Observer-based trajectory tracking for anaerobic digestion process

Khadidja Chaib Draa, Holger Voos, Marouane Alma, Ali Zemouche, Mohamed Darouach

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K. CHAIB DRAA, H. VOOS, M. ALMA, A. ZEMOUCHE and M. DAROUACH

***Abstract*— A novel control strategy is proposed for the Anaerobic digestion (AD) process to track a reference trajectory. This is motivated by the aim to control the produced biogas quality and quantity. The control scheme is composed from an exponential nonlinear observer which remedies for the lack of measurements and a feedback control which accounts for all the process dynamics. By using the LPV techniques, the Lyapunov analysis and the well-known Barbalat's lemma, global stability conditions are synthesized in the form of Linear Matrix Inequalities (LMIs). The feasibility of the obtained LMIs is enhanced due to the introduction of a non diagonal multiplier matrix coming from a convenient use of Young's inequality. Finally, the simulation results are provided to validate the effectiveness of the proposed control strategy in this note.**

***Index Terms*— Anaerobic digestion, LMI approach, Observer Design, Reference trajectory tracking.**

I. INTRODUCTION

AD is an energy-efficient and environmentally beneficial technology to convert organic waste into a useful energy such as biogas. The later is a mixture of gaseous, generally composed of $45-65\%$ methane, $36-41\%$ carbon dioxide, up to 17% nitrogen, $< 1\%$ oxygen, $32-169.1$ ppm hydrogen sulphide, and traces of other gases [1], which can be used in many domains and replace the use of fossil fuel. The more biogas contains methane the more it is energetic. Hence, its quality is quite important to control. Moreover, the biogas can be regarded as a solution to compensate fluctuations in energy production coming from the weather dependent technologies. Indeed, the produced biogas from the AD process can be either converted into electricity and heat directly after production, either stored and used when it is required. Actually, storing biogas in gas tanks costs much cheaper than storing electricity in electrical batteries. However, usually the storing capacities at BPs are limited and thus it is of big interest to control the biogas quantity to avoid additional costs in upgrading the storing capacities.

We may find in the literature different control strategies proposed for the AD processes. They, usually, differer depending on the model complexity, available measurements and the desired criteria (pollutant minimization, product maximisation or digester stabilisation). Among the designed controls for analytical two step models (acidogenesis-methanogenesis), we may cite the control of bicarbonate alkalinity concentration in the digester by the mean of an additional control input to the model in [2], [3] and [4]. In [2], the linearizing control was used to enhance the biogas quality while in [3] and [4] the input to output linearising control was used to stabilize the digester. Whether in the first or the second control strategies, the magnitude of the added input was assumed to be very small so that it could be excluded from the dynamics of the model state variables other than the alkalinity concentration. This assumption makes the control easy to design. Especially, the input to output linearising control, where this assumption relaxes the complexity of the nonlinear transformation of the system. However, even if the control becomes easier when neglecting the effect of the added input in the dynamics of the model, this is not very consistent.

Recently, another control strategy using the Model Predective Control (MPC) has been proposed in [5] to control the biogas production for a demand-driven electricity production. The idea is to optimize the plant feeding according to a fluctuating timetable of energy demand. The control was applied to a full scale research plant and has shown satisfactory results. However, although the satisfactory results, the analytical proof for the stability of the closed loop system is yet difficult to prove.

Combining the ideas from [2], [3], [4] and [5], where the alkalinity addition is used to stabilize the reactor and enhance the biogas quality, and the plant feeding is used to optimize the production, we will propose in our turn to control the system so it tracks an admissible reference. Therefore, we will propose a simple observer-based reference tracking control. We point out that in our design we will not neglect the effect of the additional input in the model dynamics. Moreover, we will integrate a state observer in the control strategy to cope with the luck of measurements in AD process. Indeed, it is known that the bacteria measurement is difficult and costly to process. Hence, difference software sensors have been proposed to cope with this issue. Among the designed observers in the literature, we may cite the the asymptotic observer [6] which is quite simple to design and does not require the knowledge of some specific nonlinear functions. However, it is very sensitive to model uncertainties and its convergence rate depends on the operational conditions. Hence, to enhance the convergence rate, the Kalman filter has been designed frequentely in the literature [7], [8], and has shown suitable results in different chemical applications, but unfortunately convergence of the estimation errors to zero is not guaranteed. We can cite also the high gain observer [9], [10] whose convergence rate is fast, however, its synthesis is complex and it is very sensitive to noise [11]. Therefore, in

Khadidja CHAIB DRAA and Holger VOOS are with the Interdisciplinary Centre for Security, Reliability and Trust (SnT)- Université du Luxembourg. khadidja.chaibdraa@uni.lu

Marouane ALMA and Mohamed DAROUACH are with the CRAN-CNRS, Université de Lorraine, IUT de Longwy.

Ali ZEMOUCHE is with University of Lorraine, CRAN CNRS UMR 7039.1, 54400 Cosnes et Romain, France, and with EPI INRIA DISCO, Laboratoire des Signaux et SystÂèmes, CNRS-Supélec, 9.1119.12 Gif-sur-Yvette, France.

this note, we will use the nonlinear observer proposed in [12] due to its systematic implementation and fast convergence.

The rest of this note is organised as follows. In Section II, we will present the AD model. Then, in Section III, we will pose the problem of observer-based reference tracking control. Further, in Section IV, we will give the stability conditions which ensure the exponential convergence to zero of the estimation error and the $\mathcal{H}_\infty$ asymptotic stability of the tracking error. In Section V, we will provide some simulation results to illustrate the effectiveness of the proposed control scheme. Finally, we will conclude this note in Section VI.

A. Notation and Preliminaries

The following notations and preliminaries will be used throughout this note:

- $(\star)$ is used for the blocks induced by symmetry;
- A^T represents the transposed matrix of A;
- $\mathbb{I}_r$ represents the identity matrix of dimension r;
- for a square matrix S, $S > 0$ ($S < 0$) means that this matrix is positive definite (negative definite);
- the set $Co(x, y) = \{\lambda x + (1-\lambda)y, 0 \leq \lambda \leq 1\}$ is the convex hull of $\{x, y\}$;
- $e_s(i) = \big(\underbrace{0, ..., 0, \overbrace{1}^{i \text{ th}}, 0, ..., 0}_{s \text{ components}}\big)^T \in \mathbb{R}^s, s \geq 1$ is a vector of the canonical basis of $\mathbb{R}^s$.

Lemma 1.1 (*a variant of Lipschitz reformulation):* Let $\varphi : \mathbb{R}^n \to \mathbb{R}^q$ a differentiable function on $\mathbb{R}^n$. Then, the following items are equivalent [13]:

- φ is a globally γ_φ-Lipschitz function;
- there exist finite and positive scalar constants a_{ij}, b_{ij} so that for all $x, y \in \mathbb{R}^n$ there exist $z_i \in Co(x, y), z_i \neq x, z_i \neq y$ and functions ψ_{ij}: $\mathbb{R}^n \to \mathbb{R}$ satisfying the following:

$$\varphi(x) - \varphi(y) = \sum_{i,j=1}^{q,n} \psi_{ij}(z_i)\mathcal{H}_{ij}\Big(x - y\Big) \tag{1}$$

$$a_{ij} \leq \psi_{ij}\Big(z_i\Big) \leq b_{ij}, \tag{2}$$

where

$$\psi_{ij}(z_i) = \frac{\partial \varphi_i}{\partial x_j}(z_i),\ \mathcal{H}_{ij} = e_q(i)e_n^T(j).$$

Notice that this lemma has been introduced in order to simplify the presentation of our design methodology. Indeed, for our technique, we will exploit (1)-(2) instead of a direct use of the Lipschitz property.

Lemma 1.2 (*[13]):* Let X and Y be two given matrices of appropriate dimensions. Then, for any symmetric positive definite matrix S of appropriate dimension, the following inequality holds:

$$X^TY + Y^TX \leq \frac{1}{2}\Big[X + SY\Big]^T S^{-1}\Big[X + SY\Big]. \tag{3}$$

This lemma will be very useful to enhance the feasibility of the LMI conditions.

II. Mathematical Model of the AD Process

AD modelling has a long track in the literature. Often, the designed models are driven by the application objectives, the available data and their reliability. In this note, being motivated by the control and observer design for the AD process, we will use the same model considered in [12]. Its structure reads

$$\left\{\begin{array}{lcl} \dot{x}_1 &=& -k_1\mu_1(x_1)x_2 + u_1S_{1in} - ux_1 \\ \dot{x}_2 &=& (\mu_1(x_1) - \alpha)ux_2 \\ \dot{x}_3 &=& k_2\mu_1(x_1)x_2 - k_3\mu_2(x_3)x_4 + u_1S_{2in} - ux_3 \\ \dot{x}_4 &=& (\mu_2(x_3) - \alpha)ux_4 \\ \dot{x}_5 &=& k_4\mu_1(x_1)x_2 + k_5\mu_2(x_3)x_4 + u_1C_{in} - \\ && ux_5 - q_c(x) \\ \dot{x}_6 &=& u_1Z_{in} + u_2Z_{ad} - ux_6 \end{array}\right. \tag{4}$$

$$\left\{\begin{array}{l} y_1 = q_c(x) \\ y_2 = [x_1, x_3, x_6]^T \end{array}\right. \tag{5}$$

with

$$\mu_1(x_1) = \overline{\mu}_1\frac{x_1}{x_1 + k_{s_1}}, \quad \mu_2(x_3) = \overline{\mu}_2\frac{x_3}{x_3 + k_{s2} + \frac{x_3^2}{k_{i2}}} \tag{6}$$

$$co_2 = x_5 + x_3 - x_6, \quad \phi = co_2 + K_HP_T + \frac{k_6}{k_{La}\mu_2(x_3)x_4} \tag{7}$$

$$q_c(x) = k_{La}[co_2 - K_HP_c(x)], \quad q_m(x) = k_6\mu_2(x_3)x_4 \tag{8}$$

$$P_c(x) = \frac{\phi - \sqrt{\phi^2 - 4K_HP_Tco_2}}{2K_H} \tag{9}$$

where x_1 (g/l) is the organic substrate concentration to be consumed by the acidogenic bacteria x_2 (g/l) for growth and production of volatile fatty acids x_3 $(\mathrm{mmol/l})$ (assumed to behave like pure acetate), x_4 (g/l) is the methanogenic bacteria concentration, x_5 $(\mathrm{mmol/l})$ represents the inorganic carbon concentration and x_6 $(\mathrm{mmol/l})$ the alkalinity concentration. The related fed concentration to the digester S_{1in}, S_{2in}, C_{in} and Z_{in} are supposed to be known and constant. The control inputs are $u_1 = \frac{F_{1in}}{v}$ $(1/day)$ and $u_2 = \frac{F_{2in}}{v}$ $(1/day)$, where F_{1in} is the waste feeding rate and F_{2in} is the input flow rate of the added alkalinity (Z_{ad}) to the digester. Since the later volume (v) is constant, then the output flow rate $u = u_1 + u_2$. The produced biogas is assumed to be composed of methane $q_m(x)$ and co_2 $q_c(x)$ gaseous. The later partial pressure is computed by $P_c(x)$. The rest of the used parameters in the model are defined in Table I.

III. Formulation of the Problem

In reference trajectory tracking problem (this is the case with most control design problems), the state of the system, $x(t)$, is generally not available for feedback. That is why often a state observer is required.

Among the designed observers for the AD processes, we will consider the nonlinear state observer proposed in [12] due to its systematic implementation and fast convergence. Thus, to keep the design usable and applicable for other nonlinear systems belonging to the same class of systems as

TABLE I

MODEL PARAMETERS

Acronyms	Definition	Units	Value
k_1	Yield for substrate (x_1) degradation	$g/(g$ of $x_2)$	42.1
k_2	Yield for VFA (x_3) production	mmol$/(g$ of $x_2)$	116.5
k_3	Yield for VFA consumption	mmol$/(g$ of $x_4)$	268
k_4	Yield for co_2 production	mmol$/g$	50.6
k_5	Yield for co_2 production	mmol$/g$	343.6
k_6	Yield for ch_4 production	mmol$/g$	453
$\overline{\mu}_1$	Maximum acidogenic bacteria (x_2) growth rate	1/day	1.25
$\overline{\mu}_2$	Maximum methanogenic bacteria (x_4) growth rate	1/day	0.74
k_{s_1}	Half saturation constant associated with x_1	g/l	7.1
k_{s2}	Half saturation constant associated with x_3	mmol$/l$	9.1.28
k_{i2}	Inhibition constant associated with x_3	mmol$/l$	256
k_b	Acidity constant of bicarbonate	mol$/l$	6.5 10^{-7}
K_H	Henry's constant	mmol$/(l.atm)$	27
P_T	Total preasure	atm	1.013
k_{La}	Liquid/gas transfer constant	1/day	19.1.8

the AD process (4)-(5) we will present the results in a general way. Hence, we rewrite the model (4)-(5) as the following

$$\left\{ \begin{array}{l} \dot{x} = A(\rho^u)x + G\gamma(x) + Bu \\ y = Cx \end{array} \right. \tag{10}$$

where the state vector $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^q$ and the linear output measurements $y \in \mathbb{R}^p$. The parameter $\rho \in \mathbb{R}^s$ is an $\mathcal{L}_\infty$ bounded and known parameter and the affine matrix $A(\rho)$ is expressed under the form

$$A(\rho) = A_0 + \sum_{j=1}^{s} \rho_j A_j \tag{11}$$

with

$$\rho_{\min} \leq \rho^u \leq \rho_{\max} \tag{12}$$

which means that the parameter ρ belongs to a bounded convex set for which the set of 2^s vertices can be defined by:

$$\mathbb{V}_\rho = \Big\{ \varrho \in \mathbb{R}^s : \varrho_j \in \{\rho_{j,\min}, \rho_{j,\max}\} \Big\}. \tag{13}$$

The matrices $A_i \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are constant. The nonlinear function $\gamma: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is assumed to be globally Lipschitz and can always be written under the detailed form:

$$G\gamma(x) = \sum_{i=1}^{m} G_i \gamma_i(\overbrace{H_i x}^{\vartheta_i}) \tag{14}$$

where $H_i \in \mathbb{R}^{n_i \times n}$ and G_i refers to the i^{th} column of the matrix G.

Remark 1: We refer the reader to Appendix I to see how to write the AD model (4)-(5) under the form (10) and how to obtain (14).

To estimate the unmeasurable sate variables of the model (10), we use the following observer scheme:

$$\dot{\hat{x}} = A(\rho^u)\hat{x} + \sum_{i=1}^{m} G_i \gamma_i(\hat{\vartheta}_i) + Bu + L(\rho^u)\Big(y - C\hat{x}\Big) \tag{15}$$

with

$$\hat{\vartheta}_i = H_i \hat{x} + \mathcal{K}_i(\rho)(y - C\hat{x}), \tag{16}$$

and

$$L(\rho) = L_0 + \sum_{j=1}^{s} \rho_j L_j, \;\; \mathcal{K}_i(\rho) = \mathcal{K}_{i0} + \sum_{j=1}^{s} \rho_j \mathcal{K}_{ij}. \tag{17}$$

where $\hat{x}$ is the estimate of x. The matrices $L_i \in \mathbb{R}^{n\times p}$ and $\mathcal{K}_{ij} \in \mathbb{R}^{n_i \times p}$ are the observer gains.

Since $\gamma(.)$ is globally Lipschitz, then from Lemma 1.1 there exist $z_i \in Co(\vartheta_i, \hat{\vartheta}_i)$, functions

$$\phi_{ij} \;:\; \mathbb{R}^{n_i} \longrightarrow \mathbb{R}$$

and constants a_{ij}, b_{ij}, so that

$$G(\gamma(x) - \gamma(\hat{x})) = \sum_{i,j=1}^{m,n_i} \phi_{ij}(z_i)\mathcal{H}_{ij}\Big(\vartheta_i - \hat{\vartheta}_i\Big) \tag{18}$$

and

$$a_{ij} \leq \phi_{ij}(z_i) \leq b_{ij}, \tag{19}$$

where

$$\phi_{ij}(z_i) = \frac{\partial \gamma_i}{\partial \vartheta_i^j}(z_i), \;\; \mathcal{H}_{ij} = G_i e_{n_i}(j).$$

For shortness, we set $\phi_{ij} \triangleq \phi_{ij}(z_i)$. Without loss of generality, we assume that $a_{ij} = 0$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$. For more details about this, we refer the reader to [14].

Since $\vartheta_i - \hat{\vartheta}_i = (H_i - \mathcal{K}_i(\rho)C)\, e$, then the dynamic equation of the estimation error can be obtained as

$$\dot{e} = \left(\mathbb{A}_L(\rho) + \sum_{i,j=1}^{m,n_i} \phi_{ij}\mathcal{H}_{ij}\Big(H_i - \mathcal{K}_i(\rho)C\Big) \right) e \tag{20}$$

with

$$\mathbb{A}_L = A(\rho) - L(\rho)C \tag{21}$$

Now, the objective is to use the observer (15) in the control design for tracking the trajectory of the following desired system

$$\dot{x}_d = A(\rho^{u_d})x_d + G\gamma(x_d) + Bu_d \tag{22}$$

That is the tracking control is given by

$$u = -K(\rho^{u_d})(\hat{x} - x_d) + u_d \tag{23}$$

where

$$K(\rho^{u_d}) = K_0 + \sum_{j=1}^{s} \rho_j^{u_d} K_j \tag{24}$$

Let us define the tracking error by

$$\tilde{x} = x - x_d \tag{25}$$

Its dynamic can be easily obtained as

$$\begin{aligned}\dot{\tilde{x}} = &\left(A(\rho^{u_d}) - BK(\rho^{u_d}) + \sum_{i,j=1}^{m,n_i} \varphi_{ij}(t)\mathcal{H}_{ij}H_i\right)\tilde{x}\\ &+BK(\rho^{u_d})e + \underbrace{\Big(A(\rho^{u}) - A(\rho^{u_d})\Big)x_d}_{\omega(t)}\end{aligned} \tag{26}$$

where $\varphi_{ij} = \frac{\partial \gamma_{ci}}{\partial x_j}(\nu_i)$, with $\nu_i \in Co(x, x_d)$ and

$$\underline{\varphi}_{ij} \leq \varphi_{ij} \leq \overline{\varphi}_{ij}. \tag{27}$$

The aim consists in finding the controller and observer gain matrices, so that the tracking error $\tilde{x}$ satisfies the following $\mathcal{H}_\infty$ criterion

$$\|\tilde{x}\|_{\mathcal{L}_2^n} \leq \sqrt{\mu\|\omega\|^2_{\mathcal{L}_2^n} + \nu\|\tilde{x}_0\|^2} \tag{28}$$

where $\mu > 0$ is the gain from ω to $\tilde{x}$, and $\nu > 0$ is to be determined. In the next section, we will provide the stability conditions to satisfy our objective.

IV. Stability analysis

In this section we present a kind of separation principle for nonlinear systems. Since the dynamics (20) do not depend on the reference tracking error $\tilde{x}$ and the functions ϕ_{ij} are bounded, then we can study the convergence of the estimation error e separately, and will use it in the dynamics of the tracking error as a bounded disturbance exponentially converging towards zero. The following theorem provides the synthesis conditions expressed in term of LMIs.

Theorem 4.1: The closed-loop system (26) is $\mathcal{H}_\infty$ asymptotically stabilizable by the observer-based feedback (23), if there exist symmetric positive definite matrices $\mathbb{P}$, $\mathbb{Q}$, $\mathcal{Z}_{ij}$, $\mathcal{S}_{ij}$, $i,j = 1,\ldots,n$, and matrices $\mathbb{Y}_i, \mathbb{X}_i, \mathcal{X}_{ij}$ of appropriate dimensions such that for given positive scalar β, the LMI conditions (29) are fulfilled and the convex optimization problem (38) is solvable.

1) LMIs for the observer gains:

$$\begin{bmatrix} \mathbb{A}\Big(\mathbb{Q},\mathbb{X},\varrho\Big) + \beta\mathbb{Q} & \overbrace{\begin{bmatrix}\Pi_1 & \ldots & \Pi_m\end{bmatrix}}^{\Pi} \\ (\star) & -\Lambda\mathbb{S}\end{bmatrix} \leq 0, \quad \forall \varrho \in \mathbb{V}_\rho \tag{29}$$

with

$$\begin{aligned}\mathbb{A}\Big(\mathbb{Q},\mathbb{X},\varrho\Big) = &A_0^T\mathbb{Q} + \mathbb{Q}A_0 - C^T\mathbb{X}_0 - \mathbb{X}_0^T C +\\ &\sum_{j=1}^{s}\varrho_j\Big(A_j^T\mathbb{Q} + \mathbb{Q}A_j - C^T\mathbb{X}_j - \mathbb{X}_j^T C\Big)\end{aligned} \tag{30}$$

and

$$\Pi_i = \Big[\mathcal{M}_i^1\Big(\mathbb{Q},\mathcal{S}_{i1}\Big)\ldots\ \mathcal{M}_i^{n_i}\Big(\mathbb{Q},\mathcal{S}_{in_i}\Big)\Big] \tag{31}$$

$$\mathcal{M}_i^j\Big(\mathbb{Q},\mathcal{S}_{ij}\Big) = \mathbb{Q}\mathcal{H}_{ij} + H_i^T\mathcal{S}_{ij} - C^\top\mathcal{X}_{ij} \tag{32}$$

$$\Lambda = \text{block-diag}\Big(\Lambda_1, ..., \Lambda_m\Big) \tag{33}$$

$$\Lambda_i = \text{block-diag}\left(\frac{2}{\bar{\varphi}_{i1}}\mathbb{I}_{n_i},\ldots,\frac{2}{\bar{\varphi}_{in_i}}\mathbb{I}_{n_i}\right) \tag{34}$$

$$\mathbb{S} = \text{block-diag}\Big(\mathbb{S}_1,\ldots,\mathbb{S}_m\Big) \tag{35}$$

$$\mathbb{S}_i = \text{block-diag}\Big(\mathcal{S}_{i1},\ldots,\mathcal{S}_{in_i}\Big) \tag{36}$$

The observer gains L_j and $\mathcal{K}_{ij}$ are computed as

$$L_j = \mathbb{Q}^{-1}\mathbb{X}_j^T, \quad \mathcal{K}_{ij} = \mathcal{S}_{ij}^{-1}\mathcal{X}_{ij}^T. \tag{37}$$

2) Optimization problem for the controller gains:

$$\min(\mu) \quad \text{subject to} \quad (39) \tag{38}$$

$$\begin{bmatrix} \Theta & \overbrace{\begin{bmatrix}\Sigma_1 & \ldots & \Sigma_m\end{bmatrix}}^{\Sigma} \\ (\star) & -\Lambda\mathbb{Z}\end{bmatrix} \leq 0, \ \forall \varrho \in \mathbb{V}_\rho \tag{39}$$

with

$$\Theta = \begin{bmatrix}\Theta_{11} & \begin{bmatrix}\mathbb{P}\\0\end{bmatrix}\\ (\star) & -\mathbb{I}_n\end{bmatrix}, \quad \Theta_{11} = \begin{bmatrix}\mathbb{A}\Big(\mathbb{P},\mathbb{Y},\varrho\Big) & \mathbb{I}_n\\ (\star) & -\mu\mathbb{I}_n\end{bmatrix}$$

$$\begin{aligned}\mathbb{A}\Big(\mathbb{P},\mathbb{Y},\varrho\Big) \ = \ &\mathbb{P}A_0^T + A_0\mathbb{P} - \mathbb{Y}_0B^\top - B\mathbb{Y}_0^\top\\ &+\sum_{j=1}^{s}\varrho_j\Big(\mathbb{P}A_j^T + A_0\mathbb{P} - \mathbb{Y}_jB^\top - B\mathbb{Y}_j^\top\Big)\end{aligned} \tag{40}$$

$$\Sigma_i = \Big[\mathcal{N}_i^1\Big(\mathbb{P},\mathbb{Y},\mathcal{Z}_{i1}\Big)\ldots\mathcal{N}_i^{n_i}\Big(\mathbb{P},\mathbb{Y}_i,\mathcal{Z}_{in_i}\Big)\Big] \tag{41}$$

$$\mathcal{N}_i^j\Big(\mathbb{P},\mathbb{Y},\mathcal{Z}_{ij}\Big) = \begin{bmatrix}\mathbb{P}H_i^T\\0\\0\end{bmatrix} + \begin{bmatrix}\mathcal{H}_{ij}\\0\\0\end{bmatrix}\mathbb{Z}_{ij} \tag{42}$$

$$\Lambda = \text{block-diag}\Big(\Lambda_1, ..., \Lambda_m\Big) \tag{43}$$

$$\Lambda_i = \text{block-diag}\left(\frac{2}{\bar{\varphi}_{i1}}\mathbb{I}_{n_i},\ldots,\frac{2}{\bar{\varphi}_{in_i}}\mathbb{I}_{n_i}\right) \tag{44}$$

$$\mathbb{Z} = \text{block-diag}\Big(\mathbb{Z}_1,\ldots,\mathbb{Z}_m\Big) \tag{45}$$

$$\mathbb{Z}_i = \text{block-diag}\Big(\mathcal{Z}_{i1},\ldots,\mathcal{Z}_{in_i}\Big) \tag{46}$$

Thus, the $\mathcal{H}_\infty$ criterion (28) is satisfied with the tracking controller gains

$$K_j = \mathbb{Y}_j^\top \mathbb{P}^{-1}, \; j = 1, \dots, s.$$

The disturbance attenuation level μ is the minimum value returned by (38), and $\nu = \lambda_{\max}(\mathbb{P})$.

Proof: The proof is based on the use of the Barbalat's lemma since the dynamics of the augmented system with the state $\begin{bmatrix}\tilde{x}\\ e\end{bmatrix}$ has a triangular structure. For more details, we refer the reader to [15]. For the observer convergence we use the Lyapunov function $V_1(e)$, and for the tracking error we use $V_2(\tilde{x})$ and the $\mathcal{H}_\infty$ criterion (28), where

$$V_1(e) = e^\top \mathbb{Q} e, \; V_2(\tilde{x}) = \tilde{x}^\top \mathbb{P}^{-1} \tilde{x} \tag{47}$$

■

It is useless to reproduce all steps of the convergence analysis. We refer the reader to [12] to see how the LMIs (29) ensure the exponential convergence of the estimation error towards zero.

V. Simulation Results

In this section, we will present an numerical example where the plant designer targets to enhance the produced biogas quality. To run the simulation, we use the parameter values given in Table I, and we take $S_{1in} = 16 \; g/l$, $S_{2in} = 170$ mmol/l, $C_{in} = 76.15$ mmol/l, $Z_{in} = 200$ mmol/l, $Z_{ad} = 700$ mmol/l. We initialize the system and the observer by $x(0) = [1.8, 0.4, 12, 0.7, 109.1.15, 55]^T$ and $\hat{x}_0 = [1.8, 0.6, 12, 0.3, 45, 55]^T$, respectively. The objective in this example is to track the desired reference given by $x_d = [1.9.1572, 0.6058, 5.4, 1.389.13, 242.8, 240.3413]^T$ and $u_d = [0.49.166, 0.0436]^T$, which corresponds to an enhanced quality of biogas at steady state. In order to solve the LMI conditions given by Theorem 4.1, we put $\rho_{min} = 0.1$ (1/day) and $\rho_{max} = 0.8$ (1/day). After solving the LMI conditions (29) and the optimization problem (38), which have been found feasible by using the LMI Toolbox of Matlab, we have obtained the results depicted in Figures 1-9. As it can be seen from these figures, although the large initial estimation error, the observer is converging to the simulated state vector of the system and the closed loop system tracks the desired reference trajectory. Moreover, the behaviour of the controller remains smooth and very acceptable.

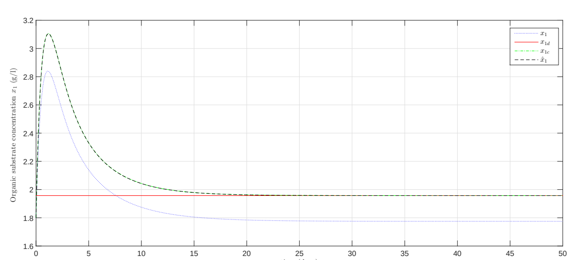

Fig. 1. Organic substrate concentration x_1 (g/l).

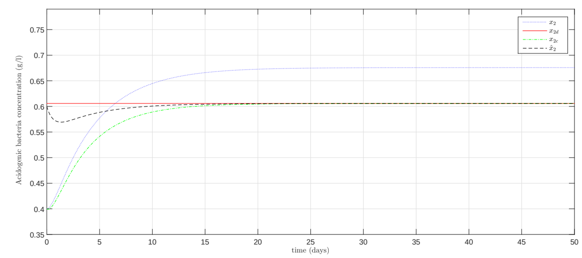

Fig. 2. Acidogenic bacteria concentration (g/l).

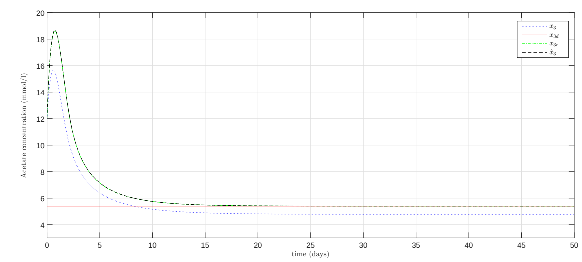

Fig. 3. Acetate concentration (mmol/l).

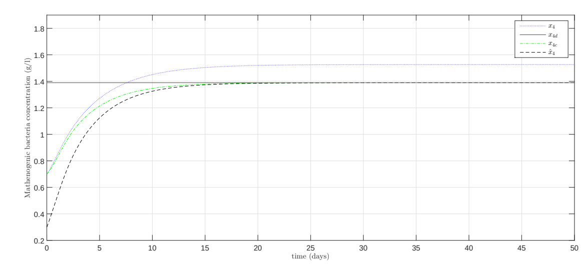

Fig. 4. Mathenogenic bacteria concentration (g/l).

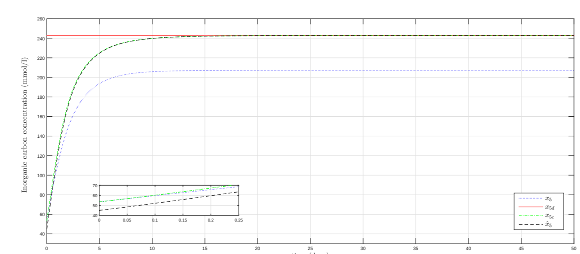

Fig. 5. Inorganic carbon concentration (mmol/l).

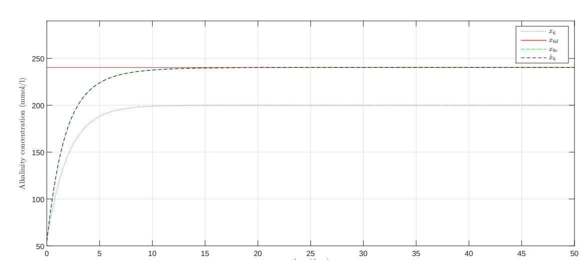

Fig. 6. Alkalinity concentration (mmol/l).

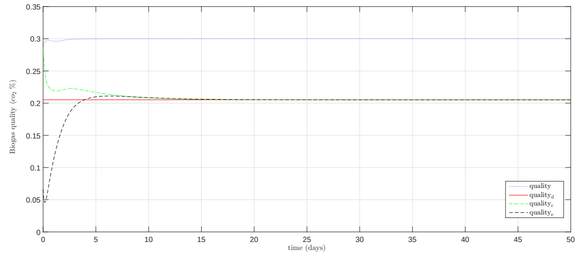

Fig. 7. Biogas quality ($co_2\%$).

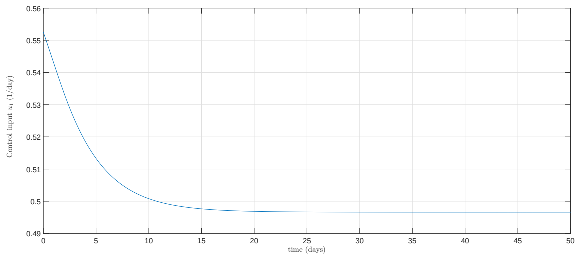

Fig. 8. Control input u_1 (1/day).

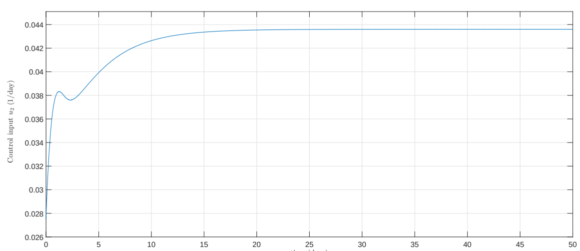

Fig. 9. Control input u_2 (1/day).

VI. Conclusions

In this note, we have designed an observer-based reference tracking control for the AD process. The control scheme is composed from an exponential nonlinear observer and a feedback control. The stability conditions have been provided in the form of enhanced end easily tractable LMIs due to the use of an adequate reformulation of the Young's inequality and Lipschitz property. In order to extend the use of our technique and make it applicable for other systems belonging to the same class of systems as the AD process, we have presented the results in a general way. Moreover, we have provided a numerical simulation to illustrate the effectiveness of the proposed approach. In view of the satisfactory results, we target in the near future to extend the design methodology for saturation constraints on the control inputs.

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Appendix I

The system (4)-(5) can be easily written under the form (10) using the following parameters

$$\rho = u, \quad A_0 = 0, \quad A_1 = -\text{block-diag}(1, \alpha, 1, \alpha, 1, 1)$$

$$G = \begin{bmatrix} -k_1 & 1 & k_2 & 0 & k_4 & 0 \\ 0 & 0 & -k_3 & 1 & k_5 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}^T$$

$$\gamma(x) = \begin{bmatrix} \mu_1(x_1)x_2, & \mu_2(x_3)x_4, & q_c(x) \end{bmatrix}^T$$

$$B = \begin{bmatrix} S_{1in} & 0 & S_{2in} & 0 & C_{in} & Z_{in} \\ 0 & 0 & 0 & 0 & 0 & Z_{ad} \end{bmatrix}^T$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$