A METHOD WITH PENALIZED RIGHT HAND SIDE FOR THE NUMERICAL RESOLUTION OF THE CURL CURL EQUATION

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ABSTRACT. A new method is proposed to solve numerically the curl curl equation by means of edge finite elements. In a first step, a compatible right hand side that belongs exactly to the range of the singular curl curl matrix is computed by penalization. In a second step, a Conjugate Gradient solver computes a discretely gauged solution of the curl curl equation. Convergence results are proven both in terms of the penalization parameter and the mesh size. Numerical simulations in two dimensions corroborate the theoretical results.

1. INTRODUCTION

This paper deals with the numerical resolution of the curl curl equation in a bounded domain $\Omega \subset \mathbb{R}^3$ with a perfectly conducting boundary condition on $\Gamma := \partial\Omega$. We assume that Ω is a simply connected Lipschitz domain with connected boundary. For a given field $\mathbf{f}$, we consider the following problem: find $\mathbf{u}$ such that

$$\operatorname{curl}\operatorname{curl}\mathbf{u} = \mathbf{f} \text{ in } \Omega, \tag{1a}$$

$$\mathbf{u} \times \mathbf{n} = 0 \text{ on } \Gamma, \tag{1b}$$

where $\mathbf{n}$ is the outward unit normal to Γ and $\mathbf{f}$ is assumed to be divergence free.

When solving the curl curl problem, one is confronted to two main difficulties. The first one is that the vector solution is not unique but defined up to the gradient of a scalar function and thus a gauge condition has to be added. The second difficulty is to ensure the compatibility of the right hand side (r.h.s.) of the curl curl equation. This implies that the source term has to be divergence free. From a discrete point of view, this means that the r.h.s. of the linear system (corresponding to the discretization of (1)) belongs to the range of the curl curl matrix. A solution proposed in [3, 9] is to express the field $\mathbf{f}$ by the curl of a source field Ψ and to project this source field onto the space of the *curl* of first order edge elements. Here, we adopt an approach without computing the vector potential Ψ explicitly. The novelty in the present paper is to combine the resolution of the curl curl problem with a penalization method which computes the gauged component of the source term $\mathbf{f}$ according to the Helmholtz-decomposition of the latter. The penalized r.h.s. can be shown to satisfy the compatibility condition and belongs to the (larger) space of first order edge elements. The computation of the divergence free component of a vector field *via* penalization has been addressed in [1] for fields with vanishing normal trace. Here, we focus on perfect conducting

(M. Darbas, J. Heleine) LAMFA UMR CNRS 7352, UNIVERSITÉ DE PICARDIE JULES VERNE, AMIENS, FRANCE

(S. Lohrengel) LMR EA 4535, UNIVERSITÉ DE REIMS-CHAMPAGNE ARDENNE, MOULIN DE LA HOUSSE, REIMS, FRANCE

E-mail addresses: marion.darbas@u-picardie.fr, jeremy.heleine@u-picardie.fr, stephanie.lohrengel@univ-reims.fr.

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boundary conditions. We give a full mathematical and numerical description of the method that includes convergence proofs with respect to the penalty parameter as well as to the mesh parameter. To the best of our knowledge, this point has not been dealt with in the existing literature.

2. Setting of the problem and presentation of the method

On Ω, we introduce the vector space related to the curl-operator with perfect conducting boundary condition, $\mathcal{H}_0(\mathrm{curl};\Omega) = \left\{\mathbf{f} \in \mathbf{L}^2(\Omega) \,\middle|\, \mathrm{curl}\,\mathbf{f} \in \mathbf{L}^2(\Omega),\ \mathbf{f}\times\mathbf{n} = 0 \text{ on } \Gamma\right\}$. Here, and below, bold-faced symbols refer to spaces of vector fields, e.g. $\mathbf{L}^2(\Omega) \stackrel{\mathrm{def}}{=} \left(L^2(\Omega)\right)^3$. We further introduce the sub-space of $\mathcal{H}_0(\mathrm{curl};\Omega)$ of divergence-free vector fields, $\mathbf{X}_0 = \{\mathbf{u} \in \mathcal{H}_0(\mathrm{curl};\Omega) \mid \mathrm{div}\,\mathbf{u} = 0 \text{ in } \Omega\}$. On $\mathbf{X}_0$, the variational formulation of problem (1) reads

$$(\mathcal{P}(\mathrm{curl})) \qquad \begin{cases} \text{Find } \mathbf{u} \in \mathbf{X}_0 \text{ such that} \\ (\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{w}) = (\mathbf{f}, \mathbf{w})\ \forall \mathbf{w} \in \mathbf{X}_0. \end{cases}$$

We can state the following properties for the space $\mathbf{X}_0$ (see e.g. [4]):

Proposition 1. *Let Ω be a Lipschitz domain. Then, the imbedding of $\mathbf{X}_0$ into $\mathbf{L}^2(\Omega)$ is compact and the following Poincaré-like inequality holds true:*

$$\exists C_P > 0 :\ \forall \mathbf{v} \in \mathbf{X}_0,\ \|\mathbf{v}\|_{0,\Omega} \le C_P \left(\|\,\mathrm{curl}\,\mathbf{v}\|_{0,\Omega}\right). \tag{2}$$

It follows from inequality (2) that $(\mathcal{P}(\mathrm{curl}))$ has a unique solution in $\mathbf{X}_0$ for any $\mathbf{f} \in \mathbf{L}^2(\Omega)$ that is divergence free. However, the discretization of problem $(\mathcal{P}(\mathrm{curl}))$ by finite elements is challenging since no conforming elements are available for the space $\mathbf{X}_0$. We therefore consider edge elements which are conforming in $\mathcal{H}(\mathrm{curl};\Omega)$. To this end, let $\mathcal{T}_h$ be a mesh of Ω and denote by $\mathbf{V}_h \subset \mathcal{H}(\mathrm{curl};\Omega)$ the discretization space built of finite edge elements of the first order (see [8]). Let $\mathbf{X}_h = \mathbf{V}_h \cap \mathcal{H}_0(\mathrm{curl};\Omega)$ be the space of discrete fields that satisfy the boundary condition $\mathbf{v}_h \times \mathbf{n} = 0$ on Γ. Assume that the source field $\mathbf{f}$ is approximated by a field $\mathbf{f}_h$ which could be, for example, the interpolate of $\mathbf{f}$. Then, the discrete problem reads

$$(\mathcal{P}_h(\mathrm{curl})) \qquad \begin{cases} \text{Find } \mathbf{u}_h \in \mathbf{X}_h \text{ such that} \\ (\mathrm{curl}\,\mathbf{u}_h, \mathrm{curl}\,\mathbf{w}_h) = (\mathbf{f}_h, \mathbf{w}_h)\ \forall \mathbf{w}_h \in \mathbf{X}_h. \end{cases}$$

In matrix form, we get

$$\mathbb{K}U = \mathbb{M}F \tag{3}$$

where the matrix $\mathbb{K}$ of the above linear system is related to the bilinear form $(\mathrm{curl}\,\cdot, \mathrm{curl}\,\cdot)$, and the mass matrix $\mathbb{M}$ corresponds to the canonical scalar product in $L^2(\Omega)$. In particular, $\mathbb{K}$ is singular and (3) admits a solution if and only if the r.h.s. $\mathbb{M}F$ belongs exactly to the image of $\mathbb{K}$. This compatibility condition is in general not satisfied even if the source term $\mathbf{f}$ is divergence free. Indeed, $\mathbf{f}_h$ is obtained by interpolation or numerical integration of $\mathbf{f}$, and does not necessarily fulfill a discrete gauge condition. We are going to propose a method to ensure this condition and to solve (1) numerically.

It has been stated in [3, 9] that the r.h.s. of $(\mathcal{P}(\mathrm{curl}))$ should be computed numerically by the identity $(\mathbf{f}, \mathbf{w}) = (\mathrm{curl}\,\Psi, \mathbf{w}) = (\Psi, \mathrm{curl}\,\mathbf{w})$ where Ψ is the vector potential of the field $\mathbf{f}$. Discretization by means of edge finite elements leads to a system with a new right hand side $(\Psi, \mathrm{curl}\,\mathbf{w}_h)$ that belongs to the image of the matrix $\mathbb{K}$, and the curl curl operator is implicitly gauged by the CG solver according to the results in [9]. The analytic computation of the vector potential Ψ can be achieved for some particular geometries. In general, however, Ψ has to be computed numerically. In [2], this is for example achieved via the projection of $\mathbf{f}$ onto the space $\operatorname{curl} \mathbf{X}_h$ which is of polynomial order 0. Here, we propose to compute an appropriate r.h.s. corresponding to an approximation of $\mathbf{f}$ in $\mathbf{X}_h$ through a penalization method [1]. Our method consists in two steps:

Step 1: Solve a penalized problem

$$\varepsilon \mathbb{M} F^\varepsilon + \mathbb{K} F^\varepsilon = \mathbb{K} F \tag{4}$$

with a penalization parameter $\varepsilon > 0$, using a direct or iterative solver.

Step 2: Solve the linear system

$$\mathbb{K} U^\varepsilon = \mathbb{M} F^\varepsilon \tag{5}$$

with the CG solver.

We deduce from (4) that the r.h.s. satisfies the property

$$\mathbb{M} F^\varepsilon = \frac{1}{\varepsilon} \mathbb{K}(F - F^\varepsilon) \in \operatorname{Im}(\mathbb{K}),$$

and thus belongs exactly to the range of $\mathbb{K}$. The vector F^ε is associated with a discrete vector field $\mathbf{f}_h^\varepsilon$ that converges in $\mathcal{H}(\mathrm{curl}; \Omega)$ to the source field $\mathbf{f}$ provided the penalization parameter ε and the mesh size h are chosen in an appropriate way in the Step 1. Then, the vector U^ε yields an approximation $\mathbf{u}_h^\varepsilon$ of the solution $\mathbf{u}$ of problem $(\mathcal{P}(\mathrm{curl}))$. Furthermore, any iterate of the CG Algorithm satisfies a discrete gauge condition provided this condition is satisfied by the initial vector and the right hand side [9].

Remark 1. We assume that Ω is filled with a material with constant magnetic permeability $\mu = 1$. All the following results can be extended to the case of a non-constant permeability.

3. Approximation of the divergence free vector potential

In this section, we are concerned with the approximation of the divergence free component of $\mathbf{f}$, denoted by $\mathbf{f}_\psi$. We do not assume that the field $\mathbf{f}$ is divergence free since this condition is in general not exactly fulfilled by the numerical representation of $\mathbf{f}$. Hence, all the following results hold true whether or not the condition $\operatorname{div} \mathbf{f} = 0$ is satisfied. Classical results yield the Helmholtz-decomposition of $\mathbf{f}$: there is a unique vector field $\mathbf{f}_\psi \in \mathbf{X}_0$ such that $\mathbf{f} = \mathbf{f}_\psi + \mathbf{f}_\phi$ where $\mathbf{f}_\psi = \operatorname{curl} \Psi$ with $\Psi \in \mathcal{H}(\mathrm{curl}; \Omega)$ such that $\operatorname{div} \Psi = 0$ in Ω and $\Psi \cdot \mathbf{n} = 0$ on Γ, whereas $\mathbf{f}_\phi = \nabla \phi$ with $\phi \in H_0^1(\Omega)$. Obviously, we have $\operatorname{curl} \mathbf{f}_\psi = \operatorname{curl} \mathbf{f}$, i.e. $\mathbf{f}_\psi$ realizes the projection of $\mathbf{f}$ onto the subspace of divergence free fields. Consider then the penalized problem

$$(\mathcal{P}^\varepsilon) \qquad \begin{cases} \text{Find } \mathbf{f}_\psi^\varepsilon \in \mathbf{X}_0 \text{ such that} \\ \varepsilon(\mathbf{f}_\psi^\varepsilon, \mathbf{w}) + (\operatorname{curl} \mathbf{f}_\psi^\varepsilon, \operatorname{curl} \mathbf{w}) = (\operatorname{curl} \mathbf{f}, \operatorname{curl} \mathbf{w}) \ \forall \mathbf{w} \in \mathbf{X}_0, \end{cases}$$

where $\varepsilon > 0$ is the penalization parameter. The bilinear form $a_\varepsilon(\cdot,\cdot) = \varepsilon(\cdot,\cdot) + (\operatorname{curl} \cdot, \operatorname{curl} \cdot)$ is still coercive on $\mathbf{X}_0$ with a coercivity constant independent from ε according to (2) and since the fields in $\mathbf{X}_0$ are divergence free. Hence, $(\mathcal{P}^\varepsilon)$ has a unique solution.

An error estimate for the penalized problem is given in the following theorem. A similar result for fields in $\mathcal{H}(\mathrm{curl}; \Omega)$ has been stated in [1]. Here, we give in addition the full proof.

Theorem 2. *Assume that* $\mathbf{f} \in \mathcal{H}_0(\mathrm{curl}; \Omega)$. *For* $\varepsilon > 0$, *denote by* $\mathbf{f}_\psi^\varepsilon \in \mathbf{X}_0$ *the solution of* $(\mathcal{P}^\varepsilon)$. *There is a constant* c *independent from* ε *such that*

$$\|\mathbf{f}_\psi - \mathbf{f}_\psi^\varepsilon\|_{0,\Omega} + \|\operatorname{curl}(\mathbf{f}_\psi - \mathbf{f}_\psi^\varepsilon)\|_{0,\Omega} \le c\varepsilon \|\mathbf{f}_\psi\|_{0,\Omega} \ \forall \varepsilon > 0, \tag{6}$$

where $\mathbf{f}_\psi \in \mathbf{X}_0$ *is the unique solution of the following variational problem*

$$(\mathcal{P}^0) \qquad \begin{cases} \text{Find } \mathbf{f}_\psi \in \mathbf{X}_0 \text{ such that} \\ (\operatorname{curl} \mathbf{f}_\psi, \operatorname{curl} \mathbf{w}) = (\operatorname{curl} \mathbf{f}, \operatorname{curl} \mathbf{w}) \ \forall \mathbf{w} \in \mathbf{X}_0. \end{cases}$$

Proof. First, taking $\mathbf{w} = \mathbf{f}_\psi^\varepsilon$ in $(\mathcal{P}^\varepsilon)$, we get

$$\| \operatorname{curl} \mathbf{f}_\psi^\varepsilon \|_{0,\Omega} \leq \| \operatorname{curl} \mathbf{f} \|_{0,\Omega}$$

and it follows from inequality (2) that

$$\| \mathbf{f}_\psi^\varepsilon \|_{0,\Omega} \leq C_P \| \operatorname{curl} \mathbf{f}^\varepsilon \|_{0,\Omega} \leq C_P \| \operatorname{curl} \mathbf{f} \|_{0,\Omega}.$$

Hence, the sequence $(\mathbf{f}_\psi^\varepsilon)_{\varepsilon>0}$ is bounded in $\mathbf{X}_0$ and there is a subsequence, still denoted by $(\mathbf{f}_\psi^\varepsilon)_\varepsilon$, and a field $\mathbf{f}^* \in \mathbf{X}_0$ such that $\mathbf{f}_\psi^\varepsilon$ converges weakly to $\mathbf{f}^*$ in $\mathbf{X}_0$. Taking the limit in $(\mathcal{P}^\varepsilon)$ as $\varepsilon \to 0$ yields $\mathbf{f}^* = \mathbf{f}_\psi$ due to the uniqueness of the solution of problem $(\mathcal{P}^0)$ in $\mathbf{X}_0$. We get the strong convergence of the sequence $(\mathbf{f}_\psi^\varepsilon)_\varepsilon$ to $\mathbf{f}_\psi$ in $\mathbf{L}^2(\Omega)$ since the embedding of $\mathbf{X}_0$ in $\mathbf{L}^2(\Omega)$ is compact. Next, we see from the definition of problems $(\mathcal{P}^0)$ and $(\mathcal{P}^\varepsilon)$ that

$$(\operatorname{curl}(\mathbf{f}_\psi - \mathbf{f}_\psi^\varepsilon), \operatorname{curl} \mathbf{w}) = \varepsilon(\mathbf{f}_\psi^\varepsilon, \mathbf{w}) \ \forall \mathbf{w} \in \mathbf{X}_0.$$

Then, taking $\mathbf{w} = \mathbf{f}_\psi - \mathbf{f}_\psi^\varepsilon$ in the above equation, we get with the help of (2)

$$\| \operatorname{curl}(\mathbf{f}_\psi - \mathbf{f}_\psi^\varepsilon) \|_{0,\Omega} \leq C_P \varepsilon \| \mathbf{f}_\psi^\varepsilon \|_{0,\Omega}. \tag{7}$$

Since $\mathbf{f}_\psi^\varepsilon$ converges strongly to $\mathbf{f}_\psi$ in $\mathbf{L}^2(\Omega)$, the above inequality implies that $\operatorname{curl}(\mathbf{f}_\psi - \mathbf{f}_\psi^\varepsilon)$ tends to zero in $\mathbf{L}^2(\Omega)$. Hence the sequence $(\mathbf{f}_\psi^\varepsilon)_\varepsilon$ converges strongly in $\mathbf{X}_0$ to $\mathbf{f}_\psi$ and estimate (6) follows from (7) and inequality (2). □

Consider now the discretization of problem $(\mathcal{P}^\varepsilon)$ by edge elements of order 1. With the notations in Section 2, the discrete penalized problem is given by

$$(\mathcal{P}_h^\varepsilon) \qquad \begin{cases} \text{Find } \mathbf{f}_{\psi,h}^\varepsilon \in \mathbf{X}_h \text{ such that} \\ \varepsilon(\mathbf{f}_{\psi,h}^\varepsilon, \mathbf{w}_h) + (\operatorname{curl} \mathbf{f}_{\psi,h}^\varepsilon, \operatorname{curl} \mathbf{w}_h) = (\operatorname{curl} \mathbf{f}, \operatorname{curl} \mathbf{w}_h) \ \forall \mathbf{w}_h \in \mathbf{X}_h. \end{cases}$$

The bilinear form $a_\varepsilon(\cdot,\cdot)$ is still coercive on $\mathbf{X}_h$ and $(\mathcal{P}_h^\varepsilon)$ has a unique solution. However, since $\mathbf{X}_h$ is not included in $\mathbf{X}_0$, the coercivity constant now depends on ε. More precisely, we have

$$a_\varepsilon(\mathbf{w}_h, \mathbf{w}_h) \geq \varepsilon \| \mathbf{w}_h \|^2_{\mathcal{H}(\operatorname{curl};\Omega)} \ \forall \mathbf{w}_h \in \mathbf{X}_h,$$

which yields the following abstract error estimate using classical arguments of Galerkin theory:

$$\| \mathbf{f}_\psi^\varepsilon - \mathbf{f}_{\psi,h}^\varepsilon \|_{\mathcal{H}(\operatorname{curl};\Omega)} \leq \frac{1}{\varepsilon} \inf_{\mathbf{w}_h \in \mathbf{X}_h} \| \mathbf{f}_\psi^\varepsilon - \mathbf{w}_h \|_{\mathcal{H}(\operatorname{curl};\Omega)}. \tag{8}$$

The next Theorem follows from standard error estimates (see [7]) for the first order edge elements and estimate (6) for the error between $\mathbf{f}_\psi$ and the solution of the penalized problem $(\mathcal{P}^\varepsilon)$.

Theorem 3. *Assume that the field* $\mathbf{f}$ *belongs to* $\mathcal{H}_0(\operatorname{curl};\Omega)$*. Let* $\mathbf{f}_\psi \in \mathbf{X}_0$ *and* $\mathbf{f}_{\psi,h}^\varepsilon \in \mathbf{X}_h$ *denote the respective solutions of problems* $(\mathcal{P}^0)$ *and* $(\mathcal{P}_h^\varepsilon)$*. Assume that* $\mathbf{f}_\psi$ *belongs to* $\mathbf{H}^1(\operatorname{curl};\Omega) = \{ \mathbf{f} \in \mathbf{H}^1(\Omega) \mid \operatorname{curl} \mathbf{f} \in \mathbf{H}^1(\Omega) \}$*. Then, the following error estimate holds true:*

$$\| \mathbf{f}_\psi - \mathbf{f}_{\psi,h}^\varepsilon \|_{\mathcal{H}(\operatorname{curl};\Omega)} \leq \left(C_1 \varepsilon + C_2 \frac{h}{\varepsilon} \right) \| \mathbf{f}_\psi \|_{\mathbf{H}^1(\operatorname{curl};\Omega)}. \tag{9}$$

From estimate (9) we see that ε and h should be chosen such that $\varepsilon = \mathcal{O}(\sqrt{h})$ in order to get a global convergence rate of $h^{1/2}$. In view of realistic applications, this prevents from chosing ε too small in order to avoid heavy computational costs.

4. Back to the curl curl problem and numerical results

Let $\mathbf{f}$ be the source field of problem (1) which we assume to belong to $\mathbf{X}_0$. Step 2 of our method is the matrix formulation of the following discrete problem:

$$(\mathcal{P}_h^\varepsilon(\text{curl})) \qquad \begin{cases} \text{Find } \mathbf{u}_h^\varepsilon \in \mathbf{X}_h \text{ such that} \\ (\operatorname{curl} \mathbf{u}_h^\varepsilon, \operatorname{curl} \mathbf{v}_h) = (\mathbf{f}_{\psi,h}^\varepsilon, \mathbf{v}_h) \ \forall \mathbf{v}_h \in \mathbf{X}_h, \end{cases}$$

where $\mathbf{f}_{\psi,h}^\varepsilon \in \mathbf{X}_h$ is the solution of the discrete penalized problem $(\mathcal{P}_h^\varepsilon)$.

Using Theorem 3, we can state the following error estimate: there is $h_0 > 0$, such that for all $0 < h < h_0$,

$$\|\mathbf{u} - \mathbf{u}_h^\varepsilon\|_{\mathcal{H}(\text{curl};\Omega)} \leq \left(C_1 \varepsilon + C_2 \frac{h}{\varepsilon} \right) \|\mathbf{f}\|_{\mathbf{H}^1(\text{curl};\Omega)} + C_3 h \|\mathbf{u}\|_{\mathbf{H}^1(\text{curl};\Omega)}. \tag{10}$$

where the constants $C_1, C_2, C_3 > 0$ are independent of h, $\mathbf{u}$ and $\mathbf{u}_h^\varepsilon$. The proof is based on the ideas of [7] and assumes that $\mathbf{u}_h^\varepsilon$ is discrete divergence free, i.e. $(\mathbf{u}_h^\varepsilon, \nabla \xi_h) = 0 \ \forall \xi_h \in S_h$, where $S_h = \left\{ \xi_h \in H_0^1(\Omega \,\middle|\, \xi_{h|K} \in \mathbb{P}_1(K) \ \forall K \in \mathcal{T}_h \right\}$ is the discretization space of scalar P1-Lagrange elements.

We illustrate now the efficiency of the method with penalized r.h.s. for the resolution of the curl curl equation. We consider a uniform mesh of size h of $\Omega = [-1,1]^2$. Edge elements of order 1 are used to discretize the problem. We use Simpson's rule to approximate the line integrals. We consider the field $\boldsymbol{u}(x,y) = (-\pi \cos(\pi x)\sin(\pi y),\ \pi \sin(\pi x)\cos(\pi y))^t$ as the exact solution of (1). Let $\mathbf{f} = 2\pi^2 \mathbf{u}$ the corresponding source term. We simulate some numerical perturbations of the vector F of the degrees of freedom of $\mathbf{f}$ by adding a small random term. We compute $F_{\boldsymbol{\eta}} = F(1 + \boldsymbol{\eta})$ where the coefficients of $\boldsymbol{\eta}$ are equally distributed random numbers between -10^{-2} and 10^{-2}.

First, we compute the discrete solution of the curl curl problem with this perturbed source term through a standard discretization by edge elements without any special treatment of the r.h.s.. To this end, we solve the linear system (3) using the CG algorithm with a tolerance of 10^{-13}. The results are summarized in Table 1 where the approximation of $\mathbf{u}$ is denoted by $\mathbf{u}_h$. We report the error in L^2-norm and $\mathcal{H}(\text{curl})$-seminorm as well as the number of iterations needed by the CG method to reach a minimal residual. We observe that the errors are about 10^{-1} independently from the mesh size. Moreover, the CG method stagnated for any tested mesh size. This agrees with the observations in [9] for a r.h.s. that is not compatible with the matrix of the linear system.

Table 1. Errors in the approximation of the solution of the curl curl problem over h (without penalization).

h	$\|\mathbf{u} - \mathbf{u}_h\|_0$	$\|\operatorname{curl}(\mathbf{u} - \mathbf{u}_h)\|_0$	nb. iter.	residual
1/32	2.59e-01	1.16e-01	86	1.37e-02
1/64	2.74e-01	1.00e-01	176	2.00e-02
1/128	2.70e-01	9.35e-02	349	2.83e-02
1/256	2.68e-01	9.15e-02	692	3.99e-02

Then, we tested the first step of our method with a penalization parameter of $\varepsilon = \sqrt{h}$ and observed the predicted convergence rate of 0.5 for the error (cf. Table 2).

Table 2. Errors in the approximation of the divergence free field over h with $\varepsilon = \sqrt{h}$.

h	$\|\mathbf{f}_\psi - \mathbf{f}^\varepsilon_{\psi,h}\|_0$	$\|\operatorname{curl}(\mathbf{f}_\psi - \mathbf{f}^\varepsilon_{\psi,h})\|_0$	τ_0	τ_{curl}
1/32	2.09e-02	9.29e-02	·	·
1/64	1.49e-02	6.62e-02	0.49	0.49
1/128	1.06e-02	4.70e-02	0.49	0.49
1/256	7.49e-03	3.33e-02	0.50	0.50

Finally, we applied the two steps of the method with penalized r.h.s.. We compute the divergence free component of the perturbed source field by solving the penalized problem $(\mathcal{P}^\varepsilon_h)$ with $\varepsilon = 10^{-5}$ and use the result as r.h.s. of the curl curl problem that is solved by a CG solver with a tolerance of 10^{-13} (cf. Table 3). This test shows global numerical rates of 2 (resp. 3/2) in the L^2-norm (resp. in the $\|\operatorname{curl}(\cdot)\|_0$-seminorm). Notice that the CG solver does not converge with the desired tolerance of $\eta = 10^{-13}$, but yields a residual between 10^{-6} and 10^{-8} corresponding roughly speaking to the order η/ε.

Notice that the condition number of the penalized matrix $\varepsilon\mathbb{M} + \mathbb{K}$ behaves like $\mathcal{O}(h^{-2}\varepsilon^{-1})$. In the present 2D study, this does not prevent the convergence of the CG-algorithm in a reasonable number of iterations. Nevertheless, a possible solution is to use an ILU-preconditioning in order to speed up the convergence.

5. Concluding remarks

We have presented a method for the numerical resolution of the curl curl equation (1) in the context of edge element discretization. The approach pays careful attention to the discretization of the source term. It consists in two steps. First, we solve a penalized problem which computes a discrete divergence free component of the source term $\mathbf{f}$. This provides a compatible r.h.s. for the linear system corresponding to the discretization of the curl curl problem. Then, the associated linear system can be solved by the CG algorithm. We have proved convergence results for the method with respect to both the penalty parameter and the mesh size. These results indicate how to choose the parameters of the method.

Two-dimensional numerical results corroborate the theoretical convergence rates and attest the efficiency of this method with penalized r.h.s.. A more complete numerical study in complex configurations is the aim of ongoing work.

The particular application that we have in mind is the numerical computation of a boundary control by the Hilbert Uniqueness Method (H.U.M.) [6] for the exact boundary controllability

Table 3. Errors in the approximation of the solution of the curl curl problem over h with $\varepsilon = 10^{-5}$.

h	$\|\mathbf{u} - \mathbf{u}_h\|_0$	$\|\operatorname{curl}(\mathbf{u} - \mathbf{u}_h)\|_0$	nb. iter.	residual	τ_0	τ_{curl}
1/32	1.98e-02	5.61e-02	207	2.84e-08	·	·
1/64	5.29e-03	1.65e-02	393	1.14e-07	1.91	1.76
1/128	1.53e-03	5.86e-03	665	6.08e-07	1.79	1.50
1/256	2.86e-04	2.36e-03	973	3.53e-06	2.42	1.32

of the second-order Maxwell equations. More precisely, a Bi-Grid preconditioned Conjugate Gradient is applied in order to inverse the H.U.M. operator where the computation of the residual at each iteration requires the resolution of a curl curl equation [5]. This application has motivated our research, but the proposed numerical method could be employed in many other ones such as magnetic resonance imaging or insulating materials.

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