum_{\delta \leq \kappa, \kappa \in \mathcal{B}_k} K^{(k)}_{\kappa,\delta} s^{(k)}_\kappa$$

for any δ in $\mathcal{B}_k$.

[1]So $\mathcal{B}_k$ *should not be confused* with the subgraph of the Young graph with vertices the k-bounded partitions.

Proposition 2.5 (Pieri rule for k-Schur functions)**.** *For any $r \leq k$ and any $\kappa \in \mathcal{B}_k$ we have*

$$h_r s_\kappa^{(k)} = \sum_{\varkappa \in \mathcal{B}_k} s_\varkappa^{(k)} \tag{1}$$

where the sum is over all the k-bounded partitions $\varkappa$ such that $\varkappa/\kappa$ is a weak horizontal strip of size r in $\mathcal{B}_k$.

When $r = 1$, the multiplication by h_1 is easily described by considering all the possible k-bounded partitions at distance 1 from κ in $\mathcal{B}_k$. Thanks to a geometric interpretation of the k-Schur functions in terms of the homology of affine Grassmannians, Lam showed that the product of two k-Schur functions is k-Schur positive:

Theorem 2.6. [6] *Given κ and δ two k-bounded partitions, we have*

$$s_\kappa^{(k)} s_\delta^{(k)} = \sum_{\nu \in \mathcal{B}_k} c_{\lambda,\delta}^{\nu(k)} s_\nu^{(k)}$$

with $c_{\lambda,\delta}^{\nu(k)} \in \mathbb{Z}_{\geq 0}$.

2.3. **Recollection of properties of k-Schur functions.** The k-conjugation operation ω_k can be read directly at the level of k-partitions without using the ordinary conjugation operations on the l-cores (see (1.9)) in [9]). To do this, start with a k-partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ and decompose it into its chains $\{c_1, c_2, \ldots, c_r\}$ where each chain is a sequence of parts of λ obtained recursively as follows. The procedure is such that any partition λ_i is in the same chain as the partition $\lambda_{i+k-\lambda_i+1}$ when $i + k - \lambda_i + 1 \leq r$ (from the part λ_i one jumps $k - \lambda_i$ parts to get the following part of the chain). Observe in particular that all the parts with length k belong to the same chain for in this case we jump 0 parts. Once the chains c_i are determined, λ^{ω_k} is the partitions with k-columns whose lengths are the sums of the c_i's.

Example 2.7. *Consider the* 5*-partition* $\lambda = (\mathbf{5}, \mathbf{5}, \mathbf{5}, \mathbf{4}, \mathit{4}, \mathbf{3}, \mathit{3}, 3, \mathbf{2}, \mathit{2}, 1)$*. Then, we get* $c_1 = \{5, 5, 5, 4, 3, 2\}$ *next* $c_2 = \{4, 3, 2\}$, $c_3 = \{3, 1\}$, $c_4 = \emptyset$ *and* $c_5 = \emptyset$*. So* λ^{ω_5} *is the partition with columns of heights* 24, 9 *and* 4*.*

The following facts will be useful.

1. Any partition λ of rank at most k is a k-partition and is then equal to its associated l-core (because λ has no hook of length $l = k + 1$!).
2. The lattice $\mathcal{B}_k$ coincides with the ordinary Young lattice on the partitions of rank at most k. On this subset ω_k is the ordinary conjugation.
3. For any partition λ of rank at most k, the k-Schur function coincides with the ordinary Schur function that is $s_\lambda^{(k)} = s_\lambda$. In particular, the homogeneous functions $h_1, \ldots, h_k$ and the elementary functions $e_1, \ldots, e_k$ are the k-Schur functions corresponding to the rows and columns partitions with at most k boxes, respectively.

The $k = 2$ case is easily tractable because the lattice of 2-bounded partitions we consider has a simple structure. One verifies easily that for any 2-partition $\lambda = (2^a, 1^{n-2a})$, we get in that case

$$s_\lambda^{(2)} = \begin{cases} h_2^a e_2^{\frac{n}{2}-a} \text{ when } n \text{ even,} \\ h_2^a e_2^{\frac{n-1}{2}-a} e_1 \text{ when } n \text{ is odd.} \end{cases}$$

When $k > 2$, the structure of the graph $\mathcal{B}_k$ becomes more complicated. Given a k-bounded partition λ one can first precise where it is possible to add a box in the Young diagram of λ to get an arrow in $\mathcal{B}_k$. Assume we add a box on the row λ_i of λ to get the k-partition μ, denote by $c = \{\lambda_{i_1}, \ldots, \lambda_{i_r}\}$ the chain containing λ_i where we have $\lambda_i = \lambda_{i_a}$ with $a \in \{1, \ldots, r\}$. Observe we can add components equal to zero to c if needed since λ is defined up to an arbitrary zero parts. The following lemma permits to avoid the use of ω_k in the construction of $\mathcal{B}_k$.

Lemma 2.8. *There is an arrow $\lambda \to \mu$ in $\mathcal{B}_k$ if and only if $\lambda_{i_b-1} = \lambda_{i_b}$ for any $b = a+1, \dots, r$, that is if each part located up to λ_i in the chain containing λ_i is preceded by a part with the same size.*

Proof. One verifies that if the previous condition is not satisfied, μ^{ω_k} and λ^{ω_k} will differ by at least two boxes and if it is satisfied by only one as desired. □

Example 2.9.

(1) *One can always add a box on the first column of λ since the parts located up to the part 0 are all equal to 0.*
(2) *Assume $k = 2$, then we can add a box on the part λ_i equal to 1 to get a part equal to 2 if and only if there is an even parts equal to 1 up to λ_i. This is equivalent to says that λ has an odd number parts equal to 1, that is that the rank of λ is odd since the other parts are equals to 2 (or 0).*

For any $a = 1, \dots, k$, let R_a be the rectangle partition $(k-a+1)^a$. The previous observations can be generalized (see [9]):

Proposition 2.10.

(1) *Assume λ is a k-bounded partition which is also a $(k+1)$-core. Then $s_\lambda^{(k)} = s_\lambda$ (that is the k-Schur and the Schur functions corresponding to λ coincide).*
(2) *In particular, for any rectangle partition R_a, we have $s_{R_a}^{(k)} = s_{R_a}$.*
(3) *For any $a = 1, \dots, k$ and any k-partition λ we have*

$$s_{R_a} s_\lambda^{(k)} = s_{\lambda \cup R_a}^{(k)}$$

where $\lambda \cup R_a$ is obtained by adding a parts equal to $k-a+1$ to λ.

Corollary 2.11. *For each k-bounded partition λ, there exists a unique irreducible k-partition[2] $\widetilde{\lambda}$ and a unique sequence of nonnegative integers $p_1, \dots, p_k$ such that*

$$s_\lambda^{(k)} = \prod_{a=1}^{k} s_{R_a}^{p_a} s_{\widetilde{\lambda}}^{(k)}.$$

In particular, the k-Schur functions are completely determined by the k-Schur functions indexed by an irreducible k-bounded partition and by the $s_{(k-a+1)^a}$, $a = 1, \dots, k$.

Remark 2.12. *Write $\mathcal{P}_{\mathrm{irr}}$ for the set of irreducible partitions. The map $\Delta : \mathcal{B}_k \to \mathcal{P}_{\mathrm{irr}} \times \mathbb{Z}_{\geq 0}^k$ which associates to each k-bounded partition λ the pair $(p, \widetilde{\lambda})$ where $p = (p_1, \dots, p_k)$ is a bijection.*

Example 2.13. *Assume $k = 4$ and $\lambda = (4, 4, 3, 3, 3, 3, 3, 2, 2, 2, 2, 1, 1, 1, 1, 1)$. Then we get $\widetilde{\lambda} = (3, 2, 1)$ and*

$$s_\lambda^{(k)} = s_{(4)}^2 s_{(3,3)}^2 s_{(2,2,2)} s_{(1,1,1,1)} s_{\widetilde{\lambda}}^{(k)}.$$

We conclude this paragraph by recalling other important properties of k-Schur functions. We have first the inclusions of algebras $\Lambda_{(k)} \subset \Lambda_{(k+1)} \subset \Lambda$.

[2]A k-bounded partition is irreducible when it contains less than a parts equal to $k-a$ for any $a = 0, \dots, k-1$.

Proposition 2.14. [9]

(1) *Each k-Schur function has a positive expansion on the basis of $(k+1)$-Schur functions.*
(2) *Each k-Schur function has a positive expansion on the basis of ordinary Schur functions.*

2.4. **Harmonic functions and minimal boundary of $\mathcal{B}_k$.**

Definition 2.15. *A function $f:\mathcal{B}_k\to\mathbb{R}$ is said harmonic when*

$$f(\lambda)=\sum_{\lambda\to\mu} f(\mu) \text{ for any } \lambda\in\mathcal{B}_k.$$

We denote by $\mathcal{H}(\mathcal{B}_k)$ the set of harmonic functions on $\mathcal{B}_k$.

Another way to understand harmonic functions is to introduce the infinite matrix $\mathcal{M}$ of the graph $\mathcal{B}_k$. The harmonic functions on $\mathcal{B}_k$ then correspond to the right eigenvectors for $\mathcal{M}$ associated to the eigenvalue 1. One can also consider t-harmonic functions which correspond to the right eigenvectors for $\mathcal{M}$ associated to the eigenvalue t. Clearly $\mathcal{H}(\mathcal{B}_k)$ is a vector space over $\mathbb{R}$. In fact, we mostly restrict ourself to the set $\mathcal{H}^+(\mathcal{B}_k)$ of positive harmonic functions for which f takes values in $\mathbb{R}_{\geq 0}$. Then, $\mathcal{H}^+(\mathcal{B}_k)$ is a cone since it is stable by addition and multiplication by a positive real. To study $\mathcal{H}^+(\mathcal{B}_k)$ we only have to consider its subset $\mathcal{H}_1^+(\mathcal{B}_k)$ of normalized harmonic functions such that $f(1)=1$. In fact, $\mathcal{H}_1^+(\mathcal{B}_k)$ is a convex set and its structure is controlled by its extremal subset $\partial\mathcal{H}^+(\mathcal{B}_k)$. We aim to characterize the extremal positive harmonic functions defined on $\mathcal{B}_k$ and obtain a simple parametrization of $\partial\mathcal{H}^+(\mathcal{B}_k)$. By using the Pieri rule on k-Schur functions, we get

$$s_\lambda s_{(1)}=\sum_{\lambda\to\mu} s_\mu$$

for any k-partitions λ and μ. This means that $\mathcal{B}_k$ is a so-called multiplicative graph with associated algebra $\Lambda_{(k)}$. Moreover, if we denote by K the positive cone spanned by the set of k-Schur functions, we can apply the ring theorem of Kerov and Vershik (see for example [12, Section 8.4]) which characterizes the extreme points $\partial\mathcal{H}^+(\mathcal{B}_k)$. Denote by $\mathrm{Mult}^+(\Lambda_{(k)})\subset(\Lambda_{(k)})^*$ the set of multiplicative functions on $\Lambda_{(k)}$ which are nonnegative on K and equal to 1 on s_1. Note that $i:\mathcal{B}_k\longrightarrow\Lambda_{(k)}$ such that $i(\lambda)=s_\lambda^{(k)}$ induces a map $i^*:(\Lambda_{(k)})^*\longrightarrow F(\mathcal{B}_k,\mathbb{R})$. Since we have $K.K\subset K$, we get the following algebraic characterization of $\partial\mathcal{H}^+(\mathcal{B}_k)$.

Proposition 2.16. *The map i^* yields an homeomorphism between* $\mathrm{Mult}^+(\Lambda_{(k)})$ *and* $\partial\mathcal{H}^+(\mathcal{B}_k)$.

Since $i(\mathcal{B}_k)$ is a basis of $\Lambda_{(k)}$, this means that $\partial\mathcal{H}(\mathcal{B}_k)$ is completely determined by the $\mathbb{R}$-algebra morphisms $\varphi:\Lambda_{(k)}\to\mathbb{R}$ such that $\varphi(s_1)=1$ and $\varphi(s_\lambda^{(k)})\geq 0$ for any k-bounded partition λ. Each function $f\in\partial\mathcal{H}^+(\mathcal{B}_k)$ can then be written $f=\varphi\circ i$.

By Corollary 2.11, the condition $\varphi(s_\lambda^{(k)})\geq 0$ for each k-bounded partition reduces in fact to test a finite number of k-Schur functions, namely $\varphi(s_{\tilde\lambda}^{(k)})\geq 0$ for each irreducible k-bounded partition (there are $(k-1)!$ such partitions) and $\varphi(s_{(k-a+1)^a})\geq 0$ for any $a=1,\dots,k$.

3. Restricted graph and irreducibility

3.1. **The matrix Φ.** By Corollary 2.11, each morphism $\varphi:\Lambda_{(k)}\to\mathbb{R}$ is uniquely determined by its values on the rectangle Schur functions $s_{R_a}, 1\leq a\leq k$ and on each $s_{\tilde\lambda}^{(k)}$ where $\tilde\lambda$ is an irreducible k-bounded partition. Set $r_a=\varphi(s_{R_a}), a=1,\dots,k$ and $\vec{r}=(r_1,\dots,r_k)$. Recall that

$\mathcal{P}_{\text{irr}}$ is the set of irreducible k-bounded partitions (including the empty partition). Then, for $\lambda \in \mathcal{P}_{\text{irr}}$,

$$\varphi(s_\lambda^{(k)})\varphi(s_{(1)}) = \sum_{\lambda \to \mu} \varphi(s_\mu^{(k)}). \tag{2}$$

By Corollary 2.11, for each k-bounded partition μ there exists a sequence $\{p_1^\mu, p_2^\mu \ldots, p_k^\mu\}$ of elements in $\{1, \ldots, k\}$ and an irreducible partition $\widetilde{\mu}$ such that

$$s_\mu^{(k)} = \prod_{a=1}^{k} s_{R_a}^{p_a^\mu} s_{\widetilde{\mu}}^{(k)} \text{ and thus } \varphi(s_\mu^{(k)}) = \prod_{a=1}^{k} r_a^{p_a^\mu} \varphi(s_{\widetilde{\mu}}^{(k)}). \tag{3}$$

Hence by setting

$$\varphi_{\lambda\nu} = \sum_{\substack{\lambda \to \mu \\ \widetilde{\mu} = \nu}} \prod_{1 \le a \le k} r_a^{p_a^\mu}$$

we get

$$\varphi(s_\lambda^{(k)}) = \sum_{\nu \in \mathcal{P}_{\text{irr}}} \varphi_{\lambda\nu} \varphi(s_\nu^{(k)}).$$

Let $\Phi_{(r_1,\ldots,r_k)} := (\varphi_{\nu\lambda})_{\lambda,\nu \in \mathcal{P}_{\text{irr}}}$ [3] and define $f \in \mathbb{R}^{\mathcal{P}_{\text{irr}}}$ as the vector $(\varphi(s_\lambda^{(k)}))_{\lambda \in \mathcal{P}_{\text{irr}}}$. When there is no risk of confusion, we simply write Φ instead of $\Phi_{(r_1,\ldots,r_k)}$. The vector f is an left eigenvector of Φ for the eigenvalue $\varphi(s_1)$ with positive entries having value 1 on $\emptyset$ and $\varphi(s_1)$ on s_1.

3.2. **Irreducibility of the matrix Φ.** Recall that a matrix $M \in M_n(\mathbb{R})$ with nonnegative entries is irreducible if and only if for each $1 \le i, j \le n$ there exists $n \ge 1$ such that $(M^n)_{ij} > 0$.

Proposition 3.1. *Assume that $\varphi(s_{R_a}) \ge 0$ for any $a = 1, \ldots, k$. Then, the matrices Φ and Φ^t associated to φ is irreducible if and only if for all $1 \le a \le k-1$, $\varphi(s_{R_a})$ or $\varphi(s_{R_{a+1}})$ is positive.*

We will prove in fact that Φ^t is irreducible. Let G be the graph with set of vertices $\mathcal{P}_{\text{irr}}$ and a directed edge from λ to ν if and only if $\Phi_{\lambda\nu} \neq 0$. The matrix Φ^t is irreducible if and only if G is strongly connected, which means that there is a (directed) path from any vertex to any other vertex of the graph. We prove Proposition 3.1 by showing that G is strongly connected. Let us first establish a preliminary lemma. We say that $\lambda \in \mathcal{P}_{\text{irr}}$ is $i$saturated when $i = 1$ or $i \ge 2$ and

$$\lambda = (\ldots, (i-1)^{k-i+1}, \ldots, 2^{k-2}, 1^{k-1}) \text{ with } \lambda \neq (\ldots, i^k, (i-1)^{k-i+1}, \ldots, 2^{k-2}, 1^{k-1}).$$

Denote by λ^1 the irreducible k-bounded partition $((k-1)^1, (k-2)^2, \ldots, 1^{k-1})$. Remark that if $\lambda \in \mathcal{P}_{\text{irr}}$ is $k$saturated if and only if $\lambda = \lambda^1$.

Lemma 3.2. *Any vertex λ of G is connected to λ^1.*

Proof. The statement of the lemma is a deduction of the four following facts:

(1) If λ is $i$saturated, then $\lambda \to \lambda^{\uparrow i-1}$, where $\lambda^{\uparrow i-1}$ is the partition obtained by adding one box to the first row of size $(i-1)$ of λ. Moreover, $\lambda^{\uparrow i-1}$ is $(i-1)$-saturated.

Proof. Let c be the chain containing the first row of size $i-1$ in λ assumed to be λ_r. Since λ is $i$saturated, a combinatorial computation shows that

$$c = \{\ldots, \lambda_r, \lambda_{r+k-i}, \lambda_{r+(k-i)+(k-i+1)}, \ldots, \lambda_{r+\sum_{s=0}^{i-1}(k-i+s)}\},$$

[3] Observe we have defined $\Phi_{(r_1,\ldots,r_k)}$ as the transpose of the matrix $(\varphi_{\lambda,\mu})_{\lambda,\nu \in \mathcal{P}_{\text{irr}}}$ to make it compatible with the multiplication by $s_{(1)}$ used in Section 4.

and for $0 \leq t \leq i-1$, each $\lambda_{r+\sum_{s=0}^{t}(k-i+s)}$ is the $(t+1)$-th row of length $(i-1-t)$. In particular, each element of c after λ_r is preceded by a row of same size, and there is less than $k-2$ rows after the last row of c (which has size one). Therefore, we can apply Lemma 2.8 to get the existence of an edge between λ and $\lambda^{\uparrow i-1}$. □

(2) Suppose that λ has less than $(k-i)$ rows of length i. Applying successively the previous operation to $\lambda, \lambda^{\uparrow(i-1)}, (\lambda^{\uparrow(i-1)})^{\uparrow(i-2)}, \ldots$ eventually yields a partition ν which is irreducible, $i$saturated, such that there is a path between λ and ν in $\mathcal{B}_k$ and such that the number of rows of length i in ν is one more than that in λ.

(3) Suppose that λ had initially $l \leq k-i-1$ rows of size i. Repeating the previous process $k-i-l$ times yields a partition κ such that λ is connected to κ in $\mathcal{B}_k$ and κ is $(i+1)$-saturated.

(4) Repeating the previous process for $i < l \leq k$ yields a path between λ and λ^1.

□

We prove Proposition 3.1 by giving necessary and sufficient conditions to have a path on G between λ^1 and $\emptyset$.

Proof of Proposition 3.1. Let us show that there is a path between λ^1 and $\emptyset$ if and only if for all $1 \leq a \leq k-1$, $\varphi(s_{R_a})$ or $\varphi(s_{R_{a+1}})$ is positive.

- Suppose first that for all $1 \leq a \leq k-1$, $\varphi(s_{R_a})$ or $\varphi(s_{R_{a+1}})$ is positive.

  For $1 \leq a \leq k-1$, let λ^a be the partition $(k-1, (k-2)^2, \ldots, a^{k-a})$, and set $\lambda^k = \emptyset$. Let us prove that for $1 \leq a \leq k-1$, there is a path on G from λ^a to λ^{a+1}. If $\varphi(s_{R_a}) > 0$, it suffices to add a part of length a at the end of λ^a, which is always possible. Suppose now that $\varphi(s_{R_a}) = 0$, which implies that $\varphi(s_{R_{a+1}}) > 0$ by the hypothesis on φ. Let i be minimal such that $\lambda^a_i = a$, and let c be the chain containing i. On the one hand, since $\lambda^a_i = a$, the part following λ^a_i in c is the part $\lambda^a_{i+k-a+1}$. On the other hand, by the definition of λ^a, $\lambda^a_{i+(k-a)} = \lambda^a_{i+(k-a)+1} = 0$, thus by Lemma 2.8, we can add a box to the part λ^a_i, which makes appear a block $(a+1)^{k-a+1}$. Thus, since $\varphi(s_{R_{a+1}}) > 0$, there is an arrow in G between λ^a and the partition $(\lambda^{a-2}, \lambda^a_{i+1}, \ldots, \lambda^a_{i+(k-a)-1})$. Similarly, we can successively add a box to part $\lambda^a_{i+1}, \ldots, \lambda^a_{i+(k-a)-1}$, which yields a path between $(\lambda^{a-2}, \lambda^a_{i+1}, \ldots, \lambda^a_{i+(k-a)-1})$ and λ^{a+1}.

  By the above results, there is a path in G from λ^a to λ^{a+1} for all $1 \leq a \leq k-1$, thus there is a path from λ^1 to $\lambda^k = \emptyset$.

- Suppose now that there exists $1 \leq a \leq k-1$ such that $\varphi(s_{R_a}) = \varphi(s_{R_{a+1}}) = 0$. Let $\mu = (\mu^1, \ldots, \mu^r)$ be a path on G starting at a^{k-a}, and denote by x_i (resp. y_i) the number of parts equal to $a+1$ (resp. a) in μ^i. Since γ is a path on G, for all $1 \leq i \leq r$, $x_i \leq k-a-1$ and $y_i \leq k-a$. Let us prove by induction on $1 \leq i \leq r$ that $x_i + y_i \geq k-a$.

  This is certainly true for $i = 1$. Suppose that $i > 1$ and that the result holds for $i-1$. Since $\varphi(s_{R_a}) = \varphi(s_{R_{a+1}}) = 0$, the only way to get $x_i + y_i < x_{i-1} + y_{i-1}$ is to add a box to the first part of length $a+1$ (if any) in μ^{i-1}. Hence if $x_{i-1} = 0$, then $x_i + y_i \geq x_{i-1} + y_{i-1} \geq k-a$. Assume now that $x_{i-1} > 0$. Let l be minimal such that $\mu^{i-1}_l = a+1$ and let c be the chain containing l. Hence, the part following μ^{i-1}_l in c is equal to $l + k - (a+1) + 1 = l + (k-a)$. Since $x_{i-1} + y_{i-1} \geq k-a$ and $x_{i-1} \leq k-a-1$, we have $\mu^{i-1}_{l+(k-a)-1} = a$. Thus, by Lemma 2.8, it is possible to add a box to the part l in μ^{i-1} if and only if $\mu^{i-1}_{l+k-a} = a$. If so, then we get $x_{i-1} + y_{i-1} \geq (l+k-a) - l + 1 \geq k-a+1$ and $x_i + y_i = x_{i-1} + y_{i-1} - 1$. Therefore, in any cases, $x_i + y_i \geq k-a$. For any path starting on G at a^{k-a}, the number of parts of length a or $a+1$ remains larger than $k-a$, thus there is no path between a^{k-a} and $\emptyset$ in G.

□

4. Field extensions and k-Schur functions

4.1. **Field extensions.** Recall that $\Lambda_{(k)}=\mathbb{R}[h_1,\dots,h_k]$. Since $h_1,\dots,h_k$ are algebraically independent over $\mathbb{R}$, we can consider the fraction field $\mathbb{L}=\mathbb{R}(h_1,\dots,h_k)$. Write $\mathbb{A}=\mathbb{R}[s_{R_1},\dots s_{R_k}]$ the subalgebra of $\Lambda_{(k)}$ generated by the rectangle Schur functions $R_a, a=1,\dots,k$. In order to introduce the fraction field of $\mathbb{A}$, we first need to check that $s_{R_1},\dots s_{R_k}$ are algebraically independent over $\mathbb{R}$. We shall use a proposition giving a sufficient condition on a family of polynomials to be algebraically independent.

Let k be a field and $k[T_1,\dots,T_m]$ the ring of polynomials in $T_1,\dots,T_m$ over k. For any $\beta\in\mathbb{Z}_{\geq 0}^m$, we set $T^\beta=T_1^{\beta_1}\cdots T_m^{\beta_m}$. We also assume we have a total order $\preceq$ on the monomials of $k[T_1,\dots,T_m]$. The leading monomial $\mathrm{lm}(P)$ of a polynomial $P\in k[T_1,\dots,T_m]$ is the monomial appearing in the support of P (that is, with a nonzero coefficient) maximal under the total order $\preceq$.

Proposition 4.1. *(See* [5]*)*

(1) *The monomial $T^{\beta^{(1)}},\dots,T^{\beta^{(l)}}$ are algebraically independent if and only if $\beta^{(1)},\dots,\beta^{(p)}$ are linearly independent over $\mathbb{Z}$.*
(2) *Consider $P_1,\dots,P_l$ polynomials in $k[T_1,\dots,T_m]$ such that $\mathrm{lm}(P_1),\dots,\mathrm{lm}(P_l)$ are algebraically independent, then $P_1,\dots,P_l$ are algebraically independent.*

With Proposition 4.1 in hand, it is then easy to check that $s_{R_1},\dots s_{R_k}$ are algebraically independent over $\mathbb{R}$. Recall that each Schur function s_λ with indeterminate set $X=\{X_1,X_2\dots\}$ decomposes on the form

$$s_\lambda = X^\lambda + \sum_{\mu<\lambda} K_{\lambda,\mu}X^\mu$$

where $\leq$ is the dominant order over finite sequences of integers, that is, $\beta<\beta'$ when $\beta-\beta'$ decomposes as a sum of $\varepsilon_i-\varepsilon_j, i<j$ with nonnegative integer coefficients. We can choose any total order $\preceq$ refining this dominance order. Then, by Assertion 1 of the previous proposition, the monomials $X^{R_1},\dots,X^{R_k}$ are algebraically independent since the rectangle partitions $R_a, a=1,\dots,k$ are linearly independent over $\mathbb{Z}$. Assertion 2 then implies that $s_{R_1},\dots s_{R_k}$ are algebraically independent over $\mathbb{R}$. We denote by $\mathbb{K}=\mathbb{R}(s_{R_1},\dots s_{R_k})$ the fraction field of the algebra $\mathbb{A}$.

Proposition 4.2.

(1) *Each $h_1,\dots,h_k$ is algebraic over $\mathbb{K}$.*
(2) *We have $\mathbb{L}=\mathbb{R}(h_1,\dots,h_k)=\mathbb{K}[h_1,\dots,h_k]$.*
(3) *$\mathbb{L}$ is an algebraic extension of $\mathbb{K}$.*
(4) *The field $\mathbb{L}$ is a finite extension of $\mathbb{K}$ with degree $[\mathbb{L}:\mathbb{K}]=k!$ and the set $\mathcal{I}=\{s_\kappa^{(k)}\mid\kappa$ is k-irreducible$\}$ is a basis of $\mathbb{L}$ over $\mathbb{K}$.*
(5) *$\Lambda_{(k)}$ is an integral extension of $\mathbb{A}$.*

Proof. 1: For any $a=1,\dots,k$, consider the evaluation morphism $\theta_a:\mathbb{K}[T]\to\mathbb{L}$ which associates to any $P\in\mathbb{K}[T]$, the polynomial $P(h_a)\in\Lambda_{(k)}=\mathbb{R}(h_1,\dots,h_k)$. We know that $\{s_\lambda^{(k)}\mid\lambda\in\mathcal{B}_k\}$ is a basis of $\Lambda_{(k)}$ over $\mathbb{R}$ thus each power $h_a^i, i\in\mathbb{Z}_{\geq 0}$ decomposes on the basis of k-Schur functions with real coefficients. By Corollary 2.11, h_a^i then decomposes on the family $\mathcal{I}=\{s_\kappa^{(k)}\mid\kappa$ is

k-irreducible} with coefficients in $\mathbb{K}$. Thus, $P(h_a)$ also decomposes on the family $\mathcal{I} = \{s_\kappa^{(k)} \mid \kappa$ is k-irreducible} with coefficients in $\mathbb{K}$. Since $\mathcal{I}$ is a finite set, this shows that $\mathrm{Im}(\theta_a)$ is a finite-dimensional $\mathbb{K}$-subspace of $\mathbb{L}$, thus h_a is algebraic over $\mathbb{K}$.

2: Since $h_1, \dots, h_k$ are algebraic over $\mathbb{K}$, we get $\mathbb{K}[h_1, \dots, h_k] = \mathbb{K}(h_1, \dots, h_k) \subset \mathbb{L}$. We also have $\mathbb{L} = \mathbb{R}(h_1, \dots, h_k) \subset \mathbb{K}(h_1, \dots, h_k)$ since $\mathbb{K}$ is an extension of $\mathbb{R}$. Thus, $\mathbb{L} = \mathbb{K}[h_1, \dots, h_k]$.

3: This easily follows from 1 and 3.

4: By using the same arguments as in the proof of 1, we get that each element $Q(h_1, \dots, h_k)$ with $Q \in \mathbb{K}[T_1, \dots, T_k]$ decomposes on the family $\mathcal{I} = \{s_\kappa^{(k)} \mid \kappa$ is k-irreducible} with coefficients in $\mathbb{K}$. Assume we have

$$\sum_{\kappa \mid k\text{-irreducible}} c_\kappa s_\kappa^{(k)} = 0$$

with $c_\kappa \in \mathbb{K} = \mathbb{R}(s_{R_1}, \dots, s_{R_k})$ for any k-irreducible partition κ. Up to multiplication, we can assume these coefficients belong in fact to $\mathbb{R}[s_{R_1}, \dots, s_{R_k}]$. Set

$$c_\kappa = \sum_{\beta \in \mathbb{Z}^k_{\geq 0}} a_\beta^{(\kappa)} s_{R_1}^{\beta_1} \cdots s_{R_k}^{\beta_k}$$

where all the coefficients $a_\beta^{(\kappa)}$ are equal to 0 up to a finite number (in which case $a_\beta^{(\kappa)}$ is real). We get

$$\sum_{\kappa \mid k\text{-irreducible}} \sum_{\beta \in \mathbb{Z}^k_{\geq 0}} a_\beta^{(\kappa)} s_{R_1}^{\beta_1} \cdots s_{R_k}^{\beta_k} s_\kappa^{(k)} = 0. \tag{4}$$

By Remark 2.12, there is a bijection between the set of k-bounded partitions and that of pairs (β, κ) with κ k-irreducible and $\beta \in \mathbb{Z}^k_{\geq 0}$. So equation (4) gives in fact a linear combination of the k-Schur functions with real coefficients which equates to 0. Since we know that the set of k-Schur functions is a basis of $\Lambda_{(k)}$, this imposes that each coefficient $a_\beta^{(\kappa)}$ is in fact equal to 0. So the family $\mathcal{I} = \{s_\kappa^{(k)} \mid \kappa$ is k-irreducible} is a $\mathbb{K}$-basis of $\mathbb{L}$ and we have $[\mathbb{L} : \mathbb{K}] = \mathrm{card}(\mathcal{I}) = k!$.

5: The characteristic polynomial of each $h_a, a = 1, \dots, k$ belongs to $\mathbb{A}[T]$ because the multiplication by h_a on the basis $\mathcal{I}$ makes only appear coefficients in $\mathbb{A}$. Thus, each h_a is an integral element of $\Lambda_{(k)} = \mathbb{A}[h_1, \dots, h_k]$ over $\mathbb{A}$. □

4.2. **Primitive element.** By Proposition 4.2, $s_1 = h_1$ is algebraic over $\mathbb{K}$. Denote by Π its minimal polynomial. Observe that Π is an irreducible polynomial of $\mathbb{K}[T]$. Also write $\mathbf{\Phi}$ for the matrix of the multiplication by s_1 in $\mathbb{L}$ in the basis $\mathcal{I} = \{s_\kappa^{(k)} \mid \kappa$ is k-irreducible} (here we assume we have fixed once for all a total order on the set $\mathcal{P}_{\mathrm{irr}}$ of k-irreducible partitions). Let $\Xi(T) = \det(TI_{k!} - \mathbf{\Phi})$ be the characteristic polynomial of the matrix $\mathbf{\Phi}$. We thus have that Π divides Ξ. Moreover both polynomials belong to $\mathbb{A}[T]$ since the entries of the matrix $\mathbf{\Phi}$ are in the ring $\mathbb{A}$. We have in fact the following stronger proposition.

Proposition 4.3.

(1) *The invariant factors of the multiplication by s_1 are all equal to Π: there exists an integer m such that $\Xi = \Pi^m$.*
(2) *The coefficients of Π and Ξ are invariant under the flips of s_{R_a} and $s_{R_{k-a+1}}$ for any $a = 1, \dots, \lfloor \frac{k}{2} \rfloor$.*

Proof. 1: Write $P_1, \dots, P_m$ for the invariant factors of $\mathbf{\Phi}$. We must have $P_1/P_2/\cdots/P_m$, $P_m = \Pi$ and $P_1P_2\cdots P_m = \Xi$. Since Π is irreducible, this imposes $P_1 = \cdots = P_m = \Pi$ which gives the assertion.

2: We apply similarly ω_k to each equality $\Pi(s_{(1)}) = 0$ and $\Xi(s_{(1)}) = 0$. □

Example 4.4. *For $k=3$, we get by listing the restricted k-partitions as $s_{\emptyset}, s_{(1)}, s_{(2)}, s_{(1,1)}, s_{(2,1)}$ and $s_{(2,1,1)}$*

$$\mathbf{\Phi}=\begin{pmatrix} 0 & 0 & s_{R_1} & s_{R_3} & s_{R_2} & 0 \\ 1 & 0 & 0 & 0 & 0 & s_{R_2} \\ 0 & 1 & 0 & 0 & 0 & s_{R_3} \\ 0 & 1 & 0 & 0 & 0 & s_{R_1} \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

This gives $\Xi(T)=T^6-2\left(s_{R_1}+s_{R_3}\right)T^3-4s_{R_2}T^2+\left(s_{R_1}-s_{R_3}\right)^2$. Observe the symmetry of $\mathbf{\Phi}$ which will be elucidated in § 6.1.

In Proposition 4.3, we have just used that $\mathbf{\Phi}$ is the matrix of the multiplication by $s_{(1)}$ which is algebraic over $\mathbb{K}$. Given any morphism $\widetilde{\varphi}:\mathbb{A}\rightarrow\mathbb{R}$ such that $\widetilde{\varphi}(s_{R_a})\geq 0$ for any $a=1,\ldots,k$, the matrix $\widetilde{\varphi}(\mathbf{\Phi})$ obtained by replacing in $\mathbf{\Phi}$ each rectangle Schur function s_{R_a} by $\widetilde{\varphi}(s_{R_a})$ coincides with the matrix Φ defined in § 3.1. Thus $\Phi=\widetilde{\varphi}(\mathbf{\Phi})$ has nonnegative integer coefficients. We can now state the main theorem of this section.

Theorem 4.5. *We have $\mathbb{L}=\mathbb{K}(s_{(1)})$, that is $s_{(1)}$ is a primitive element for $\mathbb{L}$ regarded as an extension of $\mathbb{K}$.*

Proof. It suffices to show that $\Pi=\Xi$. By Proposition 4.3, we already know that $\Xi=\Pi^m$ with $m\in\mathbb{Z}_{>0}$. Then by Frobenius reduction, there exists an invertible matrix P with coefficients in $\mathbb{K}$ such that

$$\mathbf{\Phi}=P\begin{pmatrix} \mathcal{C}_{\Pi} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{C}_{\Pi} \end{pmatrix}P^{-1}, \tag{5}$$

that is, the matrix $\mathbf{\Phi}$ is equivalent to a block diagonal matrix with k blocks equal to $\mathcal{C}_{\Pi}$ the companion matrix of the polynomial Π. By multiplying the columns of the matrix P by elements of $\mathbb{A}$, one can also assume that the coefficients of P belong to $\mathbb{A}$. Then we can write $P^{-1}=\frac{1}{\det(P)}Q$ where Q has also coefficients in $\mathbb{A}$ and $\det(P)\in\mathbb{A}$ is nonzero. Since $\det(P)\in\mathbb{A}=\mathbb{R}[s_{R_1},\ldots,s_{R_k}]$ is nonzero, there exists a nonzero polynomial $F\in\mathbb{R}[T_1,\ldots,T_k]$ such that $\det(P)=F(s_{R_1},\ldots,s_{R_k})$. Also a morphism $\widetilde{\varphi}:\mathbb{A}\rightarrow\mathbb{R}$ such that $\widetilde{\varphi}(s_{R_a})\geq 0$ for any $a=1,\ldots,k$ is characterized by the datum of the $\widetilde{\varphi}(s_{R_a})$'s. The polynomial F being nonzero, one can find $(r_1,\ldots,r_k)\in\mathbb{R}^k_{>0}$ such that $F(r_1,\ldots,r_k)\neq 0$. For such a k-tuple, let us define $\widetilde{\varphi}$ by setting $\widetilde{\varphi}(s_{R_a})=r_a$. Then $\widetilde{\varphi}(\det(P))\neq 0$ and we can apply $\widetilde{\varphi}$ to (5) which gives

$$\Phi=\widetilde{\varphi}(P)\begin{pmatrix} \mathcal{C}_{\widetilde{\varphi}(\Pi)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathcal{C}_{\widetilde{\varphi}(\Pi)} \end{pmatrix}\widetilde{\varphi}(P)^{-1}.$$

The matrix Φ has nonnegative coefficients and is irreducible by Lemma 3.1. So, by Perron Frobenius theorem, it admits a unique eigenvalue $t>0$ of maximal module and the corresponding eigenspace is one-dimensional. This eigenvalue t should also be a root of $\widetilde{\varphi}(\Pi)$, thus there is a vector $v\in\mathbb{R}^d$ with $d=\deg(\Pi)$ such that $\mathcal{C}_{\widetilde{\varphi}(\Pi)}v=tv$. Then we get m right eigenvectors of Φ

linearly independent on $\mathbb{R}^{dm}$

$$\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{pmatrix}, \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{v} \\ \vdots \\ \boldsymbol{0} \end{pmatrix}, \dots, \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{v} \end{pmatrix}.$$

Since the eigenspace considered is one-dimensional, this means that $m = 1$ and we are done. $\square$

Corollary 4.6. *There exist $\Delta \in \mathbb{A}$ and for each irreducible k-partition κ a polynomial $P_\kappa \in \mathbb{A}[T]$ such that*

$$s_\kappa^{(k)} = \frac{1}{\Delta} P_\kappa(s_{(1)}).$$

In particular, for any morphism $\varphi : \Lambda_{(k)} \to \mathbb{R}$ such that $\varphi(\Delta) \neq 0$ we have

$$\varphi(s_\kappa^{(k)}) = \frac{1}{\varphi(\Delta)} \varphi(P_\kappa)\varphi(s_{(1)}).$$

Proof. Since $s_{(1)}$ is a primitive element for $\mathbb{L}$ regarded as an extension of $\mathbb{K}$, $\{1, s_{(1)}, \dots, s_{(1)}^{k!-1}\}$ is a $\mathbb{K}$-basis of $\mathbb{L}$. It then suffices to consider the matrix M whose columns are the vectors $s_{(1)}^i, i = 0, \dots, k! - 1$ expressed on the basis $\mathcal{I} = \{s_\kappa^{(k)}, \kappa \in \mathcal{P}_{\mathrm{irr}}\}$. Its inverse can be written $M^{-1} = \frac{1}{\det M} N$ where the entries of N belongs to $\mathbb{A}$. So we have $\Delta = \det(M)$ and the entries on each columns of the matrix N give the polynomials $P_\kappa, \kappa \in \mathcal{P}_{\mathrm{irr}}$. $\square$

Remark 4.7. *In fact we get the equality of $\mathbb{A}$-modules $\Lambda_{(k)} = \frac{1}{\Delta}\mathbb{A}[s_1]$. In particular, the polynomial Δ (once assumed monic) only depends on $\Lambda_{(k)}$ and $\mathbb{A}[s_1]$ and not on the choice of the bases considered in these $\mathbb{A}$-modules. Indeed a basis change will multiply Δ by an invertible in $\mathbb{A}$, that is by a nonzero real.*

Example 4.8. *For $k = 2$ we get*

$$\boldsymbol{\Phi} = \begin{pmatrix} 0 & s_{R_1} + s_{R_2} \\ 1 & 0 \end{pmatrix} \text{ and } M = I_2.$$

Example 4.9. *For $k = 3$ and with the same convention as in Example 4.4, we get*

$$M = \begin{pmatrix} 1 & 0 & 0 & s_{R_1} + s_{R_3} & 2s_{R_2} & 0 \\ 0 & 1 & 0 & 0 & s_{R_1} + s_{R_3} & 4s_{R_2} \\ 0 & 0 & 1 & 0 & 0 & s_{R_1} + 3s_{R_3} \\ 0 & 0 & 1 & 0 & 0 & 3s_{R_1} + s_{R_3} \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix} \text{ and}$$

$$M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{s_{R_1}+s_{R_3}}{2} & -s_{R_2} \\ 0 & 1 & \frac{2s_{R_2}}{s_{R_1}-s_{R_3}} & \frac{2s_{R_2}}{s_{R_3}-s_{R_1}} & 0 & -\frac{s_{R_1}+s_{R_3}}{2} \\ 0 & 0 & \frac{3s_{R_1}+s_{R_3}}{2s_{R_1}-2s_{R_3}} & \frac{s_{R_1}+3s_{R_3}}{2s_{R_3}-2s_{R_1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{-1}{2s_{R_1}-2s_{R_3}} & \frac{-1}{2s_{R_3}-2s_{R_1}} & 0 & 0 \end{pmatrix}.$$

So in particular, $s_{(2,1,1)}^{(3)} = \frac{1}{2}s_{(1)}^4 - \frac{1}{2}(s_{R_1} + s_{R_3})s_{(1)} - s_{R_2}$.

4.3. **Algebraic variety associated to fixed values of rectangles.** Recall that $\Lambda_{(k)} = \mathbb{R}[h_1, \ldots, h_k]$ and each Schur rectangle polynomial can be written $s_{R_a} = J_a(h_1, \ldots, h_k)$ for any $a = 1, \ldots, k$ where $J_a \in \mathbb{R}[h_1, \ldots, h_k]$ is given by the Jacobi-Trudi determinantal formula. Consider $\vec{r} = (r_1, \ldots, r_k) \in \mathbb{R}^k_{\geq 0}$.

Definition 4.10. *Let $\mathcal{R}_{\vec{r}}$ be the algebraic variety of $\mathbb{R}^k$ defined by the equations $s_{R_a} = r_a$ for any $a = 1, \ldots, k$.*

We can consider the algebra $\overline{\Lambda}_{(k)} := \Lambda_{(k)}/J$ where J is the ideal generated by the relations $s_{R_a} = r_a$ for any $a = 1, \ldots, k$. Write $\overline{\varphi} : \Lambda_{(k)} \to \Lambda_{(k)}/J$ for the canonical projection obtained by specializing in $\Lambda_{(k)}$ each rectangle Schur function s_{R_a} to r_a. We shall write for short $\overline{b} = \overline{\varphi}(b)$ for any $b \in \Lambda_{(k)}$. Clearly $\overline{\Lambda}_{(k)} = \Lambda_{(k)}/J$ is a finite-dimensional $\mathbb{R}$-algebra and $\overline{\Lambda}_{(k)} = \mathrm{vect}\langle \overline{s}_\kappa \mid \kappa$ irreducible$\rangle$. The following proposition shows that the non-cancellation of Δ can be naturally interpreted as a condition for the multiplication by $\overline{s}_1$ to be a cyclic morphism in $\overline{\Lambda}_{(k)}$.

Proposition 4.11.

(1) *The algebra $\overline{\Lambda}_{(k)}$ has dimension $k!$ over $\mathbb{R}$ and $\{\overline{s}_\kappa \mid \kappa$ irreducible$\}$ is a basis of $\overline{\Lambda}_{(k)}$.*
(2) *We have $\overline{\Lambda}_{(k)} = \mathbb{R}[\overline{s}_1]$ if and only if $\overline{\Delta} \neq 0$.*

Proof. 1: Assume we have reals c_κ such that

$$\sum_\kappa c_\kappa \overline{s}_\kappa = 0. \tag{6}$$

For any k-partition λ, write $\lambda = R_1^{m_1} \sqcup \cdots \sqcup R_k^{m_k} \sqcup \kappa$ for its decomposition into rectangles and irreducible partitions. Set $u(\lambda) = r_1^{m_1} \cdots r_k^{m_k}$. Then J regarded as a $\mathbb{R}$-vector space has basis $\{s_\lambda^{(k)} - u(\lambda) \mid \lambda$ k-partition such that $\kappa = \emptyset\}$. Observe also that $\{s_\lambda^{(k)} - u(\lambda) \mid \lambda$ k-partition$\}$ is a basis of $\Lambda_{(k)}$. Then (6) can be rewritten

$$\sum_{\kappa \in \mathcal{P}_{\mathrm{irr}}} c_\kappa (s_\kappa - u(\kappa)) = \sum_{\lambda \mid \kappa = \emptyset} c_\lambda (s_\lambda^{(k)} - u(\lambda))$$

where the c_λ's are real coefficients. Since $\{s_\lambda^{(k)} - u(\lambda) \mid \lambda \in \mathcal{B}_k\}$ is a basis of $\Lambda_{(k)}$ we obtain $c_\kappa = 0$ for any irreducible partition κ. Finally we get a basis for each element of $\overline{\Lambda}_{(k)}$ decomposes as a linear combination of the $\overline{s}_\kappa$'s.

2: We have $\overline{\Delta} \neq 0$ if and only if $\{\overline{s}_1^r \mid 0 \leq r \leq k! - 1\}$ is a basis of $\overline{\Lambda}_{(k)}$ since $\{\overline{s}_\kappa \mid \kappa$ irreducible$\}$ is a basis of $\overline{\Lambda}_{(k)}$ and $\overline{\Delta}$ is then the determinant between the two bases. □

The following proposition is classical, we prove it for completion.

Proposition 4.12. *The algebraic variety $\mathcal{R}_{\vec{r}}$ is finite.*

Proof. It suffices to see that the algebraic variety $\mathcal{R}_{\vec{r}}^{\mathbb{C}}$ of $\mathbb{C}^k$ defined by the equations $R_a = r_a$ for any $a = 1, \ldots, k$ is finite. We can decompose $\mathcal{R}_{\vec{r}}^{\mathbb{C}} = V_1 \cup \cdots \cup V_m$ into its irreducible components. To each such component V_j is associated a prime ideal J_j and we have $J = J_1 \cap \cdots \cap J_m$. Therefore for any $j = 1, \ldots, m$, $\mathbb{C}[h_1, \ldots, h_k]/J_j$ is a finite-dimensional algebra which is an integral domain. So $\mathbb{C}[h_1, \ldots, h_k]/J_j$ is in fact a field and J_j is maximal in $\mathbb{C}[h_1, \ldots, h_k]/J_j$. By using Hilbert's Nullstellensatz's theorem, we obtain that each V_j reduces to a point, so $\mathcal{R}_{\vec{r}}^{\mathbb{C}}$ is finite. □

5. Nonnegative morphisms on $\Lambda_{(k)}$

5.1. **Nonnegative morphisms with Φ irreducible.** We show now that when the matrix Φ introduced in § 3.1 is irreducible, the values of φ on the rectangle Schur functions s_{R_a}, $1 \le a \le k$ determine completely the morphism φ. Denote by $\mathcal{R}_k$ the set $\{s_{R_a}\}_{1\le a\le k}$. We define an action of $\mathbb{R}_{>0}$ on $\mathcal{F}(\mathcal{R}_k, \mathbb{R}_{\ge 0})$ by

$$t \cdot \varphi(s_\lambda^{(k)}) = t^{|\lambda|}\varphi(s_{R_a}),$$

for $t > 0$, $\varphi \in \mathcal{F}(\mathcal{B}_k, \mathbb{R}_{\ge 0})$.

Theorem 5.1.

(1) *Let $\varphi : \Lambda_{(k)} \longrightarrow \mathbb{R}_{\ge 0}$ be a morphism, and suppose that φ is positive on the k-Schur functions. Then, φ is uniquely determined by its values on the s_{R_a}, $1 \le a \le k$.*

(2) *Let $\varphi \in \mathcal{F}(\mathcal{R}_k, \mathbb{R}_{\ge 0})$ and suppose that the matrix Φ associated to φ is irreducible. Then φ can be extended to a morphism $\widetilde{\varphi} : \Lambda_{(k)} \longrightarrow \mathbb{R}_{\ge 0}$ which is nonnegative on the k-Schur functions.*

Proof. 1: Since Φ is irreducible by Lemma 3.1 we can apply Perron Frobenius theorem and get the values $\varphi(s_\lambda^{(k)}), \lambda \in \mathcal{P}_{\text{irr}}$ as the coordinates of the unique positive left eigenvector of Φ with eigenvalue $\varphi(s_1)$ normalized so that $\varphi(s_\emptyset) = 1$. This proves our assertion 1.

2: Set $r_a = \varphi(s_{R_a})$ for $a = 1, \dots, k$. Assume first that $\Delta(r_1, \dots, r_k) \neq 0$. We have to show that there exists a positive morphism $\widetilde{\varphi}$ on $\Lambda_{(k)}$ such that $\widetilde{\varphi}$ is nonnegative on $s_\lambda^{(k)}, \lambda \in \mathcal{B}_k$ and $\widetilde{\varphi}(s_{R_a}) = r_a$. The set of morphisms from $\Lambda_{(k)}$ to $\mathbb{R}$ which takes values r_a on s_{R_a} for $a = 1 \dots k$ is in bijection with the set of morphisms from $\overline{\Lambda}_{(k)}$ to $\mathbb{R}$, where we recall that $\overline{\Lambda}_{(k)} = \Lambda_{(k)}/J$ with J the ideal generated by the relations $s_{R_a} = r_a$ for $a = 1, \dots, k$. Since we have assumed $\bar{\Delta} \neq 0$, Proposition 4.11 yields that $\overline{\Lambda}_{(k)} = \mathbb{R}[\overline{s}_1]$. There exists one morphism from $\mathbb{R}[\overline{s}_1]$ to $\mathbb{R}$ for each real root of the minimal polynomial of $\overline{s}_1$, which is $\overline{\Xi}$ because $\deg(\overline{\Xi}) = k! = \dim(\overline{\Lambda}_{(k)})$. Let t be the root of greatest modulus of $\overline{\Xi}$ which is positive since $\overline{\Xi}$ is the characteristic polynomial of the irreducible matrix $\Phi_{(r_1,\dots,r_k)}$. Then, the specialization $\overline{s}_1 = t$ yields a morphism from $\overline{\Lambda}_{(k)}$ to $\mathbb{R}$, and by extension a morphism $\widetilde{\varphi}$ from $\Lambda_{(k)}$ to $\mathbb{R}$. For $\lambda \in \mathcal{P}_{\text{irr}}$, set $X(\lambda) = \widetilde{\varphi}(s_\lambda^{(k)})$. For $\mu \in \mathcal{B}_k$ such that $s_\mu^{(k)} = \prod_{a=1}^k s_{R_a}^{p_a^\mu} s_{\widetilde{\mu}}^{(k)}$, we have

$$\widetilde{\varphi}(s_\mu^{(k)}) = \prod_{a=1}^{k} \varphi(s_{R_a}^{p_a^\mu}) X(\widetilde{\mu}). \tag{7}$$

By (2) fir any $\lambda \in \mathcal{P}_{\text{irr}}$

$$tX(\lambda) = \widetilde{\varphi}(s_1)\widetilde{\varphi}(s_\lambda^{(k)}) = \widetilde{\varphi}(s_1 s_\lambda^{(k)}) = \sum_{\lambda \to \mu} \Phi_{\lambda\mu} \widetilde{X}(\mu).$$

Hence, $X := (X_\lambda)_{\lambda \in \mathcal{P}_{irr}}$ is a left eigenvector of Φ with eigenvalue t. Since Φ is irreducible and t is the Perron Frobenius eigenvalue of Φ, X is an eigenvector with positive entries. Hence, by (7), $\widetilde{\varphi}$ is nonnegative.

Assume now we drop the hypothesis $\Delta = 0$. Consider $\vec{r} = (r_1, \dots, r_k)$ such that the matrix Φ is irreducible. Let $X^{\vec{r}}$ be the eigenvector of $\Phi_{(r_1,\dots,r_k)}$ corresponding to the Perron Frobenius eigenvalue $t_{\vec{r}}$ such that $X^{\vec{r}}(\emptyset) = 1$. For $\mu \in \mathcal{B}_k$, set

$$\widetilde{\varphi}_{\vec{r}}(s_\mu^{(k)}) = \prod_{a=1}^{k} r_a^{p_a^\mu} X^{\vec{r}}(\widetilde{\mu}) \text{ with } s_\mu^{(k)} = \prod_{a=1}^{k} s_{R_a}^{p_a^\mu} s_{\widetilde{\mu}}^{(k)}.$$

Then, $\widetilde{\varphi}_{\vec{r}}$ is positive on $\mathcal{B}_k$ by construction, and it just remains to prove that $\widetilde{\varphi}_{\vec{r}}$ is a morphism.

Since $t_{\vec{r}}$ and $X^{\vec{r}}$ are continuous functions of $\vec{r}$ on the set of irreducible matrices, the map $\widetilde{\varphi}_{\vec{r}}$ is a continuous function of $\vec{r}$. The hypersurface $V(\Delta) := \{\Delta(\vec{r}) = 0\}$ is Zariski closed, thus $V(\Delta)$ has empty interior in the set Θ of $\vec{r} \in \mathbb{R}^k_{\geq 0}$ such that $\Phi_{(r_1,\ldots,r_k)}$ is irreducible. By the previous arguments, for all $\vec{r} \in \Theta$ outside $V(\Delta)$, the map $\widetilde{\varphi}_{\vec{r}}$ is a morphism and $\vec{r} \mapsto \widetilde{\varphi}_{\vec{r}}$ is continuous on Θ, thus $\widetilde{\varphi}_{\vec{r}}$ is a morphism for $\vec{r} \in \overline{\Theta \setminus V(\Delta)}$. By Proposition 3.1, Θ is an open set. Let $\vec{r}_0 \in \Theta \cap V(\Delta)$. Since the interior of $V(\Delta)$ is empty, one can define a sequence $\vec{r}^{(n)} \in \Theta \setminus V(\Delta)$ which tends to $\vec{r}_0 \in \overline{\Theta \setminus V(\Delta)}$ as n goes to infinity. Finally $\widetilde{\varphi}_{\vec{r}_0}$ is a morphism and we are done. □

An immediate consequence of the latter theorem is the description of positive extremal harmonic measures. We define an action of $\mathbb{R}_{>0}$ on $\mathcal{F}(\mathcal{B}_k, \mathbb{R}_{\geq 0})$ by

$$t \cdot \varphi(s_\lambda^{(k)}) = t^{|\lambda|}\varphi(s_\lambda^{(k)}),$$

for $t > 0$, $\varphi \in \mathcal{F}(\mathcal{B}_k, \mathbb{R}_{\geq 0})$.

Corollary 5.2. (1) *Let $\varphi \in \partial\mathcal{H}^+(\mathcal{B}_k)$, and suppose that φ is positive on the k-Schur functions. Then, φ is uniquely determined by its values on the s_{R_a}, $1 \leq a \leq k$.*
(2) *Assume the matrix Φ associated to φ is irreducible. Then there exists $t > 0$ such that $t^{-1} \cdot \varphi$ can be extended to an element $\tilde{\varphi} \in \partial\mathcal{H}^+(\mathcal{B}_k)$.*

Proof. The only non-trivial statement is the second one . Suppose that Φ is irreducible. Then, by Theorem 5.1, φ can be extended to a non-negative morphism $\widetilde{\varphi}$ on $\mathcal{B}_k$. Let $t = \widetilde{\varphi}(s_1)$. Then, $t^{-1}.\widetilde{\varphi}$ is a nonnegative morphism on $\mathcal{B}_k$ such that $t^{-1}.\widetilde{\varphi}(s_1) = 1$, which corresponds to an extremal element of $\partial\mathcal{B}_k$. It is clear that $t^{-1}.\widetilde{\varphi}$ extends $t^{-1}.\varphi$. □

5.2. **Morphism defined from an irreducible matrix.** Recall we have denoted by $\mathbb{A} = \mathbb{R}[s_{R_1}, \ldots, s_{R_k}]$ the algebra generated by the k-rectangle Schur functions and we have $\Lambda_{(k)} = \mathbb{R}[h_1, \ldots, h_k]$. Also $\mathbf{\Phi}$ is the matrix of the multiplication by $s_{(1)}$ on the basis $\mathcal{I} = \{s_\kappa^{(k)} \mid \kappa \in \mathcal{P}_{\mathrm{irr}}\}$. Assume we have a nonnegative morphism $\varphi : \mathbb{A} \to \mathbb{R}$. Since the entries of $\mathbf{\Phi}$ belong to $\mathbb{A}$ one can compute $\Phi = \varphi(\mathbf{\Phi})$. When Φ is an irreducible matrix, we can extend φ to a nonnegative morphism $\varphi : \Lambda_{(k)} \to \mathbb{R}$ in only one way : we apply Perron Frobenius theorem and get $\varphi(s_1)$ as the greatest positive eigenvalue, next the other values of $\varphi(s_\kappa^{(k)})$ are given by the corresponding left eigenvector normalized so that $\varphi(s_\emptyset) = 1$. Moreover, by Theorem 5.1 the construction is bijective, in particular two different matrices Φ will give two different morphisms.

5.3. **Two parametrizations of the positive morphisms.** The more immediate parametrization of the positive morphisms $\varphi : \Lambda_{(k)} \to \mathbb{R}$ such that $\varphi(s_\lambda^{(k)}) > 0$ for any k-bounded partition is obtained from the factorization property (Corollary 2.11) of the k-Schur functions. Consider

$$V = \vec{h} = \left\{(h_1, \ldots, h_k) \in \mathbb{R}^k \mid \left\{ \begin{array}{c} J_{R_i}(h_1, \ldots, h_k) > 0, i = 1, \ldots, k \\ J_\kappa(h_1, \ldots, h_k) > 0 \ \forall \kappa \in \mathcal{P}_{\mathrm{irr}} \end{array} \right. \right\} \subset \mathbb{R}^k_{>0}$$

where we have set $s_{Ri} = J_{R_i}(h_1, \ldots, h_k)$ and $s_\kappa = J_\kappa(h_1, \ldots, h_k)$ where $J_{R_1}, \ldots, J_{R_k}$ and $J_\kappa, \kappa \in \mathcal{P}_{\mathrm{irr}}$ are polynomials in $\mathbb{R}[X_1, \ldots, X_k]$. To each point in V corresponds a unique positive morphism φ defined on $\Lambda_{(k)}$. Now define

$$U = \{\vec{r} = (r_1, \ldots, r_k) \in \mathbb{R}^k_{>0}\}.$$

By Proposition 3.1, the matrix Φ is irreducible. Thus Theorem 5.1 implies that U parametrizes the positive morphisms $\varphi : \Lambda_{(k)} \to \mathbb{R}$ such that $\varphi(s_\lambda^{(k)}) > 0$ for any $\lambda \in \mathcal{B}_k$. We can define a map $f : U \to V$ such that

$$f(r_1, \ldots, r_k) = (\varphi(h_1), \ldots, \varphi(h_k)).$$

The map f is then continuous on U since the entries of the matrix Φ are and so is its Perron Frobenius vector normalized at 1 on $s_\emptyset$. Moreover, the map f is bijective by Theorem 5.1 and we have

$$f^{-1}: \left\{ \begin{array}{c} V \to U \\ (h_1,\ldots,h_k) \mapsto (J_{R_1}(h_1,\ldots,h_k),\ldots,J_{R_k}(h_1,\ldots,h_k)) \end{array} \right.$$

where the polynomials $J_{R_1},\ldots,J_{R_k}$ are given by the Jacobi-Trudi determinantal formulas. In particular f^{-1} is continuous on V.

Lemma 5.3. *The map f is bounded on any bounded subset of U.*

Proof. Let $B \subset U$ be a bounded subset of U. By definition of f, for any $\vec{r} = (r_1,\ldots,r_k)$ in B, $\varphi(h_1)$ is the first coordinate of $f(r_1,\ldots,r_k)$ and coincides with the Perron Frobenius eigenvalue of the matrix Φ, that is with its spectral radius. Since the spectral radius of a real matrix is a bounded function of its entries, we get that $\varphi(h_1)$ is bounded when $\vec{r}$ runs over B. To conclude, observe that for any $a = 2,\ldots,k$ we have $\varphi(h_a) \leq \varphi(h_1)^a$ because $h_1 = s_1$, the map φ is multiplicative and h_a appears in the decomposition of s_1^a on the basis of k-Schur functions (which only makes appear nonnegative real coefficients). $\square$

Now set

$$\overline{U} = \{\vec{r} = (r_1,\ldots,r_k) \in \mathbb{R}^k_{\geq 0}\} \text{ and}$$
$$\overline{V} = \left\{ (h_1,\ldots,h_k) \in \mathbb{R}^k \mid \left\{ \begin{array}{c} J_{R_i}(h_1,\ldots,h_k) \geq 0, i = 1,\ldots,k \\ J_\kappa(h_1,\ldots,h_k) \geq 0 \ \forall \kappa \ k\text{-irreducible} \end{array} \right\} \right\} \subset \mathbb{R}^k_{\geq 0}$$

Since $J_{R_1},\ldots,J_{R_k}$ are polynomials, we can extend f^{-1} by continuity on $\overline{V}$ and get a continuous map $g : \overline{V} \to \overline{U}$. But this is not immediate right now that g is bijective and f can also be extended to a bijective map from $\overline{U}$ to $\overline{V}$. Observe nevertheless that if we can extend f by continuity on $\overline{U}$, the continuity of g and f will imply that $f \circ g = id_{\overline{V}}$ and $g \circ f = id_{\overline{U}}$. Therefore, to extend f by continuity will suffice to prove that $\overline{U}$ and $\overline{V}$ are homeomorphic by f.

5.4. **Extension of the map f on $\overline{U}$.** Let $\vec{r}_0 \in \mathbb{R}^k_{\geq 0}$, and denote by $A(\vec{r}_0)$ the set of limiting values of $f(\vec{r})$ as $\vec{r}$ goes to $\vec{r}_0$. Recall the notation of the previous paragraph, in particular the function g is defined and continuous on $\overline{V}$ and $g = f^{-1}$ on $f(U)$.

Lemma 5.4. *The set $A(\vec{r}_0)$ is a connected subset of $\mathcal{R}_{\vec{r}_0}$ (see Definition 4.10).*

Proof. Consider $K_n = B\left(\vec{r}_0, \frac{1}{n}\right) \cap \mathbb{R}^k_{>0}$. This is a system of decreasing bounded connected neighborhoods of $\vec{r}_0$ in $\mathbb{R}^k_{>0}$. By definition, $A(\vec{r}_0) = \bigcap_{n\geq 1} \overline{f(K_n)}$. By Lemma 5.3, we know that f is bounded on bounded subsets of $U = \mathbb{R}^k_{>0}$, therefore we get that $f(K_n)$ is bounded and thus $\overline{f(K_n)}$ is compact. Since f is continuous on U and K_n is connected, $f(K_n)$ is also connected, which implies that $\overline{f(K_n)}$ is connected. Hence, $A(\vec{r}_0)$ is a decreasing intersection of connected compact sets, and thus $A(\vec{r}_0)$ is connected.

Let $\vec{h} \in A(\vec{r}_0)$. We claim there exists a sequence $(\vec{r_n})_{n\geq 1}$ in U converging to $\vec{r}_0$ such that $\vec{h}_n := f(\vec{r_n})$ converges to $\vec{h}$ as n goes to infinity. To see this, observe we have that $\vec{h}$ belongs to $\overline{f(K_n)}$ for any integer $n \geq 1$. Therefore, for any such n, there exists $\vec{h}_n$ in $f(K_n) \subset V$ such that $|\vec{h}_n - \vec{h}| \leq \frac{1}{n}$ for any $n \geq 1$. Since $f : U \to V$ is bijective, there exists for any $n \geq 1$ a unique $\vec{r}_n$ in K_n such that $\vec{h}_n = f(\vec{r}_n)$. Now, we have $\lim_{n\to+\infty} \vec{r_n} = \vec{r}_0$ and $\lim_{n\to+\infty} f(\vec{r_n}) = \lim_{n\to+\infty} \vec{h}_n = \vec{h}$ as desired.

Since $g = f^{-1}$ on $\mathbb{R}^k_{\geq 0}$, $g(\vec{h}_n) = g \circ f(\vec{r_n}) = \vec{r_n}$ for $n \geq 1$. Moreover, since g is continuous and $(\vec{h}_n)_{n\geq 1}$ converges to $\vec{h}$ as n goes to infinity,

$$g(\vec{h}) = \lim_{n\to\infty} g(\vec{h}_n) = \lim_{n\to\infty} \vec{r_n} = \vec{r}_0$$

which implies that $\vec{h} \in \mathcal{R}_{\vec{r}_0}$. □

Theorem 5.5.

(1) *The map f is an homeomorphism from $\overline{U}$ to $\overline{V}$.*
(2) *The morphisms $\varphi : \Lambda_{(k)} \to \mathbb{R}$ nonnegative on the k-Schur functions are parametrized by $\mathbb{R}^k_{\geq 0}$.*

Proof. The set $A(\vec{r}_0)$ is a connected subset by the previous lemma and it is also finite by Proposition 4.12. Therefore, the set $A(\vec{r}_0)$ is a singleton. In particular, $f(\vec{r})$ converges to some $f(\vec{r}_0)$ as $\vec{r}$ goes to $\vec{r}_0$, and f can be extended continuously to $\mathbb{R}^k_{\geq 0}$. As explained as the end of § 5.3, this suffices to conclude that f is an homeomorphism from $\overline{U}$ to $\overline{V}$. □

Example 5.6. *For $k = 2$, we get for the matrix associated to $\vec{r} = (r_1, r_2) \in \mathbb{R}^2_{\geq 0}$*

$$\Phi = \begin{pmatrix} 0 & r_1 + r_2 \\ 1 & 0 \end{pmatrix}$$

whose greatest eigenvalue is $\sqrt{r_1 + r_2}$ with associated normalized left eigenvector $(1, \sqrt{r_1 + r_2})$. We thus get $\vec{h} = f(\vec{r}) = (\sqrt{r_1 + r_2}, r_1)$ since $h_1 = \sqrt{r_1 + r_2}$ and $h_2 = r_1$. Conversely, we have $g(\vec{h}) = (h_2, h_1^2 - h_2)$. If we assume $h_1 = 1$, we get $\partial\mathcal{H}^+(\mathcal{B}_2) = \{(1, h_2) \mid h_2 \in [0, 1]\}$.

6. Markov chains on alcoves

6.1. **Central Markov chains on alcoves from harmonic functions.** Recall the notation of § 2.1 for the notion of reduced alcove paths. A probability distribution on reduced alcove paths is said central when the probability p_π of $\pi = (A_1 = A^{(0)}, A_2, \ldots, A_m)$ only depends on m, A_1 and A_m, that is only on its length and its alcoves ends. In the situation we consider, affine Grassmannian central random paths correspond to central random paths on $\mathcal{B}_k$. Similarly, affine (non Grassmannian) central random alcove paths correspond to central random paths on the Hasse diagram $\mathcal{G}$ of the weak Bruhat order. They are determined by the positive harmonic functions on $\mathcal{B}_k$ and $\mathcal{G}$, respectively (see [4]).

More precisely any central probability distribution on the affine Grassmannian alcove paths can be written

$$p_\pi = \frac{h(\mu)}{h(\lambda)}$$

where $h \in \mathcal{H}^+(\mathcal{B}_k)$ is positive and for any path $\pi = (A_1, \ldots, A_m)$, μ and λ are the k-bounded partitions associated to A_1 and A_m. Also we then get a Markov chain on $\mathcal{B}_k$ (or equivalently on the affine Grassmannian elements) with transition matrix

$$\Pi(\lambda, \mu) = \frac{h(\mu)}{h(\lambda)}.$$

When h is extremal, it corresponds to a morphism φ on $\Lambda_{(k)}$ with $\varphi(s_{(1)}) = 1$ and nonnegative on the k-Schur functions. We get an extremal central distribution on the trajectories starting at $A^{(0)}$ verifying $p_\pi = \frac{\varphi(s_\mu^{(k)})}{\varphi(s_\lambda^{(k)})}$. The associated Markov chain has then the transition matrix $\Pi(\lambda, \mu) = \frac{\varphi(s_\mu^{(k)})}{\varphi(s_\lambda^{(k)})}$.

One can similarly determine the set $\partial\mathcal{H}^+(\mathcal{G})$ of nonnegative extremal harmonic functions of $\mathcal{G}$ and the set of extremal central distributions on the alcove paths. To do this, recall that Λ is endowed with a scalar product $\langle\cdot,\cdot\rangle$ making the basis of Schur functions orthonormal. Let us write s_1^* for the adjoint of the multiplication by s_1 with respect to $\langle\cdot,\cdot\rangle$. In [7], Lam introduced the affine Stanley symmetric functions $\widetilde{F}_w, w\in\widetilde{W}$ which have the important following properties:

(1) $\{\widetilde{F}_w \mid w \text{ is affine Grassmannian}\}$ is the dual basis of $\{s_\lambda^{(k)} \mid \lambda\in\mathcal{B}_k\}$. We shall then write $\widetilde{F}_w=\widetilde{F}_\lambda$ where λ is the k-bounded partition associated to w.
(2) $s_1^*(\widetilde{F}_w)=\sum_{w\to w'}\widetilde{F}_{w'}$ for any $w\in\widetilde{W}$.
(3) For any $w\in\widetilde{W}$, there exists nonnegative integer coefficients $a_{w,\lambda}$ such that $\widetilde{F}_w=\sum_{\lambda\in\mathcal{B}_k}a_{w,\lambda}\widetilde{F}_\lambda$.

Now if we introduce for any $w\in\widetilde{W}$, the polynomial $s_w^{(k)}:=\sum_{\lambda\in\mathcal{B}_k}a_{w,\lambda}s_\lambda^{(k)}$, we get the relation

$$s_1 s_w^{(k)}=\sum_{w\to w'}s_{w'}^{(k)}.$$

Therefore, the Hasse diagram $\mathcal{G}$ is multiplicative. Since each $s_w^{(k)}$ belongs to the cone generated by the k-Schur functions (by the above property 3), one can apply Proposition 2.4 in [12] and conclude that $\partial\mathcal{H}^+(\mathcal{G})$ corresponds to the morphisms $\Lambda_{(k)}\to\mathbb{R}$ nonnegative on the functions $s_w^{(k)}, w\in\widetilde{W}$. Since the coefficients $a_{w,\lambda}$ are nonnegative, they coincide with the morphisms nonnegative on the k-Schur functions. We thus get:

Theorem 6.1. *We have* $\partial\mathcal{H}^+(\mathcal{G})\simeq\partial\mathcal{H}^+(\mathcal{B}_k)$.

Remark 6.2. *By the previous discussion, the extremal central distributions on the alcove paths starting at w are such that $p_\pi=\frac{\varphi(s_{w'}^{(k)})}{\varphi(s_w^{(k)})}$ where π ends on the alcove $A_{w'}$.*

Involutions on the reduced walk. By Corollary 2.11, the structure of the graph $\mathcal{B}_k$ is completely determined by the matrix $\mathbf{\Phi}$ depicted in Section 3.1 with coefficients in $\mathbb{R}[s_{R_1},\dots,s_{R_k}]$. Then $\Phi_{(r_1,\dots,r_k)}$ is the matrix $\mathbf{\Phi}$ after the specialization $s_{R_1}=r_1,\dots,s_{R_k}=r_k$. We are going to see that this matrix exhibits particular symmetries coming from the underlying alcove structure.

The first symmetry is due to the action of ω_k on $\Lambda_{(k)}$which sends s_{R_a} to $s_{R_{k-a}}$ for any $a=1,\dots,k$. Since ω_k is an algebra morphism, we get for $1\le a_i\le k$ and $s\ge1$

$$\begin{cases}\Phi_{(r_1,\dots,r_k)}(\lambda,\mu)=1\Leftrightarrow\Phi_{(r_1,\dots,r_k)}(\lambda^{\omega_k},\mu^{\omega_k})=1,\\ \Phi_{(r_1,\dots,r_k)}(\lambda,\mu)=r_{a_1}+\cdots+r_{a_s}\Leftrightarrow\Phi_{(r_1,\dots,r_k)}(\lambda^{\omega_k},\mu^{\omega_k})=r_{k+1-a_1}+\cdots+r_{k+1-a_s}.\end{cases}$$

Hence, if we denote by Ω the matrix of the conjugation ω_k on the basis of irreducible partitions, we get

$$\Omega\Phi_{(r_1,\dots,r_k)}\Omega^{-1}=\Omega\Phi_{(r_1,\dots,r_k)}\Omega=\Phi_{(r_k,\dots,r_1)}. \tag{8}$$

For the second symmetry, we need some basic facts on the affine Coxeter arrangement of type $A_k^{(1)}$. For any root α and any integer, let $H_{\alpha,r}$ be the affine hyperplane

$$H_{\alpha,r}=\{v\in\mathbb{R}^k,\langle v,\alpha\rangle=r\}.$$

We denote by $s_{\alpha,r}$ the reflection with respect to this hyperplane and for β in the weight lattice P and we write t_β for the translation by β. We have then $s_{\alpha,r}=t_{r\alpha}s_{\alpha,0}$. For $w\in W$, we have the commutation relations

$$wt_{\alpha,r}=t_{w(\alpha),r}w\,,\quad ws_{\alpha,r}=s_{w(\alpha),r}w\quad\text{and}\quad t_\beta s_{\alpha,r}=s_{\alpha,r+\langle\beta,\alpha\rangle}t_\beta. \tag{9}$$

Affine Grassmannian elements are in bijection with alcoves in the dominant Weyl chamber through a map $w \mapsto A_w$ such that $w \to w'$ (that is we have a covering relation for the weak order from w to w') if and only if there is a hyperplane $H_{\alpha,r}$ such that $A_{w'} = s_{\alpha,r}(A_w)$. In this case, we write $w \xrightarrow{\alpha,r} w'$.

Write v_w for the center of the alcove A_w (defined as the mean of the its extreme weights). With these notations, $w \xrightarrow{\alpha,r} w'$ if and only if $v_{w'} = s_{\alpha,r}(v_w)$ and $r < \langle \alpha, v_{w'} \rangle < r+1$.

Any alcove A_w is completely determined by its center v_w. Let B be the set of alcoves corresponding to affine Grassmannian elements w such that $\langle v_w, \alpha_i \rangle \in]0,1[$ for any $i = 1, \dots, k$ (i.e. such that the coordinates of v_w on the basis of fundamental weights belong to $]0,1[$). Recall also there is an involution on the Dynkin diagram of affine type $A_k^{(1)}$ fixing the node 0 and sending each node $i \in \{1, \dots, k\}$ to $i^* = k+1-i$. Consider now the involution $I : \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$I(v) = t_\rho \circ w_0(v),$$

where $\rho = \sum_{i=1}^k \Lambda_i$ and w_0 the longest element of W. Observe we indeed get an involution because $w_0 \circ t_\rho \circ w_0 = t_{-\rho}$.

Lemma 6.3. *The involution I restricts to an involution on the set B .*

Proof. Consider $A \in \mathsf{B}$ with center v. Set $v = \sum_{i=1}^k \langle v_A, \alpha_i \rangle \Lambda_i$. Since $w_0(\Lambda_i) = -\Lambda_{i^*}$ we get $w_0(v) = \sum_{i=1}^k -\langle v, \alpha_i \rangle \Lambda_{i^*}$. We also have $\rho = \sum_{i=1}^k \Lambda_i$ which gives

$$I(v) = t_\rho \circ w_0(v) = \sum_{i=1}^k (1 - \langle v, \alpha_i \rangle) \Lambda_{i^*}.$$

By hypothesis, $\langle v, \alpha_i \rangle \in]0,1[$ for any $i = 1, \dots k$ and thus $1 - \langle v, \alpha_i \rangle \in]0,1[$. Now the coordinates of $I(v)$ on the basis of fundamental weights all belong to $]0,1[$. This implies that $I(v)$ is the center of an alcove in B and I restricts to an involution on B. □

Lemma 6.4. *If A, A' are two alcoves of B such that $A \xrightarrow{\alpha,r} A'$, then we have*

$$I(A') \xrightarrow{\alpha^*, \langle \alpha, \rho \rangle - r} I(A),$$

where $\alpha^* = -w_0(\alpha)$. In particular, $A \to A'$ if and only if $I(A') \to I(A)$.

Proof. We have

$$I(A') = t_\rho \circ w_0 \circ s_{\alpha,r}(A) = t_\rho s_{w_0(\alpha),r} w_0(A) = s_{w_0(\alpha), r + \langle \rho, w_0(\alpha) \rangle} t_\rho w_0(A).$$

Since $w_0(\alpha) = -\alpha^*$ and $s_{\alpha,r} = s_{-\alpha,-r}$ for any root α, we get

$$I(A) = s_{\alpha^*, \langle \alpha, \rho \rangle - r} I(A'). \tag{10}$$

Moreover, we can write

$$\langle I(v), \alpha^* \rangle = \langle t_\rho \circ w_0(v), \alpha^* \rangle = \langle \rho, \alpha^* \rangle + \langle v, w_0(\alpha^*) \rangle = \langle \rho, \alpha^* \rangle - \langle v, \alpha \rangle$$

where v is the center of A. Since $\langle \rho, \alpha \rangle = \langle \rho, \alpha^* \rangle$, this yields

$$\langle I(v), \alpha^* \rangle = \langle \rho, \alpha \rangle - \langle v, \alpha \rangle.$$

By hypothesis, $r - 1 < \langle v, \alpha \rangle < r$, thus

$$\langle \rho, \alpha \rangle - r < \langle I(v), \alpha^* \rangle < \langle \rho, \alpha \rangle - r + 1.$$

The last inequalities together with (10) implies that

$$I(A') \xrightarrow{\alpha^*, \langle \alpha, \rho \rangle - r} I(A).$$

□

Lemma 6.5. *Suppose that A, A' are two elements of B such that $A \xrightarrow{\alpha,r} t_{\Lambda_i}A'$. Then, we have $\alpha = \alpha_i, r = 1$, and*

$$I(A') \xrightarrow{\alpha_{i^*},1} t_{\Lambda_{i^*}} I(A).$$

In particular, $A \to t_{\Lambda_i}A'$ if and only if $I(A') \to t_{\Lambda_{i^*}} I(A)$.

Proof. Since $A \xrightarrow{\alpha,r} t_{\Lambda_i}A'$, the alcove $s_{\alpha,r}(A)$ does not belong to B, but belongs to the dominant Weyl chamber. Since B is delimited by the affine hyperplanes $H_{\alpha_i,0}$ and $H_{\alpha_i,1}$ for $1 \leq i \leq k$, this implies that $r = 1$ and α is a simple root. Since $t_{-\Lambda_i}s_{\alpha,1}(A)$ is contained in the dominant Weyl chamber, this yields that $\alpha = \alpha_i$. Let v and v' be the centers of A and A', respectively.

Then, $v = s_{\alpha,r} \circ t_{\Lambda_i}v'$ for $s_{\alpha,r}(v) = t_{\Lambda_i}v'$. Using (9) we so derive

$$\begin{aligned} I(v) &= t_\rho \circ w_0 \circ s_{\alpha,r} \circ t_{\Lambda_i}(v') = t_\rho \circ w_0 \circ t_{\Lambda_i} \circ s_{\alpha,r-\langle\alpha,\Lambda_i\rangle}(v') \\ &= t_\rho \circ t_{w_0(\Lambda_i)} \circ s_{w_0(\alpha),r-\langle\alpha,\Lambda_i\rangle} \circ w_0(v') = t_{w_0(\Lambda_i)} \circ s_{w_0(\alpha),r-\langle\alpha,\Lambda_i\rangle+\langle\rho,w_0(\alpha)\rangle} \circ t_\rho \circ w_0(v') \\ &= t_{w_0(\Lambda_i)} \circ s_{w_0(\alpha),r-\langle\alpha,\Lambda_i+\rho\rangle} I(v') \end{aligned}$$

where $\langle\rho, w_0(\alpha)\rangle = \langle w_0(\rho), \alpha\rangle = -\langle\rho,\alpha\rangle$ for the last equality. Since $w_0(\Lambda_i) = -\Lambda_{i^*}$, we get

$$t_{\Lambda_{i^*}} I(v) = s_{\alpha^*,\langle\alpha,\Lambda_i+\rho\rangle-r} I(v').$$

Finally, we have $\langle I(v'), \alpha^*\rangle = \langle\rho,\alpha^*\rangle - \langle v',\alpha\rangle = \langle\rho,\alpha\rangle - \langle v',\alpha\rangle$, so that the hypothesis $r < \langle t_{\Lambda_i}v', \alpha\rangle < r+1$ gives

$$\begin{aligned} r - \langle\Lambda_i,\alpha\rangle < \langle v',\alpha\rangle < r + 1 - \langle\Lambda_i,\alpha\rangle \text{ and} \\ \langle\rho+\Lambda_i,\alpha\rangle - r - 1 < \langle I(v'),\alpha^*\rangle < \langle\rho+\Lambda_i,\alpha\rangle - r. \end{aligned}$$

So

$$I(w') \xrightarrow{\alpha^*,\langle\alpha,\Lambda_i+\rho\rangle-r} t_{\Lambda_{i^*}} I(w).$$

Since $\alpha = \alpha_i$ and $r = 1$, $\alpha^* = \alpha_i^*$ and

$$\langle\alpha,\Lambda_i+\rho\rangle - r = \langle\alpha_i,\Lambda_i+\rho\rangle - 1 = 1.$$

□

Recall from Section 2.1 that $\mathcal{B}_k$ is the Hass diagram for the weak Bruhat order on affine Grassmannian elements. We also have a bijection which associates to $\lambda \in \mathcal{B}_k$ its corresponding affine Grassmannian element w_λ. Let $1 \leq a \leq k$. Since the multiplication of $s_\lambda^{(k)}$ by s_{R_a} is simply $s_{\lambda\cup R_a}^{(k)}$, there exists a map T_a on the set of alcoves in the Weyl chamber such that $T_a(A_{w_\lambda}) = A_{w_{\lambda\cup R_a}}$. By [1], interpreting k-Schur functions as elements of the affine nilCoxeter algebra yields that T_a coincides with the translation t_{Λ_a} on the alcoves of the dominant Weyl chamber. In particular, the partition λ is irreducible if and only A_{w_λ} belongs to B. By the definition of the matrix $\Phi_{(r_1,\ldots,r_k)}$ in Section 3.1 we have $\Phi_{(r_1,\ldots,r_k)}(\lambda,\mu) = 1$ if λ and μ are two irreducible partitions such that $\lambda \to \mu$ on $\mathcal{B}_k$, and $\Phi_{(r_1,\ldots,r_k)}(\lambda,\mu) = r_{a_1} + \cdots + r_{a_s}$ if and only if $\lambda \to (\mu \cup R_{a_1}), \ldots, \lambda \to (\mu \cup R_{a_s})$ on $\mathcal{B}_k$.

Proposition 6.6. *There exists an involutive permutation matrix I such that*

$$\mathsf{I}\Phi_{(r_1,\ldots,r_k)}\mathsf{I} = \Phi^t_{(r_k,\ldots,r_1)}$$

for any $(r_1,\ldots,r_k) \in \mathbb{R}^k_{\geq 0}$.

Proof. Let us write $\mathbf{\Phi} = \mathbf{\Phi}_{(R_1,\dots,R_k)}$ and $\mathbf{\Phi}_{(R_k,\dots,R_1)}$ for the matrix $\mathbf{\Phi}$ in which each R_a is flipped in R_{k-a+1}. We get that

$$\text{(11)} \qquad \begin{cases} \mathbf{\Phi}_{(R_1,\dots,R_k)}(\lambda,\mu) = 1 \Longleftrightarrow w_\lambda \to w_\mu \\ \mathbf{\Phi}_{(R_1,\dots,R_k)}(\lambda,\mu) = s_{R_{a_1}} + \cdots + s_{R_{a_s}} \Longleftrightarrow w_\lambda \to t_{\Lambda_{a_1}} w_\mu, \dots, w_\lambda \to t_{\Lambda_{a_s}} w_\mu. \end{cases}$$

By Lemma 6.3, it makes sense to consider the matrix I of the restriction of the involution I on the basis of k-bounded partitions. By Lemma 6.4 and (11), $\mathbf{\Phi}_{(R_1,\dots,R_k)}(\lambda,\mu) = 1$ if and only if $\mathbf{\Phi}^t_{(R_k,\dots,R_1)}(I(\mu), I(\lambda)) = 1$, and $\mathbf{\Phi}_{(R_1,\dots,R_k)} = s_{R_{a_1}} + \cdots + s_{R_{a_s}}$ if and only if $\mathbf{\Phi}^t_{(R_1,\dots,Rk)}(I(\mu), I(\lambda)) = s_{R_{k+1-a_1}} + \cdots + s_{R_{k+1-a_s}}$. Since nonzero entries of $\mathbf{\Phi}_{(R_1,\dots,R_k)}$ are either 1 or a sum $R_{i_1} + \cdots + R_{i_s}$ with $1 \le i_j \le k$, this shows that $\mathsf{I}\mathbf{\Phi}_{(R_1,\dots,R_k)}\mathsf{I} = \mathbf{\Phi}^t_{(R_k,\dots,R_1)}$ and thus by specializing $\mathsf{I}\Phi_{(r_1,\dots,r_k)}\mathsf{I} = \Phi^t_{(r_k,\dots,r_1)}$. □

The matrix $\Phi_{(r_1,\dots,r_k)}$ exhibits thus two symmetries relating to the k-conjugation and the involution I.

Proposition 6.7. *The matrices* I *and* Ω *commute, and*

$$(\mathsf{I}\Omega)\Phi_{(r_1,\dots,r_k)}(\mathsf{I}\Omega)^{-1} = (\mathsf{I}\Omega)\Phi_{(r_1,\dots,r_k)}(\mathsf{I}\Omega) = \Phi^t_{(r_1,\dots,r_k)}.$$

Proof. In order to show that Ω and I commute, it suffices to show that the involutions I and ω_k commute at the level of their action on alcoves in the dominant Weyl chamber. On the one hand, I is the operator $t_\rho w_0$. On the other hand, ω_k sends the $(k+1)$-core associated to λ to its usual conjugate. Hence, if we have the reduced decomposition $w_\lambda = s_{i_1} s_{i_2} \cdots s_{i_k}$, we get the reduced decomposition $w_{\omega(\lambda)} = s_{i_1^*} \cdots s_{i_k^*}$. Hence, the action of ω_k on the alcoves coincide to that of $-w_0$ which commutes with $t_\rho w_0$ and so $\mathsf{I}\Omega = \Omega\mathsf{I}$. The second part of the proposition is a direct consequence of Proposition 6.6 and (8). □

Drift under harmonic measures. Let $\mathcal{A}_k$ be the set of alcoves in the dominant Weyl chamber. We denote by $\Gamma_f(\mathcal{A}_k)$ the set of reduced finite alcove paths which start at $A^{(0)}$ and remain in the dominant Weyl chamber. For any A in $\mathcal{A}_k$, write $\lambda_A \in \mathcal{B}_k$ its corresponding k-bounded partition. Conversely recall that for any $\lambda \in \mathcal{B}_k$, $A_{w_\lambda} \in \mathcal{A}_k$ is the alcove associated to λ. Let φ be an extremal harmonic measure on $\mathcal{B}_k$ associated to $\vec{r} = (r_1, \dots, r_k) \in \mathbb{R}^k$, and let $(A_n)_{n\ge1}$ be the central Markov chain on $\mathcal{A}_k$ defined in Section 6.1. By considering for each $n \ge 1$ the center v_n of the alcove A_n, we get a genuine random walk $(v_n)_{n\ge1}$ on $\mathbb{R}^k$. The goal of this subsection is to prove the law of large numbers for this random walk. This will be obtained by using the matrix $\Phi = \Phi_{r_1,\dots,r_k}$ and a reduced version of the walk $(v_n)_{n\ge0}$. For simplicity, we will assume that Φ is irreducible. Nevertheless, by continuity arguments, Theorem 6.10 below also holds in full generality.

Observe that 1 is the maximal eigenvalue of Φ for $\vec{r} \in \mathcal{S}_k$. We denote by X the corresponding left eigenvector of Φ normalized so that $X(\emptyset) = 1$. Let $\widehat{X}$ be the right eigenvector also for the eigenvalue 1 normalized so that $(X, \widehat{X}) = 1$ (here $(\cdot,\cdot)$ is the usual scalar product on vectors).

Let $\mathcal{M}_k$ be the multigraph with set of vertices B such that for each affine reflection $s_{\alpha,r}$, we have an edge between A and A' when $A' = s_{\alpha,r}A$ or $t_{\Lambda_i}A' = s_{\alpha,r}A$. We color each edge e by colors in $\{0, 1, \dots, k\}$ so that $c(e) = i$ if $\alpha = \alpha_i$ is simple and with $c(e) = 0$ otherwise [4]. Let $(\widetilde{A}_n)_{n\ge1}$ be the Markov chain on the graph $\mathcal{M}_k$ starting on $A^{(0)}$ with transition probabilities

[4] Observe that $\mathcal{M}_k$ is the graph with adjacency matrix Φ except that each arrow with weight $r_{i_1} + \cdots + r_{i_m}$ is split in m arrows with weights $r_{i_1}, \dots, r_{i_m}$.

$\widetilde{\mathbb{P}}(A \overset{e}{\to} A') = r_{c(e)}\frac{X(\lambda_{A'})}{X(\lambda_A)}$, with the convention $r_0 = 1$. Note that $(\widetilde{A}_n)_{n\geq 1}$ is indeed a random walk, since

$$\sum_{\substack{e,A' \\ A \text{ gives } A' \text{ through } e}} r_{c(e)}\frac{X(\lambda_{A'})}{X(\lambda_A)} = \sum_{A,A'} \Phi_{\lambda_A,\lambda_{A'}}\frac{X(\lambda_{A'})}{X(\lambda_A)} = \frac{X(\lambda_A)}{X(\lambda_A)} = 1.$$

The weight $\mathrm{wt}(\gamma)$ of a path $\gamma = A_0 \xrightarrow{e_1} A_1 \xrightarrow{e_2} \dots \xrightarrow{e_n} A_n$ is defined by $\mathrm{wt}(\gamma) = \sum_{i=1}^n \Lambda_{c(e_i)}$, with the convention that $\Lambda_0 = 0$. We denote by ℓ the associated length function. Let $\Gamma_f(\mathcal{M}_k)$ and $\Gamma_f(\mathcal{A}_k)$ be respectively the sets of finite paths on $\mathcal{M}_k$ and $\mathcal{A}_k$ starting at A_0.

We define $p : \Gamma_f(\mathcal{M}_k) \longrightarrow \mathcal{A}_k$ by $p(\gamma) = \gamma(n)+\mathrm{wt}(\gamma)$, where n is the length of γ, and we extend the map p to a map $L : \Gamma_f(\mathcal{M}_k) \to \Gamma_f(\mathcal{A}_k)$ where $L(\gamma) = (p(A_0), p(A_0, A_1), \dots, p(A_0, \dots, A_n))$. Let $M : \Gamma_f(\mathcal{A}_k) \to \Gamma_f(\mathcal{M}_k)$ be the map which sends a path $(A_0 \to A_1 \cdots \to A_n)$ to the path $(\tilde{A}_0 \xrightarrow{e_1} \tilde{A}_1 \xrightarrow{e_2} \dots \xrightarrow{e_n} \tilde{A}_n)$, where e_i is the unique edge from $\tilde{A}_{i-1}$ to $\tilde{A}_i$ such that $c(e_i) = j$ if $A_i = s_{\alpha_j,k}A_{i-1}$ for some $k \in \mathbb{Z}_{>0}$ and $1 \leq j \leq k$, and e_i is the unique edge from $\tilde{A}_{i-1}$ to $\tilde{A}_i$ with color 0 if $A_i = s_{\alpha,k}A_{i-1}$ with α non-simple and $k \in \mathbb{Z}_{>0}$. It is easy to see that $LM = id_{\Gamma_f(\mathcal{A}_k)}$ and $ML = id_{\Gamma_f(\mathcal{M}_k)}$.

Lemma 6.8. *The image of the Markov chain $(\tilde{A}_n)_{n\geq 0}$ through the map L is exactly the Markov chain $(A_n)_{n\geq 0}$.*

Proof. Let γ be a finite path on $\mathcal{M}_k$ of weight $\mathrm{wt}(\gamma)$ and ending at $\tilde{A}$. By the Markov kernel defined above,

$$\widetilde{\mathbb{P}}(\gamma) = r^{\mathrm{wt}(\gamma)}X(\lambda_{\tilde{A}}),$$

where $r^{\mathrm{wt}(\gamma)} = r_1^{\beta_1}\dots r_k^{\beta_k}$ when $\mathrm{wt}(\gamma) = \beta_1\Lambda_1 + \dots + \beta_k\Lambda_k$, with $\beta_i \in \mathbb{Z}_{\geq 0}$. Since $L(\gamma)$ ends at $p(\gamma) = A + \mathrm{wt}(\gamma)$ and $X(\lambda) = \varphi(s_\lambda^{(k)})$ for any $\lambda \in \mathcal{P}_{\mathrm{irr}}$, we have

$$\widetilde{\mathbb{P}}(\gamma) = \mathbb{P}(L(\gamma)).$$

□

Recall that for any $n \geq 0$, v_n is the center of the alcove A_n. Denote by $x_i(n) = \langle v_n, \alpha_i\rangle$ the position of v_n along the direction Λ_i.

Lemma 6.9. *As n goes to infinity,*

$$\frac{1}{n}x_i(n) \longrightarrow r_i \sum_{\substack{e:A\to A'\in\mathcal{M}_k \\ c(e)=i}} \widehat{X}(\lambda_A)X(\lambda_{A'}).$$

Proof. Set $y_i(n) = \lfloor x_i(n)\rfloor$. Then,

$$\lim_{n\to\infty}\frac{1}{n}x_i(n) = \lim_{n\to\infty}\frac{1}{n}y_i(n).$$

Let $N \geq 1$ and $0 \leq n \leq N$. Suppose that $y_i(n+1) - y_i(n) = 1$. Since $x_i(n) - y_i(n) > 0$, we have $\langle v_n, \Lambda_i\rangle < y_i(n+1) < \langle v_{n+1}, \Lambda_i\rangle$. Hence, the affine hyperplane $H_{\alpha_i,y_i(n+1)}$ separates the alcoves A_n and A_{n+1}, and thus $A_{n+1} = s_{\alpha_i,y_i(n+1)}A_n$. Hence, $y_i(n+1) - y_i(n) = 1$ if and only if the n-th edge of the path $M(A_1, \dots, A_N)$ is colored by i. Hence $y_i(N)$ is the number of arrows colored by i in the trajectory $M(A_1, \dots, A_N)$. Since $M(A_1, \dots, A_N)$ is an irreducible random walk on $\mathcal{M}_k$, the ergodic theorem for random walks on finite spaces yields that for each edge $e_0 \in \mathcal{M}_k$ from A to A',

$$\frac{1}{N}\mathrm{card}(\{e \in M(A_1, \dots, A_N), e = e_0\} \xrightarrow[n\to\infty]{a.s} m(A)\widetilde{\mathbb{P}}(e),$$

where m is the invariant measure on $\mathcal{M}_k$ with respect to $\widetilde{\mathbb{P}}$. We have

$$\widetilde{\mathbb{P}}(\widetilde{A}_n = A' \mid \widetilde{A}_{n-1} = A') = \sum_{e \text{ from } A \text{ to } A'} r_{c(e)} \frac{X(\lambda_{A'})}{X(\lambda_A)} = \Phi_{\lambda_A, \lambda_{A'}} \frac{X(\lambda_{A'})}{X(\lambda_A)},$$

thus the corresponding invariant measure is the unique vector m such that $\sum_{A \in \mathcal{M}_k} m(A) = 1$ and

$$\sum_A \Phi_{\lambda_A, \lambda_{A'}} \frac{X(\lambda_{A'})}{X(\lambda_A)} m(A) = m(A').$$

We get that $\left(\frac{m(A)}{X(\lambda_A)}, A \in \mathcal{M}_k\right)$ is a left eigenvector of Φ with eigenvalue 1, thus is proportional to $\widehat{X}(\lambda_A)$. In fact it is equal to $\widehat{X}(\lambda_A)$ since m is a measure and $(X, \widehat{X}) = 1$ so $m(A) = X(\lambda_A)\widehat{X}(\lambda_A)$. This gives

$$\frac{1}{N}\mathrm{card}(\{e \in M(v_1, \dots, v_N), e = e_0\} \xrightarrow[n\to\infty]{a.s} X(\lambda_A)\widehat{X}(\lambda_A) r_{c(e_0)} \frac{X(\lambda_{A'})}{X(\lambda_A)} = \widehat{X}(\lambda_A) X(\lambda_{A'}) r_{c(e_0)}.$$

Since $y_i(N)$ is the number of arrows colored by i in the trajectory $M(A_1, \dots, A_N)$ we obtain

$$\frac{1}{N} y_i(N) \xrightarrow[n\to\infty]{a.s} r_i \sum_{\substack{e: A \to A' \\ c(e) = i}} \widehat{X}(\lambda_A) X(\lambda_{A'}).$$

$\square$

Theorem 6.10.

(1) *As n goes to infinity, the normalized random walk $\left(\frac{1}{n}v_n\right)_{n\geq 1}$ converges almost surely to a vector $v_\varphi \in \mathbb{R}^k$.*

(2) *Moreover for any $i = 1, \dots, k$ the coordinate of v_φ on Λ_i satisfies*

$$v_\varphi(i) = \varphi\left(\frac{s_{R_i}}{\sum_{A \in \mathsf{B}} s^{(k)}_{\lambda_A} s^{(k)}_{\lambda_{\mathsf{I}\Omega(A)}}} \sum_{\substack{e: A \to A' \\ c(e) = i}} s^{(k)}_{\lambda_{\mathsf{I}\Omega(A)}} s^{(k)}_{\lambda_{A'}}\right)$$

which is a rational function on $\mathbb{R}^k$.

Proof. The previous lemma proves the first part of the theorem. It also shows that

$$v_\varphi(i) = r_i \sum_{\substack{e: A \to A' \\ c(e) = i}} \widehat{X}(\lambda_A) X(\lambda_{A'}).$$

By Proposition 6.7, the coordinates of the vector $\widehat{X}$ are such that $\widehat{X}(\lambda_A) = \frac{1}{\nabla} X(\lambda_{\mathsf{I}\Omega(A)})$ for any $A \in \mathsf{B}$ where $\nabla = \sum_{A \in \mathsf{B}} X(\lambda_A) X(\lambda_{\mathsf{I}\Omega(A)})$. Since the coordinates of X are the $\varphi(s^{(k)}_\lambda)$ with λ irreducible, we can write

$$v_\varphi(i) = \frac{1}{\nabla} r_i \sum_{\substack{e: A \to A' \\ c(e) = i}} \varphi(s^{(k)}_{\lambda_{\mathsf{I}\Omega(A)}} s^{(k)}_{\lambda_{A'}}) = \varphi\left(\frac{s_{R_i}}{\sum_{A \in \mathsf{B}} s^{(k)}_{\lambda_A} s^{(k)}_{\lambda_{\mathsf{I}\Omega(A)}}} \sum_{\substack{e: A \to A' \\ c(e) = i}} s^{(k)}_{\lambda_{\mathsf{I}\Omega(A)}} s^{(k)}_{\lambda_{A'}}\right).$$

Proposition 7.1 applied with $t = 1$ will imply that $v_\varphi(i)$ is indeed a rational function in $(r_1, \ldots, r_k)$ and so v_φ is rational in $(r_1, \ldots, r_k)$. $\square$

7. Some consequences

7.1. Limit formulas in the case $\varphi(\Delta) = 0$. For any k-irreducible partition κ, we know by § 4.2 that there exists a polynomial $P_\kappa \in \mathbb{A}[T]$ such that

$$s_\kappa^{(k)} = \frac{P_\kappa(s_1)}{\Delta} \tag{12}$$

here $\Delta \in \mathbb{A}$ is the determinant of the transition matrix between the bases $\{s_{(1)}^a \mid 0 \leq a \leq k! - 1\}$ and $\{s_\kappa^{(k)} \mid \kappa\ k\text{-irreducible}\}$. For any morphism $\varphi : \Lambda_{(k)} \to \mathbb{R}$ nonnegative on the k-Schur functions and such that $\varphi(\Delta) \neq 0$ we thus get

$$\varphi(s_\kappa^{(k)}) = \frac{\varphi(P_\kappa)(\varphi(s_1))}{\varphi(\Delta)}. \tag{13}$$

Moreover, $\varphi(P_\kappa)$ and $\varphi(\Delta)$ are directly determined by the values $r_a = \varphi(s_{R_a}), a = 1, \ldots, k$ since P_κ and Δ belong to the subalgebra $\mathbb{A}$. Also $(\varphi(s_1)$ is the spectral radius of the matrix $\Phi = \varphi(\mathbf{\Phi})$.

Now assume that the morphism φ associated to $\vec{r}$ is such that $\varphi(\Delta) = 0$. Then, we can consider a sequence $(\vec{r}_n)_{n \geq 0}$ in $U = \mathbb{R}_{\geq 0}^k$ such that each morphism $\varphi_n := f(\vec{r}_n)$ satisfies $\varphi_n(\Delta) \neq 0$. By continuity of the map f we then get for any k-irreducible partition κ

$$\varphi(s_\kappa^{(k)}) = \lim_{n \to \infty} \frac{\varphi_n(P_\kappa)(\varphi_n(s_1))}{\varphi_n(\Delta)}$$

so that the formulas (13) extends by continuity. In particular we then have $\varphi(P_\kappa)(s_1) = 0$. Alternatively, one can consider for any nonnegative real s, the sets $\overline{V}_s = \{\vec{h} \in \overline{V} \mid h_1 = s\}$ and $\overline{U}_s = g(\overline{V}_s)$. For any $\vec{r} \in \overline{U}_s$ such that $\Delta(\vec{r}) = \varphi(\Delta) \neq 0$ write $\widetilde{P}_\kappa^s(\vec{r}) = \varphi(P_\kappa)(s)$. We also set $\varphi(s_\kappa^{(k)}) = s_\kappa^{(k)}(\vec{r})$.

Proposition 7.1. *For each irreducible k-partition κ, the function $\vec{r} \longmapsto s_\kappa^{(k)}(\vec{r})$ is continuous on $\overline{U}_s$ and rational. We have*

$$s_\kappa^{(k)}(\vec{r}) = \frac{\widetilde{P}_\kappa^s(\vec{r})}{\Delta(\vec{r})}. \tag{14}$$

In particular, the coordinates of f are continuous rational functions on each $\overline{U}_s$.

7.2. The minimal boundary of $\mathcal{B}^{(3)}$. For $k = 3$, one can easily picture the domains $\overline{V}_1$. The condition to get a positive morphism φ indeed reduces to $\varphi(s_1) \geq 0, \varphi(s_2) = \varphi(h_2) \geq 0, \varphi(s_3) = \varphi(h_3) \geq 0, \varphi(s_{(1,1)}) = \varphi(e_2) \geq 0, \varphi(s_{(1,1,1)}) = \varphi(e_3) \geq 0, \varphi(s_{(2,1)}) \geq 0, \varphi(s_{(2,1,1)}^{(3)}) \geq 0$ and $\varphi(s_{(2,2)}) \geq 0$. We get moreover by a simple computation

$$s_{(2,1,1)}^{(3)} = s_{(2)} s_{(1,1)}$$

thus $\varphi(s_{(2,1,1)}^{(3)}) \geq 0$ does not add any new constraint. We also have $\varphi(h_1) = \varphi(s_1) = 1$ and the Jacobi-Trudi relations $e_2 = h_1^2 - h_2, e_3 = h_1^3 + h_3 - 2h_2h_1, s_{(2,1)} = h_2h_1 - h_3$ and $s_{(2,2)} = h_2^2 - h_3h_1$. Now by using that $\varphi(s_1) = 1$ one can see that the previous inequalities are equivalent to

$$h_1 = 1,\ 0 \leq h_2 \leq 1,\ 0 \leq h_3 \leq h_2^2 \text{ and } 2h_2 - h_3 \leq 1. \tag{15}$$

By setting $x = \varphi(h_2)$ and $y = \varphi(h_3)$. This gives the domain $\overline{V}_1 = \partial\mathcal{H}^+(\mathcal{B}_3)$ delimited in the picture below by the x abscise, the blue line and the red parabola.

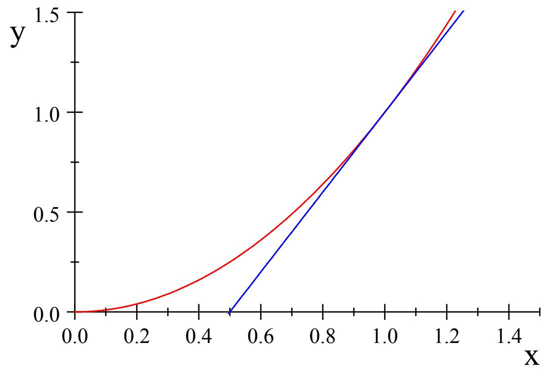

FIGURE 2. Region $\overline{V}_1$ in $x = h_2$ and $y = h_3$ coordinates delimited by the three curves $h_3 = 0$, $s_{(2,2)} = 0$ and $e_3 = 0$.

Remark 7.2. *If we consider the points of $\overline{V}_1 = \partial\mathcal{H}^+(\mathcal{B}_3)$ such that $h_3 = 0$, we get the domain $\{(1, h_2, 0) \mid h_2 \in [0, \frac{1}{2}])$. From example 5.6, one sees that its projection in $\mathbb{R}^2$ is only strictly contained in $\partial\mathcal{H}^+(\mathcal{B}_2) = \{(1, h_2) \mid h_2 \in [0, 1]\}$ (see § 7.4).*

7.3. **Minimal boundary of $\mathcal{B}_k$.** By homogeneity of the Schur functions, one gets that for any $\vec{r} = (r_1, r_2, \ldots, r_k) \in \mathbb{R}^k_{\geq 0}$ and any positive real t

$$f(t^k r_1, t^{2(k-1)} r_2, \ldots, t^k r_k) = (th_1, \ldots, t^k h_k). \tag{16}$$

Also with the notation of § 7.1, we obtain that $\partial\mathcal{H}^+(\mathcal{B}_k) = \overline{V}_1$ is homeomorphic to $\overline{U}_1$. It follows from (16) that for each nonnegative real s, the sets $\overline{U}_s$ and $\overline{V}_s$ are completely determined by $\overline{U}_1$ and $\overline{V}_1$, respectively. Also, we can associate to any element $\vec{r} \in \mathbb{R}^k_{\geq 0}$ the element in $\partial\mathcal{H}^+(\mathcal{B}_k)$ obtained by computing $\vec{h} = f(\vec{r})$ and next renormalizing it according to (16) so that its first coordinate becomes equal to 1. We also have the following description of the minimal boundary:

Proposition 7.3. *$\partial\mathcal{H}^+(\mathcal{B}_k)$ is homeomorphic to $\mathcal{S}_k = \{(r_1, \ldots, r_k) \in \mathbb{R}^k_{\geq 0} \mid r_1 + \cdots + r_k = 1\}$.*

Proof. We already know that $\partial\mathcal{H}^+(\mathcal{B}_k) = \overline{V}_1$ is homeomorphic to $\overline{U}_1$. Also any $\vec{r} = (r_1, \ldots, r_k)$ in $\overline{U}_1$ is nonzero. There thus exists a unique positive real $t(\vec{r})$ such that $t^k r_1 + t^{2(k-1)} r_2 + \cdots + t^k r_k = 1$. This follows from the fact that the polynomial function $p(t) = t^k r_1 + t^{2(k-1)} r_2 + \cdots + t^k r_k$ strictly increases on $\mathbb{R}_{>0}$ with $p(0) = 0$ and $\lim_{t\to+\infty} = +\infty$. Then, $t(\vec{r})$ is the unique real root of the polynomial $p(T) - 1$. The function $t : \vec{r} \to t(\vec{r})$ is continuous on $\overline{U}_1$, therefore the function $u : \overline{U}_1 \to \mathcal{S}_k$ defined by

$$u(r_1, \ldots, r_k) = (t(\vec{r})^k r_1, t(\vec{r})^{2(k-1)} r_2, \ldots, t(\vec{r})^k r_k)$$

is well-defined and continuous. If $u(\vec{r}) = u(\vec{R})$ with $\vec{r}$ and $\vec{R}$ in $\partial\mathcal{H}^+(\mathcal{B}_k)$, we have by applying f

$$f(u(\vec{r})) = (t(\vec{r})1, t(\vec{r})^2 h_2, \ldots, t(\vec{r})^k h_k) = (t(\vec{R})1, t(\vec{R})^2 h_2, \ldots, t(\vec{R})^k h_k) = f(u(\vec{R})).$$

Thus $t(\vec{r}) = t(\vec{R})$ and we get $\vec{r} = \vec{R}$ so that u is injective. Now given any $\vec{r} = (r_1, \ldots, r_k) \in \mathcal{S}_k$, there exists a positive real s such that $\vec{r}_s = (s^k r_1, s^{2(k-1)} r_2, \ldots, s^k r_k)$ belongs to $\partial\mathcal{H}^+(\mathcal{B}_k)$. We then have $u(\vec{r}_s) = \vec{r}$. □

By observing that $\Lambda_{(k)} = \mathbb{R}[h_1, \ldots, h_k] = \mathbb{R}[e_1, \ldots, e_k]$ is in fact isomorphic to the algebra $\Lambda[X_1, \ldots, X_k]$ of symmetric polynomials in k variables $X_1, \ldots, X_k$ over $\mathbb{R}$, we can also get informations on the values taken by these variables for each point of $\partial\mathcal{H}^+(\mathcal{B}_k)$. For any $r = 1, \ldots, k$, write for short $E_r = \frac{\widetilde{P}^1_{(1^k)}(\vec{r})}{\Delta(\vec{r})}$. Each E_r is a rational function on $\overline{U}_1$ which associates to each element of $\overline{U}_1$ the value of $\varphi(e_r)$ for the associated morphism φ.

Proposition 7.4. *For each $\vec{h} \in \partial\mathcal{H}^+(\mathcal{B}_k)$, there exists a unique $\vec{x} = (x_1, \ldots, x_k) \in \mathbb{C}^k$ such that the associated morphism $\varphi : \Lambda_{(k)} \to \mathbb{R}$, nonnegative on the k-Schur functions, coincides with the specialization*

$$\varphi(P(X_1, \ldots, X_r)) = P(x_1, \ldots, x_r).$$

Moreover $\vec{x}$ is determined by the roots of the polynomial

$$\zeta(T) = \prod_{r=1}^{k}(1 + Tx_i) = 1 + t + \sum_{r=2}^{k-1} E_r T^{r-1} + r_k T^k$$

where $E_1, \ldots, E_r$ are rational continuous functions on $\overline{U}_1$.

Example 7.5.

(1) *For $k = 2$, we have $E_1 = 1$ and $E_2 = r_2$ so that*

$$\zeta(T) = 1 + t + t^2 r_2.$$

(2) *For $k = 3$, we get by resuming Example 4.9 and using the equality $\Xi(1) = 1 - 2(r_1 + r_3) - 4r_2 + (r_1 - r_3)^2 = 0$.*

$$E_1 = 1 \text{ and } E_2 = \frac{1}{2}(r_3 - r_1 + 1).$$

This gives

$$\zeta(T) = 1 + T + \frac{1}{2}(r_3 - r_1 + 1)T^2 + r_3 T^3.$$

In that simple case we get in fact polynomial functions independent of r_2.

Remark 7.6. *The previous proposition does not mean that $\partial\mathcal{H}^+(\mathcal{B}_k)$ is parametrized by the roots of all the polynomials $\zeta(T)$. This is only true for the roots of the polynomials $\zeta(T)$ corresponding to a point in $\overline{U}_1$.*

7.4. **Embedding and projective limit of the minimal boundaries.** By Proposition 2.14, each morphism $\varphi : \Lambda_{(k+1)} \to \mathbb{R}$ nonnegative on the $(k+1)$-Schur functions yields by restriction to $\Lambda_{(k)} \subset \Lambda_{(k+1)}$ a morphism nonnegative on the k-Schur functions. Here we use the natural embedding $\Lambda_{(k)} \subset \Lambda_{(k+1)}$ corresponding to the specialization $h_{k+1} = 0$. Unfortunately, this will not give us a projection of $\partial\mathcal{H}^+(\mathcal{B}_{k+1})$ on $\partial\mathcal{H}^+(\mathcal{B}_k)$ (see Remark 7.2). Nevertheless, we can define such a projection $\pi_k : \partial\mathcal{H}^+(\mathcal{B}_{k+1}) \to \partial\mathcal{H}^+(\mathcal{B}_k)$ by setting

$$\pi_k(h_1, \ldots, h_k, h_{k+1}) = \pi_k \circ f(r_1, \ldots, r_k, r_{k+1}) = f(r_1, \ldots, r_k)$$

where $f(r_1, \ldots, r_k, r_{k+1}) = (h_1, \ldots, h_k, h_{k+1})$. This indeed yields a surjective map since for any $(h'_1, \ldots, h'_k) \in \partial\mathcal{H}^+(\mathcal{B}_k)$, we can set $(h'_1, \ldots, h'_k) = \pi_k \circ f(r'_1, \ldots, r'_k, 0)$ where $(r'_1, \ldots, r'_k) = g(h'_1, \ldots, h'_k)$.

Proposition 7.7.

(1) *The map π_k is continuous and surjective from $\partial\mathcal{H}^+(\mathcal{B}_{k+1})$ to $\partial\mathcal{H}^+(\mathcal{B}_k)$.*
(2) *The inverse limit $\varprojlim \mathcal{B}_k$ is homeomorphic to the minimal boundary of the ordinary Young lattice, that is to the Thoma simplex.*

7.5. **Rietsch parametrization of Toeplitz matrices.** Consider the variety $T_{\geq 0} \subset \mathbb{R}^k_{>0}$ of totally nonnegative unitriangular Toeplitz $(k+1) \times (k+1)$ matrices

$$M = \begin{bmatrix} 1 & & & & & \\ h_1 & 1 & & & & \\ \vdots & h_1 & \ddots & & & \\ \vdots & \vdots & \ddots & \ddots & & \\ h_{k-1} & \vdots & \vdots & \ddots & \ddots & \\ h_k & h_{k-1} & \cdots & \cdots & h_1 & 1 \end{bmatrix}.$$

The set $T_{>0}$ of totally positive unitriangular Toeplitz $(k+1) \times (k+1)$ matrices is defined as the subset of $T_{\geq 0}$ of matrices M whose minors with no row and no column in the upper part of M are positive. By Theorem 3.2.1 in [2], M is totally positive if and only if for $a = 1, \dots, k$, the $a \times a$, initial minors obtained by selecting a rows of M arbitrary and then the first a columns of M are positive.

Lemma 7.8.

(1) *The previous initial minors are equal to Schur functions s_λ, where the maximal hook of the partition λ has length less or equal to k.*
(2) *We have $\overline{T}_{>0} = T_{\geq 0}$ that is, each totally nonnegative unitriangular Toeplitz matrix is the limit of a sequence of totally positive unitriangular Toeplitz matrices.*

Proof. Let $L = \{i_1, \dots, i_a\}$ be a subset of $\{1, \dots, k\}$ such that $i_1 < \cdots < i_k$ and consider the minor Δ_L corresponding to the determinant of the submatrix $M_{L \times [1,a]}$. The diagonal of $M_{L \times [1,a]}$ is $(h_{i_1}, h_{i_2 - 1}, \cdots, h_{i_k - k + 1})$ where $i_k - k + 1 \geq \cdots \geq i_2 - 1 \geq i_1$. Thus, by using the Jacobi-Trudi formula we have $\Delta_L = s_{(i_k - k + 1, \dots, i_2 - 1, i_1)}$. The maximal hook length of the partition $\lambda = (i_k - k + 1, \dots, i_2 - 1, i_1)$ is equal to $(i_k - k + 1) + (k - 1) = i_k \leq k$ which proves assertion 1.

To get Assertion 2, consider $M \in T_{\geq 0}$ and $U \in T_{>0}$. For any real $t > 0$ let $U(t)$ be the matrix obtained by replacing each real h_a by $t^a h_a$ in U. Then $U(t)$ belongs to $T_{>0}$. Indeed, with the previous notation, if the minor Δ_L associated to U is equal to the Schur function s_λ, then the corresponding minor in $U(t)$ is equal to $t^{|\lambda|} s_\lambda$. The set $T_{\geq 0}$ is stable by matrix multiplication and we moreover get from Proposition 10 in [3] that the product matrix $U(t)M$ is totally positive. Since $U(t)$ tends to the identity matrix when t tends to 0, we obtain that $U(t)M$ tends to M as desired. $\square$

Observe in particular that for any $a = 1, \dots, k$, the initial minor $\Delta_{[k-a+1,k]}$ gives the value r_a of the rectangle Schur function s_{R_a} evaluated in $(h_1, \dots, h_k)$. In [16], Rietsch obtained the following parametrization of $T_{\geq 0}$ by using the quantum cohomology of partial flag varieties.

Theorem 7.9. *The map*

$$\begin{cases} T_{\geq 0} \to \overline{U} \\ (h_1, \dots, h_k) \longmapsto (r_1, \dots, r_k) \end{cases}$$

is a homeomorphism.

We now reprove this theorem from our preceding results.

Theorem 7.10. *We have $T_{>0} = V$ and $T_{\geq 0} = \overline{V}$, in particular the map $g : T_{\geq 0} \to \overline{U}$ is a homeomorphism.*

Proof. Observe first we have $V \subset T_{>0}$. Indeed we know that each k-Schur function $s_\lambda^{(k)}$ evaluated in $\vec{h} = (h_1, \dots, h_k)$ in V is positive. This is in particular true when λ is a partition with maximal hook length less or equal to k but then, we get by Assertion 1 of the previous Lemma that the associated Toeplitz matrix is totally positive because such k-Schur functions coincide with ordinary Schur functions. Next consider a sequence $\vec{h}_n, n \geq 0$ in V which converges to a limit $\vec{h} \in T_{>0}$. Since $\vec{h} \in T_{>0}$, each $r_a = \Delta_{[k-a+1,k]}(\vec{h}), a = 1, \ldots, k$ is positive. Thus $\vec{r} = (r_1, \ldots, r_k)$ belongs to U. Now $\vec{h}$ belongs to $\overline{V}$ and we have $g(\vec{h}) = \vec{r}$ by definition of g. Theorem 5.5 then implies that $\vec{h} \in V$ so V is closed in $T_{>0}$. Now V is open in $T_{>0}$ because each $\vec{h} \in V$ admits a neighborhood contained in $V \subset T_{>0}$ (V is an intersection of open subsets by definition). We also have that $T_{>0}$ is connected (see for example the proof of Proposition 12.2 in [16]). So V is nonempty both open and closed in $T_{>0}$ and we therefore have $T_{>0} = V$. The second assertion of Lemma 7.8 then gives $T_{\geq 0} = \overline{T}_{>0} = \overline{V}$. □

Remarks 7.11.

(1) *Since $T_{>0} = V$, we get by using the initial minors of M and Assertion 1 of Lemma 7.8 that $\vec{h}$ belongs to V if and only if the Schur functions s_λ with λ of maximal hook length less or equal to k evaluated at $\vec{h}$ are positive. Thus the criterion to test the positivity of our morphisms Φ reduces to Schur functions and can be performed without using the k-Schur functions.*

(2) *By Theorem 5.5 we are able to compute $g = f^{-1}$ from the Perron Frobenius vectors of the matrices Φ. So our Theorem 7.10 permits in fact to compute the nonnegative Toeplitz matrix associated to any point of $\overline{U}$ (i.e. to reconstruct M from the datum of the minors $r_1, \ldots, r_k$).*

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