A toy nonlinear model in kinetic theory

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ABSTRACT. This note is concerned with the study of a toy nonlinear model in kinetic theory. It consists in a non-linear kinetic Fokker-Planck equation whose diffusion in the velocity variable is proportional to the mass of the solution and steady states are Maxwellian. Solutions are constructed by combining energy estimates, well-designed hypoelliptic Schauder estimates, and the hypoelliptic extension of the De Giorgi-Nash Hölder estimates obtained recently by Golse, Vasseur and the two authors (2017).

CONTENTS

1. INTRODUCTION

1.1. **The equation and main result.** We consider the equation

$$\partial f + v \cdot \nabla_x f = \rho[f] \nabla_v \left(\nabla_v f + v f \right) \tag{1.1}$$

supplemented with the initial condition $f(t=0) = f_{in}$, and where $\rho[f] := \int_v f \, \mathrm{d}v$, for an unknown $0 \le f = f(t,x,v)$, $x \in \mathbb{T}^d$ (torus with unit volume), $v \in \mathbb{R}^d$, $d \ge 1$. Observe that the unique steady state of this equation is $\mu(v) := (2\pi)^{-d/2} e^{-|v|^2/2}$, where uniqueness means unique within solutions in $L^2(\mathrm{d}x \, \mathrm{d}\mu^{-1}(v))$. We now state the main result of this note. In the following statement $H^k(\mathbb{T}^d \times \mathbb{R}^d)$ denotes the standard L^2-based Sobolev space.

Theorem 1.1. *Let $\mathcal{Q} := (0,+\infty) \times \mathbb{T}^d \times \mathbb{R}^d$ and two constants $0 < C_1 < C_2$. There exists $\alpha \in (0,1)$, only depending on C_1, C_2 and d, such that, for all initial data f_{in} such that $f_{in}/\sqrt{\mu} \in H^k(\mathbb{T}^d \times \mathbb{R}^d)$ with $k > d/2$ and satisfying $C_1 \mu \le f_{in} \le C_2 \mu$, there exists a unique global-in-time solution f of (1.1) in $\mathcal{Q}$ satisfying $f(0,x,v) = f_{in}(x,v)$ everywhere in $\mathbb{T}^d \times \mathbb{R}^d$ and $f(t)/\sqrt{\mu} \in H^k(\mathbb{T}^d \times \mathbb{R}^d)$ for all time $t > 0$ and $C_1 \mu \le f \le C_2 \mu$. Moreover this solution is C^∞ for $t > 0$.*

Remark 1.2. A key step of the proof is the Schauder estimate. It gives the following additional information on this solution: the *hypoelliptic Hölder norm* $\mathcal{H}^\alpha$ (defined below) of $f/\sqrt{\mu}$ is uniformly bounded in terms of the L^2 norm of $f_{in}/\sqrt{\mu}$ for times away from 0. This norm is defined on a given open connected set $\mathcal{Q}$ by

$$\|g\|_{\mathcal{H}^\alpha(\mathcal{Q})} := \sup_{\mathcal{Q}} |g| + \sup_{\mathcal{Q}} |(\partial_t + v \cdot \nabla_x) g| + \sup_{\mathcal{Q}} |D_v^2 g| + [(\partial_t + v \cdot \nabla_x) g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})} + [D_v^2 g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})}$$

where $[\cdot]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})}$ denotes the Hölder anisotropic semi-norm in Definition 2.3 along the scaling $(r^2 t, r^3 x, r v)$.

Remark 1.3. We did not intend to obtain the optimal lowest initial regularity for the local and therefore global well-posedness and leave this question to further investigations. However as can be seen in Section 4 in the proof of this theorem, the initial Sobolev can slightly be reduced to k derivatives in x and ℓ derivatives in v with $1 \le \ell \le k$ and $k > d/2$.

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The equation can be rewritten with the scaled unknown $g := f\mu^{-1/2}$:

$$\partial_t g + v \cdot \nabla_x g = \mathcal{R}[g] U[g] \tag{1.2}$$

with $\mathcal{R}[g] := \int_v g\mu^{1/2}\,\mathrm{d}v$ and the operator

$$U[g] := \Delta_v g + \left(\frac{d}{2} - \frac{|v|^2}{4}\right) g = \mu^{-1/2}\nabla_v(\mu\nabla_v(\mu^{-1/2}g)). \tag{1.3}$$

After this rescaling the natural space of symmetry for the collision operator is now $L^2(\mathrm{d}x\,\mathrm{d}v)$, without weight. And observe that the unique steady state in $L^2(\mathrm{d}x\,\mathrm{d}v)$ is now $\sqrt{\mu}$.

In contrast with (1.1), this operator has no first order term in the velocity variable. When constructing solutions for this equivalent non-linear problem, the difficulty is that the coefficient $(d/2 - |v|^2/4)$ is unbounded. We overcome it using first the fact that g stays in between two Maxwellians (by maximum principle) and second that the Hölder semi-norm encodes decay in the v variable [17] (see also [39]).

We construct solutions to the non-linear problem using energy estimates, and establish to that purpose Schauder estimates for the associated linear evolution problem. Numerous Schauder estimates for linear kinetic equations are known [30, 28, 15, 11, 27, 31, 22]. However we were not able to find the Schauder estimates proven in this paper (for instance the scaling method developed in [36] does not apply because of the first order operator $v \cdot \nabla_x$), although they might not be new. In any case we propose a new simple method of proof inspired by Krylov [25]. The main difference with the parabolic case treated in [25] is in the proof of the so-called *gradient bounds*, see Proposition 3.3. We combine Bernstein's method as in [25] with ideas and techniques borrowed from the hypocoercivity theory [38].

1.2. **Motivation and background.** The Cauchy problem of the Boltzmann equation $\partial_t f + v\cdot\nabla_x f = Q(f,f)$ and its counter-part for plasma physics, the Landau-Coulomb equation when $Q(f,f)$ takes a particular nonlinear drift-diffusion form, are still poorly understood mathematically. In the case of short-range interactions, the situation can be compared to that of the incompressible three-dimensional Navier-Stokes equations: (1) there exist some partial theories when invariances are imposed on the solutions (in the case of the Boltzmann: spatially homogeneous solutions [13, 14, 5, 6, 29]), (2) perturbative solutions around the homogeneous equilibrium have been constructed [37, 21, 19], (3) some weak solutions have been constructed without perturbative or invariance conditions but without known uniqueness [16] (in a similar way as the Leray solutions). However in the case of long-distance interactions, the collision operator Q enjoys ellipticity property [26, 1, 4], of order 2 for the Landau-Coulomb operator and of fractional order for the long-distance Boltzmann collision operator. Note that the perturbative theory was also extended to the case of long-distance interactions [18, 3, 2]. More recently, Silvestre described a new regularization mechanism for the Boltzmann equation [34] by looking at it as a integro-differential equation in "non-divergence" form and by applying the recent regularity result [33] obtained with Schwab. He also obtained pointwise upper bounds for the homogeneous Landau equation [35], once again by looking at it in "non-divergence" form and through a delicate construction of barriers. He then treated the inhomogeneous case with Cameron and Snelson [12] by using the Harnack estimate from [17]. A nice contribution of the latter work is the identification of a change of variables ensuring that ellipticity constants do not degenerate for large velocities. With the Hölder estimate from [17], the change of variables and the decay estimates from [12], Henderson and Snelson [22] derived the C^∞ smoothing effect for the Landau equation provided hydrodynamic quantities are finite. It required them to use appropriate Schauder estimates. In this article, we consider the previous toy model and prove unconditional well-posedness with an approach based again on De Giorgi and Schauder theories. We make use of the Hölder estimates from [17], we have good decay estimates and non-degenerating ellipticity constant "for free" by the maximum principles, and we develop *ad hoc* Schauder estimates to get global well-posedness in Sobolev spaces. The main differences are (1) we consider different Hölder spaces, (2) the proof of the Schauder estimate follows the original idea of Safonov presented in Krylov's book [25], (3) global well-posedness is proved in Sobolev spaces; in particular, we use the specific structure of the toy model when estimating the evolution of Sobolev norms.

Our motivation is therefore to contribute to the development of tools inspired from the parabolic and elliptic theories of equations with rough or Hölder coefficients to the kinetic context. In this paper, in the line of [17, 23], we want specifically to understand how such De Giorgi-Nash-Schauder type estimates can help with the control of the supercriticality in the Cauchy problem, as examplified in this toy model.

1.3. **Organisation of the article.** In Section 2, we introduce some anisotropic Hölder spaces appropriate to our equation. In Section 3, we derive Schauder estimates for a class of linear equations with bounded coefficients. Finally in Section 4, we construct local solutions of the non-linear problem in Sobolev spaces and use the Schauder estimate to extend these solutions globally in time.

1.4. **Notation.** We use the notation $g_1 \lesssim g_2$ when there exists a constant $C > 0$ independent of the parameters of interest such that $g_1 \le C g_2$ (we analogously define $g_1 \gtrsim g_2$). Similarly, we use the notation $g_1 \approx g_2$ when there exists $C > 0$ such that $C^{-1} g_2 \le g_1 \le C g_2$. We sometimes use the notation $g_1 \lesssim_\delta g_2$ if we want to emphasize that the implicit constant depends on some parameter δ.

2. Functional spaces

2.1. **Lie group structure, scalings and cylinders.** We construct cylinders adapted to the scaling of the equation in this subsection. Define for $r > 0$:

$$z := (t, x, v), \qquad rz := (r^2 t, r^3 x, rv). \tag{2.1}$$

Observe that $(\partial_t + v \cdot \nabla_x - \Delta_v)[g(r^2 t, r^3 x, rv)] = r^2 \{(\partial_t + v \cdot \nabla_x - \Delta_v)[g]\}(r^2 t, r^3 x, rv)$, i.e. if g satisfies the Kolmogorov equation so does the function $g^\sharp(z) := g(rz)$. Define the Lie group (non-commutative) product

$$z_1 \circ z_2 = (t_1, x_1, v_1) \circ (t_2, x_2, v_2) := (t_1 + t_2, x_1 + x_2 + t_2 v_1, v_1 + v_2)$$

with inverse element denoted $z^{-1} := (-t, -x + tv, -v)$ for $z := (t, x, v)$. Observe that given $z_0 = (t_0, x_0, v_0) \in \mathbb{R}^{2d+1}$ and $r > 0$, one has

$$\begin{aligned} &(\partial_t + v \cdot \nabla_x - \Delta_v)[g(t_0 + r^2 t, x_0 + r^3 x + r^2 t v_0, v_0 + rv)] \\ &= r^2 \{(\partial_t + v \cdot \nabla_x - \Delta_v)[g]\}(t_0 + r^2 t, x_0 + r^3 x + r^2 t v_0, v_0 + rv), \end{aligned}$$

i.e. if g satisfies the Kolmogorov equation so does $g^\sharp(z) := g(z_0 \circ (rz))$.

Define the unit cylinder $Q_1 = (-1, 0] \times B_1 \times B_1$ and, given $z_0 \in \mathbb{R}^{2d+1}$ and $r > 0$, the general cylinder (the base point is omitted when $z_0 = (0, 0, 0)$)

$$Q_r(z_0) := \left\{ z : \frac{1}{r}(z_0^{-1} \circ z) \in Q_1 \right\} = \left\{ (t, x, v) : t_0 - r^2 < t \le t_0,\ |x - x_0 - (t - t_0) v_0| < r^3,\ |v - v_0| < r \right\}.$$

2.2. **The Green function.** Consider the equation

$$\partial_t g + v \cdot \nabla_x g = \Delta_v g + S \tag{2.2}$$

where S is a bounded source term.

The Green function G of (2.2) (when $S \equiv 0$) was constructed in [24]:

$$G(z) = \begin{cases} \left(\frac{\sqrt{3}}{2\pi t^2} \right)^d e^{-\frac{3\left|x + \frac{t}{2} v\right|^2}{t^3}} e^{-\frac{|v|^2}{4t}} & \text{if } t > 0, \\ 0 & \text{if } t \le 0. \end{cases} \tag{2.3}$$

Proposition 2.1 (Properties of the Green function in $x \in \mathbb{R}^d$)**.** *Given $S \in L^\infty(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$ with compact support in time, the function*

$$\begin{aligned} g(t, x, v) &= \int_{\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d} G(\tilde z^{-1} \circ z) S(\tilde z) \, \mathrm{d}\tilde t \, \mathrm{d}\tilde x \, \mathrm{d}\tilde v \qquad (\textit{with } z := (t, x, v) \textit{ and } \tilde z := (\tilde t, \tilde x, \tilde v)) \\ &= \int_{\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d} G(t - \tilde t, x - \tilde x - (t - \tilde t) \tilde v, v - \tilde v) S(\tilde t, \tilde x, \tilde v) \, \mathrm{d}\tilde t \, \mathrm{d}\tilde x \, \mathrm{d}\tilde v =: (G \star S)(z) \end{aligned}$$

satisfies (2.2) in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$. For all $z_0 = (t_0, x_0, v_0) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and $r > 0$

$$\| G \star \mathbf{1}_{Q_r(z_0)} \|_{L^\infty(Q_r(z_0))} \lesssim_d r^2.$$

Proof. The proof of [25, Lemma 8.4.1, p. 115] can be adapted. □

To deduce the fundamental solution G_p in the torus $x \in \mathbb{T}^d$ it is enough to consider a periodic source term S and use the integrable decay of G to obtain the formula

$$G_p(t,x,v) := \sum_{x \in \mathbb{Z}^d} G(t, x+n, v)$$

and one has easily the following statement as a consequence of Proposition 2.1:

Proposition 2.2 (Properties of the Green function in $x \in \mathbb{T}^d$)**.** *Given $S \in L^\infty(\mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^d)$ with compact support in time, the function*

$$g(t,x,v) := G_p \star S(t,x,v) = \int_{\mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^d} G_p(\tau, y, w) S(t-\tau, x-y-(t-\tau)w, v-w) \, \mathrm{d}\tau \, \mathrm{d}y \, \mathrm{d}w$$

satisfies (2.2) in $\mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^d$. For all $z_0 = (t_0, x_0, v_0) \in \mathbb{R} \times \mathbb{T}^d \times \mathbb{R}^d$ and $r > 0$

$$\|G_p \star \mathbf{1}_{Q_r(z_0)}\|_{L^\infty(Q_r(z_0))} \lesssim_d r^2.$$

2.3. Hypoelliptic Hölder spaces.

Definition 2.3 (Hypoelliptic Hölder spaces)**.** Given a domain (open connected set) $\mathcal{Q} \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and $\alpha \in (0,1]$, we say that a function $g : \mathcal{Q} \to \mathbb{R}$ lies in $\mathcal{C}^{0,\alpha}(\mathcal{Q})$ (hypoelliptic Hölder space) if it is bounded and there is $C > 0$ s.t.

$$\forall \, z_0 \in \mathcal{Q}, \ r > 0 \ \text{ s.t. } \ Q_r(z_0) \subset \mathcal{Q}, \quad \|g - g(z_0)\|_{L^\infty(Q_r(z_0))} \leq C r^\alpha.$$

The smallest such constant C is denoted by $[g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})}$. The $\mathcal{C}^{0,\alpha}$-norm of g is then $\|g\|_{\mathcal{C}^{0,\alpha}(\mathcal{Q})} := \|g\|_{L^\infty(\mathcal{Q})} + [g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})}$.

We then define the following higher order hypoelliptic Hölder space: a function g lies in $\mathcal{H}^\alpha(\mathcal{Q})$ if $h(t,x,v) := g(t, x+tv, v)$ is differentiable in t and $g(t,x,v)$ is twice differentiable in v, and $\partial_t g + v \cdot \nabla_x g$, $D^2_v g \in \mathcal{C}^{0,\alpha}(\mathcal{Q})$. The semi-norm $[g]_{\mathcal{H}^\alpha(\mathcal{Q})}$ is defined as

$$[g]_{\mathcal{H}^\alpha(\mathcal{Q})} := [\partial_t g + v \cdot \nabla_x g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})} + [D^2_v g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})}$$

and the norm $\|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}$ is defined as

$$\|g\|_{\mathcal{H}^\alpha(\mathcal{Q})} := \|g\|_{L^\infty(\mathcal{Q})} + \|(\partial_t + v \cdot \nabla_x) g\|_{L^\infty(\mathcal{Q})} + \|D^2_v g\|_{L^\infty(\mathcal{Q})} + [g]_{\mathcal{H}^\alpha(\mathcal{Q})}.$$

Finally in the time-independent case, a function $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ lies in $\mathcal{H}^\alpha(\mathbb{R}^{2d})$ if the function $\bar{g}(t,x,v) := g(x,v)$ lies in $\mathcal{H}^\alpha(\mathbb{R}^{2d+1})$.

Remark 2.4. Observe that the $\mathcal{C}^{0,\alpha}(\mathcal{Q})$ regularity for some α is implied by some Hölder regularity in the variables t, x, v in the usual sense, and reciprocally implies some Hölder regularity in the usual sense, however with lower exponents; see for instance [30].

Remark 2.5. When $\mathcal{Q} = \mathbb{R}^{2d+1}$, we simply write $\|\cdot\|_{L^\infty}$, $[\cdot]_{\mathcal{C}^{0,\alpha}}$, $\|\cdot\|_{\mathcal{C}^{0,\alpha}}$, $[\cdot]_{\mathcal{H}^\alpha}$, $\|\cdot\|_{\mathcal{H}^\alpha}$.

Lemma 2.6. *Give a domain $\mathcal{Q} \subset \mathbb{R}^{2d+1}$, the spaces $\mathcal{C}^{0,\alpha}(\mathcal{Q})$ and $\mathcal{H}^\alpha(\mathcal{Q})$ are Banach spaces.*

Proof. This follows from combining the following facts: (1) the standard Hölder space is a Banach space, (2) the pointwise limit agrees with the distributional limit when they both exist, (3) the Hölder regularity on the distributional derivative implies the differentiability. □

It is natural question whether the norm $\mathcal{H}^\alpha$ controls regularity in the missing directions t and x, which is the object of the following lemma. The proof uses commutator estimates à la Hörmander at the level of trajectories.

Lemma 2.7 (Hypoelliptic Hölder estimate)**.** *Let $g \in \mathcal{H}^\alpha(\mathcal{Q})$ then*

$$[g]_{\mathcal{C}^1(\mathcal{Q})} \leq \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}, \tag{2.4}$$

$$[\nabla_v g]_{\mathcal{C}^1(\mathcal{Q})} \leq \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}. \tag{2.5}$$

Remark 2.8. The core of (2.4) is to prove that if $(\partial_t + v \cdot \nabla_x) f \in \mathcal{C}^{0,\alpha}$ and $D^2_v f \in C^{0,\alpha}$, then $f \in \mathcal{C}^{0,\frac{2}{3}}_x$. This result can be seen as the counterpart in Hölder spaces of the following result in [10] in the spirit of Hörmander's hypoellipticity theory: if $(\partial_t + v \cdot \nabla_x) f \in L^2$ and $D^2_v f \in L^2$, then $|D|^{\frac{2}{3}}_x f \in L^2$.

Proof. The difficulty is to obtain the Hölder regularity on the x and t directions from the higher regularity along the directions $\partial_t + v \cdot \nabla_x$ and ∇_v, which is an hypoelliptic commutator estimate.

Take two points $z_1 \in Q_r(z_0) \subset \mathcal{Q}$ with $z_1 = z_0 + (0, r^3 u, 0)$ and $z_0 = (t, x, v)$ with $|u| \le 1$ and $r > 0$. We shall follow the following trajectories schematically:

$$\begin{tikzcd}
z_1 \arrow[r, "\text{forward along } v"] & z_2 \arrow[r, "\text{backward along transport}"] & z_3 \arrow[d, "\text{backward along } v"] \\
& z_0 \arrow[ul, "\text{forward along } x"] & z_4 \arrow[l, "\text{forward along transport}"]
\end{tikzcd}$$

Observe crucially that all four points

$$\begin{aligned}
z_1 &= (t, x + r^3 u, v) = z_0 \circ r(0, u, 0), \\
z_2 &= (t, x + r^3 u, v + ru) = z_0 \circ r(0, u, u), \\
z_3 &= (t - r^2, x - r^2 v, v + ru) = z_0 \circ r(-1, 0, u)), \\
z_4 &= (t - r^2, x - r^2 v, v) = z_0 \circ r(-1, 0, 0)
\end{aligned}$$

belong to $Q_r(z_0)$ with $z_0 = (t, x, v)$.

Compute first (Taylor expansion in the v direction)

$$g(z_1) = g(t, x + r^3 u, v + ru) - ru \cdot \nabla_v g(t, x + r^3 u, v + ru) + \frac{r^2}{2} u \cdot D_v^2 g(t, x + r^3 u, v + \theta_1 ru) \cdot u$$

for some $\theta_1 \in (0, 1)$. Then second (Taylor expansion along free streaming)

$$\begin{aligned}
g(z_1) = {} & g(t - r^2, x + r^3 u - r^2(v + ru), v + ru) \\
& + r^2 [\partial_t g + (v + ru) \cdot \nabla_x g](t - \theta_2 r^2, x + r^3 u - \theta_2 r^2 (v + ru), v + ru) \\
& - ru \cdot \nabla_v g(t, x + r^3 u, v + ru) + \frac{r^2}{2} u \cdot D_v^2 g(t, x + r^3 u, v + \theta_1 ru) \cdot u
\end{aligned}$$

for some $\theta_2 \in (0, 1)$. Compute third (Taylor expansion in the v direction backwards)

$$\begin{aligned}
g(z_1) = {} & g(t - r^2, x - r^2 v, v) + ru \cdot \nabla_v g(t - r^2, x - r^2 v, v) \\
& + \frac{r^2}{2} u \cdot D_v^2 g(t - r^2, x - r^2 v, v + \theta_3 ru) \cdot u \\
& + r^2 [\partial_t g + (v + ru) \cdot \nabla_x g](t - \theta_2 r^2, x + r^3 u - \theta_2 r^2 (v + ru), v + ru) \\
& - ru \cdot \nabla_v g(t, x + r^3 u, v + ru) + \frac{r^2}{2} u \cdot D_v^2 g(t, x + r^3 u, v + \theta_1 ru) \cdot u
\end{aligned}$$

for some $\theta_3 \in (0, 1)$. Fourth (Taylor expansion in the free streaming direction)

$$\begin{aligned}
g(z_1) = {} & g(z_0) - r^2 [\partial_t g + v \cdot \nabla_x g](t - \theta_4 r^2, x - r\theta_4 v, v) + ru \cdot \nabla_v g(t - r^2, x - r^2 v, v) \\
& + \frac{r^2}{2} u \cdot D_v^2 g(t - r^2, x - r^2 v, v + \theta_3 ru) \cdot u \\
& + r^2 [\partial_t g + (v + ru) \cdot \nabla_x g](t - \theta_2 r^2, x + r^3 u - \theta_2 r^2 (v + ru), v + ru) \\
& - ru \cdot \nabla_v g(t, x + r^3 u, v + ru) + \frac{r^2}{2} u \cdot D_v^2 g(t, x + r^3 u, v + \theta_1 ru) \cdot u
\end{aligned}$$

for some $\theta_4 \in (0, 1)$. Using the Hölder regularity on $(\partial_t + v \cdot \nabla_x) g$ and $D_v^2 g$ for points in $Q_r(z_0)$, this yields

$$|g(z_1) - g(z_0)| \lesssim r \left| \nabla_v g(t - r^2, x - r^2 v, v) - \nabla_v g(t, x + r^3 u, v + ru) \right| + r^2 \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}. \tag{2.6}$$

Observe that equation (2.6) is enough to get 1/3-Hölder regularity in the x-variations, using that

$$\|\nabla_v g\|_{L^\infty(\mathcal{Q})} \lesssim \|g\|_{L^\infty(\mathcal{Q})} + \|D_v^2 g\|_{L^\infty(\mathcal{Q})}.$$

Let us however push further the argument to get the (conjectured) optimal Hölder regularity. The right hand side depends on variations of $\nabla_v g$ along all directions. We thus estimate now these variations, first along x then v then free streaming. A difficulty then is that our estimate for the variations of $\nabla_v g$ along x itself depends on the variations along x of g, but tuning the scale R in what follows will solve this. In other words, we are going to establish an interpolation estimate for the variation of $\nabla_v g$ in terms of variations of g (with an arbitrary small constant) and the L^∞ bound on second order v derivatives.

With the following shorthands (depending on u and r),

$$A := \sup\left\{ |g(z_1) - g(z_0)| \,:\, z_0 \in \mathcal{Q} \;\middle|\; Q_r(z_0) \subset \mathcal{Q} \right\}$$
$$B := \sup\left\{ |\nabla_v g(z_4) - \nabla_v g(z_2)| \,:\, z_0 \in \mathcal{Q} \;\middle|\; Q_r(z_0) \subset \mathcal{Q} \right\}$$

we can rewrite (2.6) as

$$A \lesssim rB + r^2 \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}. \tag{2.7}$$

For $R > 0$ and $w \in \mathbb{S}^{d-1}$, define

$$\iota_R[g](z) := R^{-1}(g(t,x,v+Rw) - g(t,x,v))$$

and write for some $R_1 > 0$

$$|\nabla_v g(z_1) - \nabla_v g(z_0)| \lesssim |\iota_{R_1}[g](z_1) - \iota_{R_1}[g](z_0)| + R_1 \|D_v^2 g\|_{L^\infty(\mathcal{Q})}.$$

Rearrange to get

$$\begin{aligned} &|\nabla_v g(z_1) - \nabla_v g(z_0)| \\ &\lesssim \frac{1}{R_1} |g(z_1 \circ (0,0,R_1 w)) - g(z_0 \circ (0,0,R_1 w))| + \frac{1}{R_1} |g(z_1) - g(z_0)| + R_1 \|D_v^2 g\|_{L^\infty(\mathcal{Q})} \\ &\lesssim \frac{2}{R_1} A + R_1 \|D_v^2 g\|_{L^\infty(\mathcal{Q})}. \end{aligned} \tag{2.8}$$

Concerning variations in v: by Taylor expansion in v only

$$|\nabla_v g(z_1) - \nabla_v g(z_2)| \lesssim r \|D_v^2 g\|_{L^\infty(\mathcal{Q})} \lesssim r \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}. \tag{2.9}$$

Finally concerning variations along free streaming (using Taylor expansion on f along free streaming):

$$\begin{aligned} |\nabla_v g(z_4) - \nabla_v g(z_0)| &\lesssim |\iota_{R_2}[g](z_4) - \iota_{R_2}[g](z_0)| + R_2 \|D_v^2 g\|_{L^\infty(\mathcal{Q})} \\ &\lesssim \frac{r^2}{R_2} \|\partial_t g + v \cdot \nabla_x g\|_{L^\infty(\mathcal{Q})} + R_2 \|D_v^2 g\|_{L^\infty(\mathcal{Q})} \end{aligned}$$

and thus optimizing in R_2:

$$|\nabla_v g(z_4) - \nabla_v g(z_0)| \lesssim r \|\partial_t g + v \cdot \nabla_x g\|_{L^\infty(\mathcal{Q})}^{1/2} \|D_v^2 g\|_{L^\infty(\mathcal{Q})}^{1/2} \lesssim r \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}. \tag{2.10}$$

Combining the three previous inequalities (2.8), (2.9), (2.10) yields

$$|\nabla_v g(z_4) - \nabla_v g(z_2)| \lesssim \frac{2}{R_1} A + R_1 \|D_v^2 g\|_{L^\infty(\mathcal{Q})} + r \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}. \tag{2.11}$$

Plugging into (2.6) gives

$$A \lesssim r \left(\frac{2}{R_1} A + R_1 \|D_v^2 g\|_{L^\infty(\mathcal{Q})} + r \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})} \right) + r^2 \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}.$$

Choose $R_1 := 4r$ to get

$$A \lesssim r \left(r \|D_v^2 g\|_{L^\infty(\mathcal{Q})} + r \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})} \right) + r^2 \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}.$$

We conclude therefore that

$$|g(z_1) - g(z_0)| \le r^2 \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}.$$

We then estimate by a single Taylor expansion the variation along the v variable: for $|w| \le 1$, we have

$$|g(t,x,v+rw) - g(t,x,v)| \le r \|\nabla_v g\|_{L^\infty(\mathcal{Q})} \le r \left(\|g\|_{L^\infty(\mathcal{Q})} + \|D_v^2 g\|_{L^\infty(\mathcal{Q})} \right) \lesssim r \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}$$

and then the variation along free streaming:

$$|g(t+r^2, x+r^2 v, v) - g(t,x,v)| \le r^2 \|\partial_t g + v \cdot \nabla_v g\|_{L^\infty(\mathcal{Q})}.$$

Combine the last three inequalities to obtain equation (2.4).

$$[g]_{\mathcal{C}^{0,1}(\mathcal{Q})} \lesssim \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}. \tag{2.12}$$

Finally to prove the last equation (2.5), use the estimates (2.8)-(2.9)-(2.10) on the variations of $\nabla_v g$ already established: along x directions one gets

$$\begin{aligned}|\nabla_v g(z_1) - \nabla_v g(z_0)| &\lesssim \frac{2}{R_1} A + R_1 \|D_v^2 g\|_{L^\infty(\mathcal{Q})} \\ &\lesssim \frac{2r^2}{R_1}[g]_{\mathcal{H}^\alpha(\mathcal{Q})} + R_1 \|g\|_{\mathcal{H}^\alpha(\mathcal{Q})} \lesssim r\|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}\end{aligned}$$

by choosing $R_1 = r$. Combined with equations (2.9)-(2.10) this yields equation (2.5). □

We next consider a second semi-norm which is based on measuring the oscillation of the difference of g with a polynomial of order 1 in time, 0 in space and 2 in the velocity variable.

Definition 2.9. The semi-norm $[\cdot]_{\mathcal{P}^\alpha(\mathcal{Q})}$ of g on $\mathcal{Q}$ is defined as the smallest constant $N > 0$ s.t.

$$\forall\, z_0 \in \mathcal{Q},\ r \in (0,1) \text{ s.t. } Q_r(z_0) \subset \mathcal{Q}, \quad \inf_{P \in \mathbb{P}} \|g - P\|_{L^\infty(Q_r(z_0))} \le N r^{2+\alpha} \tag{2.13}$$

where

$$\mathbb{P} := \left\{ P(t,v) = a + bt + q\cdot v + \frac{1}{2} Av\cdot v \ \text{ for some } \ a,b \in \mathbb{R},\ q \in \mathbb{R}^d,\ A \in \mathbb{R}^{d\times d} \right\}.$$

Lemma 2.10 (One-sided control of $\mathcal{H}^\alpha(\mathcal{Q})$ by oscillations)**.** *There exists a constant $C = C(d,\alpha)$ such that for all $g \in \mathcal{H}^\alpha(\mathcal{Q})$, we have $[g]_{\mathcal{H}^\alpha(\mathcal{Q})} \le C[g]_{\mathcal{P}^\alpha(\mathcal{Q})}$.*

Proof. Part of this inequality is proved following [25, Theorem 8.5.2]: indeed the proof of $[D_v^2 g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})} \lesssim [g]_{\mathcal{P}^\alpha(\mathcal{Q})}$ is exactly similar. What remains to be proved is

$$[(\partial_t + v\cdot\nabla_x) g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})} \le C[g]_{\mathcal{P}^\alpha(\mathcal{Q})}.$$

Consider $r > 0$ and $z = (t,x,v)$ such that z and $(t-r, x-rv, v) \in \mathcal{Q}$ and define

$$\sigma_r(g)(z) := \frac{1}{r^2}\left[g(z) - g(t-r^2, x-r^2 v, v)\right]$$

(observe that the point $(t-r^2, x-r^2v, v) = (t,x,v)\circ r(-1,0,0)$ belongs to $\mathcal{Q}_r((t,x,v))$). For all $z \in \mathcal{Q}$ and $r > 0$ so that $Q_r(z) \subset \mathcal{Q}$, there exists $\theta \in (0,1)$ such that

$$\sigma_r(g)(z) = (\partial_t + v\cdot\nabla_x) g(t - \theta r^2, x - \theta r^2 v, v).$$

In particular

$$|(\partial_t + v\cdot\nabla_x) g(z) - \sigma_r(g)(z)| \le r^\alpha [(\partial_t + v\cdot\nabla_x) g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})}.$$

Consider $z_1 \in Q_{Kr}(z_0) \subset Q_{(K+2)r}(z_0) \subset \mathcal{Q}$ for some $K > 1$ to be fixed later, and $P \in \mathbb{P}$. The function $\sigma_r(P)$ is constant and $|\sigma_r(g-P)(z_i)| \le \frac{C}{r^2}|g-P|_{L^\infty(Q_{3r}(z_0))}$ for $i = 0,1$. Hence

$$\begin{aligned}|(\partial_t + v\cdot\nabla_x) g(z_1) - (\partial_t + v\cdot\nabla_x) g(z_0)| &\le \sum_{i=0,1} \left\{ |(\partial_t + v\cdot\nabla_x) g(z_i) - \sigma_r(g)(z_i)| + |\sigma_r(g-P)(z_i)| \right\} \\ &\le 2r^\alpha [(\partial_t + v\cdot\nabla_x) g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})} + \frac{C}{r^2}\|g-P\|_{L^\infty(Q_{(K+2)r}(z_0))}.\end{aligned}$$

Taking the infimum over $P \in \mathbb{P}$ results in

$$|(\partial_t + v\cdot\nabla_x) g(z_1) - (\partial_t + v\cdot\nabla_x) g(z_0)| \le 2r^\alpha [(\partial_t + v\cdot\nabla_x) g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})} + C' r^\alpha [g]_{\mathcal{P}^\alpha(\mathcal{Q})}.$$

Finally we deduce by taking the supremum over r and z_0, z_1 that

$$[(\partial_t + v\cdot\nabla_x) g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})} \le \frac{2}{K^\alpha}[(\partial_t + v\cdot\nabla_x) g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})} + C'[g]_{\mathcal{P}^\alpha(\mathcal{Q})}$$

and choosing $K > 2^{1/\alpha}$ large enough concludes the proof. □

To understand better the semi-norm $[g]_{\mathcal{P}^\alpha(\mathcal{Q})}$, let us prove that the polynomial P realising the infimum is the expected Taylor expansion. Let us recall and denote

$$[g]_{\mathcal{P}^\alpha(\mathcal{Q})} = \sup_{\{z_0\in\mathcal{Q},\ r\in(0,1)|Q_r(z_0)\subset\mathcal{Q}\}} \frac{\inf_{P\in\mathbb{P}}\|g-P\|_{L^\infty(Q_r(z_0))}}{r^{2+\alpha}}$$

$$[g]_{\mathcal{P}_0^\alpha(\mathcal{Q})} := \sup_{\{z_0\in\mathcal{Q},\ r\in(0,1)|Q_r(z_0)\subset\mathcal{Q}\}} \frac{\|g-\mathcal{T}_{z_0}[g]\|_{L^\infty(Q_r(z_0))}}{r^{2+\alpha}}$$

where $\mathcal{T}_{z_0}[g](t,x,v) := g(z_0) + (t-t_0)[\partial_t g + v_0\cdot\nabla_x g](z_0) + v\cdot\nabla_v g(z_0) + \frac{1}{2}v^T\cdot D_v^2 g(z_0)\cdot v$ is the Taylor expansion of g at z_0 along free streaming at order one and v at order two.

Lemma 2.11 (Characterization of the oscillation semi-norm)**.** *Given $g\in\mathcal{H}^\alpha(\mathcal{Q})$, there is a constant $C\in(0,1)$ s.t. $C[g]_{\mathcal{P}_0^\alpha(\mathcal{Q})}\le[g]_{\mathcal{P}^\alpha(\mathcal{Q})}\le[g]_{\mathcal{P}_0^\alpha(\mathcal{Q})}$.*

Proof. First reduce to $z_0=0$ by the change of variables $g^\sharp(z):=g(z_0\circ z)$. We continue however to simply call the function g. The second inequality $[g]_{\mathcal{P}^\alpha(\mathcal{Q})}\le[g]_{\mathcal{P}_0^\alpha(\mathcal{Q})}$ follows from $\mathcal{T}_0[g]\in\mathcal{P}$.

To prove the first inequality, one needs to identify the minimizer $P\in\mathbb{P}$ realising $\inf_{P\in\mathbb{P}}\|g-P\|_{L^\infty(Q_r)}$ in the limit $r\to0^+$. Let $\varepsilon>0$ and consider $r_k=2^{-k}$ and $P_k\in\mathbb{P}$ s.t. $\|g-P_k\|_{L^\infty(Q_{r_k})}\le r_k^{2+\alpha}([g]_{\mathcal{P}^\alpha(\mathcal{Q})}+\varepsilon)$. Write $P_k(t,v)=:a_k+b_kt+q_k\cdot v+\frac{1}{2}v^TA_k\cdot v$. By subtraction one gets

$$\|P_{k+1}-P_k\|_{L^\infty(Q_{r_{k+1}})}\le 2r_k^{2+\alpha}([g]_{\mathcal{P}^\alpha(\mathcal{Q})}+\varepsilon)$$

which writes in terms of the coefficients

$$\left\|(a_k-a_{k+1})+(b_k-b_{k+1})\,t+(q_k-q_{k+1})\cdot v+\frac{1}{2}v^T\cdot(A_k-A_{k+1})\cdot v\right\|_{L^\infty(Q_{r_{k+1}})}\le 2r_k^{2+\alpha}\left([g]_{\mathcal{P}^\alpha(\mathcal{Q})}+\varepsilon\right).$$

Testing for $t=0$ and $v=0$ gives $|a_k-a_{k+1}|\lesssim r_k^{2+\alpha}([g]_{\mathcal{P}^\alpha(\mathcal{Q})}+\varepsilon)$. Using the latter and testing for $v=0$ and $|t|=r_{k+1}^2$ gives $|b_k-b_{k+1}|\lesssim r_k^\alpha([g]_{\mathcal{P}^\alpha(\mathcal{Q})}+\varepsilon)$. Testing for $t=0$ and summing v and $-v$ with $|v|=r_{k+1}$ in all directions gives $|A_k-A_{k+1}|\lesssim r_k^\alpha([g]_{\mathcal{P}^\alpha(\mathcal{Q})}+\varepsilon)$. Finally by difference and testing with $t=0$ and all directions of $|v|=r_k$, one gets $|q_k-q_{k+1}|\lesssim r_k^{1+\alpha}([g]_{\mathcal{P}^\alpha(\mathcal{Q})}+\varepsilon)$. This shows that the coefficients are converging with $|a_k-a_\infty|\lesssim r_k^{2+\alpha}([g]_{\mathcal{P}^\alpha(\mathcal{Q})}+\varepsilon)$ and $|b_k-b_\infty|\lesssim r_k^\alpha([g]_{\mathcal{P}^\alpha(\mathcal{Q})}+\varepsilon)$ and $|q_k-q_\infty|\lesssim r_k^{1+\alpha}([g]_{\mathcal{P}^\alpha(\mathcal{Q})}+\varepsilon)$ and $|A_k-A_\infty|\lesssim r_k^\alpha([g]_{\mathcal{P}^\alpha(\mathcal{Q})}+\varepsilon)$. These convergences and estimates imply that $\|g-P_\infty\|_{L^\infty(Q_{r_k})}\lesssim r_k^{2+\alpha}([g]_{\mathcal{P}^\alpha(\mathcal{Q})}+\varepsilon)$ which in turn implies that $a_\infty=g(0,0,0)$ and $b_\infty=\partial_t g(0,0,0)$ and $q_\infty=\nabla_v g(0,0,0)$ and $A_\infty=D_v^2g(0,0,0)$. We thus proved that $\|g-\mathcal{T}_0[g]\|_{L^\infty(Q_{r_k})}\lesssim r_k^{2+\alpha}([g]_{\mathcal{P}^\alpha(\mathcal{Q})}+\varepsilon)$ where the constant does not depend on k. This in turn implies that same inequality for any $r>0$, with a constant at most multiplied by 2, which concludes the proof since ε is arbitrarily small. □

The following interpolation inequalities are needed later in the proofs:

Lemma 2.12 (Interpolation inequalities)**.** *Let $g\in\mathcal{H}^\alpha(\mathcal{Q})$ with $\alpha\in(0,1]$ and $\varepsilon>0$.*

$$\|g\|_{\mathcal{C}^{0,\alpha}(\mathcal{Q})} \lesssim \varepsilon^{\frac{1-\alpha}{\alpha}}[g]_{\mathcal{C}^{0,1}(\mathcal{Q})}+\varepsilon^{-1}\|g\|_{L^\infty(\mathcal{Q})} \tag{2.14}$$

$$\|(\partial_t+v\cdot\nabla_x)g\|_{L^\infty(\mathcal{Q})} \le \varepsilon^\alpha[g]_{\mathcal{H}^\alpha(\mathcal{Q})}+C\varepsilon^{-2}\|g\|_{L^\infty(\mathcal{Q})}, \tag{2.15}$$

$$\|D_v^2g\|_{L^\infty(\mathcal{Q})} \le \varepsilon^\alpha[g]_{\mathcal{H}^\alpha(\mathcal{Q})}+C\varepsilon^{-2}\|g\|_{L^\infty(\mathcal{Q})}, \tag{2.16}$$

$$[g]_{\mathcal{C}^{0,\alpha}(\mathcal{Q})} \le \varepsilon^{\frac{1-\alpha}{\alpha}}\|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}+C\varepsilon^{-1}\|g\|_{L^\infty(\mathcal{Q})}, \tag{2.17}$$

$$\|\nabla_v g\|_{\mathcal{C}^{0,\alpha}(\mathcal{Q})} \le \varepsilon^{\frac{1-\alpha}{\alpha}}\|g\|_{\mathcal{H}^\alpha(\mathcal{Q})}+C\varepsilon^{-\frac{1+\alpha}{\alpha}}\|g\|_{L^\infty(\mathcal{Q})}. \tag{2.18}$$

Proof. Let C_1 denote $[g]_{\mathcal{C}^{0,1}(\mathcal{Q})}$. Then for $z\in Q_r(z_0)\subset\mathcal{Q}$ and $r\le\varepsilon^{\frac{1}{\alpha}}$, we get

$$|g(z)-g(z_0)|\le C_1r\le C_1r^\alpha\varepsilon^{\frac{1-\alpha}{\alpha}}.$$

If now $r\ge\varepsilon^{\frac{1}{\alpha}}$, then

$$|g(z)-g(z_0)|\le 2\|g\|_{L^\infty(\mathcal{Q})}\varepsilon^{-1}r^\alpha.$$

In all cases, we thus have

$$|g(z)-g(z_0)|\le(C_1\varepsilon^{\frac{1-\alpha}{\alpha}}+2\|g\|_{L^\infty(\mathcal{Q})}\varepsilon^{-1})r^\alpha$$

which yields (2.14).

We prove the next two inequalities as in [25, Theorem 8.8.1]. Consider $\varepsilon > 0$ and write

$$\begin{aligned}|(\partial_t + v\cdot\nabla_x)g(z)| &\le \left|(\partial_t + v\cdot\nabla_x)g(z) - \varepsilon^{-2}\left[g(t+\varepsilon^2, x+\varepsilon^2 v, v) - g(t,x,v)\right]\right| + 2\varepsilon^{-2}\|g\|_{L^\infty(\mathcal{Q})}\\ &\le |(\partial_t + v\cdot\nabla_x)g(z) - (\partial_t + v\cdot\nabla_x)g(t+\theta\varepsilon^2, x+\theta\varepsilon^2 v, v)| + 2\varepsilon^{-2}\|g\|_{L^\infty(\mathcal{Q})}\\ &\le \varepsilon^\alpha [g]_{\mathcal{H}^\alpha(\mathcal{Q})} + 2\varepsilon^{-2}\|g\|_{L^\infty(\mathcal{Q})}\end{aligned}$$

for some $\theta \in [0,1]$, and

$$\begin{aligned}|D_v^2 g(z)| &\le \left|D_v^2 g(z) - \varepsilon^{-2}\left[g(t,x,v+\varepsilon) + g(t,x,v-\varepsilon) - 2g(t,x,v)\right]\right| + 4\varepsilon^{-2}\|g\|_{L^\infty(\mathcal{Q})}\\ &\le \left|D_v^2 g(z) - \frac{1}{2}D_v^2 g(t,x,v+\theta\varepsilon) - \frac{1}{2}D_v^2 g(t,x,v-\theta'\varepsilon)\right| + 4\varepsilon^{-2}\|g\|_{L^\infty(\mathcal{Q})}\\ &\le \varepsilon^\alpha [g]_{\mathcal{H}^\alpha(\mathcal{Q})} + 4\varepsilon^{-2}\|g\|_{L^\infty(\mathcal{Q})}\end{aligned}$$

for some $\theta,\theta' \in [0,1]$.

Equation (2.17) is obtained by combining (2.14) with the Hölder regularity (2.4) while Equation (2.18) proceeds from (2.14) for $\nabla_v g$ and

$$\|\nabla_v g\|_{L^\infty(\mathcal{Q})} \lesssim \varepsilon^{-\frac{1}{\alpha}}\|g\|_{L^\infty(\mathcal{Q})} + \varepsilon^{\frac{1}{\alpha}}\|D_v^2 g\|_{L^\infty(\mathcal{Q})}$$

and the Hölder regularity (2.5). ☐

3. The abstract Schauder estimate

In this section, we denote $\mathcal{L} := (\partial_t + v\cdot\nabla_x) - a^{i,j}\partial^2_{v_i v_j} - b^i\partial_{v_i} - c$ and consider equations of the form $\mathcal{L}g = S$ i.e.

$$(\partial_t + v\cdot\nabla_x)g = a^{i,j}\partial^2_{v_i v_j}g + b^i\partial_{v_i}g + cg + S \tag{3.1}$$

where $S \in C^\alpha(\mathbb{R}^{2d+1})$ and the diffusion matrix $A = (a_{i,j}(t,x,v))_{i,j}$ is *strictly positive*

$$\forall\,(t,x,v) \in (0,+\infty)\times\mathbb{R}^{2d}, \quad a^{i,j}(t,x,v)\xi_i\xi_j \ge \lambda|\xi|^2. \tag{3.2}$$

Theorem 3.1 (Schauder estimate for Hölder continuous coefficients)**.** *Given $\alpha \in (0,1)$ and $g \in \mathcal{H}^\alpha(\mathbb{R}^{2d+1})$ and $a^{i,j}, b^i, c \in \mathcal{C}^{0,\alpha}(\mathbb{R}^{2d+1})$ satisfying (3.2) for some constant $\lambda > 0$, then*

$$\|g\|_{\mathcal{H}^\alpha} \le C\|\mathcal{L}g + g\|_{\mathcal{C}^{0,\alpha}}$$

where $\mathcal{L} := (\partial_t + v\cdot\nabla_x) - a^{i,j}\partial^2_{v_iv_j} - b^i\partial_{v_i} - c$ and the constant C depends on d, λ and α and $\|a\|_{\mathcal{C}^{0,\alpha}}$, $\|b\|_{\mathcal{C}^{0,\alpha}}$, $\|c\|_{\mathcal{C}^{0,\alpha}}$.

Remark 3.2. This is the counterpart to [25, Theorem 8.9.2, p. 127].

3.1. Gradient bounds for the Kolmogorov equation.

We follow and extend the method of Safonov [32] presented in Krylov's book [25]. We start with the case of constant coefficients.

Proposition 3.3 (Gradient bounds)**.** *Consider g solution to (2.2) in $Q_1 = (-1,0]\times B_1\times B_1$, then*

$$|\partial_{x_i}g(0,0,0)| + |\partial_{v_i}g(0,0,0)| \lesssim_d \|g\|_{L^\infty(Q_1)} + \|S\|_{L^\infty(Q_1)} + \|\partial_{x_i}S\|_{L^\infty(Q_1)} + \|\partial_{v_i}S\|_{L^\infty(Q_1)}.$$

Remark 3.4. See also [20, 8] for gradient estimates.

Proof. We use Bernstein's method as Krylov does in [25] in the elliptic-parabolic case, combined with methods from hypocoercivity theory (see for instance [38]) in order to control the full (x,v)-gradient of the solution: see the construction of the quadratic form w in $\partial_{x_i}g$ and $\partial_{v_i}g$ below.

Denote the Kolmogorov operator $\mathcal{L}_K g := \partial_t g + v\cdot\nabla_x g - \Delta_v g$ and compute the following defaults of distributivity of the operator[1]

$$\mathcal{L}_K(g_1g_2) = g_1\mathcal{L}_K g_2 + g_2\mathcal{L}_K g_1 - 2\nabla_v g_1\cdot\nabla_v g_2, \qquad \mathcal{L}_K(g^2) = 2g\mathcal{L}_K g - 2\left|\nabla_v g\right|^2. \tag{3.3}$$

In order to get the desired estimate, it is enough to find a cut-off function $0 \le \zeta \in C^\infty$ with support in $(-1,0]\times B_1\times B_1$ and $\zeta(0,0,0) = 1$, and $\nu_0,\nu_1 > 0$ and $0 < A \le B$ and $0 < C < AB$ such that, for any $i \in \{1,\dots,d\}$,

$$w = \nu_0 g^2 - \nu_1 t + \left[A^2\zeta^4(\partial_{x_i}g)^2 + C\zeta^3(\partial_{x_i}g)(\partial_{v_i}g) + B^2\zeta^2(\partial_{v_i}g)^2\right]$$

[1] This calculation is reminiscent of the “carré du champ” approach and Γ-calculus going back to [7].

satisfies

$$-\mathcal{L}_K w \geq 0.$$

Indeed, the maximum principle for parabolic equations then implies that $\sup_{Q_1} w = \sup_{\partial_p Q_1} w$ where $\partial_p Q_1 = \{-1\} \times B_1 \times B_1 \cup [-1,0] \times S_1 \times S_1$ (parabolic boundary). Since $\zeta \equiv 0$ in $\partial_p Q_1$ and $\zeta(0,0,0) = 1$, we get

$$\begin{aligned}
A^2 \left[(\partial_{x_i} g)^2 + (\partial_{v_i} g)^2\right](0,0,0) &\leq \left[A^2(\partial_{x_i} g)^2 + B^2(\partial_{v_i} g)^2\right](0,0,0) + 2\nu_0 g^2(0,0,0) \\
&\leq 2\left[A^2(\partial_{x_i} g)^2 + C(\partial_{x_i} g \partial_{v_i} g) + B^2(\partial_{v_i} g)^2\right](0,0,0) + 2\nu_0 g^2(0,0,0) \\
&\leq 2\left[A^2\zeta^4(\partial_{x_i} g)^2 + C\zeta^3(\partial_{x_i} g)(\partial_{v_i} g) + B^2\zeta^2(\partial_{v_i} g)^2\right](0,0,0) + 2\nu_0 g^2(0,0,0) \\
&\leq 2w(0,0,0) \leq 2\left(\nu_0 \sup_{Q_1} g^2 + \nu_1\right). \qquad (3.4)
\end{aligned}$$

Observe first that $-\mathcal{L}_K(-\nu_1 t) = \nu_1$. Compute second $-\mathcal{L}_K(\nu_0 g^2)$ using (3.3)

$$-\mathcal{L}_K(\nu_0 g^2) = 2\nu_0 |\nabla_v g|^2 - 2S\nu_0 g. \qquad (3.5)$$

Compute third $-\mathcal{L}_K(\zeta^4(\partial_{x_i} g)^2)$ using that $\mathcal{L}_K(\partial_{x_i} g) = \partial_{x_i} S$ and (3.3)

$$\begin{aligned}
-\mathcal{L}_K(\zeta^4(\partial_{x_i} g)^2) &= 2\zeta^4|\nabla_v \partial_{x_i} g|^2 - (\partial_{x_i} g)^2 \mathcal{L}_K(\zeta^4) + 2\nabla_v(\zeta^4)\cdot\nabla_v(\partial_{x_i} g)^2 - 2\zeta^4 \partial_{x_i} g \partial_{x_i} S \\
&\geq \zeta^4|\nabla_v \partial_{x_i} g|^2 + (\partial_{x_i} g)^2\left[-\mathcal{L}_K(\zeta^4) - 2\zeta^{-4}|\nabla_v(\zeta^4)|^2\right] - \zeta^3(\partial_{x_i} g)^2 - \zeta^5(\partial_{x_i} S)^2. \qquad (3.6)
\end{aligned}$$

Compute fourth $-\mathcal{L}_K(\zeta^2(\partial_{v_i} g)^2)$ using $\mathcal{L}_K(\partial_{v_i} g) = \partial_{v_i} S - \partial_{x_i} g$ and (3.3)

$$\begin{aligned}
-\mathcal{L}_K(\zeta^2(\partial_{v_i} g)^2) &= 2\zeta^2|\nabla_v \partial_{v_i} g|^2 + 2\nabla_v(\zeta^2)\cdot\nabla_v(\partial_{v_i} g)^2 - (\partial_{v_i} g)^2\mathcal{L}_K(\zeta^2) + 2\zeta^2 \partial_{v_i} g \partial_{x_i} g - 2\zeta^2 \partial_{v_i} g \partial_{v_i} s \\
&\geq \zeta^2|\nabla_v \partial_{v_i} g|^2 + (\partial_{v_i} g)^2\left[-1 - \epsilon_1^{-1}\zeta - \mathcal{L}_K(\zeta^2) - 2\zeta^{-2}|\nabla_v \zeta|^2\right] - \epsilon_1 \zeta^3(\partial_{x_i} g)^2 - \zeta^4(\partial_{v_i} S)^2 \qquad (3.7)
\end{aligned}$$

for some $\epsilon_1 > 0$.

Compute fifth $-\mathcal{L}_K[\zeta^3(\partial_{x_i} g)(\partial_{v_i} g)]$, with the intermediate step

$$-\mathcal{L}_K[(\partial_{x_i} g)(\partial_{v_i} g)] = (\partial_{x_i} g)^2 + 2\nabla_v \partial_{x_i} g \cdot \nabla_v \partial_{v_i} g - \partial_{x_i} g \partial_{v_i} S - \partial_{v_i} f \partial_{x_i} S,$$

$$\begin{aligned}
-\mathcal{L}_K[\zeta^3(\partial_{x_i} g)(\partial_{v_i} g)] =& \zeta^3\left[(\partial_{x_i} g)^2 + 2\nabla_v \partial_{x_i} g \cdot \nabla_v \partial_{v_i} g\right] - (\partial_{x_i} g)(\partial_{v_i} g)\mathcal{L}_K(\zeta^3) \\
&+ 2\nabla_v(\zeta^3)\cdot\nabla_v[(\partial_{x_i} g)(\partial_{v_i} g)] - \zeta^3 \partial_{x_i} g \partial_{v_i} S - \zeta^3 \partial_{v_i} g \partial_{x_i} S \\
\geq& \frac{1}{2}\zeta^3(\partial_{x_i} g)^2 - \epsilon_2 \zeta^4 |\nabla_v \partial_{x_i} g|^2 - \epsilon_2^{-1}\zeta^2|\nabla_v \partial_{v_i} g|^2 - (\partial_{x_i} g)(\partial_{v_i} g)\mathcal{L}_K(\zeta^3) \qquad (3.8) \\
&+ 2\nabla_v(\zeta^3)\cdot\nabla_v[(\partial_{x_i} g)](\partial_{v_i} g) + 2\nabla_v(\zeta^3)\cdot\nabla_v[(\partial_{v_i} g)](\partial_{x_i} g) \\
&- \frac{1}{2}\zeta^3(\partial_{v_i} S)^2 - \frac{1}{2}\zeta^3(\partial_{v_i} g)^2 - \frac{1}{2}\zeta^3(\partial_{x_i} S)^2
\end{aligned}$$

for some $\epsilon_2 > 0$.

To clean a little the calculations, observe that (1) any error term involving the source term is controlled –whatever power of ζ it is multiplied by– by choosing ν_1 large enough, (2) any term involving the square of a first-order v-derivative or a product of a first-order v-derivative with another derivative appearing in our positive terms is controlled by choosing ν_0 large enough, (3) the term g is controlled simply by the sup norm. Observe that then equation (3.5) is free (i.e. not involved in any constant dependency) as well as (crucially) equation (3.7) by choosing ϵ small enough so that the term $-\epsilon\zeta^3(\partial_{x_i} g)^3$ is cancelled by the last equation. Equation (3.6) has an error term of the form $-O(1)\zeta^3(\partial_{x_i} g)^2$ that must be cancelled by equation (3.8). We use here $|-\mathcal{L}_K(\zeta^4) - 2\zeta^{-4}|\nabla_v(\zeta^4)|^2| \lesssim \zeta^3$. In the last equation (3.8) we also split $-(\partial_{x_i} g)(\partial_{v_i} g)\mathcal{L}_K(\zeta^3) \lesssim \frac{1}{8}(\partial_{x_i} g)^2 + O(1)(\partial_{v_i} g)^2$, and $2C\nabla_v(\zeta^3)\cdot\nabla_v[(\partial_{x_i} g)](\partial_{v_i} g) \lesssim \epsilon_3 C\zeta^4|\nabla_v[(\partial_{x_i} g)]|^2 + \epsilon_3^{-1} O(1)(\partial_{v_i} g)^2$ for some $\epsilon_3 > 0$ and $2C\nabla_v(\zeta^3)\cdot\nabla_v[(\partial_{v_i} g)](\partial_{x_i} g) \lesssim CO(1)\zeta^2|\nabla_v[(\partial_{v_i} g)](\partial_{x_i} g)|^2 + \frac{C}{4}\zeta^3(\partial_{x_i} g)^2$, where we have used a cutoff function ζ such that its derivatives satisfy $|\nabla\zeta| \lesssim \zeta^{1/2}$.

These considerations result in the following calculations:

$$\begin{aligned}
&-\mathcal{L}_K w \geq \\
&2\nu_0|\nabla_v g|^2 + \nu_1 + A^2\zeta^4|\nabla_v \partial_{x_i} g|^2 - A^2 O(1)\zeta^3(\partial_{x_i} g)^2 - A^2 O(1)(\partial_{x_i} S)^2 \\
&+ B^2\zeta^2|\nabla_v \partial_{v_i} g|^2 - B^2 O(1)(\partial_{v_i} g)^2 - B^2\epsilon\zeta^3(\partial_{x_i} g)^2 - O(1)(\partial_{v_i} S)^2 \\
&+ \frac{C}{4}\zeta^3(\partial_{x_i} g)^2 - C(\epsilon_2 + \epsilon_3)\zeta^4|\nabla_v \partial_{x_i} g|^2 - O(1)C\zeta^2|\nabla_v \partial_{v_i} g|^2 - CO(1)(\partial_{x_i, v_i} S)^2 - CO(1)(\partial_{v_i} g)^2.
\end{aligned}$$

We finally choose (1) $A = 1$, (2) C large enough so that the first term in the third line controls the fourth term in the first line, (3) ϵ_2 and ϵ_3 small enough so that the second term of the third line is controlled by the third term in the first line, (3) B large enough so that the third term in the third line is controlled by the first term in the second line and $AB > C$ so that the quadratic form is strictly positive, (4) ϵ_1 small enough so that the third term in the third line is controlled by the first term in the third line, (5) finally ν_0 and ν_1 large enough to control all the v-gradients of g and gradients of S. This proves that $-\mathcal{L}_K \omega \geq 0$ and the desired inequality is thus obtained from (3.4), which concludes the proof. □

3.2. **Proof of Schauder estimates.**

3.2.1. *Bounds on all derivatives.* As a direct consequence of Proposition 3.3:

Corollary 3.5 (Bounds on arbitrary derivatives around the origin)**.** *Given $k \in \mathbb{N}$, there exists a constant C depending on dimension d and k such that any solution of (2.2) in Q_r with zero source terme $S \equiv 0$ satisfies for all integer $n \geq 0$ and multi-indices $\alpha, \beta \in \mathbb{N}^d$ with $|\beta| = k$,*

$$|\partial_t^n D_x^\alpha D_v^\beta g(0,0,0)| \leq \frac{C\|g\|_{L^\infty(Q_r)}}{r^{2n+3|\alpha|+|\beta|}}$$

where $|\alpha| = \sum_i |\alpha_i|$ and $|\beta| = \sum_i |\beta_i|$.

Proof. We reduce to the case $r = 1$ by rescaling: the function $g_r(t,x,v) = g(r^2 t, r^3 x, rv)$ is a solution of (2.2) in Q_1. If the result is true for $r = 1$, then we get the desired estimate for arbitrary r's. We then first treat the case $n = 0$ and argue by induction on $|\beta|$. Proposition 3.3 yields the result for $|\beta| \leq 1$ since $D_x^\alpha g$ solves (2.2) with $S \equiv 0$ for an arbitrary multi-index α. Assuming the result true for $n = 0$, any α and $|\beta| \leq k$, remark that $D_x^\alpha D_v^\beta g$ solves (2.2) with $S = \sum_{i=1,\dots,d \,|\, \beta_i \geq 1} D_x^{\alpha+\delta_i} D_v^{\beta-\delta_i} g$. Consider $\beta \in \mathbb{N}^d$ with $|\beta| = k+1$; the previous step yields controls of $\partial_{x_j} S$ and $\partial_{v_j} S$ for the source term S in the equation for $D_x^\alpha D_v^\beta g$. Proposition 3.3 then gives the control $D_x^\alpha D_v^\beta g(0,0,0)$ which completes the induction. We finally get the result for an arbitrary $n \geq 1$ by remarking that the equation allows us to control any time derivatives by space and velocity derivatives. □

3.2.2. *The core estimate.*

Theorem 3.6. *Let $\alpha \in (0,1)$ and $g \in \mathcal{H}^\alpha(\mathbb{R}^{2d+1})$. Then*

$$[g]_{\mathcal{H}^\alpha} \lesssim_{d,\alpha} [g]_{\mathcal{P}^\alpha} \lesssim_{d,\alpha} [\mathcal{L}_K g]_{\mathcal{C}^{0,\alpha}}.$$

where $\mathcal{L}_K g = (\partial_t + v \cdot \nabla_x) g - \Delta_v g$.

Remark 3.7*.* This theorem is the counterpart of [25, Theorem 8.6.1 & Lemma 8.7.1].

Proof. First reduce to the case where $g \in C_c^\infty(\mathbb{R}^{2d+1})$ by mollification and truncation (as for instance in the proof of [25, Lemma 8.7.1, p. 122]). Define $S = \partial_t g + v \cdot \nabla_x g - \Delta_v g$ and reduce to the base point $z_0 = 0$ by considering the change of unknown $g^\sharp(z) := g(z_0 \circ z)$ (we however keep on calling the unknown g). Given $r > 0$ and $K \geq 1$ to be chosen later, consider $Q_{(K+1)r}$ and a cut-off function $\zeta \in \mathbb{C}_c^\infty$ such that $\zeta \equiv 1$ in $Q_{(K+1)r}$. Denote the Kolmogorov operator $\mathcal{L}_K := \partial_t + v \cdot \nabla_x - \Delta_v$, and define $\bar{S} := \mathcal{L}_K(\zeta \mathcal{T}_0 g)$, where the Taylor polynomial $\mathcal{T}_0 g$ of g at $(0,0,0)$ is defined as before. Decompose in $Q_{(K+1)r}$:

$$g - \mathcal{T}_0 g = g - \zeta \mathcal{T}_0 g = G \star (s - \bar{s}) = h_1 + h_2 \quad \text{with} \quad \begin{cases} h_1 := G \star (s - \bar{S}) \mathbf{1}_{Q_{(K+1)r}}, \\ h_2 := G \star (S - \bar{S}) \mathbf{1}_{Q^c_{(K+1)r}} \end{cases}$$

where $Q^c_{(K+1)r} = \mathbb{R}^{2d+1} \setminus Q_{(K+1)r}$ and G is the Green function of $\partial_t g + v \cdot \nabla_x g = \Delta_v g$ studied in Proposition 2.2: hence h_1 is the solution to $\partial_t h_1 + v \cdot \nabla_x h_1 = s - s(0,0,0)$ (observe that $\bar{S}(z) = \text{cst} = S(0,0,0)$ in $z \in Q_{(K+1)r}$) and h_2 is the solution to $\partial_t h_2 + v \cdot \nabla_x h_2 = S - \bar{S}$.

We next estimate

$$\|g - \mathcal{T}_0 g - \mathcal{T}_0 h_2\|_{L^\infty(Q_r)} \leq \|h_2 - \mathcal{T}_0 h_2\|_{L^\infty(Q_r)} + \|h_1\|_{L^\infty(Q_r)}. \tag{3.9}$$

Using Proposition 2.2, we get

$$\|h_1\|_{L^\infty(Q_r)} \lesssim (K+1)^{2+\alpha} r^{2+\alpha} [S]_{\mathcal{C}^{0,\alpha}(Q_{(K+1)r})}. \tag{3.10}$$

Now for $z = (r^2t, r^3x, rv) \in Q_r$ with $(t,x,v) \in Q_1$. There exists $\theta_1, \theta_2, \theta_3 \in (0,1)$ such that

$$\begin{aligned} h_2(z) =& h_2(r^2t, r^3x, rv) \\ =& h_2(r^2t, 0, rv) + (r^3x)\cdot \nabla_x h_2(r^2t, r^3\theta_1 x, rv) \\ =& h_2(0,0,rv) + (r^3x)\cdot \nabla_x h_2(r^2t, r^3\theta_1 x, rv) + r^2 t \partial_t h_2(\theta_2 r^2 t, 0, rv) \\ =& h_2(0,0,0) + (r^3x)\cdot \nabla_x h_2(r^2t, r^3\theta_1 x, rv) + r^2 t \partial_t h_2(\theta_2 r^2 t, 0, rv) \\ & + \nabla_v h_2(0,0,0)\cdot (rv) + \frac{1}{2}(rv)^T \cdot D_v^2 h_2(0,0,r\theta_3 v)\cdot (rv). \end{aligned}$$

As a consequence

$$\begin{aligned} \|h_2 - \mathcal{T}_0 h_2\|_{L^\infty(Q_r)} \le r^2 \Big[& r\|\nabla_x h_2\|_{L^\infty(Q_{(K+1)r})} + r^2 \|\partial_t^2 h_2\|_{L^\infty(Q_{(K+1)r})} \\ & + r\|\partial_t \nabla_v h_2\|_{L^\infty(Q_{(K+1)r})} + r\|D_v^3 h_2\|_{L^\infty(Q_{(K+1)r})}\Big]. \end{aligned}$$

We remark that h_2 satisfies (2.2) with $S \equiv 0$ in $Q_{(K+1)r}$. We thus can apply Corollary 3.5 and get

$$\begin{aligned} \|h_2 - \mathcal{T}_0 h_2\|_{L^\infty(Q_r)} &\lesssim r^2 \left[\frac{r}{((K+1)r)^3} + \frac{r^2}{((K+1)r)^4} + \frac{r}{((K+1)r)^3} + \frac{r}{((K+1)r)^3}\right] \|h_2\|_{L^\infty(Q_{(K+1)r})} \\ &\lesssim (K+1)^{-3}\|h_2\|_{L^\infty(Q_{(K+1)r})}. \end{aligned}$$

Since $g - \mathcal{T}_0 g = h_1 + h_2$, we can estimate $\|h_2\|_{L^\infty(Q_{(K+1)r})}$ as follows

$$\begin{aligned} \|h_2\|_{L^\infty(Q_{(K+1)r})} &\le \|h_1\|_{L^\infty(Q_{(K+1)r})} + \|g - \mathcal{T}_0 g\|_{L^\infty(Q_{(K+1)r})} \\ &\lesssim (K+1)^{2+\alpha} r^{2+\alpha}\left([S]_{\mathcal{C}^{0,\alpha}(Q_{(K+1)r})} + [g]_{\mathcal{P}_0^\alpha(Q_{(K+1)r})}\right) \end{aligned}$$

(we used (3.10)) and get

$$\|h_2 - \mathcal{T}_0 h_2\|_{L^\infty(Q_r)} \le C \frac{r^{2+\delta}}{(K+1)^{1-\delta}} \left([S]_{\mathcal{C}^{0,\alpha}(Q_{(K+1)r})} + [g]_{\mathcal{P}_o^\alpha(Q_{(K+1)r})}\right). \tag{3.11}$$

Combining (3.9), (3.10) and (3.11) and Lemma 2.11, we get

$$\begin{aligned} &\inf_{P\in\mathbb{P}} \|g-P\|_{L^\infty(Q_r)} \le \|g - \mathcal{T}_0 g - \mathcal{T}_0 h_2\|_{L^\infty(Q_r)} \\ &\le \|h_2 - \mathcal{T}_0 h_2\|_{L^\infty(Q_r)} + \|h_1\|_{L^\infty(Q_r)} \\ &\lesssim (K+1)^{2+\alpha} r^{2+\alpha} [S]_{\mathcal{C}^{0,\alpha}(Q_{(K+1)r})} + \frac{r^{2+\delta}}{(K+1)^{1-\delta}}[S]_{\mathcal{C}^{0,\alpha}(Q_{(K+1)r})} + (K+1)^{-(1-\delta)}[g]_{\mathcal{P}_0^\alpha(Q_{(K+1)r})} \\ &\lesssim (K+1)^{2+\alpha} r^{2+\alpha} [S]_{\mathcal{C}^{0,\alpha}(Q_{(K+1)r})} + \frac{r^{2+\delta}}{(K+1)^{1-\delta}}[S]_{\mathcal{C}^{0,\alpha}(Q_{(K+1)r})} + (K+1)^{-(1-\delta)}[g]_{\mathcal{P}_0^\alpha(Q_{(K+1)r})} \end{aligned}$$

and by setting K large enough so that $C(K+1)^{-(1-\delta)} \le \frac{1}{2}$ where C is the constant in this inequality, it results into

$$\inf_{P\in\mathbb{P}} \|g-P\|_{L^\infty(Q_r)} \lesssim r^{2+\alpha}[S]_{\mathcal{C}^{0,\alpha}(Q_{(K+1)r})}$$

and thus taking the supremum on r yields $[g]_{\mathcal{P}^\alpha} \lesssim [S]_{\mathcal{C}^{0,\alpha}}$ which concludes the proof. □

3.2.3. *Maximum principle.* Combining Theorem 3.6 with the maximum principle, and interpolation inequalities (2.15)-(2.16), we also obtain the estimate of the complete $\mathcal{H}^\alpha$-norm.

Corollary 3.8 (Schauder estimate for the Kolmogorov equation)**.** *Let $\alpha \in (0,1)$ and $g \in \mathcal{H}^\alpha(\mathbb{R}^{2d+1})$. Then*

$$[g]_{\mathcal{H}^\alpha} \lesssim_{d,\alpha} [\mathcal{L}_K g]_{\mathcal{C}^{0,\alpha}} \quad \text{and} \quad \|g\|_{\mathcal{H}^\alpha} \lesssim_{d,\alpha} \|\mathcal{L}_K g + g\|_{\mathcal{C}^{0,\alpha}}$$

where $\mathcal{L}_K g = (\partial_t + v\cdot\nabla_x) g - \Delta_v g$.

Remark 3.9. This corresponds to [25, Theorem 8.7.2, p. 123].

Proof. The first inequality was proved in Theorem 3.6. Denote $S := \partial_t g + v\cdot\nabla_x g - \Delta_v g + g$ and observe that $\pm\|S\|_{L^\infty}$ are sub/super-solutions to the equation $\partial_t g + v\cdot\nabla_x g - \Delta_v g + g = S$, hence by maximum principle argument any solution g satisfies $-\|S\|_{L^\infty} \le g \le \|S\|_{L^\infty}$ and thus $\|g\|_{L^\infty(\mathcal{Q})} \le \|\partial_t g + v\cdot\nabla_x g - \Delta_v g + g\|_{L^\infty}$. The interpolation inequalities (2.15) and (2.16) then imply $\|(\partial_t + v\cdot\nabla_x)g\|_{L^\infty} + \|D_v^2 g\|_{L^\infty} \lesssim \|g\|_{L^\infty} + [g]_{\mathcal{H}^\alpha}$ which, combined with Theorem 3.6, concludes the proof. □

3.2.4. *Generalisation to constant diffusion coefficients.* Changing the diffusion coefficients is done through a change of variables.

Corollary 3.10 (Schauder estimates for Kolmogorov equations with constant diffusion coefficients)**.** *Let $A := (a^{ij})$ be a $d\times d$-matrix that satisfies (3.2) and $g \in \mathcal{H}^\alpha(\mathbb{R}^{2d+1})$ then*

$$[g]_{\mathcal{H}^\alpha} \lesssim [\mathcal{L}g]_{\mathcal{C}^{0,\alpha}} \quad \textit{and} \quad \|g\|_{\mathcal{H}^\alpha} \lesssim \|\mathcal{L}g + g\|_{\mathcal{C}^{0,\alpha}}$$

where the constant depends on d, α, $\|(a_{i,j})_{i,j}\|$ and λ in (3.2).

Remark 3.11. This is the counterpart to [25, Theorem 8.9.1, p. 127].

Proof. Consider the change of variables $\bar g(t,x,v) := g(t, M^{-1}x, M^{-1}v)$ with $M^2 = A$ where $A = (a_{ij})$. Then we have $\mathcal{L}\bar g(t,x,v) = \mathcal{L}_K g(t, M^{-1}x, M^{-1}v)$ and $(\mathcal{L}\bar g + \bar g)(t,x,v) = (\mathcal{L}_K g + g)(t, M^{-1}x, M^{-1}v)$ and the result follows from Corollary 3.8. □

3.2.5. *Proof of Theorem 3.1 (the Schauder a priori estimate).* We first treat the case where $b \equiv 0$ and $c \equiv 0$ by freezing coefficients. Let S denote $\mathcal{L}g$.

We consider a constant $\gamma > 0$ which will be fixed later, and pick two distinct $z_1, z_2 \in \mathbb{R}^{2d+1}$ such that

$$[(\partial_t + v\cdot\nabla_x)g]_{\mathcal{C}^{0,\alpha}} \le 2\frac{|(\partial_t + v\cdot\nabla_x)g(z_1) - (\partial_t + v\cdot\nabla_x)g(z_2)|}{R_{1,2}^\alpha}$$

where $R_{1,2}$ is the smallest $r > 0$ such that $z_1 \in Q_r(z_2)$.

If $R_{1,2} \ge \gamma$, then

$$[(\partial_t + v\cdot\nabla_x)g]_{\mathcal{C}^{0,\alpha}} \le 4\gamma^{-\alpha}\|(\partial_t + v\cdot\nabla_x)g\|_0 \le \frac14[g]_{\mathcal{H}^\alpha} + N(\gamma)\|g\|_{L^\infty} \le \frac14[g]_{\mathcal{H}^\alpha} + N(\gamma)\|S\|_{L^\infty}$$

with $N(\gamma)$ depending on γ and dimension. We used the interpolation inequality (2.15) to get the second inequality and the maximum principle to get the third one.

If now $R_{1,2} \le \gamma$, we consider a cut-off function $\zeta \in C_c^\infty(\mathbb{R}^{2d+1})$, $\zeta \in [0,1]$, such that $\zeta \equiv 1$ in Q_1 and $\zeta \equiv 0$ outside Q_2 and we define

$$\xi(z) = \zeta\left(\gamma^{-1} z_2^{-1} \circ z\right).$$

We now get from Corollary 3.10 that

$$\begin{aligned}
[(\partial_t + v\cdot\nabla_x)g]_{\mathcal{C}^{0,\alpha}} &\le 2\frac{|(\partial_t + v\cdot\nabla_x)g(z_1) - (\partial_t + v\cdot\nabla_x)g(z_2)|}{R_{1,2}^\alpha} \le 2[g\xi]_{\mathcal{H}^\alpha} \\
&\le C\|(\partial_t + v\cdot\nabla_x)(g\xi) - a^{ij}(z_1)\partial^2_{v_iv_j}(g\xi) + g\xi\|_{\mathcal{C}^{0,\alpha}} \\
&\le C\|(\partial_t + v\cdot\nabla_x)(g\xi) - a^{ij}\partial^2_{v_iv_j}(g\xi) + g\xi\|_{\mathcal{C}^{0,\alpha}} + C\|(a^{ij} - a^{ij}(z_1))\partial^2_{v_iv_j}(g\xi)\|_{\mathcal{C}^{0,\alpha}(Q_{2\gamma}(z_2))}
\end{aligned}$$

(we used the definition of the cut-off function ξ). We estimate successively the two terms of the right hand side. On the one hand,

$$\begin{aligned}
\|(\partial_t + v\cdot\nabla_x)(g\xi) - a^{ij}\partial^2_{v_iv_j}(g\xi) + g\xi\|_{\mathcal{C}^{0,\alpha}} \le& \|(\partial_t + v\cdot\nabla_x)(g) - a^{ij}\partial^2_{v_iv_j}g + g\|_{\mathcal{C}^{0,\alpha}} \\
&+ \|g\left\{(\partial_t + v\cdot\nabla_x)(\xi) - a^{ij}\partial^2_{v_iv_j}\xi\right\}\|_{\mathcal{C}^{0,\alpha}} + \|\nabla_v g\cdot\nabla_v\xi\|_{\mathcal{C}^{0,\alpha}} \\
&\le N(\gamma)(\|S\|_{\mathcal{C}^{0,\alpha}} + \|g\|_{\mathcal{C}^{0,\alpha}} + \|\nabla_v g\|_{\mathcal{C}^{0,\alpha}}) \\
&\le \gamma^\alpha[g]_{\mathcal{H}^\alpha} + N(\gamma)\|S\|_{\mathcal{C}^{0,\alpha}}
\end{aligned}$$

(we used the interpolation inequality (2.18)) for a constant $N(\gamma)$ depending on d, $\|a\|_{\mathcal{C}^{0,\alpha}}$, α. On the other hand,

$$\begin{aligned}
\|(a^{ij} - a^{ij}(z_1))\partial^2_{v_iv_j}(g\xi)\|_{\mathcal{C}^{0,\alpha}(Q_{2\gamma}(z_2))} &\le C\gamma^\alpha[D_v^2 g]_{\mathcal{C}^{0,\alpha}} + \|D_v^2 g\|_{L^\infty} \\
&\le C\gamma^\alpha[g]_{\mathcal{H}^\alpha} + N(\gamma)\|g\|_{L^\infty} \\
&\le C\gamma^\alpha[g]_{\mathcal{H}^\alpha} + N(\gamma)\|S\|_{L^\infty}
\end{aligned}$$

(we used the maximum principle). Combining the last three estimates yields (in the case $R_{1,2} \le \gamma$)

$$[(\partial_t + v \cdot \nabla_x) g]_{\mathcal{C}^{0,\alpha}} \le C\gamma^\alpha [g]_{\mathcal{H}^\alpha} + N(\gamma) \|S\|_{L^\infty}.$$

We thus get in both cases ($R_{1,2} \ge \gamma$ and $R_{1,2} \le \gamma$) that

$$[(\partial_t + v \cdot \nabla_x) g]_{\mathcal{C}^{0,\alpha}} \le \left(C\gamma^\alpha + \frac{1}{4} \right) [g]_{\mathcal{H}^\alpha} + N(\gamma) \|S\|_{L^\infty}.$$

Similarly, we get

$$[D_v^2 g]_{\mathcal{C}^{0,\alpha}} \le \left(C\gamma^\alpha + \frac{1}{4} \right) [g]_{\mathcal{H}^\alpha} + N(\gamma) \|S\|_{L^\infty}.$$

Summing up the two last estimates yields

$$[g]_{\mathcal{H}^\alpha} \le \left(C\gamma^\alpha + \frac{1}{2} \right) [g]_{\mathcal{H}^\alpha} + N(\gamma) \|S\|_{L^\infty}.$$

We now pick γ such that $C\gamma^\alpha + \frac{1}{2} \le \frac{3}{4}$ and we get the desired estimate in the case where $b \equiv 0$ and $c \equiv 0$.

The case $b \not\equiv 0$ and $c \not\equiv 0$ is then treated by interpolation (using inequalities (2.18) and (2.17)). The proof is now complete.

3.3. **Localization of the Schauder estimates.**

Theorem 3.12 (Localized Schauder estimate for Hölder continuous coefficients)**.** *Given $\alpha \in (0,1)$ and $g \in \mathcal{C}^{0,\alpha}(\mathbb{R}^{2d+1})$ and $a^{i,j}, b^i, c, S \in \mathcal{C}^{0,\alpha}(\mathbb{R}^{2d+1})$ satisfying (3.2) for some constant $\lambda > 0$ and $z_0 \in \mathbb{R}^{2d+1}$*

$$\|g\|_{\mathcal{H}^\alpha(Q_1(z_0))} \lesssim \|\mathcal{L}g + g\|_{\mathcal{C}^{0,\alpha}(Q_2(z_0))} + \|g\|_{L^\infty(Q_2(z_0))}.$$

where the constant depends on d, λ and α and $\|a\|_{\mathcal{C}^{0,\alpha}}$, $\|b\|_{\mathcal{C}^{0,\alpha}}$, $\|c\|_{\mathcal{C}^{0,\alpha}}$.

Proof. We use the strategy of [25, Theorem 8.11.1]. Consider $z_0 = 0$ without loss of generality and define $R_n := \sum_{j=0}^n 2^{-j}$ for $n \ge 0$. Define a cutoff function ζ_n that is smooth, one on Q_{R_n} and zero outside $Q_{R_{n+1}}$. It satisfies the controls $\|\zeta_n\|_{\mathcal{C}^{0,\alpha}}, \|v \cdot \nabla_x \zeta_n\|_{\mathcal{C}^{0,\alpha}}, \|\nabla_v \zeta_n\|_{\mathcal{C}^{0,\alpha}}, \|\nabla_v^2 \zeta_n\|_{\mathcal{C}^{0,\alpha}} \lesssim \rho^{-n}$ for some $\rho \in (0,1)$. Then apply the non-localized estimate of Theorem 3.1 to $\zeta_n g$:

$$\begin{aligned} A_n := \|g\|_{\mathcal{H}^\alpha(Q_{R_n})} \le & \|\zeta_n g\|_{\mathcal{H}^\alpha} \lesssim \|\mathcal{L}(\zeta_n g) + (\zeta_n g)\|_{\mathcal{C}^{0,\alpha}} \\ \lesssim & \|\mathcal{L}g + g\|_{\mathcal{C}^{0,\alpha}(Q_2)} + \rho^{-n} \left(\|g\|_{\mathcal{C}^{0,\alpha}(Q_{R_{n+1}})} + \|\nabla_v g\|_{\mathcal{C}^{0,\alpha}(Q_{R_{n+1}})} + \|g\|_{L^\infty(Q_{R_{n+1}})} \right) \\ & + \rho^{-n} \left(\|(\partial_t + v \cdot \nabla_x) g\|_{L^\infty(Q_{R_{n+1}})} + \|D_v^2 g\|_{L^\infty(Q_{R_{n+1}})} \right) \end{aligned}$$

and use the interpolation inequalities (2.15), (2.16), (2.17) and (2.18)

$$\begin{aligned} \Big(\|g\|_{\mathcal{C}^{0,\alpha}(Q_{R_{n+1}})} + \|\nabla_v g\|_{\mathcal{C}^{0,\alpha}(Q_{R_{n+1}})} + \|(\partial_t + v \cdot \nabla_x) g\|_{L^\infty(Q_{R_{n+1}})} + \|D_v^2 g\|_{L^\infty(Q_{R_{n+1}})} \Big) \\ \le \varepsilon A_{n+1} + \varepsilon^{-\beta} \|g\|_{L^\infty(Q_2)} \end{aligned}$$

for some $\beta > 0$. Choosing next $\varepsilon_n := \varepsilon_0 \rho^n$ for $\varepsilon_0 \in (0,1)$ small enough yields

$$A_n \lesssim \|\mathcal{L}g + g\|_{\mathcal{C}^{0,\alpha}(Q_2)} + \varepsilon_0 A_{n+1} + \varepsilon_0^{-\beta} \rho^{-\beta n} \|g\|_{L^\infty(Q_2)}$$

Consider then the geometric sum $\sum_{n \ge 0} \varepsilon_0^n A_n$, and calculate

$$\sum_{n \ge 0} \varepsilon_0^n A_n \lesssim \left(\sum_{n \ge 0} \varepsilon_0^n \right) \|\mathcal{L}g + g\|_{\mathcal{C}^{0,\alpha}(Q_2)} + \sum_{n \ge 0} \varepsilon_0^{n+1} A_{n+1} + \varepsilon_0^{-\beta} \sum_{n \ge 0} \left(\frac{\varepsilon_0}{\rho^\beta} \right)^n \|g\|_{L^\infty(Q_2)}.$$

Assuming $\varepsilon_0 < \rho^\beta < 1$ and cancelling terms gives finally:

$$A_0 \lesssim \|\mathcal{L}g + g\|_{\mathcal{C}^{0,\alpha}(Q_{R_2})} + \|g\|_{L^\infty(Q_2)}$$

which concludes the proof. ∎

4. Global existence for the toy model

We now go back to the toy model

$$\partial_t g + v \cdot \nabla_x g = \mathcal{R}[g]U[g] = \mathcal{R}[g]\left(\Delta_v g + \left(\frac{d}{2} - \frac{|v|^2}{4}\right) g\right) \tag{4.1}$$

where $\mathcal{R}[g] := \int_v g\mu^{1/2}\,\mathrm{d}v$. This section is devoted to the proof of Theorem 1.1. We first extend the Schauder estimate to this equation; it differs from the linear model case treated in the previous section by the unbounded coefficient in the right hand side. Second we prove local well-posedness in Sobolev spaces; the energy estimates in particular establish a continuation criterion governed by the $\mathcal{H}^\alpha$ norm (in the same spirit as the Beale-Kato-Majda blow-up criterion [9]). This is used in the fourth subsection to continue the solutions for all times. We finally prove the infinite regularisation for positive times in the fifth subsection.

4.1. **The Schauder estimate for the toy model.** The Schauder estimate follows from (1) the Hölder regularity established in [17], (2) the abstract Schauder estimate we just proved and (3) a Gaussian bounds on our solutions thanks to the maximum principle.

Proposition 4.1. *Consider g a weak solution to equation to*

$$\partial_t g + v \cdot \nabla_x g = \mathcal{R}[g]U[g] = \mathcal{R}[g]\left(\Delta_v g + \left(\frac{d}{2} - \frac{|v|^2}{4}\right) g\right) + S$$

in $L^\infty_t([0,T], L^2(\mathbb{T}^d \times \mathbb{R}^d)) \cap L^2_{t,x}([0,T]\times\mathbb{T}^d, H^1_v(\mathbb{R}^d))$ that satisfy $C_1\sqrt{\mu} \le g \le C_2\sqrt{\mu}$, then for any $\tau \in (0,T)$ there is $\alpha \in (0,1)$ and $\delta \in (0,1/4)$ and some constant C depending only on C_1, C_2, τ, d such that

$$\|g\|_{\mathcal{H}^\alpha([\tau,T]\times\mathbb{T}^d\times A_r)} \le Ce^{-\delta r^2} \tag{4.2}$$

where A_r is the annulus $\{v \in \mathbb{R}^d : r/2 \le |v| \le r\}$ and C depends on $\|\mu^{2\delta}S\|_{\mathcal{C}^{0,\alpha}}$.

Proof. It was proved in [17] that weak solutions $g \in L^\infty_t([0,T], L^2(\mathbb{T}^d \times \mathbb{R}^d)) \cap L^2_{t,x}([0,T]\times\mathbb{T}^d, H^1_v(\mathbb{R}^d))$ to $\partial_t g + v\cdot\nabla_x g = \nabla_v(A\nabla_v g) + S_0$ with A symmetric measurable matrix with $C_1 \le A \le C_2$ and S_0 source term in L^∞, satisfy (after rescaling)

$$\|g\|_{\mathcal{C}^{0,\alpha}(Q_r(z_0))} \lesssim r^{-\alpha}\left(\|g\|_{L^2(Q_{2r}(z_0))} + r^{-2}\|S_0\|_{L^\infty(Q_{2r}(z_0))}\right).$$

We apply this with $A = \mathcal{R}[g]\mathrm{Id}$ and $S_0 = (d/2 - |v|^2/4)g + S$: through a covering procedure, we get

$$\begin{aligned}\|g\|_{\mathcal{C}^{0,\alpha}([\tau,T]\times\mathbb{T}^d\times A_r)} \lesssim\ & r^{-\alpha}\|g\|_{L^2([\tau/2,T]\times\mathbb{T}^d\times\{r/4\le|v|\le 2r\})} \\ & + r^{-2-\alpha}\|(d/2 - |v|^2/4)g\|_{L^\infty([\tau/2,T]\times\mathbb{T}^d\times\{r/4\le|v|\le 2r\})} \\ & + r^{-2-\alpha}\|S\|_{L^\infty([\tau/2,T]\times\mathbb{T}^d\times\{r/4\le|v|\le 2r\})}.\end{aligned}$$

The Gaussian decay of g yields (4.2) for some constant C depending only on C_1, C_2, τ, d and some $\delta \in (0,1/4)$. It also yields $\mu^{-\delta} g \in \mathcal{C}^{0,\alpha}([\tau,T]\times\mathbb{T}^d\times\mathbb{R}^d)$ by reducing slightly δ.

This implies in turn that $\mu^{-\delta}S = \mu^{-\delta}(d/2 - |v|^2/4)g$ lies in $\mathcal{C}^{0,\alpha}([\tau,T]\times\mathbb{T}^d\times\mathbb{R}^d)$ and that $\mathcal{R}[g] = \int_v g\mu^{1/2}\,\mathrm{d}v \in \mathcal{C}^{0,2\alpha/3}([\tau,T]\times\mathbb{T}^d)$. Indeed,

$$\begin{aligned}|\mathcal{R}[g](t,x) - \mathcal{R}[g](s,y)| &\le \int |\mu^{-\delta_0} g(t,x,v) - \mu^{-\delta} g(s,y,v)|\mu^{\frac12+\delta_0}(v)\,\mathrm{d}v \\ &\le [\mu^{-\delta_0} g]_{\mathcal{C}^{0,\alpha}} \int (|t-s|^{\frac{\alpha}{2}} + |x-y-(t-s)v|^{\frac{\alpha}{3}})\mu^{\frac12+\delta_0}(v)\,\mathrm{d}v \\ &\lesssim [\mu^{-\delta_0} g]_{\mathcal{C}^{0,\alpha}}(|t-s|^{\frac{\alpha}{2}} + |x-y|^{\frac{\alpha}{3}} + |t-s|^{\frac{\alpha}{3}}).\end{aligned}$$

This ensures the Hölder regularity of the coefficients and source term, and we thus can apply Theorem 3.12 in cylinders covering A_r as above to conclude the proof. ☐

4.2. **Standard interpolation product inequality.** Let us recall and prove an interpolation inequality tailored to our needs; it is folklore knowledge.

Lemma 4.2. *Consider $k \geq d/2$ and two functions g_1, g_2 in $H^k(\mathbb{T}^d)$ and $\alpha, \beta \in \mathbb{N}^d$ multi-indeces s.t. $|\alpha| + |\beta| = k$ then*

$$\left\|\partial_x^\alpha g_1 \partial_x^\beta g_2\right\|_{L^2(\mathbb{T}^d)} \lesssim_k \|g_1\|_{L^\infty_x(\mathbb{T}^d)} \|g_2\|_{H^k_x(\mathbb{T}^d)} + \|g_1\|_{H^k_x(\mathbb{T}^d)} \|g_2\|_{L^\infty_x(\mathbb{T}^d)}.$$

Moreover if $\alpha \neq 0$, for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ s.t.

$$\left\|\partial_x^\alpha g_1 \partial_x^\beta g_2\right\|_{L^2(\mathbb{T}^d)} \leq \varepsilon \|g_1\|_{L^\infty_x(\mathbb{T}^d)} \|g_2\|_{H^k_x(\mathbb{T}^d)} + C_\varepsilon \|g_1\|_{H^k_x(\mathbb{T}^d)} \|g_2\|_{L^\infty_x(\mathbb{T}^d)}. \tag{4.3}$$

Proof. Recall the *Nash inequality*: given $h \in H^k(\mathbb{T}^d)$ with $k > d/2$ and any $\alpha \in \mathbb{N}^d$ with $0 \leq |\alpha| \leq k$ then for $p := 2k/|\alpha| \in [1, +\infty]$ one has

$$\|\partial_x^\alpha h\|_{L^p(\mathbb{T}^d)} \lesssim \|h\|_{L^\infty(\mathbb{T}^d)}^{1-|\alpha|/k} \|h\|_{H^k(\mathbb{T}^d)}^{|\alpha|/k}.$$

Then apply Hölder inequality with $p := 2k/|\alpha|$ and $q := 2k/|\beta|$, and then Nash inequality to get

$$\begin{aligned}
\left\|\partial_x^\alpha g_1 \partial_x^\beta g_2\right\|_{L^2(\mathbb{T}^d)} &\leq \left\|\partial_x^\alpha g_1\right\|_{L^p(\mathbb{T}^d)} \left\|\partial_x^\alpha g_2\right\|_{L^q(\mathbb{T}^d)} \\
&\leq \left\|\partial_x^\alpha g_1\right\|_{L^p(\mathbb{T}^d)} \left\|\partial_x^\alpha g_2\right\|_{L^q(\mathbb{T}^d)} \\
&\lesssim \|g_1\|_{L^\infty(\mathbb{T}^d)}^{1-|\alpha|/k} \|g_1\|_{H^k(\mathbb{T}^d)}^{|\alpha|/k} \|g_2\|_{L^\infty(\mathbb{T}^d)}^{1-|\beta|/k} \|g_2\|_{H^k(\mathbb{T}^d)}^{|\beta|/k} \\
&\lesssim \left(\|g_1\|_{L^\infty(\mathbb{T}^d)} \|g_2\|_{H^k(\mathbb{T}^d)}\right)^{1-|\alpha|/k} \left(\|g_1\|_{H^k(\mathbb{T}^d)} \|g_2\|_{L^\infty(\mathbb{T}^d)}\right)^{|\alpha|/k}.
\end{aligned}$$

We now apply $a^{1-\mu} b^\mu \leq (1-\mu)a + \mu b$ to deduce finally the two claimed inequalities. ☐

4.3. **Local well-posedness in Sobolev spaces.** Let us denote

$$\|g\|^2_{H^k_x H^\ell_v(\mathbb{T}^d \times \mathbb{R}^d)} := \|g\|^2_{L^2(\mathbb{T}^d \times \mathbb{R}^d)} + \sum_{i=1,\dots,d} \left\|\partial^k_{x_i} h\right\|^2_{L^2(\mathbb{T}^d \times \mathbb{R}^d)} + \sum_{i=1,\dots,d} \left\|\partial^\ell_{v_i} h\right\|^2_{L^2(\mathbb{T}^d \times \mathbb{R}^d)}.$$

Theorem 4.3 (Local well-posedness in H^k)**.** *Consider $g_{in} \in H^k_x H^\ell_v(\mathbb{T}^d \times \mathbb{R}^d)$ with k, ℓ non-negative integers s.t. $1 \leq \ell \leq k$ and $k > d/2$, and $C_1\sqrt{\mu} \leq g_{in} \leq C_2\sqrt{\mu}$ for $0 < C_1 < C_2$. Then there is $T > 0$ depending only on C_1, C_2 and the Sobolev controls on g_{in} such that there exists a unique local solution $g \in C^1([0,T], H^k_x H^\ell_v(\mathbb{T}^d \times \mathbb{R}^d))$ to (4.1) with initial data g_{in} with $C_1\sqrt{\mu} \leq g(t,\cdot,\cdot) \leq C_2\sqrt{\mu}$ for all $t \in [0,T]$.*

Remark 4.4. With more work (keeping track of negative coercive terms) the energy estimates in the proof below can be refined to imply global well-posedness of perturbative solutions, i.e. close in $H^k_x H^\ell_v(\mathbb{T}^d \times \mathbb{R}^d)$ to the steady state. We do not investigate it further since the continuation criterion below imply the stronger result of unconditional global well-posedness.

Proof. Consider first an a priori solution $g \in C^1([0,T], H^k_x H^\ell_v(\mathbb{T}^d \times \mathbb{R}^d))$ and compute:
$(*)$ L^2 estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} |g|^2 \,\mathrm{d}x \,\mathrm{d}v \leq -C_1 \int_{\mathbb{T}^d \times \mathbb{R}^d} |h|^2 \,\mathrm{d}x \,\mathrm{d}v \tag{4.4}$$

where we denote $h := \mu^{1/2} \nabla_v (\mu^{-1/2} g)$.
$(**)$ Estimate of v-derivatives: for any integer $\ell \geq 1$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} |\partial^\ell_{v_i} g|^2 \,\mathrm{d}x \,\mathrm{d}v = -\ell \int_{\mathbb{T}^d \times \mathbb{R}^d} (\partial^{\ell-1}_{v_i} \partial_{x_i} g) \partial^\ell_{v_i} g \,\mathrm{d}x \,\mathrm{d}v - \int_{\mathbb{T}^d \times \mathbb{R}^d} \mathcal{R}[g] \left| \nabla_v \left(\frac{\partial^\ell_{v_i} g}{\sqrt{\mu}} \right) \right|^2 \mu \,\mathrm{d}x \,\mathrm{d}v \\
+ \frac{1}{4} \binom{\ell}{1} \int_{\mathbb{T}^d \times \mathbb{R}^d} \mathcal{R}[g] \left|\partial^{\ell-1}_{v_i} g\right|^2 \,\mathrm{d}x \,\mathrm{d}v + \frac{1}{2} \binom{\ell}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} \mathcal{R}[g] \left|\partial^{\ell-1}_{v_i} g\right|^2 \,\mathrm{d}x \,\mathrm{d}v.
\end{aligned}$$

In the right hand side, the first term corresponds to the transport $v \cdot \nabla_x$, the second one to the operator U if $\mathcal{R}[g]$ was constant, the third term appears when one v-derivative applies to $|v|^2$ and the others apply to g in the product $|v|^2 g$ appearing in $U[g]$, the fourth term appears after deriving $|v|^2$ twice. Notice that integration by parts are used either to further differentiate $|v|^2$ or to make appear $|\partial^{k-1}_{v_i} g|^2$. Discarding the negative term and using the fact that $\mathcal{R}[g] \le C_2$, we get after summing over $i = 1, \dots, d$ and combining with equation (4.4)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|g(t)\|^2_{H^0_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)} \lesssim_{k,C_2} \|g\|^2_{H^\ell_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)}. \tag{4.5}$$

$(***)$ Estimate of x-derivatives: since x-derivatives commute with the operators $v\cdot\nabla_x$ and U

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial^k_{x_i} g|^2 \,\mathrm{d}x\,\mathrm{d}v = &-\int_{\mathbb{T}^d\times\mathbb{R}^d} \mathcal{R}[g] \left|\nabla_v\left(\frac{\partial^k_{x_i} g}{\sqrt{\mu}}\right)\right|^2 \mu \,\mathrm{d}x\,\mathrm{d}v \\ &- \sum_{0\le\beta<k}\binom{k}{\beta}\int_{\mathbb{T}^d\times\mathbb{R}^d} (\partial^{k-\beta}_{x_i}\mathcal{R}[g])\nabla_v\left(\frac{\partial^\beta_{x_i} g}{\sqrt{\mu}}\right)\cdot\nabla_v\left(\frac{\partial^k_{x_i} g}{\sqrt{\mu}}\right)\mu\,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

This together with the lower bound $\mathcal{R}[g] \ge C_1$ implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial^k_{x_i} g|^2 \,\mathrm{d}x\,\mathrm{d}v &\lesssim_k -C_1\int_{\mathbb{T}^d\times\mathbb{R}^d}|\partial^k_{x_i} h|^2\,\mathrm{d}x\,\mathrm{d}v - \sum_{0\le\beta<k}\binom{k}{\beta}\int_{\mathbb{T}^d\times\mathbb{R}^d}(\partial^{k-\beta}_{x_i}\mathcal{R}[g])\partial^\beta_{x_i}h\,\partial^k_{x_i}h\,\mathrm{d}x\,\mathrm{d}v \\ &\lesssim_k -C_1\int_{\mathbb{T}^d\times\mathbb{R}^d}|\partial^k_{x_i} h|^2\,\mathrm{d}x\,\mathrm{d}v + \sum_{0\le\beta<k}\int_{\mathbb{T}^d\times\mathbb{R}^d}|\partial^{k-\beta}_{x_i}\mathcal{R}[g]|\cdot|\partial^\beta_{x_i}h|\cdot|\partial^k_{x_i}h|\,\mathrm{d}x\,\mathrm{d}v. \end{aligned}\tag{4.6}$$

Observe that, given $0 \le \beta < k$, the index $\alpha := k-\beta \ne 0$ and the inequality (4.3) in Lemma 4.2 can be applied (we use below the upper bound $\mathcal{R}[g] \le C_2$):

$$\begin{aligned}&\int_{\mathbb{T}^d\times\mathbb{R}^d}|\partial^{k-\beta}_{x_i}\mathcal{R}[g]|\cdot|\partial^\beta_{x_i}h|\cdot|\partial^k_{x_i}h|\,\mathrm{d}x\,\mathrm{d}v \\ &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{T}^d}|\partial^\alpha_{x_i}\mathcal{R}[g]|\cdot|\partial^\beta_{x_i}h|\cdot|\partial^k_{x_i}h|\,\mathrm{d}x\right)\mathrm{d}v \\ &\le \int_{\mathbb{R}^d}\left(\left\|\partial^\alpha_{x_i}\mathcal{R}[g](t,\cdot)\partial^\beta_{x_i}h(t,\cdot,v)\right\|_{L^2_x(\mathbb{T}^d)}\left\|\partial^k_{x_i}h(t,\cdot,v)\right\|_{L^2_x(\mathbb{T}^d)}\right)\mathrm{d}v \\ &\le \varepsilon\int_{\mathbb{R}^d}\left(\|\mathcal{R}[g](t,\cdot)\|_{L^\infty_x(\mathbb{T}^d)}\|h(t,\cdot,v)\|^2_{H^k_x(\mathbb{T}^d)}\right)\mathrm{d}v \\ &\qquad + C_\varepsilon\int_{\mathbb{R}^d}\left(\|\mathcal{R}[g](t,\cdot)\|_{H^k_x(\mathbb{T}^d)}\|h(t,\cdot,v)\|_{L^\infty_x(\mathbb{T}^d)}\|h(t,\cdot,v)\|_{H^k_x(\mathbb{T}^d)}\right)\mathrm{d}v \\ &\le \varepsilon(C_2+1)\|h(t,\cdot,\cdot)\|^2_{H^k_xH^0_v(\mathbb{T}^d\times\mathbb{R}^d)} + C'_\varepsilon\|\mathcal{R}[g](t,\cdot)\|^2_{H^k_x(\mathbb{T}^d)}\int_{\mathbb{R}^d}\|h(t,\cdot,v)\|^2_{L^\infty_x(\mathbb{T}^d)}\,\mathrm{d}v\end{aligned}$$

for any $\varepsilon > 0$ and some corresponding constant $C_\varepsilon, C'_\varepsilon > 0$. Use then

$$\|\mathcal{R}[g](t,\cdot)\|_{H^k_x(\mathbb{T}^d)} \le \|g(t,\cdot,\cdot)\|_{H^k_xH^0_v(\mathbb{T}^d\times\mathbb{R}^d)}$$

and equation (4.4) to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|g\|^2_{H^k_xH^0_v(\mathbb{T}^d\times\mathbb{R}^d)} \lesssim_{k,C_2} &- C_1\|h(t,\cdot,\cdot)\|^2_{H^k_xH^0_v(\mathbb{T}^d\times\mathbb{R}^d)} \\ &+ \varepsilon\|h(t,\cdot,\cdot)\|^2_{H^k_xH^0_v(\mathbb{T}^d\times\mathbb{R}^d)} \\ &+ \left(\int_{\mathbb{R}^d}\|h(t,\cdot,v)\|^2_{L^\infty_x(\mathbb{T}^d)}\,\mathrm{d}v\right)\|g(t,\cdot,\cdot)\|^2_{H^k_xH^0_v(\mathbb{T}^d\times\mathbb{R}^d)}.\end{aligned}$$

Finally choose ε small enough (in terms of absolute constants, independently of the solution) to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|g\|^2_{H^k_xH^0_v(\mathbb{T}^d\times\mathbb{R}^d)} \lesssim_{k,C_1,C_2}\left(\int_{\mathbb{R}^d}\|h(t,\cdot,v)\|^2_{L^\infty_x(\mathbb{T}^d)}\,\mathrm{d}v\right)\|g(t,\cdot,\cdot)\|^2_{H^k_xH^0_v(\mathbb{T}^d\times\mathbb{R}^d)}. \tag{4.7}$$

The combination of equations (4.5) and (4.7) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|g(t,\cdot,\cdot)\|^2_{H^k_xH^l_v(\mathbb{T}^d\times\mathbb{R}^d)} \lesssim_{k,C_1,C_2} &\|g(t,\cdot,\cdot)\|^2_{H^\ell_xH^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)} \\ &+ \left(\int_{\mathbb{R}^d}\|h(t,\cdot,v)\|^2_{L^\infty_x(\mathbb{T}^d)}\,\mathrm{d}v\right)\|g(t,\cdot,\cdot)\|^2_{H^k_xH^0_v(\mathbb{T}^d\times\mathbb{R}^d)}.\end{aligned}\tag{4.8}$$

Now observe that (using $g \le C_2 \mu^{\frac12}$, $k > d/2$ and Sobolev embedding in $\mathbb{T}^d$)

$$\int_{\mathbb{R}^d} \|h(t,\cdot,v)\|^2_{L^\infty_x(\mathbb{T}^d)} \,\mathrm{d}v \lesssim C_2^2 + \int_{\mathbb{R}^d} \|\nabla_v g\|^2_{L^\infty_x(\mathbb{T}^d)}\,\mathrm{d}v \lesssim C_2^2 + \|g\|^2_{H^k_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)}$$

to conclude finally that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac12 \|g(t,\cdot,\cdot)\|^2_{H^k_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)} \lesssim_{k,C_1,C_2} \|g(t,\cdot,\cdot)\|^2_{H^k_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)} + \|g(t,\cdot,\cdot)\|^4_{H^k_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)} \tag{4.9}$$

which is the first main a priori estimate, that shows that the $H^k_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)$ norm remains finite on a short time interval (whose length depends on the size of the initial data) thanks to Gronwall's lemma.

Consider the difference of two solutions $g_1, g_2 \in H^k_x H^\ell(\mathbb{T}^d\times\mathbb{R}^d)$ that satisfies

$$\partial_t(g_1-g_2) + v\cdot\nabla_x(g_1-g_2) = r[g_1-g_2]U[g_1] + r[g_2]U[g_1-g_2],$$

and perform strictly similar calculations to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac12 \|(g_1-g_2)(t,\cdot,\cdot)\|^2_{H^k_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)} &\lesssim_{k,C_1,C_2} \|(g_1-g_2)(t,\cdot,\cdot)\|^2_{H^k_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)} \\ &+ \left(\|g_1(t,\cdot,\cdot)\|^2_{H^k_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)} + \|g_2(t,\cdot,\cdot)\|^2_{H^k_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)}\right) \|(g_1-g_2)(t,\cdot,\cdot)\|^2_{H^k_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)}\end{aligned} \tag{4.10}$$

which implies uniqueness in the space $H^k_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)$. Finally, it is standard that the two a priori estimates (4.9) and (4.10) imply the existence of solutions constructed through the iterative scheme

$$\partial_t g_{n+1} + v\cdot\nabla_x g_{n+1} = r[g_n]U[g_{n+1}].$$

This concludes the proof. □

4.4. **From local-in-time to global-in-time.** To continue the solutions to all times it is enough to prove that the $H^k_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)$ norm remains finite. And since the part $H^0_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)$ was already controlled linearly in terms of itself and $H^k_x H^0_v(\mathbb{T}^d\times\mathbb{R}^d)$, it remains only to prove that $H^k_x H^0_v(\mathbb{T}^d\times\mathbb{R}^d)$ remains finite. Consider again the a priori estimate (4.8) (satisfied by the solutions constructed on their time of existence):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac12 \|g(t,\cdot,\cdot)\|^2_{H^k_x H^0_v(\mathbb{T}^d\times\mathbb{R}^d)} &\lesssim_{k,C_1,C_2} \|g(t,\cdot,\cdot)\|^2_{H^0_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)} \\ &+ \left(\int_{\mathbb{R}^d} \|h(t,\cdot,v)\|^2_{L^\infty_x(\mathbb{T}^d)}\,\mathrm{d}v\right) \|g(t,\cdot,\cdot)\|^2_{H^k_x H^0_v(\mathbb{T}^d\times\mathbb{R}^d)}.\end{aligned}$$

Observe that $h = \nabla_v g + (v/2)g$ and thus

$$\left(\int_{\mathbb{R}^d} \|h(t,\cdot,v)\|^2_{L^\infty_x(\mathbb{T}^d)}\,\mathrm{d}v\right) \lesssim \left(\int_{\mathbb{R}^d} \|\nabla_v g(t,\cdot,v)\|^2_{L^\infty_x(\mathbb{T}^d)}\,\mathrm{d}v\right) + C_2$$

and finally applying Proposition 4.1:

$$\left(\int_{\mathbb{R}^d} \|\nabla_v g(t,\cdot,v)\|^2_{L^\infty_x(\mathbb{T}^d)}\,\mathrm{d}v\right) \lesssim \|\mu^{-\delta}\nabla_v g\|^2_{L^\infty([\tau,T]\times\mathbb{T}^d\times\mathbb{R}^d)} \lesssim \|\mu^{-\delta} g\|^2_{\mathcal{H}^\alpha([\tau,T]\times\mathbb{T}^d\times\mathbb{R}^d)} \lesssim C.$$

It shows finally that after some arbitrarily small time $\tau > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac12 \|g(t,\cdot,\cdot)\|^2_{H^k_x H^0_v(\mathbb{T}^d\times\mathbb{R}^d)} \lesssim_{k,C_1,C_2,\tau} \|g(t,\cdot,\cdot)\|^2_{H^0_x H^\ell_v(\mathbb{T}^d\times\mathbb{R}^d)}$$

which shows that the solutions are global-in-time by the Gronwall lemma.

4.5. **Infinite regularisation for positive times.** We only sketch the argument here; it is a variation along standard techniques.

There are two issues that make the problem different from the standard iterative scheme for regularisation in the parabolic theory as in [25]: (1) an asymmetry between the x-derivatives and the v-derivatives due to hypoellipticity, (2) in this asymmetry the hypoelliptic gain of regularity on the x variable is $2/3 < 1$ and is not enough for differentiating the equation. As a result we start with the gain on x variable, and use finite-difference fractional derivation as an intermediate step.

Consider the global solution built in $g \in H^k(\mathbb{T}^d \times \mathbb{R}^d)$ with $C_1\sqrt{\mu} \le g \le C_2\sqrt{\mu}$, for some given $k > d/2$. Proposition 4.1 implies that $\mu^{-\delta} g \in C^{2/3}_x$ for positive times. We then consider for $u \in \mathbb{R}^d$ the finite difference $G := D_u g := |u|^{-\nu}[g(t,x+u,v) - g(t,x,v)]$ with $\nu \in (1/3, 2/3)$. It satisfies the equation

$$\partial_t G + v \cdot \nabla_x G = \mathcal{R}[g]U[G] + \mathcal{R}[G]U[g(\cdot, \cdot + u, \cdot)].$$

and $G \in L^2_{t,x,v} \cap L^2_{t,x}H^1_v$ to qualify as a weak solution, and the source term $S := \mathcal{R}[G]U[g(\cdot, \cdot+u, \cdot)] \in \mathcal{C}^{0,\alpha}$ for $\alpha > 0$ small enough thanks to the $\mathcal{H}^\alpha$ bound. From Schauder (Proposition 4.1 with a source term), we deduce a bound $\|G\|_{\mathcal{H}^\alpha} \lesssim \|g\|_{\mathcal{H}^\alpha}$. Letting $u \to 0$ in all directions we deduce a control $\mu^{-\delta} g \in C^1_x$ (by summing the gain of Hölder regularity $\nu + 2/3$). Hence we can now differentiate the equation and $G := \partial_{x_i} g$ satisfies

$$\partial_t G + v \cdot \nabla_x G = \mathcal{R}[g]U[G] + \mathcal{R}[g]U[g].$$

Again $G \in L^2_{t,x,v} \cap L^2_{t,x}H^1_v$ and qualifies as a weak solution, and the source term $S := \mathcal{R}[G]U[g] \in \mathcal{C}^{0,\alpha}$ thanks to the $\mathcal{H}^\alpha$ bound. We are now in the same situation as when we started. This iterative argument hence shows that on any initial time layer, all derivatives in x are gained. Then the gain of v derivatives is easier: define $G := \partial_{v_i} g$ satisfies

$$\partial_t G + v \cdot \nabla_x G = \mathcal{R}[g]U[G] - \partial_{x_i} g - \mathcal{R}[g]\frac{v_i}{2} g.$$

This new unknown G qualifies as a weak solution and the source term $S := -\partial_{x_i} g - \mathcal{R}[g]\frac{v_i}{2} g \in \mathcal{C}^{0,\alpha}$ from the previous step. Hence $G \in \mathcal{H}^\alpha$ and one can iterate as before to gain all derivatives in v.

As a conclusion, for any $\tau \in (0,1)$, the solution $g \in H^\infty(\mathbb{T}^d \times \mathbb{R}^d)$, and from this time on, the energy estimates in Sobolev show the propagation of all Sobolev norms. The solution is thus smooth for all times.

References

[1] Alexandre, R., Desvillettes, L., Villani, C., and Wennberg, B. Entropy dissipation and long-range interactions. *Arch. Ration. Mech. Anal. 152*, 4 (2000), 327–355.

[2] Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., and Yang, T. The Boltzmann equation without angular cutoff in the whole space: II, Global existence for hard potential. *Anal. Appl. (Singap.) 9*, 2 (2011), 113–134.

[3] Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., and Yang, T. The Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential. *J. Funct. Anal. 262*, 3 (2012), 915–1010.

[4] Alexandre, R., and Villani, C. On the Boltzmann equation for long-range interactions. *Comm. Pure Appl. Math. 55*, 1 (2002), 30–70.

[5] Arkeryd, L. On the Boltzmann equation. I. Existence. *Arch. Rational Mech. Anal. 45* (1972), 1–16.

[6] Arkeryd, L. On the Boltzmann equation. II. The full initial value problem. *Arch. Rational Mech. Anal. 45* (1972), 17–34.

[7] Bakry, D., and Émery, M. Diffusions hypercontractives. In *Séminaire de probabilités, XIX, 1983/84*, vol. 1123 of *Lecture Notes in Math.* Springer, Berlin, 1985, pp. 177–206.

[8] Baudoin, F. Bakry-Émery meet Villani. Tech. rep., arXiv 1308.4938, June 2017.

[9] Beale, J. T., Kato, T., and Majda, A. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys. 94*, 1 (1984), 61–66.

[10] Bouchut, F. Hypoelliptic regularity in kinetic equations. *J. Math. Pures Appl. (9) 81*, 11 (2002), 1135–1159.

[11] Bramanti, M., and Brandolini, L. Schauder estimates for parabolic nondivergence operators of Hörmander type. *J. Differential Equations 234*, 1 (2007), 177–245.

[12] Cameron, S., Silvestre, L., and Snelson, S. Global a priori estimates for the inhomogeneous landau equation with moderately soft potentials. *arXiv preprint arXiv:1701.08215* (2017).

[13] Carleman, T. Sur la théorie de l'équation intégrodifférentielle de Boltzmann. *Acta Math. 60*, 1 (1933), 91–146.

[14] Carleman, T. *Problèmes mathématiques dans la théorie cinétique des gaz.* Publ. Sci. Inst. Mittag-Leffler. 2. Almqvist & Wiksells Boktryckeri Ab, Uppsala, 1957.

[15] Di Francesco, M., and Polidoro, S. Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov-type operators in non-divergence form. *Adv. Differential Equations 11*, 11 (2006), 1261–1320.

[16] DiPerna, R. J., and Lions, P.-L. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2) 130*, 2 (1989), 321–366.

[17] Golse, F. F., Imbert, C., Mouhot, C., and Vasseur, A. F. Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation. 31 pages, 4 figures., 2017.

[18] Gressman, P. T., and Strain, R. M. Global classical solutions of the Boltzmann equation without angular cut-off. *J. Amer. Math. Soc. 24*, 3 (2011), 771–847.

[19] Gualdani, M., Mischler, S., and Mouhot, C. *Factorization for non-symmetric operators and exponential H-theorem.* Mémoire de la Société Mathématique de France. SMF, 2017.

[20] Guillin, A., and Wang, F.-Y. Degenerate Fokker-Planck equations: Bismut formula, gradient estimate and Harnack inequality. *J. Differential Equations 253*, 1 (2012), 20–40.

[21] Guo, Y. Classical solutions to the Boltzmann equation for molecules with an angular cutoff. *Arch. Ration. Mech. Anal. 169*, 4 (2003), 305–353.
[22] Henderson, C., and Snelson, S. C^∞ smoothing for weak solutions of the inhomogeneous Landau equation. arxiv Preprint 1707.05710, 2017.
[23] Imbert, C., and Silvestre, L. Weak Harnack inequality for the Boltzmann equation without cut-off. Arxiv Preprint 1608.07571, 2017.
[24] Kolmogoroff, A. Zufällige Bewegungen (zur Theorie der Brownschen Bewegung). *Ann. of Math. (2) 35*, 1 (1934), 116–117.
[25] Krylov, N. V. *Lectures on elliptic and parabolic equations in Hölder spaces*, vol. 12 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996.
[26] Lions, P.-L. Régularité et compacité pour des noyaux de collision de Boltzmann sans troncature angulaire. *C. R. Acad. Sci. Paris Sér. I Math. 326*, 1 (1998), 37–41.
[27] Lunardi, A. Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in $\mathbf{R}^n$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 24*, 1 (1997), 133–164.
[28] Manfredini, M. The Dirichlet problem for a class of ultraparabolic equations. *Adv. Differential Equations 2*, 5 (1997), 831–866.
[29] Mouhot, C., and Villani, C. Regularity theory for the spatially homogeneous Boltzmann equation with cut-off. *Arch. Ration. Mech. Anal. 173*, 2 (2004), 169–212.
[30] Polidoro, S. On a class of ultraparabolic operators of Kolmogorov-Fokker-Planck type. *Matematiche (Catania) 49*, 1 (1994), 53–105 (1995).
[31] Radkevich, E. V. Equations with nonnegative characteristic form. II. *Sovrem. Mat. Prilozh.*, 56, Differentsial′nye Uravneniya s Chastnymi Proizvodnymi (2008), 3–147.
[32] Safonov, M. V. The classical solution of the elliptic Bellman equation. *Dokl. Akad. Nauk SSSR 278*, 4 (1984), 810–813.
[33] Schwab, R. W., and Silvestre, L. Regularity for parabolic integro-differential equations with very irregular kernels. *Anal. PDE 9*, 3 (2016), 727–772.
[34] Silvestre, L. A new regularization mechanism for the Boltzmann equation without cut-off. *Comm. Math. Phys. 348*, 1 (2016), 69–100.
[35] Silvestre, L. Upper bounds for parabolic equations and the Landau equation. *J. Differential Equations 262*, 3 (2017), 3034–3055.
[36] Simon, L. Schauder estimates by scaling. *Calc. Var. Partial Differential Equations 5*, 5 (1997), 391–407.
[37] Ukai, S. On the existence of global solutions of mixed problem for non-linear Boltzmann equation. *Proc. Japan Acad. 50* (1974), 179–184.
[38] Villani, C. Hypocoercivity. *Mem. Amer. Math. Soc. 202*, 950 (2009), iv+141.
[39] Wang, W., and Zhang, L. The C^α regularity of a class of non-homogeneous ultraparabolic equations. *Sci. China Ser. A 52*, 8 (2009), 1589–1606.

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