Bohr's correspondence principle for the renormalized Nelson model

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BOHR'S CORRESPONDENCE PRINCIPLE FOR THE RENORMALIZED NELSON MODEL.

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Abstract. In the mid Sixties Edward Nelson proved the existence of a consistent quantum field theory that describes the Yukawa-like interaction of a non-relativistic nucleon field with a relativistic meson field. Since then it is thought, despite the renormalization procedure involved in the construction, that the quantum dynamics should be governed in the classical limit by a Schrödinger-Klein-Gordon system with Yukawa coupling. In the present paper we prove this fact in the form of a Bohr correspondence principle.

Contents

Date: January 18, 2018.

2010 *Mathematics Subject Classification.* 35Q40, 81T16, 81Q20.

Key words and phrases. Nelson model, Schrödinger-Klein-Gordon system, Yukawa interaction, Classical limit.

The second author has been partially supported by the Centre Henri Lebesgue (programme "Investissements d'avenir" ANR-11-LABX-0020-01).

1. Introduction.

Modern theoretical physics explains how matter interacts with radiation and proposes phenomenological models of quantum field theory that in principle describe such fundamental interaction. Giving a firm mathematical ground to these models is known to be a difficult task related to renormalization theory [19, 36, 53, 54, 58, 66]. Since the fifties there were spectacular advances in these problems culminating with the perturbative renormalization of quantum electrodynamics, the birth of the renormalization group method and the renormalizability of gauge field theories. Nevertheless, conceptual mathematical difficulties remain as well as outstanding open problems, see [61, 74]. The purpose of the present article is to study the quantum-classical correspondence for a simple renormalized model of particles interacting with a scalar field: the Nelson model. We believe that the study of the relationship between classical and quantum nonlinear field theories sheds light on the mathematical foundation of renormalization theory. In particular, in the case considered here the renormalization procedure turns out to be related to a normal form implemented by nonlinear symplectic transformations on the classical phase-space. Interested reader may found a formal discussion concerning the possibility of a different point of view on renormalization in an extended version of this article [6].

The so-called Nelson model is a system of Quantum Field Theory that has been widely studied from a mathematical standpoint [see e.g. 1, 13–16, 21, 30, 34, 39, 43, 43–45, 67, 71, 78]. It consists of nonrelativistic spin zero particles interacting with a scalar boson field, and can be used to model various systems of physical interest, such as nucleons interacting with a meson field. In the mid sixties Edward Nelson rigorously constructed a quantum dynamic for this model free of ultraviolet (high energy) cutoffs in the particle-field coupling, see [68]. This is done by means of a renormalization procedure: roughly speaking, we need to subtract a divergent quantity from the Hamiltonian, so the latter can be defined as a self-adjoint operator in the limit of the ultraviolet cutoff. The quantum dynamics is rather singular in this case (renormalization is necessary); and the resulting generator has no explicit form as an operator though it is unitarily equivalent to an explicit one. Since the work of Gross [56] and Nelson [68] it is believed, but never proved, that the renormalized dynamics is generated by a canonical quantization of the Schrödinger-Klein-Gordon (S-KG) system with Yukawa coupling. In other words, the quantum fluctuations of the particle-field system are centered around the classical trajectories of the Schrödinger-Klein-Gordon system at certain scale and the renormalization procedure preserves the suitable quantum-classical correspondence as well as being necessary to define the quantum dynamics. We give a mathematical formulation of such result in Theorem 1.1 in the form of a Bohr correspondence principle. Consequently, our result justifies in some sense the use of the Schrödinger-Klein-Gordon system as a model of nucleon-meson interaction [see e.g. 17, 18, 27, 40, 50, 70].

Recently, the authors of this paper have studied the classical limit of the Nelson model, in its regularized version [5, 32]. We have proved that the quantum dynamic converges, when an effective semiclassical parameter $\varepsilon \to 0$, towards a non-linear Hamiltonian flow on a classical phase space. This flow is governed by a Schrödinger-Klein-Gordon system, with a regularized Yukawa-type coupling. To extend the classical-quantum correspondence to the system without ultraviolet cutoff, we rely on the recent techniques elaborated in the mean-field approximation of many-body Schrödinger dynamics in [8–11] as well as the result with cutoff [5]. As a matter of fact the renormalization procedure, implemented by a dressing transform, generates a many-body Schrödinger dynamics in a mean-field scaling (see for instance the recent results [24, 25, 72] and references therein). So it was convenient that the mean-field approximation was derived with the same general techniques that allow to prove equally the classical approximation for QFT models. The result is further discussed in Subsection 1.2, and all the details and proofs are provided in Section 4.

For the sake of presentation, we collected the notations and basic definitions—used throughout the paper—in the Subsection 1.1 below. In subsection 1.2 we present our main result on the classical-quantum correspondence principle. The rest of the paper is organized as follows: in Section 2 we review the basic properties of the quantum system and the usual procedure of renormalization with some crucial uniform estimates; in Section 3 we analyze the classical S-KG dynamics and the classical dressing transformation; in Section 4 we study in detail the classical limit of the renormalized Nelson model, and prove our main Theorem 1.1.

1.1. Notations and general definitions.

* We fix once and for all $\bar{\varepsilon}, m_0, M > 0$. We also define the function $\omega(k) = \sqrt{k^2 + m_0^2}$.
* The effective (semiclassical) parameter will be denoted by $\varepsilon \in (0, \bar{\varepsilon})$.
* Let $\mathcal{Z}$ be a Hilbert space; then we denote by $\Gamma_s(\mathcal{Z})$ the symmetric Fock space over $\mathcal{Z}$. We have that
$$\Gamma_s(\mathcal{Z}) = \bigoplus_{n=0}^{\infty} \mathcal{Z}^{\otimes_s n} \quad \text{with} \quad \mathcal{Z}^{\otimes_s 0} = \mathbb{C} ,$$
where $\mathcal{Z}^{\otimes_s n}$ is the n-fold symmetrized tensor product.
* Let X be an operator on a Hilbert space $\mathcal{Z}$. We will usually denote by $D(X) \subset \mathcal{Z}$ its domain of definition, and by $Q(X) \subset \mathcal{Z}$ the domain of definition of the corresponding quadratic form.
* Let $S : \mathcal{Z} \supseteq D(S) \to \mathcal{Z}$ be a densely defined self-adjoint operator on $\mathcal{Z}$. Its second quantization $d\Gamma(S)$ is the self-adjoint operator on $\Gamma_s(\mathcal{Z})$ defined by
$$d\Gamma(S)\big|_{D(S)^{\otimes_s^{alg} n}} = \varepsilon \sum_{k=1}^{n} 1 \otimes \cdots \otimes \underbrace{S}_{k} \otimes \cdots \otimes 1 ,$$
where $D(S)^{\otimes_s^{alg} n}$ denotes the algebraic tensor product. In particular, the operator $d\Gamma(1)$ is the scaled number operator which we simply denote by N without stressing the ε-dependence.
* We denote by $\mathcal{C}_0^\infty(N)$ the subspace of finite particle vectors:
$$\mathcal{C}_0^\infty(N) = \{\psi \in \Gamma_s(\mathcal{Z})\; ;\; \exists \bar{n} \in \mathbb{N}, \psi\big|_{\mathcal{Z}^{\otimes_s n}} = 0 \;\forall n > \bar{n}\} .$$
* Let U be a unitary operator on $\mathcal{Z}$. We define $\Gamma(U)$ to be the unitary operator on $\Gamma_s(\mathcal{Z})$ given by
$$\Gamma(U)|_{\mathcal{Z}^{\otimes_s n}} = \bigotimes_{k=1}^{n} U .$$
If $U = e^{itS}$ is a one parameter group of unitary operators on $\mathcal{Z}$, $\Gamma(e^{itS}) = e^{i\frac{t}{\varepsilon} d\Gamma(S)}$.

* On $\Gamma_s(\mathcal{Z})$, we define the annihilation/creation operators $a^{\#}(g)$, $g \in \mathcal{Z}$, by their action on $f^{\otimes n} \in \mathcal{Z}^{\otimes_s n}$ (with $a(g)f_0 = 0$ for any $f_0 \in \mathcal{Z}^{\otimes_s 0} = \mathbb{C}$):
$$a(g)f^{\otimes n} = \sqrt{\varepsilon n}\,\langle g, f\rangle_{\mathcal{Z}}\; f^{\otimes(n-1)} \;;$$
$$a^*(g)f^{\otimes n} = \sqrt{\varepsilon(n+1)}\; g \otimes_s f^{\otimes n} \;.$$
They satisfy the Canonical Commutation Relations (CCR), $[a(f), a^*(g)] = \varepsilon\langle f, g\rangle_{\mathcal{Z}}$.
If $\mathcal{Z} = L^2(\mathbb{R}^d)$ it is useful to introduce the operator valued distributions $a^{\#}(x)$ defined by
$$a(g) = \int_{\mathbb{R}^d} \bar{g}(x)a(x)dx \quad , \quad a^*(g) = \int_{\mathbb{R}^d} g(x)a^*(x)dx \;.$$
* $\mathcal{H} = \Gamma_s(L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)) \simeq \Gamma_s(L^2(\mathbb{R}^3)) \otimes \Gamma_s(L^2(\mathbb{R}^3))$. We denote by $\psi^{\#}(x)$ and N_1 the annihilation/creation and number operators corresponding to the nucleons (conventionally taken to be the first Fock space), by $a^{\#}(k)$ and N_2 the annihilation/creation and number operators corresponding to the meson scalar field (second Fock space). In particular, we will always use the following ε-dependent representation of the CCR if not specified otherwise:
$$[\psi(x), \psi^*(x')] = \varepsilon\delta(x - x') \quad , \quad [a(k), a^*(k')] = \varepsilon\delta(k - k') \;.$$
* We will sometimes use the following decomposition:
$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n, \text{ with } \mathcal{H}_n = \bigl(L^2(\mathbb{R}^3)\bigr)^{\otimes_s n} \otimes \Gamma_s(L^2(\mathbb{R}^3)) \;.$$
We denote by $T^{(n)} := T\big|_{\mathcal{H}_n}$ the restriction to $\mathcal{H}_n$ of any operator T on $\mathcal{H}$
* On $\mathcal{H}$, the Segal quantization of $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \ni \xi = \xi_1 \oplus \xi_2$ is given by
$$R(\xi) = \bigl(\psi^*(\xi_1) + \psi(\xi_1) + a^*(\xi_2) + a(\xi_2)\bigr)/\sqrt{2} \;,$$
and therefore the Weyl operator becomes
$$W(\xi) = e^{\frac{i}{\sqrt{2}}\left(\psi^*(\xi_1)+\psi(\xi_1)\right)}\, e^{\frac{i}{\sqrt{2}}\left(a^*(\xi_2)+a(\xi_2)\right)} \;.$$
* Given a Hilbert space $\mathcal{Z}$, we denote by $\mathcal{L}(\mathcal{Z})$ the C^*-algebra of bounded operators; by $\mathcal{K}(\mathcal{Z}) \subset \mathcal{L}(\mathcal{Z})$ the C^*-algebra of compact operators; and by $\mathcal{L}^1(\mathcal{Z}) \subset \mathcal{K}(\mathcal{Z})$ the trace-class ideal.
* We denote classical Hamiltonian flows by boldface capital letters (e.g. $\mathbf{E}(\cdot)$); their corresponding energy functional by script capital letters (e.g. $\mathscr{E}$).
* Let $f \in \mathscr{S}'(\mathbb{R}^d)$ be a tempered distribution. We denote by $\mathcal{F}(f)(k)$ its Fourier transform
$$\mathcal{F}(f)(k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x)e^{-ik\cdot x}dx \;.$$
* We denote by $\mathcal{C}_0^\infty(\mathbb{R}^d)$ the infinitely differentiable functions of compact support. We denote by $H^s(\mathbb{R}^d)$ the non-homogeneous Sobolev space:
$$H^s(\mathbb{R}^d) = \Bigl\{f \in \mathscr{S}'(\mathbb{R}^d) \;,\; \int_{\mathbb{R}^d} (1 + |k|^2)^s |\mathcal{F}(f)(k)|^2 dk < +\infty\Bigr\} \;;$$
and its “Fourier transform”
$$\mathcal{F}H^s(\mathbb{R}^d) = \Bigl\{f \in \mathscr{S}'(\mathbb{R}^d) \,,\, \mathcal{F}^{-1}f \in H^s(\mathbb{R}^d)\Bigr\} \;.$$

* Let $\mathcal{Z}$ be a Hilbert space. We denote by $\mathfrak{P}(\mathcal{Z})$ the set of Borel probability measures on $\mathcal{Z}$.

1.2. **The classical limit of the renormalized Nelson model.** The Schrödinger-Klein-Gordon equations with Yukawa-like coupling is a widely studied system of non-linear PDEs in three dimension [see e.g. 17, 18, 27, 40–42, 50, 70]. This system can be written as:

$$\begin{cases} i\partial_t u = -\dfrac{\Delta}{2M}u + Vu + Au \\ (\Box + m_0^2)A = -|u|^2 \end{cases} ;$$

where $m_0, M > 0$ are real parameters and V is a non-negative potential that is confining or equal to zero. Using the complex field α as a dynamical variable instead of A (see Equation (47) of Section 3), the aforementioned dynamics can be seen as a Hamilton equation generated by the following energy functional, densely defined on[1] $L^2 \oplus L^2$:

$$\mathscr{E}(u,\alpha) := \Big\langle u, \Big(-\tfrac{\Delta}{2M} + V\Big)u\Big\rangle_2 + \langle \alpha, \omega\alpha\rangle_2 + \tfrac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \tfrac{1}{\sqrt{2\omega(k)}} \Big(\bar{\alpha}(k)e^{-ik\cdot x} + \alpha(k)e^{ik\cdot x}\Big)|u(x)|^2 dxdk \ .$$

With suitable assumptions on the external potential V, one proves the global existence of the associated flow $\mathbf{E}(t)$; a detailed discussion can be found in Subsection 3.3 where the precise condition on V is given by Assumption (A_V). So there is a Hilbert space[2] $\mathcal{D} = Q(-\Delta + V) \oplus \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$ densely imbedded in $L^2 \oplus L^2$ such that there exists a classical flow $\mathbf{E} : \mathbb{R} \times \mathcal{D} \to \mathcal{D}$ that solves the Schrödinger-Klein-Gordon equation (S-KG$_\alpha$[Y]) written using the complex field α.

A question of significant interest, both mathematically and physically, is whether it is possible to quantize the Schrödinger-Klein-Gordon dynamics with Yukawa coupling as a consistent theory that describes quantum mechanically the particle-field interaction. As mentioned previously, E. Nelson rigorously constructed a self-adjoint operator satisfying in some sense the above requirement. Afterwards the model was proved to satisfy some of the main properties that are familiar in the axiomatic approach to quantum fields, see [22]. Furthermore, asymptotic completeness was proved to be true in [4]. The problem of quantization of such infinite dimensional nonlinear dynamics is related to constructive quantum field theory. The general framework is as follows.

Let $\mathcal{Z}$ be a complex Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$. We define the associated symplectic structure $\Sigma(\mathcal{Z})$ as the pair $\{\mathcal{Y}, B(\,\cdot\,,\,\cdot\,)\}$ where $\mathcal{Y}$ is $\mathcal{Z}$ considered as a real Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle_r = \mathrm{Re}\langle\,\cdot\,,\,\cdot\,\rangle$, and $B(\,\cdot\,,\,\cdot\,)$ is the symplectic form defined by $B(\,\cdot\,,\,\cdot\,) = \mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle$. Following Segal [77], we define a (bosonic) quantization of the structure $\Sigma(\mathcal{Z})$ any linear map $R(\cdot)$ from $\mathcal{Y}$ to self-adjoint operators on a complex Hilbert space such that:

* The Weyl operator $W(z) = e^{iR(z)}$ is weakly continuous when restricted to any finite dimensional subspace of $\mathcal{Y}$;
* $W(z_1)W(z_2) = e^{-\frac{i}{2}B(z_1,z_2)}W(z_1 + z_2)$ for any $z_1, z_2 \in \mathcal{Y}$ (Weyl's relations).

When the dimension of $\mathcal{Z}$ is not finite, there are uncountably many irreducible unitarily inequivalent Segal quantizations of $\Sigma(\mathcal{Z})$ (or representation of Weyl's relations). A representation of particular relevance in physics is the so-called Fock representation [29, 35] on the symmetric Fock space $\Gamma_s(\mathcal{Z})$. Once this representation is considered there is a natural way to quantize polynomial functionals on $\mathcal{Z}$ into quadratic forms on $\Gamma_s(\mathcal{Z})$ according to the Wick or normal order (we briefly outline the essential

[1]Sometimes the shorthand notation $L^2 \oplus L^2$ is used instead of $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, if no confusion arises.
[2]If $V = 0$, $\mathcal{D}$ could be the whole space $L^2 \oplus L^2$.

features of Wick quantization on Section 4.3, the reader may refer to [8, 20, 31] for a more detailed presentation).

Following these rules, the formal quantization of the classical energy $\mathscr{E}$ yields a quadratic form h on the Fock space $\Gamma_s(L^2 \oplus L^2)$ which plays the role of a quantum energy. The difficulty now lies on the fact that the quadratic form h do not define straightforwardly a dynamical system (i.e. h may not be related to a self-adjoint operator). Nevertheless, according to the work of Nelson it is possible in our case to define for any $\sigma_0 \in \mathbb{R}_+$, a renormalized self-adjoint operator $H_{\mathrm{ren}}(\sigma_0)$ associated in some specific sense to h (see Section 2 for details). However, the relationship between the classical and the quantum theory at hand is obscured by the renormalization procedure and it is unclear even formally if the quantum dynamics generated by $H_{\mathrm{ren}}(\sigma_0)$ are still related to the original Schrödinger-Klein-Gordon equation. Therefore, we believe that it is mathematically interesting to verify Bohr's correspondence principle for this model.

> *Bohr's principle*: "The quantum system should reproduce, in the limit of large quantum numbers, the classical behavior."

This principle may be reformulated as follows. We make the quantization procedure dependent on a effective parameter ε, that would converge to zero in the limit. The physical interpretation of ε is of a quantity of the same order of magnitude as the Planck's constant, that becomes negligible when large energies and orbits are considered. In the Fock representation, we introduce the ε-dependence in the annihilation and creation operator valued distributions $\psi^{\#}(x)$ and $a^{\#}(k)$, whose commutation relations then become $[\psi(x), \psi^*(x')] = \varepsilon\delta(x - x')$ and $[a(k), a^*(k')] = \varepsilon\delta(k - k')$. If in the limit $\varepsilon \to 0$ the quantum unitary dynamics converges towards the Hamiltonian flow generated by the Schrödinger-Klein-Gordon equation with Yukawa interaction, Bohr's principle is satisfied.

If the phase space $\mathcal{Z}$ is finite dimensional, the quantum-classical correspondence has been proved in the context of semiclassical or microlocal analysis, with the aid of pseudo-differential calculus, Wigner measures or coherent states [see e.g. 3, 26, 28, 46, 47, 57, 59, 60, 63, 65, 75]. If $\mathcal{Z}$ is infinite dimensional, the situation is more complicated, and there are fewer results for systems with unconserved number of particles [5, 8, 12, 37, 38, 51]. The approach we adopt here makes use of the infinite-dimensional Wigner measures introduced by Ammari and Nier [8, 9, 10, 11]. Remark that Wigner measures are related to phase-space analysis and are in general an effective tool for the study of the classical limit. Given a family of normal quantum states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ on the Fock space, we say that a Borel probability measure μ on $\mathcal{Z}$ is a Wigner measure associated to it if there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0, \bar{\varepsilon})$ such that $\varepsilon_k \to 0$ and[3]

$$\lim_{k\to\infty} \mathrm{Tr}[\varrho_{\varepsilon_k} W(\xi)] = \int_{\mathcal{Z}} e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle_{\mathcal{Z}}} d\mu(z) \ , \ \forall\xi \in \mathcal{Z} \ . \tag{1}$$

We denote by $\mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0, \bar{\varepsilon})\big)$ the set of Wigner measures associated to $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$. Let $e^{-i\frac{t}{\varepsilon}H_{\mathrm{ren}}(\sigma_0)}$ be the quantum dynamics on $\Gamma_s(\mathcal{Z})$, $\mathcal{Z} = L^2 \oplus L^2$, then the time-evolved quantum states can be written as $(e^{-i\frac{t}{\varepsilon}H_{\mathrm{ren}}(\sigma_0)}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_{\mathrm{ren}}(\sigma_0)})_{\varepsilon\in(0,\bar{\varepsilon})}$. Bohr's principle is satisfied if Wigner measures of time evolved quantum states are exactly the push-forward, by the classical flow $\mathbf{E}(t)$, of the initial Wigner

[3] $W(\xi)$ is the ε_k-dependent Weyl operator explicitly defined by (66).

measures at time $t = 0$; i.e.

$$\mathcal{M}\Big(e^{-i\frac{t}{\varepsilon}H_{\rm ren}(\sigma_0)}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_{\rm ren}(\sigma_0)}, \varepsilon\in(0,\bar\varepsilon)\Big) = \Big\{\mathbf{E}(t)_{\#}\mu\,,\ \mu\in\mathcal{M}(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon))\Big\}\,. \tag{2}$$

To ensure that $\mathcal{M}\big(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon)\big)$ is not empty, it is sufficient to assume that there exist $\delta>0$ and $C>0$ such that, for any $\varepsilon\in(0,\bar\varepsilon)$, $\mathrm{Tr}[\varrho_\varepsilon N^\delta]<C$; where N is the number operator of the Fock space $\Gamma_s(\mathcal{Z})$ with $\mathcal{Z}=L^2\oplus L^2$. Actually, we make the following more restrictive assumptions: Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\Gamma_s(L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3))$, then

$$\exists\mathfrak{C}>0\,,\ \forall\varepsilon\in(0,\bar\varepsilon)\,,\ \forall k\in\mathbb{N}\,,\ \mathrm{Tr}[\varrho_\varepsilon N_1^k]\le\mathfrak{C}^k\ ; \tag{A_0}$$

$$\exists C>0\,,\ \forall\varepsilon\in(0,\bar\varepsilon)\,,\ \mathrm{Tr}[\varrho_\varepsilon(N+U_\infty^* H_0U_\infty)]\le C\ ; \tag{A_ρ}$$

where N_1 is the nucleonic number operator, $N=N_1+N_2$ the total number operator, H_0 is the free Hamiltonian defined by Equation (5) and U_∞ is the unitary quantum dressing defined in Lemma 2.3. As a matter of fact, it is possible in principle to remove Assumption (A_0), but it has an important role in connection with the parameter σ_0 related to the renormalization procedure. This condition restricts the considered states ϱ_ε to be at most with $[\mathfrak{C}/\varepsilon]$ nucleons.

We are now in a position to state precisely our result: *the Bohr's correspondence principle holds between the renormalized quantum dynamics of the Nelson model generated by $H_{\rm ren}(\sigma_0)$ and the Schrödinger-Klein-Gordon classical flow generated by $\mathscr{E}$.* The operator $H_{\rm ren}(\sigma_0)$ is constructed in Subsection 2.3 according to Definition 2.13. Recall that $\mathcal{D}=Q(-\Delta+V)\oplus\mathcal{F}H^{\frac12}(\mathbb{R}^3)$.

Theorem 1.1. *Let $\mathbf{E}:\mathbb{R}\times\mathcal{D}\to\mathcal{D}$ be the Schrödinger-Klein-Gordon flow provided by Theorem 3.15 and solving the equation (S-KG$_\alpha$[Y]) with a potential V satisfying Assumption (A_V). Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states in $\Gamma_s\big(L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\big)$ that satisfies Assumptions (A_0) and (A_ρ). Then:*

- *(i) There exists a $\sigma_0\in\mathbb{R}_+$ such that the dynamics $e^{-i\frac{t}{\varepsilon}H_{\rm ren}(\sigma_0)}$ is non-trivial on the states ϱ_ε.*
- *(ii) $\mathcal{M}\big(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon)\big)\neq\emptyset$.*
- *(iii) For any $t\in\mathbb{R}$,*

$$\mathcal{M}\Big(e^{-i\frac{t}{\varepsilon}H_{\rm ren}(\sigma_0)}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}H_{\rm ren}(\sigma_0)}, \varepsilon\in(0,\bar\varepsilon)\Big) = \Big\{\mathbf{E}(t)_{\#}\mu\,,\ \mu\in\mathcal{M}\big(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon)\big)\Big\}\,. \tag{3}$$

Furthermore, let $(\varepsilon_k)_{k\in\mathbb{N}}\subset(0,\bar\varepsilon)$ be a sequence such that $\lim_{k\to\infty}\varepsilon_k=0$ and $\mathcal{M}\big(\varrho_{\varepsilon_k},k\in\mathbb{N}\big)=\{\mu\}$, i.e.: for any $\xi\in L^2\oplus L^2$,

$$\lim_{k\to\infty}\mathrm{Tr}[\varrho_{\varepsilon_k}W(\xi)]=\int_{L^2\oplus L^2}e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}d\mu(z).$$

Then for any $t\in\mathbb{R}$, $\mathcal{M}\big(e^{-i\frac{t}{\varepsilon_k}H_{\rm ren}(\sigma_0)}\varrho_\varepsilon e^{i\frac{t}{\varepsilon_k}H_{\rm ren}(\sigma_0)},k\in\mathbb{N}\big)=\{\mathbf{E}(t)_{\#}\mu\}$, i.e.:

$$\lim_{k\to\infty}\mathrm{Tr}\Big[e^{-i\frac{t}{\varepsilon_k}H_{\rm ren}(\sigma_0)}\varrho_{\varepsilon_k}e^{i\frac{t}{\varepsilon_k}H_{\rm ren}(\sigma_0)}W(\xi)\Big]=\int_{L^2\oplus L^2}e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}d(\mathbf{E}(t)_{\#}\mu)(z)\ ,\ \forall\xi\in L^2\oplus L^2\ . \tag{4}$$

Remark 1.2.

* The choice of σ_0 is related to our Definition 2.13 of the renormalized dynamics and the localization of states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ satisfying Assumption (A_0) (see Lemma 4.2). Actually, one can take any $\sigma_0\ge 2K(\mathfrak{C}+1+\bar\varepsilon)$ where $K>0$ is a constant given in Theorem 2.10.

* We remark that every Wigner measure $\mu \in \mathcal{M}(\varrho_\varepsilon, \varepsilon \in (0,\bar{\varepsilon}))$, with $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfying Assumption (A_ρ) is a Borel probability measure on $\mathcal{D}$ equipped with its graph norm, hence the push-forward by means of the classical flow $\mathbf{E}$ is well defined (see Section 4.4).
* Adopting a shorthand notation, the last assertion of the above theorem can be written as:

$$\varrho_{\varepsilon_k} \to \mu \Leftrightarrow \left(\forall t \in \mathbb{R} \ , \ e^{-i\frac{t}{\varepsilon_k}H_{\mathrm{ren}}(\sigma_0)}\varrho_{\varepsilon_k}e^{i\frac{t}{\varepsilon_k}H_{\mathrm{ren}}(\sigma_0)} \to \mathbf{E}(t)_\#\mu\right) .$$

2. The quantum system: Nelson Hamiltonian.

In this section we define the quantum system of "nucleons" interacting with a meson field, and briefly review the standard renormalization procedure due to Nelson [68]. Since we are interested in the classical limit and our original and dressed Hamiltonians depend on an effective parameter $\varepsilon \in (0,\bar{\varepsilon})$, it is necessary to check that several known estimates of the quantum theory are uniform with respect to ε. This step is crucial and motivated our brief revisit of the Nelson renormalization procedure.

On $\mathcal{H} = \Gamma_s(L^2(\mathbb{R}^3)) \otimes \Gamma_s(L^2(\mathbb{R}^3))$ we define the following free Hamiltonian as the positive self-adjoint operator given by:

$$H_0 = \int_{\mathbb{R}^3} \psi^*(x)\Big(-\tfrac{\Delta}{2M} + V(x)\Big)\psi(x)dx + \int_{\mathbb{R}^3} a^*(k)\omega(k)a(k)dk = d\Gamma(-\tfrac{\Delta}{2M} + V) + d\Gamma(\omega) \ , \tag{5}$$

where $V \in L^2_{loc}(\mathbb{R}^3, \mathbb{R}_+)$. We denote its domain of self-adjointness by $D(H_0)$. We denote by $d\Gamma$ the second quantization acting either on the first or second Fock space, when no confusion arises.

Now let $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^3)$; $0 \leq \chi \leq 1$ and $\chi \equiv 1$ if $|k| \leq 1$, $\chi \equiv 0$ if $|k| \geq 2$. Then, for all $\sigma > 0$ define $\chi_\sigma(k) = \chi(k/\sigma)$; it will play the role of an ultraviolet cutoff in the interaction. The Nelson Hamiltonian with cutoff has thus the form:

$$H_\sigma = H_0 + \tfrac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \psi^*(x)\Big(a^*\big(\tfrac{e^{-ik\cdot x}}{\sqrt{2\omega}}\chi_\sigma\big) + a\big(\tfrac{e^{-ik\cdot x}}{\sqrt{2\omega}}\chi_\sigma\big)\Big)\psi(x)dx \ . \tag{6}$$

We will denote the interaction part by $H_I(\sigma) = H_\sigma - H_0$.

Remark 2.1. There is no loss of generality in the choice of χ as a radial function [see 4, Proposition 3.9].

The following proposition shows the self-adjointness of H_σ, see e.g. [5, Proposition 2.5] or [33].

Proposition 2.2. *For any $\sigma > 0$, H_σ is essentially self-adjoint on $D(H_0) \cap \mathcal{C}_0^\infty(N)$.*

To obtain a meaningful limit when $\sigma \to \infty$, we use a dressing transformation, introduced in the physics literature by Greenberg and Schweber [55] following the work of van Hove [80, 81]. The dressing and the renormalization procedure are described in Sections 2.1 and 2.2 respectively. In Section 2.3 we discuss a possible extension of the renormalized Hamiltonian on $\mathcal{H}_n$ to the whole Fock space $\mathcal{H}$. The extension we choose is not the only possible one, however the choice is motivated by two facts: other extensions should provide the same classical limit, and our choice $\hat{H}_{\mathrm{ren}}(\sigma_0)$ is, in our opinion, more consistent with the quantization procedure of the classical energy functional.

2.1. **Dressing.** The dressing transform was introduced as an alternative way of doing renormalization in the Hamiltonian formalism, and has been utilized in a rigorous fashion in various situations [see e.g.

48, 52, 58, 68]. For the Nelson Hamiltonian, it consists of a unitary transformation that singles out the singular self-energy.

From now on, let $0 < \sigma_0 < \sigma$, with σ_0 fixed. Then define:

$$g_\sigma(k) = -\frac{i}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(k)}} \frac{\chi_\sigma(k) - \chi_{\sigma_0}(k)}{\frac{k^2}{2M} + \omega(k)} \; ; \tag{7}$$

$$E_\sigma = \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{\omega(k)} \frac{(\chi_\sigma - \chi_{\sigma_0})^2(k)}{\frac{k^2}{2M} + \omega(k)} dk - \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\chi_\sigma(k)}{\omega(k)} \frac{(\chi_\sigma - \chi_{\sigma_0})(k)}{\frac{k^2}{2M} + \omega(k)} dk \; . \tag{8}$$

The dressing transformation is the unitary operator generated by (the dependence on σ_0 will be usually omitted):

$$T_\sigma = \int_{\mathbb{R}^3} \psi^*(x) \Big(a^*(g_\sigma e^{-ik\cdot x}) + a(g_\sigma e^{-ik\cdot x}) \Big) \psi(x) dx \; . \tag{9}$$

The function $g_\sigma \in L^2(\mathbb{R}^3)$ for all $\sigma \leq \infty$; therefore it is possible to prove the following Lemma, e.g. utilizing the criterion of [33].

Lemma 2.3. *For any $\sigma \leq \infty$, T_σ is essentially self-adjoint on $\mathcal{C}_0^\infty(N)$. We denote by $U_\sigma(\theta)$ the corresponding one-parameter unitary group $U_\sigma(\theta) = e^{-i\frac{\theta}{\varepsilon}T_\sigma}$.*

For the sake of brevity, we will write $U_\sigma := U_\sigma(-1)$. We remark that T_σ and H_σ preserve the number of "nucleons", i.e.: for any $\sigma \leq \infty$, $\sigma' < \infty$:

$$[T_\sigma, N_1] = 0 = [H_{\sigma'}, N_1] \; . \tag{10}$$

The above operators also commute in the resolvent sense. We are now in position to define the dressed Hamiltonian

$$\hat{H}_\sigma := U_\sigma \big(H_\sigma - \varepsilon N_1 E_\sigma \big) U_\sigma^* \; . \tag{11}$$

The operator $\hat{H}_\sigma$ is self-adjoint for any $\sigma < \infty$, since H_σ and N_1 are commuting self-adjoint operators and U_σ is unitary. The purpose is to show that the quadratic form associated with $\hat{H}_\sigma\big|_{\mathcal{H}_n}$ satisfies the hypotheses of KLMN theorem, even when $\sigma = \infty$, so it is possible to define uniquely a self-adjoint operator $\hat{H}_\infty$. In order to do that, we need to study in detail the form associated with $\hat{H}_\sigma^{(n)}$.

By Equation (11), it follows immediately that

$$\hat{H}_\sigma^{(n)} = \varepsilon U_\sigma^{(n)} \left(\frac{H_\sigma^{(n)}}{\varepsilon} - (\varepsilon n) E_\sigma \right) (U_\sigma^{(n)})^* \; . \tag{12}$$

A suitable calculation [4, 68] yields:

$$\begin{aligned} \hat{H}_\sigma^{(n)} = H_{\sigma_0}^{(n)} + \varepsilon^2 \sum_{i<j} V_\sigma(x_i - x_j) + \frac{\varepsilon}{2M} \sum_{j=1}^n \Bigg(& \Big(a^*(r_\sigma e^{-ik\cdot x_j})^2 + a(r_\sigma e^{-ik\cdot x_j})^2 \Big) \\ & + 2a^*(r_\sigma e^{-ik\cdot x_j}) a(r_\sigma e^{-ik\cdot x_j}) - 2\Big(D_{x_j} a(r_\sigma e^{-ik\cdot x_j}) + a^*(r_\sigma e^{-ik\cdot x_j}) D_{x_j} \Big) \Bigg) \; ; \end{aligned} \tag{13}$$

where $D_{x_j} = -i\nabla_{x_j}$ and

$$r_\sigma(k) = -ikg_\sigma(k) \, ,$$

$$V_\sigma(x) = 2\mathrm{Re}\int_{\mathbb{R}^3} \omega(k)|g_\sigma(k)|^2 e^{-ik\cdot x} dk - 4\mathrm{Im}\int_{\mathbb{R}^3} \frac{\bar{g}_\sigma(k)}{(2\pi)^{3/2}} \frac{\chi_\sigma(k)}{\sqrt{2\omega(k)}} e^{-ik\cdot x} dk \ . \tag{14}$$

It is also possible to write $\hat{H}_\sigma$ in its second quantized form as:

$$\hat{H}_\sigma = H_0 + \hat{H}_I(\sigma) \ ; \tag{15}$$

$$\begin{aligned} \hat{H}_I(\sigma) = H_I(\sigma_0) + \frac{1}{2}\int_{\mathbb{R}^6} \psi^*(x)\psi^*(y)V_\sigma(x-y)\psi(x)\psi(y)dxdy \\ +\tfrac{1}{2M}\int_{\mathbb{R}^3} \psi^*(x)\Bigl(\Bigl(a^*(r_\sigma e^{-ik\cdot x})^2 + a(r_\sigma e^{-ik\cdot x})^2\Bigr) + 2a^*(r_\sigma e^{-ik\cdot x})a(r_\sigma e^{-ik\cdot x}) \\ -2\Bigl(D_x a(r_\sigma e^{-ik\cdot x}) + a^*(r_\sigma e^{-ik\cdot x})D_x\Bigr)\Bigr)\psi(x)dx \ . \end{aligned} \tag{16}$$

Remark 2.4. The dressed interaction Hamiltonian $\hat{H}_I(\sigma)$ contains a first term analogous to the undressed interaction with cutoff, a second term of two-body interaction between nucleons, and a more singular term that can be only defined as a form when $\sigma = \infty$.

2.2. **Renormalization.** We will now define the renormalized self-adjoint operator $\hat{H}_\infty^{(n)}$. A simple calculation shows that $E_\sigma \to -\infty$ when $\sigma \to +\infty$; hence the subtraction of the self-energy in the definition (11) of $\hat{H}_\sigma$ is necessary. It is actually the only renormalization necessary for this system. We prove that the quadratic form associated with $\hat{H}_\sigma^{(n)}$ of Equation (13) has meaning for any $\sigma \le \infty$, and the KLMN theorem [see 73, Theorem X.17] can be applied, with a suitable choice of σ_0, and bounds that are uniform with respect to $\varepsilon \in (0, \bar{\varepsilon})$. Let us start with some preparatory lemmas:

Lemma 2.5. *For any $0 \le \sigma \le \infty$, the symmetric function V_σ satisfies:*

(i) $V_\sigma(1-\Delta)^{-1/2} \in \mathcal{L}(L^2(\mathbb{R}^3))$;

(ii) $(1-\Delta)^{-1/2}V_\sigma(1-\Delta)^{-1/2} \in \mathcal{K}(L^2(\mathbb{R}^3))$.

In particular, $V_\sigma \in L^s(\mathbb{R}^3) \cap L^{3,\infty}(\mathbb{R}^3)$, for any $s \in [2, +\infty[$.

Proof. It is sufficient to show [11, Corollary D.6] that $V_\sigma \in L^{3,\infty}(\mathbb{R}^3)$ (weak-L^p spaces). Write $V_\sigma = V_\sigma^{(1)} + V_\sigma^{(2)}$,

$$V_\sigma^{(1)}(x) = 2\mathrm{Re}\int_{\mathbb{R}^3} \omega(k)|g_\sigma(k)|^2 e^{-ik\cdot x} dk = 2(2\pi)^{3/2}\mathrm{Re}\mathcal{F}\Bigl(\omega|g_\sigma|^2\Bigr)(x) \ ; \tag{17}$$

$$V_\sigma^{(2)}(x) = -2\sqrt{2}\mathrm{Im}\int_{\mathbb{R}^3} \frac{\bar{g}_\sigma(k)}{(2\pi)^{3/2}} \frac{\chi_\sigma(k)}{\sqrt{\omega(k)}} e^{-ik\cdot x} dk = -2\sqrt{2}\mathrm{Im}\mathcal{F}\Bigl(\bar{g}_\sigma \frac{\chi_\sigma}{\sqrt{\omega}}\Bigr)(x) \ . \tag{18}$$

- $\bigl[V_\sigma^{(1)}\bigr]$. For any $\sigma \le \infty$, $\omega|g_\sigma|^2 \in L^{s'}(\mathbb{R}^3)$, $1 \le s' \le 2$. Then $V_\sigma^{(1)} \in L^s(\mathbb{R}^3)$ for any $s \in [2, +\infty]$; furthermore $V_\sigma^{(1)} \in \mathcal{C}_0(\mathbb{R}^3)$ (the space of continuous functions converging to zero at infinity). Hence $V_\sigma^{(1)} \in L^{3,\infty}(\mathbb{R}^3)$.
- $\bigl[V_\sigma^{(2)}\bigr]$. For any $\sigma \le \infty$, $\bar{g}_\sigma \frac{\chi_\sigma}{\sqrt{\omega}} \in L^{s'}(\mathbb{R}^3)$, $1 < s' \le 2$. Therefore $V_\sigma^{(2)} \in L^s(\mathbb{R}^3)$ for any $s \in [2, +\infty[$. It remains to show that $V_\sigma^{(2)} \in L^{3,\infty}(\mathbb{R}^3)$. Define $f(k) \in L^2(\mathbb{R}^3)$:

$$f(k) := \frac{\chi_\sigma(k)}{\omega(k)} \frac{\bigl(\chi_\sigma - \chi_{\sigma_0}\bigr)(k)}{\frac{k^2}{2M} + \omega(k)} \ . \tag{19}$$

Then there is a constant $c > 0$ such that $|V_\sigma^{(2)}(x)| \le c|\mathcal{F}(f)(x)|$, where the Fourier transform is intended to be on $L^2(\mathbb{R}^3)$. The function f is radial, so we introduce the spherical coordinates

$(r,\theta,\phi)\equiv k\in\mathbb{R}^3$, such that the z-axis coincides with the vector x. We then obtain:

$$\begin{aligned}\lim_{R\to+\infty}\int_{B(0,R)} f(k)e^{-ik\cdot x}dk &= \lim_{R\to+\infty}\int_0^R dr\int_0^\pi d\theta\int_0^{2\pi} d\phi\; r^2 f(r)e^{-ir|x|\cos\theta}\sin\theta\\ &= 2\pi\lim_{R\to+\infty}\int_0^R dr\int_{-1}^1 dy\; r^2 f(r)e^{-ir|x|y} = \frac{4\pi}{|x|}\lim_{R\to+\infty}\int_0^R f(r)r\sin(r|x|)dr\;.\end{aligned}$$

Since for any $\sigma\leq+\infty$, $f(r)r\in L^1(\mathbb{R})$ we can take the limit $R\to+\infty$ and conclude:

$$\mathcal{F}(f)(x)=\frac{4\pi}{|x|}\int_0^{+\infty} f(r)r\sin(r|x|)dr\;. \tag{20}$$

Therefore, for any $x\in\mathbb{R}^3\setminus\{0\}$, there exists a $0<\tilde{c}\leq 4\pi c\|f(r)r\|_{L^1(\mathbb{R})}$ such that:

$$|V_\sigma^{(2)}(x)|\leq\frac{\tilde{c}}{|x|}\;. \tag{21}$$

Let λ be the Lebesgue measure in $\mathbb{R}^3$. Since $\{x:|V_\sigma^{(2)}|>t\}\subset\{x:\frac{\tilde{c}}{|x|}>t\}$, there is a positive C such that:

$$\lambda\Big\{x:|V_\sigma^{(2)}(x)|>t\Big\}\leq\lambda\Big\{x:\frac{\tilde{c}}{|x|}>t\Big\}\leq\frac{C}{t^3}\;. \tag{22}$$

Finally (22) implies $V_\sigma^{(2)}\in L^{3,\infty}(\mathbb{R}^3)$.

⊣

Lemma 2.6. *There exists $c>0$ such that for any $\varepsilon\in(0,\bar{\varepsilon})$, $\sigma\leq+\infty$:*

$$\Big\|\big[(H_0+1)^{-1/2}D_{x_j}a(r_\sigma e^{-ik\cdot x_j})(H_0+1)^{-1/2}\big]^{(n)}\Big\|_{\mathcal{L}(\mathcal{H}_n)}\leq\frac{c}{\sqrt{n\varepsilon}}\|\omega^{-1/2}r_\sigma\|_2\;; \tag{23}$$

$$\Big\|\big[(H_0+1)^{-1/2}a^*(r_\sigma e^{-ik\cdot x_j})D_{x_j}(H_0+1)^{-1/2}\big]^{(n)}\Big\|_{\mathcal{L}(\mathcal{H}_n)}\leq\frac{c}{\sqrt{n\varepsilon}}\|\omega^{-1/2}r_\sigma\|_2\;. \tag{24}$$

Moreover, (23) holds if we replace the left H_0 by $d\Gamma(-\frac{\Delta}{2M}+V)$ and the right H_0 by $d\Gamma(\omega)$ and similarly (24) holds if we replace the left H_0 by $d\Gamma(\omega)$ and the right H_0 by $d\Gamma(-\frac{\Delta}{2M}+V)$.

Proof. Let $S_n\equiv S_n\otimes 1$ be the symmetrizer on $\mathcal{H}_n$ (acting only on the $\{x_1,\dots,x_n\}$ variables) and $\Psi_n\in\mathcal{H}_n$ with $n>0$. Then:

$$\langle\Psi_n,d\Gamma(-\Delta)\Psi_n\rangle=\langle\Psi_n,(n\varepsilon)S_n(D_{x_1})^2\otimes 1^{n-1}\Psi_n\rangle=(n\varepsilon)\langle\Psi_n,(D_{x_j})^2\Psi_n\rangle\;.$$

Hence $(n\varepsilon)\|D_{x_j}\Psi_n\|^2\leq\big\|\big(d\Gamma(-\Delta)+1\big)^{1/2}\Psi_n\big\|^2$. It follows that

$$\Big\|\big[D_{x_j}\big(d\Gamma(-\Delta)+1\big)^{-1/2}\big]^{(n)}\Big\|_{\mathcal{L}(\mathcal{H}_n)}\leq\frac{1}{\sqrt{n\varepsilon}}\;;\;\Big\|\big[\big(d\Gamma(-\Delta)+1\big)^{-1/2}D_{x_j}\big]^{(n)}\Big\|_{\mathcal{L}(\mathcal{H}_n)}\leq\frac{1}{\sqrt{n\varepsilon}}\;. \tag{25}$$

Using (25) we obtain for any $\Psi_n\in\mathcal{H}_n$, with $\|\Psi_n\|=1$:

$$\begin{aligned}\Big\|(H_0+1)^{-1/2}D_{x_j}a(r_\sigma e^{-ik\cdot x_j})(H_0+1)^{-1/2}\Psi_n\Big\| &\leq \frac{c}{\sqrt{n\varepsilon}}\Big\|a(r_\sigma e^{-ik\cdot x_j})\big(d\Gamma(\omega)+1\big)^{-1/2}\Psi_n\Big\|\\ &\leq \frac{c}{\sqrt{n\varepsilon}}\|\omega^{-1/2}r_\sigma\|_2\;;\end{aligned}$$

where the last inequality follows from standard estimates on the Fock space [see 5, Lemma 2.1]. The bound (24) is obtained by adjunction. ⊣

Lemma 2.7. *There exists $c>0$ such that for any $\varepsilon\in(0,\bar{\varepsilon})$, $\sigma\leq+\infty$:*

$$\Big\|\big[(H_0+1)^{-1/2}a^*(r_\sigma e^{-ik\cdot x_j})a(r_\sigma e^{-ik\cdot x_j})(H_0+1)^{-1/2}\big]^{(n)}\Big\|_{\mathcal{L}(\mathcal{H}_n)}\leq c\|\omega^{-1/2}r_\sigma\|_2^2\;;\tag{26}$$

$$\Big\|\big[(H_0+1)^{-1/2}\big(a^*(r_\sigma e^{-ik\cdot x_j})\big)^2(H_0+1)^{-1/2}\big]^{(n)}\Big\|_{\mathcal{L}(\mathcal{H}_n)}\leq c\|\omega^{-1/4}r_\sigma\|_2^2\;;\tag{27}$$

$$\Big\|\big[(H_0+1)^{-1/2}\big(a(r_\sigma e^{-ik\cdot x_j})\big)^2(H_0+1)^{-1/2}\big]^{(n)}\Big\|_{\mathcal{L}(\mathcal{H}_n)}\leq c\|\omega^{-1/4}r_\sigma\|_2^2\;.\tag{28}$$

The same bounds hold if H_0 is replaced by $d\Gamma(\omega)$.

Proof. First of all observe that, since $m_0>0$, there exists $c>0$ such that, uniformly in $\varepsilon\in(0,\bar{\varepsilon})$:

$$\Big\|(H_0+1)^{-1/2}\big(d\Gamma(\omega)+1\big)^{1/2}\Big\|_{\mathcal{L}(\mathcal{H})}\leq c\;;\;\Big\|(H_0+1)^{-1/2}(N_2+1)^{1/2}\Big\|_{\mathcal{L}(\mathcal{H})}\leq c\;.$$

Equation (26) is easy to prove:

$$\begin{aligned}\Big\|\big[(H_0+1)^{-1/2}a^*(r_\sigma e^{-ik\cdot x_j})a(r_\sigma e^{-ik\cdot x_j})(H_0+1)^{-1/2}\big]^{(n)}\Big\|_{\mathcal{L}(\mathcal{H}_n)}\leq c\Big\|\big[\big(d\Gamma(\omega)+1\big)^{-1/2}\\ a^*(r_\sigma e^{-ik\cdot x_j})\big]^{(n)}\Big\|_{\mathcal{L}(\mathcal{H}_n)}\cdot\Big\|\big[a(r_\sigma e^{-ik\cdot x_j})\big(d\Gamma(\omega)+1\big)^{-1/2}\big]^{(n)}\Big\|_{\mathcal{L}(\mathcal{H}_n)}\leq c\|\omega^{-1/2}r_\sigma\|_2^2\;.\end{aligned}$$

For the proof of (27) the reader may refer to [4, Lemma 3.3 (iv)]. Finally (28) follows from (27) by adjunction. ⊣

Lemma 2.8. *There exists $c(\sigma_0)>0$ such that for any $\varepsilon\in(0,\bar{\varepsilon})$ and $\lambda\geq1$:*

$$\Big\|\big[\big(H_0+\lambda\big)^{-1/2}H_I(\sigma_0)\big(H_0+\lambda\big)^{-1/2}\big]^{(n)}\Big\|_{\mathcal{L}(\mathcal{H}_n)}\leq c(\sigma_0)\lambda^{-1/2}(n\varepsilon)\;;\tag{29}$$

$$\Big\|\big[\big(H_0+\lambda\big)^{-1/2}\varepsilon^2\sum_{i<j}V_\sigma(x_i-x_j)\big(H_0+\lambda\big)^{-1/2}\big]^{(n)}\Big\|_{\mathcal{L}(\mathcal{H}_n)}\leq c(\sigma_0)\lambda^{-1/2}\sqrt{n\varepsilon(1+n\varepsilon)}\;.\tag{30}$$

Proof. The inequality (29) can be proved by a standard argument on the Fock space [see e.g. 32, Proposition IV.1].

To prove (30) we proceed as follows. First of all, by means of (i), Lemma 2.5 we can write:

$$\begin{aligned}\Big\|\big(-\Delta_{x_i}+\lambda\big)^{-1/2}V_\sigma(x_i-x_j)\big(-\Delta_{x_i}+\lambda\big)^{-1/2}\Big\|_{\mathcal{L}(\mathcal{H}_n)}\leq\lambda^{-1/2}\Big\|V_\sigma(x_i)\big(-\Delta_{x_i}+\lambda\big)^{-1/2}\Big\|_{\mathcal{L}(\mathcal{H}_n)}\\ \leq c(\sigma_0)\lambda^{-1/2}\;.\end{aligned}$$

Therefore $V_\sigma(x_i-x_j)\leq c(\sigma_0)\lambda^{-1/2}\big(-\Delta_{x_i}+\lambda\big)$. Let $\Psi_n\in\mathcal{H}_n$; using its symmetry, and some algebraic manipulations we can write:

$$\begin{aligned}\big\langle\Psi_n,\varepsilon^2\sum_{i<j}V_\sigma(x_i-x_j)\Psi_n\big\rangle&\leq c(\sigma_0)(n\varepsilon)^2\big\langle\Psi_n,\big(\lambda^{-1/2}(D_{x_1})^2+\lambda^{1/2}\big)\Psi_n\big\rangle\\ &=c(\sigma_0)\big\langle\Psi_n,N_1\big(\lambda^{-1/2}d\Gamma(D_x^2)+\lambda^{1/2}N_1\big)\Psi_n\big\rangle\\ \leq c(\sigma_0)\lambda^{-1/2}\Big[\Big\|N_1^{1/2}\big(d\Gamma(D_x^2)+\lambda\big)^{1/2}\Psi_n\Big\|^2&+\Big\|N_1\big(d\Gamma(D_x^2)+\lambda\big)^{1/2}\Psi_n\Big\|^2\Big]\\ &\leq c(\sigma_0)\lambda^{-1/2}\big\langle\Psi_n,\big(N_1+N_1^2\big)\big(d\Gamma(D_x^2)+\lambda\big)\Psi_n\big\rangle\;;\end{aligned}$$

where the constant $c(\sigma_0)$ is redefined in each inequality. The result follows since N_1 commutes with $d\Gamma(D_x^2)$. ⊣

Combining Lemmas 2.6, 2.7 and 2.8 together, we can prove easily the following proposition.

Proposition 2.9. *There exist $c > 0$ and $c(\sigma_0) > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$, $\lambda \geq 1$, $\sigma_0 < \sigma \leq +\infty$ and for any $\Psi \in D(N_1)$:*

$$\begin{aligned}\Big\| \big(H_0+\lambda\big)^{-1/2} \hat{H}_I(\sigma) \big(H_0+\lambda\big)^{-1/2}\Psi\Big\| \leq \Big[& c\big(\|\omega^{-1/2} r_\sigma\|_2^2 + \|\omega^{-1/4} r_\sigma\|_2^2 + \|\omega^{-1/2} r_\sigma\|_2\big) \\ & + c(\sigma_0)\lambda^{-1/2}\Big] \cdot \Big\|(N_1+1)\Psi\Big\| \ . \end{aligned} \tag{31}$$

Consider now $\hat{H}_I(\sigma)^{(n)}$. It follows easily from Equation (31) above that for any $\sigma_0 < \sigma \leq +\infty$, and $\Psi_n \in D(H_0^{1/2}) \cap \mathcal{H}_n$:

$$\begin{aligned}\Big|\big\langle \Psi_n, \hat{H}_I(\sigma)^{(n)}\Psi_n\big\rangle\Big| \leq & \Big[c(n\varepsilon+1)\big(\|\omega^{-1/2} r_\sigma\|_2^2 + \|\omega^{-1/4} r_\sigma\|_2^2 + \|\omega^{-1/2} r_\sigma\|_2\big) \\ & + c(\sigma_0)(n\varepsilon+1)\lambda^{-1/2}\Big]\big\langle \Psi_n, H_0^{(n)}\Psi_n\big\rangle \\ & + \lambda\Big[c(n\varepsilon+1)\big(\|\omega^{-1/2} r_\sigma\|_2^2 + \|\omega^{-1/4} r_\sigma\|_2^2 + \|\omega^{-1/2} r_\sigma\|_2\big) \\ & + c(\sigma_0)(n\varepsilon+1)\lambda^{-1/2}\Big]\big\langle \Psi_n, \Psi_n\big\rangle \ . \end{aligned} \tag{32}$$

Consider now the term $\big(\|\omega^{-1/2} r_\sigma\|_2^2 + \|\omega^{-1/4} r_\sigma\|_2^2 + \|\omega^{-1/2} r_\sigma\|_2\big)$; by definition of r_σ, there exists $c > 0$ such that, uniformly in $\sigma \leq +\infty$:

$$\|\omega^{-1/2} r_\sigma\|_2^2 + \|\omega^{-1/4} r_\sigma\|_2^2 + \|\omega^{-1/2} r_\sigma\|_2 \leq c\big(\sigma_0^{-2} + \sigma_0^{-1}\big) \ . \tag{33}$$

Hence for any $\sigma_0 \geq 1$ there exist $K > 0$ ($K = 2c$), $c(\sigma_0) > 0$ and $C(n, \varepsilon, \lambda, \sigma_0) > 0$ such that (32) becomes:

$$\Big|\big\langle \Psi_n, \hat{H}_I(\sigma)^{(n)}\Psi_n\big\rangle\Big| \leq \Big[\tfrac{K(n\varepsilon+1)}{\sigma_0} + c(\sigma_0)(n\varepsilon+1)\lambda^{-1/2}\Big]\big\langle \Psi_n, H_0^{(n)}\Psi_n\big\rangle + C(n,\varepsilon,\lambda,\sigma_0)\big\langle \Psi_n, \Psi_n\big\rangle \ . \tag{34}$$

Therefore choosing

$$\sigma_0 > 2K(n\varepsilon+1) \tag{35}$$

and then $\lambda > \big(2c(\sigma_0)(n\varepsilon+1)\big)^2$, we obtain the following bound for any $\Psi_n \in D(H_0^{1/2}) \cap \mathcal{H}_n$, with $a < 1$, $b > 0$ and uniformly in $\sigma_0 < \sigma \leq +\infty$:

$$\Big|\big\langle \Psi_n, \hat{H}_I(\sigma)^{(n)}\Psi_n\big\rangle\Big| \leq a\big\langle \Psi_n, H_0^{(n)}\Psi_n\big\rangle + b\big\langle \Psi_n, \Psi_n\big\rangle \ . \tag{36}$$

Applying KLMN theorem, (36) proves the following result [see e.g. 4, 68, for additional details].

Theorem 2.10. *There exists $K > 0$ such that, for any $n \in \mathbb{N}$, and $\varepsilon \in (0, \bar{\varepsilon})$ the following statements hold:*

(i) For any $\big(2K(n\varepsilon+1)\big) < \sigma_0 < \sigma \leq +\infty$, there exists a unique self-adjoint operator $\hat{H}_\sigma^{(n)}$ with domain $\hat{D}_\sigma^{(n)} \subset D\big((H_0^{(n)})^{1/2}\big) \subset \mathcal{H}_n$ associated to the symmetric form $\hat{h}_\sigma^{(n)}(\cdot,\cdot)$, defined for any $\Psi, \Phi \in D\big((H_0^{(n)})^{1/2}\big)$ as:

$$\hat{h}_\sigma^{(n)}(\Psi,\Phi) = \big\langle \Psi, H_0^{(n)}\Phi\big\rangle + \big\langle \Psi, \hat{H}_I(\sigma)^{(n)}\Phi\big\rangle \ . \tag{37}$$

The operator $\hat{H}_\sigma^{(n)}$ is bounded from below, with bound $-b_{\sigma_0}(\sigma)$ (where $|b_{\sigma_0}(\sigma)|$ is a bounded increasing function of σ).

(ii) The following convergence holds in the norm topology of $\mathcal{L}(\mathcal{H}_n)$:

$$\lim_{\sigma\to+\infty} (z - \hat{H}_\sigma^{(n)})^{-1} = (z - \hat{H}_\infty^{(n)})^{-1} \ , \quad \textit{for all} \quad z \in \mathbb{C} \setminus \mathbb{R}. \tag{38}$$

(iii) For any $t \in \mathbb{R}$, the following convergence holds in the strong topology of $\mathcal{L}(\mathcal{H}_n)$:

$$s - \lim_{\sigma \to +\infty} e^{-i\frac{t}{\varepsilon}\hat{H}_\sigma^{(n)}} = e^{-i\frac{t}{\varepsilon}\hat{H}_\infty^{(n)}} \ . \tag{39}$$

Remark 2.11. The operator $\hat{H}_\infty^{(n)}$ can be decomposed only in the sense of forms, i.e.

$$\hat{H}_\infty^{(n)} = H_0^{(n)} \dotplus \hat{H}_I^{(n)}(\infty) \ ; \tag{40}$$

where $\dotplus$ has to be intended as the form sum.

2.3. **Extension of $\hat{H}_\infty^{(n)}$ to $\mathcal{H}$.** We have defined the self-adjoint operator $\hat{H}_\infty^{(n)}$ which depends in σ_0 for each $n \in \mathbb{N}$. Now we are interested in extending it to the whole space $\mathcal{H}$. This can be done in at least two different ways. However, we choose the one that is more suitable to interpret $\hat{H}_\infty$ as the Wick quantization of a classical symbol.

Let K be defined by Theorem 2.10. Then define $\mathfrak{N}(\varepsilon, \sigma_0) \in \mathbb{N}$ by:

$$\mathfrak{N}(\varepsilon, \sigma_0) = \left[\frac{\sigma_0 - 2K}{2K\varepsilon} - 1\right] \ ; \tag{41}$$

where the square brackets mean that we take the integer part if the number within is positive, zero otherwise.

Definition 2.12 ($\hat{H}_{\mathrm{ren}}(\sigma_0)$)**.** *Let $0 \leq \sigma_0 < +\infty$ be fixed. Then we define $\hat{H}_{\mathrm{ren}}(\sigma_0)$ on $\mathcal{H}$ by:*

$$\hat{H}_{ren}(\sigma_0)\big|_{\mathcal{H}_n} = \begin{cases} \hat{H}_\infty^{(n)} & \text{if } n \leq \mathfrak{N}(\varepsilon, \sigma_0) \\ 0 & \text{if } n > \mathfrak{N}(\varepsilon, \sigma_0) \end{cases} \tag{42}$$

where $\mathfrak{N}(\varepsilon, \sigma_0)$ is defined by (41). We may also write $\hat{H}_{\mathrm{ren}}(\sigma_0) = H_0 \dotplus \hat{H}_{ren,I}(\sigma_0)$ as a sum of quadratic forms.

The operator $\hat{H}_{\mathrm{ren}}(\sigma_0)$ is self-adjoint on $\mathcal{H}$, with domain of self adjointness:

$$\hat{D}_{ren}(\sigma_0) = \left\{\Psi \in \mathcal{H} \ , \ \Psi\big|_{\mathcal{H}_n} \in \hat{D}_\infty^{(n)} \text{ for any } n \leq \mathfrak{N}(\varepsilon, \sigma_0)\right\} \ . \tag{43}$$

Acting with the dressing operator U_∞ defined in Lemma 2.3 (with the same fixed σ_0 as for $\hat{H}_{ren}(\sigma_0)$), we can also define the undressed extension $H_{\mathrm{ren}}(\sigma_0)$.

Definition 2.13 ($H_{\mathrm{ren}}(\sigma_0)$)**.** *Let $0 \leq \sigma_0 < +\infty$ be fixed. Then we define the following operator on $\mathcal{H}$:*

$$H_{ren}(\sigma_0) = U_\infty^*(\sigma_0)\hat{H}_{ren}(\sigma_0)U_\infty(\sigma_0) \ . \tag{44}$$

The operator $H_{\mathrm{ren}}(\sigma_0)$ is self-adjoint on $\mathcal{H}$, with domain of self adjointness:

$$D_{ren}(\sigma_0) = \left\{\Psi \in \mathcal{H} \ , \ \Psi\big|_{\mathcal{H}_n} \in e^{-\frac{i}{\varepsilon}T_\infty^{(n)}}\hat{D}_\infty^{(n)} \text{ for any } n \leq \mathfrak{N}(\varepsilon, \sigma_0)\right\} \ . \tag{45}$$

Remark 2.14. Let $\sigma_0 \geq 0$ be fixed. Then the $\hat{H}_\sigma$ given by (11) defines, in the limit $\sigma \to \infty$, a symmetric quadratic form $\hat{h}_\infty$ on $D(H_0^{1/2}) \subset \mathcal{H}$. Also $\hat{H}_{\mathrm{ren}}(\sigma_0)$ defines a quadratic form $\hat{h}_{\mathrm{ren}}$. We have[4]:

$$\hat{h}_\infty(\mathbb{1}_{[0,\mathfrak{N}]}(N_1)\,\cdot\,,\,\cdot\,) = \hat{h}_{\mathrm{ren}}(\mathbb{1}_{[0,\mathfrak{N}]}(N_1)\,\cdot\,,\,\cdot\,) \ . \tag{46}$$

However, we are not able to prove that there is a self-adjoint operator on $\mathcal{H}$ associated to $\hat{h}_\infty$, and it is possible that there is none.

[4] $\mathbb{1}_{[0,\mathfrak{N}]}(N_1)$ is the orthogonal projector on $\bigoplus_{n=0}^{\mathfrak{N}} \mathcal{H}_n$.

3. The classical system: S-KG equations.

In this section we define the Schrödinger-Klein-Gordon (S-KG) system, with initial data in a suitable dense subset of $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, that describes the classical dynamics of a particle-field interaction. Then we introduce the classical dressing transformation (viewed itself as a dynamical system), and then study the transformation it induces on the Hamiltonian functional. Finally, we discuss the global existence of unique solutions of the classical equations, both in their original and dressed form.

The Yukawa coupling: The S-KG[Y] system (Schrödinger-Klein-Gordon with Yukawa interaction), or undressed classical equations, is defined by:

$$\text{(S-KG[Y])} \qquad \begin{cases} i\partial_t u = -\dfrac{\Delta}{2M}u + Vu + Au \\ (\Box + m_0^2)A = -|u|^2 \end{cases} ;$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is an external potential. If we introduce the complex field α, defined by

$$A(x) = \tfrac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \tfrac{1}{\sqrt{2\omega(k)}} \big(\bar{\alpha}(k)e^{-ik\cdot x} + \alpha(k)e^{ik\cdot x}\big)dk \; , \tag{47}$$

$$\dot{A}(x) = -\tfrac{i}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \sqrt{\tfrac{\omega(k)}{2}} \big(\alpha(k)e^{ik\cdot x} - \bar{\alpha}(k)e^{-ik\cdot x}\big)dk \; , \tag{48}$$

we can rewrite (S-KG[Y]) as the equivalent system[5]:

$$\text{(S-KG}_\alpha\text{[Y])} \qquad \begin{cases} i\partial_t u = -\dfrac{\Delta}{2M}u + Vu + Au \\ i\partial_t \alpha = \omega\alpha + \dfrac{1}{\sqrt{2\omega}}\mathcal{F}\big(|u|^2\big) \end{cases} .$$

The "dressed" coupling: The system that arises from the dressed interaction is quite complicated. We will denote it by S-KG[D], and it has the following form[6]:

(S-KG[D])

$$\begin{cases} i\partial_t u = -\dfrac{\Delta}{2M}u + Vu + (W * |u|^2)u + [(\varphi * A) + (\xi * \partial_t A)]u + \displaystyle\sum_{i=1}^{3}[(\rho^{(i)} * A)\partial_{(i)} + (\zeta^{(i)} * A)^2]u \\ (\Box + m_0^2)A = -\varphi * |u|^2 + i\displaystyle\sum_{i=1}^{3}\rho^{(i)} * [(u\partial_{(i)}u) - \sqrt{2M}(\zeta^{(i)} * \partial_t A)] \end{cases}$$

where: $V, W, \varphi : \mathbb{R}^3 \to \mathbb{R}$ with W, φ even; $\xi : \mathbb{R}^3 \to \mathbb{C}$, even; $\rho : (\mathbb{R}^3)^3 \to \mathbb{C}$, odd; and $\zeta : (\mathbb{R}^3)^3 \to \mathbb{R}$, odd. Obviously also (S-KG[D]) can be written as an equivalent system S-KG$_\alpha$[D], with unknowns u and α (omitted here). As discussed in detail in Section 3.3, with a suitable choice of W, φ, ξ, ρ and ζ the global well-posedness of (S-KG[D]) follows directly from the global well-posedness of (S-KG[Y]).

3.1. **Dressing.** We look for a classical correspondent of the dressing transformation $U_\infty(\theta)$. Since $U_\infty(\theta)$ is a one-parameter group of unitary transformations on $\mathcal{H}$, the classical counterpart of its generator is expected to induce a non-linear evolution on the phase-space $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, using the quantum-classical correspondence principle for systems with infinite degrees of freedom [see e.g.

[5] The two systems are equivalent since $(1+\omega^\varsigma)\mathrm{Re}\alpha \in L^2(\mathbb{R}^3) \Leftrightarrow A \in H^{\varsigma+1/2}(\mathbb{R}^3)$, $(1+\omega^\varsigma)\mathrm{Im}\alpha \in L^2(\mathbb{R}^3) \Leftrightarrow \partial_t A \in H^{\varsigma-1/2}(\mathbb{R}^3)$. In (S-KG$_\alpha$[Y]) the unknowns are u and α.

[6] We denote by $\partial_{(i)}$ the derivative with respect to the i-th component of the variable $x \in \mathbb{R}^3$. Analogously, we denote by $v^{(i)}$ the i-th component of a 3-dimensional vector v.

8, 49, 59]. The resulting "classical dressing" $D_{g_\infty}(\theta)$ plays a crucial role in proving our results: on one hand it is necessary to link the S-KG classical dynamics with the quantum dressed one; on the other it is at the heart of the "classical" renormalization procedure.

Let $g \in L^2(\mathbb{R}^3)$; define the following functional $\mathscr{D}_g : L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \to \mathbb{R}$,

$$\mathscr{D}_g(u,\alpha) := \int_{\mathbb{R}^6} \Big(g(k)\bar{\alpha}(k)e^{-ik\cdot x} + \bar{g}(k)\alpha(k)e^{ik\cdot x}\Big)|u(x)|^2 dx dk \; . \tag{49}$$

The functional $\mathscr{D}_g$ induces the following Hamiltonian equations of motion:

$$\begin{cases} i\partial_\theta u = A_g u \\ i\partial_\theta \alpha = gF(|u|^2) \end{cases} ; \tag{50}$$

where

$$A_g(x) = \int_{\mathbb{R}^3} \Big(g(k)\bar{\alpha}(k)e^{-ik\cdot x} + \bar{g}(k)\alpha(k)e^{ik\cdot x}\Big) dk \; , \tag{51}$$

$$F(|u|^2)(k) = \int_{\mathbb{R}^3} e^{-ik\cdot x}|u(x)|^2 dx \; . \tag{52}$$

Observe that for any $g \in L^2(\mathbb{R}^3)$ and $x \in \mathbb{R}^3$, $A_g(x) \in \mathbb{R}$. This will lead to an explicit form for the solutions of the Cauchy problem related to (50). The latter can be rewritten in integral form, for any $\theta \in \mathbb{R}$:

$$\begin{cases} u_\theta(x) = u_0(x) \exp\Big\{ -i \displaystyle\int_0^\theta (A_g)_\tau(x)\, d\tau \Big\} \\ \alpha_\theta(k) = \alpha_0(k) - ig(k) \displaystyle\int_0^\theta F(|u_\tau|^2)(k)\, d\tau \end{cases} ; \tag{53}$$

where $(A_g)_\tau$ is defined by (51) with α replaced by α_τ; analogously we define B_g by (51) with α replaced by β.

Lemma 3.1. *Let $s \geq 0$, $s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2}$; $(1 + \omega^{\frac{1}{2}})g \in L^2(\mathbb{R}^3)$. Also, let $u, v \in H^s(\mathbb{R}^3)$ and $(1+\omega^\varsigma)\alpha, (1+\omega^\varsigma)\beta \in L^2(\mathbb{R}^3)$. Then there exist constants $C_s, C_\varsigma > 0$ such that:*

$$\|(A_g - B_g)u\|_{H^s} \leq C_s \max_{w\in\{u,v\}} \|w\|_{H^s} \|(1+\omega^{\frac{1}{2}})g\|_2 \|(1+\omega^\varsigma)(\alpha - \beta)\|_2 \; , \tag{54}$$

$$\|A_g(u-v)\|_{H^s} \leq C_s \max_{\zeta\in\{\alpha,\beta\}} \|(1+\omega^\varsigma)\zeta\|_2 \|(1+\omega^{\frac{1}{2}})g\|_2 \|u-v\|_{H^s} \; , \tag{55}$$

$$\Big\| (1+\omega^\varsigma) g \int_{\mathbb{R}^3} e^{-ik\cdot x}\big((u-v)\bar{v} + (\bar{u}-\bar{v})u\big) dx \Big\|_2 \leq C_\varsigma \max_{w\in\{u,v\}} \|w\|_{H^s} \|(1+\omega^{\frac{1}{2}})g\|_2 \|u-v\|_{H^s} \; . \tag{56}$$

Proof. If $s \in \mathbb{N}$, the results follow by standard estimates, keeping in mind that $|k| \leq \omega(k) \leq |k| + m_0$. The bounds for non-integer s are then obtained by interpolation. ⊣

Proposition 3.2. *Let $\theta \in \mathbb{R}$, $(u_0, \alpha_0) \in L^2 \oplus L^2$. If $(u_\theta, \alpha_\theta) \in \mathcal{C}^0(\mathbb{R}, L^2 \oplus L^2)$ is a solution of (53), then it is unique, i.e. any $(v_\theta, \beta_\theta) \in \mathcal{C}^0(\mathbb{R}, L^2 \oplus L^2)$ that satisfies (53) is such that $(v_\theta, \beta_\theta) = (u_\theta, \alpha_\theta)$.*

Proof. We have:

$$\frac{i}{2}\partial_\theta \Big(\|u_\theta - v_\theta\|_2^2 + \|\alpha_\theta - \beta_\theta\|_2^2 \Big) = \mathrm{Im}\Big(\Big\langle u_\theta - v_\theta, \big((A_g)_\theta - (B_g)_\theta\big)u_\theta + (B_g)_\theta\big(u_\theta - v_\theta\big) \Big\rangle_2$$
$$+ \Big\langle \alpha_\theta - \beta_\theta, g \int_{\mathbb{R}^3} e^{-ik\cdot x}\big((u_\theta - v_\theta)\bar{v}_\theta + (\bar{u}_\theta - \bar{v}_\theta)u_\theta\big) dx \Big\rangle_2 \Big) \; .$$

The result hence is an application of the estimates of Lemma 3.1 with $s = 0$ and Gronwall's Lemma. $\dashv$

Now that we are assured that the solution of (53) is unique, we can construct it explicitly. Since $A_g(x)$ is real, it follows that for any $\theta \in \mathbb{R}$: $|u_\theta| = |u_0|$. Therefore $F(|u_\theta|^2) = F(|u_0|^2)$, and

$$\alpha_\theta(k) = \alpha_0(k) - i\theta g(k) F(|u_0|^2)(k) \; .$$

Substituting this explicit form in the expression for u_θ, we obtain the solution for any $(u_0, \alpha_0) \equiv (u, \alpha) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$:

$$\begin{cases} u_\theta(x) = u(x)\ \exp\Big\{ -i\theta A_g(x) + i\theta^2 \mathrm{Im} \displaystyle\int_{\mathbb{R}^3} F(|u|^2)(k) |g(k)|^2 e^{ik\cdot x} dk \Big\} \\ \alpha_\theta(k) = \alpha(k) - i\theta g(k) F(|u|^2)(k) \end{cases} . \tag{57}$$

This system of equations defines a non-linear symplectomorphism: the "classical dressing map" on $L^2 \oplus L^2$.

Definition 3.3. *Let $g \in L^2(\mathbb{R}^3)$. Then $\mathbf{D}_g(\cdot) : \mathbb{R} \times \big(L^2 \oplus L^2\big) \to L^2 \oplus L^2$ is defined by (57) as:*

$$\mathbf{D}_g(\theta)(u, \alpha) = (u_\theta, \alpha_\theta) \; .$$

The map $\mathbf{D}_g(\cdot)$ is the Hamiltonian flow generated by $\mathscr{D}_g$.

Using the explicit form (57) and Lemma 3.1, it is straightforward to prove some interesting properties of the classical dressing map. The results are formulated in the following proposition, after the definition of useful classes of subspaces of $L^2 \oplus L^2$.

Definition 3.4. *Let $s \geq 0$, $s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2}$. We define the spaces $H^s(\mathbb{R}^3) \oplus \mathcal{F}H^\varsigma(\mathbb{R}^3) \subseteq L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$:*

$$H^s(\mathbb{R}^3) \oplus \mathcal{F}H^\varsigma(\mathbb{R}^3) = \Big\{ (u, \alpha) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3) \; , \, u \in H^s(\mathbb{R}^3) \text{ and } \mathcal{F}^{-1}(\alpha) \in H^\varsigma(\mathbb{R}^3) \Big\} \; .$$

Proposition 3.5. *Let $s \geq 0$, $s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2}$; and $g \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$. Then*

$$\mathbf{D}_g : \mathbb{R} \times \big(H^s \oplus \mathcal{F}H^\varsigma\big) \to H^s \oplus \mathcal{F}H^\varsigma \; ;$$

i.e. the flow preserves the spaces $H^s \oplus \mathcal{F}H^\varsigma$. Furthermore, it is a bijection with inverse $\big(\mathbf{D}_g(\theta)\big)^{-1} = \mathbf{D}_g(-\theta)$. Hence the classical dressing is an Hamiltonian flow on $H^s \oplus \mathcal{F}H^\varsigma$.

Corollary 3.6. *Let $s \geq 0$, $s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2}$, $\theta \in \mathbb{R}$, and $g \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$. Then there exists a constant $C(g, \theta) > 0$ and a $\lambda(s) \in \mathbb{N}^*$ such that for any $(u, \alpha) \in H^s \oplus \mathcal{F}H^\varsigma$:*

$$\|\mathbf{D}_g(\theta)(u, \alpha)\|_{H^s \oplus \mathcal{F}H^\varsigma} \leq C(g, \theta) \|(u, \alpha)\|^{\lambda(s)}_{H^s \oplus \mathcal{F}H^\varsigma} \; . \tag{58}$$

Using the positivity of both $-\Delta$ and V, and Corollary 3.6 one also obtains the following result.

Corollary 3.7. *Let $V \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}_+)$; and let $Q(-\Delta + V) \subset L^2(\mathbb{R}^3)$ be the form domain of $-\Delta + V$. Then for any $\frac{1}{2} \leq \varsigma \leq \frac{3}{2}$, and $g \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$:*

$$\mathbf{D}_g : \mathbb{R} \times \big(Q(-\Delta + V) \oplus \mathcal{F}H^\varsigma\big) \to Q(-\Delta + V) \oplus \mathcal{F}H^\varsigma \; .$$

3.2. **Classical Hamiltonians.** In this section we define the classical Hamiltonian functionals that generate the undressed and dressed dynamics on $L^2 \oplus L^2$. Then we show that they are related by a suitable classical dressing: the quantum procedure described in Section 2.2 is reproduced, in simplified terms, on the classical level.

Definition 3.8 ($\mathscr{E}$, $\hat{\mathscr{E}}$)**.** *The undressed Hamiltonian (or energy) $\mathscr{E}$ is defined as the following real functional on $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$:*

$$\mathscr{E}(u,\alpha) := \Bigl\langle u, \Bigl(-\tfrac{\Delta}{2M} + V\Bigr)u\Bigr\rangle_2 + \langle \alpha, \omega\alpha\rangle_2 + \tfrac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \tfrac{1}{\sqrt{2\omega(k)}} \Bigl(\bar{\alpha}(k)e^{-ik\cdot x} + \alpha(k)e^{ik\cdot x}\Bigr)|u(x)|^2 dxdk \; .$$

We denote by $\mathscr{E}_0$ the free part of the classical energy, namely

$$\mathscr{E}_0(u,\alpha) = \Bigl\langle u, \Bigl(-\tfrac{\Delta}{2M} + V\Bigr)u\Bigr\rangle_2 + \langle \alpha, \omega\alpha\rangle_2 \; .$$

Let $\chi_{\sigma_0} \in L^\infty(\mathbb{R}^3) \cap \mathcal{F}H^{-1/2}(\mathbb{R}^3)$ such that $\chi_{\sigma_0}(k) = \chi_{\sigma_0}(-k)$ for any $k \in \mathbb{R}^3$. Then (again as a real functional on $L^2 \oplus L^2$) the dressed Hamiltonian $\hat{\mathscr{E}}$ is defined as[7]*:*

$$\begin{aligned}\hat{\mathscr{E}}(u,\alpha) := &\Bigl\langle u, \Bigl(-\tfrac{\Delta}{2M} + V\Bigr)u\Bigr\rangle_2 + \langle \alpha, \omega\alpha\rangle_2 + \tfrac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^6} \tfrac{\chi_{\sigma_0}(k)}{\sqrt{2\omega(k)}} \Bigl(\bar{\alpha}(k)e^{-ik\cdot x} + \alpha(k)e^{ik\cdot x}\Bigr)|u(x)|^2 dxdk \\ &+ \tfrac{1}{2M} \int_{\mathbb{R}^9} \Bigl(r_\infty(k)\bar{\alpha}(k)e^{-ik\cdot x} + \bar{r}_\infty(k)\alpha(k)e^{ik\cdot x}\Bigr)\Bigl(r_\infty(l)\bar{\alpha}(l)e^{-il\cdot x} + \bar{r}_\infty(l)\alpha(l)e^{il\cdot x}\Bigr)|u(x)|^2 dxdkdl \\ &- \tfrac{2}{M}\mathrm{Re} \int_{\mathbb{R}^6} r_\infty(k)\bar{\alpha}(k)e^{-ik\cdot x}\bar{u}(x) D_x u(x) dxdk + \tfrac{1}{2}\int_{\mathbb{R}^6} V_\infty(x-y)|u(x)|^2|u(y)|^2 dxdy \; .\end{aligned}$$

Remark 3.9. We denote by $D(\mathscr{E}) \subset L^2 \oplus L^2$ the domain of definition of $\mathscr{E}$, and by $D(\hat{\mathscr{E}}) \subset L^2 \oplus L^2$ the domain of definition of $\hat{\mathscr{E}}$. We have that $D(\mathscr{E}) \supset \mathcal{C}_0^\infty \oplus \mathcal{C}_0^\infty$ and $D(\hat{\mathscr{E}}) \supset \mathcal{C}_0^\infty \oplus \mathcal{C}_0^\infty$. Therefore both $\mathscr{E}$ and $\hat{\mathscr{E}}$ are densely defined, and $D(\mathscr{E}) \cap D(\hat{\mathscr{E}})$ is dense in $L^2 \oplus L^2$.

We are interested in the action of $\mathscr{E}$ and $\hat{\mathscr{E}}$ on $H^1 \oplus \mathcal{F}H^{\frac{1}{2}}$, since this emerges naturally as the energy space of the system, at least when $V = 0$.

Lemma 3.10. *Let $\theta \in \mathbb{R}$, $g \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$. Then for any $u \in Q(V) \cap H^1(\mathbb{R}^3)$, and $\alpha \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$: $\mathbf{D}_g(\theta)(u,\alpha) \in D(\mathscr{E})$.*

Proof. Let $u \in Q(V)$, and $\alpha \in L^2(\mathbb{R}^3)$. Then

$$\langle u_\theta, V u_\theta\rangle_2 = \langle u, Vu\rangle_2 \; ;$$

where u_θ is defined in Equation (57), and it is the first component of $\mathbf{D}_g(\theta)(u,\alpha)$. Also, for any $(u,\alpha) \in H^1 \oplus \mathcal{F}H^{\frac{1}{2}}$ we have that:

$$\begin{aligned}\Bigl|\int_{\mathbb{R}^6} |u(x)|^2 \frac{1}{\sqrt{\omega(k)}} \alpha(k) e^{ik\cdot x} dxdk\Bigr| &= C\Bigl|\int_{\mathbb{R}^3} \frac{1}{|k|\omega(k)} \bigl(\omega^{1/2}\alpha\bigr)(k) \Bigl(\int_{\mathbb{R}^3} \bigl(D_x|u(x)|^2\bigr) e^{ik\cdot x} dx\Bigr) dk\Bigr| \\ &\leq 2C \Bigl\| \frac{1}{|k|\omega(k)} \Bigr\|_2 \|\omega^{1/2}\alpha\|_2 \|u\|_2 \|u\|_{H^1} < +\infty \; .\end{aligned}$$

The result then follows since $\mathbf{D}_g(\theta)$ maps $H^1 \oplus \mathcal{F}H^{\frac{1}{2}}$ into itself by Proposition 3.5. ⊣

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[7]We recall that: $g_\infty(k) = -i\frac{(2\pi)^{-3/2}}{\sqrt{2\omega(k)}} \frac{1-\chi_{\sigma_0}(k)}{\frac{k^2}{2M}+\omega(k)}$; $V_\infty(x) = 2\mathrm{Re}\int_{\mathbb{R}^3} \omega(k)|g_\infty(k)|^2 e^{-ik\cdot x} dk - 4\mathrm{Im}\int_{\mathbb{R}^3} \frac{\bar{g}_\infty(k)}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega(k)}} e^{-ik\cdot x} dk$. Also, $D_x = -i\nabla_x$; $r_\infty(k) = -ikg_\infty(k)$.

The functional $\mathscr{E}$ is independent of g_∞, while $\hat{\mathscr{E}}$ depends on it. In addition, we know that g_∞ has been fixed, at the quantum level, to renormalize the Nelson Hamiltonian, and it is the function that appears in the generator of the dressing transformation U_∞. Hence, since we are establishing a correspondence between the classical and quantum theories, we expect it to be the function that appears in the classical dressing too. Two features of g_∞ are very important in the classical setting: the first is that $g_\infty \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$ for any $\chi_{\sigma_0} \in L^\infty \cap \mathcal{F}H^{-\frac{1}{2}}$; the second is that it is an even function, i.e. $g_\infty(k) = g_\infty(-k)$ for any $k \in \mathbb{R}^3$. Using the first fact, one shows that $\mathbf{D}_{g_\infty}(\cdot)$ maps the energy space into itself (and that will be convenient when discussing global solutions); using the second property we can simplify the explicit form of $\mathbf{D}_{g_\infty}(\cdot)$.

Lemma 3.11. *Let $\theta \in \mathbb{R}$, and $g \in L^2(\mathbb{R}^3)$. If g is an even or odd function, then the map $\mathbf{D}_g(\theta)$ defined by (57) becomes:*

$$\mathbf{D}_g(\theta)(u(x), \alpha(k)) = \Big(u(x) e^{-i\theta A_g(x)} \, , \, \alpha(k) - i\theta g(k) F(|u|^2)(k) \Big) \, . \tag{59}$$

Proof. Consider $I(x) := \int_{\mathbb{R}^3} F(|u|^2)(k) |g(k)|^2 e^{ik\cdot x} dk$. We will show that $\bar{I}(x) = I(x)$. We have that:

$$\bar{I}(x) = \int_{\mathbb{R}^6} |u(x')|^2 |g(k)|^2 e^{-ik\cdot(x-x')} dx' dk = \int_{\mathbb{R}^6} |u(x')|^2 |g(-k)|^2 e^{ik\cdot(x-x')} dx' dk \, .$$

Now if g is either even or odd, $|g(-k)| = |g(k)|$. Hence $\bar{I}(x) = I(x)$, therefore $\mathrm{Im} I(x) = 0$. ⊣

We conclude this section proving its main result: $\mathscr{E}$ and $\hat{\mathscr{E}}$ are related by the $\mathbf{D}_{g_\infty}(1)$ classical dressing[8].

Proposition 3.12. *For any $u \in Q(V) \cap H^1(\mathbb{R}^3)$, $\alpha \in \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)$, and for any $\chi_{\sigma_0} \in L^\infty(\mathbb{R}^3) \cap \mathcal{F}H^{-\frac{1}{2}}(\mathbb{R}^3)$:*

(1) $(u, \alpha) \in D(\mathscr{E})$;
(2) $(u, \alpha) \in D(\hat{\mathscr{E}})$;
(3) $\hat{\mathscr{E}}(u, \alpha) = \mathscr{E} \circ \mathbf{D}_{g_\infty}(1)(u, \alpha)$.

Remark 3.13. Relation (3) of Proposition 3.12 actually holds for any $(u, \alpha) \in \mathbf{D}_{g_\infty}(-1) D(\mathscr{E})$.

Remark 3.14. The Wick quantization of $\mathscr{E}$ yields the quadratic form $\langle \cdot \, , H_\infty^{(n)} \cdot \rangle_{\mathcal{H}_n}$, that is not closed and bounded from below for any $n \in \mathbb{N}_*$. On the other hand, if χ_{σ_0} is the ultraviolet cutoff of Section 2, then the Wick quantization of $\hat{\mathscr{E}}$ yields directly the *renormalized* quadratic form $\langle \cdot \, , \hat{H}_\infty^{(n)} \cdot \rangle_{\mathcal{H}_n}$ that is closed and bounded from below for any $n \leq \mathfrak{N}(\varepsilon, \sigma_0)$.

Proof of Proposition 3.12. The statement (1) is just an application of Lemma 3.10 when $\theta = 0$. If (3) holds formally, than (2) follows directly, since by Lemma 3.10 the right hand side of (3) is well defined. It remains to prove that the relation (3) holds formally. This is done by means of a direct calculation,

[8] We recall again that $g_\infty = -i \frac{(2\pi)^{-3/2}}{\sqrt{2\omega(k)}} \frac{1 - \chi_{\sigma_0}(k)}{\frac{k^2}{2M} + \omega(k)}$.

that we will briefly outline here.

$$\mathscr{E}\circ\mathbf{D}_{g_\infty}(1)(u,\alpha)=\Big\langle ue^{-iA_{g_\infty}},\frac{D_x}{2M}D_x\big(ue^{-iA_{g_\infty}}\big)\Big\rangle_2+\langle u,Vu\rangle_2+\langle\alpha,\omega\alpha\rangle_2$$

$$\text{(a)}\qquad +2\mathrm{Im}\langle\alpha,\omega g_\infty F_u\rangle_2+\frac{1}{(2\pi)^{3/2}}2\mathrm{Re}\int_{\mathbb{R}^6}\frac{1}{\sqrt{2\omega(k)}}\bar\alpha(k)e^{-ik\cdot x}|u(x)|^2dxdk$$

$$\text{(b)}\qquad +\|\omega g_\infty F_u\|_2^2+\frac{1}{(2\pi)^{3/2}}2\mathrm{Im}\int_{\mathbb{R}^6}\frac{1}{\sqrt{2\omega(k)}}g_\infty(k)F_u(k)e^{ik\cdot x}|u(x)|^2dxdk\ .$$

After some manipulation, taking care of the ordering, the first term on the right hand side becomes:

$$\Big\langle ue^{-iA_{g_\infty}},\frac{D_x}{2M}D_x\big(ue^{-iA_{g_\infty}}\big)\Big\rangle_2=\Big\langle u,-\frac{\Delta}{2M}u\Big\rangle_2$$

$$\text{(c)}\qquad +\frac{1}{2M}\langle A_{r_\infty}u,A_{r_\infty}u\rangle_2$$

$$\text{(d)}\qquad -i\langle u,A_{\frac{k^2}{2M}g_\infty}u\rangle_2$$

$$\text{(e)}\qquad -\frac{1}{M}\Big\langle u,\int_{\mathbb{R}^3}dk\Big(D_x\bar r_\infty(k)\alpha(k)e^{ik\cdot x}+r_\infty(k)\bar\alpha(k)e^{-ik\cdot x}D_x\Big)u\Big\rangle_2\ .$$

The proof is concluded making the following identifications (the other terms sum to the free part):

$$(\text{a})+(\text{d})=\frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^6}\frac{\chi_{\sigma_0}}{\sqrt{2\omega(k)}}\Big(\bar\alpha(k)e^{-ik\cdot x}+\alpha(k)e^{ik\cdot x}\Big)|u(x)|^2dxdk\ ;$$

$$(\text{b})=\frac{1}{2}\int_{\mathbb{R}^6}V_\infty(x-y)|u(x)|^2|u(y)|^2dxdy\ ;$$

$$(\text{c})=\frac{1}{2M}\int_{\mathbb{R}^9}\Big(r_\infty(k)\bar\alpha(k)e^{-ik\cdot x}+\bar r_\infty(k)\alpha(k)e^{ik\cdot x}\Big)\Big(r_\infty(l)\bar\alpha(l)e^{-il\cdot x}+\bar r_\infty(l)\alpha(l)e^{il\cdot x}\Big)|u(x)|^2dxdkdl\ ;$$

$$(\text{e})=-\frac{2}{M}\mathrm{Re}\int_{\mathbb{R}^6}r_\infty(k)\bar\alpha(k)e^{-ik\cdot x}\bar u(x)D_xu(x)dxdk\ .$$

$\dashv$

3.3. **Global existence results.** In this section we discuss uniqueness and global existence of the classical dynamical system: using a well-known result on the undressed dynamics, we prove uniqueness and existence also for the dressed system.

The Cauchy problem associated to $\mathscr{E}$ by the Hamilton's equations is[9] (S-KG$_\alpha$[Y]). Theorem 3.15 below is a straightforward extension of [27, 70] that includes a (confining) potential on the NLS equation. As proved in [23, 69], the quadratic potential is the maximum we can afford to still have Strichartz estimates and global existence in the energy space. Therefore we make the following standard assumption on V:

Assumption A_V**.** $V\in\mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R}_+)$, and $\partial^\alpha V\in L^\infty(\mathbb{R}^3)$ for any $\alpha\in\mathbb{N}^3$, with $|\alpha|\geq 2$ (i.e. at most quadratic positive confining potential).

Theorem 3.15 (Undressed global existence)**.** *Assume* A_V*. Then there is a unique Hamiltonian flow solving* (S-KG$_\alpha$[Y])*:*

$$\mathbf{E}:\mathbb{R}\times\big(Q(-\Delta+V)\oplus\mathcal{F}H^{\frac12}(\mathbb{R}^3)\big)\to Q(-\Delta+V)\oplus\mathcal{F}H^{\frac12}(\mathbb{R}^3)\ . \tag{60}$$

[9]The Cauchy problem associated to $\hat{\mathscr{E}}$ is equivalent to (S-KG[D]), setting: $W=V_\infty$, $\varphi=(2\pi)^{-3/2}\mathcal{F}(\chi_{\sigma_0})$, $\xi=\frac{(2\pi)^{-3/2}}{\sqrt{2M}}\big(\mathcal{F}(\frac{k^2}{\sqrt\omega}g_\infty)-\mathcal{F}(i\frac{k^2}{\omega}g_\infty)\big)$, $\rho=\frac{\sqrt2}{M}\mathcal{F}(\sqrt\omega kg_\infty)$, and $\zeta=\frac{i}{\sqrt M}\mathcal{F}(\frac{k}{\sqrt\omega}g_\infty)$.

If $V = 0$, then there is a unique Hamiltonian flow

$$\mathbf{E} : \mathbb{R} \times \big(H^s(\mathbb{R}^3) \oplus \mathcal{F}H^\varsigma(\mathbb{R}^3)\big) \to H^s(\mathbb{R}^3) \oplus \mathcal{F}H^\varsigma(\mathbb{R}^3)\ . \tag{61}$$

for any $0 \leq s \leq 1$, $s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2}$.

Theorem 3.16 (Dressed global existence)**.** *Assume A_V. Then for any $\chi_{\sigma_0} \in L^\infty(\mathbb{R}^3) \cap \mathcal{F}H^{-\frac{1}{2}}(\mathbb{R}^3)$, there is a unique Hamiltonian flow:*

$$\hat{\mathbf{E}} : \mathbb{R} \times \big(Q(-\Delta + V) \oplus \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)\big) \to Q(-\Delta + V) \oplus \mathcal{F}H^{\frac{1}{2}}(\mathbb{R}^3)\ . \tag{62}$$

If $V = 0$, then there is a unique Hamiltonian flow

$$\hat{\mathbf{E}} : \mathbb{R} \times \big(H^s(\mathbb{R}^3) \oplus \mathcal{F}H^\varsigma(\mathbb{R}^3)\big) \to H^s(\mathbb{R}^3) \oplus \mathcal{F}H^\varsigma(\mathbb{R}^3)\ . \tag{63}$$

for any $0 \leq s \leq 1$, $s - \frac{1}{2} \leq \varsigma \leq s + \frac{1}{2}$. For any V that satisfies A_V, the flows $\hat{\mathbf{E}}$ and $\mathbf{E}$ are related by:

$$\hat{\mathbf{E}} = \mathbf{D}_{g_\infty}(-1) \circ \mathbf{E} \circ \mathbf{D}_{g_\infty}(1) \quad , \quad \mathbf{E} = \mathbf{D}_{g_\infty}(1) \circ \hat{\mathbf{E}} \circ \mathbf{D}_{g_\infty}(-1) \quad . \tag{64}$$

Proof of Theorem 3.16. The theorem is a direct consequence of the global well-posedness result of Theorem 3.15, the relation $\hat{\mathscr{E}} = \mathscr{E} \circ \mathbf{D}_{g_\infty}(1)$ proved in Proposition 3.12, and the regularity properties of the dressing proved in Proposition 3.5. ⊣

3.4. **Symplectic character of $\mathbf{D}_{\chi_{\sigma_0}}$.** To complete our description of the Schrödinger-Klein-Gordon system, we explicitly prove that the classical dressing is a (non-linear) symplectomorphism for the real symplectic structure $\big\{(L^2 \oplus L^2)_{\mathbb{R}}, \mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle_{L^2\oplus L^2}\big\}$. We denote by $\mathrm{d}\mathbf{D}_g(\theta)_{(u,\alpha)} \in \mathcal{L}(L^2 \oplus L^2)$ the (Fréchet) derivative of $\mathbf{D}_g(\theta)$ at the point $(u, \alpha) \in L^2 \oplus L^2$.

Proposition 3.17. *Let $g \in L^2(\mathbb{R}^3)$ be an even or odd function. Then for any $\theta \in \mathbb{R}$, $\mathbf{D}_g(\theta)$ is differentiable at any point $(u, \alpha) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$. In addition, it satisfies for any $(v_1, \beta_1), (v_2, \beta_2) \in L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$:*

$$\mathrm{Im}\langle \mathrm{d}\mathbf{D}_g(\theta)_{(u,\alpha)}(v_1, \beta_1), \mathrm{d}\mathbf{D}_g(\theta)_{(u,\alpha)}(v_2, \beta_2)\rangle_{L^2\oplus L^2} = \mathrm{Im}\langle (v_1, \beta_1), (v_2, \beta_2)\rangle_{L^2\oplus L^2}\ . \tag{65}$$

Proof. We recall that with the assumptions on g, $\mathbf{D}_g(\theta)$ has the explicit form:

$$\mathbf{D}_g(\theta)(u(x), \alpha(k)) = \Big(\ u(x)e^{-i\theta A_g(x)}\ ,\ \alpha(k) - i\theta g(k) F(|u|^2)(k)\ \Big)\ ;$$

where A_g and F are defined by Equations (51) and (52) respectively. The Fréchet derivative of $\mathbf{D}_g(\theta)$ is easily computed, and yields

$$\begin{aligned} \mathrm{d}\mathbf{D}_g(\theta)_{(u,\alpha)}(v(x), \beta(k)) &= \Big(\ \big(v(x) - i\theta B_g(x) u(x)\big)e^{-i\theta A_g(x)}\ ,\ \beta(k) - 2i\theta g(k) \mathrm{Re}\big(F(\bar{u}v)(k)\big)\ \Big) \\ &= \Big(\ \mathrm{i}(v, \beta)\ ,\ \mathrm{ii}(v, \beta)\ \Big)\ ; \end{aligned}$$

where we recall that $B_g(x)$ is $A_g(x)$ with α substituted by β. Then we have:

$$\mathrm{Im}\langle \mathrm{i}(v_1, \beta_1), \mathrm{i}(v_2, \beta_2)\rangle_{L^2} = \mathrm{Im}\langle v_1, v_2\rangle_{L^2} + 2\theta \mathrm{Re}\Big(\langle B_g^{(1)} u, v_2\rangle_{L^2} - \langle v_1, B_g^{(2)} u\rangle_{L^2}\Big)\ ,$$

$$\mathrm{Im}\langle \mathrm{ii}(v_1, \beta_1), \mathrm{ii}(v_2, \beta_2)\rangle_{L^2} = \mathrm{Im}\langle \beta_1, \beta_2\rangle_{L^2} + 2\theta \mathrm{Re}\Big(\langle g\, \mathrm{Re}F(\bar{u}v_1), \beta_2\rangle_{L^2} - \langle \beta_1, g\, \mathrm{Re}F(\bar{u}v_2)\rangle_{L^2}\Big)\ .$$

The result then follows, noting that $\langle g\, \mathrm{Re}F(\bar{u}v_1), \beta_2\rangle_{L^2} = \langle v_1, B_g^{(2)} u\rangle_{L^2}$ and $\langle \beta_1, g\, \mathrm{Re}F(\bar{u}v_2)\rangle_{L^2} = \langle B_g^{(1)} u, v_2\rangle_{L^2}$. ⊣

4. The classical limit of the renormalized Nelson model.

In this section we discuss in detail the classical limit of the renormalized Nelson model, both dressed and undressed, and prove the main result Theorem 1.1. A schematic outline of the proof is given in Subsection 4.1 to improve readability. The Subsections from 4.2 to 4.6 are dedicated to prove the convergence of the dressed dynamics. The obtained results are summarized by Theorem 4.26. In Subsection 4.7 we study the classical limit of the dressing transformation. Finally, in Section 4.8 we put all the pieces together to prove Theorem 1.1.

4.1. **Scheme of the proof.** First of all, let us explain the main ideas that are behind our proof of Theorem 1.1. Since the explicit form of $H_{\mathrm{ren}}(\sigma_0)$ is not known, it seems a very hard task to study directly the limit of a time-evolved family of states $e^{-i\frac{t}{\varepsilon}H_{\mathrm{ren}}(\sigma_0)}\,\varrho_\varepsilon\,e^{i\frac{t}{\varepsilon}H_{\mathrm{ren}}(\sigma_0)}$, at least using established techniques. The introduction of the classical dressing, and the relation $\mathbf{E}=\mathbf{D}_{g_\infty}(1)\circ\hat{\mathbf{E}}\circ\mathbf{D}_{g_\infty}(-1)$ (Equation (64), proved in Theorem 3.16) play therefore a crucial role. Once we combine them with the convergence of the quantum dressing to the classical dressing "as a dynamical system", see Proposition 4.25, we can relate the undressed and dressed dynamics throughout the entire limit procedure. The final ingredient is the convergence of a family of states $\varrho_\varepsilon(t)=e^{-i\frac{t}{\varepsilon}\hat{H}_{\mathrm{ren}}}\,\varrho_\varepsilon\,e^{i\frac{t}{\varepsilon}\hat{H}_{\mathrm{ren}}}$ evolved with the quantum *dressed* dynamics to the corresponding Wigner measure $\hat{\mathbf{E}}(t)_{\#}\mu_0$ evolved with the classical dressed dynamics. Despite being technically demanding, the proof of the latter takes advantage of the explicit expression of the quadratic form $\hat{h}_{\mathrm{ren}}(\cdot,\cdot)=\langle\,\cdot\,,\hat{H}_{\mathrm{ren}}\,\cdot\rangle$ associated to the dressed Hamiltonian. The lengthier part of the aforementioned proof is to control each term that arises from the expansion of the quadratic form associated to $[\hat{H}_{\mathrm{ren},I},W(\tilde{\xi}_s)]$: it is necessary to prove that each associated classical symbol either is compact, or it can be approximated with a compact one.

In the light of the discussion above, the proof of Theorem 1.1 can be schematized through the following steps.

(i) (Subsection 4.2). Express the average of the Weyl operator $W(\xi)$ with respect to the dressed time-evolved state $\tilde{\varrho}_\varepsilon(t)=e^{i\frac{t}{\varepsilon}H_0}\,\varrho_\varepsilon(t)\,e^{-i\frac{t}{\varepsilon}H_0}$ (in the interaction picture) as the integral formula
$$\mathrm{Tr}\Big[\tilde{\varrho}_\varepsilon(t)W(\xi)\Big]=\mathrm{Tr}\Big[\varrho_\varepsilon W(\xi)\Big]+\frac{i}{\varepsilon}\int_0^t\mathrm{Tr}\Big[\varrho_\varepsilon(s)[\hat{H}_{ren,I},W(\tilde{\xi}_s)]\Big]\,ds\ .$$

(ii) (Subsection 4.3). Characterize the quadratic form associated to $[\hat{H}_{ren,I},W(\tilde{\xi}_s)]$, in particular prove that the associated classical symbol can be approximated with a compact symbol (Proposition 4.9).

(iii) (Subsections 4.4 and 4.5). Take the limit $\varepsilon\to 0$ in the integral formula of Step (i) (extracting a common subsequence for all times), thus obtaining a time-dependent family $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ of Wigner measures characterized by a transport equation
$$\partial_t\tilde{\mu}_t+\nabla^T\Big(\mathbf{V}(t)\tilde{\mu}_t\Big)=0\ .$$

(iv) (Subsection 4.6). The transport equation of Step (iii) is solved by $\mathbf{E}_0(-t)_{\#}\hat{\mathbf{E}}(t)_{\#}\mu_0$. Prove that the family $(\tilde{\mu}_t)_{t\in\mathbb{R}}$ can be uniquely identified with $(\mathbf{E}_0(-t)_{\#}\hat{\mathbf{E}}(t)_{\#}\mu_0)_{t\in\mathbb{R}}$ provided that $\varrho_\varepsilon(0)\to\mu_0$. This is achieved by applying a general uniqueness result for probability measure solutions of transport equations proved in [7].

(v) (Subsection 4.7). Prove that the dressed state $e^{-i\frac{\theta}{\varepsilon}T_\infty}\,\varrho_\varepsilon\,e^{i\frac{\theta}{\varepsilon}T_\infty}$ converges when $\varepsilon\to 0$ to $\mathbf{D}_{g_\infty}(\theta)_{\#}\mu$ for any $\theta\in\mathbb{R}$, provided that $\varrho_\varepsilon\to\mu$.

(vi) (Subsection 4.8). Combine the results together, and use the relation $\mathbf{E}=\mathbf{D}_{g_\infty}(1)\circ\hat{\mathbf{E}}\circ\mathbf{D}_{g_\infty}(-1)$ to prove that $\varrho_\varepsilon\to\mu$ yields

$$e^{-i\frac{t}{\varepsilon}H_{\mathrm{ren}}(\sigma_0)}\,\varrho_\varepsilon\,e^{i\frac{t}{\varepsilon}H_{\mathrm{ren}}(\sigma_0)}\to\mathbf{E}(t)_{\#}\mu\;.$$

4.2. **The integral formula for the dressed Hamiltonian.** The results of this and the next subsection are similar in spirit to the ones previously obtained in [5, Section 3] for the Nelson model with cutoff and in [11, Section 3] for the mean field problem. However, some additional care has to be taken, for in this more singular situation the manipulations below are allowed only in the sense of quadratic forms. We start with a couple of preparatory lemmas. The proof of the first can be essentially obtained following [5, Lemma 6.1]; the second is an equivalent reformulation of Assumption (A_0):

$$\exists\mathfrak{C}>0\,,\;\forall\varepsilon\in(0,\bar{\varepsilon})\,,\;\forall k\in\mathbb{N}\,,\;\mathrm{Tr}[\varrho_\varepsilon N_1^k]\leq\mathfrak{C}^k\;.$$

We recall that the Weyl operator $W(\xi)$, $L^2\oplus L^2\ni\xi=\xi_1\oplus\xi_2$, is defined as:

$$W(\xi)=e^{\frac{i}{\sqrt{2}}\left(\psi^*(\xi_1)+\psi(\xi_1)\right)}e^{\frac{i}{\sqrt{2}}\left(a^*(\xi_2)+a(\xi_2)\right)}\;. \tag{66}$$

Lemma 4.1. *For any $\xi=\xi_1\oplus\xi_2$ such that $\xi_1\in Q(-\Delta+V)\subset H^1$ and $\xi_2\in D(\omega^{1/2})\equiv\mathcal{F}H^{1/2}$, there exists $C(\xi)>0$ that depends only on $\|\xi_1\|_{H^1}$ and $\|\xi_2\|_{\mathcal{F}H^{1/2}}$, such that for any $\varepsilon\in(0,\bar{\varepsilon})$:*

$$\|H_0^{1/2}W(\xi)\Psi\|\leq C(\xi)\|(H_0+\bar{\varepsilon})^{1/2}\Psi\|\;,\;\forall\Psi\in Q(H_0)\;;$$
$$\|(H_0+1)^{1/2}(N_1+1)^{1/2}W(\xi)\Psi\|\leq C(\xi)\|(H_0+\bar{\varepsilon})^{1/2}(N_1+\bar{\varepsilon})^{1/2}\Psi\|\;,\;\forall\Psi\in Q(H_0)\cap Q(N_1)\;.$$

In an analogous fashion, for any $\xi\in L^2\oplus L^2$, $r>0$, there exist $C(\xi)>0$ that depends only on $\|\xi_1\|_2$ and $\|\xi_2\|_2$, such that for any $\varepsilon\in(0,\bar{\varepsilon})$:

$$\|(N_1+N_2)^{r/2}W(\xi)\Psi\|\leq C(\xi)\|(N_1+N_2+\bar{\varepsilon})^{r/2}\Psi\|\;,\;\forall\Psi\in Q(N_1^r)\;.$$

Lemma 4.2. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of normal states on $\mathcal{H}$. Then $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfies Assumption (A_0) if and only if for any $\varepsilon\in(0,\bar{\varepsilon})$ there exists a sequence $(\Psi_i(\varepsilon))_{i\in\mathbb{N}}$ of orthonormal vectors in $\mathcal{H}$ with non-zero components only in $\bigoplus_{n=0}^{[\mathfrak{C}/\varepsilon]}\mathcal{H}_n$ and a sequence $(\lambda_i(\varepsilon))_{i\in\mathbb{N}}\in l^1$, with each $\lambda_i(\varepsilon)>0$, such that:*

$$\varrho_\varepsilon=\sum_{i\in\mathbb{N}}\lambda_i(\varepsilon)|\Psi_i(\varepsilon)\rangle\langle\Psi_i(\varepsilon)|\;.$$

The explicit ε-dependence of Ψ_i and λ_i will be often omitted.

Proof. We start assuming (A_0). Let $\varrho_\varepsilon=\sum_{i\in\mathbb{N}}\lambda_i|\Psi_i\rangle\langle\Psi_i|$ be the spectral decomposition of ϱ_ε. Then

$$\mathrm{Tr}\big[\varrho_\varepsilon N_1^k\big]=\sum_{i\in\mathbb{N}}\lambda_i\langle\Psi_i,N_1^k\Psi_i\rangle\leq\mathfrak{C}^k\Rightarrow\sum_{i\in\mathbb{N}}\lambda_i\langle\Psi_i,(N_1/\mathfrak{C})^k\Psi_i\rangle\leq 1\;.$$

Let $1_{[L,+\infty)}(N_1)$ be the spectral projection of N_1 on the interval $[L,+\infty)$; and choose $L > \mathfrak{C}$. Then it follows that:

$$1 \geq \mathrm{Tr}\big[\varrho_\varepsilon 1_{[L,+\infty)}(N_1)(N_1/\mathfrak{C})^k\big] = \sum_{i\in\mathbb{N}} \lambda_i \langle \Psi_i, 1_{[L,+\infty)}(N_1)(N_1/\mathfrak{C})^k \Psi_i\rangle$$
$$\geq \sum_{i\in\mathbb{N}} \lambda_i (L/\mathfrak{C})^k \langle \Psi_i, 1_{[L,+\infty)}(N_1)\Psi_i\rangle \ .$$

Therefore $(L/\mathfrak{C})^k\langle \Psi_i, 1_{[L,+\infty)}(N_1)\Psi_i\rangle \leq 1$ for any $k\in\mathbb{N}$ and for any Ψ_i. Now $(L/\mathfrak{C})^k$ diverges when $k\to\infty$, while $\langle \Psi_i, 1_{[L,+\infty)}(N_1)\Psi_i\rangle$ does not depend on k, so their product is uniformly bounded if and only if $1_{[L,+\infty)}(N_1)\Psi_i = 0$ for any $L > \mathfrak{C}$. The result follows immediately, recalling that the eigenvalues of N_1 are of the form εn_1, with $n_1\in\mathbb{N}$.

The converse statement that Assumption (A_0) follows if $\varrho_\varepsilon = \sum_{i\in\mathbb{N}}\lambda_i|\Psi_i\rangle\langle\Psi_i|$, with each Ψ_i with at most $[\mathfrak{C}/\varepsilon]$ particles is trivial to prove. $\dashv$

In this subsection, we will consider only families of states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ that satisfy Assumption (A_0) and the following assumption:

$$(A'_\rho) \qquad \exists C>0\,,\ \forall\varepsilon\in(0,\bar\varepsilon)\,,\ \mathrm{Tr}[\varrho_\varepsilon(N_1+H_0)]\leq C\ .$$

Definition 4.3 ($\varrho_\varepsilon(t)$, $\tilde\varrho_\varepsilon(t)$)**.** *We define the dressed time evolution of a state ϱ_ε to be*

$$\varrho_\varepsilon(t) = e^{-i\frac{t}{\varepsilon}\hat H_{\mathrm{ren}}}\,\varrho_\varepsilon\, e^{i\frac{t}{\varepsilon}\hat H_{\mathrm{ren}}}\ ,$$

where the dependence on σ_0 of $\hat H_{\mathrm{ren}}$ is omitted, and the σ_0 is chosen such that the dynamics is non-trivial on the whole subspace with at most $[\mathfrak{C}/\varepsilon]$ nucleons (see Lemma 4.2 and the discussion in Section 1.2). We also define the dressed evolution in the interaction picture to be

$$\tilde\varrho_\varepsilon(t) = e^{i\frac{t}{\varepsilon}H_0}\,\varrho_\varepsilon(t)\,e^{-i\frac{t}{\varepsilon}H_0}\ .$$

To characterize the evolved Wigner measures corresponding to $\tilde\varrho_\varepsilon(t)$, it is sufficient to study its Fourier transform; this is done studying the evolution of $\mathrm{Tr}[\tilde\varrho_\varepsilon(t)W(\xi)]$ by means of an integral equation.

Proposition 4.4. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\mathcal{H}$ satisfying Assumptions (A_0) and (A'_ρ). Then for any $t\in\mathbb{R}$, $Q(-\Delta+V)\oplus D(\omega^{1/2})\ni\xi=\xi_1\oplus\xi_2$:*

$$\mathrm{Tr}\Big[\tilde\varrho_\varepsilon(t)W(\xi)\Big] = \mathrm{Tr}\Big[\varrho_\varepsilon W(\xi)\Big] + \frac{i}{\varepsilon}\int_0^t \mathrm{Tr}\Big[\varrho_\varepsilon(s)[\hat H_{ren,I}, W(\tilde\xi_s)]\Big]\,ds\,, \tag{67}$$

where $\tilde\xi_s = e^{is(-\Delta+V)}\xi_1\oplus e^{-is\omega}\xi_2$. The commutator $[\hat H_{ren,I}, W(\tilde\xi_s)]$ has to be intended as a densely defined quadratic form with domain $Q(H_0)$, or equivalently as an operator from $Q(H_0)$ to $Q(H_0)^*$.

Proof. The family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ satisfies Assumption (A_0), therefore by Lemma 4.2:

$$\mathrm{Tr}\big[\tilde\varrho_\varepsilon(t)W(\xi)\big] = \sum_{i\in\mathbb{N}}\lambda_i\langle e^{i\frac{t}{\varepsilon}H_0}e^{-i\frac{t}{\varepsilon}\hat H_{\mathrm{ren}}}\Psi_i, W(\xi)e^{i\frac{t}{\varepsilon}H_0}e^{-i\frac{t}{\varepsilon}\hat H_{\mathrm{ren}}}\Psi_i\rangle\ .$$

By Assumption (A'_ρ), it follows that $\Psi_i\in Q(H_0)$ for any $i\in\mathbb{N}$. Hence the right hand side is differentiable in t by Lemma 4.1, since $Q(H_0)$ is the form domain of both H_0 and $\hat H_{\mathrm{ren}}$. Using the

Duhamel formula and the fact that $e^{-i\frac{s}{\varepsilon}H_0}W(\xi)e^{i\frac{s}{\varepsilon}H_0} = W(\tilde{\xi}_s)$, we then obtain:

$$\mathrm{Tr}\big[\tilde{\varrho}_\varepsilon(t)W(\xi)\big] = \sum_{i\in\mathbb{N}}\lambda_i\bigg(\langle\Psi_i, W(\xi)\Psi_i\rangle + \frac{i}{\varepsilon}\int_0^t\langle e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i, [\hat{H}_{ren,I}, W(\tilde{\xi}_s)]e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i\rangle ds\bigg) ;$$

where $[\hat{H}_{ren,I}, W(\tilde{\xi}_s)]$ makes sense as a quadratic form on $Q(H_0)$. The result is then obtained using Lebesgue's dominated convergence theorem on the right hand side, by virtue of Assumption (A'_ρ) and Lemma 4.1. $\dashv$

4.3. **The commutator** $[\hat{H}_{ren,I}, W(\tilde{\xi}_s)]$**.** In this subsection we deal with the commutator $[\hat{H}_{ren,I}, W(\tilde{\xi}_s)]$. The goal is to show that each of its terms converges in the limit $\varepsilon \to 0$, either to zero or to a suitable phase space symbol.

For convenience, we recall some terminology related to quantization procedures in infinite dimensional phase spaces (see [8] for additional informations). Let $\mathscr{Z}$ be a Hilbert space (the classical phase space). In the language of quantization, we call a densely defined functional $\mathscr{A} : D \subset \mathscr{Z} \to \mathbb{C}$ a (classical) *symbol*. We say that A is a *polynomial symbol* if there are densely defined bilinear forms $b_{p,q}$ on $\mathscr{Z}^{\otimes_s p} \times \mathscr{Z}^{\otimes_s q}$, $0 \leq p \leq \bar{p}$, $0 \leq q \leq \bar{q}$ (with $p, \bar{p}, q, \bar{q} \in \mathbb{N}$) such that

$$\mathscr{A}(z) = \sum_{\substack{0\leq p\leq\bar{p}\\ 0\leq q\leq\bar{q}}} b_{p,q}(z^{\otimes p}, z^{\otimes q}) . \tag{68}$$

The Wick quantized quadratic form $(\mathscr{A})^{Wick}$ on $\Gamma_s(\mathscr{Z})$ is then obtained, roughly speaking, replacing each $z(\cdot)$ with the annihilation operator valued distribution $a(\cdot)$; each $\bar{z}(\cdot)$ with the creation operator valued distribution $a^*(\cdot)$; and putting all the $a^*(\cdot)$ to the left of the $a(\cdot)$. We denote, with a straightforward notation, the class of all polynomial symbols on $\mathscr{Z}$ by $\bigoplus^{alg}_{(p,q)\in\mathbb{N}^2}\mathcal{Q}_{p,q}(\mathscr{Z})$. If $\mathscr{A} : \mathscr{Z} \to \mathbb{C}$ and the bilinear forms $b_{p,q}(z^{\otimes p}, z^{\otimes q})$ in (68) can all be written as $\langle z^{\otimes q}, \tilde{b}_{p,q}z^{\otimes p}\rangle_{\mathscr{Z}^{\otimes_s q}}$ for some bounded (resp. compact) operator $\tilde{b}_{p,q} : \mathscr{Z}^{\otimes_s p} \to \mathscr{Z}^{\otimes_s q}$, we say that $\mathscr{A}$ is a bounded (resp. compact) polynomial symbol. We denote the class of all bounded (resp. compact) polynomial symbols by $\bigoplus^{alg}_{(p,q)\in\mathbb{N}^2}\mathcal{P}_{p,q}(\mathscr{Z})$ $\Big($resp. $\bigoplus^{alg}_{(p,q)\in\mathbb{N}^2}\mathcal{P}^\infty_{p,q}(\mathscr{Z})\Big)$. We remark that $\mathscr{E}$, $\hat{\mathscr{E}}$ and $\mathscr{D}_g$ defined in Section 3 are all polynomial symbols[10] on $L^2 \oplus L^2$.

Lemma 4.5. *Let* $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ *satisfy the same assumptions as in Proposition 4.4. Then there exist maps* $\mathscr{B}_j(\cdot) : Q(-\Delta + V) \oplus D(\omega^{1/2}) \to \bigoplus^{alg}_{(p,q)\in\mathbb{N}^2}\mathcal{Q}_{p,q}\big(L^2 \oplus L^2\big)$, $j = 0, \ldots, 3$, *such that for any* $t \in \mathbb{R}$, $\xi \in Q(-\Delta + V) \oplus D(\omega^{1/2})$*:*

$$\begin{aligned}\mathrm{Tr}\Big[\tilde{\varrho}_\varepsilon(t)W(\xi)\Big] &= \mathrm{Tr}\Big[\varrho_\varepsilon W(\xi)\Big] + \sum_{j=0}^3\varepsilon^j\int_0^t\mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}_s)\big(\mathscr{B}_j(\tilde{\xi}_s)\big)^{Wick}\Big]ds\\ &= \mathrm{Tr}\Big[\varrho_\varepsilon W(\xi)\Big] + \sum_{j=0}^3\varepsilon^j\int_0^t\mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}_s)B_j(\tilde{\xi}_s)\Big]ds ;\end{aligned} \tag{69}$$

where the $\big(\mathscr{B}_j(\tilde{\xi}_s)\big)^{Wick}$ *make sense as densely defined quadratic forms. To simplify the notation, we have set* $B_j(\cdot) := \big(\mathscr{B}_j(\cdot)\big)^{Wick}$.

[10]In $L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$, we adopt the notation $z = (u, \alpha)$; and to each $u(x)$ it corresponds the operator valued distribution $\psi(x)$, to each $\alpha(k)$ the distribution $a(k)$. The Wick quantization is again obtained by substituting each $\big(u^\#(x), \alpha^\#(k)\big)$ with $\big(\psi^\#(x), a^\#(k)\big)$, and using the normal ordering of creators to the left of annihilators.

Proof. We only sketch the proof here since it follows the same lines as in [11, 62]. By (67), the only thing we have to prove is that, in the sense of quadratic forms, $\frac{i}{\varepsilon}[\hat{H}_{ren,I}, W(\tilde{\xi}_s)] = \sum_{j=0}^3 W(\tilde{\xi}_s)B_j(\tilde{\xi}_s)$. First of all, we remark that $\hat{H}_{ren,I}$ is the Wick quantization of a polynomial symbol[11]; i.e. $\hat{H}_{ren,I} = \big(\hat{\mathscr{E}}_I\big)^{Wick}$, with

$$\begin{aligned}\hat{\mathscr{E}}_I(u,\alpha) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^6}\frac{\chi_{\sigma_0}}{\sqrt{2\omega(k)}}\Big(\bar{\alpha}(k)e^{-ik\cdot x}+\alpha(k)e^{ik\cdot x}\Big)|u(x)|^2dxdk\\ +\frac{1}{2M}\int_{\mathbb{R}^9}\Big(r_\infty(k)\bar{\alpha}(k)e^{-ik\cdot x}+\bar{r}_\infty(k)\alpha(k)e^{ik\cdot x}\Big)\Big(r_\infty(l)\bar{\alpha}(l)e^{-il\cdot x}\\ +\bar{r}_\infty(l)\alpha(l)e^{il\cdot x}\Big)|u(x)|^2dxdkdl\\ -\frac{2}{M}\mathrm{Re}\int_{\mathbb{R}^6}r_\infty(k)\bar{\alpha}(k)e^{-ik\cdot x}\bar{u}(x)D_xu(x)dxdk+\frac{1}{2}\int_{\mathbb{R}^6}V_\infty(x-y)|u(x)|^2|u(y)|^2dxdy\;.\end{aligned}\tag{70}$$

We also recall, according to [8, Proposition 2.10 for bounded polynomial symbols] and [62, Proposition 2.1.30 for the general case], that essentially for any $\mathscr{A}\in\bigoplus_{(p,q)\in\mathbb{N}^2}^{alg}\mathcal{Q}_{p,q}(L^2\oplus L^2)$ the following formula is true, in the sense of forms, for any suitably regular $\xi\in L^2\oplus L^2$:

$$W^*(\xi)\big(\mathscr{A}\big)^{Wick}W(\xi) = \Big(\mathscr{A}\big(\cdot+\tfrac{i\varepsilon}{\sqrt{2}}\xi\big)\Big)^{Wick}\;.\tag{71}$$

Roughly speaking, the Weyl operators $W(\xi)$ translate each creation/annihilation operator by $\mp\frac{i\varepsilon}{\sqrt{2}}\xi$. The result then follows immediately on the states $\varrho_\varepsilon(s)$:

$$[\hat{H}_{ren,I}, W(\tilde{\xi}_s)] = W(\tilde{\xi}_s)\big(W^*(\tilde{\xi}_s)\hat{H}_{ren,I}W(\tilde{\xi}_s)-\hat{H}_{ren,I}\big) = W(\tilde{\xi}_s)\Big(\hat{\mathscr{E}}_I(\cdot+\tfrac{i\varepsilon}{\sqrt{2}}\tilde{\xi}_s)-\hat{\mathscr{E}}_I(\cdot)\Big)^{Wick}\;;$$

finally we define $\sum_{j=0}^3\varepsilon^j\mathscr{B}_j(\xi)(z) = \frac{i}{\varepsilon}\Big(\hat{\mathscr{E}}_I(z+\frac{i\varepsilon}{\sqrt{2}}\xi)-\hat{\mathscr{E}}_I(z)\Big)$, to factor out the ε-dependence. ⊣

We state the next lemma without giving the tedious proof, that is based on the same type of estimates given in Section 2.2 for the full operator $\hat{H}_{ren,I}$.

Lemma 4.6. *For any $j=0,1,2,3$, $\xi\in Q(-\Delta+V)\cap D(\omega^{1/2})$ and $\mathfrak{C}>0$, there exists $C_j(\xi)>0$ such that for any $\Phi,\Psi\in D(H_0^{1/2})\cap D(N_1)$, with Φ or Ψ in $\bigoplus_{n=0}^{[\mathfrak{C}/\varepsilon]}\mathcal{H}_n$ and for any $s\in\mathbb{R}$ and $\varepsilon\in(0,\bar{\varepsilon})$:*

$$|\langle\Phi, B_j(\tilde{\xi}_s)\Psi\rangle|\leq C_j(\xi)\|(N_1+H_0+\bar{\varepsilon})^{1/2}\Phi\|\cdot\|(N_1+H_0+\bar{\varepsilon})^{1/2}\Psi\|\;.\tag{72}$$

Thanks to this lemma we are now in a position to prove that the higher order terms in ε of Equation (69) (namely those with $j>0$) vanish in the limit $\varepsilon\to0$.

Proposition 4.7. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfy Assumptions (A_0) and (A'_ρ); let $\xi\in Q(-\Delta+V)\cap D(\omega^{1/2})$. Then the following limit holds for any $t\in\mathbb{R}$:*

$$\lim_{\varepsilon\to0}\sum_{j=1}^3\varepsilon^j\int_0^t\mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}_s)B_j(\tilde{\xi}_s)\Big]ds = 0\;.\tag{73}$$

Proof. By Lemma 4.2 we can write $\varrho_\varepsilon=\sum_i\lambda_i|\Psi_i\rangle\langle\Psi_i\rangle$, where each Ψ_i has non-zero components only in the subspace $\bigoplus_{n\leq[\mathfrak{C}/\varepsilon]}\mathcal{H}_n$, and each $\lambda_i>0$. Assumption (A'_ρ) then translates on the fact that each

[11]To be precise, we are considering here the quadratic form $\hat{h}_{ren,I}$, defined and different from zero on the whole space $\mathcal{H}$, since it agrees with $\langle\,\cdot\,,\hat{H}_{ren,I}\,\cdot\rangle$ when restricted to vectors that belong to $\bigoplus_{n\leq[\mathfrak{C}/\varepsilon]}\mathcal{H}_n$ (being here the case by Lemma 4.2).

Ψ_i is on the domain $Q(H_0)\cap Q(N_1)$, and in addition $\sum_i \lambda_i\langle\Psi_i,(N_1+H_0)\Psi_i\rangle\leq C$, uniformly with respect to $\varepsilon\in(0,\bar{\varepsilon})$. Therefore we can write

$$\Bigl|\sum_{j=1}^{3}\varepsilon^j\int_0^t \mathrm{Tr}\Bigl[\varrho_\varepsilon(s)W(\tilde{\xi}_s)B_j(\tilde{\xi}_s)\Bigr]ds\Bigr|\leq\sum_{j=1}^{3}\varepsilon^j\sum_i\lambda_i\int_0^t\Bigl|\langle W^*(\tilde{\xi}_s)e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i,B_j(\tilde{\xi}_s)e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i\rangle\Bigr|ds\;.$$

Using now Lemma 4.6 and then Lemma 4.1 and the fact that N_1 commutes with $\hat{H}_{\mathrm{ren}}$ we obtain

$$\begin{aligned}\Bigl|\sum_{j=1}^{3}\varepsilon^j\int_0^t \mathrm{Tr}\Bigl[\varrho_\varepsilon(s)W(\tilde{\xi}_s)B_j(\tilde{\xi}_s)\Bigr]ds\Bigr|&\leq\sum_{j=1}^{3}\varepsilon^jC_j(\xi)\sum_i\lambda_i\int_0^t\|(N_1+H_0+\bar{\varepsilon})^{1/2}W^*(\tilde{\xi}_s)e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i\|\\&\qquad\cdot\|(N_1+H_0+\bar{\varepsilon})^{1/2}e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i\|ds\\&\leq\sum_{j=1}^{3}\varepsilon^jC(\xi)C_j(\xi)\sum_i\lambda_i\int_0^t\langle e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i,(N_1+H_0+\bar{\varepsilon})e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i\rangle ds\\&\leq\sum_{j=1}^{3}\varepsilon^jC(\xi)C_j(\xi)\sum_i\lambda_i\Bigl(t\langle\Psi_i,(N_1+\bar{\varepsilon})\Psi_i\rangle+\int_0^t\langle e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i,H_0e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i\rangle ds\Bigr)\;.\end{aligned}$$

Now we consider the term $\langle e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i,H_0e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i\rangle$. First of all we write it as

$$\begin{aligned}\langle e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i,H_0e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i\rangle&=\langle e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i,(\hat{H}_{\mathrm{ren}}-\hat{H}_{ren,I})e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i\rangle\\&=\sum_{n=0}^{[\mathfrak{C}/\varepsilon]}\langle e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)},\Bigl(\hat{H}_\infty^{(n)}-\hat{H}_I^{(n)}(\infty)\Bigr)e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)}\rangle\\&=\sum_{n=0}^{[\mathfrak{C}/\varepsilon]}\langle\Psi_i^{(n)},\hat{H}_\infty^{(n)}\Psi_i^{(n)}\rangle-\langle e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)},\hat{H}_I^{(n)}(\infty)e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)}\rangle\\&\leq\sum_{n=0}^{[\mathfrak{C}/\varepsilon]}\Bigl(\Bigl|\langle\Psi_i^{(n)},\hat{H}_\infty^{(n)}\Psi_i^{(n)}\rangle\Bigr|+\Bigl|\langle e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)},\hat{H}_I^{(n)}(\infty)e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)}\rangle\Bigr|\Bigr)\;.\end{aligned}\tag{74}$$

The idea now is to use the bound of Equation (36) on $\Bigl|\langle e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)},\hat{H}_I^{(n)}(\infty)e^{-i\frac{s}{\varepsilon}\hat{H}_\infty^{(n)}}\Psi_i^{(n)}\rangle\Bigr|$. The crucial point is that since we have chosen σ_0 such that the dynamics is non-trivial for any $n\leq[\mathfrak{C}/\varepsilon]$, it follows that there exist an $a<1$ and a $b<\infty$ *both independent of* ε *and* n such that the bound (36) holds for any $n\leq[\mathfrak{C}/\varepsilon]$. Therefore we obtain

$$\begin{aligned}\langle e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i,H_0e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i\rangle&\leq a\langle e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i,H_0e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i\rangle+b\langle\Psi_i,\Psi_i\rangle\\&\qquad+\sum_{n=0}^{[\mathfrak{C}/\varepsilon]}\Bigl|\langle\Psi_i^{(n)},\hat{H}_\infty^{(n)}\Psi_i^{(n)}\rangle\Bigr|\;.\end{aligned}\tag{75}$$

Now since $a<1$, we may take it to the left hand side and use again (36) on $\Bigl|\langle\Psi_i^{(n)},\hat{H}_\infty^{(n)}\Psi_i^{(n)}\rangle\Bigr|$:

$$\langle e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i,H_0e^{-i\frac{s}{\varepsilon}\hat{H}_{\mathrm{ren}}}\Psi_i\rangle\leq\frac{1}{1-a}\langle\Psi_i,H_0\Psi_i\rangle+\frac{2b}{1-a}\langle\Psi_i,\Psi_i\rangle\;.\tag{76}$$

Finally, since the state is normalized (i.e. $\sum_i \lambda_i\langle\Psi_i,\Psi_i\rangle = 1$), we conclude:

$$\begin{aligned}\Big|\sum_{j=1}^{3}\varepsilon^j\int_0^t \mathrm{Tr}\Big[\varrho_\varepsilon(s)W(\tilde{\xi}_s)B_j(\tilde{\xi}_s)\Big]ds\Big| &\leq t\sum_{j=1}^{3}\varepsilon^j C(\xi)C_j(\xi)\sum_i\lambda_i\Big(\langle\Psi_i,N_1\Psi_i\rangle+\tfrac{1}{1-a}\langle\Psi_i,H_0\Psi_i\rangle\\ &\qquad +(\tfrac{2b}{1-a}+\bar{\varepsilon})\langle\Psi_i,\Psi_i\rangle\Big)\\ &\leq t\sum_{j=1}^{3}\varepsilon^j C(\xi)C_j(\xi)\Big(\Big(1+\tfrac{1}{1-a}\Big)\sum_i\lambda_i\langle\Psi_i,(N_1+H_0)\Psi_i\rangle+\tfrac{2b}{1-a}+\bar{\varepsilon}\Big)\\ &\leq t\sum_{j=1}^{3}\varepsilon^j C(\xi)C_j(\xi)\Big(\Big(1+\tfrac{1}{1-a}\Big)C+\tfrac{2b}{1-a}+\bar{\varepsilon}\Big)\,.\end{aligned}$$

The right hand side has no implicit dependence on ε, so it converges to zero when $\varepsilon\to 0$. ⊣

By the same argument used from (74) to (76) above, we can prove the following useful lemma.

Lemma 4.8. *If a family of states $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfies Assumptions (A_0) and (A'_ρ), then for any $t\in\mathbb{R}$, $\big(\varrho_\varepsilon(t)\big)_{\varepsilon\in(0,\bar{\varepsilon})}$ and $\big(\tilde{\varrho}_\varepsilon(t)\big)_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfy Assumptions (A_0) and (A'_ρ). In particular, there exist $a(\mathfrak{C})<1$ and $b(\mathfrak{C})>0$ such that uniformly on $\varepsilon\in(0,\bar{\varepsilon})$:*

$$\mathrm{Tr}[\varrho_\varepsilon(t)N_1^k]\leq\mathfrak{C}^k\ ,\ \forall k\in\mathbb{N}\ ; \tag{77}$$

$$\mathrm{Tr}[\varrho_\varepsilon(t)(N_1+H_0)]\leq\frac{1}{1-a(\mathfrak{C})}C+\frac{2b(\mathfrak{C})}{1-a(\mathfrak{C})}\ ; \tag{78}$$

and the same bounds hold for $\big(\tilde{\varrho}_\varepsilon(t)\big)_{\varepsilon\in(0,\bar{\varepsilon})}$.

It remains to study the limit of the $B_0(\cdot)$-term in (69). As already pointed out in Lemma 4.5, we know that B_0 is a Wick quantization. More precisely, there exist a densely defined map from the one-particle space to polynomial symbols in $\bigoplus_{(p,q)\in\{(i,j)|0\leq i,j\leq 2;2\leq i+j\leq 3\}}\mathcal{Q}_{p,q}\big(L^2\oplus L^2\big)$. In order to apply the convergence results of Ammari and Nier [8], we need to show that the symbol of B_0 may be approximated by a compact one, with an error that vanishes in the limit $\varepsilon\to 0$.

To improve readability, we will write $B_0(\xi)$ in a schematic fashion. The precise structure of each term will be discussed and analyzed in the proof of the sequent proposition. In addition, as seen in Equation (16), the dressed interaction quadratic form $\hat{H}_I(\infty)$ can be split in three terms: the first is just the interaction term $H_I(\sigma_0)$ of the Nelson model with cutoff (with σ replaced by σ_0), whose classical limit has been analyzed by the authors in [5]; the second is a "mean-field" term for the nucleons, of the same type as the ones analyzed by Ammari and Nier in [11]; the last one has a structure similar to the interaction part of the Pauli-Fierz model [see e.g. 14–16, 79], and thus will be called of "Pauli-Fierz type". We will concentrate on the analysis of the Pauli-Fierz type terms of B_0, while for a precise treatment of the others the reader may refer to [5, 11]. In order to highlight the different parts of $B_0(\xi)=B_0(\xi_1,\xi_2)$, we will underline with different style and color the Nelson, mean-field and Pauli-Fierz type terms:

$$
\begin{aligned}
B_0(\xi_1,\xi_2) = \big(\mathscr{B}_0(\xi_1,\xi_2)\big)^{Wick} &= \underline{(a^*+a)(\xi_1\psi^* - \bar{\xi}_1\psi) + \mathrm{Im}(\xi_2)\psi^*\psi} + \bar{\xi}_1\psi^*\psi\psi - \xi_1\psi^*\psi^*\psi \\
&+(a^*a^* + aa + a^*a)(\xi_1\psi^* - \bar{\xi}_1\psi) + (\xi_2 a^* - \bar{\xi}_2 a)\psi^*\psi + (a^*D_x + D_x a)(\psi^*\xi_1 - \bar{\xi}_1\psi) \\
&+\psi^* D_x\xi_2\psi - \psi^*\bar{\xi}_2 D_x\psi \ .
\end{aligned} \tag{79}
$$

Proposition 4.9. *There exists a family of maps $(\mathscr{B}_0^{(m)})_{m\in\mathbb{N}}$ such that:*

- *For any $m\in\mathbb{N}$*

$$
\mathscr{B}_0^{(m)}(\cdot) : Q(-\Delta+V)\oplus D(\omega^{3/4}) \to \bigoplus_{(p,q)\in\{(i,j)|0\leq i,j\leq 2;2\leq i+j\leq 3\}} \mathcal{P}^\infty_{p,q}\big(L^2\oplus L^2\big) \ ;
$$

- *For any $\xi\in Q(-\Delta+V)\oplus D(\omega^{3/4})$, there exist a sequence $\big(C^{(m)}(\xi)\big)_{m\in\mathbb{N}}$ that depends only on $\|\xi\|_{Q(-\Delta+V)\oplus D(\omega^{3/4})}$ such that $\lim_{m\to\infty} C^{(m)} = 0$; and such that for any two vectors $\Phi,\Psi\in\mathcal{H}\cap D(N_1)$, and for any $\varepsilon\in(0,\bar{\varepsilon})$:*

$$
\begin{aligned}
\Big|\Big\langle (H_0+1)^{-1/2}\Phi, \big(\mathscr{B}_0(\xi) - \mathscr{B}_0^{(m)}(\xi)\big)^{Wick}(H_0+1)^{-1/2}\Psi\Big\rangle\Big| &\leq C^{(m)}(\xi)\big\|(N_1+\bar{\varepsilon})^{1/2}\Phi\big\| \\
&\cdot\big\|(N_1+\bar{\varepsilon})^{1/2}\Psi\big\| \ .
\end{aligned} \tag{80}
$$

Remark 4.10. Contrarily to what it was previously assumed throughout Section 4, in this proposition we need additional regularity on ξ_2, namely $\xi_2\in D(\omega^{3/4})\subset D(\omega^{1/2})$. This will not be a problem in the following, since we will extend our results to any $\xi\in L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)$ by a density argument, and $D(\omega^{3/4})$ is still dense in $L^2(\mathbb{R}^3)$.

Proof of Proposition 4.9. To prove the proposition, we need to analyze each term of Equation (79), and prove that either it has a compact symbol or it can be approximated by one, in a way that (80) holds. The analysis for Nelson terms has been carried out in [5, Proposition 3.11 and Lemma 3.15]. In addition, using Lemma 2.5 we see that V_∞ satisfies the hypotheses of the mean field potentials in [11], therefore (80) holds also for the mean-field terms [see in particular Section 3.2 of 11]. For the sake of completeness, we explicitly write the Nelson and mean field part of Equation (79):

$$
\underline{(a^*+a)(\xi_1\psi^*-\bar{\xi}_1\psi)} = -\tfrac{1}{\sqrt{2}(2\pi)^{3/2}}\int_{\mathbb{R}^3}\Big(a^*\big(\tfrac{e^{-ik\cdot x}}{\sqrt{2\omega}}\chi_{\sigma_0}\big) + a\big(\tfrac{e^{-ik\cdot x}}{\sqrt{2\omega}}\chi_{\sigma_0}\big)\Big)\Big(\xi_1(x)\psi^*(x) - \bar{\xi}_1(x)\psi(x)\Big)dx \ ;
$$

$$
\underline{\mathrm{Im}(\xi_2)\psi^*\psi} = -\tfrac{1}{\sqrt{2}(2\pi)^{3/2}}\int_{\mathbb{R}^6}\bigg(\frac{\chi_{\sigma_0}(k)}{\sqrt{2\omega(k)}}\Big(\xi_2(k)e^{ik\cdot x} - \bar{\xi}_2(k)e^{-ik\cdot x}\Big)\bigg)\psi^*(x)\psi(x)dxdk \ ;
$$

$$
\bar{\xi}_1\psi^*\psi\psi - \xi_1\psi^*\psi^*\psi = \tfrac{1}{\sqrt{2}}\int_{\mathbb{R}^6} V_\infty(x-y)\Big(\bar{\xi}_1(y)\psi^*(x)\psi(x)\psi(y) - \xi_1(y)\psi^*(x)\psi^*(y)\psi(x)\Big)dxdy \ .
$$

It remains to study the terms of Pauli-Fierz type. This is done in six parts; in each part we group terms that are either adjoint of each other, or can be treated in a similar fashion.

Part 1 ($\bar{\xi}_1 aa\psi$, $\xi_1 a^*a^*\psi^*$)**.**

$$
\bar{\xi}_1 aa\psi = -\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^3}\Big(a\big(r_\infty e^{-ik\cdot x}\big)\Big)^2\bar{\xi}_1(x)\psi(x)dx \ .
$$

We recall that $r_\infty\sim kg_\infty$, where g_σ is defined by (7) for any $\sigma\leq\infty$. Let $\bar{\xi}_1\alpha\alpha u$ be the symbol[12] associated to $\bar{\xi}_1 aa\psi$, i.e. $\bar{\xi}_1 aa\psi = (\bar{\xi}_1\alpha\alpha u)^{Wick}$. Now, since $r_\infty\notin L^2(\mathbb{R}^3)$, we cannot expect that

[12]We recall that for the Nelson model $\mathcal{Z} = L^2(\mathbb{R}^d)\oplus L^2(\mathbb{R}^d)$, thus we denote the variable z by $u\oplus\alpha$.

$\bar{\xi}_1\alpha\alpha u$ is defined for any $u,\alpha\in L^2(\mathbb{R}^3)$, and therefore that it is a compact symbol. We introduce the approximated symbol $\bar{\xi}_1\alpha\alpha u^{(m)}$ defined by

$$\bar{\xi}_1 aa\psi^{(m)} = (\bar{\xi}_1\alpha\alpha u^{(m)})^{Wick} = -\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^3}\Bigl(a\bigl(r_{\sigma_m}e^{-ik\cdot x}\bigr)\Bigr)^2\bar{\xi}_1(x)\psi(x)dx\;,$$

with $(\sigma_m)_{m\in\mathbb{N}}\subset\mathbb{R}$ such that $\lim_{m\to\infty}\sigma_m=\infty$. First of all, we prove that (80) holds for $\bar{\xi}_1 aa\psi-\bar{\xi}_1 aa\psi^{(m)}$:

$$\begin{aligned}\Bigl|\Bigl\langle(H_0+1)^{-1/2}\Phi,(\bar{\xi}_1 aa\psi-\bar{\xi}_1 aa\psi^{(m)})(H_0+1)^{-1/2}\Psi\Bigr\rangle\Bigr| &\leq \tfrac{1}{2\sqrt{2}M}\|\xi_1\|_2\bigl\|(d\Gamma(\omega)+1)^{1/2}\\ &\qquad (H_0+1)^{-1/2}\Phi\bigr\|\\ \cdot\sup_{x\in\mathbb{R}^3}\Bigl\|(d\Gamma(\omega)+1)^{-1/2}&\Bigl(a\bigl((r_\infty-r_{\sigma_m})e^{-ik\cdot x}\bigr)\Bigr)^2(d\Gamma(\omega)+1)^{-1/2}\\ &(d\Gamma(\omega)+1)^{1/2}(H_0+1)^{-1/2}(N_1+\varepsilon)^{1/2}\Psi\Bigr\|\;.\end{aligned}$$

We use (28) of Lemma 2.7 and the fact that $(d\Gamma(\omega)+1)^{1/2}(H_0+1)^{-1/2}$ is bounded with norm smaller than one to obtain

$$\begin{aligned}\Bigl|\Bigl\langle(H_0+1)^{-1/2}\Phi,(\bar{\xi}_1 aa\psi-\bar{\xi}_1 aa\psi^{(m)})(H_0+1)^{-1/2}\Psi\Bigr\rangle\Bigr| &\leq \tfrac{c}{2\sqrt{2}M}\|\xi_1\|_2\|\omega^{-1/4}(r_\infty-r_{\sigma_m})\|_2^2\bigl\|\Phi\bigr\|\\ &\qquad\cdot\bigl\|(N_1+\bar{\varepsilon})^{1/2}\Psi\bigr\|\\ &\leq C^{(m)}(\xi_1)\bigl\|(N_1+\bar{\varepsilon})^{1/2}\Phi\bigr\|\cdot\bigl\|(N_1+\bar{\varepsilon})^{1/2}\Psi\bigr\|\;,\end{aligned}$$

with $C^{(m)}(\xi_1)=C(\bar{\varepsilon},\xi_1)\|\omega^{-1/4}(r_\infty-r_{\sigma_m})\|_2^2$ for some $C(\bar{\varepsilon},\xi_1)>0$. The sequence $\bigl(C^{(m)}(\xi_1)\bigr)_{m\in\mathbb{N}}$ converges to zero since by our choice of $(\sigma_m)_{m\in\mathbb{N}}$:

$$\lim_{m\to\infty}\|\omega^{-1/4}(r_\infty-r_{\sigma_m})\|_2^2=0\;.$$

It remains to show that $\bar{\xi}_1\alpha\alpha u^{(m)}$ is a compact symbol. Such symbol can be written as

$$\bar{\xi}_1\alpha\alpha u^{(m)}=-\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^9}\bar{\xi}_1(x)\bar{r}_{\sigma_m}(k)\bar{r}_{\sigma_m}(k')e^{i(k+k')\cdot x}\alpha(k)\alpha(k')u(x)dxdkdk'\;.$$

Now we can define an operator $\tilde{b}_{\alpha\alpha u}:\bigl(L^2\oplus L^2\bigr)^{\otimes_s 3}\to\mathbb{C}$ in the following way. Let the maps $\pi_1,\pi_2:L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ be the projections on the first and second space respectively. Then we define the operator $\tilde{b}_{\alpha\alpha u}$ as:

$$\tilde{b}_{\alpha\alpha u}:(u,\alpha)^{\otimes 3}\in\bigl(L^2\oplus L^2\bigr)^{\otimes_s 3}\underset{\pi_2\otimes\pi_2\otimes\pi_1}{\longrightarrow}\alpha(k)\alpha(k')u(x)\in L^2(\mathbb{R}^9)\longrightarrow\langle f,\alpha\alpha u\rangle_{L^2(\mathbb{R}^9)}\in\mathbb{C}\;;$$

where $f(k,k',x)=-\tfrac{1}{2\sqrt{2}M}\bar{\xi}_1(x)\bar{r}_{\sigma_m}(k)\bar{r}_{\sigma_m}(k')e^{i(k+k')\cdot x}\in L^2(\mathbb{R}^9)$. Therefore $\tilde{b}_{\alpha\alpha u}$ is bounded and finite rank, and therefore compact. The proof for the corresponding adjoint term

$$\xi_1 a^*a^*\psi^*=-\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^3}\Bigl(a^*\bigl(r_\infty e^{-ik\cdot x}\bigr)\Bigr)^2\xi_1(x)\psi^*(x)dx$$

can be obtained directly from the above, using the following approximation with compact symbol:

$$\xi_1 a^*a^*\psi^{*(m)}=(\xi_1\bar{\alpha}\bar{\alpha}\bar{u}^{(m)})^{Wick}=-\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^3}\Bigl(a^*\bigl(r_{\sigma_m}e^{-ik\cdot x}\bigr)\Bigr)^2\xi_1(x)\psi^*(x)dx\;.$$

Part 2 ($\xi_1 aa\psi^*$, $\bar{\xi}_1 a^*a^*\psi$)**.**

$$\xi_1 aa\psi^*=-\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^3}\Bigl(a\bigl(r_\infty e^{-ik\cdot x}\bigr)\Bigr)^2\xi_1(x)\psi^*(x)dx\;.$$

Again we approximate this term by

$$\xi_1 aa\psi^{*(m)} = (\xi_1\alpha\alpha\bar{u}^{(m)})^{Wick} = -\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^3}\Bigl(a\bigl(r_{\sigma_m}e^{-ik\cdot x}\bigr)\Bigr)^2\xi_1(x)\psi^*(x)dx$$

as above. The proof that it satisfies (80) is perfectly analogous as the one for the previous term. Therefore we only prove that $\xi_1\alpha\alpha\bar{u}^{(m)}$ is a compact symbol. We define an operator $b_{\alpha\alpha\bar{u}} : \bigl(L^2\oplus L^2\bigr)^{\otimes_s 2}\to L^2\oplus L^2$ by

$$\begin{aligned}\tilde{b}_{\alpha\alpha\bar{u}} : (u,\alpha)^{\otimes 2}\in\bigl(L^2\oplus L^2\bigr)^{\otimes_s 2} &\underset{\pi_2\otimes\pi_2}{\longrightarrow}\alpha(k)\alpha(k')\in L^2(\mathbb{R}^6)\\ &\underset{\tilde{c}_{\alpha\alpha\bar{u}}}{\longrightarrow}\biggl(\int_{\mathbb{R}^6}\bar{f}(k,k',\cdot)\alpha(k)\alpha(k')dkdk'\,\oplus\,0\biggr)\in L^2\oplus L^2\ ;\end{aligned}$$

where $f(k,k',x) = -\tfrac{1}{2\sqrt{2}M}\xi_1(x)r_{\sigma_m}(k)r_{\sigma_m}(k')e^{-i(k+k')\cdot x}$. By definition, we have that

$$\xi_1\alpha\alpha\bar{u}^{(m)} = \langle(u,\alpha),\tilde{b}_{\alpha\alpha\bar{u}}(u,\alpha)^{\otimes 2}\rangle_{L^2\oplus L^2}\ .$$

It is easily seen that the operator $\tilde{c}_{\alpha\alpha\bar{u}}$ is bounded. It is in fact compact: let $\beta_j\rightharpoonup\beta$ in $L^2(\mathbb{R}^6)$ be a weakly convergent (bounded) sequence such that $\max\bigl\{\bigl(\sup_j\|\beta_j\|_{L^2(\mathbb{R}^6)}\bigr),\|\beta\|_{L^2(\mathbb{R}^6)}\bigr\} = X<\infty$; then

$$\bigl\|\tilde{c}_{\alpha\alpha\bar{u}}(\beta-\beta_j)\bigr\|_{L^2\oplus L^2} = \Bigl\|\langle f(k,k',x),(\beta-\beta_j)(k,k')\rangle_{L^2_{k,k'}(\mathbb{R}^6)}\Bigr\|_{L^2_x(\mathbb{R}^3)}\underset{j\to\infty}{\longrightarrow}0\ ,$$

by Lebesgue's dominated convergence theorem, using the uniform bound

$$\begin{aligned}\Bigl|\langle f(k,k',x),(\beta-\beta_j)(k,k')\rangle^2_{L^2_{k,k'}(\mathbb{R}^6)}\Bigr| &\leq\|f(k,k',x)\|^2_{L^2_{k,k'}(\mathbb{R}^6)}\Bigl(\|\beta\|^2_{L^2(\mathbb{R}^6)}+\|\beta_j\|^2_{L^2(\mathbb{R}^6)}\Bigr)\\ &\leq\tfrac{2X}{8M^2}\|r_{\sigma_m}\|_2^4|\xi_1(x)|^2\in L^1_x(\mathbb{R}^3)\ .\end{aligned}$$

Therefore, since $\tilde{c}_{\alpha\alpha\bar{u}}$ is compact and $\pi_2\otimes\pi_2$ is bounded, it follows that $\tilde{b}_{\alpha\alpha\bar{u}}$ is compact. Again, that implies the result holds also for the adjoint term

$$\bar{\xi}_1 a^*a^*\psi = -\tfrac{1}{2\sqrt{2}M}\int_{\mathbb{R}^3}\Bigl(a^*\bigl(r_\infty e^{-ik\cdot x}\bigr)\Bigr)^2\bar{\xi}_1(x)\psi(x)dx\ .$$

Part 3 ($\bar{\xi}_1 a^*a\psi$, $\xi_1 a^*a\psi^*$)**.**

$$\begin{aligned}\bar{\xi}_1 a^*a\psi &= -\tfrac{1}{\sqrt{2}M}\int_{\mathbb{R}^3}a^*\bigl(r_\infty e^{-ik\cdot x}\bigr)a\bigl(r_\infty e^{-ik\cdot x}\bigr)\bar{\xi}_1(x)\psi(x)dx\ ,\\ \xi_1 a^*a\psi^* &= -\tfrac{1}{\sqrt{2}M}\int_{\mathbb{R}^3}a^*\bigl(r_\infty e^{-ik\cdot x}\bigr)a\bigl(r_\infty e^{-ik\cdot x}\bigr)\xi_1(x)\psi^*(x)dx\ .\end{aligned}$$

The proof for this couple of terms goes on exactly like the previous one, i.e. approximating r_∞ with r_{σ_m} and showing that the corresponding operator $\tilde{c}_{\bar{\alpha}\alpha u}$ is compact, for it maps weakly convergent sequences into strongly convergent ones.

Part 4 ($\bar{\xi}_2 a\psi^*\psi$, $\xi_2 a^*\psi^*\psi$)**.**

$$\bar{\xi}_2 a\psi^*\psi = -\tfrac{\sqrt{2}i}{M}\int_{\mathbb{R}^6}\mathrm{Im}\bigl(\xi_2(k')\bar{r}_\infty(k')e^{ik'\cdot x}\bigr)a\bigl(r_\infty e^{-ik\cdot x}\bigr)\psi^*(x)\psi(x)dxdk'\ .$$

We approximate it by the symbol $\bar{\xi}_2\alpha\bar{u}u^{(m)}$ defined by:

$$\begin{aligned}\bar{\xi}_2 a\psi^*\psi^{(m)} = (\bar{\xi}_2\alpha\bar{u}u^{(m)})^{Wick} = -\tfrac{\sqrt{2}i}{M}\int_{\mathbb{R}^6}\psi^*(x)\chi_m(D_x)\mathrm{Im}\bigl(\xi_2(k')\bar{r}_\infty(k')e^{ik'\cdot x}\bigr)a\bigl(r_{\sigma_m}e^{-ik\cdot x}\bigr)\\ \psi(x)dxdk'\ ;\end{aligned}$$

where χ_m is the smooth cut-off function defined at the beginning of Section 2, while r_{σ_m} is the usual regularization of r_∞ defined above. First of all we check that the approximation satisfies (80). By the chain rule, two parts have to be checked:

$$\Big|\Big\langle (H_0+1)^{-1/2}\Phi, (\bar{\xi}_2 a\psi^*\psi - \bar{\xi}_2 a\psi^*\psi^{(m)})(H_0+1)^{-1/2}\Psi\Big\rangle\Big| \leq \tfrac{\sqrt{2}(2\pi)^{3/2}}{M}\Bigg($$
$$\Big|\Big\langle (H_0+1)^{-1/2}\Phi, \int_{\mathbb{R}^3} dx\, \psi^*(x)\big(1-\chi_m(D_x)\big)\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)(x)a\big(r_\infty e^{-ik\cdot x}\big)\psi(x)(H_0+1)^{-1/2}\Psi\Big\rangle\Big|$$
$$+\Big|\Big\langle (H_0+1)^{-1/2}\Phi, \int_{\mathbb{R}^3} dx\, \psi^*(x)\chi_m(D_x)\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)(x)a\big((r_\infty - r_{\sigma_m})e^{-ik\cdot x}\big)\psi(x)(H_0+1)^{-1/2}\Psi\Big\rangle\Big|\Bigg)\ ;$$

and we will consider them separately. For the first part we have:

$$\begin{aligned}
&\Big|\Big\langle (H_0+1)^{-1/2}\Phi, \int_{\mathbb{R}^3} dx\, \psi^*(x)\big(1-\chi_m(D_x)\big)\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)(x)a\big(r_\infty e^{-ik\cdot x}\big)\psi(x)(H_0+1)^{-1/2}\Psi\Big\rangle\Big|\\
&\leq \sum_{n=0}^{\infty} n\varepsilon\Big|\Big\langle (H_0^{(n)}+1)^{-1/2}\Phi_n, (1-\chi_m(D_{x_1}))\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)(x_1)a\big(r_\infty e^{-ik\cdot x_1}\big)(H_0^{(n)}+1)^{-1/2}\Psi_n\Big\rangle_{\mathcal{H}_n}\Big|\\
&\leq \sum_{n=0}^{\infty} n\varepsilon\big\|(1-D_x^2)^{-1/2}\big(1-\chi_m(D_x)\big)\big\|_{\mathcal{L}(L^2(\mathbb{R}^3))}\cdot\big\|\mathcal{F}^{-1}(\xi_2\bar{r}_\infty)\big\|_\infty\cdot\big\|\omega^{-1/2}r_\infty\big\|_2\\
&\qquad\cdot\big\|(1-D_{x_1}^2)^{1/2}(H_0^{(n)}+1)^{-1/2}\Phi_n\big\|_{\mathcal{H}_n}\cdot\big\|d\Gamma(\omega)^{1/2}(H_0^{(n)}+1)^{-1/2}\Psi_n\big\|_{\mathcal{H}_n}\\
&\leq (1+\bar{\varepsilon})\|\xi_2\|_{\mathcal{F}H^{1/2}}\cdot\big\|\omega^{-1/2}r_\infty\big\|_2^2\cdot\big\|(1-D_x^2)^{-1/2}\big(1-\chi_m(D_x)\big)\big\|_{\mathcal{L}(L^2(\mathbb{R}^3))}\cdot\big\|(N_1+\bar{\varepsilon})^{1/2}\Phi\big\|\cdot\big\|N_1^{1/2}\Psi\big\|\ ;
\end{aligned}$$

where in the last inequality we have utilized the following bound:

$$\begin{aligned}
n\varepsilon\big\|(1-D_{x_1}^2)^{1/2}(H_0^{(n)}+1)^{-1/2}\Phi_n\big\|_{\mathcal{H}_n}^2 &= \big\langle \Phi_n, (H_0^{(n)}+1)^{-1/2}d\Gamma(1-\Delta)(H_0^{(n)}+1)^{-1/2}\Phi_n\big\rangle_{\mathcal{H}_n}\\
&\leq \big\|N_1^{1/2}\Phi_n\big\|_{\mathcal{H}_n} + \big\|d\Gamma(-\Delta)^{1/2}(H_0^{(n)}+1)^{-1/2}\Phi_n\big\|_{\mathcal{H}_n} \leq (1+\bar{\varepsilon})\big\|(N_1+\bar{\varepsilon})^{1/2}\Phi_n\big\|_{\mathcal{H}_n}\ .
\end{aligned}$$

So the first part satisfies (80), since

$$\lim_{m\to\infty}\big\|(1-D_x^2)^{-1/2}\big(1-\chi_m(D_x)\big)\big\|_{\mathcal{L}(L^2(\mathbb{R}^3))} = 0\ .$$

A similar procedure for the second part yields

$$\begin{aligned}
&\Big|\Big\langle (H_0+1)^{-1/2}\Phi, \int_{\mathbb{R}^3} dx\, \psi^*(x)\chi_m(D_x)\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)(x)a\big((r_\infty - r_{\sigma_m})e^{-ik\cdot x}\big)\psi(x)(H_0+1)^{-1/2}\Psi\Big\rangle\Big|\\
&\qquad\leq \|\xi_2\|_{\mathcal{F}H^{1/2}}\cdot\big\|\omega^{-1/2}r_\infty\big\|_2\cdot\big\|\omega^{-1/2}(r_\infty - r_{\sigma_m})\big\|_2\big\|N_1^{1/2}\Phi\big\|\cdot\big\|N_1^{1/2}\Psi\big\|\ ;
\end{aligned}$$

i.e. it satisfies (80), for $\lim_{m\to\infty}\|\omega^{-1/2}(r_\infty - r_{\sigma_m})\|_2 = 0$. Now it remains to show that $\bar{\xi}_2\alpha\bar{u}u^{(m)}$ is a compact symbol:

$$\bar{\xi}_2\alpha\bar{u}u^{(m)} = -\tfrac{(2\pi)^{3/2}\sqrt{2}i}{M}\int_{\mathbb{R}^6}\bar{u}(x)\chi_m(D_x)\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)(x)\bar{r}_{\sigma_m}(k)e^{ik\cdot x}\alpha(k)u(x)dxdk\ .$$

As for the previous terms, we define an operator $b_{\alpha\bar{u}u} : \big(L^2\oplus L^2\big)^{\otimes_s 2}\to L^2\oplus L^2$ by

$$\tilde{b}_{\alpha\bar{u}u} : (u,\alpha)^{\otimes 2}\in\big(L^2\oplus L^2\big)^{\otimes_s 2}\underset{\pi_2\otimes\pi_1}{\longrightarrow}\alpha(k)u(x)\in L^2(\mathbb{R}^6)\underset{\tilde{c}_{\alpha\bar{u}u}}{\longrightarrow}\Big(f'(x,D_x)u(x)\oplus 0\Big)\in L^2\oplus L^2\ ;$$

where $f'(x,D_x) = -\frac{(2\pi)^3\sqrt{2}i}{M}\chi_m(D_x)\mathcal{F}^{-1}\big(\bar{r}_{\sigma_m}\alpha\big)(x)\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big)(x)$. We can easily prove that $f' : L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ is a compact operator. The cutoff function $\chi_m\in L_0^\infty(\mathbb{R}^3)$ by hypothesis[13].

[13] We denote by $L_0^\infty(\mathbb{R}^3)$ the set of bounded functions on $\mathbb{R}^3$ that vanish at infinity.

Now both $\bar{r}_{\sigma_m}\alpha$ and $\xi_2\bar{r}_\infty$ belong to $L^1(\mathbb{R}^3)$, since $r_{\sigma_m}, \alpha, \omega^{1/2}\xi_2, \omega^{-1/2}r_\infty \in L^2(\mathbb{R}^3)$. Therefore $\mathcal{F}^{-1}\big(\bar{r}_{\sigma_m}\alpha\big)\mathrm{Im}\mathcal{F}^{-1}\big(\xi_2\bar{r}_\infty\big) \in L_0^\infty(\mathbb{R}^3)$, hence $f'(x, D_x) \in \mathcal{K}(L^2(\mathbb{R}^3))$. It immediately follows that $\tilde{b}_{\alpha\bar{u}u}$ is compact, and the proof is complete. As usual, this result implies the one for the adjoint term

$$\xi_2 a^*\psi^*\psi = -\tfrac{\sqrt{2}i}{M}\int_{\mathbb{R}^6}\mathrm{Im}\big(\xi_2(k')\bar{r}_\infty(k')e^{ik'\cdot x}\big)a^*\big(r_\infty e^{-ik\cdot x}\big)\psi^*(x)\psi(x)dxdk' \ .$$

Part 5 ($D_x a\bar{\xi}_1\psi$, $a^*D_x\psi^*\xi_1$, $D_x a\psi^*\xi_1$, $a^*D_x\bar{\xi}_1\psi$)**.**

$$D_x a\bar{\xi}_1\psi = \tfrac{1}{\sqrt{2M}}\int_{\mathbb{R}^3}\bar{\xi}_1(x)D_x a\big(r_\infty e^{-ik\cdot x}\big)\psi(x)dx \ .$$

The approximated symbol $D_x a\bar{\xi}_1\psi^{(m)}$ is given by

$$D_x a\bar{\xi}_1\psi^{(m)} = \tfrac{1}{\sqrt{2M}}\int_{\mathbb{R}^3}\bar{\xi}_1(x)D_x a\big(r_{\sigma_m} e^{-ik\cdot x}\big)\psi(x)dx \ .$$

First of all we prove that (80) is satisfied. Given $\Phi \in \mathcal{H}$, we denote by $\Phi_{n,p}$ its restriction to the subspace $\mathcal{H}_{n,p} = \Big(L^2(\mathbb{R}^3)\Big)^{\otimes_s n} \otimes \Big(L^2(\mathbb{R}^3)\Big)^{\otimes_s p}$ with n nucleons and p mesons. We also denote by $X_n = \{x_1, \dots, x_n\}$ a set of variables, $dX_n = dx_1\cdots dx_n$ the corresponding Lebesgue measure (and analogously for K_p, dK_p). The proof is obtained by a direct calculation on the Fock space as follows:

$$\begin{aligned}
&\Big|\Big\langle (H_0+1)^{-1/2}\Phi, \int_{\mathbb{R}^3}\bar{\xi}_1(x)D_x a\big((r_\infty - r_{\sigma_m})e^{-ik\cdot x}\big)\psi(x)dx(H_0+1)^{-1/2}\Psi\Big\rangle\Big| \\
&= \Big|\sum_{n,p=0}^{\infty}\varepsilon\sqrt{(n+1)(p+1)}\int_{\mathbb{R}^{(n+p+2)d}}\overline{\Big((H_0+1)^{-1/2}\Phi\Big)}_{n,p}(X_n;K_p)\bar{\xi}_1(x)D_x(\bar{r}_\infty - \bar{r}_{\sigma_m})(k)e^{ik\cdot x} \\
&\qquad\qquad \overline{\Big((H_0+1)^{-1/2}\Psi\Big)}_{n+1,p+1}(x, X_n; k, K_p)dxdX_ndkdK_p\Big| \\
&\leq \sum_{n,p=0}^{\infty}\sqrt{\varepsilon(n+1)}\Big|\int_{\mathbb{R}^{(n+p+2)d}}\overline{\Big((H_0+1)^{-1/2}\Phi\Big)}_{n,p}(X_n;K_p)\overline{D_x\xi_1}(x)\overline{\tfrac{r_\infty - r_{\sigma_m}}{\sqrt{\omega}}}(k)e^{ik\cdot x}\sqrt{\varepsilon(p+1)\omega(k)} \\
&\qquad\qquad \overline{\Big((H_0+1)^{-1/2}\Psi\Big)}_{n+1,p+1}(x, X_n; k, K_p)dxdX_ndkdK_p\Big| \\
&\leq \sum_{n,p=0}^{\infty}\sqrt{\varepsilon(n+1)}\big\|(-\Delta+V)^{1/2}\xi_1\big\|_2\cdot\big\|\omega^{-1/2}(r_\infty - r_{\sigma_m})\big\|_2\cdot\big\|(H_0+1)^{-1/2}\Phi_{n,p}\big\|_{\mathcal{H}_{n,p}} \\
&\qquad\qquad \cdot\big\|e^{ik\cdot x}\sqrt{\varepsilon(p+1)\omega(k_1)}(H_0+1)^{-1/2}\Psi_{n+1,p+1}(X_{n+1};K_{p+1})\big\|_{\mathcal{H}_{n+1,p+1}} \\
&\qquad\qquad \leq \big\|(-\Delta+V)^{1/2}\xi_1\big\|_2\cdot\big\|\omega^{-1/2}(r_\infty - r_{\sigma_m})\big\|_2\cdot\big\|(N_1+\bar{\varepsilon})^{1/2}\Phi\big\|\cdot\big\|\Psi\big\| \ ;
\end{aligned}$$

where in the last bound we have used Schwarz's inequality and the fact that $p\omega(k_1) \equiv \sum_{j=1}^{p}\omega(k_j)$ when acting on vectors of $\mathcal{H}_{n,p}$. Now, since $\lim_{m\to\infty}\big\|\omega^{-1/2}(r_\infty - r_{\sigma_m})\big\|_2 = 0$, Equation (80) holds with $C^{(m)}(\xi_1) = \frac{1}{\sqrt{2M}}\big\|(-\Delta+V)^{1/2}\xi_1\big\|_2\cdot\big\|\omega^{-1/2}(r_\infty - r_{\sigma_m})\big\|_2$. It remains to show that the classical symbol

$$D_x\alpha\bar{\xi}_1u^{(m)} = \tfrac{1}{\sqrt{2M}}\int_{\mathbb{R}^6}\bar{\xi}_1(x)D_x\alpha(k)\bar{r}_{\sigma_m}(k)e^{ik\cdot x}u(x)dxdk$$

is compact. Here we have written $D_x\alpha\bar{\xi}_1u^{(m)} = \langle\xi_1, D_xv\rangle_2$, with $v_x(x) = \frac{(2\pi)^{3/2}}{\sqrt{2M}}\mathcal{F}^{-1}\big(\alpha\bar{r}_{\sigma_m}\big)(x)u(x)$; and that is defined for any $v \in \dot{H}^1(\mathbb{R}^3)$. However, since $\xi_1 \in Q(-\Delta+V) \subset H^1(\mathbb{R}^3)$ and D_x is self-adjoint, we can write $D_x\alpha\bar{\xi}_1u^{(m)} = \langle D_x\xi_1, v\rangle_2$ for any $v \in L^2(\mathbb{R}^3)$. It follows that $D_x\alpha\bar{\xi}_1u^{(m)}$ is defined for any $u, \alpha \in L^2(\mathbb{R}^3)$, since $\alpha, r_{\sigma_m} \in L^2$ implies $\alpha\bar{r}_{\sigma_m} \in L^1$, and therefore $\mathcal{F}^{-1}\big(\alpha\bar{r}_{\sigma_m}\big) \in L^\infty$.

It follows that the operator $\tilde{b}_{D_x\alpha u} : \big(L^2 \oplus L^2\big)^{\otimes_s 2} \to \mathbb{C}$ defined as

$$\tilde{b}_{D_x\alpha u} : (u,\alpha)^{\otimes 2} \in \big(L^2 \oplus L^2\big)^{\otimes_s 2} \underset{\pi_2\otimes\pi_1}{\longrightarrow} \alpha(k)u(x) \in L^2(\mathbb{R}^6) \longrightarrow \langle f'', \alpha u\rangle_{L^2(\mathbb{R}^6)} \in \mathbb{C} \; ,$$

with $f''(x,k) = \frac{1}{\sqrt{2M}}(D_x\xi_1)(x) r_{\sigma_m}(k)e^{-ik\cdot x}$, is bounded and finite rank, and therefore compact.

$$a^* D_x \bar{\xi}_1 \psi = \frac{1}{\sqrt{2M}} \int_{\mathbb{R}^3} \bar{\xi}_1(x) a^*\big(r_\infty e^{-ik\cdot x}\big) D_x\psi(x) dx \; .$$

Again, the approximated symbol $a^* D_x \bar{\xi}_1 \psi$ is given by

$$a^* D_x \bar{\xi}_1 \psi^{(m)} = \frac{1}{\sqrt{2M}} \int_{\mathbb{R}^3} \bar{\xi}_1(x) a^*\big(r_{\sigma_m} e^{-ik\cdot x}\big) D_x\psi(x) dx \; .$$

Equation (80) is satisfied, and the proof follows the same guidelines as the one for the previous term $D_x a\bar{\xi}_1\psi$. We give the compactness proof for the symbol

$$\bar{\alpha} D_x \bar{\xi}_1 u^{(m)} = \frac{1}{\sqrt{2M}} \int_{\mathbb{R}^6} \bar{\xi}_1(x) \bar{\alpha}(k) r_{\sigma_m}(k) e^{-ik\cdot x} D_x u(x) dx dk \; .$$

We rewrite it as $\bar{\alpha} D_x \bar{\xi}_1 u^{(m)} = \langle (u,\alpha), \tilde{b}_{\bar{\alpha} D_x u}(u,\alpha)\rangle_{L^2\oplus L^2}$, with $\tilde{b}_{\bar{\alpha} D_x u} : L^2 \oplus L^2 \to L^2 \oplus L^2$ defined as

$$\tilde{b}_{\bar{\alpha} D_x u} : (u,\alpha) \in L^2 \oplus L^2 \underset{\pi_1}{\longrightarrow} u(x) \in L^2(\mathbb{R}^3) \underset{\tilde{c}_{\bar{\alpha} D_x u}}{\longrightarrow} \Big(0 \,\oplus\, f'''(k)\Big) \in L^2 \oplus L^2 \; ,$$

where $f'''(k) = \frac{1}{\sqrt{2M}} r_{\sigma_m}(k)\Big(k\langle e^{ik\cdot x}\xi_1, u\rangle_{L^2_x} + \langle e^{ik\cdot x} D_x\xi_1, u\rangle_{L^2_x}\Big)$. Now suppose that $u_j \rightharpoonup u$ is a weakly convergent (bounded) sequence with bound X. It follows that, uniformly in j,

$$\begin{aligned} |f_j'''(k)|^2 &= \Big|\tfrac{1}{\sqrt{2M}} r_{\sigma_m}(k)\Big(k\langle e^{ik\cdot x}\xi_1, u_j\rangle_{L^2_x} + \langle e^{ik\cdot x} D_x\xi_1, u_j\rangle_{L^2_x}\Big)\Big|^2 \\ &\leq \tfrac{1}{2M^2} X^2 |r_{\sigma_m}(k)|^2 (k^2+1) \|\xi_1\|^2_{H^1} \in L^1_k(\mathbb{R}^3) \; . \end{aligned}$$

In addition, $\lim_{j\to\infty} |f'''(k) - f_j'''(k)|^2 = 0$; therefore $\tilde{c}_{\bar{\alpha} D_x u}$ is a compact operator by Lebesgue's dominated convergence theorem. So $\tilde{b}_{\bar{\alpha} D_x u}$ is compact. The proofs above extend immediately to the adjoint terms

$$\begin{aligned} a^* D_x \psi^* \xi_1 &= \frac{1}{\sqrt{2M}} \int_{\mathbb{R}^3} \psi^*(x) a^*\big(r_\infty e^{-ik\cdot x}\big) D_x \xi_1(x) dx \; ; \\ D_x a \psi^* \xi_1 &= \frac{1}{\sqrt{2M}} \int_{\mathbb{R}^3} \psi^*(x) D_x a\big(r_\infty e^{-ik\cdot x}\big) \xi_1(x) dx \; . \end{aligned}$$

Part 6 ($\psi^* D_x \xi_2 \psi$, $\psi^* \bar{\xi}_2 D_x \psi$)**.**

$$\psi^* D_x \xi_2 \psi = \frac{(2\pi)^{3/2}}{\sqrt{2M}} \int_{\mathbb{R}^3} \psi^*(x) D_x \mathcal{F}^{-1}\Big(\xi_2 \bar{r}_\infty\Big)(x)\psi(x) dx \; .$$

The approximated symbol, as for the terms of point 4, contains $\chi_m(D_x)$:

$$\psi^* D_x \xi_2 \psi^{(m)} = \frac{(2\pi)^{3/2}}{\sqrt{2M}} \int_{\mathbb{R}^3} \psi^*(x) \chi_m(D_x) D_x \mathcal{F}^{-1}\Big(\xi_2 \bar{r}_\infty\Big)(x)\psi(x) dx \; .$$

As usual, we start proving that (80) holds. We remark that this is the only term where we need $\xi_2 \in D(\omega^{3/4})$ instead of $D(\omega^{1/2})$.

$$\begin{aligned}
&\Big|\Big\langle (H_0+1)^{-1/2}\Phi, \int_{\mathbb{R}^3} \psi^*(x)\big(1-\chi_m(D_x)\big) D_x \mathcal{F}^{-1}\Big(\xi_2 \bar{r}_\infty\Big)(x)\psi(x)dx (H_0+1)^{-1/2}\Psi\Big\rangle\Big| \\
&\quad\leq \sum_{n=0}^{\infty} n\varepsilon \Big|\Big\langle (H_0+1)^{-1/2}\Phi_n, \big(1-\chi_m(D_{x_1})\big) D_{x_1}\mathcal{F}^{-1}\Big(\xi_2\bar{r}_\infty\Big)(x_1)(H_0+1)^{-1/2}\Psi_n\Big\rangle\Big| \\
&\leq \sum_{n=0}^{\infty} n\varepsilon \big\|(1-\Delta)^{-1/2}\big(1-\chi_m(D_x)\big)\big\|_{\mathcal{L}(L^2(\mathbb{R}^3))}\cdot\Big(\big\|\mathcal{F}^{-1}\Big(\xi_2\bar{r}_\infty\Big)\big\|_\infty + \big\|\mathcal{F}^{-1}\Big(k\xi_2\bar{r}_\infty\Big)\big\|_\infty\Big) \\
&\cdot \big\|(1-\Delta_{x_1})^{1/2}(H_0+1)^{-1/2}\Phi_n\big\|_{\mathcal{H}_n}\cdot\Big(\big\|D_{x_1}(H_0+1)^{-1/2}\Psi_n\big\|_{\mathcal{H}_n} + \big\|(H_0+1)^{-1/2}\Psi_n\big\|_{\mathcal{H}_n}\Big) \\
&\qquad\leq 2\big\|(1-D_x^2)^{-1/2}\big(1-\chi_m(D_x)\big)\big\|_{\mathcal{L}(L^2(\mathbb{R}^3))}\cdot\big\|\omega^{3/4}\xi_2\big\|_2\cdot\big\|\omega^{-1/4}r_\infty\big\|_2\cdot\big\|\Phi\big\|\cdot \\
&\qquad\qquad\Big(\big\|(N_1+\bar{\varepsilon})^{1/2}\Psi\big\| + \big\|\Psi\big\|\Big)\ ;
\end{aligned}$$

hence the result follows with

$$C^{(m)}(\xi_2) = \tfrac{2\sqrt{2}(2\pi)^{3/2}}{M}\big\|(1-D_x^2)^{-1/2}\big(1-\chi_m(D_x)\big)\big\|_{\mathcal{L}(L^2(\mathbb{R}^3))}\big\|\omega^{3/4}\xi_2\big\|_2\big\|\omega^{-1/4}r_\infty\big\|_2\ ,$$

since $\lim_{m\to\infty}\big\|(1-D_x^2)^{-1/2}\big(1-\chi_m(D_x)\big)\big\|_{\mathcal{L}(L^2(\mathbb{R}^3))} = 0$. It remains to show that the symbol

$$\bar{u}D_x\xi_2 u^{(m)} = \tfrac{(2\pi)^{3/2}}{\sqrt{2}M}\int_{\mathbb{R}^3}\bar{u}(x)\chi_m(D_x)D_x\mathcal{F}^{-1}\Big(\xi_2\bar{r}_\infty\Big)(x)u(x)dx$$

is compact. We introduce the operator $\tilde{b}_{\bar{u}D_x u} : L^2\oplus L^2 \to L^2\oplus L^2$ such that $\bar{u}D_x\xi_2 u^{(m)} = \langle (u,\alpha), \tilde{b}_{\bar{u}D_x u}(u,\alpha)\rangle_{L^2\oplus L^2}$:

$$\tilde{b}_{\bar{u}D_x u} : (u,\alpha)\in L^2\oplus L^2 \underset{\pi_1}{\longrightarrow} u(x)\in L^2(\mathbb{R}^3) \underset{\tilde{c}_{\bar{u}D_x u}}{\longrightarrow} \Big(f''''(x,D_x)u(x)\,\oplus\,0\Big)\in L^2\oplus L^2\ ,$$

where $f''''(x,D_x) = \frac{(2\pi)^{3/2}}{\sqrt{2}M}D_x\,\chi_m(D_x)\mathcal{F}^{-1}\Big(\xi_2\bar{r}_\infty\Big)(x)$. Now $f''''(x,D_x)$ is a compact operator: both $x\,\chi_m(x)$ and $\mathcal{F}^{-1}\Big(\xi_2\bar{r}_\infty\Big)(x)$ are in $L_0^\infty(\mathbb{R}^3)$. Therefore $\tilde{b}_{\bar{u}D_x u}$ is compact. The proof extends immediately to the adjoint term

$$\psi^*\bar{\xi}_2 D_x\psi = \tfrac{(2\pi)^{3/2}}{\sqrt{2}M}\int_{\mathbb{R}^3}\psi^*(x)\mathcal{F}\Big(\bar{\xi}_2 r_\infty\Big)(x)D_x\psi(x)dx\ .$$

$\dashv$

4.4. **Defining the time-dependent family of Wigner measures.** The last tool we need in order to take the limit $\varepsilon\to 0$ of the integral formula (69) are Wigner measures. Throughout this section, we will leave some statements unproven; the reader may refer to [8, Section 6] for the proofs, and a detailed discussion of Wigner measures properties. We recall the definition of a Wigner measure associated with a family of states on $\mathcal{H} = \Gamma_s\big(L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\big)$.

Definition 4.11. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}\subset\mathcal{L}^1\big(\mathcal{H}\big)$ be a family of normal states; $\mu\in\mathfrak{P}\big(L^2\oplus L^2\big)$ a Borel probability measure. We say that μ is a Wigner (or semiclassical) measure associated to $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$, or in symbols $\mu\in\mathcal{M}\big(\varrho_\varepsilon,\varepsilon\in(0,\bar{\varepsilon})\big)$, if there exist a sequence $(\varepsilon_k)_{k\in\mathbb{N}}\subset(0,\bar{\varepsilon})$ such that $\lim_{k\to\infty}\varepsilon_k = 0$ and*

$$\lim_{k\to\infty}\mathrm{Tr}\Big[\varrho_{\varepsilon_k}W(\xi)\Big] = \int_{L^2\oplus L^2} e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle_{L^2\oplus L^2}}d\mu(z)\ ,\ \forall\xi\in L^2\oplus L^2\ . \tag{81}$$

We remark that the right-hand side is essentially the Fourier transform of the measure μ, so considering the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ there is at most one probability measure that could satisfy (81). If (81) is satisfied, we say that to the sequence $(\varrho_{\varepsilon_k})_{k\in\mathbb{N}}$ corresponds a single Wigner (or semiclassical) measure μ, or simply $\varrho_{\varepsilon_k} \to \mu$.

First of all, it is necessary to ensure that such a definition of Wigner measures is meaningful, i.e. that under suitable conditions the set of Wigner measures $\mathcal{M}$ associated to a family of states is not empty. Since $m_0 > 0$, it turns out that Assumption (A'_ρ) is sufficient. Assumption (A_ρ) would be sufficient as well, even if we will not use it for the moment.

Lemma 4.12. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\mathcal{H}$, that satisfies Assumptions (A'_ρ) and (A_0). Then for any $t \in \mathbb{R}$:*

(i) $\mathcal{M}\big(\varrho_\varepsilon(t), \varepsilon \in (0,\bar\varepsilon)\big) \neq \emptyset$; $\mathcal{M}\big(\tilde\varrho_\varepsilon(t), \varepsilon \in (0,\bar\varepsilon)\big) \neq \emptyset$.

(ii) Any $\mu \in \mathcal{M}\big(\varrho_\varepsilon(t), \varepsilon \in (0,\bar\varepsilon)\big)$ *or in* $\mathcal{M}\big(\tilde\varrho_\varepsilon(t), \varepsilon \in (0,\bar\varepsilon)\big)$[14] *satisfies:*

$$\mu\Big(B_u(0,\sqrt{\mathfrak{C}}) \cap Q(-\Delta+V) \oplus D(\omega^{1/2})\Big) = 1$$

(iii) Moreover

$$\int_{z=(u,\alpha)\in L^2\oplus L^2} \|(-\Delta+V)^{1/2}u\|_2^2 + \|\alpha\|^2_{\mathcal{F}H^{1/2}}\, d\mu(z) < +\infty \ .$$

We recall that $B_u(0,\sqrt{\mathfrak{C}}) = \Big\{(u,\alpha) \in L^2 \oplus L^2, \|u\|_2 \leq \sqrt{\mathfrak{C}}\Big\}$.

Proof. By (78) of Lemma 4.8, we see that $\varrho_\varepsilon(t)$ and $\tilde\varrho_\varepsilon(t)$ satisfy (A_0) and (A'_ρ) at any time. Now (i) follows by [8, Theorem 6.2] and (ii) by (iii) and [10, Lemma 2.14]. The third point is essentially a consequence of [11, Lemma 3.12]. However the latter result requires more regularity on the states ϱ_ε. So we indicate here how to adapt the argument to our case. It is enough to assume $t = 0$ and $\{\mu\} = \mathcal{M}\big(\varrho_\varepsilon, \varepsilon \in (0,\bar\varepsilon)\big)$. The operators $-\frac{\Delta}{2M} + V$ and ω are positive (self-adjoint). So one can find non-decreasing sequences of finite rank operators A_k and B_k that converge weakly to $-\frac{\Delta}{2M} + V$ and ω respectively. In particular

$$b_k^{Wick} = d\Gamma(A_k)\otimes 1 + 1\otimes d\Gamma(B_k) \leq d\Gamma(-\frac{\Delta}{2M}+V)\otimes 1 + 1\otimes d\Gamma(\omega) = H_0\,,$$

where $b_k(u,\alpha) = \langle u, A_k u\rangle + \langle \alpha, B_k\alpha\rangle \in \mathcal{P}^\infty_{1,1}(L^2\oplus L^2)$. Let P_k and Q_k be the orthogonal projections on $\mathrm{Ran}(A_k)$ and $\mathrm{Ran}(B_k)$ respectively. Using the Fock space decomposition $\Gamma_s(L^2\oplus L^2) \equiv \Gamma_s(P_kL^2 \oplus Q_kL^2)\otimes\Gamma_s(P_k^\perp L^2 \oplus Q_k^\perp L^2)$ where $P_k^\perp = 1 - P_k$ and $Q_k^\perp = 1 - Q_k$; one can write $b_k^{Wick} \equiv (b_k)^{Wick}_{|\Gamma_s(P_kL^2\oplus Q_kL^2)} \otimes 1_{\Gamma_s(P_k^\perp L^2\oplus Q_k^\perp L^2)}$ and $\varrho_\varepsilon \equiv \hat\varrho_\varepsilon$. Hence

$$\mathrm{Tr}\Big[\varrho_\varepsilon b_k^{Wick}\Big] = \mathrm{Tr}\Big[\hat\varrho_\varepsilon b^{Wick}_{k|\Gamma_s(P_kL^2\oplus Q_kL^2)} \otimes 1_{\Gamma_s(P_k^\perp L^2\oplus Q_k^\perp L^2)}\Big] = \mathrm{Tr}_{\Gamma_s(P_kL^2\oplus Q_kL^2)}\Big[\varrho^k_\varepsilon b_k^{Wick}\Big]\,,$$

where $\varrho^k_{\varepsilon_j}$ is a given reduced density matrix which is trace-class in $\Gamma_s(P_kL^2\oplus Q_kL^2)$. So the problem is in some sense reduced to finite dimension. Now using Wick calculus (in finite dimension) b_k^{Wick} can be written as an Anti-Wick operator by moving all the a^* to the right of a. So, one obtains $b_k^{Wick} = b_k^{A-Wick} + \varepsilon T$ with $T(d\Gamma(P_k\oplus Q_k)+1)^{-1}$ is bounded uniformly with respect to $\varepsilon \in (0,\bar\varepsilon)$.

[14]In this section, we have used mostly the notation $D(\omega^{1/2})$; however $D(\omega^{1/2}) = \mathcal{F}H^{1/2}$, where the latter is defined in Definition 3.4.

Hence

$$\overline{\lim_{\varepsilon\to 0}}\,\mathrm{Tr}_{\Gamma_s(P_kL^2\oplus Q_kL^2)}\Big[\varrho_\varepsilon^k b_k^{A-Wick}\Big] = \overline{\lim_{\varepsilon\to 0}}\,\mathrm{Tr}_{\Gamma_s(P_kL^2\oplus Q_kL^2)}\Big[\varrho_\varepsilon^k b_k^{Wick}\Big] \leq \overline{\lim_{\varepsilon\to 0}}\,\mathrm{Tr}\Big[\varrho_\varepsilon H_0\Big] \leq C\,.$$

For details on the Anti-Wick quantization we refer the reader to [8]; in particular it is a positive quantization (see e.g. [8, Proposition 3.6]). Hence, we see that

$$\mathrm{Tr}_{\Gamma_s(P_kL^2\oplus Q_kL^2)}\Big[\varrho_\varepsilon^k (b_{k,\chi})^{A-Wick}\Big] \leq \mathrm{Tr}_{\Gamma_s(P_kL^2\oplus Q_kL^2)}\Big[\varrho_\varepsilon^k b_k^{A-Wick}\Big]$$

where $b_{k,\chi}(u,\alpha) = \chi(u)\langle u, A_k u\rangle + \chi(\alpha)\langle \alpha, B_k\alpha\rangle$ for any cutoff function $\chi \in C_0^\infty(\mathbb{R}^3)$, $0\leq\chi\leq 1$. Finally [8, Theorem 6.2] gives

$$\int_{z=(u,\alpha)\in L^2\oplus L^2} b_{k,\chi}(u,\alpha)\,d\mu(z) = \lim_{\varepsilon\to 0}\mathrm{Tr}\Big[\varrho_\varepsilon (b_{k,\chi})^{A-Wick}\Big] = \lim_{\varepsilon\to 0}\mathrm{Tr}_{\Gamma_s(P_kL^2\oplus Q_kL^2)}\Big[\varrho_\varepsilon^k b_k^{A-Wick}\Big] \leq C\,,$$

and the monotone convergence theorem proves (iii). ⊣

As we said above, our aim is to take the limit $\varepsilon_k\to 0$ on the integral equation (69), for a suitable sequence contained in $(0,\bar\varepsilon)$. We may suppose that the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ is chosen in such a way that there exist $\mu_0\in\mathcal{M}\big(\varrho_\varepsilon,\varepsilon\in(0,\bar\varepsilon)\big)$ such that (81) holds, i.e. $\mathcal{M}\big(\varrho_{\varepsilon_k}, k\in\mathbb{N}\big) = \{\mu_0\}$. However, nothing a priori ensures that the sequence, or one of its subsequences $(\varepsilon_{k_i})_{i\in\mathbb{N}}\subset(\varepsilon_k)_{k\in\mathbb{N}}$, is such that *for any* $t\in\mathbb{R}$*:*

$$\lim_{i\to\infty}\mathrm{Tr}\Big[\tilde\varrho_{\varepsilon_{k_i}}(t)W(\xi)\Big] = \int_{L^2\oplus L^2} e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}d\tilde\mu_t(z)\ ,\ \forall\xi\in L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3)\ ;$$

where $\tilde\mu_t:\mathbb{R}\to\mathfrak{P}\big(L^2\oplus L^2\big)$ is a map such that $\tilde\mu_0=\mu_0$. The possibility of extracting such a common subsequence is crucial, since the integral equation involves all measures from zero to an arbitrary time t. To prove it is possible, we exploit the uniform continuity properties of $\mathrm{Tr}\Big[\tilde\varrho_\varepsilon(t)W(\xi)\Big]$ in both t and ξ, proved in the following lemma.

Lemma 4.13. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of quantum states on $\mathcal{H}$ that satisfies Assumptions (A_0) and (A'_ρ). Then the family of functions $(t,\xi)\mapsto\tilde G_\varepsilon(t,\xi) := \mathrm{Tr}\Big[\tilde\varrho_\varepsilon(t)W(\xi)\Big]$ is uniformly equicontinuous on bounded subsets of $\mathbb{R}\times\big(Q(-\Delta+V)\oplus D(\omega^{1/2})\big)$.*

Proof. Let $(t,\xi),(s,\eta)\in\mathbb{R}\times\big(Q(-\Delta+V)\oplus D(\omega^{1/2})\big)$. Without loss of generality, we may suppose that $s\leq t$. We write

$$\Big|\tilde G_\varepsilon(t,\xi)-\tilde G_\varepsilon(s,\eta)\Big| \leq \Big|\tilde G_\varepsilon(t,\eta)-\tilde G_\varepsilon(s,\eta)\Big| + \Big|\tilde G_\varepsilon(t,\xi)-\tilde G_\varepsilon(t,\eta)\Big|\ ;$$

and define $X_1 := \Big|\tilde G_\varepsilon(t,\eta)-\tilde G_\varepsilon(s,\eta)\Big|$, $X_2 := \Big|\tilde G_\varepsilon(t,\xi)-\tilde G_\varepsilon(t,\eta)\Big|$. Consider X_1; we get by standard manipulations and Lemma 4.2:

$$X_1 \leq \sum_{j=0}^3\varepsilon^j\sum_{i\in\mathbb{N}}\lambda_i\int_s^t\Big|\big\langle e^{-i\frac{s}{\varepsilon}\hat H_{\mathrm{ren}}}\Psi_i, W\big(\tilde{(\eta)}_s\big)B_j\big(\tilde{(\eta)}_s\big)e^{-i\frac{s}{\varepsilon}\hat H_{\mathrm{ren}}}\Psi_i\big\rangle\Big|ds\ .$$

Now using Lemma 4.6 we obtain

$$\begin{aligned}X_1 \leq \sum_{j=0}^3\varepsilon^j C_j(\eta)\sum_{i\in\mathbb{N}}\lambda_i\int_s^t&\Big\|(N_1+H_0+\bar\varepsilon)^{1/2}W^*\big(\tilde{(\eta)}_s\big)e^{-i\frac{s}{\varepsilon}\hat H_{\mathrm{ren}}}\Psi_i\Big\|\\ &\cdot\Big\|(N_1+H_0+\bar\varepsilon)^{1/2}e^{-i\frac{s}{\varepsilon}\hat H_{\mathrm{ren}}}\Psi_i\Big\|ds\ ;\end{aligned}$$

then using Lemma 4.1, and the fact that $\big\|(\tilde{\eta_1})_s\big\|_{H^1} = \big\|\eta_1\big\|_{H^1}$, $\big\|(\tilde{\eta_2})_s\big\|_{\mathcal{F}H^{1/2}} = \big\|\eta_2\big\|_{\mathcal{F}H^{1/2}}$ we get

$$\begin{aligned} X_1 &\leq C(\eta)\sum_{j=0}^{3}\varepsilon^j C_j(\eta)\int_s^t \mathrm{Tr}\Big[\varrho_\varepsilon(s)(N_1+H_0+\bar{\varepsilon})\Big]ds \\ &\leq |t-s| C(\eta)\sum_{j=0}^{3}\bar{\varepsilon}^j C_j(\eta)\Big(\tfrac{C}{1-a(\mathfrak{C})}+\tfrac{2b(\mathfrak{C})}{1-a(\mathfrak{C})}+\bar{\varepsilon}\Big)\ ; \end{aligned}$$

where in the last inequality we used Equation (78) of Lemma 4.8. Now let's consider X_2; a standard manipulation using Weyl's relation yields

$$X_2 \leq \Big\|\big(e^{i\frac{\varepsilon}{2}\mathrm{Im}\langle\xi,\eta\rangle_{L^2\oplus L^2}}W(\xi-\eta)-1\big)(N_1+N_2+1)^{-1}\Big\|_{\mathcal{L}\left(\Gamma_s(L^2\oplus L^2)\right)}\mathrm{Tr}\Big[\tilde{\varrho}_\varepsilon(t)(N_1+N_2+1)\Big]\ .$$

Now we use the estimate in [8, Lemma 3.1] and obtain

$$\begin{aligned} X_2 &\leq \|\xi-\eta\|_{L^2\oplus L^2}\Big(\bar{\varepsilon}\|\eta\|_{L^2\oplus L^2}+1\Big)\mathrm{Tr}\Big[\tilde{\varrho}_\varepsilon(t)(N_1+N_2+1)\Big] \\ &\leq \|\xi-\eta\|_{L^2\oplus L^2}\Big(\bar{\varepsilon}\|\eta\|_{L^2\oplus L^2}+1\Big)\Big(\tfrac{C}{1-a(\mathfrak{C})}+\tfrac{2b(\mathfrak{C})}{1-a(\mathfrak{C})}+1\Big)\ , \end{aligned}$$

where in the last inequality we used again Equation (78) of Lemma 4.8, keeping in mind that $N_2 \leq d\Gamma(\omega) \leq H_0$. ⊣

Now using Lemma 4.13 with the estimates on X_1, X_2 above and a diagonal extraction argument, we prove the following proposition. We omit the proof since it is similar to [11, Proposition 3.9].

Proposition 4.14. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of quantum states on $\mathcal{H}$ that satisfies Assumptions (A_0) and (A'_ρ). Then for any sequence $(\varepsilon_k)_{k\in\mathbb{N}}\subset(0,\bar{\varepsilon})$ with $\lim_{k\to\infty}\varepsilon_k=0$, there exists a subsequence $(\varepsilon_{k_i})_{i\in\mathbb{N}}$ such that there exists a map $\mu_t:\mathbb{R}\to\mathfrak{P}\big(L^2\oplus L^2\big)$ verifying the following statements:*

$$\varrho_{\varepsilon_{k_i}}(t)\to\mu_t\ ,\ \forall t\in\mathbb{R}\ ; \tag{82}$$

$$\tilde{\varrho}_{\varepsilon_{k_i}}(t)\to\tilde{\mu}_t\ ,\ \forall t\in\mathbb{R}\ ,\ \textit{with}\ \tilde{\mu}_t=\mathbf{E}_0(-t)_{\#}\mu_t\ ; \tag{83}$$

$$\varrho_{\varepsilon_{k_i}}(t)W(\tilde{\xi}_t)\to\mu_{\xi,t}\ ,\ \forall t\in\mathbb{R}\ \textit{and}\ \forall\xi\in L^2\oplus L^2\ ,\ \textit{with}\ d\mu_{\xi,t}(z)=e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi}_t,z\rangle}d\mu_t(z)\ ; \tag{84}$$

where $\mathbf{E}_0(t)z=e^{-it(-\Delta+V)}u\oplus e^{-it\omega}\alpha$ is the Hamiltonian flow associated with the free classical energy $\mathscr{E}_0$, and $\tilde{\xi}_t=\mathbf{E}_0(-t)\xi$. Moreover, μ_t and $\tilde{\mu}_t$ are both Borel probability measures on $Q(-\Delta+V)\oplus D(\omega^{1/2})$.

4.5. **The classical limit of the integral formula.** We are finally ready to discuss the limit $\varepsilon\to 0$ of the integral formula (69). As a final preparation, we state a couple of preliminary lemmas. The first is a slight improvement of [8, Theorem 6.13]. The second can be easily proved by standard estimates on the symbol $\mathscr{B}_0^{(m)}(\xi)$ which we recall for convenience:

(85)
$$\begin{aligned}\mathscr{B}_0^{(m)}(\xi)(u,\alpha) = 2i\sqrt{2}\Big\langle \mathrm{Re}\mathcal{F}\big(\tfrac{\chi_{\sigma_0}\bar{\alpha}}{\sqrt{2\omega}}\big)(x), \mathrm{Im}\big(\bar{\xi}_1 u\big)(x)\Big\rangle_2 + i\sqrt{2}\Big\langle u(x), \chi_m(D_x)\mathrm{Im}\Big(\mathcal{F}\big(\tfrac{\chi_{\sigma_0}\bar{\xi}_2}{\sqrt{2\omega}}\big)\Big)(x)u(x)\Big\rangle_2 \\ + i\sqrt{2}\mathrm{Im}\Big\langle u(x), \big(\chi_m(D_{(\cdot)})V_\infty * \bar{\xi}_1 u\big)(x)u(x)\Big\rangle_2 + \tfrac{i(2\pi)^{3/2}}{2M}\mathrm{Im}\Big\langle \xi_1(x), \Big(\mathcal{F}^{-1}(\bar{r}_{\sigma_m}\alpha)^2 + \mathcal{F}(r_{\sigma_m}\bar{\alpha})^2 \\ + \mathcal{F}^{-1}(\bar{r}_{\sigma_m}\alpha)\mathcal{F}(r_{\sigma_m}\bar{\alpha})\Big)(x)u(x)\Big\rangle_2 \\ - \tfrac{2\sqrt{2}(2\pi)^3}{M}\mathrm{Im}\Big\langle u(x), \chi_m(D_x)\mathrm{Im}\Big(\mathcal{F}^{-1}(\bar{r}_\infty\xi_2)\Big)(x)\mathcal{F}^{-1}(\bar{r}_{\sigma_m}\alpha)(x)u(x)\Big\rangle_2 \\ - \tfrac{i\sqrt{2}(2\pi)^{3/2}}{M}\mathrm{Im}\Big\langle \xi_1(x), D_x\mathcal{F}^{-1}(\bar{r}_{\sigma_m}\alpha)(x)u(x)\Big\rangle_2 - \tfrac{i\sqrt{2}(2\pi)^{3/2}}{M}\mathrm{Im}\Big\langle \xi_1(x), \mathcal{F}(r_{\sigma_m}\bar{\alpha})(x)D_x u(x)\Big\rangle_2 \\ + \tfrac{i\sqrt{2}(2\pi)^{3/2}}{M}\mathrm{Im}\Big\langle u(x), \chi_m(D_x)D_x\mathcal{F}^{-1}(\bar{r}_\infty\xi_2)(x)u(x)\Big\rangle_2 .\end{aligned}$$

Lemma 4.15. *Let $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,\bar{\varepsilon})$, $\lim_{j\to\infty}\varepsilon_j = 0$, and $\delta > 0$. Furthermore, let $(\varrho_{\varepsilon_j})_{j\in\mathbb{N}}$ be a sequence of normal states in $\mathcal{H}$ such that for some $C(\delta) > 0$,*

$$\Big\|(N_1+N_2)^{\delta/2}\varrho_{\varepsilon_j}(N_1+N_2)^{\delta/2}\Big\|_{\mathcal{L}^1(L^2\oplus L^2)} \leq C(\delta)\,, \tag{86}$$

uniformly in $\varepsilon \in (0,\bar{\varepsilon})$. Suppose that $\varrho_{\varepsilon_j} \to \mu \in \mathfrak{P}(L^2 \oplus L^2)$ then the following statement is true:

$$\Big(\forall \mathscr{A} \in \bigoplus_{\substack{(p,q)\in\mathbb{N}^2\\ p+q<2\delta}} \mathcal{P}^\infty_{p,q}\big(L^2\oplus L^2\big)\ ,\ \lim_{j\to\infty}\mathrm{Tr}\Big[\varrho_{\varepsilon_j}(\mathscr{A})^{Wick}\Big] = \int_{L^2\oplus L^2}\mathscr{A}(z)d\mu(z)\Big)\,.$$

Proof. By linearity it is enough to assume $\mathscr{A} \in \mathcal{P}^\infty_{p,q}\big(L^2\oplus L^2\big)$ for $(p,q)\in\mathbb{N}^2$ with $p+q<2\delta$. Let $(P_R)_{R>0}$ be an increasing family of finite rank orthogonal projections on L^2 such that the strong limit $s-\lim_{R\to+\infty}P_R = 1$ holds. Let $\mathscr{A}_R(z) := \mathscr{A}(P_R\oplus P_R z)$ for any $z\in L^2\oplus L^2$. One writes

$$\Big|\mathrm{Tr}\Big[\varrho_{\varepsilon_j}(\mathscr{A})^{Wick}\Big] - \int_{L^2\oplus L^2}\mathscr{A}(z)d\mu(z)\Big| \leq \Big|\mathrm{Tr}\Big[\varrho_{\varepsilon_j}(\mathscr{A})^{Wick}\Big] - \mathrm{Tr}\Big[\varrho_{\varepsilon_j}(\mathscr{A}_R)^{Wick}\Big]\Big| \tag{87}$$

$$+ \Big|\mathrm{Tr}\Big[\varrho_{\varepsilon_j}(\mathscr{A}_R)^{Wick}\Big] - \int_{L^2\oplus L^2}\mathscr{A}_R(z)d\mu(z)\Big| \tag{88}$$

$$+ \Big|\int_{L^2\oplus L^2}\mathscr{A}_R(z)d\mu(z) - \int_{L^2\oplus L^2}\mathscr{A}(z)d\mu(z)\Big|\,. \tag{89}$$

Using standard number estimates and the regularity of the states $(\varrho_{\varepsilon_j})_j$, one shows

$$\Big|\mathrm{Tr}\Big[\varrho_{\varepsilon_j}(\mathscr{A}-\mathscr{A}_R)^{Wick}\Big]\Big| \leq ||(N_1+N_2)^{\delta/2}\varrho_{\varepsilon_j}(N_1+N_2)^{\delta/2}||_{\mathcal{L}^1(L^2\oplus L^2)}\ \ ||\tilde{\mathscr{A}} - \tilde{\mathscr{A}}_R||\,,$$

where $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{A}}_R$ denote the compact operators satisfying $\mathscr{A}(z) = \langle z^{\otimes q}, \tilde{\mathscr{A}} z^{\otimes p}\rangle$ and $\mathscr{A}_R(z) = \langle z^{\otimes q}, \tilde{\mathscr{A}}_R z^{\otimes p}\rangle$ respectively. Since $\tilde{\mathscr{A}}_R = (P_R\oplus P_R)^{\otimes q}\tilde{\mathscr{A}}(P_R\oplus P_R)^{\otimes p}$ and $\tilde{\mathscr{A}}$ is compact, one shows that $\lim_{R\to+\infty}||\tilde{\mathscr{A}} - \tilde{\mathscr{A}}_R|| = 0$. So the right hand side of (87) can be made arbitrary small by choosing R large enough. According to [8, Theorem 6.2], the regularity of $(\varrho_{\varepsilon_j})_j$ insures the bound

$$\int_{L^2\oplus L^2}||z||^{2\delta}_{L^2\oplus L^2}\,d\mu(z) \leq C(\delta)\,.$$

Hence by dominated convergence the right hand side of (89) can also be made arbitrary small when R is large enough since $\mathscr{A}(z)$ and $\mathscr{A}_R(z)$ are both bounded by $c||z||^{p+q}_{L^2\oplus L^2}$ and $\mathscr{A}_R(z)$ converges pointwise to $\mathscr{A}(z)$.

To handle the right hand side of (88), we use a further regularization. Let $\chi \in C_0^\infty(\mathbb{R})$, $0 \leq \chi \leq 1$, $\chi(x) = 1$ in a neighborhood of 0 and $\chi_m(x) = \chi(\frac{x}{m})$ for $m > 0$. Recall that the Fock space has the decomposition $\Gamma_s(L^2 \oplus L^2) \equiv \Gamma_s(P_R L^2 \oplus P_R L^2) \otimes \Gamma_s(P_R^\perp L^2 \oplus P_R^\perp L^2)$ where $P_R^\perp = 1 - P_R$. In this representation $\mathscr{A}_R^{Wick} \equiv (\mathscr{A}_R)^{Wick}_{|\Gamma_s(P_R L^2 \oplus P_R L^2)} \otimes 1_{\Gamma_s(P_R^\perp L^2 \oplus P_R^\perp L^2)}$ and $\varrho_{\varepsilon_j} \equiv \hat{\varrho}_{\varepsilon_j}$. Hence using reduced density matrices $\varrho_{\varepsilon_j}^R$ that are normalized positive trace-class operators in $\Gamma_s(P_R L^2 \oplus P_R L^2)$, one writes

$$\mathrm{Tr}\Big[\varrho_{\varepsilon_j}(\mathscr{A}_R)^{Wick}\Big] = \mathrm{Tr}\Big[\hat{\varrho}_{\varepsilon_j}(\mathscr{A}_R)^{Wick}_{|\Gamma_s(P_R L^2 \oplus P_R L^2)} \otimes 1_{\Gamma_s(P_R^\perp L^2 \oplus P_R^\perp L^2)}\Big] = \mathrm{Tr}_{\Gamma_s(P_R L^2 \oplus P_R L^2)}\Big[\varrho_{\varepsilon_j}^R(\mathscr{A}_R)^{Wick}\Big].$$

As in the proof of Lemma 4.12, the Wick calculus gives that $(\mathscr{A}_R)^{Wick}$ can be written as an Anti-Wick operator by moving all the a^* to the right of a. So, one obtains $(\mathscr{A}_R)^{Wick} = (\mathscr{A}_R)^{A-Wick} + \varepsilon T$ with $T(d\Gamma(P_R \oplus P_R) + 1)^{-\frac{p+q}{2}}$ is bounded uniformly with respect to $\varepsilon \in (0, \bar{\varepsilon})$. We refer the reader to [8] where Weyl and Anti-Wick quantization are explained for "cylindrical" symbols. Hence

$$\lim_{j\to\infty} \mathrm{Tr}\Big[\varrho_{\varepsilon_j}(\mathscr{A}_R)^{Wick}\Big] = \lim_{j\to\infty} \mathrm{Tr}_{\Gamma_s(P_R L^2 \oplus P_R L^2)}\Big[\varrho_{\varepsilon_j}^R(\mathscr{A}_R)^{Wick}\Big] = \lim_{j\to\infty} \mathrm{Tr}_{\Gamma_s(P_R L^2 \oplus P_R L^2)}\Big[\varrho_{\varepsilon_j}^R(\mathscr{A}_R)^{A-Wick}\Big].$$

Now we define $\chi_{m,R}(z) := \chi_m(|P_R \oplus P_R z|^2)$ and $\varrho_{\varepsilon_j}^{R,m} := \chi_{m,R}(z)^{Weyl}\, \varrho_{\varepsilon_j}^R\, \chi_{m,R}(z)^{Weyl}$. So one writes

$$\left|\mathrm{Tr}\Big[\varrho_{\varepsilon_j}^R(\mathscr{A})^{A-Wick}\Big] - \int_{L^2 \oplus L^2} \mathscr{A}(z) d\mu(z)\right| \leq \left|\mathrm{Tr}\Big[(\varrho_{\varepsilon_j}^R - \varrho_{\varepsilon_j}^{R,m})(\mathscr{A})^{A-Wick}\Big]\right| \tag{90}$$

$$+ \left|\mathrm{Tr}\Big[\varrho_{\varepsilon_j}^{R,m}(\mathscr{A}_R)^{A-Wick}\Big] - \int \chi_{m,R}^2(z)\mathscr{A}_R(z) d\mu(z)\right| \tag{91}$$

$$+ \left|\int \chi_{m,R}^2(z)\mathscr{A}_R(z) d\mu(z) - \int \mathscr{A}_R(z) d\mu(z)\right|, \tag{92}$$

where the traces are on the Fock space $\Gamma_s(P_R L^2 \oplus P_R L^2)$ and the integrals are over $L^2 \oplus L^2$. By dominated convergence the right hand side of (92) tends to 0 when $m \to \infty$ at fixed R. The right hand side of (90) can be made arbitrary small when $m \to \infty$ using the following decomposition

$$(\varrho_{\varepsilon_j}^{R,m} - \varrho_{\varepsilon_j}^R) = \underbrace{(\chi_{m,R}^{Weyl} - 1)\, \varrho_{\varepsilon_j}^R\, \chi_{m,R}^{Weyl}}_{(A)} + \underbrace{\varrho_{\varepsilon_j}^R\, (\chi_{m,R}^{Weyl} - 1)}_{(B)},$$

which gives $\mathrm{Tr}\Big[(A)\,(\mathscr{A}_R)^{A-Wick}\Big] = \mathrm{Tr}\Big[T_1 T_2 T_3 T_4\Big]$ and a similar expression for (B) with

$$T_1 = (N_R + 1)^{\frac{p+q}{4}}(\chi_{m,R}^{Weyl} - 1)(N_R + 1)^{-\frac{\delta}{2}}, \qquad T_2 = (N_R + 1)^{\frac{\delta}{2}}\varrho_{\varepsilon_j}^R(N_R + 1)^{\frac{\delta}{2}}$$
$$T_3 = (N_R + 1)^{-\frac{\delta}{2}}\chi_{m,R}^{Weyl}(N_R + 1)^{\frac{p+q}{4}}, \qquad T_4 = (N_R + 1)^{-\frac{p+q}{4}}(\mathscr{A}_R)^{A-Wick}(N_R + 1)^{-\frac{p+q}{4}},$$

where $N_R = d\Gamma(P_R \oplus P_R)$. The Weyl-Hörmander Pseudo-differential calculus gives that $T_1 \to_{m\to\infty} 0$ in norm (since $\delta > p + q$) and that T_i, $i = 2, 3, 4$, are uniformly bounded with respect $j \in \mathbb{N}$ and $m > 0$ at fixed R (see e.g. [8, Proposition 3.2 and 3.3]).

To complete the proof, we remark that $\mathrm{Tr}\Big[\varrho_{\varepsilon_j}^{R,m}(\mathscr{A}_R)^{A-Wick}\Big] = \mathrm{Tr}\Big[\varrho_{\varepsilon_j}^R\ \chi_{m,R}^{Weyl}\ (\mathscr{A}_R)^{A-Wick}\ \chi_{m,R}^{Weyl}\Big]$. So again by pseudo-differential calculus we know that $(\mathscr{A}_R)^{A-Wick} = (\mathscr{A}_R)^{Weyl} + \varepsilon\, b(\varepsilon)^{Weyl}$ with $b(\varepsilon)$ belonging to the Weyl–Hörmander class symbol $S_{P_R \oplus P_R}(\langle z\rangle^{p+q-2}, \frac{dz^2}{\langle z\rangle^2})$ uniformly in ε (see [8, Section 3.2 and 3.4]). Therefore

$$\lim_{j\to\infty} \mathrm{Tr}\Big[\varrho_{\varepsilon_j}^{R,m}(\mathscr{A}_R)^{A-Wick}\Big] = \lim_{j\to\infty} \mathrm{Tr}\Big[\varrho_{\varepsilon_j}^R\ \chi_{m,R}^{Weyl}(\mathscr{A}_R)^{Weyl}\ \chi_{m,R}^{Weyl}\Big],$$

since $(d\Gamma(P_R\oplus P_R)+1)^{-(q+p)/2}\, b(\varepsilon)^{Weyl}\, (d\Gamma(P_R\oplus P_R)+1)^{-(p+q)/2}$ is uniformly bounded with respect to ε. The Weyl-Hörmander pseudo-differential calculus gives $\chi_{m,R}^{Weyl}(\mathscr{A}_R)^{Weyl}\,\chi_{m,R}^{Weyl} = (\chi_{m,R}^2\mathscr{A}_R)^{Weyl} + \varepsilon\, c(\varepsilon)^{Weyl}$ with $c(\varepsilon)\in S_{P_R\oplus P_R}(1,dz^2)$ uniformly in ε (see e.g. [8, Proposition 3.2]). Hence, according to [8, Theorem 6.2] one obtains

$$\lim_{j\to\infty}\mathrm{Tr}\Big[\varrho_{\varepsilon_j}^{R,m}(\mathscr{A}_R)^{A-Wick}\Big] = \lim_{j\to\infty}\mathrm{Tr}\Big[\varrho_{\varepsilon_j}\,(\chi_{m,R}^2\,\mathscr{A}_R)^{Weyl}\Big] = \int_{L^2\oplus L^2}\chi_{m,R}^2(z)\mathscr{A}_R(z)d\mu(z)\,.$$

This yields the intended bound on (88) and completes the proof. $\dashv$

Lemma 4.16. *There exists $C(\sigma_0)>0$ depending only on $\sigma_0\in\mathbb{R}_+$ such that the following bound holds for $\mathscr{B}_0^{(m)}$ uniformly in $m\in\mathbb{N}$:*

(93)
$$\Big|\mathscr{B}_0^{(m)}(\xi)(u,\alpha)\Big| \le C(\sigma_0)\|\xi\|_{L^2\oplus L^2}\Big(\|u\|_2^2+\|(-\Delta+V)^{1/2}u\|_2^2+\|\alpha\|_{\mathcal{F}H^{1/2}}^2+\|u\|_2\cdot\|(-\Delta+V)^{1/2}u\|_2^2 + \|u\|_2\cdot\|\alpha\|_{\mathcal{F}H^{1/2}}^2+\|u\|_2\cdot\|(-\Delta+V)^{1/2}u\|_2\cdot\|\alpha\|_{\mathcal{F}H^{1/2}}\Big)\,.$$

It follows that:

* *For any $\xi\in L^2\oplus L^2$, for any $(u,\alpha)\in Q(-\Delta+V)\oplus D(\omega^{1/2})$, $\lim_{m\to\infty}\mathscr{B}_0^{(m)}(\xi)(u,\alpha) = \mathscr{B}_0(\xi)(u,\alpha)$; and therefore the bound (93) holds also for $\mathscr{B}_0$.*
* *For any $m\in\mathbb{N}$, $\mathscr{B}_0^{(m)}(\cdot),\mathscr{B}_0(\cdot)$ are are jointly continuous with respect to $\xi\in L^2\oplus L^2$ and $(u,\alpha)\in Q(-\Delta+V)\oplus D(\omega^{1/2})$.*

Recall that for any $\sigma_0\ge 2K(\mathfrak{C}+1+\bar\varepsilon)$ there exists $b>0$ such that the operator $\hat H_{\mathrm{ren}}(\sigma_0)+b$ is non-negative uniformly for $\varepsilon\in(0,\bar\varepsilon)$. Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states on $\Gamma_s(L^2(\mathbb{R}^3)\oplus L^2(\mathbb{R}^3))$, we consider the additional assumption:

$$(A''_\rho)\qquad \exists C>0\,,\ \forall\varepsilon\in(0,\bar\varepsilon)\,,\ \mathrm{Tr}[\varrho_\varepsilon\,(\hat H_{\mathrm{ren}}(\sigma_0)+b)^2]\le C\,;$$

Proposition 4.17. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}\subset\mathcal{L}^1(\mathcal{H})$ be a family of normal states that satisfy Assumptions (A_0), (A'_ρ) and (A''_ρ) such that[15] $\sigma_0\ge 2K(\mathfrak{C}+1+\bar\varepsilon)$. Then:*
(i) For any sequence $(\varepsilon_k)_{k\in\mathbb{N}}\subset(0,\bar\varepsilon)$ converging to zero, there exist a subsequence $(\varepsilon_{k_\iota})_{\iota\in\mathbb{N}}$ and a map $\mu_t:\mathbb{R}\to\mathfrak{P}\big(L^2\oplus L^2\big)$ such that $\varrho_{\varepsilon_{k_\iota}}(t)\to\mu_t$ and $\tilde\varrho_{\varepsilon_{k_\iota}}(t)\to\tilde\mu_t=\mathbf{E}_0(-t)_\#\mu_t$, for any $t\in\mathbb{R}$.
(ii) The action of $e^{-i\frac{t}{\varepsilon}\hat H_{\mathrm{ren}}(\sigma_0)}$ is non-trival on the states ϱ_ε.
(iii) The Fourier transform of $\tilde\mu_{(\cdot)}$ satisfies the following transport equation $\forall\xi\in L^2\oplus L^2$:
(94)
$$\int_{L^2\oplus L^2}e^{i\sqrt2\mathrm{Re}\langle\xi,z\rangle}d\tilde\mu_t(z)=\int_{L^2\oplus L^2}e^{i\sqrt2\mathrm{Re}\langle\xi,z\rangle}d\mu_0(z)+\int_0^t\Big(\int_{L^2\oplus L^2}\mathscr{B}_0(\tilde\xi_s)(z)e^{i\sqrt2\mathrm{Re}\langle\tilde\xi_s,z\rangle}d\mu_s(z)\Big)ds\,;$$
where the right hand side makes sense since $\mathscr{B}_0(\tilde\xi_t)(z)e^{i\sqrt2\mathrm{Re}\langle\tilde\xi_t,z\rangle}\in L_t^\infty\Big(\mathbb{R},L_z^1\big[L^2\oplus L^2,d\mu_t(z)\big]\Big)$ for any $\xi\in L^2\oplus L^2$.

Proof. The first part of the proposition $(i)-(ii)$ is just a partial restatement of Proposition 4.14. We discuss the last assertion in (iii) about $\mathscr{B}_0(\tilde\xi_t)(z)e^{i\sqrt2\mathrm{Re}\langle\tilde\xi_t,z\rangle}$, before proving (94). Recall the fact that

[15] We recall that $\mathfrak{C}$ appears in Assumption (A_0) and σ_0 in Definition 2.12 of $\hat H_{\mathrm{ren}}(\sigma_0)$. The condition $\sigma_0\ge K(\mathfrak{C}+1)$ ensures that the dressed dynamics is non-trivial on $\bigoplus_{n=0}^{[\mathfrak{C}/\varepsilon]}\mathcal{H}_n$ and hence non-trivial on the state ϱ_ε according to Lemma 4.2.

for any $\xi \in L^2 \oplus L^2$ and for any $t \in \mathbb{R}$, $\|\tilde{\xi}_t\|_{L^2\oplus L^2} = \|\xi\|_{L^2\oplus L^2}$. Using bound (93) of Lemma 4.16 we obtain, setting $Q(-\Delta + V) \oplus D(\omega^{1/2}) \ni z = (u, \alpha)$:

$$\Bigl|\mathscr{B}_0(\tilde{\xi}_t)(z)e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi}_t,z\rangle}\Bigr| \leq C(\sigma_0)\|\xi\|_{L^2\oplus L^2}\Bigl(\|u\|_2^2 + \|(-\Delta+V)^{1/2}u\|_2^2 + \|\alpha\|^2_{\mathcal{F}H^{1/2}} + \|u\|_2 \cdot \|(-\Delta+V)^{1/2}u\|_2^2 + \|u\|_2 \cdot \|\alpha\|^2_{\mathcal{F}H^{1/2}} + \|u\|_2 \cdot \|(-\Delta+V)^{1/2}u\|_2 \cdot \|\alpha\|_{\mathcal{F}H^{1/2}}\Bigr)\ .$$

Now $\mu_t \in \mathcal{M}\bigl(\varrho_\varepsilon(t), \varepsilon \in (0, \bar{\varepsilon})\bigr)$, therefore by Lemma 4.12, $\mu_t\Bigl(B_u(0, \sqrt{\mathfrak{C}}) \cap Q(-\Delta+V) \oplus D(\omega^{1/2})\Bigr) = 1$ for any $t \in \mathbb{R}$. Then it follows that there exists $C(\mathfrak{C}) > 0$ such that

$$\Bigl|\int_{L^2\oplus L^2} \mathscr{B}_0(\tilde{\xi}_t)(z)e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi}_t,z\rangle}d\mu_t(z)\Bigr| \leq C(\mathfrak{C})\|\xi\|_{L^2\oplus L^2}\int_{L^2\oplus L^2}\Bigl(\|(-\Delta+V)^{1/2}u\|_2^2 + \|\alpha\|^2_{\mathcal{F}H^{1/2}}\Bigr)d\mu_t(z) \leq C(\mathfrak{C})\|\xi\|_{L^2\oplus L^2}J(t)\ ;$$

where $J(t) < \infty$ by Lemma 4.12. Actually, using the fact that the bound (78) is independent of t, it is easily proved that $J(t)$ does not depend on t as well, i.e. $J(t) \in L^\infty(\mathbb{R})$.

We prove (94) by successive approximations. Consider $\mathrm{Tr}\Bigl[\tilde{\varrho}_{\varepsilon_{k_\iota}}(t)W(\xi)\Bigr]$, $\xi \in L^2 \oplus L^2$. We can approximate ξ with $(\xi^{(l)})_{l\in\mathbb{N}} \subset Q(-\Delta+V) \oplus D(\omega^{3/4})$, since the latter is dense in $L^2 \oplus L^2$, and $\lim_{l\to\infty}\mathrm{Tr}\Bigl[\tilde{\varrho}_{\varepsilon_{k_\iota}}(t)\bigl(W(\xi) - W(\xi^{(l)})\bigr)\Bigr] = 0$ uniformly in ε_{k_ι} by Lemma 4.13. Now, for $\mathrm{Tr}\Bigl[\tilde{\varrho}_{\varepsilon_{k_\iota}}(t)W(\xi^{(l)})\Bigr]$ the integral equation (69) holds. Proposition 4.14 implies that $\tilde{\varrho}_{\varepsilon_{k_\iota}}(t) \to \tilde{\mu}_t = \mathbf{E}_0(t)_\#\mu_t$, for any $t \in \mathbb{R}$. Therefore the left-hand side of (69) converges when $\iota \to \infty$ to $\int_{L^2\oplus L^2} e^{i\sqrt{2}\mathrm{Re}\langle\xi^{(l)},z\rangle}d\tilde{\mu}_t(z)$; and that in turn converges when $l \to \infty$ to $\int_{L^2\oplus L^2} e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}d\tilde{\mu}_t(z)$ by dominated convergence theorem. In addition,

$$\lim_{\iota\to\infty}\sum_{j=1}^{3}\varepsilon^j\int_0^t \mathrm{Tr}\Bigl[\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)B_j(\tilde{\xi^{(l)}}_s)\Bigr]ds = 0\ ;$$

by Proposition 4.7. It remains to show the convergence of the B_0 term in (69). We approximate $\mathscr{B}_0$ by the compact $\mathscr{B}_0^{(m)}$, because using Lemma 4.2 and (80) of Proposition 4.9 we obtain

$$\begin{aligned}\Bigl|\mathrm{Tr}\Bigl[\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)\Bigl(B_0(\tilde{\xi^{(l)}}_s) - B_0^{(m)}(\tilde{\xi^{(l)}}_s)\Bigr)\Bigr]\Bigr| &\leq \sum_{i\in\mathbb{N}}\lambda_i\Bigl|\Bigl\langle W^*(\tilde{\xi^{(l)}}_s)e^{-i\frac{s}{\varepsilon_{k_\iota}}\hat{H}_{\mathrm{ren}}}\Psi_i, \Bigl(B_0(\tilde{\xi^{(l)}}_s) - B_0^{(m)}(\tilde{\xi^{(l)}}_s)\Bigr)e^{-i\frac{s}{\varepsilon_{k_\iota}}\hat{H}_{\mathrm{ren}}}\Psi_i\Bigr\rangle\Bigr| \\ &\leq \sum_{i\in\mathbb{N}}\lambda_i C^{(m)}(\tilde{\xi^{(l)}}_s)\Bigl\|(H_0+1)^{1/2}(N_1+\bar{\varepsilon})^{1/2}W^*(\tilde{\xi^{(l)}}_s)e^{-i\frac{s}{\varepsilon_{k_\iota}}\hat{H}_{\mathrm{ren}}}\Psi_i\Bigr\| \\ &\quad\cdot\Bigl\|(H_0+1)^{1/2}(N_1+\bar{\varepsilon})^{1/2}e^{-i\frac{s}{\varepsilon_{k_\iota}}\hat{H}_{\mathrm{ren}}}\Psi_i\Bigr\|\ .\end{aligned}$$

Now, using the fact that $C^{(m)}(\tilde{\xi^{(l)}}_s)$ depends only on $\|\tilde{\xi^{(l)}}_s\|_{Q(-\Delta+V)\oplus D(\omega^{3/4})} = \|\xi^{(l)}\|_{Q(-\Delta+V)\oplus D(\omega^{3/4})}$ and Lemma 4.1 we obtain

$$\Bigl|\mathrm{Tr}\Bigl[\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)\Bigl(B_0(\tilde{\xi^{(l)}}_s) - B_0^{(m)}(\tilde{\xi^{(l)}}_s)\Bigr)\Bigr]\Bigr| \leq \sum_{i\in\mathbb{N}}\lambda_i C^{(m)}(\xi^{(l)})C(\xi^{(l)})\Bigl\|(H_0+1)^{1/2}e^{-i\frac{s}{\varepsilon_{k_\iota}}\hat{H}_{\mathrm{ren}}}(N_1+\bar{\varepsilon})^{1/2}\Psi_i\Bigr\|^2\ .$$

We then use Equation (78) of Lemma 4.8:

$$\Bigl|\mathrm{Tr}\Bigl[\varrho_{\varepsilon_{k_\iota}}(s)W(\tilde{\xi^{(l)}}_s)\Bigl(B_0(\tilde{\xi^{(l)}}_s) - B_0^{(m)}(\tilde{\xi^{(l)}}_s)\Bigr)\Bigr]\Bigr| \leq \sum_{i\in\mathbb{N}}\lambda_i C^{(m)}(\xi^{(l)})C(\xi^{(l)})(\mathfrak{C}+\bar{\varepsilon})\tfrac{1}{1-a(\mathfrak{C})}C + \tfrac{2b(\mathfrak{C})}{1-a(\mathfrak{C})}\ .$$

The right hand side goes to zero when $m \to \infty$ uniformly with respect to ε_{k_ι} and s by Proposition 4.9, and therefore

$$\lim_{m\to\infty} \int_0^t \mathrm{Tr}\Big[\varrho_{\varepsilon_{k_\iota}}(s) W(\tilde{\xi^{(l)}}_s)\Big(B_0(\tilde{\xi^{(l)}}_s) - B_0^{(m)}(\tilde{\xi^{(l)}}_s)\Big)\Big] ds = 0 \,.$$

So the next step is to prove

$$\lim_{\iota\to\infty} \mathrm{Tr}\Big[\varrho_{\varepsilon_{k_\iota}}(s) W(\tilde{\xi^{(l)}}_s)\Big(\mathscr{B}_0^{(m)}(\tilde{\xi^{(l)}}_s)\Big)^{Wick}\Big] = \int_{L^2\oplus L^2} \mathscr{B}_0^{(m)}(\tilde{\xi}_s^{(l)})(z)\, e^{i\sqrt{2}\mathrm{Re}\langle \tilde{\xi}_s^{(l)}, z\rangle}\, d\mu_s(z).$$

This statement follows by applying Lemma 4.15 with $\delta = 2$ and by checking the assumption

$$||(N_1+N_2)\,\varrho_{\varepsilon_{k_\iota}}(s) W(\tilde{\xi^{(l)}}_s)\,(N_1+N_2)||_{\mathcal{L}^1(L^2\oplus L^2)} \leq C\,, \tag{95}$$

uniformly in k_ι for some $C > 0$. In fact (95) holds true by Assumptions (A_0)-(A''_ρ), the Higher order estimate of Proposition A.4 and Lemma 4.1. Remark that while $\varrho_{\varepsilon_{k_\iota}}(s) W(\tilde{\xi^{(l)}}_s)$ is not a non-negative trace-class operator, one can still apply Lemma 4.15. In fact, one can write

$$\mathrm{Tr}\Big[\varrho_{\varepsilon_{k_\iota}}(s) W(\tilde{\xi^{(l)}}_s)\; B_0^{(m)}(\tilde{\xi^{(l)}}_s)\Big] = \mathrm{Tr}\Big[W(\eta)\varrho_{\varepsilon_{k_\iota}}(s) W(\eta)\; \mathscr{A}^{Wick}\Big]\,,$$

for some $\mathscr{A} \in \bigoplus_{p+q<4} \mathcal{P}^\infty_{p,q}\big(L^2\oplus L^2\big)$ and with $\eta = \frac{1}{2}\tilde{\xi^{(l)}}_s$. Remark now that $W(\eta)\varrho_{\varepsilon_{k_\iota}}(s)W(\eta)$ decomposes explicitly into a linear combination of non-negative trace-class operators satisfying all the assumption (86) of Lemma 4.15. Note that the Wigner measures of $\varrho_{\varepsilon_{k_\iota}}(s) W(\tilde{\xi^{(l)}}_s)$ are identified through (84). Hence the dominated convergence theorem yields:

$$\lim_{\iota\to\infty} \int_0^t \mathrm{Tr}\Big[\varrho_{\varepsilon_{k_\iota}}(s) W(\tilde{\xi^{(l)}}_s) B_0^{(m)}(\tilde{\xi^{(l)}}_s)\Big] ds = \int_0^t \bigg(\int_{L^2\oplus L^2} \mathscr{B}_0^{(m)}(\tilde{\xi^{(l)}}_s)(z) e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi^{(l)}}_s, z\rangle}\, d\mu_s(z)\bigg) ds\,.$$

By Lemma 4.16, $\lim_{m\to\infty} \mathscr{B}_0^{(m)}(\tilde{\xi^{(l)}}_s)(z) = \mathscr{B}_0(\tilde{\xi^{(l)}}_s)(z)$, so by dominated convergence theorem

$$\lim_{m\to\infty} \int_0^t \bigg(\int_{L^2\oplus L^2} \mathscr{B}_0^{(m)}(\tilde{\xi^{(l)}}_s)(z) e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi^{(l)}}_s, z\rangle}\, d\mu_s(z)\bigg) ds = \int_0^t \bigg(\int_{L^2\oplus L^2} \mathscr{B}_0(\tilde{\xi^{(l)}}_s)(z) e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi^{(l)}}_s, z\rangle}\, d\mu_s(z)\bigg) ds\,.$$

Above it is possible to apply the dominated convergence theorem due to a reasoning analogous to the one done at the beginning of this proof: roughly speaking, we have that $\mathscr{B}_0^{(m)}(\tilde{\xi^{(l)}}_t)(z) e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi^{(l)}}_t, z\rangle} \in L^\infty_t\Big(\mathbb{R}, L^1_z\big[L^2\oplus L^2, d\mu_t(z)\big]\Big)$ uniformly with respect to $m \in \mathbb{N}$. In an analogous fashion we finally obtain

$$\lim_{l\to\infty} \int_0^t \bigg(\int_{L^2\oplus L^2} \mathscr{B}_0(\tilde{\xi^{(l)}}_s)(z) e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi^{(l)}}_s, z\rangle}\, d\mu_s(z)\bigg) ds = \int_0^t \bigg(\int_{L^2\oplus L^2} \mathscr{B}_0(\tilde{\xi}_s)(z) e^{i\sqrt{2}\mathrm{Re}\langle\tilde{\xi}_s, z\rangle}\, d\mu_s(z)\bigg) ds\,.$$

$\dashv$

Corollary 4.18. *The transport equation (94) may be rewritten as*

$$\begin{aligned} \int_{L^2\oplus L^2} e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle} d\tilde{\mu}_t(z) \quad &= \quad \int_{L^2\oplus L^2} e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle} d\mu_0(z) \\ &\quad + i\sqrt{2}\int_0^t \bigg(\int_{L^2\oplus L^2} e^{i\sqrt{2}\mathrm{Re}\langle\xi,z\rangle}\mathrm{Re}\langle\xi, \mathbf{V}(s)(z)\rangle_{L^2\oplus L^2}\; d\tilde{\mu}_s(z)\bigg) ds\,; \end{aligned} \tag{96}$$

with the vector field $\mathbf{V}(t)(z) = -i\mathbf{E}_0(-t)\circ\partial_{\bar{z}}\big(\hat{\mathscr{E}} - \mathscr{E}_0\big)\circ\mathbf{E}_0(t)(z)$. *In addition* $\tilde{\mu}_t = \mathbf{E}_0(-t)_\#\hat{\mathbf{E}}(t)_\#\mu_0$ *is a solution of Equation (96).*

Proof. It is proved by direct calculation, since $\mu_t\big(Q(-\Delta+V)\oplus\mathcal{F}H^{1/2}\big)=1$ for any $t\in\mathbb{R}$ by Lemma 4.12; and $\hat{\mathbf{E}}(t),\mathbf{E}_0(t)$ are globally well-defined on this space (for $\hat{\mathbf{E}}(t)$, it is proved in Theorem 3.16; for $\mathbf{E}_0(t)$ it is trivial). The second point is proved by differentiating with respect to time and using Lemma 4.16 and Lemma 4.12 (*iii*). ⊣

4.6. **Uniqueness of solutions for the transport equation.** As discussed in Corollary 4.18, the dressed flow yields in the classical limit a solution of the transport equation (96). The second part of the same corollary suggests that it is important to study uniqueness properties of (96): it is by means of uniqueness that we can close the argument and reach a satisfactory characterization of the dynamics of classical states (Wigner measures). This subsection is devoted to prove that the family of Wigner measures $\tilde{\mu}_t$ of Proposition 4.17 satisfies sufficient conditions, induced by the properties of $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$, to be uniquely identified with $\mathbf{E}_0(-t)_\#\hat{\mathbf{E}}(t)_\#\mu_0$. We use an optimal transport technique initiated by Ambrosio, Gigli, and Savaré [2] then extended by Ammari and Nier [11] to propagation of Wigner measures; and imporved recently by Ammari and Liard [7] (see also [62, 64]).

In order to do that, we need to introduce a suitable topology on $\mathfrak{P}\big(L^2\oplus L^2\big)$. Let $(e_j)_{j\in\mathbb{N}}\subset L^2\oplus L^2$ be an orthonormal basis. Then

$$d_w(z_1,z_2)=\Bigg(\sum_{j\in\mathbb{N}}\frac{\lvert\langle z_1-z_2,e_j\rangle_{L^2\oplus L^2}\rvert^2}{(1+j)^2}\Bigg)^{1/2}\,, \tag{97}$$

where $z_1,z_2\in L^2\oplus L^2$, defines a distance on $L^2\oplus L^2$. The topology induced by $\big(L^2\oplus L^2,d_w\big)$ is homeomorphic to the weak topology on bounded sets.

Definition 4.19 (Weak narrow convergence of probability measures)**.** *Let $(\mu_i)_{i\in\mathbb{N}}\subset\mathfrak{P}\big(L^2\oplus L^2\big)$. Then $(\mu_i)_{i\in\mathbb{N}}$ weakly narrowly converges to $\mu\in\mathfrak{P}\big(L^2\oplus L^2\big)$, in symbols $\mu_i\xrightharpoonup{n}\mu$, if*

$$\forall f\in\mathcal{C}_b\Big(\big(L^2\oplus L^2,d_w\big),\mathbb{R}\Big)\;,\;\lim_{i\to\infty}\int_{L^2\oplus L^2}f(z)d\mu_i(z)=\int_{L^2\oplus L^2}f(z)d\mu(z)\;;$$

where $\mathcal{C}_b\Big(\big(L^2\oplus L^2,d_w\big),\mathbb{R}\Big)$ is the space of bounded continuous real-valued functions on $\big(L^2\oplus L^2,d_w\big)$.

It is actually more convenient to use cylindrical functions to prove narrow continuity properties. We define below two useful spaces of smooth cylindrical functions on $L^2\oplus L^2$.

Definition 4.20 (Spaces of cylindrical functions)**.** *Let $f:L^2\oplus L^2\to\mathbb{R}$. Then $f\in\mathcal{S}_{cyl}\big(L^2\oplus L^2\big)$ if there exists an orthogonal projection $\mathbf{p}:L^2\oplus L^2\to L^2\oplus L^2$, $\dim(\operatorname{Ran}\mathbf{p})=d<\infty$, and a rapidly decrease function g in the Schwartz space $\mathcal{S}(\operatorname{Ran}\mathbf{p})$, such that*

$$\forall z\in L^2\oplus L^2\,,\;f(z)=g(\mathbf{p}z)\;.$$

Analogously, if $g\in\mathcal{C}_0^\infty(\operatorname{Ran}\mathbf{p})$, then $f\in\mathcal{C}_{0,cyl}^\infty\big(L^2\oplus L^2\big)$, the cylindrical smooth functions with compact support.

We remark that neither $\mathcal{S}_{cyl}\big(L^2\oplus L^2\big)$ nor $\mathcal{C}_{0,cyl}^\infty\big(L^2\oplus L^2\big)$ possess a vector space structure. Finally, for cylindrical Schwartz functions we define the Fourier transform:

$$\mathcal{F}[f](\eta)=\int_{\operatorname{Ran}\mathbf{p}}e^{-2\pi i\operatorname{Re}\langle\eta,z\rangle_{L^2\oplus L^2}}f(z)dL_{\mathbf{p}}(z)\;,$$

where $dL_{\mathbf{p}}$ denotes integration with respect to the Lebesgue measure on $\mathrm{Ran}\,\mathbf{p}$. The inversion formula is then

$$f(z) = \int_{\mathrm{Ran}\,\mathbf{p}} e^{2\pi i \mathrm{Re}\langle \eta, z\rangle_{L^2\oplus L^2}} \mathcal{F}[f](\eta) dL_{\mathbf{p}}(\eta) \ .$$

With these definitions in mind, we can prove the following lemma.

Lemma 4.21. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)} \subset \mathcal{L}^1(\mathcal{H})$ be a family of normal states that satisfies Assumptions (A_0), (A'_ρ) and (A''_ρ); $\tilde\mu_t : \mathbb{R} \to \mathfrak{P}\big(L^2 \oplus L^2\big)$ such that for any $t \in \mathbb{R}$, $\tilde\mu_t \in \mathcal{M}\big(\tilde\varrho_\varepsilon(t), \varepsilon \in (0,\bar\varepsilon)\big)$. If, in addition, $\tilde\mu_t$ satisfies the integral equation (96), then the following statements are true:*

* *For any $t \in \mathbb{R}$, and for any $(t_i)_{i\in\mathbb{R}} \subset \mathbb{R}$ such that $\lim_{i\to\infty} t_i = t$,*

$$\tilde\mu_{t_i} \xrightharpoonup{n} \tilde\mu_t \ ;$$

  i.e. $\tilde\mu_t$ is a weakly narrowly continuous map in $\mathfrak{P}(L^2 \oplus L^2)$.
* *The map $\tilde\mu_t$ solves the transport equation* [16]

$$\partial_t \tilde\mu_t + \nabla^T \Big(\mathbf{V}(t)\tilde\mu_t\Big) = 0$$

  in the weak sense, i.e.

$$\forall f \in \mathcal{C}^\infty_{0,cyl}\Big(\mathbb{R} \times \big(L^2\oplus L^2\big)\Big) \ , \ \int_{\mathbb{R}} \int_{L^2\oplus L^2} \Big(\partial_t f + \mathrm{Re}\langle \nabla f, \mathbf{V}(t)\rangle_{L^2\oplus L^2}\Big) d\tilde\mu_t dt = 0 \ . \tag{98}$$

Proof. Let $f \in \mathcal{S}_{cyl}\big(L^2\oplus L^2\big)$. Fubini's theorem gives

$$\int_{L^2\oplus L^2} f(z) d\tilde\mu_t(z) = \int_{\mathrm{Ran}\,\mathbf{p}} \mathcal{F}[f](\xi) \left(\int_{L^2\oplus L^2} e^{2\pi i \mathrm{Re}\langle\xi,z\rangle} d\tilde\mu_t(z)\right) dL_{\mathrm{Ran}\,\mathbf{p}}(\xi) \ ,$$

where $dL_{\mathrm{Ran}\,\mathbf{p}}$ is the Lebesgue measure on $\mathrm{Ran}\,\mathbf{p}$ and $\mathcal{F}(f)(\xi) = \int_{\mathrm{Ran}\,\mathbf{p}} f(z) e^{-2\pi i\mathrm{Re}\langle\xi,z\rangle} dL_{\mathrm{Ran}\,\mathbf{p}}(z)$. Now we define $\tilde G_0(t,\xi) := \int_{L^2\oplus L^2} e^{2\pi i \mathrm{Re}\langle\xi,z\rangle} d\tilde\mu_t(z)$. Hence Equation (94) of Proposition 4.17 gives

$$\tilde G_0(t,\xi) - \tilde G_0(s,\xi) = \int_s^t \left(\int_{L^2\oplus L^2} \mathscr{B}_0(\tilde\xi_\tau)(z) e^{i\sqrt{2}\mathrm{Re}\langle\tilde\xi_\tau, z\rangle} d\mu_\tau(z)\right) d\tau \ ; \tag{99}$$

and this proves that $t \mapsto \tilde G_0(t,\xi)$ is continuous for any $\xi \in L^2\oplus L^2$ since the integrand in the right hand side of (99) is bounded with respect to τ by Proposition 4.17. Remark that $\tilde G_0(t,\xi)$ is bounded by one for any $(t,\xi) \in \mathbb{R}\times(L^2\oplus L^2)$. Therefore the map $t \mapsto \int_{L^2\oplus L^2} f(z) d\tilde\mu_t(z)$ is continuous for any $f \in \mathcal{S}_{cyl}\big(L^2\oplus L^2\big)$. Finally, by an argument analogous to the one used at the beginning of the proof of Proposition 4.17, it is easy to prove that $\int_{L^2\oplus L^2} \|z\|^2_{L^2\oplus L^2} d\tilde\mu_t(z) \in L^\infty_t(\mathbb{R})$. In fact, we know that $\tilde\mu_t\Big(B_u(0,\sqrt{\mathfrak{C}}) \cap Q(-\Delta+V) \oplus D(\omega^{1/2})\Big) = 1$ by Lemma 4.12; and if $z = (u,\alpha)$ then the functions $\alpha \mapsto \|\alpha\|_2^2 \le \|\alpha\|^2_{\mathcal{F}H^{1/2}}$, belong to $L^1_z\Big[L^2\oplus L^2, d\tilde\mu_t(z)\Big]$ uniformly in t by Lemmas 4.12 and 4.8. Then it follows that $\tilde\mu_t$ is weakly narrowly continuous by [2, Lemma 5.1.12 - f], thus proving the first point.

Now we prove the second point by a similar argument as in [11] which we reproduce here for completeness. Let $g \in \mathcal{C}^\infty_{0,cyl}\big(L^2\oplus L^2\big)$; we integrate Equation (96) with respect to the measure $\mathcal{F}[g](\eta) dL_{\mathbf{p}}$ obtaining

$$\int_{L^2\oplus L^2} g(z) d\tilde\mu_t(z) = \int_{L^2\oplus L^2} g(z) d\tilde\mu_0(z) + 2\pi i \int_0^t \int_{\mathrm{Ran}\,\mathbf{p}} \left(\int_{L^2\oplus L^2} \mathrm{Re}\langle \eta, \mathbf{V}(s)(z)\rangle_{L^2\oplus L^2} d\tilde\mu_s(z)\right) \mathcal{F}[g](\eta) dL_{\mathbf{p}}(\eta) ds \ .$$

[16] Recall that $\mathbf{V}(t)(z) = -i\mathbf{E}_0(-t)\circ\partial_{\bar z}\big(\hat{\mathscr{E}} - \mathscr{E}_0\big)\circ\mathbf{E}_0(t)(z)$.

Let ∇g be the differential of $g : L^2 \oplus L^2 \to \mathbb{R}$, where here $L^2 \oplus L^2$ is considered as a real Hilbert space with scalar product $\mathrm{Re}\langle \cdot, \cdot \rangle_{L^2 \oplus L^2}$. Then, by Fubini's theorem and the properties of the Fourier transform, we get

$$\int_{L^2\oplus L^2} g(z) d\tilde{\mu}_t(z) = \int_{L^2\oplus L^2} g(z) d\tilde{\mu}_0(z) + \int_0^t \int_{L^2\oplus L^2} \mathrm{Re}\langle \nabla g(z), \mathbf{V}(s)(z)\rangle_{L^2\oplus L^2} d\tilde{\mu}_s(z) ds \,.$$

By Lebesgue's differentiation theorem (with respect to t), we obtain

$$\partial_t \int_{L^2\oplus L^2} g(z) d\tilde{\mu}_t(z) - \int_{L^2\oplus L^2} \mathrm{Re}\langle \nabla g(z), \mathbf{V}(t)(z)\rangle_{L^2\oplus L^2} d\tilde{\mu}_t(z) = 0 \,.$$

Equation (98) is then obtained for $f(t,z) = \varphi(t) g(z)$, multiplying by $\varphi(t) \in \mathcal{C}_0^\infty(\mathbb{R},\mathbb{R})$, integrating with respect to t, and finally using integration by parts. The result for a generic $f \in \mathcal{C}^\infty_{0,cyl}\Big(\mathbb{R} \times (L^2 \oplus L^2)\Big)$ follows immediately: $f(t,z) = g(t, \mathbf{p}z)$ for some $g \in \mathcal{C}_0^\infty\Big(\mathbb{R} \times \mathrm{Ran}\,\mathbf{p}\Big)$, and the latter can be approximated by a sequence $\Big(g_j(t,\mathbf{p}z)\Big)_{j\in\mathbb{N}} \subset \mathcal{C}_0^\infty(\mathbb{R}) \overset{alg}{\otimes} \mathcal{C}_0^\infty(\mathrm{Ran}\,\mathbf{p})$. ⊣

We need to check an hypothesis on the vector field $\mathbf{V}(t)(z) = -i\mathbf{E}_0(-t) \circ \partial_{\bar{z}}\big(\hat{\mathscr{E}} - \mathscr{E}_0\big) \circ \mathbf{E}_0(t)(z)$ to prove the sought uniqueness result. This is done in the following lemma.

Lemma 4.22. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \subset \mathcal{L}^1(\mathcal{H})$ be a family of normal states that satisfies Assumptions (A_0) and (A'_ρ); $\tilde{\mu}_t : \mathbb{R} \to \mathfrak{P}\big(L^2 \oplus L^2\big)$ such that for any $t \in \mathbb{R}$, $\tilde{\mu}_t \in \mathcal{M}\big(\tilde{\varrho}_\varepsilon(t), \varepsilon \in (0,\bar{\varepsilon})\big)$. Then $\|\mathbf{V}(t)(z)\|_{L^2\oplus L^2} \in L^\infty_t\Big(\mathbb{R}, L^1_z\big[L^2 \oplus L^2, d\mu_t(z)\big]\Big)$, i.e. the norm of the vector field is integrable with respect to $\tilde{\mu}_t$, uniformly in $t \in \mathbb{R}$.*

Proof. By Equation (93) of Lemma 4.16 and the definition of $\mathbf{V}(t)$ we have that for any $\xi \in L^2 \oplus L^2$:

$$\Big|\mathrm{Re}\langle \xi, \mathbf{V}(t)(z)\rangle\Big| \leq C(\sigma_0)\|\xi\|_{L^2\oplus L^2}\Big(\|u\|_2^2 + \|(-\Delta+V)^{1/2}u\|_2^2 + \|\alpha\|^2_{\mathcal{F}H^{1/2}} + \|u\|_2 \cdot \|(-\Delta+V)^{1/2}u\|_2^2$$
$$+\|u\|_2 \cdot \|\alpha\|^2_{\mathcal{F}H^{1/2}} + \|u\|_2 \cdot \|(-\Delta+V)^{1/2}u\|_2 \cdot \|\alpha\|_{\mathcal{F}H^{1/2}}\Big) \,.$$

It is easy to prove an equivalent bound for the imaginary part, and hence obtain for any $\xi \in L^2 \oplus L^2$:

$$\Big|\langle \xi, \mathbf{V}(t)(z)\rangle\Big| \leq C(\sigma_0)\|\xi\|_{L^2\oplus L^2}\Big(\|u\|_2^2 + \|(-\Delta+V)^{1/2}u\|_2^2 + \|\alpha\|^2_{\mathcal{F}H^{1/2}} + \|u\|_2 \cdot \|(-\Delta+V)^{1/2}u\|_2^2$$
$$+\|u\|_2 \cdot \|\alpha\|^2_{\mathcal{F}H^{1/2}} + \|u\|_2 \cdot \|(-\Delta+V)^{1/2}u\|_2 \cdot \|\alpha\|_{\mathcal{F}H^{1/2}}\Big) \,.$$

Therefore it follows immediately that

$$\|\mathbf{V}(t)(z)\|_{L^2\oplus L^2} \leq C(\sigma_0)\Big(\|u\|_2^2 + \|(-\Delta+V)^{1/2}u\|_2^2 + \|\alpha\|^2_{\mathcal{F}H^{1/2}} + \|u\|_2 \cdot \|(-\Delta+V)^{1/2}u\|_2^2$$
$$+\|u\|_2 \cdot \|\alpha\|^2_{\mathcal{F}H^{1/2}} + \|u\|_2 \cdot \|(-\Delta+V)^{1/2}u\|_2 \cdot \|\alpha\|_{\mathcal{F}H^{1/2}}\Big) \,.$$

The right hand side of the above equation is in $L^\infty_t\Big(\mathbb{R}, L^1_z\big[L^2\oplus L^2, d\mu_t(z)\big]\Big)$, as shown at the beginning of the proof of Proposition 4.17. ⊣

At this stage, we appeal to a result proved in [7, Proposition 4.1] concerning the uniqueness of measure-valued solutions of the Liouville equation (98), which we briefly recal in the Appendix B.

Proposition 4.23. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})} \subset \mathcal{L}^1(\mathcal{H})$ be a family of normal states that satisfies Assumptions (A_0),(A'_ρ) and (A''_ρ). In addition, let $\tilde{\mu}_t : \mathbb{R} \to \mathfrak{P}\big(L^2 \oplus L^2\big)$ such that for any $t \in \mathbb{R}$, $\tilde{\mu}_t \in$*

$\mathcal{M}\big(\tilde{\varrho}_{\varepsilon_k}(t), k \in \mathbb{N}\big)$ *for some sequence* $(\varepsilon_k)_{k\in\mathbb{N}}$ *with* $\varepsilon_k \to 0$ *and* $\tilde{\mu}_t$ *satisfies the integral equation* (96). *Then* $\tilde{\mu}_t = (\mathbf{E}_0(-t)\circ \hat{\mathbf{E}}(t))_{\#}\mu_0$.

Proof. Observe that Lemma 4.21, Lemma 4.22 and Lemma 4.12 (*ii*)-(*iii*) are sufficient to apply Proposition B.1 with $v(t,z) = \mathbf{V}(t)(z)$ and $(\tilde{\mu}_t)_{t\in\mathbb{R}}$. Hence, we obtain the existence of a probability measure η verifying the properties (i)-(ii) in the Appendix B. The next step is to show that η is concentrated on solutions of the dressed equation (S-KG[D]) written in the interaction representation. For simplicity one can take the interval I such that $[0,T]\subset I$, for some $T>0$.

By Hölder inequality, Lemma 4.12 (*iii*) and Proposition B.1-(*ii*),

$$\int_{\mathfrak{X}}\bigg(\int_I ||\gamma(t)||^2_{H^1\oplus\mathcal{F}H^{1/2}}dt\bigg)^{1/2} d\eta(x,\gamma) \le \bigg(\int_I\int_{H^1\oplus\mathcal{F}H^{1/2}} ||z||^2_{H^1\oplus\mathcal{F}H^{1/2}}d\tilde{\mu}_t\bigg)^{1/2} < \infty\,.$$

This means that $\gamma\in L^2(I,H^1\oplus\mathcal{F}H^{1/2})$ for η-a.e. So we conclude that there exists a η-negligible set $\mathscr{N}$ such that for any $(x,\gamma)\in\mathfrak{X}\setminus\mathscr{N}$, $\gamma\in W^{1,1}(I,L^2\oplus L^2)$, satisfy the equation

$$\gamma(t) = x + \int_0^t \mathbf{V}(s)(\gamma(s))ds\,, \quad \forall t\in I\,;$$

and furthermore $\gamma\in L^2(I,H^1\oplus\mathcal{F}H^{1/2})\cap L^\infty(I,L^2\oplus L^2)$ and $\mathbf{V}(\cdot)(\gamma(\cdot))\in L^1(I,L^2\oplus L^2)$. Remember that $\mathbf{D}_{g_\infty}(-1)$ and $\mathbf{E}_0(t)$ preserve the spaces $H^1\oplus\mathcal{F}H^{1/2}$ and $L^2\oplus L^2$ (see Proposition 3.5). So by a simple computation one checks that for any γ as before, the curve

$$t\to\tilde{\gamma}(t) := \mathbf{D}_{g_\infty}(1)\circ\mathbf{E}_0(t)(\gamma(t))\in L^2(I,H^1\oplus\mathcal{F}H^{1/2})\cap L^\infty(I,L^2\oplus L^2)$$

and satisfies the Duhamel formula,

$$\tilde{\gamma}(t) = \mathbf{E}_0(t)\circ\mathbf{D}_{g_\infty}(1)x - i\int_0^t \mathbf{E}_0(t-s)\partial_{\bar{z}}(\mathscr{E}-\mathscr{E}_0)(\tilde{\gamma}(s))ds\,, \quad \forall t\in I\,,$$

which is the original Cauchy problem (S-KG[Y]) with the energy $\mathscr{E}$ given by Definition 3.8. Remember that we have already checked that $\mathbf{D}_{g_\infty}(\theta)$ are nonlinear symplectomorphisms on the phase space $L^2\oplus L^2$ (see Proposition 3.17). Now appealing to the result [27, Theorem 1.3], wee need to show that $\tilde{\gamma}_1\in L^{10/3}([0,T],L^{10/3}(\mathbb{R}^3))\cap L^8([0,T],L^{12/5}(\mathbb{R}^3))$ where $\tilde{\gamma}=(\tilde{\gamma}_1,\tilde{\gamma}_2)$ in order to conclude that $\tilde{\gamma}$ is actually the unique strong and global solution of the S-KG equation with initial condition $\gamma(0)=x\in H^1\oplus\mathcal{F}H^{1/2}$ and belonging to $C(\mathbb{R},H^1\oplus\mathcal{F}H^{1/2})$. The last statement follows by Strichartz estimates since $\tilde{\gamma}_1\in L^2([0,T],L^6(\mathbb{R}^3)\cap L^\infty([0,T],L^2(\mathbb{R}^3)$. So going back to γ, we conclude that $\gamma(t)=\mathbf{E}_0(-t)_{\#}\hat{\mathbf{E}}(t)(x)$. Hence, for any Borel bounded function φ on $L^2\oplus L^2$ and $t\in\mathbb{R}$,

$$\int_{L^2\oplus L^2}\varphi(x)d\tilde{\mu}_t = \int_{\mathfrak{X}}\varphi(\gamma(t))d\eta = \int_{\mathfrak{X}}\varphi\circ\mathbf{E}_0(-t)\circ\hat{\mathbf{E}}(t)(x)d\eta = \int_{L^2\oplus L^2}\varphi(\mathbf{E}_0(-t)\circ\hat{\mathbf{E}}(t)(x))\,d\mu_0(x)\,.$$

$\dashv$

4.7. **The classical limit of the dressing transformation.** Let's consider now the dressing transformation $U_\infty(\theta)=e^{-i\frac{\theta}{\varepsilon}T_\infty}$ on $\mathcal{H}$, with self-adjoint generator:

$$T_\infty = \big(\mathscr{D}_{g_\infty}\big)^{Wick} = \int_{\mathbb{R}^3}\psi^*(x)\Big(a^*(g_\infty e^{-ik\cdot x}) + a(g_\infty e^{-ik\cdot x})\Big)\psi(x)dx\,;$$

$$g_\infty(k) = -\frac{i}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega(k)}}\frac{1-\chi_{\sigma_0}(k)}{\frac{k^2}{2M}+\omega(k)}\in L^2(\mathbb{R}^3)\,.$$

The family $\left(e^{-i\frac{\theta}{\varepsilon}T_\infty}\right)_{\theta\in\mathbb{R}} \subset \mathcal{L}(\mathcal{H})$ is a strongly continuous unitary group, and therefore can be seen as a dynamical system acting on quantum states. Therefore, given a family $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ of normal quantum states on $\mathcal{H}$, we could determine the Wigner measures of

$$\hat{\varrho}_\varepsilon(\theta) = e^{-i\frac{\theta}{\varepsilon}T_\infty}\,\varrho_\varepsilon\, e^{i\frac{\theta}{\varepsilon}T_\infty}\ . \tag{100}$$

Since $T_\infty = \left(\mathscr{D}_{g_\infty}\right)^{Wick}$, where $\mathscr{D}_{g_\infty}$ is the classical dressing generator defined in Section 3.1, we expect that under suitable assumptions, $\Big(\varrho_{\varepsilon_k} \to \mu \Rightarrow \hat{\varrho}_{\varepsilon_k}(\theta) \to \mathbf{D}_{g_\infty}(\theta)_{\#}\mu\Big)$, where $\mathbf{D}_{g_\infty}(\theta)$ is the classical dressing transformation. The last assertion is indeed true, as explained in the following. Observe that the dressing generator T_∞ is equal to the interaction part $H_I(\sigma)$ of the Nelson model with cutoff, where $\frac{\chi_\sigma}{\sqrt{2\omega}}$ is replaced by g_∞, i.e. $T_\infty = H_I(\sigma)\big|_{\frac{\chi_\sigma}{\sqrt{2\omega}}=g_\infty}$. The classical limit of the Nelson model with cutoff has been treated by the authors in [5], thus the results below can be immediately deduced by the results in [5, $d = 3$, $H_0 = 0$ and $\frac{\chi}{\sqrt{\omega}} = g_\infty$]. We recall also that g_∞, and therefore also T_∞ and $\mathscr{D}_{g_\infty}$, depends on $\sigma_0 \in \mathbb{R}_+$.

Lemma 4.24. *Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of normal (quantum) states on $\mathcal{H}$ that satisfies Assumptions (A_0) and (A_ρ). Then for any $\sigma_0 \in \mathbb{R}_+$, $(\hat{\varrho}_\varepsilon(-1))_{\varepsilon\in(0,\bar{\varepsilon})}$ satisfies Assumptions (A_0) and (A'_ρ).*

Proposition 4.25. *Let $\mathbf{D}_{g_\infty} : \mathbb{R}\times Q(-\Delta+V)\oplus\mathcal{F}H^{1/2} \to Q(-\Delta+V)\oplus\mathcal{F}H^{1/2}$ be the classical dressing transformation. Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of normal quantum states on $\mathcal{H}$ that satisfies Assumption (A_0) and Assumption (A_ρ) or (A'_ρ). Then $\mathcal{M}\big(\varrho_\varepsilon, \varepsilon\in(0,\bar{\varepsilon})\big) \neq \emptyset$; and for any $\sigma_0 \in \mathbb{R}_+$ and $\theta\in\mathbb{R}$,*

$$\mathcal{M}\Big(\hat{\varrho}_\varepsilon(\theta), \varepsilon\in(0,\bar{\varepsilon})\Big) = \Big\{\mathbf{D}_{g_\infty}(\theta)_{\#}\mu\,,\ \mu\in\mathcal{M}\big(\varrho_\varepsilon,\varepsilon\in(0,\bar{\varepsilon})\big)\Big\}\ . \tag{101}$$

Furthermore, let $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\bar{\varepsilon})$ be a sequence such that $\lim_{k\to\infty}\varepsilon_k = 0$. Then the following statement is true:

$$\varrho_{\varepsilon_k} \to \mu \Leftrightarrow \Big(\forall\theta\in\mathbb{R}\ ,\ \forall\sigma_0\in\mathbb{R}_+\ ,\ \hat{\varrho}_{\varepsilon_k}(\theta) \to \mathbf{D}_{g_\infty}(\theta)_{\#}\mu\Big)\ . \tag{102}$$

4.8. **Overview of the results: linking the dressed and undressed systems.** Since as discussed in the previous subsection we can treat the dressing as a dynamical transformation with its own "time" parameter θ; we are able to link the classical limit of the dressed and undressed quantum dynamics via the classical dressing. In this way we are able to recover the expected classical S-KG dynamics for the undressed dynamics, and finally prove Theorem 1.1.

First of all, we put together the results proved from Section 4.2 to Section 4.6 on the renormalized dressed dynamics and remove the Assumption (A''_ρ) with the help of an approximation argument worked out in [10]. This is done in the following theorem.

Theorem 4.26. *Let $\hat{\mathbf{E}} : \mathbb{R}\times Q(-\Delta+V)\oplus\mathcal{F}H^{1/2} \to Q(-\Delta+V)\oplus\mathcal{F}H^{1/2}$ be the dressed S-KG flow associated to $\hat{\mathscr{E}}$. Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar{\varepsilon})}$ be a family of normal states in $\mathcal{H}$ that satisfies Assumptions (A_0) and (A'_ρ). Then for any $\sigma_0 \geq 2K(\mathfrak{C}+1+\bar{\varepsilon})$ the dynamics $e^{-i\frac{t}{\varepsilon}\hat{H}_{\mathrm{ren}}(\sigma_0)}$ is non-trivial on every relevant sector with fixed nucleons of the state ϱ_ε; $\mathcal{M}\big(\varrho_\varepsilon,\varepsilon\in(0,\bar{\varepsilon})\big)\neq\emptyset$; and for any $t\in\mathbb{R}$*

$$\mathcal{M}\Big(e^{-i\frac{t}{\varepsilon}\hat{H}_{\mathrm{ren}}(\sigma_0)}\varrho_\varepsilon e^{i\frac{t}{\varepsilon}\hat{H}_{\mathrm{ren}}(\sigma_0)}, \varepsilon\in(0,\bar{\varepsilon})\Big) = \Big\{\hat{\mathbf{E}}(t)_{\#}\mu\,,\ \mu\in\mathcal{M}\big(\varrho_\varepsilon,\varepsilon\in(0,\bar{\varepsilon})\big)\Big\}\ . \tag{103}$$

Furthermore, let $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,\bar\varepsilon)$ be a sequence such that $\lim_{k\to\infty}\varepsilon_k = 0$. Then the following statement is true:

$$\varrho_{\varepsilon_k} \to \mu \Leftrightarrow \left(\forall t\in\mathbb{R}\,,\; e^{-i\frac{t}{\varepsilon_k}\hat H_{\mathrm{ren}}(\sigma_0)}\varrho_{\varepsilon_k} e^{i\frac{t}{\varepsilon_k}\hat H_{\mathrm{ren}}(\sigma_0)} \to \hat{\mathbf{E}}(t)_{\#}\mu\right). \tag{104}$$

Proof. Thanks to the argument briefly sketched below, we no longer need Assumption (A''_ρ). Let $\chi\in C_0^\infty(\mathbb{R})$ such that $0\le\chi\le 1$, $\chi\equiv 1$ in a neighbourhood of 0 and $\chi_R(x)=\chi(\frac{x}{R})$. The approximation

$$\varrho_{\varepsilon,R} = \frac{\chi_R(\hat H_{\mathrm{ren}}(\sigma_0))\;\varrho_\varepsilon\;\chi_R(\hat H_{\mathrm{ren}}(\sigma_0))}{\mathrm{Tr}\left[\chi_R(\hat H_{\mathrm{ren}}(\sigma_0))\;\varrho_\varepsilon\;\chi_R(\hat H_{\mathrm{ren}}(\sigma_0))\right]}\,,$$

satisfies the Assumptions (A_0), (A'_ρ), (A''_ρ) and the property

$$||e^{-i\frac{t}{\varepsilon}\hat H_{\mathrm{ren}}(\sigma_0)}\,(\varrho_\varepsilon-\varrho_{\varepsilon,R})\,e^{i\frac{t}{\varepsilon}\hat H_{\mathrm{ren}}(\sigma_0)}||_{\mathcal{L}^1(\mathscr{H})} = ||\varrho_\varepsilon-\varrho_{\varepsilon,R}||_{\mathcal{L}^1(\mathcal{H})}\le\nu(R)\,,$$

where $\nu(R)$ is independent of ε and $\lim_{R\to\infty}\nu(R)=0$. The last claim follows by Assumption (A'_ρ), Theorem 2.10 and Definition 2.12. Up to extracting a sequence which a priori depends on R and t, we can suppose that $\mathscr{M}(\varrho_{\varepsilon_n,R}, n\in\mathbb{N})=\{\mu_{0,R}\}$, $\mathscr{M}(\varrho_{\varepsilon_n}, n\in\mathbb{N})=\{\mu_0\}$ and $\mathcal{M}(\varrho_{\varepsilon_n}(t), n\in\mathbb{N})=\{\mu_t\}$. In particular, applying Proposition 4.23 we obtain

$$\mathcal{M}(e^{-i\frac{t}{\varepsilon_n}\hat H_{\mathrm{ren}}(\sigma_0)}\varrho_{\varepsilon_n,R}e^{i\frac{t}{\varepsilon_n}\hat H_{\mathrm{ren}}(\sigma_0)}, n\in\mathbb{N}) = \left\{\hat{\mathbf{E}}(t)_{\#}\mu_{0,R}\right\}.$$

A general estimate proved in [10, Proposition 2.10] compares the total variation distance of Wigner (probability) measures with the trace distance of their associated quantum states. In our case, it implies

$$\int_{L^2\oplus L^2}|\mu_t-\hat{\mathbf{E}}(t)_{\#}\mu_{0,R}| \le \liminf_{n\to\infty}||e^{-i\frac{t}{\varepsilon_n}\hat H_{\mathrm{ren}}(\sigma_0)}\,(\varrho_{\varepsilon_n}-\varrho_{\varepsilon_n,R})\,e^{i\frac{t}{\varepsilon_n}\hat H_{\mathrm{ren}}(\sigma_0)}||_{\mathcal{L}^1(\mathcal{H})}\le\nu(R)\,,$$
$$\int_{L^2\oplus L^2}|\mu_0-\mu_{0,R}| \le \liminf_{n\to\infty}||\varrho_{\varepsilon_n}-\varrho_{\varepsilon_n,R}||_{\mathcal{L}^1(\mathcal{H})}\le\nu(R)\,,$$

where the left hand side denotes the total variation of the signed measures $\mu_t-\hat{\mathbf{E}}(t)_{\#}\mu_{0,R}$ and $\mu_0-\mu_{0,R}$ respectively. Hence by the triangle inequality, we obtain

$$\int_{L^2\oplus L^2}|\mu_t-\hat{\mathbf{E}}(t)_{\#}\mu_0| \le \int_{L^2\oplus L^2}|\mu_t-\hat{\mathbf{E}}(t)_{\#}\mu_{0,R}| + \int_{L^2\oplus L^2}|\mu_{0,R}-\mu_0|\le 2\nu(R)\,.$$

This proves that

$$\left\{\hat{\mathbf{E}}(t)_{\#}\mu_0\right\}\subset\mathcal{M}(e^{-i\frac{t}{\varepsilon_n}\hat H_{\mathrm{ren}}(\sigma_0)}\varrho_{\varepsilon_n}e^{i\frac{t}{\varepsilon_n}\hat H_{\mathrm{ren}}(\sigma_0)}, n\in\mathbb{N})\,,$$

By reversing time and utilizing the analogue inclusion above, we prove (104). ⊣

Proof of Theorem 1.1: Observe that using the definition of the renormalized dressed evolution $\varrho_\varepsilon(t)$ (Definition 4.3) and the definition of the "dressing dynamics" $\hat\varrho_\varepsilon(\theta)$ (Equation (100)), we obtain:

$$e^{-i\frac{t}{\varepsilon}H_{\mathrm{ren}}(\sigma_0)}\,\varrho_\varepsilon\,e^{i\frac{t}{\varepsilon}H_{\mathrm{ren}}(\sigma_0)} = e^{-\frac{i}{\varepsilon}T_\infty}e^{-i\frac{t}{\varepsilon}\hat H_{\mathrm{ren}}(\sigma_0)}e^{\frac{i}{\varepsilon}T_\infty}\,\varrho_\varepsilon\,e^{-\frac{i}{\varepsilon}T_\infty}e^{i\frac{t}{\varepsilon}\hat H_{\mathrm{ren}}(\sigma_0)}e^{\frac{i}{\varepsilon}T_\infty} = \Big(\big(\hat\varrho_\varepsilon(-1)\big)(t)\Big)^{\hat{}}(1)\,.$$

Let $(\varrho_\varepsilon)_{\varepsilon\in(0,\bar\varepsilon)}$ be a family of normal states in $\mathcal{H}$ that satisfies Assumptions (A_0) and (A_ρ). In addition, as usual, let $(\varepsilon_k)_{k\in\mathbb{N}}\subset(0,\bar\varepsilon)$ be a sequence such that $\lim_{k\to\infty}\varepsilon_k=0$. Then we can use Lemma 4.24, Proposition 4.25 and Theorem 4.26 to prove the following statement:

$$\varrho_{\varepsilon_k} \to \mu \Leftrightarrow \Bigl(\forall t \in \mathbb{R}\,,\ e^{-i\frac{t}{\varepsilon_k}H_{\mathrm{ren}}(\sigma_0)}\varrho_{\varepsilon_k}e^{i\frac{t}{\varepsilon_k}H_{\mathrm{ren}}(\sigma_0)} \to \mathbf{D}_{g_\infty}(1)_{\#}\hat{\mathbf{E}}(t)_{\#}\mathbf{D}_{g_\infty}(-1)_{\#}\mu$$
$$= [\mathbf{D}_{g_\infty}(1)\circ\hat{\mathbf{E}}(t)\circ\mathbf{D}_{g_\infty}(-1)]_{\#}\mu\Bigr)\,.$$

Therefore Theorem 1.1 is proved, since by Equation (64) of Theorem 3.16, $\mathbf{D}_{g_\infty}(1)\circ\hat{\mathbf{E}}(t)\circ\mathbf{D}_{g_\infty}(-1) = \mathbf{E}(t)$. To be more precise, we use the following chain of inferences:

$$\Bigl(\varrho_{\varepsilon_k} \to \mu\Bigr) \overset{\substack{\text{Lem. } 4.24\\ \text{Prop. } 4.25}}{\Longrightarrow} \Bigl(\forall\sigma_0 \in \mathbb{R}_+\,,\ \hat{\varrho}_{\varepsilon_k}(-1) \to \mathbf{D}_{g_\infty}(-1)_{\#}\mu \text{ and } \bigl(\hat{\varrho}_{\varepsilon_k}(-1)\bigr)_{k\in\mathbb{N}}$$
$$\text{satisfies Ass. } (A_0),\ (A'_\rho)\Bigr)$$
$$\overset{\substack{\text{Thm. } 4.26\\ \text{Lem. } 4.8}}{\Longrightarrow} \Bigl(\exists\sigma_0 \in \mathbb{R}_+\,,\ \forall t\in\mathbb{R}\,,\ \bigl(\hat{\varrho}_{\varepsilon_k}(-1)\bigr)(t) \to \hat{\mathbf{E}}(t)_{\#}\mathbf{D}_{g_\infty}(-1)_{\#}\mu$$
$$\text{and } \Bigl(\bigl(\hat{\varrho}_{\varepsilon_k}(-1)\bigr)(t)\Bigr)_{k\in\mathbb{N}} \text{ satisfies Ass. } (A_0),\ (A'_\rho)\Bigr)$$
$$\overset{\text{Prop. } 4.25}{\Longrightarrow} \Bigl(\forall t\in\mathbb{R}\,,\ \Bigl(\bigl(\hat{\varrho}_{\varepsilon_k}(-1)\bigr)(t)\Bigr)\hat{}\,(1) \to \mathbf{D}_{g_\infty}(1)_{\#}\hat{\mathbf{E}}(t)_{\#}\mathbf{D}_{g_\infty}(-1)_{\#}\mu\Bigr)$$
$$\overset{\text{Thm. } 3.16}{\Longrightarrow} \Bigl(\forall t\in\mathbb{R}\,,\ e^{-i\frac{t}{\varepsilon}H_{\mathrm{ren}}(\sigma_0)}\varrho_{\varepsilon_k}e^{i\frac{t}{\varepsilon}H_{\mathrm{ren}}(\sigma_0)} \to \mathbf{E}(t)_{\#}\mu\Bigr)\,.$$

The inference in the opposite sense is trivial.

As it has become evident with the above discussion, we do not prove Theorem 1.1 directly; and it would be very difficult to do so, due to the fact that we do not know the explicit form of the generator $H_{\mathrm{ren}}(\sigma_0)$ of the undressed dynamics. We know instead how the dressed generator $\hat{H}_{\mathrm{ren}}(\sigma_0)$ acts as a quadratic form, and that is sufficient to characterize its dynamics in the classical limit, and obtain the results of Theorem 4.26. The properties of the dressing transformation and of its classical counterpart are then crucial to translate the results on the dressed dynamics to the corresponding results on the undressed one.

Appendix A. Uniform higher-order estimate

We prove in this section a higher-order estimate that bounds the meson number operator N_2 by the dressed Hamiltonian $\hat{H}_\sigma^{(n)}$ uniformly with respect to the effective (semiclassical) parameter ε and the cut-off parameter σ. Such type of estimates rely on the pull-through formula and they are known for the $P(\varphi)_2$ model [76] and for the Nelson model [4]. However, since the dependence of the dressed Hamiltonian $\hat{H}_\sigma^{(n)}$ on ε is somewhat nontrivial, we briefly indicate in this appendix how to obtain an uniform estimate.

Lemma A.1. *For any $\varepsilon \in (0,\bar{\varepsilon})$ and any $\psi \in D(N_2) \subset \mathcal{H}$,*

$$\bigl\|N_2\psi\bigr\|^2 = \int_{\mathbb{R}^3}\bigl\|(N_2+\varepsilon)^{\frac{1}{2}}a(k)\psi\bigr\|^2\,dk\,.$$

Proof. Recall that N_2 and $a(k)$ depends in the parameter ε according to the notations of Subsection 1.1. Taking care of domain issues as in [4, Lemma 2.1] one proves

$$\left\|N_2\psi\right\|^2 = \langle N_2^{\frac{1}{2}}\psi, \int_{\mathbb{R}^3} a^*(k)a(k)\,dk\, N_2^{\frac{1}{2}}\psi\rangle = \int_{\mathbb{R}^3}\left\|a(k)N_2^{\frac{1}{2}}\psi\right\|^2 dk = \int_{\mathbb{R}^3}\left\|(N_2+\varepsilon)^{\frac{1}{2}}a(k)\psi\right\|^2 dk\,.$$

$\dashv$

Recall that the interaction term $\hat{H}_I(\sigma)^{(n)}$ is given by (13). A simple computation yields

$$[a(k), \hat{H}_I(\sigma)^{(n)}] = \varepsilon^2\Bigg[\sum_{j=1}^{n}\frac{1}{2\sqrt{(2\pi)^3}}\frac{\chi_\sigma(k)}{\sqrt{\omega(k)}}e^{-ik.x_j} + \frac{1}{M}\sum_{j=1}^{n} r_\sigma(k)e^{-ik\cdot x_j}a^*(r_\sigma e^{-ik\cdot x_j}) + r_\sigma(k)e^{-ik\cdot x_j}a(r_\sigma e^{-ik\cdot x_j}) - r_\sigma(k)e^{-ik\cdot x_j}D_{x_j}\Bigg].$$

Lemma A.2. *For any $\mathfrak{C}>0$ and $\sigma_0 \geq 2K(\mathfrak{C}+1+\bar{\varepsilon})$ there exist $c, b>0$ such that for any $\varepsilon\in(0,\bar{\varepsilon})$, $\sigma_0<\sigma\leq+\infty$ and $n\in\mathbb{N}$ such that $n\varepsilon\leq\mathfrak{C}$, we have*

$$\left\|(b+\varepsilon\omega(k)+\hat{H}_\sigma^{(n)})^{-\frac{1}{2}}[a(k),\hat{H}_I(\sigma)^{(n)}](b+\hat{H}_\sigma^{(n)})^{-\frac{1}{2}}\right\| \leq c\left(\left|\frac{\chi_\sigma(k)}{\sqrt{\omega(k)}}\right| + |r_\sigma(k)|\omega(k)^{-1/4}\right).$$

Proof. According to Proposition 2.9 and Theorem 2.10, $\hat{H}_I(\sigma)^{(n)}$ is $H_0^{(n)}$-form bounded with small bound that is uniform with respect to $\varepsilon\in(0,\bar{\varepsilon})$, $\sigma_0<\sigma\leq+\infty$ and $n\in\mathbb{N}$ such that $n\varepsilon\leq\mathfrak{C}$. Hence $(H_0^{(n)})^{\frac{1}{2}}(b+\hat{H}_\sigma^{(n)})^{-\frac{1}{2}}$ is uniformly bounded for some $b>0$. So it is enough to prove the claimed bound with $H_0^{(n)}$ instead of $\hat{H}_\sigma^{(n)}$. Now using similar estimates as in Lemma 2.6 and the fact that $\sqrt{\varepsilon\omega(k)}(b+\varepsilon\omega(k)+\hat{H}_\sigma^{(n)})^{-\frac{1}{2}}$ is uniformly bounded one correctly bounds all the terms of the commutator except the one with a^*. Remark that the commutator contains the power ε^2 that controls the sum over $1\leq j\leq n$ and the factor $1/\sqrt{\varepsilon\omega(k)}$. In order to bound the term with a^*, one uses the type of estimate in [4, Lemma 3.3 (ii)] with $s=1/2$. Remark that one gets an ε-dependent estimate from [4, Lemma 3.3 (ii)] by noticing that $\varepsilon^{1/4}(H_0^{(n)}+1)^{-1/4}(d\Gamma_1(\omega)+1)^{1/4}$ and $\varepsilon^{1/4}(N_2+1)^{-1/4}(d\Gamma_1(1)+1)^{1/4}$ are uniformly bounded[17] and that a^* contains $\sqrt{\varepsilon}$ that cancels the latter $\varepsilon^{-1/4}\cdot\varepsilon^{-1/4}$. $\dashv$

Let $\mathfrak{C}>0$ and $\sigma_0\geq 2K(\mathfrak{C}+1+\bar{\varepsilon})$ as in the above lemma. In particular $\hat{H}_\sigma^{(n)}$ is a self-adjoint operator for any $\varepsilon\in(0,\bar{\varepsilon})$, $\sigma_0<\sigma\leq+\infty$ and $n\in\mathbb{N}$ such that $n\varepsilon\leq\mathfrak{C}$.

Lemma A.3 (The pull-through formula)**.** *The following identity holds true for some $b<0$, any $\phi\in D(N_2^{\frac{1}{2}})\cap\mathcal{H}_n$ and k almost everywhere in $\mathbb{R}^3$,*

$$\begin{aligned} a(k)(b-\hat{H}_\sigma^{(n)})^{-1}\phi &= (b-\varepsilon\omega(k)-\hat{H}_\sigma^{(n)})^{-1}a(k)\phi \\ &\quad +(b-\varepsilon\omega(k)-\hat{H}_\sigma^{(n)})^{-1}\,[a(k),\hat{H}_I(\sigma)^{(n)}]\,(b-\hat{H}_\sigma^{(n)})^{-1}\phi\,. \end{aligned}$$

Proof. According to [4, Lemma 4.4] there exists $\psi\in(H_0^{(n)}+1)^{-1}D(N^{\frac{1}{2}})$ such that $\phi=(b-\hat{H}_\sigma^{(n)})\psi$ for some $b<0$. So the claimed formula is equivalent to

$$(b-\varepsilon\omega(k)-\hat{H}_\sigma^{(n)})a(k)\psi = a(k)(b-\hat{H}_\sigma^{(n)})\psi + [a(k),\hat{H}_I(\sigma)^{(n)}]\psi\,.$$

The latter identity follows by a simple computation. $\dashv$

[17] $d\Gamma_1(\cdot)$ is the ε-independent second quantization operator in [4]

Proposition A.4. *For any $\mathfrak{C} > 0$ and $\sigma_0 \geq 2K(\mathfrak{C} + 1 + \bar{\varepsilon})$ there exist $c, b > 0$ such that the operator $\hat{H}_\sigma^{(n)}$ is self-adjoint and the following bound holds true:*

$$\|N_2\psi\| \leq c\|(\hat{H}_\sigma^{(n)} + b)\psi\|, \quad \forall \psi \in D(\hat{H}_\sigma^{(n)}),$$

for any $\varepsilon \in (0, \bar{\varepsilon}), \sigma \in (\sigma_0, +\infty], n \in \mathbb{N}$ such that $n\varepsilon \leq \mathfrak{C}$.

Proof. The operator $\hat{H}_\sigma^{(n)}$ is uniformly bounded from below. So by choosing $b > 0$ large enough one can take $\psi = (-b - \hat{H}_\sigma^{(n)})^{-1}\phi$. Now it is enough to prove the estimate for $\phi \in (H_0^{(n)} + 1)^{-1/2} D(N_2^{\frac{1}{2}})$. Using Lemma A.1 and Lemma A.3,

$$\begin{aligned}
\|N_2\psi\|^2 &= \int_{\mathbb{R}^3} \|(N_2 + \varepsilon)^{\frac{1}{2}} a(k)(b + \hat{H}_\sigma^{(n)})^{-1}\phi\|^2 \, dk \\
&\leq 2\int_{\mathbb{R}^3} \|(N_2 + \varepsilon)^{\frac{1}{2}} (b + \varepsilon\omega(k) + \hat{H}_\sigma^{(n)})^{-1} a(k)\phi\|^2 \, dk \qquad (105) \\
&\quad +2\int_{\mathbb{R}^3} \|(N_2 + \varepsilon)^{\frac{1}{2}} (b + \varepsilon\omega(k) + \hat{H}_\sigma^{(n)})^{-1} [a(k), \hat{H}_I(\sigma)^{(n)}] \, (b + \hat{H}_\sigma^{(n)})^{-1}\phi\|^2 \, dk \,. \qquad (106)
\end{aligned}$$

Since $(N_2 + \varepsilon)^{\frac{1}{2}} (b + \varepsilon\omega(k) + \hat{H}_\sigma^{(n)})^{-1/2}$ is uniformly bounded, by Lemma A.2 one shows

$$(106) \leq c \int_{\mathbb{R}^3} |\tfrac{\chi_\sigma(k)}{\sqrt{\omega(k)}}| + |r_\sigma(k)|\omega(k)^{-1/4} \, dk \;\cdot\; \|(b + \hat{H}_\sigma^{(n)})^{-1/2}\phi\|^2$$

For simplicity we denote by c any constant. In the same way, one also shows

$$\begin{aligned}
(105) &\leq c \int_{\mathbb{R}^3} \|(b + \varepsilon\omega(k) + \hat{H}_\sigma^{(n)})^{-1/2} a(k)\phi\|^2 \, dk \\
&\leq c \int_{\mathbb{R}^3} \|(b + \varepsilon\omega(k) + H_0^{(n)})^{-1/2} a(k)\phi\|^2 \, dk = c\|N_2^{1/2}(b + H_0^{(n)})^{-1/2}\phi\|^2 \,.
\end{aligned}$$

The last equality follows by a similar argument as in the proof of Lemma A.1. Hence, one obtains

$$\begin{aligned}
\|N_2\psi\|^2 &\leq c\left(\|\phi\|^2 + \|(b + H_0^{(n)})^{-1/2}\phi\|^2\right) = c\left(\|(b + \hat{H}_\sigma^{(n)})\psi\|^2 + \|(b + H_\sigma^{(n)})^{1/2}\psi\|^2\right) \\
&\leq c\|(b + \hat{H}_\sigma^{(n)})\psi\|^2 \,.
\end{aligned}$$

The last inequality is a consequence of the uniform boundedness of the operator $(b + H_0^{(n)})^{-1/2}(b + \hat{H}_\sigma^{(n)})^{-1/2}$ with respect to ε, σ and $n \in \mathbb{N}$ such that $n\varepsilon \leq \mathfrak{C}$. $\dashv$

Appendix B. Probabilistic representation

For any open bounded interval I, we denote by Γ_I the space of all continuous curves from $\bar{I}$ into $(L^2 \oplus L^2, ||\cdot||_{L^2\oplus L^2})$ and define the following metric space

$$\mathfrak{X} = \left(L^2 \oplus L^2 \times \Gamma_I, ||\cdot||_{(L^2\oplus L^2, d_w)} + \sup_{t\in\bar{I}} ||\cdot||_{(L^2\oplus L^2, d_w)}\right) \qquad (107)$$

where the norm $||\cdot||_{(L^2\oplus L^2, d_w)}$ is associated to the distance introduced in (97). For each $t \in I$, we define the continuous evaluation map,

$$e_t : (x, \gamma) \in E \times \Gamma_I(E) \mapsto \gamma(t) \in E \,.$$

Consider the transport or Liouville equation,

$$\partial_t\mu_t + \nabla^T(v.\mu_t) = 0 \,,$$

understood in a weak sense as the integral equation,

$$\int_I \int_{L^2\oplus L^2} \partial_t \varphi(t,x) + \mathrm{Re}\langle v(t,x), \nabla\varphi(t,x)\rangle_{L^2\oplus L^2}\, d\mu_t(x)\, dt = 0, \quad \forall \varphi \in \mathcal{C}^\infty_{0,cyl}(I\times L^2\oplus L^2)\,. \tag{108}$$

The following result is an adaptation of [7, Propositon 4.1].

Proposition B.1. *Let $v : \mathbb{R}\times H^1\oplus\mathcal{F}H^{1/2} \to L^2\oplus L^2$ be a Borel vector field such that v is bounded on bounded sets. Let $t\in I \to \mu_t \in \mathfrak{P}(H^1\oplus\mathcal{F}H^{1/2})$ be a weakly narrowly continuous solution in $\mathfrak{P}(L^2\oplus L^2)$ of the Liouville equation (108) defined on an open bounded interval I with the following estimate satisfied,*

$$\int_I \int_{H^1\oplus\mathcal{F}H^{1/2}} \|v(t,x)\|_{L^2\oplus L^2}\, d\mu_t(x)\, dt < \infty\,.$$

Then there exists a Borel probability measure η, on the space $\mathfrak{X}$ given in (107), satisfying:

- (i) *η is concentrated on the set of $(x,\gamma)\in H^1\oplus\mathcal{F}H^{1/2}\times\Gamma_I$ such that $\gamma\in W^{1,1}(I, L^2\oplus L^2)$ and γ are solutions of the initial value problem $\dot\gamma(t) = v(t,\gamma(t))$ for a.e. $t\in I$ and $\gamma(t)\in H^1\oplus\mathcal{F}H^{1/2}$ for a.e. $t\in I$ with $\gamma(s) = x$ for some fixed $s\in I$.*
- (ii) *$\mu_t = (e_t)_\sharp\eta$ for any $t\in I$.*

Here $W^{1,1}(I, L^2\oplus L^2)$ is the Sobolev space of functions in $L^1(I, L^2\oplus L^2)$ with distributional first derivatives in $L^1(I, L^2\oplus L^2)$. In particular, functions in $W^{1,1}(I, L^2\oplus L^2)$ are absolutely continuous curves in $L^2\oplus L^2$.

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