Categorified cyclic operads

Pierre-Louis Curien, Jovana Obradovic

Categorified cyclic operads

Pierre-Louis Curien[1] and Jovana Obradović[2]
IRIF, Université Paris Diderot and πr^2 team, Inria, France
[1]curien@pps.univ-paris-diderot.fr [2]jovana@pps.univ-paris-diderot.fr

Abstract

The purpose of this paper is to establish a notion of categorified cyclic operad for set-based cyclic operads with symmetries, based on individual composition operations. The categorifications we introduce are obtained by replacing sets (of operations of the same arity) with categories, by relaxing certain defining axioms (like associativity and commutativity) to isomorphisms, while leaving the equivariance strict, and by formulating coherence conditions for these isomorphisms. The coherence theorem that we prove has the form "all diagrams of canonical isomorphisms commute". For entries-only categorified cyclic operads, our proof is of syntactic nature and relies on the coherence of categorified non-symmetric operads established by Došen and Petrić. We prove the coherence of exchangeable-output categorified cyclic operads by "lifting to the categorified setting" the equivalence between entries-only and exchangeable-output cyclic operads, set up by the second author.

Introduction

Categorical structures whose axioms are replaced by some kind of "weakened" (or categorified) versions of the same axioms have been of great interest in higher-dimensional category theory. Examples of structures obtained by categorification begin with bicategories of Bénabou [B67], in which the usual associativity and unit laws for composition of morphisms

$$(f \circ g) \circ h = f \circ (g \circ h), \qquad 1_A \circ f = f \qquad \text{and} \qquad f \circ 1_B = f$$

are replaced by the existence of coherent 2-isomorphisms

$$\beta : (f \circ g) \circ h \to f \circ (g \circ h), \qquad i_l : 1_A \circ f \to f \qquad \text{and} \qquad i_r : f \circ 1_B \to f.$$

Bicategories are just a first step towards various propositions for the definition of a weak n-category, which is currently a subject of active investigation. Closely related are monoidal categories of Mac Lane [ML98] (i.e. bicategories with a single object), which are categorifications of monoids, and which themselves admit various weakenings (promonoidal categories, lax monoidal categories, multicategories). For the purposes of higher-dimensional category theory and homotopy theory, categorification recently also emerged in operad theory, where at least three definitions of *categorified operads* have been proposed. In [DS01], Day and Street define *pseudo-operads* by categorifying the original "monoidal" definition of operads of Kelly [K05], which led to an algebraic, "one-line" characterisation of the form: *a pseudo-operad is a pseudo-monoid in a certain monoidal 2-category.* In [DP15], Došen and Petrić introduce the notion of *weak Cat-operad* by categorifying the componential definition of non-symmetric operads (i.e. the definition based on individual composition operations $\circ_i$), which led them to an equational axiomatic definition, in the style of Mac Lane's definition of a monoidal category. In [DV15], Dehling and Vallette, through curved Koszul duality theory, obtain *higher homotopy (symmetric) operads*, for which the equivariance is also relaxed.

In this paper, we propose categorifications of the two principal ways to apprehend the componential notion of *cyclic operad* with symmetries. With respect to the aforementioned notions of categorified operads, the style of our definitions corresponds to the one of [DP15], except that we also consider the action of the symmetric group. Yet, our categorified cyclic operads are not exactly *cyclic operads up to the first level of homotopy* in the language of [DV15], as we keep equivariance strict.

In the original approach of Getzler and Kapranov [GK95, Theorem 2.2], cyclic operads are seen as enrichments of operads with simultaneous composition, determined by adding to the action of permuting the inputs of an operation an action of interchanging its output with one of the inputs, in a way compatible with operadic composition. In [M08, Proposition 42], Markl gave an adaptation of the definition of Getzler and Kapranov, by considering underlying operads with partial composition. Both of these definitions are *skeletal*, meaning that the labeling of inputs of operations comes from the *skeleton* $\mathbf{\Sigma}$ of the category **Bij** of finite sets and bijections. The non-skeletal variant of Markl's definition, obtained by passing from $\mathbf{\Sigma}$ to **Bij** and redeveloping appropriatelly the remaining of the structure, has been given in [O17, Definition 3.16]. We suggestively refer to these three definitions as the *exchangeable-output* definitions of cyclic operads. The fact that two operations of a cyclic operads can be composed along inputs that "used to be outputs" and outputs that "used to be inputs" led to another point of view on cyclic operads, in which an operation, instead of having inputs and an (exchangeable) output, now has only "entries", and can be composed with another operation along any of them. Such *entries-only* definitions are [O17, Definition 3.2] (unital) and [M16, Definition 48] (non-unital). Both of these definitions are naturally non-skeletal, as they involve a commutativity axiom which is itself based on the commutativity of the union of two disjoint sets.

The categorified cyclic operads that we introduce are obtained by categorifying the entries-only definition [O17, Definition 3.2] and the exchangeable-output definition [O17, Definition 3.16] of set-based cyclic operads. For the sake of simplicity, we do not consider units in neither of these definitions. Our process of categorification, like the one of [DP15], is the most common one: we replace sets (of operations of the same arity) with categories, obtaining in this way the intermediate notion of cyclic operad enriched over **Cat**, followed by relaxing certain defining axioms of cyclic operads from equalities to isomorphisms, and exhibiting the conditions which make these isomorphisms coherent. In particular, the coherence theorem has the form "all diagrams made of canonical isomorphisms commute".

Concretely, for entries-only cyclic operads, the associativity and commutativity axioms

$$(f\,{}_x\circ_{\underline{x}}\,g)\,{}_y\circ_{\underline{y}}\,h = f\,{}_x\circ_{\underline{x}}\,(g\,{}_y\circ_{\underline{y}}\,h) \quad \text{and} \quad f\,{}_x\circ_y\,g = g\,{}_y\circ_x\,f$$

become the *associator* and *commutator* isomorphims, with instances

$$\beta^{x,\underline{x};y,\underline{y}}_{f,g,h} : (f\,{}_x\circ_{\underline{x}}\,g)\,{}_y\circ_{\underline{y}}\,h \to f\,{}_x\circ_{\underline{x}}\,(g\,{}_y\circ_{\underline{y}}\,h) \quad \text{and} \quad \gamma^{x,y}_{f,g} : f\,{}_x\circ_y\,g \to g\,{}_y\circ_x\,f,$$

respectively.

At first glance, thanks to the (non-skeletal) equivariance axiom which "distributes" the action of the symmetric group from the composite of two operations to operations themselves, the coherence of the obtained notion seems easily reducible to the coherence of symmetric monoidal categories of Mac Lane (see [ML98, Section XI.1]): all diagrams made of instances of associator and commutator are required to commute. However, in the setting of cyclic operads, where the existence of operations is restricted, these instances do not exist for all possible indices, as opposed to the framework of symmetric monoidal categories. As a consequence, the coherence conditions that Mac Lane established for symmetric monoidal categories do not solve the coherence problem of categorified entries-only cyclic operads. In particular, the hexagon of Mac Lane is *not well-defined* in the setting of categorified entries-only cyclic operads. The coherence conditions that we do take from Mac Lane are the pentagon and the requirement that the commutator isomorphism is involutive. However, we need much more than this in order to ensure coherence. Borrowing the terminology from [DP15], we need two more *mixed* coherence conditions (i.e. coherence conditions that involve both associator and commutaor), a hexagon (which is *not* the hexagon of Mac Lane) and a decagon, as well as three more conditions which deal with the action of the symmetric group on *morphisms* of categories of operations of the same arity.

The approach we take to treat the coherence problem is of syntactic, term-rewriting spirit, as in [ML98] and [DP15], and relies on the coherence result of [DP15]. The proof of the coherence theorem consists of three faithful reductions, each restricting the coherence problem to a smaller class of diagrams, in order to finally reach diagrams that correspond exactly to diagrams of canonical isomorphisms of categorified non-symmetric skeletal operads, i.e. weak Cat-operads of [DP15]. Intuitively speaking, the first reduction excludes the action of the symmetric group, the second (and the most important) one removes "cyclicity", and the last one replaces non-skeletality with skeletality.

For exchangeable-output cyclic operads, the two associativity axioms of the underlying operad $\mathcal{O}$ become the *sequential associator* and *parallel associator* isomorphisms, with instances

$$\beta^{x,y}_{f,g,h} : (f \circ_x g) \circ_y h \to f \circ_x (g \circ_y h) \quad \text{and} \quad \theta^{x,y}_{f,g,h} : (f \circ_x g) \circ_y h \to (f \circ_y h) \circ_x g,$$

respectively. Therefore, the operadic part of the obtained structure is the non-skeletal and symmetric counterpart of a weak Cat-operad of [DP15]. However, in order to carry over the *equivalence between the entries-only and exchangeable-output cyclic operads*, set up in [O17, Theorem 2], to the categorified setting (and, therefore, obtain, in the appropriate sense, the *correct* notion of categorified exchangeable-output cyclic operads), an axiom of the extra structure (accounting for the input-output exchange) must additionally be weakened. This leads to a third isomorphism, caled the *exchange*, whose instances are

$$\alpha^{z,x;v}_{f,g} : D_z(f \circ_x g) \to D_{zv}(g) \circ_v D_{xz}(f),$$

where $D_z(X) : \mathcal{O}(X) \to \mathcal{O}(X)$ is the endofunctor that "exchanges the input $z \in X$ with the output", and $D_{zy}(X) : \mathcal{O}(X) \to \mathcal{O}(X \backslash \{z\} + \{y\})$ is the functor that "exchanges the input $z \in X$ with the output and then renames it to y". We establish the coherence of this notion by "lifting" the proof of [O17, Theorem 2], thanks to the coherence of categorified entries-only cyclic operads.

The non-skeletal notion of exchangeable-output categorified cyclic operad described above can be straightforwardly coerced to a skeletal notion. In this way, a categorification of [M08, Proposition 42] is obtained. The coherence of the latter notion follows by "lifting" to the categorified setting the equivalence between non-skeletal and skeletal operads, established in [MSS02, Theorem 1.61], and extending it to the corresponding structures of categorified cyclic operads. We shall provide details for the proof of this theorem, which was omitted in [MSS02], and leave the categorification part as an exercise.

Related to the coherence of skeletal exchangeable-output categorified cyclic operads, the skeletality requirement, combined with the presence of symmetries, causes an interesting issue, pointed to us by Petrić, which arises if one tries to give a coherence proof by means of rewriting. Namely, as opposed to non-skeletal equivariance, for skeletal equivariance *it is not possible* to "distribute" the action of the symmetric group from the composite of two operations to operations themselves. This makes the exclusion of symmetries (i.e. the first reduction mentioned earlier), at the very least, problematic. Therefore, as far as we can tell, the proof of skeletal coherence requires the transition to the non-skeletal framework (i.e. the equivalence of [MSS02, Theorem 1.61]), which shows that, when it comes to coherence with symmetries, the choice of non-skeletal framework is no longer a matter of convenience, but a matter of necessity. We end this work by illustrating this issue, and by pointing out certain other merits of the non-skeletal operadic framework.

Layout. The paper is organised as follows. In Section 1, we recall the entries-only definition [O17, Definition 3.2] and the exchangeable-output definition [O17, Definition 3.16]. In Section 2, we introduce our definition of categorified entries-only cyclic operads. We examine their "operadic" properties, essential for reducing the coherence problem to the coherence of weak Cat-operads. The largest part of the section will be devoted to the proof of the coherence theorem. Section 3 deals with the exchangeable-output categorified cyclic operads. We give a proof of the equivalence between the (non-skeletal) exchangeable-output and the entries-only categorified cyclic operads, which establishes the coherence of the former notion. We also give a proof of the equivalence between skeletal and non-skeletal operads, which makes the core of the coherence of skeletal exchangeable-output cyclic operads. We finish this section with a comment on the benefits of the non-skeletal operadic framework.

Notation and conventions. *About finite sets and bijections.* In this paper, union will always be *ordinary* union of *already* disjoint sets. For disjoint finite sets X and Y, $X+Y$ shall stand for the union of X and Y. For a bijection $\sigma : X' \to X$ and $Y \subseteq X$, we shall denote with $\sigma|^Y$ the corestriction of σ on $\sigma^{-1}(Y)$. If $\sigma(x') = x$, we shall denote with $\sigma[y/x]$ the bijection defined in the same way as σ, except that the pair $(x', x) \in \sigma$ is replaced with (y, y). If $\tau : Y' \to Y$ is a bijection such that $X' \cap Y' = X \cap Y = \emptyset$, then $\sigma + \tau : X' + Y' \to X + Y$ denotes the bijection defined as σ on X' and as τ on Y'. If $\kappa : X\backslash\{x\} + \{x'\} \to X$ is identity on $X\backslash\{x\}$ and $\kappa(x') = x$, we say that κ renames x to x' (notice the contravariant nature of this convention). If a bijection $\kappa : X \to X$ renames x to y and y to x, we say that it exchanges x and y.

About cyclic operads. This paper is about cyclic operads without units. For a functor $\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{C}$, a bijection $\sigma : Y \to X$ and an object f of $\mathcal{C}(X)$, we write f^σ for $\mathcal{C}(\sigma)(f)$. We shall use latin letters for operations of a categorified cyclic operad, and greek letters for morphisms between them.

1 Cyclic operads

This section is a reminder on the two componential definitions of cyclic operads with symmetries. These are the definitions whose categorifications we introduce in the following two sections. From the opposite point of view, these definitions are decategorifications of the appropriate definitions from the following two sections.

1.1 The entries-only definition

We recall below [O17, Definition 3.2]. We omit the structure of units.

Definition 1. An *entries-only cyclic operad* is a functor $\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Set}$, together with a family of functions

$$ {}_x\circ_y : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X\backslash\{x\} + Y\backslash\{y\}), $$

indexed by arbitrary non-empty finite sets X and Y and elements $x \in X$ and $y \in Y$, such that $X\backslash\{x\} \cap Y\backslash\{y\} = \emptyset$. These data must satisfy the axioms given below.

Sequential associativity. For $f \in \mathcal{C}(X)$, $g \in \mathcal{C}(Y)$ and $h \in \mathcal{C}(Z)$, the following equality holds:

(A1) $(f\ {}_x\circ_y\ g)\ {}_u\circ_z\, h = f\,{}_x\circ_z\ (g\ {}_u\circ_z\, h)$, where $x \in X$, $y, u \in Y$, $z \in Z$.

Commutativity. For $f \in \mathcal{C}(X)$, $g \in \mathcal{C}(Y)$, $x \in X$ and $y \in Y$, the following equality holds:

(CO) $f\ {}_x\circ_y\ g = g\ {}_y\circ_x\ f$.

Equivariance. For bijections $\sigma_1 : X' \to X$ and $\sigma_2 : Y' \to Y$, and $f \in \mathcal{C}(X)$ and $g \in \mathcal{C}(Y)$, the following equality holds:

(EQ) $f^{\sigma_1}\ {}_{\sigma_1^{-1}(x)}\circ_{\sigma_2^{-1}(y)}\ g^{\sigma_2} = (f\,{}_x\circ_y\ g)^{\sigma}$, where $\sigma = \sigma_1|^{X\backslash\{x\}} \cup \sigma_2|^{Y\backslash\{y\}}$.

For $f \in \mathcal{C}(X)$, we say that the elements of X are the *entries* of f. An entries-only cyclic operad $\mathcal{C}$ is constant-free if $\mathcal{C}(\emptyset) = \mathcal{C}(\{x\}) = \emptyset$, for all singletons $\{x\}$. □

1.2 The exchangeable-output definition

We now recall [O17, Definition 3.16], leaving out again the structure of units.

Definition 2. An *exchangeable-output cyclic operad* is an operad $\mathcal{O} : \mathbf{Bij}^{op} \to \mathbf{Set}$ (defined as in [O17, Definition 2.3], with units omitted), enriched with actions

$$ D_x : \mathcal{O}(X) \to \mathcal{O}(X), $$

defined for all $x \in X$ and subject to the axioms given below, wherein, for each of the axioms, we assume that $f \in \mathcal{O}(X)$.

Inverse. For $x \in X$,

[DIN] $D_x(D_x(f)) = f$.

Equivariance. For $x \in X$ and an arbitrary bijection $\sigma : Y \to X$,

[DEQ] $D_x(f)^\sigma = D_{\sigma^{-1}(x)}(f^\sigma)$.

Exchange. For $x, y \in X$ and a bijection $\sigma : X \to X$ that exchanges x and y,

[DEX] $D_x(f)^\sigma = D_x(D_y(f))$.

Compatibility with operadic compositions. For $g \in \mathcal{O}(Y)$, the following equality holds:

[DC1] $D_y(f \circ_x g) = D_y(f) \circ_x g$, where $y \in X \backslash \{x\}$, and

[DC2] $D_y(f \circ_x g) = D_y(g)^{\sigma_1} \circ_v D_x(f)^{\sigma_2}$, where $y \in Y$, $\sigma_1 : Y\backslash\{y\} + \{v\} \to Y$ renames y to v and $\sigma_2 : X\backslash\{x\} + \{y\} \to X$ renames x to y.

For $f \in \mathcal{O}(X)$, we say that the elements of X are the *inputs* of f. An exchangeable-output cyclic operad $\mathcal{O}$ is constant-free if $\mathcal{O}(\emptyset) = \emptyset$. □

2 Categorified entries-only cyclic operads

This section deals with categorified entries-only cyclic operads. The categorification is made by relaxing the axioms (A1) and (CO) of Definition 1. The axiom (EQ) remains strict. In the first part of the section, we introduce the categorified notion and exhibit important properties. The second one is dedicated to the proof of the coherence theorem.

2.1 The definition and properties

The quest for coherence led us to the following definition.

Definition 3. A *categorified entries-only cyclic operad* is a functor $\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Cat}$, together with

- a family of bifunctors

$$ {}_x\circ_y : \mathcal{C}(X) \times \mathcal{C}(Y) \to \mathcal{C}(X\backslash\{x\} + Y\backslash\{y\}), $$

  called *partial composition operations of* $\mathcal{C}$, indexed by arbitrary non-empty finite sets X and Y and elements $x \in X$ and $y \in Y$, such that $X\backslash\{x\} \cap Y\backslash\{y\} = \emptyset$, which are subject to the equivariance axiom (EQ), and

- two natural isomorphisms, β and γ, called the *associator* and the *commutator*, whose respective components

$$ \beta^{x,\underline{x};y,\underline{y}}_{f,g,h} : (f\, {}_x\circ_{\underline{x}}\, g)\, {}_y\circ_{\underline{y}}\, h \to f\, {}_x\circ_{\underline{x}}\, (g\, {}_y\circ_{\underline{y}}\, h) \quad \text{and} \quad \gamma^{x,y}_{f,g} : f\, {}_x\circ_y\, g \to g\, {}_y\circ_x\, f\,, $$

  are natural in f, g and h, and are subject to the following coherence conditions:

- (β-pentagon)

$$((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k \xrightarrow{\beta^{x,\underline{x};y,\underline{y}}_{f,g,h}\,{}_{z}\circ_{\underline{z}}\,1_k} (f\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{y}\circ_{\underline{y}}\,h))\,{}_{z}\circ_{\underline{z}}\,k \xrightarrow{\beta^{x,\underline{x};z,\underline{z}}_{f,g_{y}\circ_{\underline{y}}h,k}} f\,{}_{x}\circ_{\underline{x}}\,((g\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k) \xrightarrow{1_{f}\,{}_{x}\circ_{\underline{x}}\,\beta^{y,\underline{y};z,\underline{z}}_{g,h,k}} f\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{y}\circ_{\underline{y}}\,(h\,{}_{z}\circ_{\underline{z}}\,k))$$

$$((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k \xrightarrow{\beta^{y,\underline{y};z,\underline{z}}_{f_{x}\circ_{\underline{x}}g,h,k}} (f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,(h\,{}_{z}\circ_{\underline{z}}\,k) \xrightarrow{\beta^{x,\underline{x};y,\underline{y}}_{f,g,h_{z}\circ_{\underline{z}}k}} f\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{y}\circ_{\underline{y}}\,(h\,{}_{z}\circ_{\underline{z}}\,k))$$

- ($\beta\gamma$-hexagon)

$$(f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,h \xrightarrow{\beta^{x,\underline{x};y,\underline{y}}_{f,g,h}} f\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{y}\circ_{\underline{y}}\,h) \xrightarrow{\gamma^{x,\underline{x}}_{f,g_{y}\circ_{\underline{y}}h}} (g\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{\underline{x}}\circ_{x}\,f \xrightarrow{\gamma^{y,\underline{y}}_{g,h}\,{}_{\underline{x}}\circ_{x}\,1_f} (h\,{}_{\underline{y}}\circ_{y}\,g)\,{}_{\underline{x}}\circ_{x}\,f \xrightarrow{\beta^{\underline{y},y;\underline{x},x}_{h,g,f}} h\,{}_{\underline{y}}\circ_{y}\,(g\,{}_{\underline{x}}\circ_{x}\,f)$$

$$(f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,h \xrightarrow{\gamma^{x,\underline{x}}_{f,g}\,{}_{y}\circ_{\underline{y}}\,1_h} (g\,{}_{\underline{x}}\circ_{x}\,f)\,{}_{y}\circ_{\underline{y}}\,h \xrightarrow{\gamma^{y,\underline{y}}_{g_{\underline{x}}\circ_{x}f,h}} h\,{}_{\underline{y}}\circ_{y}\,(g\,{}_{\underline{x}}\circ_{x}\,f)$$

- ($\beta\gamma$-decagon)

$$((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k \xrightarrow{\gamma^{y,\underline{y}}_{f_{x}\circ_{\underline{x}}g,h}\,{}_{z}\circ_{\underline{z}}\,1_k} (h\,{}_{\underline{y}}\circ_{y}\,(f\,{}_{x}\circ_{\underline{x}}\,g))\,{}_{z}\circ_{\underline{z}}\,k \xrightarrow{\beta^{\underline{y},y;z,\underline{z}}_{h,f_{x}\circ_{\underline{x}}g,k}} h\,{}_{\underline{y}}\circ_{y}\,((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{z}\circ_{\underline{z}}\,k) \xrightarrow{\gamma^{\underline{y},y}_{h,(f_{x}\circ_{\underline{x}}g)_{z}\circ_{\underline{z}}k}} ((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{z}\circ_{\underline{z}}\,k)\,{}_{y}\circ_{\underline{y}}\,h$$

$$((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{z}\circ_{\underline{z}}\,k)\,{}_{y}\circ_{\underline{y}}\,h \xrightarrow{\beta^{x,\underline{x};z,\underline{z}}_{f,g,k}\,{}_{y}\circ_{\underline{y}}\,1_h} (f\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{z}\circ_{\underline{z}}\,k))\,{}_{y}\circ_{\underline{y}}\,h \xrightarrow{\beta^{x,\underline{x};y,\underline{y}}_{f,g_{z}\circ_{\underline{z}}k,h}} f\,{}_{x}\circ_{\underline{x}}\,((g\,{}_{z}\circ_{\underline{z}}\,k)\,{}_{y}\circ_{\underline{y}}\,h)$$

$$((f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k \xrightarrow{\beta^{x,\underline{x};y,\underline{y}}_{f,g,h}\,{}_{z}\circ_{\underline{z}}\,1_k} (f\,{}_{x}\circ_{\underline{x}}\,(g\,{}_{y}\circ_{\underline{y}}\,h))\,{}_{z}\circ_{\underline{z}}\,k \xrightarrow{\beta^{x,\underline{x};z,\underline{z}}_{f,g_{y}\circ_{\underline{y}}h,k}} f\,{}_{x}\circ_{\underline{x}}\,((g\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k)$$

$$f\,{}_{x}\circ_{\underline{x}}\,((g\,{}_{y}\circ_{\underline{y}}\,h)\,{}_{z}\circ_{\underline{z}}\,k) \xrightarrow{1_{f}\,{}_{x}\circ_{\underline{x}}\,(\gamma^{y,\underline{y}}_{g,h}\,{}_{z}\circ_{\underline{z}}\,1_k)} f\,{}_{x}\circ_{\underline{x}}\,((h\,{}_{\underline{y}}\circ_{y}\,g)\,{}_{z}\circ_{\underline{z}}\,k) \xrightarrow{1_{f}\,{}_{x}\circ_{\underline{x}}\,\beta^{\underline{y},y;z,\underline{z}}_{h,g,k}} f\,{}_{x}\circ_{\underline{x}}\,(h\,{}_{\underline{y}}\circ_{y}\,(g\,{}_{z}\circ_{\underline{z}}\,k)) \xrightarrow{1_{f}\,{}_{x}\circ_{\underline{x}}\,\gamma^{\underline{y},y}_{h,g_{z}\circ_{\underline{z}}k}} f\,{}_{x}\circ_{\underline{x}}\,((g\,{}_{z}\circ_{\underline{z}}\,k)\,{}_{y}\circ_{\underline{y}}\,h)$$

- (γ-involution)

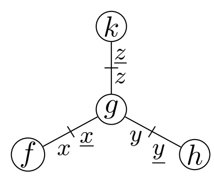
$$f\,{}_{x}\circ_{\underline{x}}\,g \xrightarrow{\gamma^{x,\underline{x}}_{f,g}} b\,{}_{\underline{x}}\circ_{x}\,g \xrightarrow{\gamma^{\underline{x},x}_{g,f}} f\,{}_{x}\circ_{\underline{x}}\,g, \qquad f\,{}_{x}\circ_{\underline{x}}\,g \xrightarrow{1_{f_{x}\circ_{\underline{x}}g}} f\,{}_{x}\circ_{\underline{x}}\,g$$

where $1_{(-)}$ denotes the identity morphism for $(-)$, as well as the following conditions which involve the action of $\mathcal{C}(\sigma)$, where $\sigma : Y \to X$, on the morphisms of $\mathcal{C}(X)$:

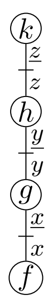
- `(`$\beta\sigma$`)` if the equality $((f\,{}_x\circ_{\underline{x}}\,g)\,{}_y\circ_{\underline{y}}\,h)^\sigma = (f^{\sigma_1}\,{}_{x'}\circ_{\underline{x}'}\,g^{\sigma_2})\,{}_{y'}\circ_{\underline{y}'}\,h^{\sigma_3}$ holds by `(EQ)`, then
$$(\beta^{x,\underline{x};y,\underline{y}}_{f,g,k})^\sigma = \beta^{x',\underline{x}';y',\underline{y}'}_{f^{\sigma_1},g^{\sigma_2},h^{\sigma_3}},$$
- `(`$\gamma\sigma$`)` if the equality $(f\,{}_x\circ_{\underline{x}}\,g)^\sigma = f^{\sigma_1}\,{}_{x'}\circ_{\underline{x}'}\,g^{\sigma_2}$ holds by `(EQ)`, then
$$(\gamma^{x,\underline{x}}_{f,g})^\sigma = \gamma^{x',\underline{x}'}_{f^{\sigma_1},g^{\sigma_2}},$$
- `(EQ-mor)` if the equality $(f\,{}_x\circ_{\underline{x}}\,g)^\sigma = f^{\sigma_1}\,{}_{x'}\circ_{\underline{x}'}\,g^{\sigma_2}$ holds by `(EQ)`, and if $\varphi : f \to f'$ and $\psi : g \to g'$, then
$$(\varphi\,{}_x\circ_{\underline{x}}\,\psi)^\sigma = \varphi^{\sigma_1}\,{}_{x'}\circ_{\underline{x}'}\,\psi^{\sigma_2}.$$

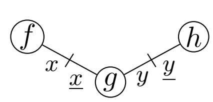
For $f \in \mathcal{C}(X)$, we say that the elements of X are the entries of f. A categorified entries-only cyclic operad $\mathcal{C}$ is constant-free if $\mathcal{C}(\emptyset) = \mathcal{C}(\{x\}) = \emptyset$, for all singletons $\{x\}$. □

Remark 1. *The nodes of the diagrams of Definition 3 can be viewed as formal expressions built over operations $f, g, \ldots$ and their entries $x, \underline{x}, y, \underline{y}, \ldots$. For each diagram, the rules for assembling correctly these expressions are determined by the "origin of entries", i.e. by the uniquely determined relation between the involved operations and entries, whose instances have the form "x is an entry of f". For example, in* `(`$\beta\gamma$`-decagon)`*, the legitimacy of all the nodes in the diagram witnesses that x is entry of f, $\underline{x}$, y and z are entries of g, $\underline{y}$ is the entry of h and $\underline{z}$ is the entry of k. From the tree-wise perspective, these data can be encoded by the unrooted tree*

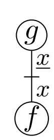


This tree also illustrates the fact that the morphism, say,
$$\beta^{\underline{x},x;y,\underline{y}}_{g,f,h} : (g\,{}_{\underline{x}}\circ_x\,f)\,{}_y\circ_{\underline{y}}\,h \to g\,{}_{\underline{x}}\circ_x\,(f\,{}_y\circ_{\underline{y}}\,h)$$
does not exist (for these particular f, g and h), since its codomain is not well-formed, which exemplifies the difference between the setting of symmetric monoidal categories, where an instance of the associator exists for any (ordered) triple of objects*. The trees corresponding to* `(`β`-pentagon)`*,* `(`$\beta\gamma$`-hexagon)` *and* `(`γ`-involution)` *are*





and



respectively. In §2.4.2, we shall introduce a formal tree-wise representation of the operations of a categorified cyclic operads, based on this intuition. Until then, we shall continue to omit the data about the "origin of entries" whenever possible.

Remark 2. *Observe that, for a categorified cyclic operad* $\mathcal{C}$ *and a finite set* X*, both the objects and the morphisms of* $\mathcal{C}(X)$ *enjoy equivariance: at the level of objects, this is ensured by* `(EQ)`*, and at the level of morphisms, by* `(EQ-mor)`*.*

In the remainder of the section, we shall work with a fixed categorified entries-only cyclic operad $\mathcal{C}$. In the remark that follows, we list the equalities on objects and morphisms of $\mathcal{C}(X)$ which are implicitly imposed by the structure of $\mathcal{C}$.

Remark 3. *For an arbitrary finite set* X*, the following equalities hold in* $\mathcal{C}(X)$*:*

1. *the categorical equations:*
   a) $\varphi \circ 1_f = \varphi = 1_g \circ \varphi$, *for* $\varphi : f \to g$,
   b) $(\varphi \circ \phi) \circ \psi = \varphi \circ (\phi \circ \psi)$,
2. *the equations imposed by the bifunctoriality of* ${}_x\circ_{\underline{x}}$*:*
   a) $1_f \,{}_x\circ_{\underline{x}}\, 1_g = 1_{f \,{}_x\circ_{\underline{x}}\, g}$,
   b) $(\varphi_2 \circ \varphi_1) \,{}_x\circ_{\underline{x}}\, (\psi_2 \circ \psi_1) = (\varphi_2 \,{}_x\circ_{\underline{x}}\, \psi_2) \circ (\varphi_1 \,{}_x\circ_{\underline{x}}\, \psi_1)$,
3. *the naturality equations for* β *and* γ*:*
   a) $\beta^{x,\underline{x};y,\underline{y}}_{f_2,g_2,h_2} \circ ((\varphi \,{}_x\circ_{\underline{x}}\, \phi) \,{}_y\circ_{\underline{y}}\, \psi) = (\varphi \,{}_x\circ_{\underline{x}}\, (\phi \,{}_y\circ_{\underline{y}}\, \psi)) \circ \beta^{x,\underline{x};y,\underline{y}}_{f_1,g_1,h_1}$,
   b) $\gamma^{x,y}_{f_2,g_2} \circ (\varphi \,{}_x\circ_y\, \phi) = (\phi \,{}_y\circ_x\, \varphi) \circ \gamma^{x,y}_{f_1,g_1}$,
5. *the equations imposed by the functoriality of* $\mathcal{C}$*:*
   a) $\mathcal{C}(1_X) = 1_{\mathcal{C}(X)}$,
   b) $(f^\sigma)^\tau = f^{\sigma\circ\tau}$,
   c) $(\varphi^\sigma)^\tau = \varphi^{\sigma\circ\tau}$
6. *the equations imposed by the functoriality of* $\mathcal{C}(\sigma)$*:*
   a) $1_f^\sigma = 1_{f^\sigma}$,
   b) $(\varphi \circ \psi)^\sigma = \varphi^\sigma \circ \psi^\sigma$.

2.1.1 "Parallel associativity" in $\mathcal{C}$

We introduce an important abbreviation: we define a natural isomorphism ϑ, called *parallel associativity*, by taking

$$\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h} = \gamma^{\underline{x},x}_{g,f \,{}_y\circ_{\underline{y}}\, h} \circ \beta^{\underline{x},x;y,\underline{y}}_{g,f,h} \circ (\gamma^{x,\underline{x}}_{f,g} \,{}_y\circ_{\underline{y}}\, 1_h) : (f \,{}_x\circ_{\underline{x}}\, g) \,{}_y\circ_{\underline{y}}\, h \longrightarrow (f \,{}_y\circ_{\underline{y}}\, h) \,{}_x\circ_{\underline{x}}\, g \tag{2.1}$$

for its components. Here are first observations about the natural isomorphism ϑ.

Remark 4. *The natural isomorphism ϑ appears in* ($\beta\gamma$`-hexagon`) *and* ($\beta\gamma$`-decagon`)*.*

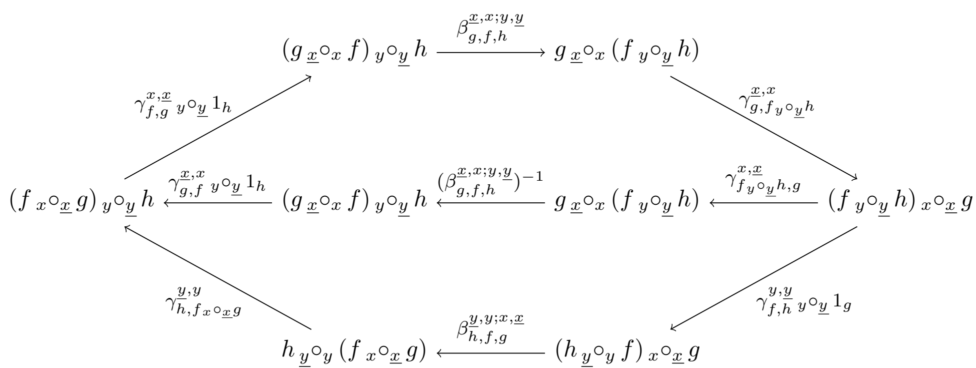
1. *An isomorphism with the same source and target as $\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h}$ could be introduced as the composition*
$$(\gamma^{\underline{y},y}_{h,f}\ {}_{x}{\circ}_{\underline{x}}\ 1_g) \circ (\beta^{\underline{y},y;x,\underline{x}}_{h,f,g})^{-1} \circ \gamma^{y,\underline{y}}_{f\,{}_{x}\circ_{\underline{x}}g,h}$$
which is as "natural" as the composition which we have fixed to be the definition of $\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h}$. With this in mind, ($\beta\gamma$`-hexagon`) *can be read as: the two possible (and equally natural) definitions of $\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h}$ are equal.*

2. *Also, ϑ appears twice in* ($\beta\gamma$`-decagon`)*, turning it into a hexagon by using explicilty the abbreviations $\vartheta^{y,\underline{y};z,\underline{z}}_{f\,{}_{x}\circ_{\underline{x}}g,h,k}$ (for the top horizontal sequence of arrows) and $1_f\ {}_{x}{\circ}_{\underline{x}}\ \vartheta^{y,\underline{y};z,\underline{z}}_{g,h,k}$ (for the bottom horizontal sequence of arrow).*

In the following two lemmas we show that ϑ is subject to certain *nice* coherence conditions. We first show that the isomorphism $\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h}$ has $\vartheta^{y,\underline{y};x,\underline{x}}_{f,h,g}$ as inverse.

Lemma 1. *The equality $\vartheta^{y,\underline{y};x,\underline{x}}_{f,h,g} \circ \vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h} = 1_{(f\,{}_{x}\circ_{\underline{x}}\,g)\,{}_{y}\circ_{\underline{y}}\,h}$ holds.*

Proof. The equality follows by the commutation of the (outer part of) diagram



in which the upper hexagon commutes by (γ`-involution`) and the lower hexagon commutes as an instance of ($\beta\gamma$`-hexagon`). ■

The following lemma shows two more laws satifsied by ϑ.

Lemma 2. *The following two equalities hold:*

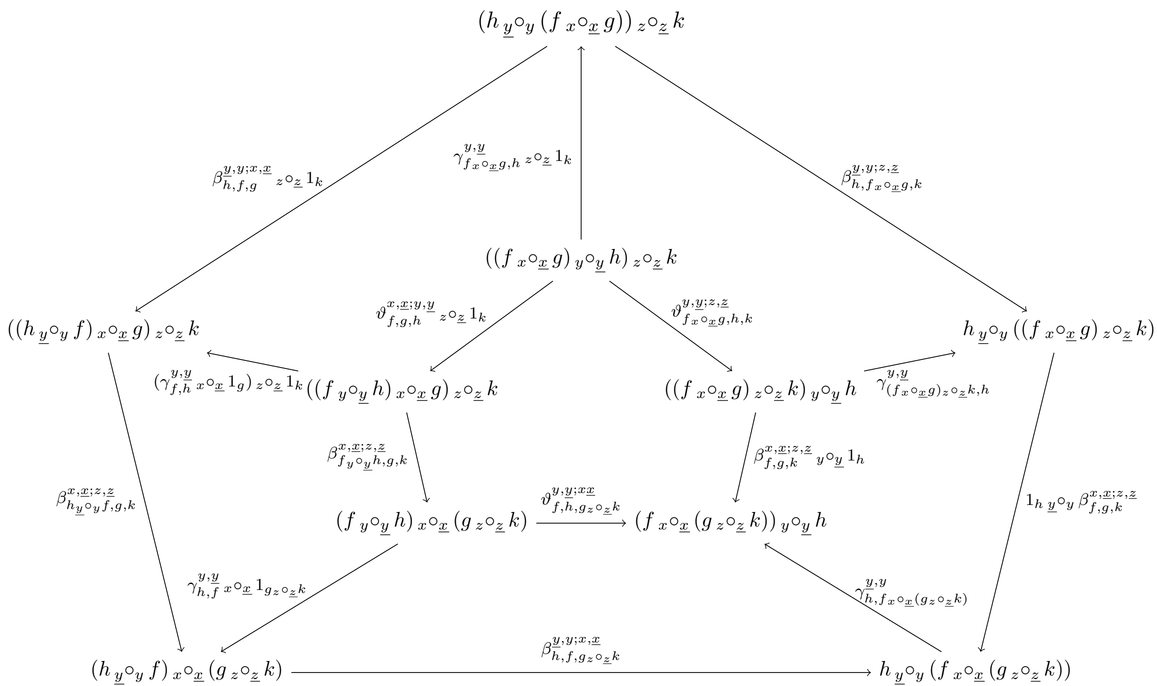
- ($\beta\vartheta$`-pentagon`)

$$\vartheta^{y,\underline{y};x\underline{x}}_{f,h,g\,{}_{z}\circ_{\underline{z}}k} \circ \beta^{x,\underline{x};z,\underline{z}}_{f\,{}_{y}\circ_{\underline{y}}h,g,k} \circ (\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h}\ {}_{z}\circ_{\underline{z}}\ 1_k) = (\beta^{x,\underline{x};z,\underline{z}}_{f,g,k}\ {}_{y}\circ_{\underline{y}}\ 1_h) \circ \vartheta^{y,\underline{y};z,\underline{z}}_{f\,{}_{x}\circ_{\underline{x}}g,h,k},$$

- (ϑ`-hexagon`)

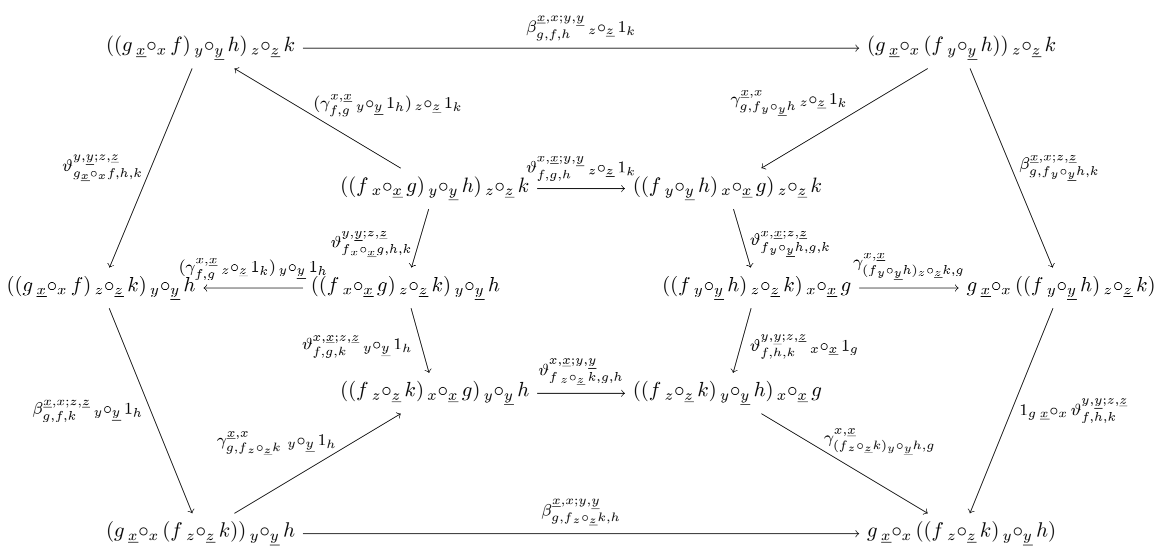
$$\vartheta^{x,\underline{x};y,\underline{y}}_{f\,{}_{z}\circ_{\underline{z}}k,g,h} \circ (\vartheta^{x,\underline{x};z,\underline{z}}_{f,g,k}\ {}_{y}\circ_{\underline{y}}\ 1_h) \circ \vartheta^{y,\underline{y};z,\underline{z}}_{f\,{}_{x}\circ_{\underline{x}}g,h,k} = (\vartheta^{y,\underline{y};z,\underline{z}}_{f,h,k}\ {}_{x}\circ_{\underline{x}}\ 1_g) \circ \vartheta^{x,\underline{x};z,\underline{z}}_{f\,{}_{y}\circ_{\underline{y}}h,g,k} \circ (\vartheta^{x,\underline{x};y,\underline{y}}_{f,g,h}\ {}_{z}\circ_{\underline{z}}\ 1_k).$$

Proof. For the first claim, consider the diagram



whose "inner" pentagon is (βϑ-pentagon) and whose "outer" pentagon commutes as an instance of (β-pentagon). The claim follows by the commutations of all the diagrams "between" the two pentagons (two naturality squares for β and three squares expressing the definition of ϑ).

We have an analogous picture for the second claim. The "inner" hexagon in the diagram



is (ϑ-hexagon) from the claim, and the "outer" hexagon is an instance of (βγ-decagon), and the claim follows by the commutations of all the diagrams "between" the two hexagons (these are the four naturality squares for ϑ and two squares which express the definition of ϑ). ■

2.2 Canonical diagrams and the coherence theorem

The coherence theorem that we shall prove has the form: *all diagrams of canonical arrows commute in* $\mathcal{C}(X)$. In order to formulate it rigorously, we shall first specify what a diagram of canonical arrows is exactly. Denoting with $\underline{\mathcal{C}}$ the underlying functor of the categorified entries-only cyclic operad $\mathcal{C}$, in this part we essentially introduce a syntax for the free categorified entries-only cyclic operad built over $\underline{\mathcal{C}}$. However, since the purpose of the syntax is solely to distinguish the canonical arrows of $\mathcal{C}(X)$, the formalism will be left without any equations.

2.2.1 The syntax $\mathtt{Free}_{\underline{\mathcal{C}}}$

Let

$$P_{\underline{\mathcal{C}}} = \{a \,|\, a \in \underline{\mathcal{C}}(X) \text{ for some finite set } X\} \tag{2.2}$$

be the collection of *parameters of* $\underline{\mathcal{C}}$, $V = \{x, y, z, \underline{x}, \underline{y}, \underline{z}, \dots\}$ the collection of variables and $\Sigma = \{\sigma, \tau, \kappa, \nu, \dots\}$ the collection of bijections of finite sets, respectively.

The syntax $\mathtt{Free}_{\underline{\mathcal{C}}}$ *of canonical diagrams (or, of* $\beta\gamma\sigma$*-arrows) of* $\mathcal{C}$, contains two kinds of typed expressions, the *object terms* and the *arrow terms* (as all the other formal systems that we shall introduce in the remaining of the section).

The syntax of object terms is obtained from raw (i.e. not *yet* typed) object terms

$$\boxed{\mathcal{W} ::= \underline{a} \mid (\mathcal{W}_x\square_y\mathcal{W}) \mid \mathcal{W}^{\sigma}}$$

where $a \in P_{\underline{\mathcal{C}}}$, $x, y \in V$, and $\sigma \in \Sigma$, by typing them as $\mathcal{W} : X$, where X ranges over finite sets. The assignment of types is done by the following rules:

$$\frac{a \in \mathcal{C}(X)}{\underline{a} : X} \qquad \frac{\mathcal{W}_1 : X \quad \mathcal{W}_2 : Y \quad x \in X \;\; y \in Y \quad X\backslash\{x\} \cap Y\backslash\{y\} = \emptyset}{\mathcal{W}_1 \,{}_x\square_y\, \mathcal{W}_2 : X\backslash\{x\} + Y\backslash\{y\}} \qquad \frac{\mathcal{W} : X \quad \sigma : Y \to X}{\mathcal{W}^{\sigma} : Y}$$

Remark 5. *The notation* ${}_x\square_y$ *(rather then* ${}_x\circ_y$*) for the syntax of partial composition operations is chosen merely to avoid confusion with the symbol* $\circ$*, used to denote the (usual) composition of morphisms in a category.*

To the syntax of object terms we add the syntax of arrow terms, obtained from raw arrow terms

$$\Phi ::= \begin{cases} 1_{\mathcal{W}} \mid \beta^{x,\underline{x};y,\underline{y}}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} \mid \beta^{x,\underline{x};y,\underline{y}\;\;-1}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} \mid \gamma^{x,y}_{\mathcal{W}_1,\mathcal{W}_2} \\ \varepsilon1^{\sigma}_{\underline{a}} \mid \varepsilon1^{\sigma\;-1}_{\underline{a}} \mid \varepsilon2_{\mathcal{W}} \mid \varepsilon2^{-1}_{\mathcal{W}} \mid \varepsilon3^{\sigma,\tau}_{\mathcal{W}} \mid \varepsilon3^{\sigma,\tau\;-1}_{\mathcal{W}} \mid \varepsilon4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma} \mid \varepsilon4^{x,y;x',y'\;-1}_{\mathcal{W}_1,\mathcal{W}_2;\sigma} \\ \Phi \circ \Phi \mid \Phi \,{}_x\square_y\, \Phi \mid \Phi^{\sigma}, \end{cases}$$

by assigning them types in the form of ordered pairs $(\mathcal{W}_1, \mathcal{W}_2)$ of object terms, denoted by $\mathcal{W}_1 \to \mathcal{W}_2$, as follows:

$$\frac{}{1_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}}$$

$$\frac{}{\beta^{x,\underline{x};y,\underline{y}}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} : (\mathcal{W}_1{}_x\Box_{\underline{x}}\mathcal{W}_2)_y\Box_{\underline{y}}\mathcal{W}_3 \to \mathcal{W}_1{}_x\Box_{\underline{x}}(\mathcal{W}_2{}_y\Box_{\underline{y}}\mathcal{W}_3)}$$

$$\frac{}{\beta^{x,\underline{x};y,\underline{y}\ -1}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} : \mathcal{W}_1{}_x\Box_{\underline{x}}(\mathcal{W}_2{}_y\Box_{\underline{y}}\mathcal{W}_3) \to (\mathcal{W}_1{}_x\Box_{\underline{x}}\mathcal{W}_2)_y\Box_{\underline{y}}\mathcal{W}_3}$$

$$\frac{}{\gamma^{x,y}_{\mathcal{W}_1,\mathcal{W}_2} : \mathcal{W}_1\,{}_x\Box_y\,\mathcal{W}_2 \to \mathcal{W}_2\,{}_y\Box_x\,\mathcal{W}_1}$$

$$\frac{}{\varepsilon1^{\sigma}_{\underline{a}} : \underline{a}^{\sigma} \to \underline{a^{\sigma}}} \qquad \frac{}{\varepsilon1^{\sigma\ -1}_{\underline{a}} : \underline{a^{\sigma}} \to \underline{a}^{\sigma}} \qquad \frac{}{\varepsilon2_{\mathcal{W}} : \mathcal{W}^{id_X} \to \mathcal{W}} \qquad \frac{}{\varepsilon2^{-1}_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}^{id_X}}$$

$$\frac{}{\varepsilon3^{\sigma,\tau}_{\mathcal{W}} : (\mathcal{W}^{\sigma})^{\tau} \to \mathcal{W}^{\sigma\circ\tau}} \qquad \frac{}{\varepsilon3^{\sigma,\tau\ -1}_{\mathcal{W}} : \mathcal{W}^{\sigma\circ\tau} \to (\mathcal{W}^{\sigma})^{\tau}}$$

$$\frac{\begin{array}{c} \sigma : Z \to X\backslash\{x\}+Y\backslash\{y\} \\ \sigma_1 : \sigma^{-1}[X\backslash\{x\}]+\{x'\} \to X \quad \sigma_1|^{X\backslash\{x\}}=\sigma|^{X\backslash\{x\}} \quad \sigma_1(x')=x \\ \sigma_2 : \sigma^{-1}[Y\backslash\{y\}]+\{y'\} \to Y \quad \sigma_2|^{Y\backslash\{y\}}=\sigma|^{Y\backslash\{y\}} \quad \sigma_2(y')=y \end{array}}{\varepsilon4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma} : (\mathcal{W}_1\,{}_x\Box_y\,\mathcal{W}_2)^{\sigma} \to \mathcal{W}_1^{\sigma_1}\,{}_{x'}\Box_{y'}\,\mathcal{W}_2^{\sigma_2}}$$

$$\frac{\begin{array}{c} \sigma_1 : X' \to X \quad \sigma_1(x')=x \\ \sigma_2 : Y' \to Y \quad \sigma_2(y')=y \\ \sigma : X'\backslash\{x'\}+Y'\backslash\{y'\} \to X\backslash\{x\}+Y\backslash\{y\} \quad \sigma=\sigma_1|^{X'\backslash\{x'\}}+\sigma_2|^{Y'\backslash\{y'\}} \end{array}}{\varepsilon4^{x,y;x',y'\ -1}_{\mathcal{W}_1,\mathcal{W}_2;\sigma} : \mathcal{W}_1^{\sigma_1}\,{}_{x'}\Box_{y'}\,\mathcal{W}_2^{\sigma_2} \to (\mathcal{W}_1\,{}_x\Box_y\,\mathcal{W}_2)^{\sigma}}$$

$$\frac{\Phi_1 : \mathcal{W}_1 \to \mathcal{W}_2 \quad \Phi_2 : \mathcal{W}_2 \to \mathcal{W}_3}{\Phi_2 \circ \Phi_1 : \mathcal{W}_1 \to \mathcal{W}_3} \qquad \frac{\Phi_1 : \mathcal{W}_1 \to \mathcal{W}_1' \quad \Phi_2 : \mathcal{W}_2 \to \mathcal{W}_2'}{\Phi_1\,{}_x\Box_y\,\Phi_2 : \mathcal{W}_1\,{}_x\Box_y\,\mathcal{W}_2 \to \mathcal{W}_1'\,{}_x\Box_y\,\mathcal{W}_2'}$$

$$\frac{\Phi : \mathcal{W}_1 \to \mathcal{W}_2 \quad \sigma : Y \to X}{\Phi^{\sigma} : \mathcal{W}_1^{\sigma} \to \mathcal{W}_2^{\sigma}}$$

where it is also (implicitly) assumed that all the object terms that appear in the types of the arrow terms are well-formed. Given an arrow term $\Phi : \mathcal{U} \to \mathcal{V}$, we call the object term $\mathcal{U}$ *the source of* Φ and the object term $\mathcal{V}$ *the target of* Φ.

Remark 6. *Observe that, for all well-typed arrow terms $\Phi : \mathcal{U} \to \mathcal{V}$ of* $\texttt{Free}_{\underline{\mathcal{C}}}$*, the object terms $\mathcal{U}$ and $\mathcal{V}$ have the same type.*

The collection of object terms of type X, together with the collection of arrow terms whose source and target have type X, will be denoted by $\texttt{Free}_{\underline{\mathcal{C}}}(X)$.

2.2.2 The interpretation of $\texttt{Free}_{\underline{\mathcal{C}}}$ in $\mathcal{C}$

The semantics of $\texttt{Free}_{\underline{\mathcal{C}}}$ in $\mathcal{C}$ is what distinguishes canonical arrows (or $\beta\gamma\sigma$-arrows) of $\mathcal{C}(X)$: they will be precisely the interpretations of the arrow terms of $\texttt{Free}_{\underline{\mathcal{C}}}(X)$. Given that the axiom `(EQ)` remains strict in the transition from Definition 1 to Definition 3, the interpretations of the arrow terms whose denotations contain ε (and which all encode the properties of the action of the symmetric group) will be identities.

The interpretation function

$$[[-]]_X : \texttt{Free}_{\underline{\mathcal{C}}}(X) \to \mathcal{C}(X)$$

is defined recursively as follows:

$$[[\underline{a}]]_X = a, \qquad [[\mathcal{W}_1\,{}_x\square_y\,\mathcal{W}_2]]_X = [[\mathcal{W}_1]]_{X_1}\,{}_x\circ_y\,[[\mathcal{W}_2]]_{X_2}, \qquad [[\mathcal{W}^\sigma]]_X = ([[\mathcal{W}]]_Y)^\sigma,$$

and

- ◇ $[[1_{\mathcal{W}}]]_X = 1_{[[\mathcal{W}]]_X}$,
- ◇ $[[\beta^{x,\underline{x};y,\underline{y}}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3}]]_X = \beta^{x,\underline{x};y,\underline{y}}_{[[\mathcal{W}_1]]_{X_1},[[\mathcal{W}_2]]_{X_2},[[\mathcal{W}_3]]_{X_3}}$,
- ◇ $[[\beta^{x,\underline{x};y,\underline{y}\ -1}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3}]]_X = \beta^{x,\underline{x};y,\underline{y}\ -1}_{[[\mathcal{W}_1]]_{X_1},[[\mathcal{W}_2]]_{X_2},[[\mathcal{W}_3]]_{X_3}}$,
- ◇ $[[\gamma^{x,y}_{\mathcal{W}_1,\mathcal{W}_2}]]_X = \gamma^{x,y}_{[[\mathcal{W}_1]]_{X_1},[[\mathcal{W}_2]]_{X_2}}$,
- ◇ $[[\varepsilon 1^{\sigma}_{\underline{a}}]]_X = 1_{[[\underline{a}^\sigma]]_X}$, $[[\varepsilon 1^{\sigma\ -1}_{\underline{a}}]]_X = 1_{[[\underline{a}^\sigma]]_X}$,
- ◇ $[[\varepsilon 2_{\mathcal{W}}]]_X = 1_{[[\mathcal{W}^{id_X}]]_X}$, $[[\varepsilon 2^{-1}_{\mathcal{W}}]]_X = 1_{[[\mathcal{W}]]_X}$,
- ◇ $[[\varepsilon 3^{\sigma,\tau}_{\mathcal{W}}]]_X = 1_{[[(\mathcal{W}^\sigma)^\tau]]_X}$, $[[\varepsilon 3^{\sigma,\tau\ -1}_{\mathcal{W}}]]_X = 1_{[[\mathcal{W}^{\sigma\circ\tau}]]_X}$
- ◇ $[[\varepsilon 4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma}]]_X = 1_{[[(\mathcal{W}_1\,{}_x\square_y\mathcal{W}_2)^\sigma]]_X}$, $[[\varepsilon 4^{x,y;x',y'\ -1}_{\mathcal{W}_1,\mathcal{W}_2;\sigma}]]_X = 1_{[[\mathcal{W}_1^{\sigma_1}\,{}_{x'}\square_{y'}\mathcal{W}_2^{\sigma_2}]]_X}$,
- ◇ $[[\Phi_2 \circ \Phi_1]]_X = [[\Phi_2]]_X \circ [[\Phi_1]]_X$,
- ◇ $[[\Phi_1\,{}_x\square_y\Phi_2]]_X = [[\Phi_1]]_{X_1}\,{}_x\circ_y[[\Phi_2]]_{X_2}$, and
- ◇ $[[\Phi^\sigma]]_X = ([[\Phi]]_Y)^\sigma$.

Lemma 3. *The interpretation function $[[-]]_X : \texttt{Free}_{\underline{\mathcal{C}}}(X) \to \mathcal{C}(X)$ is well-defined, in the sense that, for an arrow term $\Phi : \mathcal{U} \to \mathcal{V}$ of $\texttt{Free}_{\underline{\mathcal{C}}}(X)$, we have that $[[\Phi]]_X : [[\mathcal{U}]]_X \to [[\mathcal{V}]]_X$.*

Proof. The claim holds thanks to the axiom `(EQ)` for $\mathcal{C}$. ■

Relying on Lemma 3, we say that a canonical diagram in $\mathcal{C}(X)$ is a pair of parallel morphisms (i.e. morphisms that share the same source and target) arising as interpretations of two arrow terms of the same type of $\texttt{Free}_{\underline{\mathcal{C}}}$.

2.2.3 The coherence theorem

We can now state precisely the coherence theorem for $\mathcal{C}$.

Coherence Theorem. *For any finite set X and for any pair of arrow terms $\Phi, \Psi : \mathcal{W}_1 \to \mathcal{W}_2$ of the same type in $\texttt{Free}_{\underline{\mathcal{C}}}(X)$, we have $[[\Phi]]_X = [[\Psi]]_X$ in $\mathcal{C}(X)$.*

In the remaining of Section 2, we prove the coherence theorem. We do this by making three faithful reductions, each restricting the coherence problem to a smaller class of diagrams, in such a way that the coherence problem is ultimately reduced to the coherence of weak Cat-operads of [DP15].

2.3 The first reduction: getting rid of symmetries

Intuitively, the first reduction cuts down the coherence problem of $\mathcal{C}$ to the problem of commutation of all *diagrams of $\beta\gamma$-arrows* of $\mathcal{C}$. We introduce first the syntax of these diagrams.

2.3.1 The syntax $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$

The syntax we are about to introduce is obtained by removing the term constructor $(-)^{\sigma}$ from the list of raw object and raw arrow terms of $\mathtt{Free}_{\underline{\mathcal{C}}}$, as well as all the arrow terms of $\mathtt{Free}_{\underline{\mathcal{C}}}$ whose denotation contains ε. Let $P_{\underline{\mathcal{C}}}$ and V be like before (see (2.2)).

The *$\beta\gamma$-reduction* of $\mathtt{Free}_{\underline{\mathcal{C}}}$, denoted by $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$, is specified as follows. The collection of object terms of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$ is determined by raw object terms

$$\boxed{W ::= \underline{a} \,|\, W \;{}_{x}\square_{y}\, W}$$

where $a \in P_{\underline{\mathcal{C}}}$ and $x, y \in V$, and typing rules

$$\frac{a \in \mathcal{C}(X)}{\underline{a} : X} \qquad \frac{W_1 : X \quad W_2 : Y \quad x \in X \;\; y \in Y \quad (X\backslash\{x\}) \cap (Y\backslash\{y\}) = \emptyset}{W_1 \;{}_{x}\square_{y}\, W_2 : X\backslash\{x\} + Y\backslash\{y\}}$$

The collection of arrow terms of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$ is obtained from the raw arrow terms

$$\boxed{\varphi ::= 1_W \mid \beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3} \mid {\beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}}^{-1} \mid \gamma^{x,y}_{W_1,W_2} \mid \varphi \circ \varphi \mid \varphi \;{}_{x}\square_{y}\, \varphi}$$

by typing them with pairs of object terms as follows:

$$\frac{}{1_W : W \to W}$$

$$\frac{}{\beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3} : (W_1{}_{x}\square_{\underline{x}}W_2){}_{y}\square_{\underline{y}}W_3 \to W_1{}_{x}\square_{\underline{x}}(W_2{}_{y}\square_{\underline{y}}W_3)}$$

$$\frac{}{{\beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}}^{-1} : W_1{}_{x}\square_{\underline{x}}(W_2{}_{y}\square_{\underline{y}}W_3) \to (W_1{}_{x}\square_{\underline{x}}W_2){}_{y}\square_{\underline{y}}W_3}$$

$$\frac{}{\gamma^{x,y}_{W_1,W_2} : W_1{}_{x}\square_{y}W_2 \to W_2\,{}_{y}\square_{x}W_1}$$

$$\frac{\varphi_1 : W_1 \to W_2 \quad \varphi_2 : W_2 \to W_3}{\varphi_2 \circ \varphi_1 : W_1 \to W_3} \qquad \frac{\varphi_1 : W_1 \to W_1' \quad \varphi_2 : W_2 \to W_2'}{\varphi_1 \;{}_{x}\square_{y}\, \varphi_2 : W_1 \;{}_{x}\square_{y}\, W_2 \to W_1' \;{}_{x}\square_{y}\, W_2'}$$

Analogously as before, we shall denote the collection of object terms of type X, together with the collection of arrow terms whose source and target are object terms of type X, by $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$.

Remark 7. *Notice that the type of an arrow term φ of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$ is determined completely by φ only, that is, by the* indices *of φ and their* order of appearance *in φ. This allows us to write $W_s(\varphi)$ and $W_t(\varphi)$ for the source and target of φ, respectively.*

Furthermore, observe that, for an arbitrary arrow term $\varphi : U \to V$, the parameters and variables that appear in U are exactly *the parameters and variables that appear in V.*

2.3.2 The interpretation of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$ in $\mathcal{C}$

The semantics of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$ in $\mathcal{C}$ is what distinguishes $\beta\gamma$-*arrows* of $\mathcal{C}(X)$ from all other canonical arrows of $\mathcal{C}(X)$.

The interpretation function

$$[-]_X : \underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \to \mathcal{C}(X)$$

is defined recursively as follows:

$$[\underline{a}]_X = a, \qquad [W_1 \,{}_x\square_y\, W_2]_X = [W_1]_{X_1} \,{}_x\circ_y\, [W_2]_{X_2},$$

and

- ◇ $[1_W]_X = 1_{[W]_X}$,
- ◇ $[\beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}]_X = \beta^{x,\underline{x};y,\underline{y}}_{[W_1]_{X_1},[W_2]_{X_2},[W_3]_{X_3}}$,
- ◇ $[\beta^{x,\underline{x};y,\underline{y}\;\;-1}_{W_1,W_2,W_3}]_X = \beta^{x,\underline{x};y,\underline{y}\;\;-1}_{[W_1]_{X_1},[W_2]_{X_2},[W_3]_{X_3}}$,
- ◇ $[\gamma^{x,y}_{W_1,W_2}]_X = \gamma^{x,y}_{[W_1]_{X_1},[W_2]_{X_2}}$,
- ◇ $[\varphi_2 \circ \varphi_1]_X = [\varphi_2]_X \circ [\varphi_1]_X$, and
- ◇ $[\varphi_1 \,{}_x\square_y\, \varphi_2]_X = [\varphi_1]_{X_1} \,{}_x\circ_y\, [\varphi_2]_{X_2}$.

Lemma 4. *The interpretation function $[-]_X : \underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \to \mathcal{C}(X)$ is well-defined, in the sense that, for an arrow term $\varphi : U \to W$ of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$, we have that $[\varphi]_X : [U]_X \to [V]_X$.*

2.3.3 An auxiliary typing system for the raw arrow terms of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$

In this part, we introduce a slightly more permissive typing system for the raw arrow terms of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$, by "relaxing" the rule for typing the composition $\varphi_2 \circ \varphi_1$. More precisely, the new formal system, which we shall denote with $\underset{\sim}{\mathtt{Free}}_{\underline{\mathcal{C}}}$, will be the same as $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$, except for the composition rule for arrow terms, where we add a degree of freedom by allowing the composition not only "along" the *same* typed object term, but also "along" the α-*equivalent* ones.

In order to define α-equivalence on object terms of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$, we introduce some terminology. For a parameter $a \in \underline{\mathcal{C}}(X)$ of $P_{\underline{\mathcal{C}}}$, we say that X is the set of *free variables* of a, and we write $FV(a) = X$. For an object term $W : Y$, we shall denote with $P_{\underline{\mathcal{C}}}(W)$ the set of all parameters of $P_{\underline{\mathcal{C}}}$ that appear in W. The α-equivalence on object terms of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$ is the smallest equivalence relation $\equiv$ generated by the rule

$$\frac{\begin{array}{c} W_1{:}X \quad W_2{:}Y \quad x{\in}X \quad y{\in}Y \quad X\backslash\{x\}{\cap}Y\backslash\{y\}{=}\emptyset \quad x',y'{\notin}X\backslash\{x\}{+}Y\backslash\{y\} \quad x'{\neq}y' \\ a{\in}P_{\underline{\mathcal{C}}}(W_1) \quad FV(a){=}X_1 \quad x{\in}X_1{\cap}X \\ b{\in}P_{\underline{\mathcal{C}}}(W_2) \quad FV(b){=}Y_1 \quad y{\in}Y_1{\cap}Y \\ \tau_1{:}X_1\backslash\{x\}{+}\{x'\}{\to}X_1 \quad \tau_1|_{X_1\backslash\{x\}}{=}id_{X_1\backslash\{x\}} \quad \tau_1(x'){=}x \\ \tau_2{:}Y_1\backslash\{y\}{+}\{y'\}{\to}Y_1 \quad \tau_2|_{Y_1\backslash\{y\}}{=}id_{Y_1\backslash\{y\}} \quad \tau_2(y'){=}y \end{array}}{W_1 \,{}_x\square_y\, W_2 \equiv W_1[\underline{a^{\tau_1}}/\underline{a}] \,{}_{x'}\square_{y'}\, W_2[\underline{b^{\tau_2}}/\underline{b}]}$$

where $W_1[\underline{a^{\tau_1}}/\underline{a}]$ (resp. $W_2[\underline{b^{\tau_2}}/\underline{b}]$) denotes the result of the substitution of the parameter $\underline{a^{\tau_1}}$ (resp. $\underline{b^{\tau_2}}$) for the parameter $\underline{a}$ (resp. $\underline{b}$) in W_1 (resp. W_2), which is, moreover, congruent with respect to ${}_x\square_y$. The intuition is simpler than it might seem: the rule defining $\equiv$ formalises a particular case of equivariance on objects (see Remark 2). Here is an example.

EXAMPLE 1. Returning to the syntax $\mathtt{Free}_{\underline{\mathcal{C}}}$, which encompases terms of the form $\mathcal{W}^\sigma$, observe that, fixing $\sigma = id_{X\backslash\{x\}+Y\backslash\{y\}}$, by (EQ), we have

$$\begin{aligned}
[[\underline{a}\;{}_x\square_y\;\underline{b}]]_{X\backslash\{x\}+Y\backslash\{y\}} &= ([[\underline{a}]]_X\;{}_x\square_y\;[[\underline{b}]]_Y)^{id_{X\backslash\{x\}+Y\backslash\{y\}}}_{X\backslash\{x\}+Y\backslash\{y\}}\\
&= [[\underline{a}]]^{\tau_1}_{X\backslash\{x\}+\{x'\}}\;{}_{x'}\square_{y'}\;[[\underline{b}]]^{\tau_2}_{Y\backslash\{y\}+\{y'\}}\\
&= [[\underline{a^{\tau_1}}]]_{X\backslash\{x\}+\{x'\}}\;{}_{x'}\square_{y'}\;[[\underline{b^{\tau_2}}]]_{Y\backslash\{y\}+\{y'\}}\\
&= [[\underline{a^{\tau_1}}\;{}_{x'}\square_{y'}\;\underline{b^{\tau_2}}]]_{X\backslash\{x\}+Y\backslash\{y\}}.
\end{aligned}$$

The first and the last object term in this sequence of equalities of interpretations are object terms of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$ and they are α-equivalent.

The substitution of parameters of object terms canonically induces substitution of parameters of arrow terms of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$. For an arrow term $\varphi : U \to V$ of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$, $\underline{a} \in P_{\underline{\mathcal{C}}}(U)$ and $\underline{a^\tau} \notin P_{\underline{\mathcal{C}}}(U)$, such that $U[\underline{a^\tau}/\underline{a}]$ (and thus also $V[\underline{a^\tau}/\underline{a}]$) is well-typed, the arrow term $\varphi[\underline{a^\tau}/\underline{a}] : U[\underline{a^\tau}/\underline{a}] \to V[\underline{a^\tau}/\underline{a}]$ is defined straightforwardly by modifying the indices of φ as dictated by the substitution $U[\underline{a^\tau}/\underline{a}]$.

EXAMPLE 2. If $\varphi = \beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}$, where $x \in X_1$, $a \in \underline{\mathcal{C}}(X_1)$ and $\underline{a} \in P_{\underline{\mathcal{C}}}(W_1)$, then

$$\beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}[\underline{a^\tau}/\underline{a}] = \beta^{x',\underline{x};y,\underline{y}}_{W_1[\underline{a^\tau}/\underline{a}],W_2,W_3},$$

where $x' = \tau^{-1}(x)$. □

We shall need the following property of the "interpretation of substitution".

Lemma 5. *Let W be an object term of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$ and let $x \in X$. Let $\underline{a} \in P_{\underline{\mathcal{C}}}(W)$ be such that $x \in FV(a)$, and suppose that $\tau : FV(a)\backslash\{x\} + \{x'\} \to FV(a)$ renames x to x'. We then have*

$$[\,W[\underline{a^\tau}/\underline{a}]\,] = [\,W\,]^\sigma,$$

where $\sigma : X\backslash\{x\} + \{x'\} \to X$ renames x to x'. Additionally, for any arrow term φ of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$ such that $W_s(\varphi) = W$, we have

$$[\,\varphi[\underline{a^\tau}/\underline{a}]\,] = [\,\varphi\,]^\sigma.$$

Proof. By easy inductions, thanks to (EQ), ($\beta\sigma$), ($\gamma\sigma$), (EQ-mor) and Remark 3.6. ■

Lemma 6. *If $W_1 \equiv W_2$, then $[W_1]_X = [W_2]_X$.*

Proof. By induction on the proof of $W_1 \equiv W_2$ and Lemma 5. ■

We now specify the syntax $\underset{\sim}{\mathtt{Free}}_{\underline{\mathcal{C}}}$. The object terms and the raw arrow terms of $\underset{\sim}{\mathtt{Free}}_{\underline{\mathcal{C}}}$ are exactly the object terms and the raw arrow terms of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$. The type of an arrow term φ of $\underset{\sim}{\mathtt{Free}}_{\underline{\mathcal{C}}}$ is again a pair of object terms, which we shall denote with $\vdash \varphi : U \to V$. The typing rules for arrow terms are the same as the typing rules for arrow terms of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$, except for the composition rule, for which we now set:

$$\frac{\vdash \varphi_1 : W_1 \to W_2 \quad \vdash \varphi_2 : W_2' \to W_3 \quad W_2 \equiv W_2'}{\vdash \varphi_2 \circ \varphi_1 : W_1 \to W_3}$$

As usual, with $\underset{\sim}{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$ we shall denote the collection of object terms of type X, together with the collection of arrow terms whose source and target are objects terms of type X.

The interpretation of $\underset{\sim}{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$ in $\mathcal{C}(X)$, is defined (and denoted) exactly as the interpretation $[-]_X$. In particular, the interpretation of the "relaxed" composition is defined by $[\varphi_2 \circ \varphi_1]_X = [\varphi_2]_X \circ [\varphi_1]_X$. The following lemma is a direct consequence of Lemma 6.

Lemma 7. *The interpretation function* $[-]_X : \underset{\sim}{\mathtt{Free}}_{\underline{\mathcal{C}}}(X) \to \mathcal{C}(X)$ *is well-defined.*

Lemma 8. *If* $\vdash \varphi : U \to V$ *is an arrow term of* $\underset{\sim}{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$ *and if* $U \equiv U'$, *then there exists an arrow term* $\varphi^{U'} : U' \to W_t(\varphi^{U'})$ *of* $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$, *such that*

$$W_t(\varphi^{U'}) \equiv V \qquad \text{and} \qquad [\varphi]_X = [\varphi^{U'}]_X.$$

Proof. By induction on the structure of φ.

- If $\varphi = 1_U$, then $\varphi^{U'} = 1_{U'}$. We conclude by (EQ) and Remark 3.6(a), for $\sigma = id_X$.
- Suppose that $\varphi = \beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}$. The source of φ is then $U = (W_1 \,{}_{x}\square_{\underline{x}}\, W_2)\,{}_{y}\square_{\underline{y}}\, W_3$. If the parameters $a_1 \in P_{\underline{\mathcal{C}}}(W_1)$, $a_{21}, a_{22} \in P_{\underline{\mathcal{C}}}(W_2)$ and $a_3 \in P_{\underline{\mathcal{C}}}(W_3)$ are such that $x \in FV(a_1)$, $\underline{x}, y \in FV(a_2)$ and $\underline{y} \in FV(a_3)$, then $U' = (W'_1{}_{x'}\square_{\underline{x}'} W'_2)_{y'}\square_{\underline{y}'} W'_3$, where

$$W_1[\underline{a_1^{\tau_1}}/\underline{a_1}] \equiv W'_1,\ \ W_2[\underline{a_{21}^{\tau_{21}}}/\underline{a_{21}}][\underline{a_{22}^{\tau_{22}}}/\underline{a_{22}}] \equiv W'_2 \ \ \text{and} \ \ W_3[\underline{a_3^{\tau_3}}/\underline{a_3}] \equiv W'_3$$

  and τ_1, τ_{21}, τ_{22} and τ_3 rename x to x', $\underline{x}$ to $\underline{x}'$, y to y' and $\underline{y}$ to $\underline{y}'$. We set

$$\varphi^{U'} = \beta^{x',\underline{x}';y',\underline{y}'}_{W'_1,W'_2,W'_3}.$$

  We conclude by (EQ) and $(\beta\sigma)$, for $\sigma = id_X$.
- If $\varphi = {\beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}}^{-1}$, then U' has the shape $W'_1\,{}_{x'}\square_{\underline{x}'}\,(W'_2\,{}_{y'}\square_{\underline{y}'}\,W'_3)$ (where W'_i and x', $\underline{x}'$, y' and $\underline{y}'$ are as in the previous case), and we set

$$\varphi^{U'} = {\beta^{x',\underline{x}';y',\underline{y}'}_{W'_1,W'_2,W'_3}}^{-1}.$$

  We conclude by (EQ), $(\beta\sigma)$ and Remark 3.6(a), for $\sigma = id_X$.
- Suppose that $\varphi = \gamma^{x,y}_{W_1,W_2}$. The source of φ is then $U = W_1\,{}_{x}\square_{y}\,W_2$. If the parameters $a_1 \in P_{\underline{\mathcal{C}}}(W_1)$ and $a_2 \in P_{\underline{\mathcal{C}}}(W_2)$ are such that $x \in FV(a_1)$ and $y \in FV(a_2)$, then $U' = W'_1{}_{x'}\square_{y'}W'_2$, where,

$$W_1[\underline{a_1^{\tau_1}}/\underline{a_1}] \equiv W'_1 \ \ \text{and} \ \ W_2[\underline{a_2^{\tau_2}}/\underline{a_2}] \equiv W'_2$$

  and τ_1 and τ_2 rename x to x' and y to y', respectively. We set

$$\varphi^{U'} = \gamma^{x',y'}_{W'_1,W'_2}$$

  and conclude by (EQ) and $(\gamma\sigma)$, for $\sigma = id_X$.

- Suppose that $\vdash \varphi_1 : U \to W$, $\vdash \varphi_2 : W' \to V$ and that $W \equiv W'$, and let $\varphi = \varphi_2 \circ \varphi_1$. By the induction hypothesis for φ_1 and U', there exist an arrow term
$$\varphi_1^{U'} : U' \to W_t(\varphi_1^{U'}),$$
such that $W_t(\varphi_1^{U'}) \equiv W$ and $[\varphi_1]_X = [\varphi_1^{U'}]_X$. Since $W \equiv W'$, by the transitivity of $\equiv$, we get $W_t(\varphi_1^{U'}) \equiv W'$. By the induction hypothesis for φ_2 and $W_t(\varphi_1^{U'})$, there exists an arrow term
$$\varphi_2^{W_t(\varphi_1^{U'})} : W_t(\varphi_1^{U'}) \to W_t(\varphi_2^{W_t(\varphi_1^{U'})}),$$
such that $W_t(\varphi_2^{W_t(\varphi_1^{U'})}) \equiv V$ and $[\varphi_2]_X = [\varphi_2^{W_t(\varphi_1^{U'})}]_X$. We define
$$\varphi^{U'} = \varphi_2^{W_t(\varphi_1^{U'})} \circ \varphi_1^{U'}.$$

- Suppose that $\vdash \varphi_1 : U_1 \to V_1$, $\vdash \varphi_2 : U_2 \to V_2$, and let $\varphi = \varphi_1 \,{}_x\Box_y\, \varphi_2$. In this case, the source of φ is $U = U_1 \,{}_x\Box_y\, U_2$ and we have two possibilities for the shape of U'.
  - $U' = U'_1 \,{}_{x'}\Box_{y'}\, U'_2$, where, assuming that $a_1 \in P_{\underline{\mathcal{C}}}(U_1)$ and $a_2 \in P_{\underline{\mathcal{C}}}(U_2)$ are such that $x \in FV(a_1)$ and $y \in FV(a_2)$, $U_1[\underline{a_1^{\tau_1}}/\underline{a_1}] \equiv U'_1$ and $U_2[\underline{a_2^{\tau_2}}/\underline{a_2}] \equiv U'_2$. Since $\underline{a_1^{\tau_1}} \in P_{\underline{\mathcal{C}}}(U'_1)$ and $\underline{a_2^{\tau_2}} \in P_{\underline{\mathcal{C}}}(U'_2)$, this means that, symmetrically, we have $U'_1[\underline{a_1}/\underline{a_1^{\tau_1}}] \equiv U_1$ and $U'_2[\underline{a_2}/\underline{a_2^{\tau_2}}] \equiv U_2$. By the induction hypothesis for φ_1 and $U'_1[\underline{a_1}/\underline{a_1^{\tau_1}}]$, as well as φ_2 and $U'_2[\underline{a_2}/\underline{a_2^{\tau_2}}]$, we get arrow terms
  $$\varphi_1^{U'_1[\underline{a_1}/\underline{a_1^{\tau_1}}]} : U'_1[\underline{a_1}/\underline{a_1^{\tau_1}}] \to W_t(\varphi_1^{U'_1[\underline{a_1}/\underline{a_1^{\tau_1}}]})$$
  and
  $$\varphi_2^{U'_2[\underline{a_2}/\underline{a_2^{\tau_2}}]} : U'_2[\underline{a_2}/\underline{a_2^{\tau_2}}] \to W_t(\varphi_2^{U'_2[\underline{a_2}/\underline{a_2^{\tau_2}}]}),$$
  such that
  $$W_t(\varphi_1^{U'_1[\underline{a_1}/\underline{a_1^{\tau_1}}]}) \equiv V_1, \quad \text{and} \quad [\varphi_1]_X = [\varphi_1^{U'_1[\underline{a_1}/\underline{a_1^{\tau_1}}]}]_X$$
  and
  $$W_t(\varphi_2^{U'_2[\underline{a_2}/\underline{a_2^{\tau_2}}]}) \equiv V_2 \quad \text{and} \quad [\varphi_2]_X = [\varphi_2^{U'_2[\underline{a_2}/\underline{a_2^{\tau_2}}]}]_X.$$
  By means of substitution on arrow terms, we define
  $$\varphi^{U'} = \varphi_1^{U'_1[\underline{a_1}/\underline{a_1^{\tau_1}}]}[\underline{a_1^{\tau_1}}/\underline{a_1}] \,{}_{x'}\Box_{y'}\, \varphi_2^{U'_2[\underline{a_2}/\underline{a_2^{\tau_2}}]}[\underline{a_2^{\tau_2}}/\underline{a_2}].$$
  - $U' = U'_1 \,{}_x\Box_y\, U'_2$, where $U_1 \equiv U'_1$ and $U_2 \equiv U'_2$. In this case, we define
  $$\varphi^{U'} = \varphi_1^{U'_1} \,{}_x\Box_y\, \varphi_2^{U'_2}.$$

We conclude by Lemma 6. ■

2.3.4 The first reduction

We make the first reduction in two steps. We first define a (non-deterministic) rewriting algorithm $\rightsquigarrow$ on $\mathtt{Free}_{\underline{\mathcal{C}}}(X)$ with outputs in $\utilde{\mathtt{Free}}_{\underline{\mathcal{C}}}$, in such a way that the interpretation of a term of $\mathtt{Free}_{\underline{\mathcal{C}}}$ matches the interpretations of (all) its "normal forms" relative to $\rightsquigarrow$. We then use Lemma 8 to move from $\utilde{\mathtt{Free}}_{\underline{\mathcal{C}}}$ to $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$, while preserving the equality of interpretations from the first step. This allows us to reduce the proof of the coherence theorem, which concerns all $\beta\gamma\sigma$-diagrams, to the consideration of parallel $\beta\gamma$-arrows in $\mathcal{C}(X)$ only.

We first define the rewriting algorithm $\rightsquigarrow$ on object terms of $\mathtt{Free}_{\underline{\mathcal{C}}}$. The algorithm $\rightsquigarrow$ takes an object term $\mathcal{W}$ of $\mathtt{Free}_{\underline{\mathcal{C}}}$ and returns (non-deterministically) an object term W of $\utilde{\mathtt{Free}}_{\underline{\mathcal{C}}}$, which we denote by $\mathcal{W} \rightsquigarrow W$, in the way specified by the following rules:

$$\frac{}{\underline{a} \rightsquigarrow \underline{a}} \qquad \frac{\mathcal{W}_1 \rightsquigarrow W_1 \qquad \mathcal{W}_2 \rightsquigarrow W_2}{\mathcal{W}_1 \,{}_x\square_y\, \mathcal{W}_2 \rightsquigarrow W_1 \,{}_x\square_y\, W_2}$$

$$\frac{}{\underline{a^\sigma} \rightsquigarrow \underline{a^\sigma}} \qquad \frac{\mathcal{W} \rightsquigarrow W}{\mathcal{W}^{id_X} \rightsquigarrow W} \qquad \frac{\mathcal{W}^{\sigma\circ\tau} \rightsquigarrow W}{(\mathcal{W}^{\sigma})^{\tau} \rightsquigarrow W}$$

$$\frac{\begin{array}{ccc} \sigma : Z \to X\backslash\{x\}+Y\backslash\{y\} & x',y' \notin X\backslash\{x\}+Y\backslash\{y\} & x' \neq y' \\ \sigma_1 : \sigma^{-1}[X\backslash\{x\}]+\{x'\} \to X & \sigma_1|^{X\backslash\{x\}} = \sigma|^{X\backslash\{x\}} & \sigma_1(x')=x \\ \sigma_2 : \sigma^{-1}[Y\backslash\{y\}]+\{y'\} \to Y & \sigma_2|^{Y\backslash\{y\}} = \sigma|^{Y\backslash\{y\}} & \sigma_2(y')=y \\ \multicolumn{3}{c}{\mathcal{W}_1^{\sigma_1} \rightsquigarrow W_1 \quad \mathcal{W}_2^{\sigma_2} \rightsquigarrow W_2} \end{array}}{(\mathcal{W}_1 \,{}_x\square_y\, \mathcal{W}_2)^{\sigma} \rightsquigarrow W_1 \,{}_{x'}\square_{y'}\, W_2}$$

The formal system defined above obviously has a termination property, in the sense that for all object terms $\mathcal{W}$ of $\mathtt{Free}_{\underline{\mathcal{C}}}$ there exists an object term W of $\utilde{\mathtt{Free}}_{\underline{\mathcal{C}}}$, such that $\mathcal{W} \rightsquigarrow W$. Notice also that the last rule is non-deterministic, as it involves a choice of x' and y'. In what follows, for an arbitrary object term $\mathcal{W}$ of $\mathtt{Free}_{\underline{\mathcal{C}}}$, we shall say that the outputs of the algorithm $\rightsquigarrow$ applied on $\mathcal{W}$ are *normal forms of* $\mathcal{W}$. We shall denote the collection of all normal forms of $\mathcal{W}$ with $\mathtt{NF}(\mathcal{W})$.

The formal system $(\mathtt{Free}_{\underline{\mathcal{C}}}, \rightsquigarrow)$ satisfies the following confluence-like property.

Lemma 9. *If* $W_1, W_2 \in \mathtt{NF}(\mathcal{W})$, *then* $W_1 \equiv W_2$.

Proof. Suppose that $(\mathcal{W}_1 \,{}_x\square_y\, \mathcal{W}_2)^{\sigma} \rightsquigarrow W_1 \,{}_{x'}\square_{y'}\, W_2$ is obtained from $\mathcal{W}_1^{\sigma_1} \rightsquigarrow W_1$ and $\mathcal{W}_2^{\sigma_2} \rightsquigarrow W_2$, and $(\mathcal{W}_1 \,{}_x\square_y\, \mathcal{W}_2)^{\sigma} \rightsquigarrow W_1' \,{}_{x''}\square_{y''}\, W_2'$ from $\mathcal{W}_1^{\tau_1} \rightsquigarrow W_1'$ and $\mathcal{W}_2^{\tau_2} \rightsquigarrow W_2'$.

Let $a \in P_{\underline{\mathcal{C}}}(W_1)$ and $b \in P_{\underline{\mathcal{C}}}(W_2)$ be such that $FV(a) = X_1$, $FV(b) = Y_1$, $x \in X_1$ and $y \in Y_1$, and let $\kappa_1 : X_1\backslash\{x'\} + \{x''\} \to X_1$ be the renaming of x' to x'' and $\kappa_2 : Y_1\backslash\{y'\} + \{y''\} \to Y_1$ the renaming of y' to y''. It is then easy to show that $\mathcal{W}_1^{\tau_1} \rightsquigarrow W_1[\underline{a^{\kappa_1}}/\underline{a}]$ and $\mathcal{W}_2^{\tau_2} \rightsquigarrow W_2[\underline{a^{\kappa_2}}/\underline{a}]$.

By the definition of $\equiv$, and the induction hypothesis for $\mathcal{W}_1^{\tau_1}$ (that reduces to both W_1' and $W_1[\underline{a^{\kappa_1}}/\underline{a}]$) and $\mathcal{W}_2^{\tau_2}$ (that reduces to both W_2' and $W_2[\underline{b^{\kappa_2}}/\underline{b}]$), we then have

$$W_1 \,{}_{x'}\square_{y'}\, W_2 \equiv W_1[\underline{a^{\kappa_1}}/\underline{a}] \,{}_{x''}\square_{y''}\, W_2[\underline{b^{\kappa_2}}/\underline{b}] = W_1' \,{}_{x''}\square_{y''}\, W_2'.$$

■

Lemma 10. *For an arbitrary object term* $\mathcal{W} : X$ *of* $\mathtt{Free}_{\underline{\mathcal{C}}}$ *and an arbitrary* $W \in \mathtt{NF}(\mathcal{W})$, *we have* $[[\mathcal{W}]]_X = [W]_X$.

Proof. By induction on the structure of $\mathcal{W}$.

- If $\mathcal{W} = \underline{a}$, we trivially have $[[\underline{a}]]_X = a = [\underline{a}]_X$.
- If $\mathcal{W} = \mathcal{W}_1{}_x\square_y\mathcal{W}_2$, where $\mathcal{W}_1 : X$ and $\mathcal{W}_2 : Y$, then, for any $W_1 \in \mathtt{NF}(\mathcal{W}_1)$ and $W_2 \in \mathtt{NF}(\mathcal{W}_2)$, $W_1\,{}_x\square_y\,W_2 \in \mathtt{NF}(\mathcal{W})$. Hence, by Lemma 9, we have that $W \equiv W_1\,{}_x\square_y\,W_2$. By the induction hypothesis for $\mathcal{W}_1$ and $\mathcal{W}_2$, we have that $[[\mathcal{W}_1]]_X = [W_1]_X$ and $[[\mathcal{W}_2]]_Y = [W_2]_Y$, and, by Lemma 6, we get

$$\begin{array}{rcl}[[\,\mathcal{W}_1{}_x\square_y\mathcal{W}_2\,]]_{X\backslash\{x\}+Y\backslash\{y\}} & = & [[\mathcal{W}_1]]_X\,{}_x\circ_y\,[[\mathcal{W}_2]]_Y \\ & = & [W_1]_X\,{}_x\circ_y\,[W_2]_Y \\ & = & [W_1\,{}_x\square_y\,W_2]_{X\backslash\{x\}+Y\backslash\{y\}} \\ & = & [W]_{X\backslash\{x\}+Y\backslash\{y\}}.\end{array}$$

- Suppose that $\mathcal{W} = \mathcal{V}^\sigma$, where $\mathcal{V} : X$ and $\sigma : Y \to X$. We proceed by case analysis relative to the shape of $\mathcal{V}$ (and σ).
  - If $\mathcal{V} = \underline{a}$, for some $a \in P_{\underline{\mathcal{C}}}$, then $[[\underline{a}^\sigma]]_Y = [[\underline{a}]]_X^\sigma = [\underline{a}]_X^\sigma = a^\sigma = [\underline{a^\sigma}]_Y$.
  - If $\sigma = id_X$, and if $V \in \mathtt{NF}(\mathcal{V})$, then $V \in \mathtt{NF}(\mathcal{W})$, and, by Lemma 9, we have that $W \equiv V$. By the induction hypothesis for $\mathcal{V}$ and Lemma 6, we get

$$[[\mathcal{V}^{id_X}]]_X = [[\mathcal{V}]]_X^{id_X} = [[\mathcal{V}]]_X = [V]_X = [W]_X.$$

  - If $\mathcal{V} = \mathcal{V}_1{}_x\square_y\mathcal{V}_2$, and if $V_1 \in \mathtt{NF}(\mathcal{V}_1^{\sigma_1})$ and $V_2 \in \mathtt{NF}(\mathcal{V}_2^{\sigma_2})$, then $V_1{}_{x'}\square_{y'}V_2 \in \mathtt{NF}(\mathcal{W})$, and, by Lemma 9, $W \equiv V_1{}_{x'}\square_{y'}V_2$. By the induction hypothesis for $\mathcal{V}_1^{\sigma_1}$ and $\mathcal{V}_2^{\sigma_2}$ and Lemma 6, we get

$$\begin{array}{rcl}[[(\mathcal{V}_1\,{}_x\square_y\,\mathcal{V}_2)^\sigma]]_Y & = & [[\mathcal{V}_1\,{}_x\square_y\,\mathcal{V}_2]]_X^\sigma \\ & = & ([[\mathcal{V}_1]]_{X_1}\,{}_x\square_y\,[[\mathcal{V}_2]]_{X_2})^\sigma \\ & = & [[\mathcal{V}_1]]_{X_1}^{\sigma_1}\,{}_{x'}\square_{y'}\,[[\mathcal{V}_2]]_{X_2}^{\sigma_2} \\ & = & [[\mathcal{V}_1^{\sigma_1}]]_{Y_1}\,{}_{x'}\square_{y'}\,[[\mathcal{V}_2^{\sigma_2}]]_{Y_2} \\ & = & [V_1]_{Y_1}\,{}_{x'}\square_{y'}\,[V_2]_{Y_2} \\ & = & [V_1\,{}_{x'}\square_{y'}\,V_2]_Y \\ & = & [W]_Y\end{array}$$

  - If $\mathcal{V} = \mathcal{U}^\tau$, and if $U \in \mathtt{NF}(\mathcal{U}^{\tau\circ\sigma})$, then $U \in \mathtt{NF}(\mathcal{W})$, and, by Lemma 9, $W \equiv U$. By the induction hypothesis for $\mathcal{U}^{\tau\circ\sigma}$ and Lemma 6, we get

$$[[(\mathcal{U}^\tau)^\sigma]]_Y = ([[\mathcal{U}]]_X^\tau)^\sigma = [[\mathcal{U}]]_X^{\tau\circ\sigma} = [[\mathcal{U}^{\tau\circ\sigma}]]_Y = [U]_Y = [W]_Y.$$ ∎

We move on to the first step of the first reduction of arrow terms of $\mathtt{Free}_{\underline{\mathcal{C}}}$: we define a (non-deterministic) rewriting algorithm $\rightsquigarrow$, which "normalises" arrow terms of $\mathtt{Free}_{\underline{\mathcal{C}}}$:

$$\frac{U \in \mathtt{NF}(\mathcal{U})}{1_{\mathcal{U}} \rightsquigarrow 1_U}$$

$$\frac{W_i \in \mathtt{NF}(\mathcal{W}_i) \quad i \in \{1,2,3\}}{\beta^{x,\underline{x};y,\underline{y}}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} \rightsquigarrow \beta^{x,\underline{x};y,\underline{y}}_{W_1,W_2,W_3}} \qquad \frac{W_i \in \mathtt{NF}(\mathcal{W}_i) \quad i \in \{1,2,3\}}{\beta^{x,\underline{x};y,\underline{y}\;-1}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3} \rightsquigarrow \beta^{x,\underline{x};y,\underline{y}\;-1}_{W_1,W_2,W_3}}$$

$$\frac{W_i \in \mathtt{NF}(\mathcal{W}_i) \quad i \in \{1,2\}}{\gamma^{x,y}_{\mathcal{W}_1,\mathcal{W}_2} \rightsquigarrow \gamma^{x,y}_{W_1,W_2}}$$

$$\frac{}{\varepsilon 1^{\sigma}_{\underline{a}} \rightsquigarrow 1_{\underline{a^{\sigma}}}} \qquad \frac{}{\varepsilon 1^{\sigma\;-1}_{\underline{a}} \rightsquigarrow 1_{\underline{a^{\sigma}}}} \qquad \frac{W \in \mathtt{NF}(\mathcal{W})}{\varepsilon 2_{\mathcal{W}} \rightsquigarrow 1_W} \qquad \frac{W \in \mathtt{NF}(\mathcal{W})}{\varepsilon 2^{-1}_{\mathcal{W}} \rightsquigarrow 1_W}$$

$$\frac{W \in \mathtt{NF}(\mathcal{W}^{\sigma\circ\tau})}{\varepsilon 3^{\sigma,\tau}_{\mathcal{W}} \rightsquigarrow 1_W} \qquad \frac{W \in \mathtt{NF}(\mathcal{W}^{\sigma\circ\tau})}{\varepsilon 3^{\sigma,\tau\;-1}_{\mathcal{W}} \rightsquigarrow 1_W}$$

$$\frac{W \in \mathtt{NF}((\mathcal{W}_1{}_x\square_y\mathcal{W}_2)^{\sigma})}{\varepsilon 4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma} \rightsquigarrow 1_W} \qquad \frac{W_1 \in \mathtt{NF}(\mathcal{W}_1^{\sigma_1}) \quad W_2 \in \mathtt{NF}(\mathcal{W}_2^{\sigma_2})}{\varepsilon 4^{x,y;x',y'\;-1}_{\mathcal{W}_1,\mathcal{W}_2;\sigma_1,\sigma_2} \rightsquigarrow 1_W}$$

$$\frac{\Phi_1 \rightsquigarrow \varphi_1 \quad \Phi_2 \rightsquigarrow \varphi_2}{\Phi_2 \circ \Phi_1 \rightsquigarrow \varphi_2 \circ \varphi_1} \qquad \frac{\Phi_1 \rightsquigarrow \varphi_1 \quad \Phi_2 \rightsquigarrow \varphi_2}{\Phi_1 \,{}_x\square_y\, \Phi_2 \rightsquigarrow \varphi_1 \,{}_x\square_y\, \varphi_2}$$

$$\frac{}{1^{\sigma}_{\underline{a}} \rightsquigarrow 1_{\underline{a^{\sigma}}}}$$

$$\frac{W_i \in \mathtt{NF}(\mathcal{W}_i^{\sigma_i})}{(\beta^{x,\underline{x};y,\underline{y}}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3})^{\sigma} \rightsquigarrow \beta^{x',\underline{x}';y',\underline{y}'}_{W_1,W_2,W_3}} \qquad \frac{W_i \in \mathtt{NF}(\mathcal{W}_i^{\sigma_i})}{(\beta^{x,\underline{x};y,\underline{y}\;-1}_{\mathcal{W}_1,\mathcal{W}_2,\mathcal{W}_3})^{\sigma} \rightsquigarrow \beta^{x',\underline{x}';y',\underline{y}'\;-1}_{W_1,W_2,W_3}}$$

$$\frac{W_i \in \mathtt{NF}(\mathcal{W}_i^{\sigma_i}) \quad i \in \{1,2\}}{(\gamma^{x,y}_{\mathcal{W}_1,\mathcal{W}_2})^{\sigma} \rightsquigarrow \gamma^{x',y'}_{W_1,W_2}}$$

$$\frac{}{(\varepsilon 1^{\sigma}_{\underline{a}})^{\kappa} \rightsquigarrow 1_{\underline{a^{\sigma\circ\kappa}}}} \qquad \frac{}{(\varepsilon 1^{\sigma\;-1}_{\underline{a}})^{\kappa} \rightsquigarrow 1_{\underline{a^{\sigma\circ\kappa}}}} \qquad \frac{W \in \mathtt{NF}(\mathcal{W}^{\kappa})}{(\varepsilon 2_{\mathcal{W}})^{\kappa} \rightsquigarrow 1_W} \qquad \frac{W \in \mathtt{NF}(\mathcal{W}^{\kappa})}{(\varepsilon 2^{-1}_{\mathcal{W}})^{\kappa} \rightsquigarrow 1_W}$$

$$\frac{W \in \mathtt{NF}(\mathcal{W}^{\sigma\circ\tau\circ\kappa})}{(\varepsilon 3^{\sigma,\tau}_{\mathcal{W}})^{\kappa} \rightsquigarrow 1_W} \qquad \frac{W \in \mathtt{NF}(\mathcal{W}^{\sigma\circ\tau\circ\kappa})}{(\varepsilon 3^{\sigma,\tau\;-1}_{\mathcal{W}})^{\kappa} \rightsquigarrow 1_W}$$

$$\frac{W \in \mathtt{NF}((\mathcal{W}_1{}_x\square_y\mathcal{W}_2)^{\sigma\circ\kappa})}{(\varepsilon 4^{x,y;x',y'}_{\mathcal{W}_1,\mathcal{W}_2;\sigma})^{\kappa} \rightsquigarrow 1_W} \qquad \frac{W_1 \in \mathtt{NF}(\mathcal{W}_1^{\sigma_1\circ\kappa_1}) \quad W_2 \in \mathtt{NF}(\mathcal{W}_2^{\sigma_2\circ\kappa_2})}{(\varepsilon 4^{x,y;x',y'\;-1}_{\mathcal{W}_1,\mathcal{W}_2;\sigma})^{\kappa} \rightsquigarrow 1_W}$$

$$\frac{\Phi \rightsquigarrow \varphi}{\Phi^{id_X} \rightsquigarrow \varphi} \qquad \frac{\Phi^{\sigma\circ\tau} \rightsquigarrow \varphi}{(\Phi^{\sigma})^{\tau} \rightsquigarrow \varphi} \qquad \frac{\Phi_1^{\sigma_1} \rightsquigarrow \varphi_1 \quad \Phi_2^{\sigma_2} \rightsquigarrow \varphi_2}{(\Phi_1 \,{}_x\square_y\, \Phi_2)^{\sigma} \rightsquigarrow \varphi_1 \,{}_{x'}\square_{y'}\, \varphi_2} \qquad \frac{\Phi_1^{\sigma} \rightsquigarrow \varphi_1 \quad \Phi_2^{\sigma} \rightsquigarrow \varphi_2}{(\Phi_2 \circ \Phi_1)^{\sigma} \rightsquigarrow \varphi_2 \circ \varphi_1}$$

We make first observations about this rewriting algorithm.

Remark 8. *Notice that, if $\Phi : \mathcal{U} \to \mathcal{V}$ and if $\Phi \rightsquigarrow \varphi$, then $\vdash \varphi : U \to V$, for some $U \in \mathtt{NF}(\mathcal{U})$ and $V \in \mathtt{NF}(\mathcal{V})$. Also, in the rule defining $(\Phi_2 \circ \Phi_1)^{\sigma} \rightsquigarrow \varphi_2 \circ \varphi_1$, the arrow term $\varphi_2 \circ \varphi_1$ is not well-typed in $\mathtt{Free}_{\underline{\mathcal{C}}}$ in general.*

As it was the case for the algorithm on object terms, this formal system is terminating. Therefore, the algorithm gives us, for each arrow term $\Phi : \mathcal{U} \to \mathcal{V}$, the set $\mathtt{NF}(\Phi)$ of normal forms of Φ, which are arrow terms of $\mathtt{Free}_{\underline{\mathcal{C}}}$. Here is the most important property of these normal forms.

Lemma 11. *For arbitrary arrow term Φ of $\mathtt{Free}_{\underline{\mathcal{C}}}(X)$ and $\varphi \in \mathtt{NF}(\Phi)$, we have $[[\Phi]]_X = [\varphi]_X$.*

Proof. By induction on the structure of Φ and Lemma 10. ■

2.3.5 The first reduction

Suppose that, for all object terms $\mathcal{W}$ of $\mathtt{Free}_{\underline{\mathcal{C}}}$, a normal form $\mathtt{red_1}(\mathcal{W}) \in \mathtt{NF}(\mathcal{W})$ in $\utilde{\mathtt{Free}}_{\underline{\mathcal{C}}}$ has been fixed, and that, *independently of that choice*, for all arrow terms Φ of $\mathtt{Free}_{\underline{\mathcal{C}}}$ a normal form $\mathtt{red_1}(\Phi) \in \mathtt{NF}(\Phi)$ in $\utilde{\mathtt{Free}}_{\underline{\mathcal{C}}}$ has been fixed.

We define the *first reduction function* $\mathtt{Red_1} : \mathtt{Free}_{\underline{\mathcal{C}}} \to \underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$ by

$$\mathtt{Red_1}(\mathcal{W}) = \mathtt{red_1}(\mathcal{W}) \quad \text{and} \quad \mathtt{Red_1}(\Phi) = \mathtt{red_1}(\Phi)^{\mathtt{red_1}(\mathcal{U})},$$

where $\Phi : \mathcal{U} \to \mathcal{V}$. Observe that, in the definition of $\mathtt{Red_1}(\Phi)$, we used the construction of Lemma 8, which indeed turns $\mathtt{red_1}(\Phi)$ (which is an arrow term of $\utilde{\mathtt{Free}}_{\underline{\mathcal{C}}}$) into an arrow term of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$. Also, for an arrow term $\Phi : \mathcal{U} \to \mathcal{V}$ of $\mathtt{Free}_{\underline{\mathcal{C}}}$, we have that $\mathtt{Red_1}(\Phi) : \mathtt{Red_1}(\mathcal{U}) \to V$, where, in general, $V \neq \mathtt{Red_1}(\mathcal{V})$. However, the following important property holds.

Lemma 12. *For any two arrow terms $\Phi, \Psi : \mathcal{U} \to \mathcal{V}$ of the same type in $\mathtt{Free}_{\underline{\mathcal{C}}}$, $\mathtt{Red_1}(\Phi)$ and $\mathtt{Red_1}(\Psi)$ are arrow terms of the same type in $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$.*

Proof. That $\mathtt{Red_1}(\Phi)$ and $\mathtt{Red_1}(\Psi)$ have the same source is clear by the definition. We prove the equality $W_t(\mathtt{Red_1}(\Phi)) = W_t(\mathtt{Red_1}(\Psi))$ by induction on the proof of $W_t(\mathtt{Red_1}(\Phi)) \equiv W_t(\mathtt{Red_1}(\Psi))$. Suppose that

$$W_t(\mathtt{Red_1}(\Phi)) = W_1 \,{}_x\square_y\, W_2 \quad \text{and} \quad W_t(\mathtt{Red_1}(\Psi)) = W_1[\underline{a^{\tau_1}}/\underline{a}] \,{}_{x'}\square_{y'}\, W_2[\underline{b^{\tau_2}}/\underline{b}].$$

If, moreover, at least one of τ_1 and τ_2 is not the identitiy, i.e. if, say, $x' \neq x$, then, by Remark 7, it cannot be the case that $\mathtt{Red_1}(\Phi)$ and $\mathtt{Red_1}(\Psi)$ have the same source. ■

The following theorem, essential for the proof of the coherence theorem, is simply an instance of Lemma 10 and Lemma 11.

Theorem 1. *For an arbitrary object term $\mathcal{W}$ and an arbitrary arrow term Φ of $\mathtt{Free}_{\underline{\mathcal{C}}}$, the following equalities of interpretations hold*

$$[[\mathcal{W}]]_X = [\mathtt{Red_1}(\mathcal{W})]_X \quad \textit{and} \quad [[\Phi]]_X = [\mathtt{Red_1}(\Phi)]_X.$$

2.4 The second reduction: getting rid of the cyclicity

Intuitively, this reduction goes from "cyclic operadic" to just "operadic", which cuts down the problem of commutation of all $\beta\gamma$-diagrams of $\mathcal{C}(X)$ to the problem of commutation of all $\beta\vartheta$-diagrams of $\mathcal{C}(X)$ (see (2.1)). As the "removal of cyclicity" is based on a transition from unrooted to rooted trees, we shall use a tree representation of our syntax, more convenient for "visualising" this reduction.

2.4.1 Unrooted trees

We first recall the formalism of unrooted trees, introduced in [CO16, Section 1.2.1] as a formalism of unrooted trees used in the definition of the free cyclic operad built over $\underline{\mathcal{C}}$. We omit the part of the syntax of unrooted trees which accounts for units of cyclic operads. Also, as the purpose of the formalism is to provide a representation of the terms of $\underline{\texttt{Free}}_{\underline{\mathcal{C}}}$, which do not encode symmetries, the unrooted trees will *not* be quotiented with α-equivalence, as it was the case in [CO16, Section 1.2.1].

A *corolla* is a term $a(x, y, z, \dots)$, where $a \in \underline{\mathcal{C}}(X)$ and $X = \{x, y, z, \dots\}$. We call the elements of X the *free variables* of $a(x, y, z, \dots)$, and we write $FV(a) = X$ to denote this set.

A *unrooted graph* $\mathcal{V}$ is a non-empty, finite set of corollas with mutually disjoint free variables, together with an involution σ on the set

$$V(\mathcal{V}) = \bigcup_{i=1}^{k} FV(a_i)$$

of all variables occuring in $\mathcal{V}$. We write

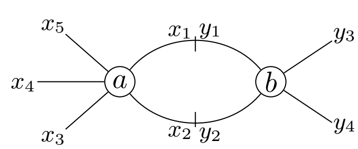
$$\mathcal{V} = \{a_1(x_1, \dots, x_n), \dots, a_k(y_1, \dots y_m); \sigma\}.$$

We shall denote with $Cor(\mathcal{V})$ the set of all corollas of $\mathcal{V}$, and we shall refer to a corolla by its parameter. The set of edges $Edge(\mathcal{V})$ of $\mathcal{V}$ consists of pairs (x, y) of variables such that $\sigma(x) = y$. Next, $FV(\mathcal{V})$ will denote the set of fixpoints of σ. Finally, with $FCor(\mathcal{V})$ we shall denote the subset of $Cor(\mathcal{V})$ consisting of corollas f of $\mathcal{V}$ for which $FV(f) \cap FV(\mathcal{V}) \neq \emptyset$.

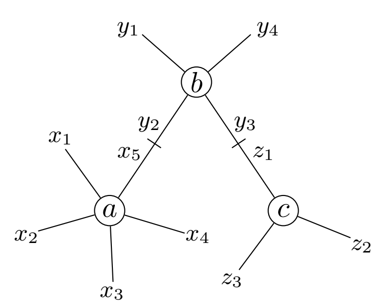
A graph is an unrooted tree if it is connected and if it does not contain loops, multiple edges and cycles. A subtree of an unrooted tree $\mathcal{T}$ is any non-empty connected subgraph of $\mathcal{T}$.

To give some intuition, here is an example.

Example 3. The graph $\mathcal{V} = \{a(x_1, x_2, x_3, x_4, x_5), b(y_1, y_2, y_3, y_4); \tau\}$, where $\tau(x_1) = y_1$, $\tau(x_2) = y_2$ and τ is identity otherwise, is *not* an unrooted tree, since it has two edges between a and b, which can be visualised as



The graph $\mathcal{T} = \{a(x_1, x_2, x_3, x_4, x_5), b(y_1, y_2, y_3, y_4), c(z_1, z_2, z_3); \sigma\}$, where $\sigma(x_5) = y_2$, $\sigma(y_3) = z_1$ and σ is identity otherwise, is an unrooted tree. It can be visualised as



□

Let $\mathcal{T}$, $\mathcal{T}_1$ and $\mathcal{T}_2$ be unrooted trees with involutions σ, σ_1 and σ_2, respectively. We say that $\mathcal{T}_1$ and $\mathcal{T}_2$ make a decomposition of $\mathcal{T}$ if $Cor(\mathcal{T}) = Cor(\mathcal{T}_1) + Cor(\mathcal{T}_2)$ and there exist $x \in FV(\mathcal{T}_1)$ and $y \in FV(\mathcal{T}_2)$ such that

$$\sigma(v) = \begin{cases} \sigma_1(v), & \text{if } v \in V(\mathcal{T}_1)\backslash\{x\} \\ \sigma_2(v), & \text{if } v \in V(\mathcal{T}_2)\backslash\{y\} \\ y, & \text{if } v = x. \end{cases}$$

We write $\mathcal{T} = \{\mathcal{T}_1\,(xy)\,\mathcal{T}_2\}$. We say that two subtrees $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathcal{T}$ are adjacent, and we write $a_{\mathcal{T}}(\mathcal{S}_1, \mathcal{S}_2) = 1$, if there exist $u \in FV(\mathcal{S}_1)$ and $v \in FV(\mathcal{S}_2)$, such that $\sigma(u) = v$. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are not adjacent, we write $a_{\mathcal{T}}(\mathcal{S}_1, \mathcal{S}_2) = 0$. If a subtree $\mathcal{S}$ of $\mathcal{T}$ is a corolla, say $a(x_1, \dots, a_n)$, we shall refer to $\mathcal{S}$ simply by a.

We shall denote with $\underline{\mathtt{T}_{\mathcal{C}}}$ (resp. $\underline{\mathtt{T}_{\mathcal{C}}}(X)$) the collection of unrooted trees whose corollas belong to $P_{\underline{\mathcal{C}}}$ (resp. whose corollas belong to $P_{\underline{\mathcal{C}}}$ and whose free variables are given by the set X).

2.4.2 A tree-wise representation of the terms of $\underline{\mathtt{Free}_{\mathcal{C}}}$

We introduce the syntax of parenthesised words generated by $P_{\underline{\mathcal{C}}}$, as

$$\boxed{w ::= \underline{a} \,|\, ww}$$

where $a \in P_{\underline{\mathcal{C}}}$. We shall denote the collection of all terms obtained in this way by $\mathtt{PWords}_{\underline{\mathcal{C}}}$.

For an unrooted tree $\mathcal{T}$, we next introduce the $\mathcal{T}$-*admissibility* relation on $\mathtt{PWords}_{\underline{\mathcal{C}}}$. Intuitively, w is $\mathcal{T}$-admissible if it represents a gradual composition of the corollas of $\mathcal{T}$. Formally, the predicate *w is $\mathcal{T}$-admissible* is defined by the following two clauses:

- ⋄ $\underline{a}$ is $\mathcal{T}$-admissible if $Cor(\mathcal{T}) = \{a\}$, and
- ⋄ if $\mathcal{T} = \{\mathcal{T}_1\,(xy)\,\mathcal{T}_2\}$, w_1 is $\mathcal{T}_1$-admissible and w_2 is $\mathcal{T}_2$-admissible, then $w_1 w_2$ is $\mathcal{T}$-admissible.

We shall denote the set of all $\mathcal{T}$-admissible terms of $\mathtt{PWords}_{\underline{\mathcal{C}}}$ with $A(\mathcal{T})$.

Remark 9. *Notice that, if w is $\mathcal{T}$-admissible, then, since all the corollas of $\mathcal{T}$ are mutually distinct, w does not contain repetitions of letters from $P_{\underline{\mathcal{C}}}$.*

A parenthesised word can be admissible with respect to more than one unrooted tree. In the second clause above, $w_1 w_2$ is admissible with respect to any *tree formed by $\mathcal{T}_1$ and $\mathcal{T}_2$.*

We introduce the syntax of *unrooted trees with grafting data induced by $\underline{\mathcal{C}}$*, denoted by $\underline{\mathtt{T}^+_{\mathcal{C}}}$, as follows. The collection of object terms of $\underline{\mathtt{T}^+_{\mathcal{C}}}$ is obtained by combining the syntax $\underline{\mathtt{T}_{\mathcal{C}}}$ and the syntax $\mathtt{PWords}_{\underline{\mathcal{C}}}$, by means of the $\mathcal{T}$-admissibility relation: we take for object terms of $\underline{\mathtt{T}^+_{\mathcal{C}}}$ all the pairs $(\mathcal{T}, w)$, typed as

$$\boxed{\frac{\mathcal{T} \in \underline{\mathtt{T}_{\mathcal{C}}}(X) \quad w \in \mathtt{PWords}_{\underline{\mathcal{C}}} \quad w \in A(\mathcal{T})}{(\mathcal{T}, w) : X}}$$

The arrow terms of $\underline{\mathtt{T}^+_{\mathcal{C}}}$ are obtained from raw terms

$$\varphi ::= 1_{(\mathcal{T},w)} \mid \beta^{x,\underline{x};y,\underline{y}}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)} \mid \beta^{x,\underline{x};y,\underline{y}\ \ -1}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)} \mid \gamma^{x,y}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)} \mid \varphi \circ \varphi \mid \varphi\ {}_x\square_y\ \varphi$$

by typing them as follows:

$$\frac{}{1_{(\mathcal{T},w)} : (\mathcal{T},w) \to (\mathcal{T},w)}$$

$$\frac{\mathcal{T} = \{\{\mathcal{T}_1\,(x\underline{x})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\} \quad y \in FV(\mathcal{T}_2)}{\beta^{x,\underline{x};y,\underline{y}}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)} : (\mathcal{T},(w_1w_2)w_3\,) \to (\mathcal{T},w_1(w_2w_3)\,)}$$

$$\frac{\mathcal{T} = \{\mathcal{T}_1\,(x\underline{x})\,\{\mathcal{T}_2\,(y\underline{y})\,\mathcal{T}_3\}\} \quad \underline{x} \in FV(\mathcal{T}_2)}{\beta^{x,\underline{x};y,\underline{y}\ \ -1}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)} : (\mathcal{T},w_1(w_2w_3)\,) \to (\mathcal{T},(w_1w_2)w_3\,)}$$

$$\frac{\mathcal{T} = \{\mathcal{T}_1\,(x\underline{x})\,\mathcal{T}_2\}}{\gamma^{x,y}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)} : (\mathcal{T},w_1w_2\,) \to (\mathcal{T},w_2w_1\,)}$$

$$\frac{\varphi_1 : (\mathcal{T},w_1) \to (\mathcal{T},w_2) \quad \varphi_2 : (\mathcal{T},w_2) \to (\mathcal{T},w_3)}{\varphi_2 \circ \varphi_1 : (\mathcal{T},w_1) \to (\mathcal{T},w_3)}$$

$$\frac{\varphi_1 : (\mathcal{T}_1,w_1) \to (\mathcal{T}_1,w_1') \quad \varphi_2 : (\mathcal{T}_2,w_2) \to (\mathcal{T}_2,w_2')}{\varphi_1\ {}_x\square_y\ \varphi_2 : (\{\mathcal{T}_1\,(xy)\,\mathcal{T}_2\},w_1w_2) \to (\{\mathcal{T}_1\,(xy)\,\mathcal{T}_2\},w_1'w_2')}$$

We shall denote the class of object terms of $\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}$ whose type is X, together with the class of arrow terms whose types are pairs of object terms of type X, by $\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}(X)$.

Lemma 13. *The terms of* $\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}(X)$ *are in one-to-one corresponcence with the terms of* $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$.

Proof. The correspondence $\Delta_X : \underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}(X) \to \underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$ is defined recursively as follows:

- ⋄ $\Delta_X((\{a(x_1,\dots,x_n);id\},\underline{a})) = \underline{a}$,
- ⋄ if $\Delta_X((\mathcal{T}_1,w_1)) = W_1$ and $\Delta_Y((\mathcal{T}_2,w_2)) = W_2$, and if $\mathcal{T} = \{\mathcal{T}_1(xy)\mathcal{T}_2\}$, then
$$\Delta_{X\backslash\{x\}+Y\backslash\{y\}}((\mathcal{T},w_1w_2)) = W_1\ {}_x\square_y\ W_2,$$
- ⋄ $\Delta_X(1_{(\mathcal{T},w)}) = 1_{\Delta_X((\mathcal{T},w))}$,
- ⋄ $\Delta_X(\beta^{x,\underline{x};y,\underline{y}}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)}) = \beta^{x,\underline{x};y,\underline{y}}_{\Delta_{X_1}((\mathcal{T}_1,w_1)),\Delta_{X_2}((\mathcal{T}_2,w_2)),\Delta_{X_2}((\mathcal{T}_3,w_3))}$,
- ⋄ $\Delta_X(\beta^{x,\underline{x};y,\underline{y}\ \ -1}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)}) = \beta^{x,\underline{x};y,\underline{y}\ \ -1}_{\Delta_{X_1}((\mathcal{T}_1,w_1)),\Delta_{X_2}((\mathcal{T}_2,w_2)),\Delta_{X_2}((\mathcal{T}_3,w_3))}$,
- ⋄ $\Delta_X(\gamma^{x,y}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)}) = \gamma^{x,y}_{\Delta_{X_1}((\mathcal{T}_1,w_1)),\Delta_{X_2}((\mathcal{T}_2,w_2))}$,
- ⋄ $\Delta_X(\varphi_2 \circ \varphi_1) = \Delta_X(\varphi_2) \circ \Delta_X(\varphi_1)$,
- ⋄ $\Delta_X(\varphi_1\ {}_x\square_y\ \varphi_2) = \Delta_{X_1}(\varphi_1)\ {}_x\square_y\ \Delta_{X_2}(\varphi_2)$.

■

We define the interpretation function $\lfloor - \rfloor_X : \underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}(X) \to \mathcal{C}(X)$ to be the composition $[-]_X \circ \Delta_X$. The following lemma is an immediate consequence of the definition of $\lfloor - \rfloor_X$.

Lemma 14. *For arbitrary object term W and arrow term φ of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}(X)$, the following equalities of interpretations hold*

$$[W]_X = \lfloor \Delta_X^{-1}(W) \rfloor_X \quad \textit{and} \quad [\varphi]_X = \lfloor \Delta_X^{-1}(\varphi) \rfloor_X.$$

Lemma 13 and Lemma 14 justify the representation of terms of $\underline{\mathtt{Free}}_{\underline{\mathcal{C}}}$ by means of unrooted trees with grafting data.

2.4.3 "Rooting" the syntax $\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}$

In this part, we introduce the syntax of *rooted trees with grafting data induced by* $\underline{\mathcal{C}}$, denoted by $\mathtt{r}\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}$, as follows.

For a pair $(\mathcal{T}, x)$ of an unrooted tree $\mathcal{T} \in \underline{\mathtt{T}}_{\underline{\mathcal{C}}}(X)$ and $x \in X$, we first introduce the $(\mathcal{T}, x)$-*admissibility* relation on $\mathtt{PWords}_{\underline{\mathcal{C}}}$. The predicate *$w$ is $(\mathcal{T}, x)$-admissibile* is defined by the following two clauses:

- $\diamond$ $\underline{a}$ is $(\mathcal{T}, x)$-admissibile if $Cor(\mathcal{T}) = \{a\}$, and
- $\diamond$ if $\mathcal{T} = \{\mathcal{T}_1 \,(zy)\, \mathcal{T}_2\}$, $x \in FV(\mathcal{T}_1)$ (without loss of generality), w_1 is $(\mathcal{T}_1, x)$-admissible and w_2 is $(\mathcal{T}_2, y)$-admissible, then $w_1 w_2$ is $(\mathcal{T}, x)$-admissible.

We shall denote the set of all $(\mathcal{T}, x)$-admissible terms of $\mathtt{PWords}_{\underline{\mathcal{C}}}$ with $A(\mathcal{T}, x)$.

Intuitively, w is $(\mathcal{T}, x)$-admissible if it is $\mathcal{T}$-admissible and it is an *operadic word* with respect to the rooted tree determined by considering x as the root of $\mathcal{T}$. As a matter of fact, $(\mathcal{T}, w)$ enjoys the following normalisation property, inherent to (formal terms which describe) *operadic* operations: all β^{-1}-reduction sequences starting from $(\mathcal{T}, w)$ end with an object term $(\mathcal{T}, w')$, such that all pairs of parentheses of w' are associated to the left.

The object terms of $\mathtt{r}\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}$ are triplets $(\mathcal{T}, x, w)$, typed as

$$\frac{\mathcal{T} \in \underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}(X) \qquad x \in X \qquad w \in A(\mathcal{T}, x)}{(\mathcal{T}, x, w) : X}$$

The class of arrow terms of $\mathtt{r}\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}$ is obtained from raw terms

$$\chi ::= \begin{cases} 1_{(\mathcal{T},x,w)} \mid \beta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} \mid \beta^{z;y\;\;-1}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} \\ \theta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} \mid \chi \circ \chi \mid \chi \,{}_{x}\square_{y}\, \chi \end{cases}$$

by typing them as follows:

$$\overline{1_{(\mathcal{T},x,w)} : (\mathcal{T},x,w) \to (\mathcal{T},x,w)}$$

$$\frac{\mathcal{T} = \{\{\mathcal{T}_1\,(z\underline{z})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\} \quad y \in FV(\mathcal{T}_2) \quad x \in X \cap FV(\mathcal{T}_1)}{\beta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} : (\mathcal{T},x,(w_1w_2)w_3\,) \to (\mathcal{T},x,w_1(w_2w_3)\,)}$$

$$\frac{\mathcal{T} = \{\mathcal{T}_1\,(z\underline{z})\,\{\mathcal{T}_2\,(y\underline{y})\,\mathcal{T}_3\}\} \quad \underline{z} \in FV(\mathcal{T}_2) \quad x \in X \cap FV(\mathcal{T}_1)}{\beta^{z;y\;\,-1}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} : (\mathcal{T},x,w_1(w_2w_3)\,) \to (\mathcal{T},x,(w_1w_2)w_3\,)}$$

$$\frac{\mathcal{T} = \{\{\mathcal{T}_1\,(z\underline{z})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\} \quad y \in FV(\mathcal{T}_1) \quad x \in X \cap FV(\mathcal{T}_1)}{\theta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} : (\mathcal{T},x,(w_1w_2)w_3\,) \to (\mathcal{T},x,(w_1w_3)w_2\,)}$$

$$\frac{\chi_1 : (\mathcal{T},x,w_1) \to (\mathcal{T},x,w_2) \quad \chi_2 : (\mathcal{T},x,w_2) \to (\mathcal{T},x,w_3)}{\chi_2 \circ \chi_1 : (\mathcal{T},x,w_1) \to (\mathcal{T},x,w_3)}$$

$$\frac{\chi_1 : (\mathcal{T}_1,x,w_1) \to (\mathcal{T}_1,x,w'_1) \;\; \chi_2 : (\mathcal{T}_2,y,w_2) \to (\mathcal{T}_2,y,w'_2) \;\; z \in FV(\mathcal{T}_1) \;\; z \neq x}{\chi_1\ {}_z\square_y\ \chi_2 : (\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},x,w_1w_2) \to (\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},x,w'_1w'_2)}$$

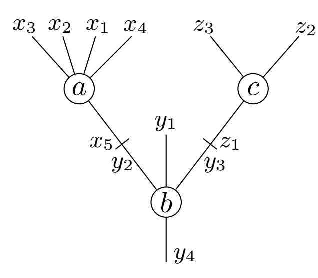
We shall denote the class of object terms of $\mathtt{r}\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}$ whose type is X, together with the class of arrow terms whose types are pairs of object terms of type X, by $\mathtt{r}\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}(X)$.

Notice that, for an object term $(\mathcal{T},x,w)$ of $\mathtt{r}\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}(X)$, the choice of $x \in X$ as the root of $\mathcal{T}$ determines the roots of all subtrees of $\mathcal{T}$, and, in particular, of all corollas of $\mathcal{T}$. In other words, this choice allows us to speak about the inputs and the output of any subtree of $\mathcal{T}$.

Formally, for a subtree $\mathcal{S}$ of $\mathcal{T}$ and a variable $x \in FV(\mathcal{T})$, we define the set $\mathtt{inp}_{(\mathcal{T},x)}(\mathcal{S})$ of inputs of $\mathcal{S}$ and the output $\mathtt{out}_{(\mathcal{T},x)}(\mathcal{S})$ of $\mathcal{S}$, induced by x, as follows:

- ⋄ $\mathtt{inp}_{(\mathcal{T},x)}(\mathcal{T}) = FV(\mathcal{T})\backslash\{x\}$ and $\mathtt{out}_{(\mathcal{T},x)}(\mathcal{T}) = x$,
- ⋄ if $\mathcal{S} \neq \mathcal{T}$, if $a \in Cor(\mathcal{T})$ is such that $x \in FV(a)$, if $c \in Cor(\mathcal{S})$ is the corolla of $\mathcal{S}$ with the smallest distance from a, and if p is the sequence of half-edges from c to a, then $\mathtt{inp}_{(\mathcal{T},x)}(\mathcal{S}) = FV(\mathcal{S})\backslash\{z\}$, where $z \in FV(c) \cap p$, and $\mathtt{out}_{(\mathcal{T},x)}(\mathcal{S}) = z$.

Example 4. For unrooted tree $\mathcal{T}$ from example 3, the choice of, say, $y_4 \in X$, turns $\mathcal{T}$ into a rooted tree, which can be visualised as



We have $\mathtt{inp}_{(\mathcal{T},y_4)}(b) = \{y_1,y_2,y_3\}$, $\mathtt{out}_{(\mathcal{T},y_4)}(b) = y_4$, $\mathtt{inp}_{(\mathcal{T},y_4)}(a) = \{x_1,x_2,x_3,x_4\}$, $\mathtt{out}_{(\mathcal{T},y_4)}(a) = x_5$ and $\mathtt{inp}_{(\mathcal{T},y_4)}(c) = \{z_2,z_3\}$, $\mathtt{out}_{(\mathcal{T},y_4)}(c) = z_1$.

Observe that, among all paranthesised words admisslible with respect to $\mathcal{T}$, only $(\underline{ba})\underline{c}$ and $(\underline{bc})\underline{a}$ are operadic, relative to the choice of y_4 as the root of $\mathcal{T}$. □

2.4.4 The interpretation of $\mathtt{r}\underline{\mathtt{T}}^+_{\mathcal{C}}$ in $\mathcal{C}$

We define the interpretation function

$$\lceil - \rceil_X : \mathtt{r}\underline{\mathtt{T}}^+_{\mathcal{C}}(X) \to \mathcal{C}(X)$$

recursively as follows:

- $\diamond$ $\lceil (\{a(x_1,\dots,x_n)\}, x_i, \underline{a}) \rceil_X = a$,
- $\diamond$ $\lceil (\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\}, x, w_1w_2) \rceil_X = \lceil (\mathcal{T}_1, x, w_1) \rceil_{X_1\,z}\circ_y \lceil (\mathcal{T}_2, y, w_2) \rceil_{X_2}$,

and

- $\diamond$ $\lceil 1_{(\mathcal{T},x,w)} \rceil_X = 1_{\lceil (\mathcal{T},x,w) \rceil_X}$,
- $\diamond$ $\lceil \beta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} \rceil_X = \beta^{z,\underline{z};y,\underline{y}}_{\lceil (\mathcal{T}_1,x,w_1)\rceil_{X_1},\lceil (\mathcal{T}_2,\underline{z},w_2)\rceil_{X_2},\lceil (\mathcal{T}_3,\underline{z},w_3)\rceil_{X_3}}$,
- $\diamond$ $\lceil \beta^{z;y\ \ -1}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} \rceil_X = \beta^{z,\underline{z};y,\underline{y}\ \ -1}_{\lceil (\mathcal{T}_1,x,w_1)\rceil_{X_1},\lceil (\mathcal{T}_2,\underline{z},w_2)\rceil_{X_2},\lceil (\mathcal{T}_3,\underline{z},w_3)\rceil_{X_3}}$,
- $\diamond$ $\lceil \theta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)} \rceil_X = \vartheta^{z,\underline{z};y,\underline{y}}_{\lceil (\mathcal{T}_1,x,w_1)\rceil_{X_1},\lceil (\mathcal{T}_2,\underline{z},w_2)\rceil_{X_2},\lceil (\mathcal{T}_3,\underline{y},w_3)\rceil_{X_3}}$ (see (2.1)),
- $\diamond$ $\lceil \chi_2 \circ \chi_1 \rceil_X = \lfloor \chi_2 \rfloor_X \circ \lfloor \chi_1 \rfloor_X$, and
- $\diamond$ $\lceil \chi_1\ {}_z\square_y\ \chi_2 \rceil_X = \lceil \chi_1 \rceil_{X_1\,z}\circ_y \lceil \chi_2 \rceil_{X_2}$.

Remark 10. *Notice that $\lceil \chi \rceil_X$ is an arrow in $\mathcal{C}(X)$ all of whose instances of the isomorphism γ are "hidden" by using explicitly the abbreviation ϑ. In other words, the semantics of arrow terms of $\mathtt{r}\underline{\mathtt{T}}^+_{\mathcal{C}}$ is what distinguishes $\beta\vartheta$-arrows of $\mathcal{C}(X)$.*

2.4.5 The second reduction

We define the familly of *second reduction functions*

$$\mathtt{Red}_2(X,x) : \underline{\mathtt{T}}^+_{\mathcal{C}}(X) \to \mathtt{r}\underline{\mathtt{T}}^+_{\mathcal{C}}(X),$$

where $x \in X$, as follows. For the object terms of $\underline{\mathtt{T}}^+_{\mathcal{C}}(X)$, we set

$$\mathtt{Red}_2(X,x)((\mathcal{T},w)) = (\mathcal{T}, x, w^{\cdot x}),$$

where $w^{\cdot x}$ is the $(\mathcal{T},x)$-admissible parenthesised word defined recursively by the following clauses:

- $\diamond$ if $w = \underline{a}$, then $w^{\cdot x} = \underline{a}$,
- $\diamond$ if $\mathcal{T} = \{\mathcal{T}_1\,(x_1x_2)\,\mathcal{T}_2\}$, $w = w_1w_2$, $w_i \in A(\mathcal{T}_i)$, $(\mathcal{T}_i, w_i) : X_i$ $i = 1,2$, then
  - if $x \in X_1$, then $w^{\cdot x} = w_1^{\cdot x} w_2^{\cdot x_2}$,
  - if $x \in X_2$, then $w^{\cdot x} = w_2^{\cdot x} w_1^{\cdot x_1}$.

Observe that the successive commutations which transform w into the operadic word $w^{\cdot x}$ are witnessed in $\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}$ by the arrow term

$$\kappa_{(\mathcal{T},w,x)} : (\mathcal{T}, w) \to (\mathcal{T}, w^{\cdot x}),$$

defined recursively as follows:

- ◇ if $w = \underline{a}$, then $\kappa_{(\mathcal{T},w,x)} = 1_{(\mathcal{T},w)}$,
- ◇ if $\mathcal{T} = \{\mathcal{T}_1(x_1x_2)\mathcal{T}_2\}$, $w = w_1w_2$, $w_i \in A(\mathcal{T}_i)$ and $(\mathcal{T}_i, w_i) : X_i$ $i = 1, 2$, then
  - if $x \in X_1$, then $\kappa_{(\mathcal{T},w,x)} = \kappa_{(\mathcal{T}_1,w_1,x)}\ {}_{x_1}\square_{x_2}\ \kappa_{(\mathcal{T}_2,w_2,x_2)}$,
  - if $x \in X_2$, then $\kappa_{(\mathcal{T},w,x)} = \big(\kappa_{(\mathcal{T}_2,w_2,x)}\ {}_{x_2}\square_{x_1}\ \kappa_{(\mathcal{T}_1,w_1,x_1)}\big) \circ \gamma^{x_1,x_2}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)}$.

Before we rigorously define the second reduction of arrow terms, we illustrate the idea behind it with a toy example.

Example 5. Consider the object term $(\mathcal{T}, (\underline{a}\,\underline{b})\underline{c}) : X$, where $\mathcal{T}$ is defined as in Example 3. The arrow term

$$\beta^{x_i,y_{j_1};y_{j_2},z_l}_{(\mathcal{T}_1,\underline{a}),(\mathcal{T}_2,\underline{b}),(\mathcal{T}_3,\underline{c})} : (\mathcal{T}, (\underline{a}\,\underline{b})\underline{c}) \to (\mathcal{T}, \underline{a}(\underline{b}\,\underline{c}))$$

where $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are the subtrees of $\mathcal{T}$ determined by corollas a, b and c, respectively, is then well-typed and, by choosing $y_4 \in X$ (as we did in Example 4), we have

$$\mathtt{Red}_2(X, y_4)((\mathcal{T}, (\underline{a}\,\underline{b})\underline{c})) = (\mathcal{T}, y_4, (\underline{b}\,\underline{a})\underline{c}) \quad \text{and} \quad \mathtt{Red}_2(X, y_4)((\mathcal{T}, \underline{a}(\underline{b}\,\underline{c}))) = (\mathcal{T}, y_4, (\underline{b}\,\underline{c})\underline{a}).$$

For the two reductions of object terms, the arrow term

$$\theta^{y_2;y_3}_{(\mathcal{T}_2,y_4,\underline{b}),(\mathcal{T}_1,x_5,\underline{a}),(\mathcal{T}_3,z_1,\underline{c})} : (\mathcal{T}, y_4, (\underline{b}\,\underline{a})\underline{c}) \to (\mathcal{T}, y_4, (\underline{b}\,\underline{c})\underline{a})$$

is well-typed and it will be exactly the second reduction of $\beta^{x_i,y_{j_1};y_{j_2},z_l}_{(\mathcal{T}_1,\underline{a}),(\mathcal{T}_2,\underline{b}),(\mathcal{T}_3,\underline{c})}$. □

Formally, for an arrow term $\varphi : (\mathcal{T}, u) \to (\mathcal{T}, v)$ of $\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}(X)$, $\mathtt{Red}_2(X, x)(\varphi)$ is the arrow term defined recursively, as follows:

- ◇ $\mathtt{Red}_2(X, x)(1_{(\mathcal{T},w)}) = 1_{\mathtt{Red}_2(X,x)((\mathcal{T},w))}$,
- ◇ if $\varphi = \beta^{z,\underline{z};y,\underline{y}}_{(\mathcal{T}_1,w),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)}$, where $(\mathcal{T}_1, w_1) : X_i$, and
  - if $x \in X_1$, then $\mathtt{Red}_2(X, x)(\varphi) = \beta^{z;y}_{\mathtt{Red}_2(X_1,x)((\mathcal{T}_1,w_1)),\mathtt{Red}_2(X_2,\underline{z})((\mathcal{T}_2,w_2)),\mathtt{Red}_2(X_3,\underline{y})((\mathcal{T}_3,w_3))}$,
  - if $x \in X_2$, then $\mathtt{Red}_2(X, x)(\varphi) = \theta^{\underline{z};y}_{\mathtt{Red}_2(X_2,x)((\mathcal{T}_2,w_2)),\mathtt{Red}_2(X_1,z)((\mathcal{T}_1,w_1)),\mathtt{Red}_2(X_3,\underline{y})((\mathcal{T}_3,w_3))}$,
  - if $x \in X_3$, then $\mathtt{Red}_2(X, x)(\varphi) = \beta^{\underline{y};\underline{z}\ -1}_{\mathtt{Red}_2(X_3,x)((\mathcal{T}_3,w_3)),\mathtt{Red}_2(X_2,y)((\mathcal{T}_2,w_2)),\mathtt{Red}_2(X_1,z)((\mathcal{T}_1,w_1))}$,
- ◇ if $\varphi = \beta^{z,\underline{z};y,\underline{y}\ -1}_{(\mathcal{T}_1,w),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)}$, where $(\mathcal{T}_1, w_1) : X_i$, and
  - if $x \in X_1$, then $\mathtt{Red}_2(X, x)(\varphi) = \beta^{z;y\ -1}_{\mathtt{Red}_2(X_1,x)((\mathcal{T}_1,w_1)),\mathtt{Red}_2(X_2,\underline{z})((\mathcal{T}_2,w_2)),\mathtt{Red}_2(X_3,\underline{y})((\mathcal{T}_3,w_3))}$,

– if $x\in X_2$, then $\mathtt{Red}_2(X,x)(\varphi)=\theta^{y;\underline{z}}_{\mathtt{Red}_2(X_2,x)((\mathcal{T}_2,w_2)),\mathtt{Red}_2(X_3,\underline{y})((\mathcal{T}_3,w_3)),\mathtt{Red}_2(X_1,z)((\mathcal{T}_1,w_1))}$,

– if $x\in X_3$, then $\mathtt{Red}_2(X,x)(\varphi)=\beta^{\underline{y};\underline{z}}_{\mathtt{Red}_2(X_3,x)((\mathcal{T}_3,w_3)),\mathtt{Red}_2(X_2,y)((\mathcal{T}_2,w_2)),\mathtt{Red}_2(X_1,z)((\mathcal{T}_1,w_1))}$,

◇ if $\varphi=\gamma^{z,y}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)}$, then $\mathtt{Red}_2(X,x)(\varphi)=1_{\mathtt{Red}_2(X,x)((\{\mathcal{T}_1(zy)\mathcal{T}_2\},w_1w_2))}$

◇ if $\varphi=\varphi_2\circ\varphi_1$, then $\mathtt{Red}_2(X,x)(\varphi)=\mathtt{Red}_2(X,x)(\varphi_2)\circ\mathtt{Red}_2(X,x)(\varphi_1)$,

◇ if $\varphi=\varphi_1\;{}_z\square_y\;\varphi_2$, where $\varphi_1:(\mathcal{T}_1,w_1)\rightarrow(\mathcal{T}_1',w_1')$, $\varphi_2:(\mathcal{T}_2,w_2)\rightarrow(\mathcal{T}_2',w_2')$ and $(\mathcal{T}_i,w_i):X_i$, then

– if $x\in X_1$, then $\mathtt{Red}_2(X,x)(\varphi)=\mathtt{Red}_2(X_1,x)(\varphi_1)\;{}_z\square_y\;\mathtt{Red}_2(X_2,y)(\varphi_2)$,

– if $x\in X_2$, then $\mathtt{Red}_2(X,x)(\varphi)=\mathtt{Red}_2(X_2,x)(\varphi_2)\;{}_y\square_z\;\mathtt{Red}_2(X_1,z)(\varphi_1)$.

Remark 11. *Notice that, for $\varphi:(\mathcal{T},u)\rightarrow(\mathcal{T},v)$, the type of $\mathtt{Red}_2(X,x)(\varphi)$ is*

$$\mathtt{Red}_2(X,x)(\varphi):\mathtt{Red}_2(X,x)((\mathcal{T},u))\rightarrow\mathtt{Red}_2(X,x)((\mathcal{T},v)).$$

Therefore, the second reduction of a pair of arrow terms of the same type *in $\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}(X)$ is a pair of arrow terms of the* same type *in $\mathtt{r}\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}(X)$.*

The following theorem is the core of the coherence theorem. Intuitively, it says that the coherence of non-symmetric non-skeletal cyclic operads can be reduced to the coherence of non-symmetric non-skeletal operads[1]. As it will be clear from its proof, ($\beta\gamma$-hexagon) is the key coherence condition that makes this reduction possible.

Theorem 2. *For an arbitrary object term $(\mathcal{T},w)$ and an arbitrary arrow term $\varphi:(\mathcal{T},u)\rightarrow(\mathcal{T},v)$ of $\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}$, the following equality of interpretations holds:*

$$\lfloor\kappa_{(\mathcal{T},v,x)}\rfloor_X\circ\lfloor\varphi\rfloor_X=\lceil\mathtt{Red}_2(X,x)(\varphi)\rceil_X\circ\lfloor\kappa_{(\mathcal{T},u,x)}\rfloor_X.$$

Proof. By the definition of the interpretation function $\lfloor-\rfloor_X$, the equality of interpretations of arrow terms that we need to prove is

$$[\Delta_X(\kappa_{(\mathcal{T},v,x)})]_X\circ[\Delta_X(\varphi)]_X=\lceil\mathtt{Red}_2(X,x)(\varphi)\rceil_X\circ[\Delta_X(\kappa_{(\mathcal{T},u,x)})]_X. \tag{2.3}$$

We proceed by induction on the structure of φ.

- If $\varphi=1_{(\mathcal{T},w)}$, then

$$\begin{aligned}[\Delta_X(\kappa_{(\mathcal{T},w,x)})]_X\circ[\Delta_X(1_{(\mathcal{T},w)})]_X &= [\Delta_X(\kappa_{(\mathcal{T},w,x)})]_X\\ &= 1_{(\mathcal{T},x,w\cdot x)}\circ[\Delta_X(\kappa_{(\mathcal{T},w,x)})]_X\\ &= \lceil\mathtt{Red}_2(X,x)(1_{(\mathcal{T},w)})\rceil_X\circ[\Delta_X(\kappa_{(\mathcal{T},w,x)})]_X.\end{aligned}$$

[1]Although it might seem that the syntax $\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}$ encodes canonical diagrams of non-symmetric categorified cyclic operads, this is not the case: non-symmetric cyclic operads still contain *cyclic actions*, while $\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}$ does not encode any action of the symmetric group. For the definition of a non-symmetric cyclic operad, see [CGR14, Section 3.2] (exchangeable-output, skeletal) and [M16, Sections 1,2,3] (entries-only, non-skeletal).

- Suppose that $\varphi = \beta^{z,\underline{z};y,\underline{y}}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)}$, where $(\mathcal{T}_i, w_i) : X_i$.
  - If $x \in X_1$, then
$$\kappa_{(\mathcal{T},u,x)} = \big(\kappa_{(\mathcal{T}_1,w_1,x)}\ {}_x\square_{\underline{x}}\ \kappa_{(\mathcal{T}_2,w_2,\underline{z})}\big)\ {}_y\square_{\underline{y}}\ \kappa_{(\mathcal{T}_3,w_3,\underline{y})}$$
and
$$\kappa_{(\mathcal{T},v,x)} = \kappa_{(\mathcal{T}_1,w_1,x)}\ {}_x\square_{\underline{x}}\ \big(\kappa_{(\mathcal{T}_2,w_2,\underline{z})}\ {}_y\square_{\underline{y}}\ \kappa_{(\mathcal{T}_3,w_3,\underline{y})}\big).$$
Denote
$$\begin{array}{lll} \kappa_1 = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,w_1,x)})]_{X_1} & \kappa_2 = [\Delta_{X_2}(\kappa_{(\mathcal{T}_2,w_2,\underline{z})})]_{X_2} & \kappa_3 = [\Delta_{X_3}(\kappa_{(\mathcal{T}_3,w_3,\underline{y})})]_{X_3} \\ f_1 = [\Delta_{X_1}((\mathcal{T}_1,w_1))]_{X_1} & f_2 = [\Delta_{X_2}((\mathcal{T}_2,w_2))]_{X_2} & f_3 = [\Delta_{X_3}((\mathcal{T}_3,w_3))]_{X_3} \\ f_1^{\bullet} = \lceil \texttt{Red}_2(X_1,x)((\mathcal{T}_1,w_1))\rceil_{X_1} & f_2^{\bullet} = \lceil \texttt{Red}_2(X_2,\underline{z})(\mathcal{T}_2,w_2)\rceil_{X_2} & f_3^{\bullet} = \lceil \texttt{Red}_2(X_3,\underline{y})(\mathcal{T}_3,w_3))\rceil_{X_3} \end{array}$$
The left-hand side and the right-hand side of (2.3) then correspond exactly to the top-right side and the left-bottom side, respectively, of the commuting diagram
$$\begin{array}{ccc} (f_1\ {}_z\circ_{\underline{z}}\ f_2)\ {}_y\circ_{\underline{y}}\ f_3 & \xrightarrow{\beta^{z,\underline{z};y,\underline{y}}_{f_1,f_2,f_3}} & f_1\ {}_z\circ_{\underline{z}}\ (f_2\ {}_y\circ_{\underline{y}}\ f_3) \\ {\scriptstyle (\kappa_1\ {}_z\circ_{\underline{z}}\ \kappa_2)\ {}_y\circ_{\underline{y}}\ \kappa_3}\downarrow & & \downarrow{\scriptstyle \kappa_1\ {}_z\circ_{\underline{z}}\ (\kappa_2\ {}_y\circ_{\underline{y}}\ \kappa_3)} \\ (f_1^{\bullet}\ {}_z\circ_{\underline{z}}\ f_2^{\bullet})\ {}_y\circ_{\underline{y}}\ f_3^{\bullet} & \xrightarrow{\beta^{z,\underline{z};y,\underline{y}}_{f_1^{\bullet},f_2^{\bullet},f_3^{\bullet}}} & f_1^{\bullet}\ {}_z\circ_{\underline{z}}\ (f_2^{\bullet}\ {}_y\circ_{\underline{y}}\ f_3^{\bullet}) \end{array}$$
which is a naturality diagram for β.
  - If $x \in X_2$, then
$$\kappa_{(\mathcal{T},u,x)} = \big(\big(\kappa_{(\mathcal{T}_2,w_2,x)}\ {}_{\underline{z}}\square_z\ \kappa_{(\mathcal{T}_1,w_1,z)}\big)\ {}_y\square_{\underline{y}}\ \kappa_{(\mathcal{T}_3,w_3,\underline{y})}\big) \circ \big(\gamma^{z,\underline{z}}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)}\ {}_y\square_{\underline{y}}\ 1_{(\mathcal{T}_3,w_3)}\big)$$
and
$$\kappa_{(\mathcal{T},v,x)} = \big(\big(\kappa_{(\mathcal{T}_2,w_2,x)}\ {}_y\square_{\underline{y}}\ \kappa_{(\mathcal{T}_3,w_3,\underline{y})}\big)\ {}_{\underline{z}}\square_z\ \kappa_{(\mathcal{T}_1,w_1,z)}\big) \circ \big(\gamma^{z,\underline{z}}_{(\mathcal{T}_1,w_1),(\{\mathcal{T}_2(y\underline{y})\mathcal{T}_3\},w_2w_3)}\big).$$
Denote
$$\begin{array}{lll} \kappa_1 = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,w_1,z)})]_{X_1} & \kappa_2 = [\Delta_{X_2}(\kappa_{(\mathcal{T}_2,w_2,x)})]_{X_2} & \kappa_3 = [\Delta_{X_3}(\kappa_{(\mathcal{T}_3,w_3,\underline{y})})]_{X_3} \\ f_1 = [\Delta_{X_1}((\mathcal{T}_1,w_1))]_{X_1} & f_2 = [\Delta_{X_2}((\mathcal{T}_2,w_2))]_{X_2} & f_3 = [\Delta_{X_3}((\mathcal{T}_3,w_3))]_{X_3} \\ f_1^{\bullet} = \lceil \texttt{Red}_2(X_1,z)((\mathcal{T}_1,w_1))\rceil_{X_1} & f_2^{\bullet} = \lceil \texttt{Red}_2(X_2,x)((\mathcal{T}_2,w_2))\rceil_{X_2} & f_3^{\bullet} = \lceil \texttt{Red}_3(X_3,\underline{y})((\mathcal{T}_3,w_3))\rceil_{X_3} \end{array}$$
The left-hand side and the right-hand side of (2.3) then correspond exactly to the top-right side and the left-bottom side, respectively, of the commuting diagram
$$\begin{array}{ccc} (f_1\ {}_z\circ_{\underline{z}}\ f_2)\ {}_y\circ_{\underline{y}}\ f_3 & \xrightarrow{\beta^{z,\underline{z};y,\underline{y}}_{f_1,f_2,f_3}} & f_1\ {}_z\circ_{\underline{z}}\ (f_2\ {}_y\circ_{\underline{y}}\ f_3) \\ {\scriptstyle \gamma^{z,\underline{z}}_{f_1,f_2}\ {}_y\circ_{\underline{y}}\ 1_{f_3}}\downarrow & & \downarrow{\scriptstyle \gamma^{z,\underline{z}}_{f_1,f_2\ {}_y\circ_{\underline{y}}\ f_3}} \\ (f_2\ {}_{\underline{z}}\circ_z\ f_1)\ {}_y\circ_{\underline{y}}\ f_3 & \xrightarrow{\vartheta^{\underline{z},z;y,\underline{y}}_{f_2,f_1,f_3}} & (f_2\ {}_y\circ_{\underline{y}}\ f_3)\ {}_{\underline{z}}\circ_z\ f_1 \\ {\scriptstyle (\kappa_2\ {}_{\underline{z}}\circ_z\ \kappa_1)\ {}_y\circ_{\underline{y}}\ \kappa_3}\downarrow & & \downarrow{\scriptstyle (\kappa_2\ {}_y\circ_{\underline{y}}\ \kappa_3)\ {}_{\underline{z}}\circ_z\ \kappa_1} \\ (f_2^{\bullet}\ {}_{\underline{z}}\circ_z\ f_1^{\bullet})\ {}_y\circ_{\underline{y}}\ f_3^{\bullet} & \xrightarrow{\vartheta^{\underline{z},z;y,\underline{y}}_{f_2^{\bullet},f_1^{\bullet},f_3^{\bullet}}} & (f_2^{\bullet}\ {}_y\circ_{\underline{y}}\ f_3^{\bullet})\ {}_{\underline{z}}\circ_z\ f_1^{\bullet} \end{array}$$

in which the upper square commutes by the definition of the isomorphism ϑ (see (2.1)) and the lower square is a naturality diagram for ϑ.

- If $x \in X_3$, then

$$\kappa_{(\mathcal{T},u,x)} = \big(\kappa_{(\mathcal{T}_3,w_3,x)}\ {}_{\underline{y}}\square_{y}\ \big(\kappa_{(\mathcal{T}_2,w_2,y)}\ {}_{\underline{z}}\square_{z}\ \kappa_{(\mathcal{T}_1,w_1,z)}\big)\big)\circ$$
$$\big(1_{(\mathcal{T}_3,w_3)}\ {}_{\underline{y}}\square_{y}\ \gamma^{z,\underline{z}}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)}\big) \circ \gamma^{y,\underline{y}}_{(\{\mathcal{T}_1(z\underline{z})\mathcal{T}_2\},w_1w_2),(\mathcal{T}_3,w_3)}$$

and

$$\kappa_{(\mathcal{T},v,x)} = \big(\big(\kappa_{(\mathcal{T}_3,w_3,x)}\ {}_{\underline{y}}\square_{y}\ \kappa_{(\mathcal{T}_2,w_2,y)}\big)\ {}_{\underline{z}}\square_{z}\ \kappa_{(\mathcal{T}_1,w_1,z)}\big)\circ$$
$$\big(\gamma_{(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)}\ {}_{\underline{z}}\square_{z}\ 1_{(\mathcal{T}_1,w_1)}\big) \circ \gamma^{z,\underline{z}}_{(\mathcal{T}_1,w_1),(\{\mathcal{T}_2(y\underline{y})\mathcal{T}_3\},w_2w_3)}.$$

Denote

$$\begin{array}{lll}
\kappa_1 = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,w_1,z)})]_{X_1} & \kappa_2 = [\Delta_{X_2}(\kappa_{(\mathcal{T}_2,w_2,y)})]_{X_2} & \kappa_3 = [\Delta_{X_3}(\kappa_{(\mathcal{T}_3,w_3,x)})]_{X_3} \\
f_1 = [\Delta_{X_1}((\mathcal{T}_1,w_1))]_{X_1} & f_2 = [\Delta_{X_2}((\mathcal{T}_2,w_2))]_{X_2} & f_3 = [\Delta_{X_3}((\mathcal{T}_3,w_3))]_{X_3} \\
f_1^\bullet = \lceil \mathtt{Red}_2(X_1,z)((\mathcal{T}_1,w_1))\rceil_{X_1} & f_2^\bullet = \lceil \mathtt{Red}_2(X_2,y)((\mathcal{T}_2,w_2))\rceil_{X_2} & f_3^\bullet = \lceil \mathtt{Red}_2(X_3,x)((\mathcal{T}_3,w_3))\rceil_{X_3}
\end{array}$$

The left-hand side and the right-hand side of (2.3) then correspond exactly to the top-right side and the left-bottom side, respectively, of the commuting diagram

$$\begin{array}{ccc}
(f_1\ {}_z\circ_{\underline{z}}\ f_2)\ {}_y\circ_{\underline{y}}\ f_3 & \xrightarrow{\ \beta^{z,\underline{z};y,\underline{y}}_{f_1,f_2,f_3}\ } & f_1\ {}_z\circ_{\underline{z}}\ (f_2\ {}_y\circ_{\underline{y}}\ f_3) \\
\Big\downarrow {\scriptstyle \gamma^{y,\underline{y}}_{f_1\ {}_z\circ_{\underline{z}}\ f_2,f_3}} & & \Big\downarrow {\scriptstyle \gamma^{z,\underline{z}}_{f_1,f_2\ {}_y\circ_{\underline{y}}\ f_3}} \\
f_3\ {}_{\underline{y}}\circ_y\ (f_1\ {}_z\circ_{\underline{z}}\ f_2) & & (f_2\ {}_y\circ_{\underline{y}}\ f_3)\ {}_{\underline{z}}\circ_z\ f_1 \\
\Big\downarrow {\scriptstyle 1_{f_3}\ {}_{\underline{y}}\circ_y\ \gamma^{z,\underline{z}}_{f_1,f_2}} & & \Big\downarrow {\scriptstyle \gamma^{y,\underline{y}}_{f_2,f_3}\ {}_{\underline{z}}\circ_z\ 1_{f_1}} \\
f_3\ {}_{\underline{y}}\circ_y\ (f_2\ {}_{\underline{z}}\circ_z\ f_1) & \xrightarrow{\ \beta^{\underline{y},y;\underline{z},z\ -1}_{f_3,f_2,f_1}\ } & (f_3\ {}_{\underline{y}}\circ_y\ f_2)\ {}_{\underline{z}}\circ_z\ f_1 \\
\Big\downarrow {\scriptstyle \kappa_3\ {}_{\underline{y}}\circ_y\ (\kappa_2\ {}_{\underline{z}}\circ_z\ \kappa_1)} & & \Big\downarrow {\scriptstyle (\kappa_3\ {}_{\underline{y}}\circ_y\ \kappa_2)\ {}_{\underline{z}}\circ_z\ \kappa_1} \\
f_3^\bullet\ {}_{\underline{y}}\circ_y\ (f_2^\bullet\ {}_{\underline{z}}\circ_z\ f_1^\bullet) & \xrightarrow{\ \beta^{\underline{y},y;\underline{z},z\ -1}_{f_3^\bullet,f_2^\bullet,f_1^\bullet}\ } & (f_3^\bullet\ {}_{\underline{y}}\circ_y\ f_2^\bullet)\ {}_{\underline{z}}\circ_z\ f_1^\bullet
\end{array}$$

in which the upper square commutes as an instance of ($\beta\gamma$`-hexagon`) and the bottom square commutes by the naturality of β^{-1}.

- The proof for the case $\varphi = \beta^{z,\underline{z};y,\underline{y}\,-1}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2),(\mathcal{T}_3,w_3)}$ follows directly from the previous item.
- Suppose now that $\varphi = \gamma^{z,y}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)}$, where $(\mathcal{T}_i,w_i) : X_i$.
  - If $x \in X_1$, then

$$\kappa_{(\mathcal{T},u,x)} = \kappa_{(\mathcal{T}_1,w_1,x)}\ {}_z\square_y\ \kappa_{(\mathcal{T}_2,w_2,y)}$$

and

$$\kappa_{(\mathcal{T},v,x)} = \big(\kappa_{(\mathcal{T}_1,w_1,z)}\ {}_z\square_y\ \kappa_{(\mathcal{T}_2,w_2,x)}\big) \circ \gamma^{y,z}_{(\mathcal{T}_2,w_2),(\mathcal{T}_1,w_1)}.$$

Denote

$$\begin{array}{ll}\kappa_1 = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,w_1,x)})]_{X_1} & \kappa_2 = [\Delta_{X_2}(\kappa_{(\mathcal{T}_2,w_2,y)})]_{X_2} \\ f_1 = [\Delta_{X_1}((\mathcal{T}_1,w_1))]_{X_1} & f_2 = [\Delta_{X_2}((\mathcal{T}_2,w_2))]_{X_2} \\ f_1^{\bullet} = \lceil \mathtt{Red}_2(X_1,x)((\mathcal{T}_1,w_1))\rceil_{X_1} & f_2^{\bullet} = \lceil \mathtt{Red}_2(X_2,y)((\mathcal{T}_2,w_2))\rceil_{X_2}\end{array}$$

By (γ-`involution`), we then have

$$\begin{aligned}[\Delta_X(\kappa_{(\mathcal{T},v,x)})]_X \circ [\Delta_X(\gamma^{z,y}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)})]_X &= \\ ((\kappa_1\ {}_z\!\circ_y\, \kappa_2) \circ \gamma^{z,x}_{f_2,f_1}) \circ \gamma^{z,y}_{f_1,f_2} &= \\ \kappa_1\ {}_z\!\circ_y\, \kappa_2 &= \\ 1_{f_1^{\bullet}\,{}_z\circ_y f_2^{\bullet}} \circ (\kappa_1\ {}_z\!\circ_y\, \kappa_2) &= \\ \lceil \mathtt{Red}_2(X,x)(\gamma^{z,y}_{(\mathcal{T}_1,w_1),(\mathcal{T}_2,w_2)})\rceil_X \circ [\Delta_X(\kappa_{(\mathcal{T},u,x)})]_X.\end{aligned}$$

– The proof goes symmetrically if $x \in X_2$.

- If $\varphi = \varphi_2 \circ \varphi_1$, where $\varphi_1 : (\mathcal{T},u) \to (\mathcal{T},w)$ and $\varphi_2 : (\mathcal{T},w) \to (\mathcal{T},v)$, then, by the induction hypothesis for φ_1 and φ_2, we get

$$\begin{aligned}[\Delta_X(\kappa_{(\mathcal{T},v,x)})]_X \circ [\Delta_X(\varphi_2 \circ \varphi_1)]_X &= \\ [\Delta_X(\kappa_{(\mathcal{T},v,x)})]_X \circ [\Delta_X(\varphi_2)]_X \circ [\Delta_X(\varphi_1)]_X &= \\ \lceil \mathtt{Red}_2(X,x)(\varphi_2)\rceil_X \circ [\Delta_X(\kappa_{(\mathcal{T},w,x)})]_X \circ [\Delta_X(\varphi_1)]_X &= \\ \lceil \mathtt{Red}_2(X,x)(\varphi_2)\rceil_X \circ \lceil \mathtt{Red}_2(X,x)(\varphi_1)\rceil_X \circ [\Delta_X(\kappa_{(\mathcal{T},v,x)})]_X &= \\ \lceil \mathtt{Red}_2(X,x)(\varphi_2 \circ \varphi_1)\rceil_X \circ [\Delta_X(\kappa_{(\mathcal{T},u,x)})]_X.\end{aligned}$$

- Finally, suppose that $\varphi = \varphi_1\ {}_z\Box_y\, \varphi_2$, where $\varphi_1 : (\mathcal{T}_1,u_1) \to (\mathcal{T}_1,v_1)$, $\varphi_2 : (\mathcal{T}_2,u_2) \to (\mathcal{T}_2,v_2)$, and $(\mathcal{T}_i,u_i) : X_i$.

  – If $x \in X_1$, then

$$\kappa_{(\mathcal{T},u,x)} = \kappa_{(\mathcal{T}_1,u_1,x)}\ {}_z\Box_y\, \kappa_{(\mathcal{T}_2,u_2,y)}$$

and

$$\kappa_{(\mathcal{T},v,x)} = \kappa_{(\mathcal{T}_1,v_1,x)}\ {}_z\Box_y\, \kappa_{(\mathcal{T}_2,v_2,y)}.$$

Denote

$$\begin{array}{ll}\kappa_{u_1} = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,u_1,x)})]_{X_1} & \kappa_{u_2} = [\Delta_{X_2}(\kappa_{(\mathcal{T}_2,u_2,y)})]_{X_2} \\ \kappa_{v_1} = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,v_1,x)})]_{X_1} & \kappa_{v_2} = [\Delta_{X_2}(\kappa_{(\mathcal{T}_2,v_2,y)})]_{X_2}\end{array}$$

By Remark 3.2(b) and the induction hypothesis for φ_1 and φ_2, we get

$$\begin{aligned}[\Delta_X(\kappa_{(\mathcal{T},v,x)})]_X \circ [\Delta_X(\varphi_1\ {}_z\Box_y\, \varphi_2)]_X &= \\ (\kappa_{v_1}\ {}_z\!\circ_y\, \kappa_{v_2}) \circ ([\Delta_{X_1}(\varphi_1)]_{X_1}\ {}_z\!\circ_y\, [\Delta_{X_2}(\varphi_2)]_{X_2}) &= \\ (\kappa_{v_1} \circ [\Delta_{X_1}(\varphi_1)]_{X_1})\ {}_z\!\circ_y\, (\kappa_{v_2} \circ [\Delta_{X_2}(\varphi_2)]_{X_2}) &= \\ (\lceil \mathtt{Red}_2(X_1,x)(\varphi)\rceil_{X_1} \circ \kappa_{u_1})\ {}_z\!\circ_y\, (\lceil \mathtt{Red}_2(X_2,y)(\varphi_2)\rceil_{X_2} \circ \kappa_{u_2}) &= \\ (\lceil \mathtt{Red}_2(X_1,x)(\varphi)\rceil_{X_1}\ {}_z\!\circ_y\, \lceil \mathtt{Red}_2(X_2,y)(\varphi_2)\rceil_{X_2}) \circ (\kappa_{u_1}\ {}_z\!\circ_y\, \kappa_{u_2}) &= \\ \lceil \mathtt{Red}_2(X,x)(\varphi_1\ {}_z\Box_y\, \varphi_2)\rceil_X \circ [\Delta_X(\kappa_{(\mathcal{T},u,x)})]_X.\end{aligned}$$

– If $x \in X_2$, then

$$\kappa_{(\mathcal{T},u,x)} = (\kappa_{(\mathcal{T}_2,u_2,x)}\;{}_y\square_z\;\kappa_{(\mathcal{T}_1,u_1,z)}) \circ \gamma^{z,y}_{(\mathcal{T}_1,u_1),(\mathcal{T}_2,u_2)}$$

and

$$\kappa_{(\mathcal{T},v,x)} = (\kappa_{(\mathcal{T}_2,v_2,x)}\;{}_y\square_z\;\kappa_{(\mathcal{T}_1,v_1,z)}) \circ \gamma^{z,y}_{(\mathcal{T}_1,v_1),(\mathcal{T}_2,v_2)}.$$

Denote

$$\begin{array}{ll}
\kappa_{u_1} = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,u_1,z)})]_{X_1} & \kappa_{u_2} = [\Delta_{X_2}(\kappa_{(\mathcal{T}_2,u_2,x)})]_{X_2} \\
\kappa_{v_1} = [\Delta_{X_1}(\kappa_{(\mathcal{T}_1,v_1,z)})]_{X_1} & \kappa_{v_2} = [\Delta_{X_2}([\kappa_{(\mathcal{T}_2,v_2,x)})]_{X_2} \\
f_{u_1} = [\Delta_{X_1}((\mathcal{T}_1,u_1))]_{X_1} & f_{u_2} = [\Delta_{X_2}((\mathcal{T}_2,u_2))]_{X_2} \\
f^\bullet_{u_1} = \lceil \mathtt{Red}_2(X_1,z)((\mathcal{T}_1,u_1))\rceil_{X_1} & f^\bullet_{u_2} = \lceil \mathtt{Red}_2(X_2,x)((\mathcal{T}_2,u_2))\rceil_{X_2} \\
f_{v_1} = [\Delta_{X_1}((\mathcal{T}_1,v_1))]_{X_1} & f_{v_2} = [\Delta_{X_2}((\mathcal{T}_2,v_2))]_{X_2} \\
f^\bullet_{v_1} = \lceil \mathtt{Red}_2(X_1,z)((\mathcal{T}_1,v_1))\rceil_{X_1} & f^\bullet_{v_2} = \lceil \mathtt{Red}_2(X_2,x)((\mathcal{T}_2,v_2))\rceil_{X_2}
\end{array}$$

By Remark 3.2(b), naturality of γ and the induction hypothesis for φ_1 and φ_2, we get

$$\begin{array}{rl}
[\Delta_X(\kappa_{(\mathcal{T},v,x)})]_X \circ [\Delta_X(\varphi_1\;{}_z\square_y\;\varphi_2)]_X & = \\
(\kappa_{v_2}\;{}_y\circ_z\;\kappa_{v_1}) \circ \gamma^{z,y}_{f_{v_1},f_{v_2}} \circ ([\Delta_{X_1}(\varphi_1)]_{X_1}\;{}_z\circ_y\;[\Delta_{X_2}(\varphi_2)]_{X_2}) & = \\
\gamma^{z,y}_{f^\bullet_{v_1},f^\bullet_{v_2}} \circ (\kappa_{v_1}\;{}_z\circ_y\;\kappa_{v_2}) \circ ([\Delta_{X_1}(\varphi_1)]_{X_1}\;{}_z\circ_y\;[\Delta_{X_2}(\varphi_2)]_{X_2}) & = \\
\gamma^{z,y}_{f^\bullet_{v_1},f^\bullet_{v_2}} \circ ((\kappa_{v_1} \circ [\Delta_{X_1}(\varphi_1)]_{X_1})\;{}_z\circ_y\;(\kappa_{v_2} \circ [\Delta_{X_2}(\varphi_2)]_{X_2})) & = \\
\gamma^{z,y}_{f^\bullet_{v_1},f^\bullet_{v_2}} \circ ((\lceil \mathtt{Red}_2(X_1,z)(\varphi_1)\rceil_{X_1} \circ \kappa_{u_1})\;{}_z\circ_y\;(\lceil \mathtt{Red}_2(X_2,x)(\varphi_2)\rceil_{X_2} \circ \kappa_{u_2})) & = \\
((\lceil \mathtt{Red}_2(X_2,x)(\varphi_2)\rceil_{X_2} \circ \kappa_{u_2})\;{}_y\circ_z\;(\lceil \mathtt{Red}_2(X_1,z)(\varphi_1)\rceil_{X_1} \circ \kappa_{u_1})) \circ \gamma^{x,y}_{f_{u_1},f_{u_2}} & = \\
(\lceil \mathtt{Red}_2(X_2,x)(\varphi_2)\rceil_{X_2}\;{}_y\circ_z\;\lceil \mathtt{Red}_2(X_1,z)(\varphi_1)\rceil_{X_1}) \circ (\kappa_{u_2}\;{}_y\circ_z\;\kappa_{u_1}) \circ \gamma^{z,y}_{f_{u_1},f_{u_2}} & = \\
\lceil \mathtt{Red}_2(X,x)(\varphi_1\;{}_z\square_y\;\varphi_2)\rceil_X \circ [\Delta_X(\kappa_{(\mathcal{T},u,z)})]_X. &
\end{array}$$

■

The following result is a direct consequence of Theorem 2.

Corollary 1. *For arrow terms φ_1 and φ_2 of the same type in $\underline{\mathtt{T}}^+_{\mathcal{C}}(X)$, the equality $\lfloor \varphi_1 \rfloor_X = \lfloor \varphi_2 \rfloor_X$ follows from the equality $\lceil \mathtt{Red}_2(X,x)(\varphi_1)\rceil_X = \lceil \mathtt{Red}_2(X,x)(\varphi_2)\rceil_X$.*

2.5 The third reduction: establishing skeletality

Intuitively, in the third reduction we pass from the non-skeletal to the skeletal operadic framework. This will reduce the problem of commutation of all $\beta\vartheta$-diagrams of $\mathcal{C}(X)$ to the problem of commutation of all diagrams of canonical arrows of the skeletal non-symmetric categorified operad $\mathcal{O}_{\mathcal{C}}$, constructed from $\mathcal{C}$ in the appropriate way.

2.5.1 The skeletal non-symmetric categorified operad $\mathcal{O}_{\mathcal{C}}$

Starting from $\mathcal{C}$, we first define a *skeletal non-symmetric categorified operad* $\mathcal{O}_{\mathcal{C}} = \{\mathcal{O}_{\mathcal{C}}(n)\}_{n\in\mathbb{N}}$, i.e. a weak Cat-operad in the sense of [DP15], as follows.

- The objects of the category $\mathcal{O}_{\mathcal{C}}(n)$ are quadruplets (X, x, σ, f), where $|X| = n+1$, $x \in X$, $f \in \mathcal{C}(X)$ and $\sigma : [n] \to X\backslash\{x\}$ is a bijection (inducing a total order on $X\backslash\{x\}$).

- The morphisms of $\mathcal{O}_{\mathcal{C}}(n)[(X,x,\sigma,f),(X,x,\sigma,g)]$ are quadruplets (X,x,σ,φ), such that φ is a morphism of $\mathcal{C}(X)[f,g]$ (in particular, $\mathcal{O}_{\mathcal{C}}(n)[(X,x,\sigma,f),(Y,y,\tau,g)]$ is empty for $(X,x,\sigma)\neq(Y,y,\tau)$). The identity morphism for (X,x,σ,f) is $(X,x,\sigma,1_f)$. The composition of morphisms is canonically induced from the composition of morphisms in $\mathcal{C}(X)$.

- The composition operation $\circ_i : \mathcal{O}_{\mathcal{C}}(n)\times\mathcal{O}_{\mathcal{C}}(m)\to\mathcal{O}_{\mathcal{C}}(n+m-1)$ on objects is defined by

$$(X,x,\sigma_1,f)\circ_i(Y,y,\sigma_2,g)=(X+Y\backslash\{y\},x,\sigma,f\,{}_{\sigma_1(i)}{\circ}_y\,g),$$

  and on morphisms by

$$(X,x,\sigma_1,\varphi)\circ_i(Y,y,\sigma_2,\psi)=(X+Y\backslash\{y\},x,\sigma,\varphi\,{}_{\sigma_1(i)}{\circ}_y\,\psi),$$

  where $\sigma:[n+m-1]\to X\backslash\{x\}+Y\backslash\{y\}$ is a bijection defined by

$$\sigma(j)=\begin{cases}\sigma_1(j) & \text{for } j\in\{1,\dots,i-1\}\\ \sigma_2(j-i+1) & \text{for } j\in\{i,\dots,i+m-1\}\\ \sigma_1(j-m) & \text{for } j\in\{i+m,\dots,n+m-1\}.\end{cases}\tag{2.4}$$

- For $\tilde{f}=(X,x,\sigma_1,f)$, $\tilde{g}=(Y,y,\sigma_2,g)$ and $\tilde{h}=(Z,z,\sigma_3,h)$, where $\sigma_1:[n]\to X\backslash\{x\}$, $\sigma_2:[m]\to Y\backslash\{y\}$ and $\sigma_3:[k]\to Z\backslash\{z\}$, the components

$$\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}:(\tilde{f}\circ_i\tilde{g})\circ_j\tilde{h}\to\tilde{f}\circ_i(\tilde{g}\circ_j\tilde{h})\quad\text{and}\quad\theta^{i;k}_{\tilde{f},\tilde{g},\tilde{h}}:(\tilde{f}\circ_i\tilde{g})\circ_k\tilde{h}\to(\tilde{f}\circ_k\tilde{h})\circ_i\tilde{h}$$

  of natural isomorphisms β and θ are distinguished among the morphisms of $\mathcal{O}_{\mathcal{C}}(n)$ as the quadruplets arising from the appropriate components of β and ϑ of $\mathcal{C}$, as follows:

$$\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}=(X{+}Y\backslash\{y\}{+}Z\backslash\{z\},x,\sigma,\beta^{\sigma_1(i),y;\sigma_2(j),z}_{f,g,h})$$

  and

$$\theta^{i;k}_{\tilde{f},\tilde{g},\tilde{h}}=(X{+}Y\backslash\{y\}{+}Z\backslash\{z\},x,\sigma',\vartheta^{\sigma_1(i),y;\sigma_1(k),z}_{f,g,h}),$$

  where σ and σ' are the bijections induced in the appropriate way from σ_1,σ_2 and σ_3.

In the following lemma, we show that the structure $\mathcal{O}_{\mathcal{C}}=\{\mathcal{O}_{\mathcal{C}}(n)\}_{n\in\mathbb{N}}$ indeed verifies the axioms of weak Cat-operads given in [DP15, Section 7].

Lemma 15. *For an arbitrary $n\in\mathbb{N}$, the following equations hold in $\mathcal{O}_{\mathcal{C}}(n)$:*

1. *the categorical equations:*
   a) $\varphi\circ 1_{\tilde{f}}=\varphi=1_{\tilde{g}}\circ\varphi$, *for* $\varphi:\tilde{f}\to\tilde{g}$,
   b) $(\varphi\circ\phi)\circ\psi=\varphi\circ(\phi\circ\psi)$,
2. *the bifunctoriality equations:*
   a) $1_{\tilde{f}}\circ_i 1_{\tilde{g}}=1_{\tilde{f}\circ_i\tilde{g}}$,
   b) $(\varphi_2\circ\varphi_1)\circ_i(\psi_2\circ\psi_1)=(\varphi_2\circ_i\psi_2)\circ(\varphi_1\circ_i\psi_1)$,

3. the naturality equations:

a) $\beta^{i;j}_{\tilde{f}_2,\tilde{g}_2,\tilde{h}_2}\circ((\varphi\circ_i\phi)\circ_j\psi)=(\varphi\circ_i(\phi\circ_j\psi))\circ\beta^{i;j}_{\tilde{f}_1,\tilde{g}_1,\tilde{h}_1}$,

b) $\theta^{i;j}_{\tilde{f}_2,\tilde{g}_2,\tilde{h}_2}\circ((\varphi\circ_i\phi)\circ_j\psi)=((\varphi\circ_j\psi)\circ_i\phi)\circ\theta^{i;j}_{\tilde{f}_1,\tilde{g}_1,\tilde{h}_1}$,

5. the equations concernig inverse isomorphisms:

a) $\beta^{i;j\ -1}_{\tilde{f},\tilde{g},\tilde{h}}\circ\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}=1_{(\tilde{f}\circ_i\tilde{g})\circ_j\tilde{h}}$, $\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}\circ\beta^{i;j\ -1}_{\tilde{f},\tilde{g},\tilde{h}}=1_{\tilde{f}\circ_i(\tilde{g}\circ_j\tilde{h})}$,

b) $\theta^{j;i}_{\tilde{f},\tilde{h},\tilde{g}}\circ\theta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}=1_{(\tilde{f}\circ_i\tilde{g})\circ_j\tilde{h}}$,

6. the coherence conditions:

a) $(1_{\tilde{f}}\circ_i\beta^{j;l}_{\tilde{g},\tilde{h},\tilde{k}})\circ\beta^{i;l}_{\tilde{f},\tilde{g}\circ_j\tilde{h},\tilde{k}}\circ(\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}\circ_l 1_{\tilde{k}})=\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}\circ_l\tilde{k}}\circ\beta^{j;l}_{\tilde{f}\circ_i\tilde{g},\tilde{h},\tilde{k}}$,

b) $(1_{\tilde{f}}\circ_i\theta^{j;l}_{\tilde{g},\tilde{h},\tilde{k}})\circ\beta^{i;l}_{\tilde{f},\tilde{g}\circ_j\tilde{h},\tilde{k}}\circ(\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}\circ_l 1_k)=\beta^{i;j}_{\tilde{f},\tilde{g}\circ_l\tilde{h},\tilde{k}}\circ(\beta^{i;l}_{\tilde{f},\tilde{g},\tilde{k}}\circ_j 1_{\tilde{h}})\circ\theta^{j;l}_{\tilde{f}\circ_i\tilde{g},\tilde{h},\tilde{k}}$,

c) $\theta^{i;l}_{\tilde{f},\tilde{g}\circ_j\tilde{h},\tilde{k}}\circ(\beta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}\circ_l 1_{\tilde{k}})=\beta^{i;j}_{\tilde{f}\circ_l\tilde{k},\tilde{g},\tilde{h}}\circ(\theta^{i;l}_{\tilde{f},\tilde{g},\tilde{k}}\circ_j 1_{\tilde{h}})\circ\theta^{j;l}_{\tilde{f}\circ_i\tilde{g},\tilde{h},\tilde{k}}$,

d) $\theta^{i;j}_{\tilde{f}\circ_l\tilde{k},\tilde{g},\tilde{h}}\circ(\theta^{i;l}_{\tilde{f},\tilde{g},\tilde{k}}\circ_j 1_{\tilde{h}})\circ\theta^{j;l}_{\tilde{f}\circ_i\tilde{g},\tilde{h},\tilde{k}}=(\theta^{j;l}_{\tilde{f},\tilde{h},\tilde{k}}\circ_i 1_{\tilde{g}})\circ\theta^{i;l}_{\tilde{f}\circ_j\tilde{h},\tilde{g},\tilde{k}}\circ(\theta^{i;j}_{\tilde{f},\tilde{g},\tilde{h}}\circ_l 1_{\tilde{k}})$.

Proof. The first two groups of equations, as well as the equation 3.(a), are verified straightforwardly by the corresponding groups of equations for $\mathcal{C}$, given in Remark 3. The equation 3.(b) follows by the naturality of ϑ (see (2.1)). The equations 5.(a) holds by the analogous equations for $\mathcal{C}$. The equation 5.(b) holds by Lemma 1. The equation 6.(a) holds by (β-`pentagon`), 6.(b) by ($\beta\gamma$-`decagon`) and Remark 4.(b), and 6.(c) and 6.(d) by Lemma 2. ∎

2.5.2 "Skeletalisation" of the syntax $\mathtt{rT}^{+}_{\underline{\mathcal{C}}}$

In order to correctly apply the coherence result of [DP15], which is established for formal diagrams encoding the canonical diagrams of the skeletal non-symetric categorified operad $\mathcal{O}_{\mathcal{C}}$, we introduce the syntax of these diagrams. Intuitively, this syntax is a "skeletalisation" of the syntax $\mathtt{rT}^{+}_{\underline{\mathcal{C}}}$.

Let $\mathcal{T}$ be an unrooted tree. Suppose that $FV(\mathcal{T})=X$ and let $x\in X$. For a corolla $c\in Cor(\mathcal{T})$, such that $|\mathtt{inp}_{(\mathcal{T},x)}(c)|=n$ (see the end of §2.4.3), we define the *set of skeletalisations of c (relative to $\mathcal{T}$ and x)* as

$$\Sigma_{(\mathcal{T},x)}(c)=\mathbf{Bij}[\,[n],\mathtt{inp}_{(\mathcal{T},x)}(c)\,].$$

We set

$$\Sigma(\mathcal{T},x)=\prod_{c\in Cor(\mathcal{T})}\Sigma_{(\mathcal{T},x)}(c).$$

We shall denote the elements of $\Sigma(\mathcal{T},x)$ with $\overrightarrow{\sigma}$.

Remark 12. *Notice that $\overrightarrow{\sigma_1}\in\Sigma(\mathcal{T}_1,x)$ and $\overrightarrow{\sigma_2}\in\Sigma(\mathcal{T}_2,y)$ determine "by concatenation" an element of $\overrightarrow{\sigma}\in\Sigma(\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},x)$, and that, symmetrically, any $\overrightarrow{\sigma}\in\Sigma(\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},x)$ can be "split" into $\overrightarrow{\sigma_1}\in\Sigma(\mathcal{T}_1,x)$ and $\overrightarrow{\sigma_2}\in\Sigma(\mathcal{T}_2,y)$. We shall denote this decomposition of $\overrightarrow{\sigma}$ with $\overrightarrow{\sigma_1\cdot\sigma_2}$.*

The *skeletalisation of the syntax* $\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$ is the syntax ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$, obtained as follows.

The objects terms of ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$ are quadruplets $(\mathcal{T}, x, \overrightarrow{\sigma}, w)$, typed by the rule

$$\frac{\mathcal{T} \in \underline{\mathtt{T}}^{+}_{\mathcal{C}}(X) \quad x \in X \quad \overrightarrow{\sigma} \in \Sigma(\mathcal{T}, x) \quad w \in A(\mathcal{T}, x)}{(\mathcal{T}, x, \overrightarrow{\sigma}, w) : X\backslash\{x\}}$$

The arrow terms of ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$ are obtained from raw terms

$$\chi ::= \begin{cases} 1_{(\mathcal{T},x,\overrightarrow{\sigma},w)} \mid \beta^{z;y}_{(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)} \mid \beta^{z;y\ \ -1}_{(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)} \\ \theta^{z;y}_{(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)} \mid \chi \circ \chi \mid \chi\ {}_{z}\square_{y}\ \chi \end{cases}$$

by typing them as follows:

$$\frac{}{1_{(\mathcal{T},x,\overrightarrow{\sigma},w)} : (\mathcal{T}, x, \overrightarrow{\sigma}, w) \to (\mathcal{T}, x, \overrightarrow{\sigma}, w)}$$

$$\frac{\begin{array}{c}\mathcal{T}=\{\{\mathcal{T}_1\,(z\underline{z})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\} \quad y\in FV(\mathcal{T}_2) \quad x\in X\cap FV(\mathcal{T}_1) \\ \overrightarrow{\sigma_1}\in\Sigma(\mathcal{T}_1,x) \quad \overrightarrow{\sigma_2}\in\Sigma(\mathcal{T}_2,\underline{z}) \quad \overrightarrow{\sigma_3}\in\Sigma(\mathcal{T}_3,\underline{y})\end{array}}{\beta^{z;y}_{(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)} : (\mathcal{T}, x, \overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3}, (w_1w_2)w_3\,) \to (\mathcal{T}, x, \overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3}, w_1(w_2w_3)\,)}$$

$$\frac{\begin{array}{c}\mathcal{T}=\{\mathcal{T}_1\,(z\underline{z})\,\{\mathcal{T}_2\,(y\underline{y})\,\mathcal{T}_3\}\} \quad \underline{z}\in FV(\mathcal{T}_2) \quad x\in X\cap FV(\mathcal{T}_1) \\ \overrightarrow{\sigma_1}\in\Sigma(\mathcal{T}_1,x) \quad \overrightarrow{\sigma_2}\in\Sigma(\mathcal{T}_2,\underline{z}) \quad \overrightarrow{\sigma_3}\in\Sigma(\mathcal{T}_3,\underline{y})\end{array}}{\beta^{z;y\ \ -1}_{(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)} : (\mathcal{T}, x, \overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3}, w_1(w_2w_3)\,) \to (\mathcal{T}, x, \overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3}, (w_1w_2)w_3\,)}$$

$$\frac{\begin{array}{c}\mathcal{T}=\{\{\mathcal{T}_1\,(z\underline{z})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\} \quad y\in FV(\mathcal{T}_1) \quad x\in X\cap FV(\mathcal{T}_1) \\ \overrightarrow{\sigma_1}\in\Sigma(\mathcal{T}_1,x) \quad \overrightarrow{\sigma_2}\in\Sigma(\mathcal{T}_2,\underline{z}) \quad \overrightarrow{\sigma_3}\in\Sigma(\mathcal{T}_3,\underline{y})\end{array}}{\theta^{z;y}_{(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)} : (\mathcal{T}, x, \overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3}, (w_1w_2)w_3\,) \to (\mathcal{T}, x, \overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3}, (w_1w_3)w_2\,)}$$

$$\frac{\chi_1 : (\mathcal{T}, x, \overrightarrow{\sigma}, w_1) \to (\mathcal{T}, x, \overrightarrow{\sigma}, w_2) \quad \chi_2 : (\mathcal{T}, x, \overrightarrow{\sigma}, w_2) \to (\mathcal{T}, x, \overrightarrow{\sigma}, w_3)}{\chi_2 \circ \chi_1 : (\mathcal{T}, x, \overrightarrow{\sigma}, w_1) \to (\mathcal{T}, x, \overrightarrow{\sigma}, w_3)}$$

$$\frac{\chi_1 : (\mathcal{T}_1, x, \overrightarrow{\sigma_1}, w_1) \to (\mathcal{T}_1, x, \overrightarrow{\sigma_1}, w'_1) \ \ \chi_2 : (\mathcal{T}_2, y, \overrightarrow{\sigma_2}, w_2) \to (\mathcal{T}_2, y, \overrightarrow{\sigma_2}, w'_2) \ \ z \in FV(\mathcal{T}_1) \ \ z \neq x}{\chi_1\ {}_{z}\square_{y}\ \chi_2 : (\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\}, x, \overrightarrow{\sigma_1\cdot\sigma_2}, w_1w_2) \to (\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\}, x, \overrightarrow{\sigma_1\cdot\sigma_2}, w'_1w'_2)}$$

As usual, we shall denote the class of object terms of ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$ with type X, together with the class of arrow terms whose types are pairs of object terms of type X, by ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}(X)$.

2.5.3 The interpretation of ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$ in $\mathcal{O}_{\mathcal{C}}$

In order to define the interpretation of ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$ in $\mathcal{O}_{\mathcal{C}}$, we first need to "order the inputs" of unrooted trees figuring in object terms $(\mathcal{T}, x, \overrightarrow{\sigma}, w)$ of ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\mathcal{C}}$.

For an unrooted tree $\mathcal{T}$, a variable $x \in FV(\mathcal{T})$ and an element $\overrightarrow{\sigma} = (\sigma_1, \ldots, \sigma_n) \in \Sigma_{(\mathcal{T},x)}$, the *total order*

$$\sigma : [\,|\mathtt{inp}_{(\mathcal{T},x)}(\mathcal{T})|\,] \to \mathtt{inp}_{(\mathcal{T},x)}(\mathcal{T})$$

on the set of inputs of $\mathcal{T}$ *(relative to* x*) induced by* $\overrightarrow{\sigma}$ is defined as follows:

- ◇ if $(\mathcal{T}, x) = (\{a(x_1, \ldots, x_n); id_X\}, x_i)$, then $\sigma = \overrightarrow{\sigma}$,
- ◇ if $(\mathcal{T}, x) = (\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\}, x)$, $x \in FV(\mathcal{T}_1)$, $|\mathtt{inp}_{(\mathcal{T}_1,x)}(\mathcal{T}_1)| = n$, $|\mathtt{inp}_{(\mathcal{T}_2,y)}(\mathcal{T}_1)| = m$, $\sigma_1 : [n] \to \mathtt{inp}_{(\mathcal{T}_1,x)}(\mathcal{T}_1)$ is the total order induced by $\overrightarrow{\sigma_1} \in \Sigma_{(\mathcal{T}_1,x)}$, $\sigma_2 : [m] \to \mathtt{inp}_{(\mathcal{T}_2,y)}(\mathcal{T}_2)$ is the total order induced by $\overrightarrow{\sigma_2} \in \Sigma_{(\mathcal{T}_2,y)}$ and $\sigma_1(i) = z$, then

  $$\sigma : [n + m - 1] \to FV(\mathcal{T})\backslash\{x\}$$

  is defined by (2.4).

The interpretation function

$$\lceil - \rceil^{\mathtt{sk}}_X : {}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}}^{+}_{\underline{\mathcal{C}}}(X) \to \mathcal{O}_{\mathcal{C}}(|X|)$$

is defined recursively as follows:

- ◇ $\lceil\,(\{a(x_1, \ldots, x_n); id_X\}, x_i, \vec{\sigma}, \underline{a})\,\rceil^{\mathtt{sk}}_{X\backslash\{x_i\}} = (\{x_1, \ldots, x_n\}, x_i, \sigma, a)$,
- ◇ $\lceil(\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\}, x, \overrightarrow{\sigma_1 \cdot \sigma_2}, w_1w_2)\rceil^{\mathtt{sk}}_{X\backslash\{x\}} = \lceil(\mathcal{T}_1, x, \overrightarrow{\sigma_1}, w_1)\rceil^{\mathtt{sk}}_{X_1\backslash\{x\}} \circ_{\sigma_1^{-1}(z)} \lceil(\mathcal{T}_2, y, \overrightarrow{\sigma_2}, w_2)\rceil^{\mathtt{sk}}_{X_2\backslash\{y\}}$,

and

- ◇ $\lceil 1_{(\mathcal{T},x,\overrightarrow{\sigma},w)}\rceil^{\mathtt{sk}}_{X\backslash\{x\}} = 1_{\lceil(\mathcal{T},x,\overrightarrow{\sigma},w)\rceil^{\mathtt{sk}}_{X\backslash\{x\}}}$,
- ◇ $\lceil\beta^{z;y}_{(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)}\rceil^{\mathtt{sk}}_{X\backslash\{x\}} =$
  $$\beta^{\sigma_1^{-1}(z);\sigma_2^{-1}(y)}_{\lceil(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1)\rceil^{\mathtt{sk}}_{X_1\backslash\{x\}},\lceil(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2)\rceil^{\mathtt{sk}}_{X_2\backslash\{\underline{z}\}},\lceil(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)\rceil^{\mathtt{sk}}_{X_3\backslash\{\underline{y}\}}},$$
- ◇ $\lceil\beta^{z;y\ \ -1}_{(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)}\rceil^{\mathtt{sk}}_{X\backslash\{x\}} =$
  $$\beta^{\sigma_1^{-1}(z);\sigma_2^{-1}(y)\ \ -1}_{\lceil(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1)\rceil^{\mathtt{sk}}_{X_1\backslash\{x\}},\lceil(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2)\rceil^{\mathtt{sk}}_{X_2\backslash\{\underline{z}\}},\lceil(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)\rceil^{\mathtt{sk}}_{X_3\backslash\{\underline{y}\}}},$$
- ◇ $\lceil\theta^{z;y}_{(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)}\rceil^{\mathtt{sk}}_{X\backslash\{x\}} =$
  $$\theta^{\sigma_1^{-1}(z);\sigma_1^{-1}(y)}_{\lceil(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1)\rceil^{\mathtt{sk}}_{X_1\backslash\{x\}},\lceil(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2)\rceil^{\mathtt{sk}}_{X_2\backslash\{\underline{z}\}},\lceil(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)\rceil^{\mathtt{sk}}_{X_3\backslash\{\underline{y}\}}},$$
- ◇ $\lceil\chi_2 \circ \chi_1\rceil^{\mathtt{sk}}_{X\backslash\{x\}} = \lceil\chi_2\rceil^{\mathtt{sk}}_{X\backslash\{x\}} \circ \lceil\chi_1\rceil^{\mathtt{sk}}_{X\backslash\{x\}}$,
- ◇ $\lceil\chi_1\,{}_z\square_y\,\chi_2\rceil^{\mathtt{sk}}_{X\backslash\{x\}} = \lceil\chi_1\rceil^{\mathtt{sk}}_{X_1\backslash\{x\}} \circ_{\sigma_1^{-1}(z)} \lceil\chi_2\rceil^{\mathtt{sk}}_{X_2\backslash\{y\}}$,

where it is assumed that every total order σ (resp. σ_i) is induced by $\overrightarrow{\sigma}$ (resp. $\overrightarrow{\sigma_i}$).

2.5.4 The third reduction

In what follows, we shall denote with $\mathtt{rT}^+_{\underline{\mathcal{C}}}(X,x,\mathcal{T})$ the subclass of $\mathtt{rT}^+_{\underline{\mathcal{C}}}(X)$ determined by the rooted tree $(\mathcal{T},x)$ (i.e. by the object terms whose first two components are given by $(\mathcal{T},x)$ and by the arrow terms among them). We define the family of *third reduction functions*

$$\mathtt{Red_3}(X,x,\mathcal{T},\overrightarrow{\sigma}):\mathtt{rT}^+_{\underline{\mathcal{C}}}(X,x,\mathcal{T})\to {}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}_{\mathcal{C}}}^+(X),$$

where $x\in X$, $\mathcal{T}$ is an unrooted tree such that $FV(\mathcal{T})=X$ and $\overrightarrow{\sigma}\in\Sigma_{(\mathcal{T},x)}$, as follows.

For object terms of $\mathtt{rT}^+_{\underline{\mathcal{C}}}(X)$, we set

$$\mathtt{Red_3}(X,x,\mathcal{T},\overrightarrow{\sigma})((\mathcal{T},x,w))=(\mathcal{T},x,\overrightarrow{\sigma},w).$$

For an arrow term χ of $\mathtt{rT}^+_{\underline{\mathcal{C}}}(X)$, $\mathtt{Red_3}(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi)$ is defined recursively as follows:

- ◇ $\mathtt{Red_3}(X,x,\mathcal{T},\overrightarrow{\sigma})(1_{(\mathcal{T},x,w)})=1_{\mathtt{Red_3}(X,x,\mathcal{T},\overrightarrow{\sigma})((\mathcal{T},x,w))}$,
- ◇ $\mathtt{Red_3}(X,x,\{\{\mathcal{T}_1\,(z\underline{z})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\},\overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3})(\beta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)})=$
$$\beta^{\sigma_1^{-1}(z);\sigma_2^{-1}(y)}_{(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)},$$
- ◇ $\mathtt{Red_3}(X,x,\{\mathcal{T}_1\,(z\underline{z})\,\{\mathcal{T}_2\,(y\underline{y})\,\mathcal{T}_3\}\},\overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3})(\beta^{z;y\;\;-1}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)})=$
$$\beta^{\sigma_1^{-1}(z);\sigma_2^{-1}(y)\;\;-1}_{(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)},$$
- ◇ $\mathtt{Red_3}(X,x,\{\{\mathcal{T}_1\,(z\underline{z})\,\mathcal{T}_2\}\,(y\underline{y})\,\mathcal{T}_3\},\overrightarrow{\sigma_1\cdot\sigma_2\cdot\sigma_3})(\theta^{z;y}_{(\mathcal{T}_1,x,w_1),(\mathcal{T}_2,\underline{z},w_2),(\mathcal{T}_3,\underline{y},w_3)})=$
$$\theta^{\sigma_1^{-1}(z);\sigma_1^{-1}(y)}_{(\mathcal{T}_1,x,\overrightarrow{\sigma_1},w_1),(\mathcal{T}_2,\underline{z},\overrightarrow{\sigma_2},w_2),(\mathcal{T}_3,\underline{y},\overrightarrow{\sigma_3},w_3)},$$
- ◇ $\mathtt{Red_3}(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi_2\circ\chi_1)=\mathtt{Red_3}(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi_2)\circ\mathtt{Red_3}(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi_1)$,
- ◇ if $\chi=\chi_1\,{}_z\square_y\,\chi_2$, where $\chi_1:(\mathcal{T}_1,x,w_1)\to(\mathcal{T}_1,x,w'_1)$ and $\chi_2:(\mathcal{T}_2,y,w_2)\to(\mathcal{T}_2,y,w'_2)$, and if $\overrightarrow{\sigma_1}\in\Sigma_{(\mathcal{T}_1,x)}$ and $\overrightarrow{\sigma_2}\in\Sigma_{(\mathcal{T}_2,y)}$, then
$$\mathtt{Red_3}(X,x,\{\mathcal{T}_1\,(zy)\,\mathcal{T}_2\},\overrightarrow{\sigma_1\cdot\sigma_2})(\chi_1\,{}_z\square_y\,\chi_2)=$$
$$\mathtt{Red_3}(X_1,x,\mathcal{T}_1,\overrightarrow{\sigma_1})(\chi_1)\;{}_{\sigma_1^{-1}(z)}\square_{\sigma_2^{-1}(y)}\,\mathtt{Red_3}(X_2,y,\mathcal{T}_2,\overrightarrow{\sigma_2})(\chi_2).$$

Remark 13. *For the third reduction of an arrow term $\chi:(\mathcal{T},x,u)\to(\mathcal{T},x,v)$, we have that*

$$\mathtt{Red_3}(X,x,\mathcal{T},\overrightarrow{\sigma})(\chi):\mathtt{Red_3}(X,x,\mathcal{T},\overrightarrow{\sigma})((\mathcal{T},x,u))\to\mathtt{Red_3}(X,x,\mathcal{T},\overrightarrow{\sigma})((\mathcal{T},x,v)).$$

Therefore, the third reduction of a pair of arrow terms of the same type *in $\mathtt{rT}^+_{\underline{\mathcal{C}}}(X)$ is a pair of arrow terms of the* same type *in ${}_{\mathtt{sk}}\mathtt{r}\underline{\mathtt{T}_{\mathcal{C}}}^+(X)$. Recall that the analogous properties hold for the first two reductions (see Lemma 12 and Remark 11).*

Theorem 3. *For an arbitrary object term $(\mathcal{T}, x, w)$ and an arbitrary arrow term χ of $\mathtt{r}\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}(X)$, the following equalities hold*

$$\lceil \mathtt{Red}_3(X, x, \mathcal{T}, \overrightarrow{\sigma})((\mathcal{T}, x, w)) \rceil^{\mathtt{sk}}_{X\backslash\{x\}} = (X, x, \sigma, \lceil (\mathcal{T}, x, w) \rceil_X)$$

and

$$\lceil \mathtt{Red}_3(X, x, \mathcal{T}, \overrightarrow{\sigma})(\chi) \rceil^{\mathtt{sk}}_{X\backslash\{x\}} = (X, x, \sigma, \lceil \chi \rceil_X),$$

where the total order σ is induced from $\overrightarrow{\sigma}$.

Proof. Easy, by induction on the proof of the $(\mathcal{T}, x)$-admissibility of w (for the first equality), and by induction on the structure of χ (for the second equality). ∎

The following result is a direct consequence of Theorem 3.

Corollary 2. *For arrow terms χ_1 and χ_2 of the same type in $\mathtt{r}\underline{\mathtt{T}}^+_{\underline{\mathcal{C}}}(X)$, the equality $\lceil \chi_1 \rceil_X = \lceil \chi_1 \rceil_X$ follows from the equality $\lceil \mathtt{Red}_3(X, x, \mathcal{T}, \overrightarrow{\sigma})(\chi_1) \rceil^{\mathtt{sk}}_{X\backslash\{x\}} = \lceil \mathtt{Red}_3(X, x, \mathcal{T}, \overrightarrow{\sigma})(\chi_2) \rceil^{\mathtt{sk}}_{X\backslash\{x\}}$.*

2.6 The proof of the coherence theorem

We finally assemble the three reductions in the proof of the coherence theorem. The proof is outlined by the two invariance properties common for all three reductions: *by reducing a pair of arrow terms of the same type,*

1. *the result is always a pair of arrow terms of the same type, and*
2. *the equality of interpretations of the two resulting arrow terms implies the equality of the interpretations of the respective starting arrow terms.*

Coherence Theorem. *For any finite set X and for any pair of arrow terms $\Phi, \Psi : \mathcal{W}_1 \to \mathcal{W}_2$ of the same type in $\mathtt{Free}_{\underline{\mathcal{C}}}(X)$, we have $[[\Phi]]_X = [[\Psi]]_X$ in $\mathcal{C}(X)$.*

Proof. By Theorem 1 (first reduction), it is enough to prove the equality

$$[\mathtt{Red}_1(\Phi)]_X = [\mathtt{Red}_1(\Psi)]_X.$$

By Lemma 13 and Lemma 14, the problem translates to showing that

$$\lfloor \Delta_X^{-1}(\mathtt{Red}_1(\Phi)) \rfloor_X = \lfloor \Delta_X^{-1}(\mathtt{Red}_1(\Psi)) \rfloor_X.$$

By Corollary 1 (second reduction), this equality follows from the equality

$$\lceil \mathtt{Red}_2(X, x)(\Delta_X^{-1}(\mathtt{Red}_1(\Phi))) \rceil_X = \lceil \mathtt{Red}_2(X, x)(\Delta_X^{-1}(\mathtt{Red}_1(\Psi))) \rceil_X,$$

where $x \in X$ is arbitrary. By Corollary 2 (third reduction), the above equality holds if, in $\mathcal{O}_{\mathcal{C}}$, we have

$$\lceil \mathtt{Red}_3(X, x, \mathcal{T}, \overrightarrow{\sigma})(\mathtt{Red}_2(X, x)(\Delta_X^{-1}(\mathtt{Red}_1(\Phi)))) \rceil^{\mathtt{sk}}_{X\backslash\{x\}} = \lceil \mathtt{Red}_3(X, x, \mathcal{T}, \overrightarrow{\sigma})(\mathtt{Red}_2(X, x)(\Delta_X^{-1}(\mathtt{Red}_1(\Psi)))) \rceil^{\mathtt{sk}}_{X\backslash\{x\}},$$

where $\mathcal{T}$ is the unrooted tree figuring in $\Delta_X^{-1}(\mathtt{Red}_1(W_s(\Phi)))$. Finally, the last equality holds by the coherence of $\mathcal{O}_{\mathcal{C}}$, established in [DP15]. ∎

3 Categorified exchangeable-output cyclic operads

In [O17, Theorem 2], the equivalence between Definition 1 and Definition 2 has been worked out in detail. In this section, by lifting that equivalence to the categorified setting, we first set up the definition of the *exchangeable-output non-skeletal categorified cyclic operads*. Then, by translating the obtained definition to the skeletal framework, we finally introduce the definition of the *exchangeable-output skeletal categorified cyclic operads*.

3.1 The exchangeable-output *non-skeletal* categorified cyclic operads

The categorification of Definition 2 is made by enriching the structure of a *categorified non-skeletal symmetric operad* $\mathcal{O}$ by *endofunctors* $D_x : \mathcal{O}(X) \to \mathcal{O}(X)$ that account for the exchange of the output with the input x, whose properties need to be such that the equivalence of [O17, Theorem 2] is not violated in the weakened setting. In other words, the decision whether some axiom of $D_x^{\mathcal{O}}$ should be weakened or not must respect the weakening made in passing from entries-only cyclic operads to their categorified version.

Before we give the resulting definition (with operadic units omitted), given that categorified operads of [DP15] are *non-symmetric* and *skeletal*, we first adapt their definition into a characterisation of categorified, *symmetric* and *non-skeletal* operads. As we did for categorified entries-only cyclic operads, we shall keep the equivariance axiom strict.

Definition 4. A *non-skeletal categorified symmetric operad* is a functor $\mathcal{O} : \mathbf{Bij}^{op} \to \mathbf{Cat}$, together with

- a family of bifunctors
$$\circ_x : \mathcal{O}(X) \times \mathcal{O}(Y) \to \mathcal{O}(X \backslash \{x\} + Y),$$
indexed by arbitrary non-empty finite sets X and Y and element $x \in X$ such that $X\backslash\{x\} \cap Y = \emptyset$, subject to the equivariance axiom:
`[EQ]` for bijections $\sigma_1 : X' \to X$ and $\sigma_2 : Y' \to Y$,
$$f^{\sigma_1} \circ_{\sigma_1^{-1}(x)} g^{\sigma_2} = (f \circ_x g)^{\sigma},$$
where $\sigma = \sigma_1|^{X\backslash\{x\}} + \sigma_2$,

- two natural isomorphisms, β and θ, called *sequential associativity* and *parallel associativity*, respectively, whose respective components
$$\beta^{x;y}_{f,g,h} : (f \circ_x g) \circ_y h \to f \circ_x (g \circ_y h) \quad \text{and} \quad \theta^{x;y}_{f,g,h} : (f \circ_x g) \circ_y h \to (f \circ_y h) \circ_x g\,,$$
are natural in f, g and h, and are subject to the following coherence conditions:
  - `[`θ`-involution]` $\theta^{y;x}_{f,h,g} \circ \theta^{x;y}_{f,g,h} = 1_{(f \circ_x g) \circ_y h}$,
  - `[`β`-pentagon]` $(1_f \circ_x \beta^{y;z}_{g,h,k}) \circ \beta^{x;z}_{f,g\circ_y h,k} \circ (\beta^{x;y}_{f,g,h} \circ_z 1_k) = \beta^{x;y}_{f,g,h\circ_z k} \circ \beta^{y;z}_{f\circ_x g,h,k}$,
  - `[`$\beta\theta$`-hexagon]` $(1_f \circ_x \theta^{y;z}_{g,h,k}) \circ \beta^{x;z}_{f,g\circ_y h,k} \circ (\beta^{x;y}_{f,g,h} \circ_z 1_k) = \beta^{x;y}_{f,g\circ_z h,k} \circ (\beta^{x;z}_{f,g,k} \circ_y 1_h) \circ \theta^{y;z}_{f\circ_x g,h,k}$,

– `[`$\beta\theta$`-pentagon]` $\theta^{x;z}_{f,g\circ_y h,k}\circ(\beta^{x;y}_{f,g,h}\circ_z 1_k)=\beta^{x;y}_{f\circ_z k,g,h}\circ(\theta^{x;z}_{f,g,k}\circ_y 1_h)\circ\theta^{y;z}_{f\circ_x g,h,k}$,

– `[`θ`-hexagon]` $\theta^{x;y}_{f\circ_z k,g,h}\circ(\theta^{x;z}_{f,g,k}\circ_y 1_h)\circ\theta^{y;z}_{f\circ_x g,h,k}=(\theta^{y;z}_{f,h,k}\circ_x 1_g)\circ\theta^{x;z}_{f\circ_y h,g,k}\circ(\theta^{x;y}_{f,g,h}\circ_z 1_k)$,

– `[`$\beta\sigma$`]` if the equality $((f\circ_x g)\circ_y h)^\sigma=(f^{\sigma_1}\circ_{x'} g^{\sigma_2})\circ_{y'} h^{\sigma_3}$ holds by `[EQ]`, then $(\beta^{x;y}_{f,g,h})^\sigma=\beta^{x';y'}_{f^{\sigma_1},g^{\sigma_2},h^{\sigma_3}}$,

– `[`$\theta\sigma$`]` if the equality $((f\circ_x g)\circ_y h)^\sigma=(f^{\sigma_1}\circ_{x'} g^{\sigma_2})\circ_{y'} h^{\sigma_3}$ holds by `[EQ]`, then $(\theta^{x;y}_{f,g,h})^\sigma=\theta^{x';y'}_{f^{\sigma_1},g^{\sigma_2},h^{\sigma_3}}$,

– `[EQ-mor]` if the equality $(f\circ_x g)^\sigma=f^{\sigma_1}\circ_{x'} g^{\sigma_2}$ holds by `[EQ]`, and if $\varphi:f\to f'$ and $\psi:g\to g'$, then $(\varphi\circ_x\psi)^\sigma=\varphi^{\sigma_1}\circ_{x'}\psi^{\sigma_2}$.

We next give the definition of non-skeletal categorified exchangeable-output cyclic operads. Below, for $f\in\mathcal{O}(X)$, $x\in X$ and $y\notin X\backslash\{x\}$, we write $D^{\mathcal{O}}_{xy}(f)$ for $D^{\mathcal{O}}_x(f)^\sigma$, where $\sigma:X\backslash\{x\}+\{y\}\to X$ renames x to y.

Definition 5. A *categorified exchangeable-output non-skeletal cyclic operad* is a (non-skeletal) categorified symmetric operad $\mathcal{O}$, together with

- a family of endofunctors
$$D_x:\mathcal{O}(X)\to\mathcal{O}(X),$$
indexed by arbitrary finite sets X and elements $x\in X$, which are subject to the following axioms, in which f and g denote operadic operations and φ and ψ morphims between operadic operations:

  `[DIN]` $D_x(D_x(f))=f$ and $D_x(D_x(\varphi))=\varphi$,

  `[DEQ]` $D_x(f)^\sigma=D_{\sigma^{-1}(x)}(f^\sigma)$ and $D_x(\varphi)^\sigma=D_{\sigma^{-1}(x)}(\varphi^\sigma)$, where $\sigma:Y\to X$ is a bijection,

  `[DEX]` $D_x(f)^\sigma=D_x(D_y(f))$ and $D_x(\varphi)^\sigma=D_x(D_y(\varphi))$, where $\sigma:X\to X$ exchanges x and y,

  `[DC1]` $D_y(f\circ_x g)=D_y(f)\circ_x g$ and $D_y(\varphi\circ_x\psi)=D_y(\varphi)\circ_x\psi$, where $y\in X\backslash\{x\}$,

  `[`$D\beta$`]` $D_z(\beta^{x;y}_{f,g,h})=\beta^{x;y}_{D_z(f),g,h}$, where $f\in\mathcal{O}(X)$, $g\in\mathcal{O}(y)$, $h\in\mathcal{O}(Z)$, $x,z\in X$ and $y\in Y$,

  `[`$D\theta$`]` $D_z(\theta^{x;y}_{f,g,h})=\theta^{x;y}_{D_z(f),g,h}$, where $f\in\mathcal{O}(X)$, $g\in\mathcal{O}(y)$, $h\in\mathcal{O}(Z)$ and $x,y,z\in X$,

- a natural isomorphism α, called the *exchange*, whose components
$$\alpha^{y,x;v}_{f,g}:D_y(f\circ_x g)\to D_{yv}(g)\circ_v D_{xy}(f),$$
are natural in f and g, and are subject to the following coherence conditions:

  - `[`$\alpha\beta\theta$`-square]` for $f\in\mathcal{O}(X)$, $g\in\mathcal{O}(y)$, $h\in\mathcal{O}(Z)$, $x\in X$ and $y,z\in Y$, the following diagram commutes

$$
\begin{array}{ccc}
D_z((f \circ_x g) \circ_y h) & = & D_z(f \circ_x g) \circ_y h \\
\Big\downarrow{\scriptstyle D_z(\beta^{x;y}_{f,g,h})} & & \Big\downarrow{\scriptstyle \alpha^{z,x;v}_{f,g} \circ_y 1_h} \\
D_z(f \circ_x (g \circ_y h)) & & (D_{zv}(g) \circ_v D_{xz}(f)) \circ_y h \\
\Big\downarrow{\scriptstyle \alpha^{z,x;v}_{f,g\circ_y h}} & & \Big\downarrow{\scriptstyle \theta^{v;y}_{D_{zv}(g),D_{xz}(f),h}} \\
D_{zv}(g \circ_y h) \circ_v D_{xz}(f) & = & (D_{zv}(g) \circ_y h) \circ_v D_{xz}(f)
\end{array}
$$

- `[`$\alpha\beta$`-hexagon]` for $f \in \mathcal{O}(X)$, $g \in \mathcal{O}(y)$, $h \in \mathcal{O}(Z)$, $x \in X$, $y \in Y$ and $z \in Z$, the following diagram commutes

$$
\begin{array}{ccc}
D_z((f \circ_x g) \circ_y h) & \xrightarrow{\alpha^{z,y;v}_{f\circ_x g,h}} & D_{zv}(h) \circ_v D_{yz}(f \circ_x g) \\
\Big\downarrow{\scriptstyle D_z(\beta^{x;y}_{f,g,h})} & & \Big\downarrow{\scriptstyle 1_{D_{zv}(h)} \circ_v (\alpha^{y,x;v}_{f,g})^{\tau}} \\
D_z(f \circ_x (g \circ_y h)) & & D_{zv}(h) \circ_v (D_{yv}(g) \circ_v D_{xy}(f))^{\sigma} \\
\Big\downarrow{\scriptstyle \alpha^{z,x;v}_{f,g\circ_y h}} & & \Big\| \\
D_{zv}(g \circ_y h) \circ_v D_{xz}(f) & & D_{zv}(h) \circ_v (D_{yv}(g) \circ_v D_{xz}(f)) \\
\Big\downarrow{\scriptstyle (\alpha^{z,y;v}_{g,h})^{\sigma} \circ_v 1_{D_{xz}(f)}} & & \Big\downarrow{\scriptstyle \beta^{v;x\;-1}_{D_{zv}(h),D_{yz}(g),D_{xz}(f)}} \\
(D_{zv}(h) \circ_v D_{yz}(g))^{\sigma} \circ_v D_{xz}(f) & = & (D_{zv}(h) \circ_v D_{yv}(g)) \circ_v D_{xz}(f)
\end{array}
$$

where σ renames z to v and τ renames y to z,

- `[`$D\alpha$`]` for $f \in \mathcal{O}(X)$, $g \in \mathcal{O}(y)$ and $z \in Y$, the following diagram commutes

$$
\begin{array}{ccc}
D_z(D_z(f \circ_x g)) & \xrightarrow{D_z(\alpha^{z,x;v}_{f,g})} & D_z(D_{zv}(g) \circ_v D_{xz}(f)) \\
\Big\| & & \Big\downarrow{\scriptstyle \alpha^{z,v;u}_{D_{zv}(g),D_{xz}(f)}} \\
f \circ_x g & = & D_{zu}(D_{xz}(f)) \circ_u D_{vz}(D_{zv}(g))
\end{array}
$$

- `[`$\alpha\sigma$`]` if the equality $(f \circ_x g)^{\sigma} = f^{\sigma_1} \circ_{\sigma_1^{-1}(x)} g^{\sigma_2}$ holds by `[EQ]`, then

$$(\alpha^{z,x;v}_{f,g})^{\sigma} = \alpha^{\sigma^{-1}(z),\sigma_1^{-1}(x);w}_{f^{\sigma_1},g^{\sigma_2}},$$

where $v \notin X\backslash\{x\} \cup Y\backslash\{z\}$ and $w \notin \sigma^{-1}[X\backslash\{x\} \cup Y\backslash\{z\}]$ are arbitrary variables. $\square$

Remark 14. *By comparing Definition 5 with Definition 2, one sees that the only axiom of D_x from Definition 2 that got weakened is* `[DC2]`. *Indeed, the proof of* [O17, Theorem 2] *testifies that all the axioms of D_x, except* `[DC2]`, *are proved by the functoriality and the equivariance of the corresponding entries-only cyclic operad, while the proof of* `[DC2]` *requires the axiom* `(CO)`. *Therefore, since* `(CO)` *gets weakened in passing from cyclic operads to categorified cyclic operads,* `[DC2]` *has to be weakened too.*

Remark 15. *Observe that, by* $[\alpha\sigma]$ *for $\sigma = id$, we have that*

$$\alpha^{y,x;u}_{f,g} = \alpha^{y,x;v}_{f,g}$$

for arbitrary variables $u, v \notin X\backslash\{x\} \cup Y\backslash\{z\}$.

We now lift the proof of [O17, Theorem 2] to the equivalence between the categorified versions of the non-skeletal entries-only and the exchangeable-output definitions of cyclic operads. This has as a consequence the coherence of the latter notion. Henceforth, we shall restrict ourselves to constant-free cyclic operads (as required by the proof of [O17, Theorem 2]).

Theorem 4. *Definition 3 and Definition 5 are equivalent definitions of categorified cyclic operads.*

Proof. [ENTRIES-ONLY $\Rightarrow$ EXCHANGEABLE-OUTPUT] Let $\mathcal{C} : \mathbf{Bij}^{op} \to \mathbf{Cat}$ be an entries-only categorified cyclic operad. The functor $\mathcal{O}_{\mathcal{C}} : \mathbf{Bij}^{op} \to \mathbf{Cat}$, underlying the corresponding exchangeable-output categorified cyclic operad, is defined by

$$\mathcal{O}_{\mathcal{C}}(X) = \mathcal{C}(X + \{*_X\}) \quad \text{and} \quad \mathcal{O}_{\mathcal{C}}(\sigma) = \mathcal{C}(\sigma^+),$$

where, for $\sigma : Y \to X$, $\sigma^+ : Y \cup \{*_Y\} \to X \cup \{*_X\}$ is defined by $\sigma^+(y) = \sigma(y)$ for $y \in Y$ and $\sigma^+(*_Y) = *_X$.

For $f \in \mathcal{O}_{\mathcal{C}}(X)$ and $g \in \mathcal{O}_{\mathcal{C}}(Y)$, the partial composition operation $\circ_x : \mathcal{O}_{\mathcal{C}}(X) \times \mathcal{O}_{\mathcal{C}}(Y) \to \mathcal{O}_{\mathcal{C}}(X\backslash\{x\} + Y)$ is defined by setting

$$f \circ_x g = f^{\sigma}\ {}_{x}\circ_{*_Y}\ g,$$

where $\sigma : X + \{*_{X\backslash\{x\}+Y}\} \to X + \{*_X\}$ renames $*_X$ to $*_{X\backslash\{x\}+Y}$.

Let $f \in \mathcal{O}_{\mathcal{C}}(X)$, $g \in \mathcal{O}_{\mathcal{C}}(Y)$, $h \in \mathcal{O}_{\mathcal{C}}(Z)$ and $x \in X$. For $y \in Y$, we set

$$\beta^{x;y}_{f,g,h} = \beta^{x,*_Y;y,*_Z}_{f^{\kappa},g,h},$$

where $\kappa : X + \{*_{X\backslash\{x\}+Y\backslash\{y\}+Z}\} \to X + \{*_X\}$ renames $*_X$ to $*_{X\backslash\{x\}+Y\backslash\{y\}+Z}$. If $y \in X$, we set

$$\theta^{x,y}_{f,g,h} = \vartheta^{x,*_Y;y,*_Z}_{f^{\kappa},g,h},$$

where $\kappa : X + \{*_{X\backslash\{x,y\}+Y+Z}\} \to X + \{*_X\}$ renames $*_X$ to $*_{X\backslash\{x,y\}+Y+Z}$.

The action $D_x : \mathcal{O}_{\mathcal{C}}(X) \to \mathcal{O}_{\mathcal{C}}(X)$ is defined as $\mathcal{C}(\sigma)$, where $\sigma : X + \{*_X\} \to X + \{*_X\}$ exchanges x and $*_X$.

Finally, for $f \in \mathcal{O}_{\mathcal{C}}(X), g \in \mathcal{O}_{\mathcal{C}}(Y)$, $x \in X$ and $y \in Y$, we set

$$\alpha^{y,x;v}_{f,g} = c^{*_X,v}_{f^{\kappa},g^{\nu}},$$

where $\kappa : X\backslash\{x\} + \{y\} + \{*_X\} \to X + \{*_X\}$ renames x to $*_X$ and $*_X$ to y, and $\nu : Y\backslash\{y\} + \{v\} + \{*_{X\backslash\{x\}+\{y\}}\} \to Y + \{*_Y\}$ renames $*_Y$ to v and y to $*_{X\backslash\{x\}+Y}$.

The coherence conditions of $\mathcal{O}_{\mathcal{C}}$ are verified as follows. We get `[`θ`-involution]` by Lemma 1, `[`β`-pentagon]` by `(`β`-pentagon)`, `[`$\beta\theta$`-hexagon]` by `(`$\beta\gamma$`-decagon)`, and `[`$\beta\theta$`-pentagon]` and `[`θ`-hexagon]` by Lemma 2. The coherence conditions `[`$\beta\sigma$`]`, `[`$\theta\sigma$`]`, `[EQ-mor]`, as well as `[`$D\beta$`]` and `[`$D\theta$`]`, hold by `(`$\beta\sigma$`)`, `(`$\gamma\sigma$`)` and `(EQ-mor)`. The equalities `[DIN]`, `[DEQ]`, `[DEX]` and `[DC1]` hold by the functoriality of $\mathcal{C}$ and `(EQ)`. The commutation of `[`$\alpha\beta\theta$`-square]` follows by the definition of ϑ in $\mathcal{C}$ (see (2.1)). By redefining `[`$\alpha\beta$`-hexagon]` in the language of the cyclic operad $\mathcal{C}$ (which is straightforward, but quite tedious), thanks to `(EQ)`, `(`$\beta\sigma$`)`, `(`$\gamma\sigma$`)` and `(EQ-mor)`, we get exactly an instance of `(`$\beta\gamma$`-hexagon)`. (We shall see how `(`$\beta\gamma$`-hexagon)` translates into `[`$\alpha\beta$`-hexagon]` in the proof of the other transition below.) The condition `[`$D\alpha$`]` follows by `(`$\gamma\sigma$`)` and `(`γ`-involution)`, and, finally, `[`$\alpha\sigma$`]` follows by `(`$\gamma\sigma$`)`.

[EXCHANGEABLE-OUTPUT $\Rightarrow$ ENTRIES-ONLY] Let now $\mathcal{O} : \mathbf{Bij}^{op} \to \mathbf{Cat}$ be an exchangeable-output categorified cyclic operad. The functor $\mathcal{C}_{\mathcal{O}} : \mathbf{Bij}^{op} \to \mathbf{Cat}$, underlying the corresponding entries-only categorified cyclic operad, is defined as

$$\mathcal{C}_{\mathcal{O}}(X) = \sum_{x\in X} \mathcal{O}(X\backslash\{x\})/_{\approx},$$

where $\approx$ is the smallest equivalence relation generated by equalities

$$(x, f) \approx (z, D_{zx}(f)) \quad \text{and} \quad (x, \varphi) \approx (z, D_{zx}(\varphi)),$$

where $z \in X\backslash\{x\}$ is arbitrary. For $[(x, f)]_{\approx} \in \mathcal{C}_{\mathcal{O}}(X)$ and a bijection $\sigma : Y \to X$, we set

$$\mathcal{C}_{\mathcal{O}}(\sigma)([(x, f)]_{\approx}) = [(\sigma^{-1}(x), \mathcal{O}(\sigma|^{X\backslash\{x\}})(f))]_{\approx}.$$

For $[(u, f)]_{\approx} \in \mathcal{C}_{\mathcal{O}}(X)$ and $[(v, g)]_{\approx} \in \mathcal{C}_{\mathcal{O}}(Y)$, the composition operation ${}_x\circ_y : \mathcal{C}_{\mathcal{O}}(X) \times \mathcal{C}_{\mathcal{O}}(Y) \to \mathcal{C}_{\mathcal{O}}(X\backslash\{x\} + Y\backslash\{y\})$ is defined as follows:

$$[(u, f)]_{\approx\,x}\circ_y[(v, g)]_{\approx} = \begin{cases} [(z, D_{zx}(f) \circ_x g)]_{\approx}, & \text{if } u = x \text{ and } v = y,\\ [(z, D_{zx}(f) \circ_x D_{yv}(g))]_{\approx}, & \text{if } u = x \text{ and } v \neq y,\\ [(u, f \circ_x g)]_{\approx}, & \text{if } u \neq x \text{ and } v = y,\\ [(u, f \circ_x D_{yv}(g))]_{\approx}, & \text{if } u \neq x \text{ and } v \neq y, \end{cases}$$

where $z \in X\backslash\{x\}$ is arbitrary. In what follows, given that $\mathcal{O}$ (and, therefore, $\mathcal{C}_{\mathcal{O}}$) is constant-free, when calculating the composition $[(u, f)]_{\approx\,x}\circ_{\underline{x}}[(v, g)]_{\approx}$, we shall always assume that $u \neq x$ and $\underline{x} \neq v$. Furthermore, when considering the composite $([(u, f)]_{\approx\,x}\circ_{\underline{x}}[(v, g)]_{\approx})_{y}\circ_{\underline{y}}[(w, h)]_{\approx}$, where $g \in \mathcal{O}(Y)$, noticing that, in the "worst case", the set Y could be reduced to $\{\underline{x}\}$, we shall assume that $v = y$.

For the definition of $\beta^{x,\underline{x};y,\underline{y}}_{[(u,f)]_{\approx},[(v,g)]_{\approx},[(w,h)]_{\approx}}$, we calculate

$$([(u, f)]_{\approx}\,{}_x\circ_{\underline{x}}\,[(y, g)]_{\approx})\,{}_y\circ_{\underline{y}}\,[(w, h)]_{\approx} = [(u, (f \circ_x D_{\underline{x}y}(g)) \circ_y D_{\underline{y}w}(h))]_{\approx}$$

and

$$[(u, f)]_{\approx}\,{}_x\circ_{\underline{x}}\,([(y, g)]_{\approx}\,{}_y\circ_{\underline{y}}\,[(w, h)]_{\approx}) = [(u, f \circ_x (D_{\underline{x}y}(g) \circ_y D_{\underline{y}w}(h)))]_{\approx}$$

and we set

$$\beta^{x,\underline{x};y,\underline{y}}_{[(x,f)]_\approx,[(y,g)]_\approx,[(w,h)]_\approx} = [(u, \beta^{x;y}_{f,D_{\underline{x}y}(g),D_{\underline{y}w}(h)})]_\approx.$$

For the definition of $\gamma^{x,y}_{[(u,f)]_\approx,[(v,g)]_\approx}$, we calculate

$$[(u,f)]_\approx {}_x\circ_y [(v,g)]_\approx = [(u, f \circ_x D_{yv}(g))]_\approx \quad \text{and} \quad [(v,g)]_\approx {}_y\circ_x [(u,f)]_\approx = [(v, g \circ_y D_{xu}(f))]_\approx.$$

Observe that, depending on the choice of the variable we take to be the common one for both classes (u or v), and by using `[`$\alpha\sigma$`]`, we can define $\gamma^{x,y}_{[(u,f)]_\approx,[(v,g)]_\approx}$ in two ways:

$$\gamma^{x,y}_{[(u,f)]_\approx,[(v,g)]_\approx} = [(u, D_{uv}(\alpha^{u,x;y}_{f,D_{yu}(g)}))]_\approx = [(v, \alpha^{u,x;y}_{f,D_{yu}(g)})]_\approx.$$

We fix the definition $\gamma^{x,y}_{[(u,f)]_\approx,[(v,g)]_\approx} = [(v, \alpha^{u,x;y}_{f,D_{yu}(g)})]_\approx$. Therefore, in calculating an instance of the commutator, for the common variable of the source and the target we shall always choose the one of the target class.

As for the coherences of $\mathcal{C}_\mathcal{O}$, `(`β`-pentagon)` holds by `[`β`-pentagon]`.

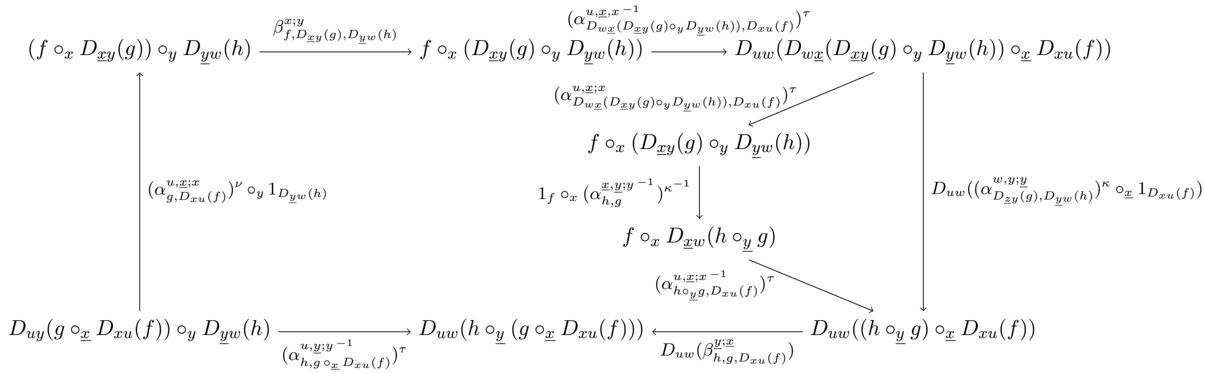
We show that `(`$\beta\gamma$`-hexagon)` holds by `[`$\alpha\beta$`-hexagon]`. For the nodes of `(`$\beta\gamma$`-hexagon)` we have

- $([(u,f)]_\approx {}_x\circ_{\underline{x}} [(y,g)]_\approx) {}_y\circ_{\underline{y}} [(w,h)]_\approx = [(u, (f \circ_x D_{\underline{x}y}(g)) \circ_y D_{\underline{y}w}(h))]_\approx$,
- $[(u,f)]_\approx {}_x\circ_{\underline{x}} ([(y,g)]_\approx {}_y\circ_{\underline{y}} [(w,h)]_\approx) = [(u, f \circ_x (D_{\underline{x}y}(g) \circ_y D_{\underline{y}w}(h)))]_\approx$,
- $([(y,g)]_\approx {}_y\circ_{\underline{y}} [(w,h)]_\approx) {}_{\underline{x}}\circ_x [(u,f)]_\approx = [(w, D_{w\underline{x}}(D_{\underline{x}y}(g) \circ_y D_{\underline{y}w}(h)) \circ_{\underline{x}} D_{xu}(f))]_\approx$,
- $([(w,h)]_\approx {}_{\underline{y}}\circ_y [(y,g)]_\approx) {}_{\underline{x}}\circ_x [(u,f)]_\approx = [(w, (h \circ_{\underline{y}} g) \circ_{\underline{x}} D_{xu}(f))]_\approx$,
- $[(w,h)]_\approx {}_{\underline{y}}\circ_y ([(y,g)]_\approx {}_{\underline{x}}\circ_x [(u,f)]_\approx) = [(w, h \circ_y (g \circ_{\underline{x}} D_{xu}(f)))]_\approx$,
- $([(y,g)]_\approx {}_{\underline{x}}\circ_x [(u,f)]_\approx) {}_y\circ_{\underline{y}} [(w,h)]_\approx = [(u, D_{uy}(g \circ_{\underline{x}} D_{xu}(f)) \circ_y D_{\underline{y}w}(h))]_\approx$.

By replacing the representatives of the classes above with the ones whose first component is u, we get

- $([(u,f)]_\approx {}_x\circ_{\underline{x}} [(y,g)]_\approx) {}_y\circ_{\underline{y}} [(w,h)]_\approx = [(u, (f \circ_x D_{\underline{x}y}(g)) \circ_y D_{\underline{y}w}(h))]_\approx$,
- $[(u,f)]_\approx {}_x\circ_{\underline{x}} ([(y,g)]_\approx {}_y\circ_{\underline{y}} [(w,h)]_\approx) = [(u, f \circ_x (D_{\underline{x}y}(g) \circ_y D_{\underline{y}w}(h)))]_\approx$,
- $([(y,g)]_\approx {}_y\circ_{\underline{y}} [(w,h)]_\approx) {}_{\underline{x}}\circ_x [(u,f)]_\approx = [(u, D_{uw}(D_{w\underline{x}}(D_{\underline{x}y}(g) \circ_y D_{\underline{y}w}(h)) \circ_{\underline{x}} D_{xu}(f)))]_\approx$,
- $([(w,h)]_\approx {}_{\underline{y}}\circ_y [(y,g)]_\approx) {}_{\underline{x}}\circ_x [(u,f)]_\approx = [(u, D_{uw}((h \circ_{\underline{y}} g) \circ_{\underline{x}} D_{xu}(f)))]_\approx$,
- $[(w,h)]_\approx {}_{\underline{y}}\circ_y ([(y,g)]_\approx {}_{\underline{x}}\circ_x [(u,f)]_\approx) = [(u, D_{uw}(h \circ_y (g \circ_{\underline{x}} D_{xu}(f))))]_\approx$,
- $([(y,g)]_\approx {}_{\underline{x}}\circ_x [(u,f)]_\approx) {}_y\circ_{\underline{y}} [(w,h)]_\approx = [(u, D_{uy}(g \circ_{\underline{x}} D_{xu}(f)) \circ_y D_{\underline{y}w}(h))]_\approx$,

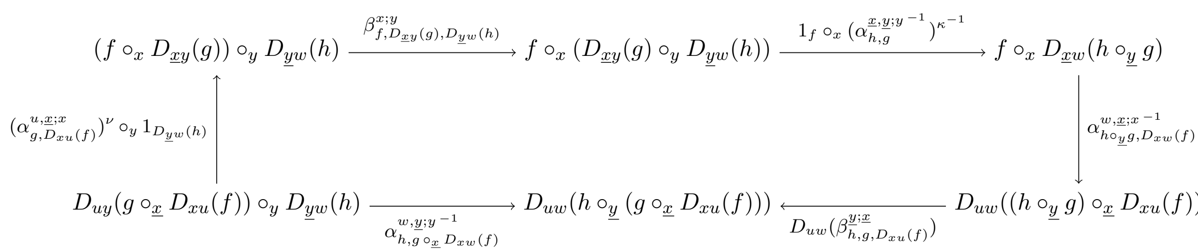
which gives us the outer part of the following diagram in $\mathcal{O}$:



where τ renames u to w, κ renames w to $\underline{x}$ and ν renames u to y, and in which the square on the right commutes by naturality of α. The equality

$$(\alpha^{u,\underline{x},x}_{D_{w\underline{x}}(D_{\underline{x}y}(g)\circ_y D_{\underline{y}w}(h)),D_{xu}(f)})^{\tau} \circ (\alpha^{u,\underline{x},x\ -1}_{D_{w\underline{x}}(D_{\underline{x}y}(g)\circ_y D_{\underline{y}w}(h)),D_{xu}(f)})^{\tau} = 1_{f\circ_x(D_{\underline{x}y}(g)\circ_y D_{\underline{y}w}(h))},$$

together with `[`$\alpha\sigma$`]`, turns the diagram above into the following instance of `[`$\alpha\beta$`-hexagon]`:



For `(`$\beta\gamma$`-decagon)`, we use `[`$\beta\theta$`-hexagon]` together with `[`$\alpha\beta\theta$`-square]`. We illustrate the proof by showing that the composition of the top three morphisms of `(`$\beta\gamma$`-decagon)` is exactly an instance of the isomorphsm θ. We have:

- $(([(u,f)]_{\approx}\ {}_x{\circ}_{\underline{x}}\ [(y,g)]_{\approx})\ {}_y{\circ}_{\underline{y}}\ [(z,h)]_{\approx})\ {}_z{\circ}_{\underline{z}}\ [(w,k)]_{\approx} = [(u,((f\circ_x D_{\underline{x}y}(g))\circ_y D_{\underline{y}z}h)\circ_z D_{\underline{z}w}(k))]_{\approx}$,
- $([(z,h)]_{\approx}\ {}_{\underline{y}}{\circ}_y\ ([(u,f)]_{\approx}\ {}_x{\circ}_{\underline{x}}\ [(y,g)]_{\approx}))\ {}_z{\circ}_{\underline{z}}\ [(w,k)]_{\approx} = [(u,D_{uz}(h\circ_{\underline{y}} D_{yu}(f\circ_x D_{\underline{x}y}(g)))\circ_z D_{\underline{z}w}(k))]_{\approx}$,
- $[(z,h)]_{\approx}\ {}_{\underline{y}}{\circ}_y\ (([(u,f)]_{\approx}\ {}_x{\circ}_{\underline{x}}\ [(y,g)]_{\approx})\ {}_z{\circ}_{\underline{z}}\ [(w,k)]_{\approx}) = [(u,D_{uz}(h\circ_{\underline{y}}(D_{yu}(f\circ_x D_{\underline{x}y}(g))\circ_z D_{\underline{z}w}(k))))]_{\approx}$,
- $(([(u,f)]_{\approx}\ {}_x{\circ}_{\underline{x}}\ [(y,g)]_{\approx})\ {}_z{\circ}_{\underline{z}}\ [(w,k)]_{\approx})\ {}_y{\circ}_{\underline{y}}\ [(z,h)]_{\approx} = [(u,((f\circ_x D_{\underline{x}y}(y))\circ_z D_{\underline{z}w}(k))\circ_y D_{\underline{y}z}(h))]_{\approx}$.

For the definitions of the top three morphisms of `(`$\beta\gamma$`-decagon)`, we have:

- $\gamma^{y,\underline{y}}_{[(u,f)]_{\approx}\ {}_x{\circ}_{\underline{x}}\ [(y,g)]_{\approx},[(z,h)]_{\approx}}\ {}_z{\circ}_{\underline{z}}\ 1_{[(w,k)]_{\approx}} = (\alpha^{u,\underline{y};y\ -1}_{h,D_{yu}(f\circ_x D_{\underline{x}y}(g))})^{\sigma}\circ_z 1_{D_{\underline{z}w}(k)}$,
- $\beta^{\underline{y},y;z,\underline{z}}_{[(z,h)]_{\approx},[(u,f)]_{\approx}\ {}_x{\circ}_{\underline{x}}\ [(y,g)]_{\approx},[(w,k)]_{\approx}} = D_u(\beta^{\underline{y};z}_{h,D_{yu}(f\circ_x D_{\underline{x}y}(g)),D_{\underline{z}w}(k)})^{\sigma}$, and
- $\gamma^{\underline{y},y}_{[(z,h)]_{\approx},([(u,f)]_{\approx}\ {}_x{\circ}_{\underline{x}}\ [(y,g)]_{\approx})\ {}_z{\circ}_{\underline{z}}\ [(w,k)]_{\approx}} = (\alpha^{u,\underline{y};y}_{h,D_{yu}(f\circ_x D_{\underline{x}y}(g))\circ_z D_{\underline{z}w}(k)})^{\sigma}$,

where σ renames u to z. By `[`$\beta\sigma$`]`, `[`$\alpha\sigma$`]` and `[DEQ]`, we get

- $(\alpha^{u,\underline{y};y\ -1}_{h,D_{yu}(f\circ_x D_{\underline{x}y}(g))})^{\sigma}\circ_z 1_{D_{\underline{z}w}(k)}=\alpha^{z,\underline{y};y\ -1}_{h,D_z(f\circ_x D_z(g^{\tau}))}\circ_z 1_{D_{\underline{z}w}(k)}$,
- $D_u(\beta^{\underline{y};z}_{h,D_{yu}(f\circ_x D_{\underline{x}y}(g)),D_{\underline{z}w}(k)})^{\sigma}=D_z(\beta^{\underline{y};z}_{h,D_z(f\circ_x D_z(g^{\tau})),D_{\underline{z}w}(k)})$, and
- $(\alpha^{u,\underline{y};y}_{h,D_{yu}(f\circ_x D_{\underline{x}y}(g))\circ_z D_{\underline{z}w}(k)})^{\sigma}=\alpha^{z,\underline{y};y}_{h,D_z(f\circ_x D_z(g^{\tau}))\circ_z D_{\underline{z}w}(k))}$

where τ renames $\underline{x}$ to z. Finally, by `[`$\alpha\beta\theta$`-square]` and `[`$\theta\sigma$`]`, we have that

$$\alpha^{z,\underline{y};y\ -1}_{h,D_z(f\circ_x D_z(g^{\tau}))}\circ_z 1_{D_{\underline{z}w}(k)}\circ D_z(\beta^{\underline{y};z}_{h,D_z(f\circ_x D_z(g^{\tau})),D_{\underline{z}w}(k)})\circ(\alpha^{z,\underline{y};y\ -1}_{h,D_z(f\circ_x D_z(g^{\tau}))}\circ_z 1_{D_{\underline{z}w}(k)})=\theta^{z;y}_{f\circ_x g,h,k}.$$

For `(`γ`-involution)`, observe that $\gamma^{y,x}_{[(v,g)]_{\approx},[(u,f)]_{\approx}}$ is defined exactly in a way which makes the composition $\gamma^{y,x}_{[(v,g)]_{\approx},[(u,f)]_{\approx}}\circ\gamma^{x,y}_{[(u,f)]_{\approx},[(v,g)]_{\approx}}$ figure favorably in `[`$D\alpha$`]`.

[THE ISOMORPHISM OF CATEGORIFIED CYCLIC OPERADS $\mathcal{C}$ AND $\mathcal{C}_{\mathcal{O}_{\mathcal{C}}}$ (AND $\mathcal{O}$ AND $\mathcal{O}_{\mathcal{C}_{\mathcal{O}}}$)] The two isomorphisms are easily defined from their corresponding decategorified versions in the proof of [O17, Theorem 2]. ■

3.2 The exchangeable-output *skeletal* categorified cyclic operads

Given that the skeletal exchangeable-output characterisation of cyclic operads is arguably most commonly seen in the literature (cf. [M08, Proposition 42]), we round up this work by indicating that the categorification of this notion is made straightforwardly by translating Definition 5 in the skeletal setting. The coherence of the obtained notion follows by lifting to the categorified setting the equivalence of non-skeletal and skeletal operads, established in [MSS02, Theorem 1.61], extended naturally so that it also includes endofunctors $D_x:\mathcal{O}(X)\to\mathcal{O}(X)$ (for non-skeletal operads) and $D_i:\mathcal{O}(n)\to\mathcal{O}(n)$ (for skeletal operads). In this section, we describe in detail the equivalence of [MSS02, Theorem 1.61].

Let $\mathcal{O}:\mathbf{\Sigma}^{op}\to\mathbf{Set}$ be a skeletal operad, defined as in [LV12, 5.3.7] (only without units). We shall write $\mathcal{O}(n)$ instead of $\mathcal{O}([n])$, and we shall denote the operadic composition operations of $\mathcal{O}$ with $\diamond_i$. Quoting [LV12, 5.3.7], the equivariance of $\mathcal{O}$ is given by the following two relations:

`[EQ1]` For any $\sigma\in\mathbb{S}_m$, we have

$$f\diamond_i g^{\sigma}=(f\diamond_i g)^{\sigma'},$$

where $\sigma'\in\mathbb{S}_{n+m-1}$ is the permutation which acts by the identity, except on the block $\{i,\dots,i+m-1\}$, on which it acts by σ.

`[EQ2]` For any $\sigma\in\mathbb{S}_n$, we have

$$f^{\sigma}\diamond_i g=(f\diamond_{\sigma(i)} g)^{\sigma''},$$

where $\sigma''\in\mathbb{S}_{n+m-1}$ is acting like σ on the block $\{1,\dots,n+m-1\}\backslash\{i,\dots,i+m-1\}$ with values in $\{1,\dots,n+m-1\}\backslash\{\sigma(i),\dots\sigma(i)+m-1\}$ and identically on the block $\{i,\dots,i+m-1\}$ with values in $\{\sigma(i),\dots\sigma(i)+m-1\}$.

As the definition of a non-skeletal operad, we fix [O17, Definition 7] (again, without units). Recall that, for a non-skeletal operad $\mathcal{O}:\mathbf{Bij}^{op}\to\mathbf{Set}$ (whose operadic composition morphisms we continue to denote with $\circ_x$), the equivariance axiom is `[EQ]` from Definition 4.

Theorem 5. *[LV12, 5.3.7] and [O17, Definition 7] are equivalent definitions of symmetric operads.*

Proof. [SKELETAL $\Rightarrow$ NON-SKELETAL] Let $\mathcal{O} : \mathbf{\Sigma}^{op} \to \mathbf{Set}$ be a skeletal operad. The functor underlying the corresponding non-skeletal operad $\mathcal{O}_{ns} : \mathbf{Bij}^{op} \to \mathbf{Set}$ is defined as

$$\mathcal{O}_{ns}(X) = \{[(f, \varphi_X)]_\approx \,|\, f \in \mathcal{O}(n) \text{ and } \varphi_X : X \to [n] \ (\varphi_X \text{ bijective})\},$$

where $|X| = n$ and $\approx$ is the smallest equivalence relation generated by

$$(f, \varphi_X) \approx (f^{\varphi_X \circ \psi_X^{-1}}, \psi_X), \tag{3.1}$$

where $\psi_X : X \to [n]$. For a bijection $\sigma : Y \to X$, $\mathcal{O}_{ns}(\sigma) : \mathcal{O}_{ns}(X) \to \mathcal{O}_{ns}(Y)$ is defined by

$$[(f, \varphi_X)]_\approx^\sigma = [(f, \varphi_X \circ \sigma)]_\approx.$$

The composition operation $\circ_x : \mathcal{O}_{ns}(X) \times \mathcal{O}_{ns}(Y) \to \mathcal{O}_{ns}(X \backslash \{x\} + Y)$ is defined as follows. Let $[(f, \varphi_X)]_\approx \in \mathcal{O}_{ns}(X)$ and $[(g, \varphi_Y)]_\approx \in \mathcal{O}_{ns}(Y)$, where $|X| = n$ and $|Y| = m$, and let $x \in X$. We set

$$[(f, \varphi_X)]_\approx \circ_x [(g, \varphi_Y)]_\approx = [(f \diamond_{\varphi_X(x)} g, \varphi_Z)]_\approx,$$

where $Z = X \backslash \{x\} + Y$ and

$$\varphi_Z(v) = \begin{cases} \varphi_X(v) & \text{for all } v \in X \text{ such that } \varphi_X(v) < \varphi_X(x) \\ \varphi_Y(v) + \varphi_X(x) - 1 & \text{for all } v \in Y \\ \varphi_X(v) + m - 1 & \text{for all } v \in X \text{ such that } \varphi_X(v) > \varphi_X(x). \end{cases} \tag{3.2}$$

The equivariance axiom of $\mathcal{O}$ ensures that the definition of $\circ_x$ does not depend on the choice of φ_X and φ_Y. Indeed, if $\varphi'_X : X \to [n]$ and $\varphi'_Y : Y \to [m]$ are different from φ_X and ψ_X, respectively, then

$$\begin{aligned} [(f^{\varphi_X \circ {\varphi'_X}^{-1}}, \varphi'_X)]_\approx \circ_x [(g^{\varphi_Y \circ {\varphi'_Y}^{-1}}, \varphi'_Y)]_\approx &= [(f^{\varphi_X \circ {\varphi'_X}^{-1}} \diamond_{\varphi'_X(x)} g^{\varphi_Y \circ {\varphi'_Y}^{-1}}, \varphi'_Z)]_\approx \\ &= [((f \diamond_{\varphi_X(x)} g)^{\varphi_Z \circ {\varphi'_Z}^{-1}}, \varphi'_Z)]_\approx \\ &= [(f \diamond_{\varphi_X(x)} g, \varphi_Z)]_\approx \\ &= [(f, \varphi_X)]_\approx \circ_x [(g, \varphi_Y)]_\approx. \end{aligned}$$

It is easily seen that the associativity axioms of $\mathcal{O}$ ensure the associativity of $\mathcal{O}_{ns}$. Finally, we show that the equivariance axiom of $\mathcal{O}_{ns}$ comes "for free". Let $[(f, \varphi_X)]_\approx \in \mathcal{O}_{ns}(X)$, $[(g, \varphi_Y)]_\approx \in \mathcal{O}_{ns}(Y)$ and $x \in X$. Then, for an arbitrary bijection $\sigma : U \to X \backslash \{x\} + Y$ and bijections $\sigma_1 : V_1 \to X$ and $\sigma_2 : V_2 \to Y$, such that $\sigma = \sigma_1|^{X \backslash x} + \sigma_2|^Y$, we have

$$\begin{aligned} ([(f, \varphi_X)]_\approx \circ_x [(g, \varphi_Y)]_\approx)^\sigma &= [(f \diamond_{\varphi_X(x)} g, \varphi_Z \circ \sigma)] \\ &= [(f, \varphi_X \circ \sigma_1)]_\approx \circ_{\sigma_1^{-1}(x)} [(g, \varphi_Y \circ \sigma_2)]_\approx \\ &= [(f, \varphi_X)]_\approx^{\sigma_1} \circ_{\sigma_1^{-1}(x)} [(g, \varphi_Y)]_\approx^{\sigma_2}. \end{aligned}$$

the key being that $\varphi_Z \circ \sigma$ coincides with the bijection built from $\varphi_X \circ \sigma_1$ and $\varphi_Y \circ \sigma_2$.

[NON-SKELETAL $\Rightarrow$ SKELETAL] Let now $\mathcal{O} : \mathbf{Bij}^{op} \to \mathbf{Set}$ be a non-skeletal operad. The functor underlying the corresponding skeletal operad $\mathcal{O}_s : \mathbf{\Sigma}^{op} \to \mathbf{Set}$ is defined by

$$\mathcal{O}_s(n) = \mathcal{O}(n) \quad \text{and} \quad \mathcal{O}_s(\sigma) = \mathcal{O}(\sigma).$$

The composition operation $\diamond_i : \mathcal{O}_s(n) \times \mathcal{O}_s(m) \to \mathcal{O}_s(n+m-1)$ is defined by:

$$f \diamond_i g = f^{\sigma_1} \circ_i g^{\sigma_2},$$

where $\sigma_1 : \{1, \dots, i\} + \{i+m, \dots, n+m-1\} \to [n]$ and $\sigma_2 : \{i, i+1, \dots, i+m-1\} \to [m]$ are defined as follows:

$$\sigma_1(j) = \begin{cases} j & \text{if } j \in \{1, \dots, i\} \\ j-m+1 & \text{if } j \in \{i+m, \dots, n+m-1\} \end{cases} \quad \text{and} \quad \sigma_2(k) = k-i+1. \tag{3.3}$$

Therefore, $\mathcal{O}_s : \mathbf{\Sigma}^{op} \to \mathbf{Set}$ is defined by restricting the data of $\mathcal{O}$ in the natural way. Notice that the proof of associativity of $\mathcal{O}_s$ requires both the associativity and the equivariance of $\mathcal{O}$. Here is the proof of `[EQ1]` of $\mathcal{O}_s$. Let $f \in \mathcal{O}_s(n)$, $g \in \mathcal{O}_s(m)$, $1 \le i \le n$ and let $\tau_2 : [m] \to [m]$ be a permutation. We then have

$$\begin{aligned} f \diamond_i g^{\tau_2} &= f^{\sigma_1} \circ_i (g^{\tau_2})^{\sigma_2} \\ &= f^{\sigma_1} \circ_i g^{\tau_2 \circ \sigma_2} \\ &= f^{\sigma_1} \circ_i g^{\sigma_2 \circ (\sigma_2^{-1} \circ \tau_2 \circ \sigma_2)} \\ &= f^{\sigma_1} \circ_i (g^{\sigma_2})^{\sigma_2^{-1} \circ \tau_2 \circ \sigma_2} \\ &= (f^{\sigma_1} \circ_i g^{\sigma_2})^{\tau} \\ &= (f \diamond_i g)^{\tau}, \end{aligned}$$

where $\sigma_1 : \{1, \dots, i\} + \{i+m, \dots, n+m-1\} \to [n]$ and $\sigma_2 : \{i, i+1, \dots, i+m-1\} \to [m]$ are defined as above, $\tau : [n+m-1] \to [n+m-1]$ is defined as

$$\tau = id_{\{1,\dots,i-1\}+\{i+m,\dots,n+m-1\}} + (\sigma_2^{-1} \circ \tau_2 \circ \sigma_2),$$

and the equality

$$f^{\sigma_1} \circ_i (g^{\sigma_2})^{\sigma_2^{-1} \circ \tau_2 \circ \sigma_2} = (f^{\sigma_1} \circ_i g^{\sigma_2})^{\tau}$$

holds by the equivariance of $\mathcal{O}$.

We now prove `[EQ2]`. If $\tau_1 : [n] \to [n]$ is a permutation, we have

$$f^{\tau_1} \diamond_i g = (f^{\tau_1})^{\sigma_1} \circ_i g^{\sigma_2} = f^{\tau_1 \circ \sigma_1} \circ_i g^{\sigma_2},$$

where $\sigma_1 : \{1, \dots, i\} + \{i+m, \dots, n+m-1\} \to [n]$ and $\sigma_2 : \{i, i+1, \dots, i+m-1\} \to [m]$ are defined like before. On the other hand, we have

$$f \diamond_{\tau_1(i)} g = f^{\kappa_1} \circ_{\tau_1(i)} g^{\kappa_2},$$

where $\kappa_1 : \{1, \dots, \tau_1(i)\} + \{\tau_1(i)+m, \dots, n+m-1\} \to [n]$ and $\kappa_2 : \{\tau_1(i), \tau_1(i)+1, \dots, \tau_1(i)+m-1\} \to [m]$ are defined as

$$\kappa_1(j) = \begin{cases} j & \text{if } j \in \{1, \dots, \tau_1(i)\} \\ j-m+1 & \text{if } j \in \{\tau_1(i)+m, \dots, n+m-1\} \end{cases} \quad \text{and} \quad \kappa_2(k) = k - \tau_1(i) + 1.$$

Let

$$\tau_2 : \{i, i+1, \ldots, i+m-1\} \to \{\tau_1(i), \tau_1(i)+1, \ldots, \tau_1(i)+m-1\}$$

be a bijection defined as

$$\tau_2 = \kappa_2^{-1} \circ \sigma_2.$$

We then have

$$\begin{array}{rcl} f^{\tau_1\circ\sigma_1} \circ_i g^{\sigma_2} & = & f^{\kappa_1\circ(\kappa_1^{-1}\circ\tau_1\circ\sigma_1)} \circ_i g^{\kappa_2\circ\tau_2} \\ & = & (f^{\kappa_1})^{\kappa_1^{-1}\circ\tau_1\circ\sigma_1} \circ_i (g^{\kappa_2})^{\tau_2} \\ & = & (f^{\kappa_1} \circ_{(\kappa_1^{-1}\circ\tau_1\circ\sigma_1)(i)} g^{\kappa_2})^{\tau} \\ & = & (f^{\kappa_1} \circ_{\tau_1(i)} g^{\kappa_2})^{\tau} \\ & = & (f \diamond_{\tau_1(i)} g)^{\tau}, \end{array}$$

where $\tau : [n+m-1] \to [n+m-1]$ is defined as

$$\tau = (\kappa_1^{-1} \circ \tau_1 \circ \sigma_1)|^{\{1,\ldots,\tau_1(i)-1\}+\{\tau_1(i)+m,\ldots,n+m-1\}} + \tau_2,$$

and the equality

$$(f^{\kappa_1})^{\kappa_1^{-1}\circ\tau_1\circ\sigma_1} \circ_i (g^{\kappa_2})^{\tau_2} = (f^{\kappa_1} \circ_{(\kappa_1^{-1}\circ\tau_1\circ\sigma_1)(i)} g^{\kappa_2})^{\tau}$$

holds by the equivariance of $\mathcal{O}$. Notice that, in the proofs of both equations, τ is acting exactly like specified by the equivariance of $\mathcal{O}_s$. This makes the equivariance established.

[THE ISOMORPHISM OF OPERADS $\mathcal{O}$ AND $(\mathcal{O}_{ns})_s$] The bijection $\phi_{[n]}$ between the sets $\mathcal{O}(n)$ and $(\mathcal{O}_{ns})_s(n) = \mathcal{O}_{ns}(n)$ is defined by

$$\phi_{[n]} : f \mapsto [(f, id_{[n]})]_{\approx}.$$

The remaining of the (skeletal) operad structure transfers via $\phi_{[n]}$ as follows:

$$\phi_{[n]}(f^{\sigma}) = [(f^{\sigma}, id)]_{\approx} = [(f^{\sigma\circ\sigma^{-1}}, \sigma)]_{\approx} = [(f, \sigma)]_{\approx} = [(f, id)]_{\approx}^{\sigma} = \phi_{[n]}(f)^{\sigma}$$

and

$$\phi_{[n]}(f \diamond_i g) = [f \circ_i g, id]_{\approx} = [(f, id)]_{\approx} \circ_i [(g, id)]_{\approx} = \psi_{[n]}(f) \circ_i \psi_{[n]}(g),$$

which shows that the natural transformation $\phi : \mathcal{O} \to (\mathcal{O}_{ns})_s$, with components $\psi_{[n]}$, is indeed an isomorphism of operads $\mathcal{O}$ and $(\mathcal{O}_{ns})_s$.

[THE ISOMORPHISM OF OPERADS $\mathcal{O}$ AND $(\mathcal{O}_s)_{ns}$] The bijection ψ_X between the sets $\mathcal{O}(X)$ and $(\mathcal{O}_s)_{ns}(X)$ is defined by

$$\psi_X : f \mapsto [(f^{\varphi_X^{-1}}, \varphi_X)]_{\approx},$$

where, assuming that $|X| = n$, $\varphi_X : X \to [n]$ is an arbitrary bijection. To see that ψ_X is well-defined, notice first that $f^{\varphi_X^{-1}} \in \mathcal{O}(n)$, i.e. that $[(f^{\varphi_X^{-1}}, \varphi_X)]_{\approx}$ is indeed an element of

$$\begin{array}{rcl} (\mathcal{O}_s)_{ns}(X) & = & \{[(g, \varphi_X)]_{\approx} \,|\, g \in \mathcal{O}_s(n) \text{ and } \varphi_X : X \to [n]\} \\ & = & \{[(g, \varphi_X)]_{\approx} \,|\, g \in \mathcal{O}(n) \text{ and } \varphi_X : X \to [n]\}, \end{array}$$

and that, by (3.1), any other choice of φ_X would lead to the same equivalence class in the definition of ψ_X. The remaining of the (non-skeletal) operad structure transfers via ϕ_X as follows: for a bijection $\sigma : Y \to X$ we have

$$\begin{aligned}\psi_X(f^\sigma) &= [((f^\sigma)^{\varphi_Y^{-1}}, \varphi_Y)]_\approx\\ &= [(f^{\sigma\circ\varphi_Y^{-1}\circ\varphi_Y\circ\sigma^{-1}\circ\varphi_X^{-1}}, \varphi_X\circ\sigma)]_\approx\\ &= [(f^{\varphi_X^{-1}}, \varphi_X\circ\sigma)]_\approx\\ &= [(f^{\varphi_X^{-1}}, \varphi_X)]_\approx^\sigma\\ &= \psi_X(f)^\sigma,\end{aligned}$$

and the composition transfers as

$$\begin{aligned}\psi_X(f)\circ_x\psi_X(g) &= [(f^{\varphi_X^{-1}}, \varphi_X)]_\approx \circ_x [(g^{\varphi_Y^{-1}}, \varphi_Y)]_\approx\\ &= [f^{\varphi_X^{-1}} \diamond_{\varphi_X(x)} g^{\varphi_Y^{-1}}, \varphi_Z]_\approx\\ &= [(f^{\varphi_X^{-1}})^{\sigma_1} \circ_{\varphi_X(x)} (g^{\varphi_Y^{-1}})^{\sigma_2}, \varphi_Z]_\approx\\ &= [((f\circ_x g)^{\varphi_Z^{-1}}, \varphi_Z)]_\approx\\ &= \psi_X(f\circ_x g),\end{aligned}$$

where $Z = X\backslash\{x\} + Y$, $\varphi_Z : X\backslash\{x\} + Y \to [n+m-1]$ is defined as in (3.2), and $\sigma_1 : \{1,\dots,i\} + \{i+m,\dots,n+m-1\} \to [n]$ and $\sigma_2 : \{i, i+1, \dots, i+m-1\} \to [m]$ are defined as in (3.3). Notice that

$$\varphi_Z^{-1} = (\varphi_X\circ\sigma_1)|^{X\backslash\{x\}} + (\varphi_Y^{-1}\circ\sigma_2),$$

which establishes the equality

$$(f^{\varphi_X^{-1}})^{\sigma_1} \circ_{\varphi_X(x)} (g^{\varphi_Y^{-1}})^{\sigma_2} = (f\circ_x g)^{\varphi_Z^{-1}}$$

as an instance of the equivariance axiom of $\mathcal{O}$. ■

3.3 The good side of non-skeletality

We end this section with comments on the advantages of the non-skeletal framework for cyclic operads. As the first benefit, we point out that, as opposed to the skeletal approach, the non-skeletal approach allows the entries-only presentation of non-symmetric cyclic operads (without the action of the symmetric group, there is no such thing as commutativity with numbered entries!). In turn, given that the entries-only definition of categorified cyclic operads is more compact than the exchangeable-output definition (compare Definition 3 and Definition 5), the non-skeletal approach is more economical in the categorified setting.

Also, non-skeletality turns out to be crucial for the rewriting involved in our proof of coherence in the presence of symmetries in Section 2. Namely, in the non-skeletal setting of (cyclic) operads, an action of the symmetric group can always be "pushed" from the composite of two operations to the operations themselves, by directing the equivariance law in the appropriate way. This was essential for the *first reduction* made in §2.3. For the skeletal setting of (cyclic) operads, this distribution of actions of the symmetric group doesn't work in general, as we illustrate in the example below.

Example 6. Let $\mathcal{O} : \Sigma^{op} \to \mathbf{Set}$ be a (skeletal) operad. Let $f, g \in \mathcal{O}(2)$, and let $\sigma : [3] \to [3]$ be a permutation defined by $\sigma(1) = 2$, $\sigma(2) = 1$ and $\sigma(3) = 3$. Notice that there is a canonical embedding

$$\mathcal{O}(n) \ni h^{\tau} \mapsto [(h, \tau)]_{\approx} \in \mathcal{O}_{ns}(n)$$

and consider the term $(f \circ_2 g)^{\sigma}$. Clearly, it is not possible to distribute σ on f and g in $\mathcal{O}(3)$. However, with the above embedding, we get

$$\mathcal{O}(3) \quad \ni \quad (f \diamond_2 g)^{\sigma} \quad \mapsto \quad [(f \diamond_2 g, \sigma)]_{\approx} = [(f \diamond_2 g, id_{[3]})]^{\sigma}_{\approx} \quad \in \quad \mathcal{O}_{ns}(3).$$

In $\mathcal{O}_{ns}(3)$, the distribution of σ works as follows:

$$[(f \diamond_2 g, id_{[3]})]^{\sigma}_{\approx} = ([(f, id_{[2]})]_{\approx} \circ_2 [(g, \tau)]_{\approx})^{\sigma} = ([(f, id_{[2]})]^{\sigma_1}_{\approx} \circ_{2'} [(g, \tau)]^{\sigma_2}_{\approx},$$

where

- $\tau : \{2, 3\} \to \{1, 2\}$ is defined by $\tau(2) = 1$ and $\tau(3) = 2$,
- $\sigma_1 : \{2, 2'\} \to \{1, 2\}$ is defined by $\sigma_1(2) = 1$ and $\sigma_1(2') = 2$,
- $\sigma_2 : \{1, 3\} \to \{2, 3\}$ is defined by $\sigma_2(1) = 2$ and $\sigma_2(3) = 3$,
- the first equality holds by the definition of the composition operation $\circ_2$ in $\mathcal{O}_{ns}(n)$, and
- the second equality holds by the equivariance of $\mathcal{O}_{ns}$.

Therefore, if $\sigma_3 = \tau \circ \sigma_2$, we have $[(f \diamond_2 g, \sigma)]_{\approx} = [(f, \sigma_1)]_{\approx} \circ_{2'} [(g, \sigma_3)]_{\approx}$. □

Additionally, we are not sure whether orienting the equivariance in the opposite direction would work for the coherence proof. As a consequence, as we pointed out in §3.2, we prove skeletal coherence in the presence of symmetries by reducing it to the non-skeletal one.

Conclusion and further study

An overview of the categorifications established in this paper is given in the table below.

	Categorified cyclic operads		
	Entries-only	Exchangeable-output non-skeletal	Exchangeable skeletal
Definitions	Definition 3	Definition 5	§3.2
Coherence proof	§2.6	Theorem 4	Theorem 5

Given the context in which operads and cyclic operads have emerged, the main idea for future work is to exhibit categorified cyclic operads "in nature" and, in particular, to determine their place in non-commutative geometry and algebraic topology. We believe that we can build an example of a categorified cyclic operad based on (generalised) profunctors of [B73]. Also, we presume that the rewriting techniques of §2.6 can be used in proving the Koszulness of coloured (cyclic) operads, encoding (cyclic) operads, of [DV15], by exhibiting their Gröbner bases. Naturally, we hope to use the categorification methods of this paper in order to categorify other variations of cyclic operads, primarily non-symmetric cyclic operads of [CGR14] and [M16], as well as modular operads of [M16]. Finally, in the spirit of pseudo-operads of [DS01], we hope to establish a "pseudo" variant of the "monoid-like" definition of cyclic operads [O17, Definition 3.8].

References and Notes

[B67] J. Bénabou, *Introduction to bicategories*, Lecture Notes in Mathematics 47, Springer, pp. 1-77, 1967.

[B73] J. Bénabou, *Les distributeurs*, Université Catholique de Louvain, Institut de Mathématique Pure et Appliquée, rapport 33, 1973.

[CGR14] E. Cheng, N. Gurski, E. Riehl, *Cyclic multicategories, multivariable adjunctions and mates*, Journal of K-Theory, 13(2):337-396, 2014.

[CO16] P. -L. Curien, J. Obradović, *A formal language for cyclic operads*, arXiv:1602.07502, 2016.

[DS01] B. Day , R. Street, *Lax monoids, pseudo-operads and convolution*, Contemporary Mathematics, Vol. 318, p.75-96, 2003.

[DV15] M. Dehling, B. Vallette, *Symmetric homotopy theory for operads*, arXiv:1503.02701, 2015.

[DP15] K. Došen, Z. Petrić, *Weak Cat-operads*, Logical Methods in Computer Science, Vol. 11(1:10), 2015.

[GK95] E. Getzler, M. Kapranov, *Cyclic operads and cyclic homology*, Geom., Top., and Phys. for Raoul Bott, International Press, Cambridge, MA, 167-201, 1995.

[K05] G. M. Kelly, *On the operads of J. P. May*, Repr. Theory Appl. Categ., 13 1-13, 2005.

[LV12] J. -L. Loday, B. Vallette, *Algebraic operads*, Springer, 2012.

[ML98] S. Mac Lane. *Categories for the Working Mathematician*, Springer-Verlag, expanded second edition, 1998.

[M08] M. Markl, *Operads and PROPs*, Elsevier, Handbook for Algebra, Vol. 5, 87-140, 2008.

[M16] M. Markl, *Modular envelopes, OSFT and nonsymmetric (non-Σ) modular operads*, J. Noncommut. Geom. 10, 775-809, 2016.

[MSS02] M. Markl, S. Schnider, J. Stasheff, Operads in Algebra, Topology and Physics, American Mathematical Society, Providence, 2002.

[O17] J. Obradović, *Monoid-like definitions of cyclic operads*, Theory and Applications of Categories, Vol. 32, No. 12, pp. 396-436, 2017.