Minimal time problem for crowd models with a localized vector field

Michel Duprez, Morgan Morancey, Francesco Rossi

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Abstract

In this work, we study the minimal time to steer a given crowd to a desired configuration. The control is a vector field, representing a perturbation of the crowd velocity, localized on a fixed control set.

We give a characterization of the minimal time both for microscopic and macroscopic descriptions of a crowd. We show that the minimal time to steer one initial configuration to another is related to the condition of having enough mass in the control region to feed the desired final configuration.

The construction of the control is explicit, providing a numerical algorithm for computing it. We finally give some numerical simulations.

1 Introduction and main results

In recent years, the study of systems describing a crowd of interacting autonomous agents has drawn a great interest from the control community. A better understanding of such interaction phenomena can have a strong impact in several key applications, such as road traffic and egress problems for pedestrians. For a few reviews about this topic, see *e.g.* [5, 6, 9, 19, 26, 28, 34, 40].

Beside the description of interactions, it is now relevant to study problems of control of crowds, *i.e.* of controlling such systems by acting on few agents, or on a small subset of the configuration space. Roughly speaking, basic problems for such models are controllability (*i.e.* reachability of a desired configuration) and optimal control (*i.e.* the minimization of a given functional). We already addressed the controllability problem in [21], thus identifying reachable configurations for crowd models. The present article deals with the subsequent step, that is the study of a classical optimal control problem: the minimal time to reach a desired configuration.

The nature of the control problem relies on the model used to describe the crowd. Two main classes are widely used. In microscopic models, the position of each agent is clearly identified; the crowd dynamics is described by an ordinary differential equation of large dimension, in which couplings of terms represent interactions between agents. For control of such models, a large literature is available, see *e.g.* reviews [8, 31, 32], as well as applications, both to pedestrian crowds [23, 33] and to road traffic [10, 25].

In macroscopic models, instead, the idea is to represent the crowd by the spatial density of agents; in this setting, the evolution in time of the density solves a partial differential equation, usually of transport type. Nonlocal terms (such as convolutions) model interactions between agents. To our knowledge, there exist few studies of control of this family of equations. In [36], the authors provide approximate alignment of a crowd described by the macroscopic Cucker-Smale model [20]. The control is the acceleration, and it

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is localized in a control region ω which moves in time. In a similar situation, a stabilization strategy has been established in [11, 12], by generalizing the Jurdjevic-Quinn method to partial differential equations. A different approach is given by mean-field type control, *i.e.* control of mean-field equations and of mean-field games modelling crowds, see *e.g.* [3, 1, 13, 24]. In this case, problems are often of optimization nature, *i.e.* the goal is to find a control minimizing a given cost, with no final constraint. A notable exception is the so-called planning problem, see [2, 38, 39]; in this case, a diffusion term is introduced too. In this article, we are interested in the minimal time to reach a specific configuration with no diffusion phenomena. For such setting, there exists no result for mean-field games, at the best of our knowledge.

This article deals with the problem of steering one initial configuration to a final one in minimal time. As stated above, we recently discussed in [21] the problem of controllability for the systems described here, which main results are recalled in Section 2.3. We proved that one can approximately steer an initial to a final configuration if they satisfy the Geometric Condition 1 recalled below. Roughly speaking, it requires that the whole initial configuration crosses the control set ω forward in time, and the final one crosses it backward in time. From now on, we will always assume that this condition is satisfied, so to ensure that the final configuration is approximately reachable.

When the controllability property is satisfied, it is then interesting to study minimal time problems. Indeed, from the theoretical point of view, it is the first problem in which optimality conditions can be naturally defined. More related to applications described above, minimal time problems play a crucial role: egress problems can be described in this setting, while traffic control is often described in terms of minimization of (maximal or average) total travel time.

For microscopic models, the dynamics can be written in terms of finite-dimensional control systems. For this reason, minimal time problems can sometimes be addressed with classical (linear or non-linear) control theory, see *e.g.* [4, 7, 29, 41]. Instead, very few results are known for macroscopic models, that can be recasted in terms of control of the transport equation. The linear case is classical, see *e.g.* [18]. Instead, more recent developments in the non-linear case (based on generalization of differential inclusions) have been recently described in [14, 15, 16].

The originality of our research lies in the constraint given on the control: it is a perturbation of the velocity vector field localized in a given region ω of the space. Such constraint is highly non-trivial, since the control problem is clearly non-linear even though the uncontrolled dynamics is. At the best of our knowledge, minimal time problems with this constraints have not been studied, neither for microscopic nor for macroscopic models. We first study a microscopic model, where the crowd is described by a vector with nd components ($n, d \in \mathbb{N}^*$) representing the positions of n agents in $\mathbb{R}^d$. The natural (uncontrolled) vector field is denoted by $v : \mathbb{R}^d \to \mathbb{R}^d$, assumed C^1 and uniformly bounded. We act on the vector field in a fixed subdomain ω of the space, which will be a nonempty open subset of $\mathbb{R}^d$. The admissible controls are thus functions of the form $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$. We then consider the following ordinary differential equation (microscopic model)

$$\left\{\begin{array}{l} \dot{x}_i(t) = v(x_i(t)) + \mathbb{1}_\omega(x_i(t))u(x_i(t), t), \\ x_i(0) = x_i^0 \end{array}\right. \qquad i = 1, \ldots, n, \tag{1}$$

where $X^0 := \{x_1^0, ..., x_n^0\}$ is the initial configuration of the crowd.

We also study a macroscopic model, where the crowd is represented by its density, that is a time-evolving probability measure $\mu(t)$ defined on the space $\mathbb{R}^d$. We consider the same natural vector field v, control region ω and admissible controls $\mathbb{1}_\omega u$. The dynamics is given by the following linear transport equation (macroscopic model)

$$\left\{\begin{array}{ll} \partial_t \mu + \nabla \cdot ((v + \mathbb{1}_\omega u)\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d, \end{array}\right. \tag{2}$$

where μ^0 is the initial density of the crowd. The function $v + \mathbb{1}_\omega u$ represents the vector field acting on μ.

To a microscopic configuration $X := \{x_1, ..., x_n\}$, we can associate the empirical measure

$$\mu := \sum_{i=1}^{n} \frac{1}{n}\delta_{x_i}. \tag{3}$$

With this notation, System (1) is a particular case of System (2). This identification can be applied only if the different microscopic agents are considered identical or interchangeable, as it is often the case for crowd models with a large number of agents. This identification will be used several times in the following, namely to approximate macroscopic models with microscopic ones.

Systems (1) and (2) are first approximations for crowd modeling, since the uncontrolled vector field v is given, and it does not describe interactions between agents. Nevertheless, it is necessary to understand control properties for such simple equations as a first step, before dealing with vector fields depending on the crowd itself. Thus, in a future work, we will study control problems for crowd models with a non-local term $v[\mu]$, based on the results for linear equations presented here. From now on, we always assume that the following geometric condition is satisfied:

Condition 1 (Geometric condition)**.** Let μ^0, μ^1 be two probability measures on $\mathbb{R}^d$ satisfying:

(i) For each $x^0 \in \text{supp}(\mu^0)$, there exists $t^0 > 0$ such that $\Phi^v_{t^0}(x^0) \in \omega$, where Φ^v_t is the *flow* associated to v (see Definition 2.3 below).

(ii) For each $x^1 \in \text{supp}(\mu^1)$, there exists $t^1 > 0$ such that $\Phi^v_{-t^1}(x^1) \in \omega$.

The Geometric Condition 1 means that each particle crosses the control region. It is illustrated in Figure 1. It is the minimal condition that we can expect to steer any initial condition to any target. Indeed, if Item (i) of the Geometric Condition 1 is not satisfied, then there exists a whole subpopulation of the measure μ^0 that never intersects the control region, thus the localized control cannot act on it.

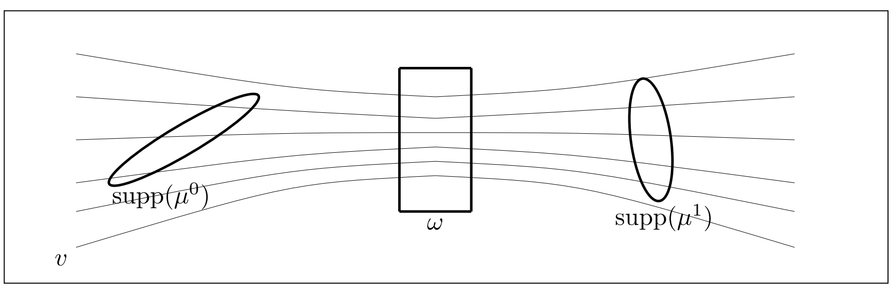


Figure 1: Geometric condition.

To ensure the well-posedness of Systems (1) and (2), we search a control $\mathbb{1}_\omega u$ satisfying the following condition:

Condition 2 (Carathéodory condition)**.** Let $\mathbb{1}_\omega u$ be such that

(i) For all $t \in \mathbb{R}$, the application $x \mapsto \mathbb{1}_\omega u(x,t)$ is Lipschitz.

(ii) For all $x \in \mathbb{R}^d$, the application $t \mapsto \mathbb{1}_\omega u(x,t)$ is measurable.

(iii) There exists $M > 0$ such that $\|\mathbb{1}_\omega u\|_\infty \leqslant M$.

For arbitrary macroscopic models, one can expect approximate controllability only, since for general measures there exists no homeomorphism sending one to another. Indeed, if we impose the Carathéodory condition, then the flow $\Phi^{v+\mathbb{1}_\omega u}_t$ is an homeomorphism (see [7, Th. 2.1.1]). Similarly, in the microscopic case, such control vector field u cannot separate points, due to uniqueness of the solution of System (1). For more details, see Remark 2.

In the microscopic model, we then assume that the initial configuration X^0 and the final one X^1 are disjoint, in the following sense:

Definition 1.1. A configuration $X = \{x_1, ..., x_n\}$ is said to be disjoint if $x_i \neq x_j$ for all $i \neq j$.

In other words, in a disjoint configuration, two agents cannot lie in the same point of the space. Since we deal with velocities $v + \mathbb{1}_\omega u$ satisfying the Carathéodory condition, if X^0 is a disjoint configuration, then the solution $X(t)$ to microscopic System (1) is also a disjoint configuration at each time $t \geqslant 0$.
In this article, we aim to study the minimal time problem. We denote by T_a the minimal time to approximately steer the initial configuration μ^0 to a final one μ^1 in the following sense: it is the infimum of times for which there exists a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ satisfying the Carathéodory condition steering μ^0 arbitrarily close to μ^1. We similarly define the minimal time T_e to exactly steer the initial configuration μ^0 to a final one μ^1. A precise definition will be given below. Since the minimal time is not always reached, we will speak about *infimum time*.

In the sequel, we will use the following notation for all $x \in \mathbb{R}^d$:

$$\begin{array}{ll} \bar{t}^0(x) := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x) \in \overline{\omega}\}, & \bar{t}^1(x) := \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x) \in \overline{\omega}\}, \\ t^0(x) := \inf\{t \in \mathbb{R}^+ : \Phi_t^v(x) \in \omega\}, & t^1(x) := \inf\{t \in \mathbb{R}^+ : \Phi_{-t}^v(x) \in \omega\}. \end{array} \tag{4}$$

The quantity $\bar{t}^0(x)$ is the infimum time at which the particle localized at point x with the velocity v belongs to $\overline{\omega}$. Idem for the other quantities. Under the Geometric Condition 1, $\bar{t}^0(x)$ and $t^0(x)$ are finite for all $x \in \text{supp}(\mu^0)$, and similarly for $\bar{t}^1(x)$ and $t^1(x)$ when $x \in \text{supp}(\mu^1)$. Moreover, it is clear that it always holds $\bar{t}^j(x) \leqslant t^j(x)$. In some situations, this inequality can be strict. For example, in Figure 2, it holds $\bar{t}^1(x_1^1) < t^1(x_1^1)$. It is also important to remark that these quantities are not continuous with respect to x. Indeed, for instance, in Figure 2, if we shift x_1^1 to the right (resp. to the left), we observe a discontinuity of $\bar{t}^1$ (resp. t^1) at the point x_1^1.

If the set ω admits everywhere an outer normal vector n, *e.g.* when it is $\mathcal{C}^1$, one can have $\bar{t}^1(x_1^1) < t^1(x_1^1)$ only if $v(\Phi^v_{-\bar{t}^1(x_1^1)}(x_1^1)) \cdot n = 0$. This means that the trajectory $\Phi_{-t}^v(x_1^1)$ touches the boundary of ω with a parallel tangent vector. The proof of such simple statement can be recovered by applying the implicit function theorem.

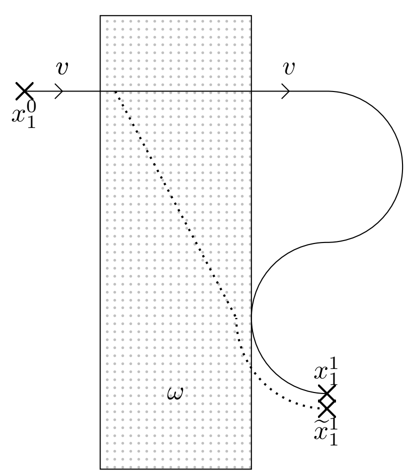


Figure 2: Example of difference between $t^1(x_1^1)$ and $\bar{t}^1(x_1^1)$.

1.1 Infimum time for microscopic models

In this section, we state the two main results about the infimum time for microscopic models. We evaluate the minimal time both for exact and approximate controllability, highlighting the different role of $t^1(x)$ and $\bar{t}^1(x)$ in the two cases.

For simplicity, we denote by

$$\bar{t}_i^0 := \bar{t}^0(x_i^0), \qquad \bar{t}_i^1 := \bar{t}^1(x_i^1), \qquad t_i^0 := t^0(x_i^0), \qquad t_i^1 := t^1(x_i^1), \tag{5}$$

for $i \in \{1, ..., n\}$. We then define

$$\begin{cases} M_e^*(X^0, X^1) := \max\{t_i^0, t_i^1 : i = 1, ..., n\}, \\ M_a^*(X^0, X^1) := \max\{t_i^0, \bar{t}_i^1 : i = 1, ..., n\}. \end{cases}$$

The value $M_e^*(X^0, X^1)$ has the following meaning for exact controllability: a time $T > M_e^*(X^0, X^1)$ is sufficiently large for each particle x_i^0 to enter ω and for each particle x_i^1 to enter it backward in time. The value $M_a^*(X^0, X^1)$ plays the same role for approximate controllability.

We now state our main result on the infimum time about exact control of System (1).

Theorem 1.2 (Infimum time for exact control of microscopic models)**.** *Let $X^0 := \{x_1^0, ..., x_n^0\}$ and $X^1 := \{x_1^1, ..., x_n^1\}$ be two disjoint configurations (see Definition 1.1). Assume that ω is convex and the empirical measures associated to X^0 and X^1 (see (3)) satisfy the Geometric Condition 1. Arrange the sequences $\{t_i^0\}_i$ and $\{t_j^1\}_j$ to be increasingly and decreasingly ordered, respectively. Then*

$$M_e(X^0, X^1) := \max_{i \in \{1,...,n\}} |t_i^0 + t_i^1| \tag{6}$$

is the infimum time $T_e(X^0, X^1)$ for exact control of System (1) in the following sense:

(i) For each $T > M_e(X^0, X^1)$, System (1) is exactly controllable from X^0 to X^1 at time T, i.e. there exists a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ satisfying the Carathéodory condition and steering X^0 exactly to X^1. Moreover, at each time $t \in [0, T]$, the configuration $X(t)$ is disjoint, i.e. $x_i(t) \neq x_j(t)$ for all $i \neq j$.

(ii) For each $T \in (M_e^*(X^0, X^1), M_e(X^0, X^1)]$, System (1) is not exactly controllable from X^0 to X^1.

(iii) There exists at most a finite number of times $T \in [0, M_e^*(X^0, X^1)]$ for which System (1) is exactly controllable from X^0 to X^1.

We give a proof of Theorem 1.2 in Section 3. As stated in item (iii), it can exist some times $T \leqslant M_e^*(X^0, X^1)$ at which it is possible to steer X^0 to X^1, but it will be not entirely thanks to the control. We give an example of this situation in Remark 5 below. We now give a characterization of the minimal time for approximate control of microscopic models.

Theorem 1.3 (Infimum time for approximate control of microscopic models)**.** *Let $X^0 := \{x_1^0, ..., x_n^0\}$ and $X^1 := \{x_1^1, ..., x_n^1\}$ be two disjoint configurations (see Definition 1.1). Assume that ω is convex and that the empirical measures associated to X^0 and X^1 (see (3)) satisfy the Geometric Condition 1. Arrange the sequences $\{t_i^0\}_i$ and $\{\bar{t}_j^1\}_j$ to be increasingly and decreasingly ordered, respectively. Then*

$$M_a(X^0, X^1) := \max_{i \in \{1,...,n\}} |t_i^0 + \bar{t}_i^1|$$

is the infimum time $T_a(X^0, X^1)$ for approximate control of System (1) in the following sense:

(i) For each $T > M_a(X^0, X^1)$, System (1) is approximately controllable from X^0 to X^1 at time T, i.e. there exists a permutation $\sigma \in \mathbb{S}_n$ and a sequence of controls $\{\mathbb{1}_\omega u_k\}_{k \in \mathbb{N}^*}$ satisfying the Carathéodory condition such that the associated solution $x_{i,k}(t)$ to System (1) satisfies $x_{i,k}(T) \xrightarrow[k \to \infty]{} x_{\sigma(i)}^1$.

(ii) *For each $T \in (M_a^*(X^0,X^1), M_a(X^0,X^1)]$, System (1) is not approximately controllable from X^0 to X^1.*

(iii) *There exists at most a finite number of times $T \in [0, M_a^*(X^0,X^1)]$ for which System (1) is approximately controllable from X^0 to X^1.*

We give a proof of Theorem 1.3 in Section 3. As for the exact controllability, one might have approximate controllability for a time $T \leqslant M_a^*(X^0,X^1)$, but not entirely due to the control. We refer also to Remark 5 below for an example.

It is well know that the notions of approximate and exact controllability are equivalent for finite dimensional linear systems when the control acts linearly, see e.g. [18]. We remark that it is not the case for System (1), which highlights the fact that we are dealing with a non-linear control problem. The difference is indeed related to the fact that for exact and approximate controllability, tangent trajectories give different behaviors. For example, in Figure 2, if we denote by $X^0 := \{x_1^0\}$ and $X^1 := \{x_1^1\}$, then it holds $M_a(X^0,X^1) < M_e(X^0,X^1)$ due to the presence of a tangent trajectory. A trajectory achieving approximate controllability is represented as dashed lines in the case $T \in (M_a(X^0,X^1), M_e(X^0,X^1))$ in Figure 2.

1.2 Infimum time for macroscopic models

In this section, we state the main result about the infimum time for approximate control of macroscopic models.

Let μ^0 and μ^1 be two probability measures, with compact support. We introduce the maps $\mathcal{F}_{\mu^0}$ and $\mathcal{B}_{\mu^1}$ defined for all $t \geqslant 0$ by

$$\begin{cases} \mathcal{F}_{\mu^0}(t) := \mu^0(\{x \in \operatorname{supp}(\mu^0) : t^0(x) \leqslant t\}), \\ \mathcal{B}_{\mu^1}(t) := \mu^1(\{x \in \operatorname{supp}(\mu^1) : t^1(x) \leqslant t\}). \end{cases}$$

The function $\mathcal{F}_{\mu^0}$ (resp. $\mathcal{B}_{\mu^1}$) gives the quantity of mass coming from μ^0 forward in time (resp. the quantity of mass coming from μ^1 backward in time) which has entered in ω at time t. Observe that we do not decrease $\mathcal{F}_{\mu^0}$ when the mass eventually leaves ω, and similarly for $\mathcal{B}_{\mu^1}$. Define the generalised inverse functions $\mathcal{F}_{\mu^0}^{-1}$ and $\mathcal{B}_{\mu^1}^{-1}$ of $\mathcal{F}_{\mu^0}$ and $\mathcal{B}_{\mu^1}$ given for all $m \in [0,1]$ by

$$\begin{cases} \mathcal{F}_{\mu^0}^{-1}(m) := \inf\{t \geq 0 : \mathcal{F}_{\mu^0}(t) \geqslant m\}, \\ \mathcal{B}_{\mu^1}^{-1}(m) := \inf\{t \geq 0 : \mathcal{B}_{\mu^1}(t) \geqslant m\}. \end{cases}$$

The function $\mathcal{F}_{\mu^0}^{-1}$ is increasing, lower semi-continuous and gives the time at which a mass m has entered in ω; similarly for $\mathcal{B}_{\mu^1}^{-1}$. Define

$$S^*(\mu^0,\mu^1) := \sup\{t^l(x) : x \in \operatorname{supp}(\mu^l) \text{ and } l \in \{0,1\}\}. \tag{7}$$

We then have the following main result about infimum time in the macroscopic case:

Theorem 1.4 (Infimum time for approximate control of macroscopic models)**.** *Let μ^0 and μ^1 be two probability measures, with compact support, absolutely continuous with respect to the Lebesgue measure and satisfying the Geometric Condition 1. Assume that ω is convex. Then*

$$S(\mu^0,\mu^1) := \sup_{m\in[0,1]} \{\mathcal{F}_{\mu^0}^{-1}(m) + \mathcal{B}_{\mu^1}^{-1}(1-m)\} \tag{8}$$

is the infimum time $T_a(\mu^0,\mu^1)$ to approximately steer μ^0 to μ^1 in the following sense:

(i) For all $T > S(\mu^0, \mu^1)$, System (2) is approximately controllable from μ^0 to μ^1 at time T with a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ satisfying the Carathéodory condition.

(ii) For all $T \in (S^*(\mu^0, \mu^1), S(\mu^0, \mu^1)]$, System (2) is not approximately controllable from μ^0 to μ^1.

We give a proof of Theorem 1.4 in Section 4. We observe that the quantity $S(\mu^0, \mu^1)$ given in (8) is the continuous equivalent of $M_e(X^0, X^1)$ in (6). Contrarily to the microscopic case, System (2) can be approximately controllable at each time $T \in (0, S^*(\mu^0, \mu^1))$. We give some examples in Remark 8 below. We do not analyse the infimum time to exactly control System (2) since it is not exactly controllable with controls satisfying the Carathéodory condition. For more details, we refer to Remark 2 below.

Remark 1. Statement (ii) in Theorem 1.4 is the only result in this article in which the uncontrolled vector field v is required to be of class C^1. All the other statements, both in the microscopic and macroscopic setting, actually hold even with v being Lipschitz.

Such difference comes from the fact that Statement ii) is based on Lemma 4.5, Statement 5, for which we use C^1- regularity of v. In a future work, we aim to investigate if such statement holds if v is Lipschitz only.

Also observe that the control $\mathbb{1}_\omega u$ is required to be Lipschitz only. In most of the constructive parts of the proof, such control will be defined as a Lipschitz spline.

This paper is organised as follows. In Section 2, we recall basic properties of the Wasserstein distance, ordinary differential equations and continuity equations. We prove our main results Theorems 1.2 and 1.3 in Section 3 and Theorem 1.4 in Section 4. We finally introduce an algorithm to compute the infimum time for approximate control of macroscopic models and give some numerical examples in Section 5.

2 Wasserstein distance, models and controllability

In this section, we highlight the connections of the microscopic model (1) and the macroscopic model (2) with the Wasserstein distance, that is the natural distance associated to these dynamics. We also recall our previous results obtained in [21] about controllability of these systems.

2.1 Wasserstein distance

In the rest of the article, we denote by $\mathcal{P}_c(\mathbb{R}^d)$ the space of probability measures in $\mathbb{R}^d$ with compact support. We also denote by "AC measures" the measures which are absolutely continuous with respect to the Lebesgue measure and by $\mathcal{P}_c^{ac}(\mathbb{R}^d)$ the subset of $\mathcal{P}_c(\mathbb{R}^d)$ of AC measures.

Definition 2.1. For $\mu,\ \nu \in \mathcal{P}_c(\mathbb{R}^d)$, we denote by $\Pi(\mu, \nu)$ the set of *transference plans* from μ to ν, *i.e.* the probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal μ and second marginal ν. Let $p \in [1, \infty)$ and $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$. Define

$$W_p(\mu,\nu) := \inf_{\pi\in\Pi(\mu,\nu)} \left\{ \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \|x-y\|^p d\pi(x,y) \right)^{1/p} \right\}, \tag{9}$$

$$W_\infty(\mu,\nu) := \inf\{\pi - \text{esssup}\ \|x-y\| : \pi \in \Pi(\mu,\nu)\}. \tag{10}$$

For $p \in [1, \infty]$, this quantity is called the **Wasserstein distance** or p-Wasserstein distance.

The idea behind this definition is the problem of *optimal transportation*, consisting in finding the optimal way to transport mass from a given measure to another. For a thorough introduction, see *e.g.* [42].

Between two microscopic configurations, we will use the following distance.

Definition 2.2. Let $p \in [1,\infty]$. Consider two configurations $X^0 := \{x_1^0, ..., x_n^0\}$ and $X^1 := \{x_1^1, ..., x_n^1\}$ of $\mathbb{R}^d$. Define the Wasserstein distance between X^0 and X^1 as follows:

$$W_p(X^0, X^1) := W_p(\mu^0, \mu^1),$$

where $\mu^0 := \frac{1}{n}\sum_i \delta_{x_i^0}$ and $\mu^1 := \frac{1}{n}\sum_i \delta_{x_i^1}$.

In this case, the Wasserstein distance can be rewritten as follows:

Proposition 1. *Let $p \in [1,\infty)$. Consider two configurations $X^0 := \{x_1^0, ..., x_n^0\}$ and $X^1 := \{x_1^1, ..., x_n^1\}$ of $\mathbb{R}^d$. It holds*

$$\begin{cases} W_p(X^0, X^1) = \inf_{\sigma\in\mathbb{S}_n} \left(\sum_{i=1}^n \frac{1}{n}\|x_i^0 - x_{\sigma(i)}^1\|^p\right)^{1/p}, \\ W_\infty(X^0, X^1) = \inf_{\sigma\in\mathbb{S}_n} \max_{i\in\{1,\dots,n\}} \|x_i^0 - x_{\sigma(i)}^1\|, \end{cases}$$

where $\mathbb{S}_n$ is the set of permutations on $\{1, ..., n\}$.

For a proof, we refer to [42, p. 5]. The idea behind this result is exactly the one of optimal transportation in the discrete setting: one has n initial and final configurations to be paired, with a given cost. The cost minimizer is the minimizer among all the permutations.

The Wasserstein distance satisfies some useful properties.

Proposition 2 (see [42, Chap. 7] and [17])**.** *It holds:*

1. *For all $\mu,\nu \in \mathcal{P}_c(\mathbb{R}^d)$, the infimum in (9) or (10) is achieved by at least one minimizer $\pi \in \Pi(\mu,\nu)$.*
2. *For $p \in [1,\infty]$, the quantity W_p is a distance on $\mathcal{P}_c(\mathbb{R}^d)$. Moreover, for $p \in [1,\infty)$, the topology induced by the Wasserstein distance W_p on $\mathcal{P}_c(\mathbb{R}^d)$ coincides with the weak topology of measures, i.e, for all sequence $\{\mu_k\}_{k\in\mathbb{N}^*} \subset \mathcal{P}_c(\mathbb{R}^d)$ and all $\mu \in \mathcal{P}_c(\mathbb{R}^d)$, the following statements are equivalent:*
   - *(i) $W_p(\mu_k, \mu) \underset{k\to\infty}{\longrightarrow} 0$.*
   - *(ii) $\mu_k \underset{k\to\infty}{\longrightarrow} \mu$ in the weak sense.*

The Wasserstein distance can be extended to all pairs of measures μ, ν compactly supported with the same total mass $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d) \neq 0$ by the formula

$$W_p(\mu,\nu) = |\mu|^{1/p} W_p\left(\frac{\mu}{|\mu|}, \frac{\nu}{|\nu|}\right).$$

In the rest of the paper, the following properties of the Wasserstein distance will be helpful.

Proposition 3. *Let μ, ν two positive measures satisfying $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$ supported in a subset X. It then holds*

$$W_{p_2}(\mu,\nu) \leqslant \mu(\mathbb{R}^d)\mathrm{diam}(X).$$

Proof. This is a direct consequence of the definition. $\square$

Proposition 4 (see [35, 42])**.** *Let μ, ρ, ν, η be four positive measures compactly supported satisfying $\mu(\mathbb{R}^d) = \nu(\mathbb{R}^d)$ and $\rho(\mathbb{R}^d) = \eta(\mathbb{R}^d)$.*

(i) For each $p \in [1,\infty]$, it holds

$$W_p^p(\mu+\rho, \nu+\eta) \leqslant W_p^p(\mu,\nu) + W_p^p(\rho,\eta).$$

(ii) For each p_1, $p_2 \in [1,\infty]$ with $p_1 \leqslant p_2$, it holds

$$\begin{cases} W_{p_1}(\mu,\nu) \leqslant W_{p_2}(\mu,\nu), \\ W_{p_2}(\mu,\nu) \leqslant \mathrm{diam}(X)^{1-p_1/p_2} W_{p_1}^{p_1/p_2}(\mu,\nu), \end{cases} \tag{11}$$

where X contains the supports of μ and ν.

2.2 Well-posedness of System (2)

In this section, we study the macroscopic model (2), together with its connections with the Wasserstein distance.

Consider the following system

$$\begin{cases} \partial_t \mu + \nabla \cdot (w\mu) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}^+, \\ \mu(\cdot, 0) = \mu^0 & \text{in } \mathbb{R}^d, \end{cases} \tag{12}$$

where $w : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ is a time-dependent vector field. This equation is called the **continuity equation**. We now introduce the flow associated to System (12).

Definition 2.3. We define the **flow** associated to a vector field $w : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ satisfying the Carathéodory condition as the application $(x^0, t) \mapsto \Phi_t^w(x^0)$ such that, for all $x^0 \in \mathbb{R}^d$, $t \mapsto \Phi_t^w(x^0)$ is the unique solution to

$$\begin{cases} \dot{x}(t) = w(x(t), t) \text{ for a.e. } t \geqslant 0, \\ x(0) = x^0. \end{cases} \tag{13}$$

It is classical that, for a vector field w satisfying the Carathéodory condition, System (13) is well-posed. See for instance [7].

We denote by Γ the set of the Borel maps $\gamma : \mathbb{R}^d \to \mathbb{R}^d$. We recall the definition of the *push-forward* of a measure.

Definition 2.4. For a $\gamma \in \Gamma$, we define the *push-forward* $\gamma \# \mu$ of a measure μ of $\mathbb{R}^d$ as follows:

$$(\gamma \# \mu)(E) := \mu(\gamma^{-1}(E)),$$

for every subset E such that $\gamma^{-1}(E)$ is μ-measurable.

We now recall a standard result linking (12) and (13), known as the method of characteristics.

Theorem 2.5 (see [42, Th. 5.34])**.** *Let $T > 0$, $\mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$ and w a vector field satisfying the Carathéodory condition. Then, System (12) admits a unique solution μ in $\mathcal{C}^0([0,T]; \mathcal{P}_c(\mathbb{R}^d))$, where $\mathcal{P}_c(\mathbb{R}^d)$ is equipped with the weak topology. Moreover:*

(i) It holds $\mu(\cdot, t) = \Phi_t^w \# \mu^0$, where Φ_t^w is the flow of w as in Definition 2.3;

(ii) If $\mu^0 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$, then $\mu(\cdot, t) \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$.

We also recall the following proposition:

Proposition 5. *Let μ, $\nu \in \mathcal{P}_c(\mathbb{R}^d)$ and $w : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be a vector field satisfying the Carathéodory condition, with a Lipschitz constant equal to L. For each $t \in \mathbb{R}$ and $p \in [1, \infty)$, it holds*

$$W_p(\Phi_t^w \# \mu, \Phi_t^w \# \nu) \leqslant e^{L|t|} W_p(\mu, \nu).$$

Proof. Consider π a minimiser of the Wasserstein distance (9) between μ and ν. Then $(\Phi_t^v, \Phi_t^v)\#\pi \in \Pi(\Phi_t^v \# \mu, \Phi_t^v \# \nu)$ and it holds by Gronwall's Lemma

$$\begin{aligned} W_p^p(\Phi_t^v \# \mu, \Phi_t^v \# \nu) &\leqslant \int_{\mathbb{R}^d \times \mathbb{R}^d} \|\Phi_t^v(x) - \Phi_t^v(y)\|^p d\pi(x,y) \\ &\leqslant e^{pL|t|} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\|^p d\pi(x,y). \end{aligned}$$

□ □

2.3 Approximate and exact controllability of System (1) and System (2)

In this section, we recall the main results of [21] about approximate and exact controllability of microscopic and macroscopic models.

We first recall the precise notions of approximate and exact controllability for System (1) and System (2):

Definition 2.6. We say that

- The microscopic system (1) (resp. the macroscopic system (2)) is **approximately controllable** from X^0 to X^1 (resp. from μ^0 to μ^1) at time T if for each $\varepsilon > 0$ there exists a sequence of controls $\{\mathbb{1}_\omega u_k\}_{k\in\mathbb{N}^*}$ satisfying the Carathéodory condition such that, denoting by X_k (resp. μ_k) the corresponding solution to System (1) (resp. System (2)), $X_k(T)$ converges to X^1 (resp. $\mu_k(T)$ converges weakly to μ^1).
- The microscopic system (1) (resp. the macroscopic system (2)) is **exactly controllable** from X^0 to X^1 (resp. from μ^0 to μ^1) at time T if there exists a control $\mathbb{1}_\omega u$ satisfying the Carathéodory condition such that the corresponding solution X (resp. μ) to System (1) (resp. System (2)) satisfies

$$X(T) = X^1 \text{ (resp. } \mu(T) = \mu^1).$$

For approximate controllability of System (2), the following result holds.

Theorem 2.7 (see [21]). *Let $\mu^0, \mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ satisfying the Geometric Condition 1. Then there exists T such that System (2) is **approximately controllable** from μ^0 to μ^1 at time T with a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ satisfying the Carathéodory condition.*

Remark 2. For exact controllability, the picture is completely different. The Carathéodory condition implies that the flow $\Phi_t^{v+\mathbb{1}_\omega u}$ is an homeomorphism. Since general μ^0, μ^1 are not homeomorphic, one needs to search a control with less regularity. The drawback is that one loses the uniqueness of the solution to System (2). This is the meaning of our recent result Theorem 2.8 below, proving that a class of controls ensuring exact controllability exists, but uniqueness is lost.

Theorem 2.8 (see [21]). *Let $\mu^0, \mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ satisfying the Geometric Condition 1. Then, there exists $T > 0$ such that System (2) is **exactly controllable** from μ^0 to μ^1 at time T in the following sense: there exists a couple $(\mathbb{1}_\omega u, \mu)$ composed of a Borel vector field $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ and a time-evolving measure μ being weak solution to System (2) and satisfying $\mu(T) = \mu^1$.*

Using Proposition 2, the approximate controllability of System (2) can be rewritten in terms of the Wasserstein distance:

Proposition 6. *The macroscopic system (2) is approximately controllable from μ^0 to μ^1 at time T if for each $\varepsilon > 0$ there exists a control $\mathbb{1}_\omega u$ satisfying the Carathéodory condition such that the corresponding solution μ to System (2) satisfies*

$$W_p(\mu^1, \mu(T)) \leqslant \varepsilon. \tag{14}$$

Remark 3. Properties (11) imply that all the Wasserstein distances W_p are equivalent for measures compactly supported and $p \in [1, \infty)$, see [42]. Thus, we can replace (14) by

$$W_1(\mu^1, \mu(T)) \leqslant \varepsilon.$$

Thus, in this work, we study approximate controllability by considering the distance W_1 only. We will use the distances W_2 and W_∞, in some other specific cases only.

3 Infimum time for microscopic models

In this section, we prove Theorem 1.2 and 1.3, *i.e.* the infimum time for the approximate and exact controllability of microscopic models. We first obtain the following result:

Proposition 7. *Let $X^0 := \{x_1^0, ..., x_n^0\} \subset \mathbb{R}^d$ and $X^1 := \{x_1^1, ..., x_n^1\} \subset \mathbb{R}^d$ be two disjoint configurations (see Definition 1.1). Assume that ω is convex and the empirical measures associated to X^0 and X^1 (see (3)) satisfy the Geometric Condition 1. Consider the sequences $\{t_i^0\}_i$ and $\{t_i^1\}_i$ given in (5). Then*

$$\widetilde{M}_e(X^0, X^1) := \min_{\sigma\in\mathbb{S}_n} \max_{i\in\{1,...,n\}} |t_i^0 + t_{\sigma(i)}^1| \tag{15}$$

is the infimum time $T_e(X^0, X^1)$ to exactly control System (1) in the sense of Theorem 1.2.

Proof. We first prove the result corresponding to **Item (i)** of Theorem 1.2. Let $T := \widetilde{M}_e(X^0, X^1) + \delta$ with $\delta > 0$. Using the Geometric Condition 1, for all $i \in \{1, ..., n\}$, there exist $s_i^0 \in (t_i^0, t_i^0 + \delta/3)$ and $s_i^1 \in (t_i^1, t_i^1 + \delta/3)$ such that

$$y_i^0 := \Phi^v_{s_i^0}(x_i^0) \in \omega \text{ and } y_i^1 := \Phi^v_{-s_i^1}(x_i^1) \in \omega.$$

This part of the proof is divided into two steps:

- In Step 1, we build a permutation σ and a flow on ω sending y_i^0 to $y_{\sigma(i)}^1$ for all $i \in \{1, ..., n\}$ with no intersection of trajectories.
- In Step 2, we define a corresponding control sending x_i^0 to $x_{\sigma(i)}^1$ for all $i \in \{1, ..., n\}$.

Item (i), Step 1: The goal is to build a flow with no intersection of the characteristic. For all $i, j \in \{1, ..., n\}$, we define the cost

$$K_{ij}(y_i^0, s_i^0, y_j^1, s_j^1) := \begin{cases} \|(y_i^0, s_i^0) - (y_j^1, T - s_j^1)\|_{\mathbb{R}^{d+1}} & \text{if } s_i^0 < T - s_j^1, \\ \infty & \text{otherwise.} \end{cases} \tag{16}$$

Consider the minimization problem:

$$\inf_{\pi\in\mathbb{B}_n} \frac{1}{n} \sum_{i,j=1}^{n} K_{ij}(y_i^0, s_i^0, y_j^1, s_j^1)\pi_{ij}, \tag{17}$$

where $\mathbb{B}_n$ is the set of the bistochastic $n \times n$ matrices, *i.e.* the matrices $\pi := (\pi_{ij})_{1\leqslant i,j\leqslant n}$ satisfying, for all $i, j \in \{1, ..., n\}$,

$$\sum_{i=1}^{n} \pi_{ij} = 1, \ \sum_{j=1}^{n} \pi_{ij} = 1, \ \pi_{ij} \geqslant 0.$$

The infimum in (17) is finite since $T > \widetilde{M}_e(X^0, X^1)$. The problem (17) is a linear minimization problem on the closed convex set $\mathbb{B}_n$. Hence, as a consequence of Krein-Milman's Theorem (see [30]), the functional (17) admits a minimum at an extremal point of $\mathbb{B}_n$, *i.e.* a permutation matrix. Let σ be a permutation, for which the associated matrix minimizes (17). Consider the straight trajectories $y_i(t)$ steering y_i^0 at time s_i^0 to $y_{\sigma(i)}^1$ at time $T - s_{\sigma(i)}^1$, that are explicitly defined by

$$y_i(t) := \frac{T - s_{\sigma(i)}^1 - t}{T - s_{\sigma(i)}^1 - s_i^0} y_i^0 + \frac{t - s_i^0}{T - s_{\sigma(i)}^1 - s_i^0} y_{\sigma(i)}^1. \tag{18}$$

We now prove by contradiction that these trajectories have no intersection: Assume that there exist i and j such that the associated trajectories $y_i(t)$ and $y_j(t)$ intersect. If we associate y_i^0 and y_j^0 to $y_{\sigma(j)}^0$ and $y_{\sigma(i)}^0$

respectively, *i.e.* we consider the permutation $\sigma \circ \mathcal{T}_{i,j}$, where $\mathcal{T}_{i,j}$ is the transposition between the i-th and the j-th elements, then the associated cost (17) is strictly smaller than the cost associated to σ. Indeed, using some geometric considerations (see Figure 3), we obtain

$$\begin{cases} \|(y_i^0, s_i^0) - (y^1_{\sigma\circ\mathcal{T}_{ij}(i)}, T - s^1_{\sigma\circ\mathcal{T}_{ij}(i)})\| = \|(y_i^0, s_i^0) - (y^1_{\sigma(j)}, T - s^1_{\sigma(j)})\| \\ \qquad\qquad\qquad\qquad\qquad\qquad < \|(y_i^0, s_i^0) - (y^1_{\sigma(i)}, T - s^1_{\sigma(i)})\|, \\ \|(y_j^0, s_j^0) - (y^1_{\sigma\circ\mathcal{T}_{ij}(j)}, T - s^1_{\sigma\circ\mathcal{T}_{ij}(j)})\| = \|(y_j^0, s_j^0) - (y^1_{\sigma(i)}, T - s^1_{\sigma(i)})\| \\ \qquad\qquad\qquad\qquad\qquad\qquad < \|(y_j^0, s_j^0) - (y^1_{\sigma(j)}, T - s^1_{\sigma(j)})\|. \end{cases}$$

This is in contradiction with the fact that σ minimizes (17).

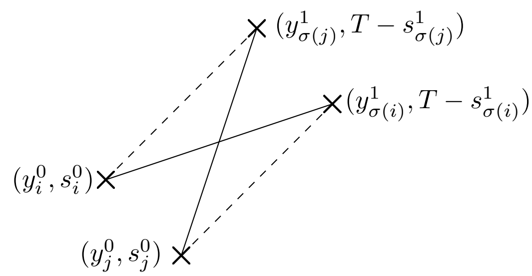


Figure 3: An optimal permutation.

Item (i), Step 2: Consider now a trajectory z_i satisfying:

$$z_i(t) := \begin{cases} \Phi_t^v(x_i^0) & \text{for all } t \in (0, s_i^0), \\ y_i(t) & \text{for all } t \in (s_i^0, T - s^1_{\sigma(i)}), \\ \Phi^v_{t-T}(x_i^1) & \text{for all } t \in (T - s^1_{\sigma(i)}, T). \end{cases}$$

The trajectories z_i have no intersection. Since ω is convex, then, using the definition of the trajectory $y_i(t)$ in (18), the points $y_i(t)$ belong to ω for all $t \in (s_i^0, T - s^1_{\sigma(i)})$. For all $i \in \{1, ..., n\}$, choose r_i, R_i satisfying $0 < r_i < R_i$ and such that it holds

$$\begin{cases} B_{r_i}(z_i(t)) \subset B_{R_i}(z_i(t)) \subset \omega & \text{for all } t \in (s_i^0, T - s^1_{\sigma(i)}) \\ B_{R_i}(z_i(t)) \cap B_{R_j}(z_j(t)) = \varnothing & \text{for all } t \in (0, T), \end{cases}$$

with $i, j \in \{1, ..., n\}$. Such radii r_i, R_i exist as a consequence of the fact that we deal with a finite number of trajectories that do not cross. The corresponding control can be chosen as a $\mathcal{C}^\infty$ function satisfying

$$u(x,t) := \begin{cases} \dfrac{y^1_{\sigma(i)} - y_i^0}{T - s^1_{\sigma(i)} - s_i^0} - v & \text{if } t \in (s_i^0, T - s^1_{\sigma(i)}) \text{ and } x \in B_{r_i}(z_i(t)), \\ 0 & \text{if } t \in (s_i^0, T - s^1_{\sigma(i)}) \text{ and } x \notin B_{R_i}(z_i(t)), \\ 0 & \text{if } t \notin (s_i^0, T - s^1_{\sigma(i)}), \\ C^\infty\text{-spline} & \text{elsewhere}. \end{cases}$$

We have then built a control satisfying the Carathéodory condition such that each i-th component of the associated solution to System (1) is $z_i(t)$, hence steering x_i^0 to $x^1_{\sigma(i)}$ in time T.

We now prove the result corresponding to **Item (ii)** of Theorem 1.2. Assume that System (1) is exactly controllable at a time $T > M_e^*(X^0, X^1)$, and consider σ the corresponding permutation defined by

$x_i(T) = x^1_{\sigma(i)}$. The idea of the proof is that the trajectory steers x^0_i to ω in time t^0_i, then it moves inside ω for a small but positive time, then it steers a point from ω to $x^i_{\sigma(i)}$ in time $t^1_{\sigma(i)}$, hence $T > t^0_i + t^1_{\sigma(i)}$.

Fix an index $i \in \{1, ..., n\}$. First recall the definition of $t^0_i, t^1_{\sigma(i)}$ and observe that it holds both $T > t^0_i$ and $T > t^1_{\sigma(i)}$. Then, the trajectory $x_i(t)$ satisfies[1] $x_i(t) \notin \omega$ for all $t \in (0, t^0_i)$, as well as $x_i(t) \notin \omega$ for all $t \in (T - t^1_{\sigma(i)}, T)$. Moreover, we prove that it exists $\tau_i \in (0, T)$ for which it holds $x_i(\tau_i) \in \omega$. By contradiction, if such τ_i does not exist, then the trajectory $x_i(t)$ never crosses the control region, hence it coincides with $\Phi^v_t(x^0_i)$. But in this case, by definition of t^0_i as the infimum of times such that $\Phi^v_t(x^0_i) \in \omega$ and recalling that $t^0_i < T$, there exists $\tau_i \in (t^0_i, T)$ such that it holds $x_i(\tau_i) = \Phi^v_{\tau_i}(x^0_i) \in \omega$. Contradiction. Also observe that ω is open, hence there exists ϵ_i such that $x_i(\tau) \in \omega$ for all $\tau \in (\tau_i - \epsilon_i, \tau_i + \epsilon)$. We merge the conditions $x_i(t) \notin \omega$ for all $t \in (0, t^0_i) \cup (T - t^1_{\sigma(i)}, T)$ with $x_i(\tau) \in \omega$ for all $\tau \in (\tau_i - \epsilon_i, \tau_i + \epsilon_i)$ with a given $\tau_i \in (0, T)$. This implies that it holds $t^0_i < \tau_i < T - t^1_{\sigma(i)}$, hence

$$T > t^0_i + t^1_{\sigma(i)}.$$

Such estimate holds for any $i \in \{1, ..., n\}$. Using the definition of $\widetilde{M}_e(X^0, X^1)$, we deduce that $T > \widetilde{M}_e(X^0, X^1)$.

We finally prove the result corresponding to **Item (iii)** of Theorem 1.2. By definition of $M^*_e(X^0, X^1)$, there exists $l \in \{0, 1\}$ and $m \in \{1, ..., n\}$ such that $M^*_e(X^0, X^1) = t^l_m$. We only study the case $l = 0$, since the case $l = 1$ can be recovered by reversing time. We consider the trajectory starting from such x^0_m only. By definition of t^0_m, the trajectory $\Phi^v_t(x^0_m)$ satisfies $\Phi^v_t(x^0_m) \notin \omega$ for all $t \in [0, M^*_e(X^0, X^1)]$. Then, for any choice of the control u localized in ω, it holds $\Phi^{v+\mathbb{1}_\omega u}_t(x^0_m) = \Phi^v_t(x^0_m)$, *i.e.* the choice of the control plays no role in the trajectory starting from x^0_m on the time interval $t \in [0, M^*_e(X^0, X^1)]$. Observe that it holds $v(\Phi^v_t(x^0_m)) \neq 0$ for all $t \in [0, M^*_e(X^0, X^1)]$, due to the fact that the vector field is time-independent and the trajectory $\Phi^v_t(x^0_m)$ enters ω for some $t > M^*_e(X^0, X^1)$.

We now prove that the set of times $t \in [0, M^*_e(X^0, X^1)]$ for which exact controllability holds is finite. A necessary condition to have exact controllability at time t is that the equation $\Phi^v_t(x^0_m) = x^1_i$ admits a solution for some time $t \in [0, M^*_e(X^0, X^1)]$ and index $i \in \{1, ..., n\}$. Then, we aim to prove that the set of times-indexes (t, i) solving such equation is finite. By contradiction, assume to have an infinite number of solutions (t, i). Since the set $i \in \{1, ..., n\}$ is finite, this implies that there exists an index I and an infinite number of (distinct) times $t_k \in [0, M^*_e(X^0, X^1)]$ such that $\Phi^v_{t_k}(x^0_m) = x^1_I$. Since v is an autonomous vector field, this implies that, for each pair t_{k_1}, t_{k_2} it holds $\Phi^v_{t_{k_1} - t_{k_2}}(x^1_i) = x^1_i$, hence $\Phi^v_t(x^1_i)$ is a periodic trajectory, with period $|t_{k_1} - t_{k_2}|$. By compactness of $[0, M^*_e(X^0, X^1)]$, there exists a converging subsequence (that we do not relabel) $t_k \to t_* \in [0, M^*_e(X^0, X^1)]$. Then $\Phi^v_t(x^1_i)$ is a periodic trajectory with arbitrarily small period, hence x^1_i is an equilibrium[2]. This contradicts the fact that for all $t \in [0, M^*_e(X^0, X^1)]$ it holds $v(\Phi^v_t(x^0_m)) \neq 0$. □ □

Remark 4. Proposition 7 can be interpreted as follows: Each particle at point x^0_i needs to be sent on a target point $x^1_{\sigma(i)}$. The time $\widetilde{M}_e(X^0, X^1)$ coincides with the infimum time for a particle at x^0_i to enter in ω and then go from ω to $x^1_{\sigma(i)}$. We are thus assuming that the particle travels with an arbitrarily large speed in ω.

Formula (15) leads to the proof of Theorem 1.2.

Proof of Theorem 1.2. Consider $\widetilde{M}_e(X^0, X^1)$ given in (15). By relabeling particles, we assume that the sequence $\{t^0_i\}_{i \in \{1,...,n\}}$ is increasingly ordered. Let σ_0 be a minimizing permutation in (15). We build recursively a sequence of permutations $\{\sigma_1, ..., \sigma_n\}$ as follows:

[1] These estimates hold even in the specific case of $x^0_i \in \omega$, for which it holds $t^0_i = 0$.

[2] Such final statement can be easily proved by contradiction: if $v(x^1_i) \neq 0$, then apply the local rectifiability theorem and prove that any eventually periodic trajectory passing through x^1_i has a strictly positive minimal period.

- Let k_1 be such that $t^1_{\sigma_0(k_1)}$ is a maximum of $\{t^1_{\sigma_0(1)}, ..., t^1_{\sigma_0(n)}\}$. We denote by $\sigma_1 := \sigma_0 \circ \mathcal{T}_{1,k_1}$, where, for all $i, j \in \{1, ..., n\}$, $\mathcal{T}_{i,j}$ is the transposition between the i-th and the j-th elements. As illustrated in Figure 4, it holds

$$\begin{cases} t^0_{k_1} + t^1_{\sigma_0(k_1)} \geqslant t^0_1 + t^1_{\sigma_0(1)}, \\ t^0_{k_1} + t^1_{\sigma_0(k_1)} \geqslant t^0_1 + t^1_{\sigma_0(k_1)} = t^0_1 + t^1_{\sigma_0 \circ \mathcal{T}_{1,k_1}(1)} = t^0_1 + t^1_{\sigma_1(1)}, \\ t^0_{k_1} + t^1_{\sigma_0(k_1)} \geqslant t^0_{k_1} + t^1_{\sigma_0(1)} = t^0_{k_1} + t^1_{\sigma_0 \circ \mathcal{T}_{1,k_1}(k_1)} = t^0_{k_1} + t^1_{\sigma_1(k_1)}. \end{cases}$$

  Thus σ_1 minimizes (15) too, since it holds

$$\max_{i \in \{1,\ldots,n\}} \{t^0_i + t^1_{\sigma_0(i)}\} \geqslant \max_{i \in \{1,\ldots,n\}} \{t^0_i + t^1_{\sigma_1(i)}\}.$$

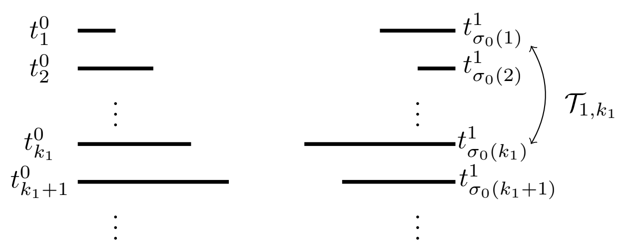


Figure 4: Computation of the infimum time

- Assume that σ_j is built. Let k_{j+1} be such that $t^1_{\sigma_j(k_{j+1})}$ is a maximum of $\{t^1_{\sigma_j(j+1)}, ..., t^1_{\sigma_j(n)}\}$. Again, we clearly have

$$\begin{cases} t^0_{k_{j+1}} + t^1_{\sigma_j(k_{j+1})} \geqslant t^0_{j+1} + t^1_{\sigma_j(j+1)}, & & \\ t^0_{k_{j+1}} + t^1_{\sigma_j(k_{j+1})} \geqslant t^0_{j+1} + t^1_{\sigma_j(k_{j+1})} & = & t^0_{j+1} + t^1_{\sigma_j \circ \mathcal{T}_{j+1,k_{J+1}}(j+1)} \\ & = & t^0_{j+1} + t^1_{\sigma_{j+1}(j+1)}, \\ t^0_{k_{j+1}} + t^1_{\sigma_j(k_{j+1})} \geqslant t^0_{k_{j+1}} + t^1_{\sigma_j(k_j)} & = & t^0_{k_{j+1}} + t^1_{\sigma_j \circ \mathcal{T}_{j+1,k_{J+1}}(k_{j+1})} \\ & = & t^0_{k_{j+1}} + t^1_{\sigma_{j+1}(k_{j+1})}. \end{cases}$$

  Thus $\sigma_{j+1} := \sigma_j \circ \mathcal{T}_{j+1,k_{j+1}}$ minimizes (15) too:

$$\max_{i \in \{1,\ldots,n\}} \{t^0_i + t^1_{\sigma_j(i)}\} \geqslant \max_{i \in \{1,\ldots,n\}} \{t^0_i + t^1_{\sigma_{j+1}(i)}\}.$$

The sequence $\{t^1_{\sigma_n(1)}, ..., t^1_{\sigma_n(n)}\}$ is then decreasing and σ_n is a minimizing permutation in (15). We deduce that $\widetilde{M_e}(X^0, X^1) = M_e(X^0, X^1)$. □

With Theorem 1.2, we give an explicit and simple expression of the infimum time for microscopic models. This result is useful in numerical simulations of Section 5 (in particular, see Algorithm 2). Let μ^0 and μ^1 be the probability measures defined by

$$\mu^0 := \sum_{i=1}^{n} \frac{1}{n} \delta_{x^0_i} \text{ and } \mu^1 := \sum_{i=1}^{n} \frac{1}{n} \delta_{x^1_i}, \tag{19}$$

where the points x^0_i (resp. x^1_i) are disjoint. We now deduce that $T_e(X^0, X^1)$ is equal to $S(\mu^0, \mu^1)$ given in (8), when μ^0 and μ^1 are given by (19).

Corollary 1. *Let $X^0 := \{x_1^0, ..., x_n^0\} \subset \mathbb{R}^d$ and $X^1 := \{x_1^1, ..., x_n^1\} \subset \mathbb{R}^d$ be two disjoint configurations. Consider μ^0 and μ^1 the corresponding measures given in (19). Assume that ω is convex and μ^0, μ^1 satisfy the Geometric Condition 1. Then the infimum time $T_e(X^0, X^1)$ is equal to $S(\mu^0, \mu^1)$ given in (8).*

Proof. Remark that if the sequences $\{t_i^0\}_{i\in\{1,...n\}}$ and $\{t_i^1\}_{i\in\{1,...n\}}$ are increasingly and decreasingly ordered respectively, then for all $m \in \left(\frac{i-1}{n}, \frac{i}{n}\right]$ it holds

$$\mathcal{F}_{\mu^0}^{-1}(m) = t_i^0 \quad \text{and} \quad \mathcal{B}_{\mu^1}^{-1}(1-m) = t_i^1.$$

Then, the result is given by identification of the expression of $M_e(X^0, X^1)$ given in (6) with the expression of $S(\mu^0, \mu^1)$ in (8). $\square$

We now prove Theorem 1.3, which characterizes the infimum time for approximate control of System (1).

Proof of Theorem 1.3. We first prove **Item (i)**. Consider $X^0 := \{x_1^0, ..., x_n^0\}$ and $X^1 := \{x_1^1, ..., x_n^1\}$ two disjoint configurations satisfying the Geometric Condition 1. We first prove that the infimum time $T_a(X^0, X^1)$ is equal to

$$\widetilde{M}_a(X^0, X^1) := \min_{\sigma\in\mathbb{S}_n} \max_{i\in\{1,...,n\}} |t_i^0 + \bar{t}_{\sigma(i)}^1|.$$

First assume that $T > \widetilde{M}_a(X^0, X^1)$. Let $\varepsilon > 0$. For each x_i^1, we need to find points y_i^1 satisfying

$$\|y_i^1 - x_i^1\| \leqslant \varepsilon \text{ and } y_i := \Phi^v_{-\bar{t}_i^1}(y_i^1) \in \omega. \tag{20}$$

For each x_i^1, observe that the Geometric Condition 1 implies that either $x_i^1 \in \omega$ or that the trajectory enters ω backward in time. In the first case, define $y_i^1 := x_i^1$. In the second case, remark that $v(\Phi^v_{-t}(x_i^1))$ is nonzero for a whole interval $t \in [0, \tilde{t}]$, with $\tilde{t} > \bar{t}_i^1$, and $\Phi^v_{-\bar{t}_i^1}(x_i^1) \in \overline{\omega}$, hence the flow $\Phi^v_{-\bar{t}_i^1}(\cdot)$ is a diffeomorphism in a neighborhood of x_i^1. Then, there exists $y_i^1 \in \mathbb{R}^d$ such that (20) is satisfied.

We denote by $Y^1 := \{y_1^1, ..., y_n^1\}$. For all $i \in \{1, ..., n\}$, since $y_i \in \omega$, then $t^1(y_i^1) \leqslant \bar{t}_i^1$, hence $\widetilde{M}_e(X^0, Y^1) \leqslant \widetilde{M}_a(X^0, X^1) < T$. Proposition 7 implies that we can exactly steer X^0 to Y^1 at time T with a control u satisfying the Carathéodory condition. Denote by $X(t)$ the solution to System (1) for the initial condition X^0 and the control u. It holds

$$W_1(X^1, X(T)) = W_1(X^1, Y^1) \leqslant \sum_{i=1}^n \frac{1}{n}\|y_i^1 - x_i^1\| \leqslant \varepsilon. \tag{21}$$

Choose now $\varepsilon = \frac{1}{k}$ for each $k \in \mathbb{N}^*$, and denote by u_k the corresponding control and with $x_{i,k}(t)$ the associated solution to System (1) for the i-th particle. By the definition of W_1 in the discrete case and (21), there exists a permutation $\sigma_k \in \mathbb{S}_n$ for which it holds

$$\left\|x_{i,k}(T) - x^1_{\sigma_k(i),k}\right\| \leq \frac{n}{k},$$

for all $i = 1, \ldots, n$. Since the space of permutations $\mathbb{S}_n$ is finite, one can extract a subsequence with σ_k constant. With such subsequence, the statement (i) is proved.

We now prove **Item (ii)**. Consider a time $T > M_a^*(X^0, X^1)$ at which System (1) is approximately controllable. We aim to prove that it satisfies $T > M_a(X^0, X^1)$. For each $k \in \mathbb{N}^*$, there exists a control u_k satisfying the Carathéodory condition such that the corresponding solution $X_k(t)$ to System (2) satisfies

$$W_1(X^1, X_k(T)) \leqslant \frac{1}{k}. \tag{22}$$

We denote by $Y_k^1 := \{y_{k,1}^1, ..., y_{k,n}^1\}$ the configuration defined by

$$y_{k,i}^1 := X_{k,i}(T),$$

where $X_{k,i}$ is the i-th component of X_k. Since X^0 is disjoint and u_k satisfies the Carathéodory condition, then Y_k^1 is disjoint too. We now prove that it holds

$$T > M_e^*(X^0, Y_k^1). \tag{23}$$

Since $T > M_a^*(X^0, X^1)$, then (23) is equivalent to $T > t_i^1(y_{k,i}^1)$ for all $i \in \{1, ..., n\}$. By contradiction, assume that there exists $j \in \{1, \dots, n\}$ such that

$$t^1(y_{k,j}^1) \geqslant T.$$

We distinguish two cases:

- If $t^1(y_{k,j}^1) > T$, then for any $t \in [0, T]$ it holds $\Phi_{-t}^v(y_{k,j}^1) \notin \omega$. Thus, the localized control does not act on the trajectory, *i.e.* for each $t \in [0, T]$ it holds $\Phi_{-t}^v(y_{k,j}^1) = \Phi_{-t}^{v+\mathbb{1}_\omega u_k}(y_{k,j}^1)$. Since $y_{k,j}^1 = \Phi_T^{v+\mathbb{1}_\omega u_k}(x_j^0) = \Phi_T^v(x_j^0)$, then $\Phi_t^v(x_j^0) \notin \omega$ for all $t \in [0, T]$. This is a contradiction with the fact that $t_j^0 \leqslant M_a^*(X^0, X^1) < T$. Thus (23) holds.
- If $t^1(y_{k,j}^1) = T$, then for all $t \in [0, T)$ it holds $\Phi_{-t}^v(y_{k,j}^1) \notin \omega$. Since ω is open, then it also holds $\Phi_{-T}^v(y_{k,j}^1) \notin \omega$. We then conclude as in the previous case.

Since $Y_k^1 = X_k(T)$, then Proposition 7 implies that

$$T > \widetilde{M}_e(X^0, Y_k^1). \tag{24}$$

For each control u_k, denote by σ_k the permutation for which $y_{k,i}^1 = \Phi_T^{v+\mathbb{1}_\omega u_k}(x_{\sigma_k(i)}^0)$. Up to extract a subsequence, for all k large enough, σ_k is equal to a permutation σ. Inequality (22) implies that for all $i \in \{1, ..., n\}$ it holds

$$y_{k,i}^1 \underset{k\to\infty}{\longrightarrow} x_{\sigma(i)}^1. \tag{25}$$

Since $t^1(y_{k,i}^1) \leqslant \widetilde{M}_e(X^0, Y_k^1) < T$, up to a subsequence, for a $s_i \geqslant 0$, it holds

$$t^1(y_{k,i}^1) \underset{k\to\infty}{\longrightarrow} s_i. \tag{26}$$

Using (25), (26) and the continuity of the flow, it holds

$$\begin{aligned}
&\|\Phi_{-t^1(y_{k,i}^1)}^v(y_{k,i}^1) - \Phi_{-s_i}^v(x_{\sigma(i)}^1)\| \\
&\qquad \leqslant \|\Phi_{-t^1(y_{k,i}^1)}^v(y_{k,i}^1) - \Phi_{-s_i}^v(y_{k,i}^1)\| + \|\Phi_{-s_i}^v(y_{k,i}^1) - \Phi_{-s_i}^v(x_{\sigma(i)}^1)\| \underset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

The fact that $\Phi_{-t^1(y_{k,i}^1)}^v(y_{k,i}^1) \in \overline{\omega}$ for each $i = 1, \dots, n$ leads to $\Phi_{-s_i}^v(x_{\sigma(i)}^1) \in \overline{\omega}$. Thus

$$\bar{t}^1(x_{\sigma(i)}^1) \leqslant \lim_{k\to\infty} t^1(y_{k,i}^1).$$

Denoting by $\delta := (T - \widetilde{M}_a(X^0, X^1))/2$ and using (24), there exists $K \in \mathbb{N}$ such that for all $k > K$ it holds

$$\begin{array}{rcl}
\widetilde{M}_a(X^0, X^1) & = & \min_{\widetilde{\sigma}\in\mathbb{S}_n} \max_{i\in\{1,\dots,n\}} |t_i^0 + \bar{t}_{\widetilde{\sigma}(i)}^1| \\
& \leqslant & \max_{i\in\{1,\dots,n\}} |t_i^0 + \bar{t}_{\sigma(i)}^1| \\
& \leqslant & \max_{i\in\{1,\dots,n\}} |t_i^0 + t^1(y_{k,\sigma(i)}^1)| + \delta \\
& = & \widetilde{M}_e(X^0, Y_k^1) + \delta < T.
\end{array}$$

We finally prove **Item (iii)** of Theorem 1.3. Let $T \in (0, M_a^*(X^0, X^1))$ be such that System (1) is approximately controllable. For any $\varepsilon > 0$, there exists u_ε such that the associated trajectory to System (1) satisfies

$$W_1(X_\varepsilon(T), X^1) < \varepsilon. \tag{27}$$

There exists $j \in \{1, \dots, n\}$ s.t. $t^0(x_j^0) = M_a^*(X^0, X^1) > T$ or $\bar{t}^1(x_j^1) = M_a^*(X^0, X^1) > T$. We distinguish these two cases:

- Assume that $t^0(x_j^0) = M_a^*(X^0, X^1) > T$.

  Define $x_{\varepsilon,j}(t) := \Phi_t^{v+\mathbb{1}_\omega u_\varepsilon}(x_j^0)$. Inequality (27) implies that there exists $k \in \{1, \ldots, n\}$ such that

  $$\|x_{\varepsilon,j}(T) - x_{k_\varepsilon}^1\| < \varepsilon. \tag{28}$$

  As $t^0(x_j^0) > T$, the trajectory $\Phi_t^v(x_j^0)$ does not cross the control set ω for $t \in [0, T)$, hence $x_{\varepsilon,j}(T) = \Phi_T^{v+\mathbb{1}_\omega u_\varepsilon}(x_j^0) = \Phi_T^v(x_j^0)$ does not depend on ε. Define $R := \frac{1}{2}\min_{p,q}\|x_p^1 - x_q^1\|$, that is strictly positive since X^1 is disjoint. For each $\epsilon < R$, Estimate (28) gives $k_\varepsilon = k$ independent on ε and

  $$x_{\varepsilon,j}(T) = \Phi_t^v(x_j^0) = x_k^1.$$

  Use now the proof of Item (iii) in Proposition 7 to prove that the equation $\Phi_t^v(x_j^0) = x_k^1$ admits a finite number of solutions (t, k) with $t \in [0, t^0(x_j^0)]$ and $k \in \{1, \ldots, n\}$.

- Assume now that $\bar{t}^1(x_j^1) = M_a^*(X^0, X^1) > T$. Again, inequality (27) implies that there exists $k_\varepsilon \in \{1, \ldots, n\}$ such that

  $$\|x_{\varepsilon,k_\varepsilon}(T) - x_j^1\| = \|\Phi_T^{v+\mathbb{1}_\omega u_\varepsilon}(x_{k_\varepsilon}^0) - x_j^1\| < \varepsilon.$$

  As $\bar{t}^1(x_j^1) > T$, the trajectory $\Phi_{-t}^v(x_j^1)$ does not cross the control set $\overline{\omega}$ for all $t \in [0, T]$. Then for all $t \in [0, T]$, there exists $r(t) > 0$ such that $B_{r(t)}(\Phi_{-t}^v(x_j^1)) \cap \omega = \varnothing$. By compactness of the set $\{\Phi_{-t}^v(x_j^1) : t \in [0, T]\}$, there exist $t_1, ..., t_N \in [0, T]$ for which it holds

  $$\{\Phi_{-t}^v(x_j^1) : t \in [0, T]\} \subset \bigcup_{i=1}^N B_{r(t_i)}(\Phi_{-t_i}^v(x_j^1)) \subset \omega^c.$$

  Thus there exists a common $r > 0$ such that $B_r(\Phi_{-t}^v(x_j^1)) \cap \omega = \varnothing$ for all $t \in [0, T]$. For $\varepsilon < r$, it holds $k_\varepsilon = k$ and

  $$x_{\varepsilon,k}(T) = \Phi_T^v(x_k^0) = x_j^1.$$

  We conclude as in the previous case.

Finally, apply the permutation method used in the proof of Theorem 1.2 to prove that it holds $\widetilde{M}_a(X^0, X^1) = M_a(X^0, X^1)$. $\square$

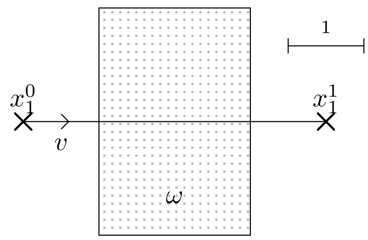
Remark 5. We illustrate Item (iii) of Theorem 1.2 by giving an example in which System (1) is never exactly controllable on $[0, M_e^*(X^0, X^1))$ and another where System (1) is exactly controllable at a time $T \in [0, M_e^*(X^0, X^1))$:

- Figure 5 (left). Consider $\omega := (-1, 1) \times (-1.5, 1.5)$, $v := (1, 0)$, $X^0 := (-2, 0)$ and $X^1 := (2, 0)$. The time $M_e^*(X^0, X^1)$ at which we can act on the particles and the minimal time $M_e(X^0, X^1)$ are respectively equal to 1 and 2. We observe that System (1) is neither exactly controllable nor approximately controllable on the interval $[0, M_e^*(X^0, X^1))$.

- Figure 5 (right). Consider $\omega := (-1, 1) \times (-1.5, 1.5)$, v be a vector field equal to $(-y, x)$ at each point (x, y) of the circle centred at $(1, 0)$ of radius 1 and X^0, X^1 given by

  $$\begin{cases} X^0 := \{(1 + \sqrt{2}/2, -\sqrt{2}/2)\}, \\ X^1 := \{(1 + \sqrt{2}/2, \sqrt{2}/2)\}. \end{cases}$$

  The time $M_e^*(X^0, X^1)$ at which we can act on the particles and the minimal time $M_e(X^0, X^1)$ are respectively equal to $3\pi/4$ and π. We remark that System (1) is exactly controllable, then approximately controllable at time $T = \pi/2 \in [0, M_e^*(X^0, X^1))$.



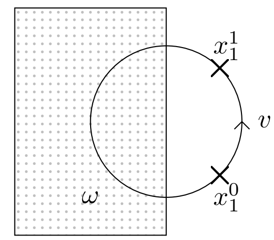


Figure 5: Examples in the case $T \in (0, M_e^*(X^0, X^1))$.

4 Infimum time for macroscopic models

This section is devoted to the proof of main Theorem 1.4 about infimum time for AC measures. We first introduce the notion of infimum time up to small mass in Section 4.1. We then give some comparisons between the microscopic and macroscopic cases in Section 4.2. We finally use the obtained results to prove Theorem 1.4 in Section 4.3.

4.1 Infimum time in the microscopic setting up to small masses

In this section, we introduce the notion of infimum time up to small mass and prove some results about this notion in the microscopic case.

From now on, we denote by $\lfloor x \rfloor$ the integer part of $x \in \mathbb{R}$.

Definition 4.1 (**Infimum time up to small mass**).

- Consider μ^0, μ^1 be two AC-measures compactly supported with the same total mass $\mu^0(\mathbb{R}^d) = \mu^1(\mathbb{R}^d) = \gamma > 0$. We denote by $T_{e,\varepsilon}(\mu^0, \mu^1)$ the infimum time to exactly steer μ^0 to μ^1 up to a mass ε. More precisely, it is the infimum of times $T \geqslant S^*(\mu^0, \mu^1)$ for which there exists a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ satisfying the Carathéodory condition and two sets R^0, R^1 of $\mathbb{R}^d$ such that $\mu^0(R^0) = \mu^1(R^1) = \varepsilon$ and
$$\Phi_T^{v+\mathbb{1}_\omega u} \# \mu^0_{|(R^0)^c} = \mu^1_{|(R^1)^c}.$$
We similarly define the minimal time $T_{a,\varepsilon}(\mu^0, \mu^1)$ to approximately steer μ^0 to μ^1 up to a mass ε.
- Let $\gamma > 0$ and $X^0 := \{x_1^0, ..., x_n^0\} \subset \mathbb{R}^d$, $X^1 := \{x_1^1, ..., x_n^1\} \subset \mathbb{R}^d$ be two disjoint configurations (see Definition 1.1). We denote by $T_{e,\varepsilon}(X^0, X^1, \gamma)$ the infimum time to exactly steer X^0 to X^1 up to a mass ε for a total mass γ. More precisely, it is the infimum of times $T \geqslant M_e^*(X^0, X^1)$ such that there exists a control $\mathbb{1}_\omega u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ satisfying the Carathéodory condition and two permutations $\sigma_0, \sigma_1 \in \mathbb{S}_n$ with $R := \lfloor n\varepsilon\gamma \rfloor$ such that
$$x_{\sigma_0(i)}(T) = x^1_{\sigma_1(i)},$$
for all $i \in \{1, ..., n-R\}$. We similarly define $T_{a,\varepsilon}(X^0, X^1, \gamma)$ the minimal time to approximately steer X^0 to X^1 up to a mass ε for a total mass γ.

Remark 6. Let $\gamma > 0$ and $X^0 := \{x_1^0, ..., x_n^0\} \subset \mathbb{R}^d$, $X^1 := \{x_1^1, ..., x_n^1\} \subset \mathbb{R}^d$ be two disjoint configurations (see Definition 1.1). We remark that
$$T_{e,\varepsilon}(X^0, X^1, \gamma) = T_{e,\varepsilon}(\mu^0, \mu^1)$$
for the measures μ^0 and μ^1 given by
$$\mu^0 := \sum_{i=1}^n \frac{\gamma}{n} \delta_{x_i^0} \text{ and } \mu^1 := \sum_{i=1}^n \frac{\gamma}{n} \delta_{x_i^1}. \tag{29}$$

For all pair of measures μ^0 and μ^1 compactly supported with the same total mass $\gamma := \mu^0(\mathbb{R}^d) = \mu^1(\mathbb{R}^d)$, we define the quantity

$$S_\varepsilon(\mu^0, \mu^1) := \sup_{m\in[0,\gamma-\varepsilon]} \{\mathcal{F}^{-1}_{\mu^0}(m) + \mathcal{B}^{-1}_{\mu^1}(\gamma - \varepsilon - m)\}. \tag{30}$$

We have the following result:

Proposition 8. *Let $\gamma > 0$ and $X^0 := \{x^0_1, ..., x^0_n\}, X^1 := \{x^1_1, ..., x^1_n\} \subset \mathbb{R}^d$ be two disjoint configurations. Assume that ω is convex and the empirical measures associated to X^0 and X^1 defined in (3) satisfy the Geometric Condition 1. Assume that the sequences $\{t^0_i\}_{i\in\{1,...n\}}$ and $\{t^1_i\}_{i\in\{1,...n\}}$ are increasingly and decreasingly ordered respectively. The infimum time $T_{e,\varepsilon}(X^0, X^1, \gamma)$ to exactly steer μ^0 to μ^1 up to a mass ε for a total mass γ is equal to*

$$M_{e,\varepsilon}(X^0, X^1, \gamma) := \max_{1\le i\le n-R} \{t^0(x^0_i) + t^1(x^1_{i+R})\},$$

where $R := \lfloor n\varepsilon\gamma \rfloor$. Moreover, it is also equal to

$$S_\varepsilon(X^0, X^1, \gamma) := S_\varepsilon(\mu^0, \mu^1), \tag{31}$$

where the measures μ^0 and μ^1 are defined in (29).

Proof. We can adapt the proof of Proposition 7 as follows. We first replace the set $\mathbb{B}_n$ of bistochastic matrices in (17) by the set $\mathbb{B}_{n,R}$ composed with the matrices satisfying

$$\sum_i \pi_{ij} \leqslant 1, \quad \sum_j \pi_{ij} \leqslant 1, \quad \sum_{ij} \pi_{ij} = n - R \text{ and } \pi_{ij} \geqslant 0.$$

The set $\mathbb{B}_{n,R}$ is closed and convex. Consider the minimization problem

$$\inf_{\pi\in\mathbb{B}_{n,R}} \frac{1}{n} \sum_{i,j=1}^{n} K_{ij}(y^0_i, s^0_i, y^1_j, s^1_j)\pi_{ij}, \tag{32}$$

where the quantities $K_{ij}(y^0_i, s^0_i, y^1_j, s^1_j)$ are defined in (16). Also in this case, the infimum in (32) is finite since $T > \widetilde{M}_{e,\varepsilon}(X^0, X^1, \gamma)$. Problem (32) is linear, hence, again as a consequence of Krein-Milman's Theorem (see [30]), some minimisers of this functional are extremal points, that are matrices composed of a permutation sub-matrix for some rows and columns, and zeros for other rows and columns. We then define

$$\widetilde{M}_{e,\varepsilon}(X^0, X^1, \gamma) := \min_{\sigma_0,\sigma_1\in\mathbb{S}_n} \max_{1\le i\le n-R} |t^0_{\sigma_0(i)} + t^1_{\sigma_1(i)}|.$$

By applying the permutation method of the proof of Theorem 1.2, we prove that $M_{e,\varepsilon}(X^0, X^1, \gamma)$ is equal to $\widetilde{M}_{e,\varepsilon}(X^0, X^1, \gamma)$.

This also implies that formula (31) holds, by using the definitions of $\mathcal{F}^{-1}_{\mu^0}$ and $\mathcal{B}^{-1}_{\mu^1}$. □ □

Proposition 8 can be also adapted to the case of approximate controllability.

Proposition 9. *Let $\gamma > 0$ and $X^0 := \{x^0_1, ..., x^0_n\}, X^1 := \{x^1_1, ..., x^1_n\} \subset \mathbb{R}^d$ be two disjoint configurations. Assume that ω is convex and the empirical measures associated to X^0 and X^1 (see (3)) satisfy the Geometric Condition 1. Assume that the sequences $\{t^0_i\}_{i\in\{1,...n\}}$ and $\{\bar{t}^1_i\}_{i\in\{1,...n\}}$ are increasingly and decreasingly ordered respectively. Denoting $R := \lfloor n\varepsilon\gamma \rfloor$, the infimum time $T_{a,\varepsilon}(X^0, X^1, \gamma)$ to approximately steer X^0 to X^1 up to a mass ε for a total mass γ is equal to*

$$M_{a,\varepsilon}(X^0, X^1, \gamma) := \max_{1\le i\le n-R} \{t^0(x^0_i) + \bar{t}^1(x^1_{i+R})\}.$$

4.2 Comparison of microscopic and macroscopic cases

The goal of this section is to give some estimates of the functions S^* and S_ε defined in (7) and (30), depending both on their arguments μ^0, μ^1 and the control region. They will be instrumental to prove Theorem 1.4 in Section 4.3. For this reason, we will specify the considered control region in the notation t^0, t^1, $\mathcal{F}_{\mu^0}$, $\mathcal{B}_{\mu^1}$, $\mathcal{F}_{\mu^0}^{-1}$ and $\mathcal{B}_{\mu^1}^{-1}$ in an index, *i.e.* t^0_ω, t^1_ω, $\mathcal{F}_{\mu^0,\omega}$, $\mathcal{B}_{\mu^1,\omega}$, $\mathcal{F}_{\mu^0,\omega}^{-1}$ and $\mathcal{B}_{\mu^1,\omega}^{-1}$. For the functions S, S_ε and S^*, the control region will be specified as follows: $S(\mu^0,\mu^1,\omega)$, $S_\varepsilon(\mu^0,\mu^1,\omega)$ and $S^*(\mu^0,\mu^1,\omega)$.

Proposition 10. *Let ω be the control set, assumed open, bounded and convex. Let $\mu^0,\mu^1 \in \mathcal{P}_c^{ac}(\mathbb{R}^d)$ satisfying the Geometric Condition 1 with respect to ω. Fix $\{f_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers converging to zero. For each $n \in \mathbb{N}$, let ω_n be defined by*

$$\omega_n := \{x \in \mathbb{R}^d : d(x,\omega^c) > f_n\}. \tag{33}$$

Let two sequences $\{\mu_n^0\}_{n\in\mathbb{N}}$, $\{\mu_n^1\}_{n\in\mathbb{N}}$ of compactly supported measures be given, satisfying the Geometric Condition 1 with respect to ω. Consider also two sequences of Borel sets $\{R_n^0\}_{n\in\mathbb{N}}$, $\{R_n^1\}_{n\in\mathbb{N}}$ of $\mathbb{R}^d$ that satisfy

$$\begin{cases} r_n := \mu^0(R_n^0) = \mu^1(R_n^1) \underset{n\to\infty}{\longrightarrow} 0,\\ \mu_n^0(\mathbb{R}^d) = \mu_n^1(\mathbb{R}^d) = 1 - r_n,\\ W_\infty(\mu^0_{|(R_n^0)^c}, \mu_n^0) < f_n,\ W_\infty(\mu^1_{|(R_n^1)^c}, \mu_n^1) < f_n. \end{cases} \tag{34}$$

Let $S_\varepsilon(\cdot,\cdot,\tilde\omega)$ be the function defined in (30) for a given control set $\tilde\omega$. Then for each $\varepsilon, \delta > 0$, there exists $N \in \mathbb{N}^*$ such that for all $n \geq N$, it holds

(i) $S_{2\varepsilon}(\mu_n^0,\mu_n^1,\omega_n) \leqslant S_\varepsilon(\mu^0,\mu^1,\omega) + \delta$.

(ii) $S_{2\varepsilon}(\mu^0,\mu^1,\omega) \leqslant S_\varepsilon(\mu_n^0,\mu_n^1,\omega_n) + \delta$.

Moreover, there exist two sequences of Borel sets $U_{0,n}, U_{1,n} \subset \mathbb{R}^d$ such that

$$\begin{cases} \mu_n^0(U_{0,n}) = \mu_n^1(U_{1,n}) \underset{n\to\infty}{\longrightarrow} 1,\\ \limsup\limits_{n\to\infty} S^*(\mu^0_{n|U_{0,n}}, \mu^1_{n|U_{1,n}}, \omega_n) \leqslant S^*(\mu^0,\mu^1,\omega). \end{cases}$$

Proof. We first prove **Item (i)**. Fix $\varepsilon,\delta > 0$. For each $n \in \mathbb{N}$, define

$$V_n^0 := \{x^0 \in \mathrm{supp}(\mu^0) : \Phi_t^v(B_{f_n}(x^0)) \subset\subset \omega_n \text{ for a } t \in [0, t^0_\omega(x^0) + \delta/2]\}.$$

We define

$$r_n^0 := \mu^0((V_n^0)^c). \tag{35}$$

Since $\{f_n\}_{n\in\mathbb{N}^*}$ is decreasing and $\{\omega_n\}_{n\in\mathbb{N}^*}$ is a sequence of increasing sets, then V_n^0 is a sequence of increasing sets. Thus r_n^0 is decreasing, hence $\lim_{n\to\infty} r_n^0$ exists. We now prove that it holds

$$\lim_{n\to\infty} r_n^0 = 0. \tag{36}$$

We prove it by contradiction. Assume that $\lim_{n\to\infty} r_n^0 > 0$. Then it holds $\mu^0((V_n^0)^c) > C > 0$ for all $n \in \mathbb{N}^*$. In particular, the limit set $\left(\cap_{n=1}^\infty (V_n^0)^c\right) \cap \mathrm{supp}(\mu^0)$ is non-empty, hence there exists $x^0 \in \mathrm{supp}(\mu^0)$ such that

$$\Phi_t^v(B_{f_n}(x^0)) \not\subset \omega_n, \tag{37}$$

for all $t \in [0, t^0_\omega(x^0) + \delta/2]$ and $n \in \mathbb{N}^*$. Due to the Geometric Condition 1, such time $t^0_\omega(x^0)$ is finite. Since ω is open, for a $t^*_\omega(x^0) \in (t^0_\omega(x^0), t^0_\omega(x^0) + \delta/2)$ and a $r(x^0) > 0$, it holds $B_{r(x^0)}(\Phi^v_{t^*_\omega(x^0)}(x^0)) \subset\subset \omega$. By continuity of Φ_t^v, there exists $\widehat{r}(x^0) > 0$ such that

$$\Phi^v_{t^*_\omega(x^0)}(B_{\widehat{r}(x^0)}(x^0)) \subset\subset \omega. \tag{38}$$

Since (37) and (38) are in contradiction, for n large enough, we conclude that (36) holds. We deduce that, for all $x^0 \in V_n^0$, it holds

$$\begin{aligned} \xi^0 \in \overline{B_{f_n}(x^0)} \quad &\Rightarrow \quad \Phi_t^v(\xi_0) \in \omega_n \quad \text{for some} \quad t \in [0, t_\omega^0(x^0) + \delta/2] \\ &\Rightarrow \quad t_{\omega_n}^0(\xi^0) \leqslant t_\omega^0(x^0) + \delta/2. \end{aligned} \tag{39}$$

For each $n \in \mathbb{N}^*$, consider an optimal transference plan π_n realizing the ∞-Wasserstein distance $W_\infty(\mu^0_{|(R_n^0)^c}, \mu_n^0)$, see Proposition 2. We remark that (34) implies that

$$|x^0 - \xi^0| < f_n \quad \text{for } \pi_n\text{-almost every } (x^0, \xi^0) \in (R_n^0)^c \cap \operatorname{supp}(\mu^0) \times \operatorname{supp}(\mu_n^0). \tag{40}$$

Thus, combining (39) and (40), it holds

$$t_{\omega_n}^0(\xi^0) \leqslant t_\omega^0(x^0) + \delta/2 \tag{41}$$

for π_n-almost every pair (x^0, ξ^0) with $x^0 \in (R_n^0)^c \cap V_n^0$ and $\xi^0 \in \operatorname{supp}(\mu_n^0)$. Using (35) and (41), we obtain

$$\begin{aligned} &\mathcal{F}_{\mu^0,\omega}(t) \leqslant \mu^0(\{x^0 \in (R_n^0)^c \cap V_n^0 : t_\omega^0(x^0) \leqslant t\}) + \mu^0(R_n^0) + \mu^0((V_n^0)^c) \\ &= \pi_n(\{(x^0,\xi^0) \in ((R_n^0)^c \cap V_n^0) \times \operatorname{supp}(\mu_n^0) : t_\omega^0(x^0) \leqslant t\}) + r_n + r_n^0 \\ &\leqslant \pi_n(\{(x^0,\xi^0) \in ((R_n^0)^c \cap V_n^0) \times \operatorname{supp}(\mu_n^0) : t_{\omega_n}^0(\xi^0) \leqslant t + \delta/2\}) + r_n + r_n^0 \\ &\leqslant \mu_n^0(\{\xi^0 \in \operatorname{supp}(\mu_n^0) : t_{\omega_n}^0(\xi^0) \leqslant t + \delta/2\}) + r_n + r_n^0 \\ &= \mathcal{F}_{\mu_n^0,\omega_n}(t + \delta/2) + r_n + r_n^0, \end{aligned}$$

for all $t \in \mathbb{R}^+$. Similarly, we have

$$\mathcal{B}_{\mu^1,\omega}(t) \leqslant \mathcal{B}_{\mu_n^1,\omega_n}(t + \delta/2) + r_n + r_n^1,$$

for all $t \in \mathbb{R}^+$, where r_n^1 is defined similarly to (35). We deduce that the generalized inverse satisfies

$$\begin{aligned} \mathcal{F}^{-1}_{\mu_n^0,\omega_n}(m) \quad &:= \quad \inf\{t \geqslant 0 : \mathcal{F}_{\mu_n^0,\omega_n}(t) \geqslant m\} \\ &\leqslant \quad \inf\{t \geqslant \delta/2 : \mathcal{F}_{\mu^0,\omega}(t - \delta/2) - r_n^0 - r_n \geqslant m\} \\ &= \quad \inf\{s \geqslant 0 : \mathcal{F}_{\mu^0,\omega}(s) \geqslant m + r_n^0 + r_n\} + \delta/2 \\ &= \quad \mathcal{F}^{-1}_{\mu^0,\omega}(m + r_n^0 + r_n) + \delta/2, \end{aligned}$$

for all $m \in (0, 1 - r_n^0 - r_n)$. Similarly, we obtain

$$\mathcal{B}^{-1}_{\mu_n^1,\omega_n}(1 - r_n - 2\varepsilon - m) \leqslant \mathcal{B}^{-1}_{\mu^1,\omega}(1 + r_n^1 - 2\varepsilon - m) + \delta/2,$$

for all $m \in (r_n^1 - 2\varepsilon, 1 - r_n - 2\varepsilon)$. For n large enough, we have

$$\begin{aligned} S_{2\varepsilon}(\mu_n^0, \mu_n^1, \omega_n) \quad &:= \quad \sup_{m \in [0, 1 - r_n - 2\varepsilon]} \{\mathcal{F}^{-1}_{\mu_n^0,\omega_n}(m) + \mathcal{B}^{-1}_{\mu_n^1,\omega_n}(1 - r_n - 2\varepsilon - m)\} \\ &\leqslant \quad \sup_{m \in [0, 1 - r_n - 2\varepsilon]} \{\mathcal{F}^{-1}_{\mu^0,\omega}(m + r_n^0 + r_n) \\ &\qquad + \mathcal{B}^{-1}_{\mu^1,\omega}(1 + r_n^1 - 2\varepsilon - m)\} + \delta \\ &\leqslant \quad \sup_{m \in [r_n^0 + r_n, 1 + r_n^0 - 2\varepsilon]} \{\mathcal{F}^{-1}_{\mu^0,\omega}(m) \\ &\qquad + \mathcal{B}^{-1}_{\mu^1,\omega}(1 + r_n^1 + r_n^0 + r_n - 2\varepsilon - m)\} + \delta. \end{aligned}$$

Observe that it holds $0 \leq r_n^0 + r_n$. For n large enough, it also holds $1 + r_n^0 - 2\varepsilon \leq 1 - \varepsilon$, as well as $r_n^1 + r_n^0 + r_n \leqslant \varepsilon$. Since $\mathcal{B}^{-1}_{\mu^1,\omega}$ is increasing, this implies

$$S_{2\varepsilon}(\mu_n^0, \mu_n^1, \omega_n) \leq \sup_{m \in [0, 1 - \varepsilon]} \{\mathcal{F}^{-1}_{\mu^0,\omega}(m) + \mathcal{B}^{-1}_{\mu^1,\omega}(1 - \varepsilon - m)\} + \delta = S_\varepsilon(\mu^0, \mu^1, \omega) + \delta$$

for n sufficiently large.

We now prove **Item (ii)**. For all $n \in \mathbb{N}$, define

$$\widetilde{V}_n^0 := \{x^0 \in \operatorname{supp}(\mu_n^0) : \Phi_t^v(B_{f_n}(x^0)) \subset\subset \omega \quad \text{for a} \ \ t \leqslant t_{\omega_n}^0(x^0) + \delta/2\}$$

and $\widetilde{r}_n^0 := \mu^0((\widetilde{V}_n^0)^c)$. Using the same argument of item (i), we can prove that

$$t_\omega^0(\xi^0) \leqslant t_{\omega_n}^0(x^0) + \delta/2 \tag{42}$$

for π_n-almost every $x^0 \in \widetilde{V}_n^0$ and $\xi^0 \in \operatorname{supp}(\mu^0) \cap (R_n^0)^c$, where π_n is a transportation plan realizing $W_\infty(\mu_n^0, \mu^0)$. Inequality (42) implies

$$\begin{aligned}
\mathcal{F}_{\mu_n^0,\omega_n}(t) &\leqslant \mu_n^0(\{x^0 \in \widetilde{V}_n^0 : t_{\omega_n}^0(x^0) \leqslant t\}) + \mu^0((\widetilde{V}_n^0)^c) \\
&= \pi_n(\{(x^0,\xi^0) \in \widetilde{V}_n^0 \times (\operatorname{supp}(\mu^0) \cap (R_n^0)^c) : t_{\omega_n}^0(x^0) \leqslant t\}) + \widetilde{r}_n^0 \\
&\leqslant \pi_n(\{(x^0,\xi^0) \in \widetilde{V}_n^0 \times (\operatorname{supp}(\mu^0) \cap (R_n^0)^c) : t_\omega^0(\xi^0) \leqslant t + \delta/2\}) + \widetilde{r}_n^0 \\
&\leqslant \mu^0(\{\xi^0 \in \operatorname{supp}(\mu^0) \cap (R_n^0)^c : t_\omega^0(\xi^0) \leqslant t + \delta/2\}) + \widetilde{r}_n^0 \\
&\leqslant \mu^0(\{\xi^0 \in \operatorname{supp}(\mu^0) : t_\omega^0(\xi^0) \leqslant t + \delta/2\}) + \mu^0(R_n^0) + \widetilde{r}_n^0 \\
&= \mathcal{F}_{\mu^0,\omega}(t + \delta/2) + r_n + \widetilde{r}_n^0,
\end{aligned}$$

for all $t \in \mathbb{R}^+$. We also have

$$\mathcal{B}_{\mu_n^1,\omega_n}(t) \leqslant \mathcal{B}_{\mu^1,\omega}(t + \delta/2) + r_n + \widetilde{r}_n^1,$$

for all $t \in \mathbb{R}^+$, where $\widetilde{r}_n^1$ is similarly defined. We conclude as before.

The last statement holds for $U_{0,n} := V_n^0$ and $U_{1,n} := V_n^1$, where

$$V_n^1 := \{x^1 \in \operatorname{supp}(\mu^1) : \Phi_{-t}^v(B_{f_n}(x^1)) \subset\subset \omega_n \text{ for a } t \leqslant t_\omega^1(x^1) + \delta/2\}.$$

□ □

4.3 Proof of Theorem 1.4

In this section, we prove Theorem 1.4. The proof is based on the results obtained in Section 4.1 and 4.2. We highlight that the idea of the proof is to approximate the macroscopic crowds with microscopic ones with a sufficiently large number of agents, find a control steering one microscopic crowd to another, then adapt such control for the macroscopic one. For this reason, this result can be seen as the limit of Proposition 8 about microscopic models.

Proof of Theorem 1.4. We first prove **Item (i)**. Fix $s > 0$ and $\varepsilon > 0$ and define the time

$$T := S(\mu^0, \mu^1, \omega) + s.$$

To prove the theorem, we aim to show that there exists an admissible control u such that

$$W_1(\mu^1, \Phi_T^{v+\mathbb{1}_\omega u} \# \mu^0) \leqslant \varepsilon.$$

We assume that the space dimension is $d := 2$, but the reader will see that the proof can be clearly adapted to any space dimension. The proof is divided into three steps.

Step 1: In this step, we discretize the measures, in four sub-steps. We first discretize uniformly in space the supports of μ^0 and μ^1. To simplify the presentation, assume that $\operatorname{supp}(\mu^0) \subset (0,1)^2$ and $\operatorname{supp}(\mu^1) \subset (0,1)^2$. For all $k := (k_1, k_2) \in \{0, ..., n-1\}^2$, consider the set $C_{n,k}$ defined by

$$C_{n,k} := \left[\frac{k_1}{n}, \frac{k_1+1}{n}\right] \times \left[\frac{k_2}{n}, \frac{k_2+1}{n}\right].$$

Then define $\mu^l_{nk} := \mu^l_{|C_{n,k}}$ for $l = 0, 1$. Remark that the number of indexes nk is exactly n^2.

We now consider only the decomposition of μ^0, while the decomposition of μ^1 is similar. To send a measure to another, these measures need to have the same total mass. With this goal, we perform the second step of our decomposition: on each set $C_{n,k}$ consider a decomposition

$$a_0 = \frac{k_1}{n} < a_1 < \ldots < a_m \leq \frac{k_1+1}{n}$$

such that it holds $\mu^0_{nk}([a_i, a_{i+1}] \times [\frac{k_2}{n}, \frac{k_2+1}{n}]) = \frac{1}{n^3}$ for all $i = 0, \ldots, m-1$, as well as $\mu^0_{nk}([a_m, \frac{k_1+1}{n}] \times [\frac{k_2}{n}, \frac{k_2+1}{n}]) < \frac{1}{n^3}$. If $\mu^0(C_{n,k}) < \frac{1}{n^3}$, we set $m = 0$. One can always perform such a decomposition, since μ_0 is absolutely continuous. Then define $C^0_{nki} := [a_i, a_{i+1}] \times [\frac{k_2}{n}, \frac{k_2+1}{n}]$ for $i = 0, \ldots, m-1$ as well as $\mu^0_{nki} := (\mu^0_{nk})_{|C^0_{nki}}$. Also define

$$\bar{\mu}^0_{nk} := (\mu^0_{nk})_{|\left[a_m, \frac{k_1+1}{n}\right] \times \left[\frac{k_2}{n}, \frac{k_2+1}{n}\right]}.$$

The meaning of such decomposition is the following: each μ^0_{nki} has mass $\frac{1}{n^3}$ and is localized in C^0_{nki}, while $\bar{\mu}^0_{nk}$ has a mass strictly smaller than $\frac{1}{n^3}$ and is localized in $C^0_{n,k}$. Remark that the number of indexes i depends on n, k. Nevertheless, the total mass not contained in the sets C^0_{nki} is smaller than $\frac{1}{n}$, hence the total number of indexes nki is between $n^3 - n^2$ and n^3.

We now consider the third step of the decomposition: each μ^0_{nki} is localized in the set $C^0_{nki} := [a_i, a_{i+1}] \times [\frac{k_2}{n}, \frac{k_2+1}{n}]$. For each index nki, define a decomposition

$$a_{i0} = \frac{k_2}{n} < a_{i1} < \ldots < a_{im} \leq \frac{k_2+1}{n}$$

such that it holds $\mu^0_{nki}([a_i, a_{i+1}] \times [a_{ij}, a_{i(j+1)}]) = \frac{1}{n^4}$ for each $j = 0, \ldots, m-1$. Then define $C^0_{nkij} := [a_i, a_{i+1}] \times [a_{ij}, a_{i(j+1)}]$ for each $j = 0, \ldots, m-1$, as well as $\mu^0_{nkij} := (\mu^0_{nki})_{|C^0_{nkij}}$. Remark that the number of indexes j depends on nki, but that the total number of indexes $nkij$ is between $n^4 - n^3$ and n^4.

If the dimension of the space is larger than 2, one simply needs to decompose the μ_{nkij} with respect to the subsequent dimension with blocks of mass $\frac{1}{n^5}$, and so on, up to cover all dimensions.

The second and third step of the decomposition are represented in Figure 6.

For more details on such discretization, we refer to [21, Prop. 3.1].

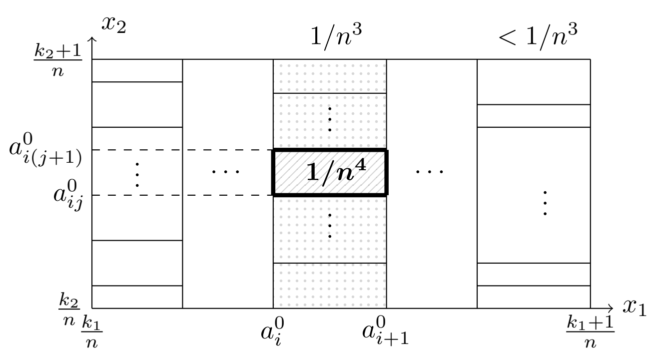


Figure 6: Example of a partition of $C_{n,k}$ in cells such as C^0_{nkij} (hashed).

We finally consider the fourth step of our decomposition: in each cell $C^0_{nkij} = [a_i, a_{i+1}] \times [a_{ij}, a_{i(j+1)}]$, it is defined an absolutely continuous measure μ^0_{nkij} with mass $\frac{1}{n^4}$. Thus, it exists a constant $\epsilon > 0$ such that the set

$$B^0_{nkij} := [a_i + \epsilon, a_{i+1} - \epsilon] \times [a_{ij} + \epsilon, a_{i(j+1)} - \epsilon] \tag{43}$$

satisfies $\mu^0_{nkij}(B^0_{nkij}) = \frac{1}{n^4} - \frac{1}{n^6}$. See Figure 7 for a graphical description of this decomposition. For more details of such construction, we also refer to [21, Prop. 3.1].

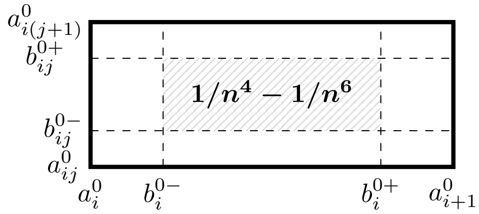


Figure 7: Example of cells B^0_{nkij} (hashed).

We finally define $\tilde{\mu}^0_{nkij} := (\mu^0_{nkij})_{|_{B^0_{nkij}}}$ and $\bar{\mu}^0_{nkij} := (\mu^0_{nkij})_{|_{C^0_{nkij}\setminus B^0_{nkij}}}$.

We discretize similarly the measure μ^1 on sets $C^1_{nki}, C^1_{nkij}, B^1_{nkij}$, then defining $\bar{\mu}^1_{nk}, \bar{\mu}^1_{nkij}, \tilde{\mu}^1_{nkij}$.

Step 2: In this step, we prove exact controllability of microscopic approximations $\hat{\mu}^0_n, \hat{\mu}^1_n$ of μ^0, μ^1. In a first step, we define such sequence of approximations $\hat{\mu}^0_n, \hat{\mu}^1_n$ of μ^0, μ^1 satisfying hypotheses of Proposition 10.

Step 2.1: We aim to define a sequence of microscopic approximations $\hat{\mu}^0_n, \hat{\mu}^1_n$ of μ^0, μ^1, that satisfy hypotheses of Proposition 10. We first observe the following: the number of $nkij$ indexes are between $n^4 - n^3$ and n^4 both for μ^0 and μ^1. We then choose exactly a set I^0_n of $n^4 - n^3$ indexes $nkij$ for μ^0, as well as a set I^1_n of $n^4 - n^3$ indexes $nkij$ for μ^1.

We then define the following measures:

$$\begin{aligned}
&\tilde{\mu}^0_n := \sum_{nkij \in I^0_n} \tilde{\mu}^0_{nkij}, \qquad \tilde{\mu}^1_n := \sum_{nkij \in I^1_n} \tilde{\mu}^1_{nkij}, \\
&\bar{\mu}^0_n := \mu^0 - \tilde{\mu}^0_n = \sum_{nk} \bar{\mu}^0_{nk} + \sum_{nki} \bar{\mu}^0_{nki} + \sum_{nkij} \bar{\mu}^0_{nkij} + \sum_{nkij \notin I^0_n} \tilde{\mu}^0_{nkij},
\end{aligned} \tag{44}$$

and similarly $\tilde{\mu}^1_n$, $\bar{\mu}^1_n = \mu^1 - \tilde{\mu}^1_n$. The idea of the notation is the following: the measure μ^0 is decomposed into a "relevant" part $\tilde{\mu}^0_n$ and a negligible one $\bar{\mu}^0_n$. Such decomposition only depends on the parameter n of the grid, as well as the choice of $n^4 - n^3$ indexes defining the set I^0_n. The same holds for the decomposition of μ^1.

We are ready to define the microscopic approximation $\hat{\mu}^0_n$. Observe that each $\tilde{\mu}^0_{nkij}$ is supported in the set B^0_{nkij} defined in (43) and has mass $\frac{1}{n^4} - \frac{1}{n^6}$. For each $nkij \in I^0_n$, choose a point x^0_{nkij} belonging to the support of μ^0_{nkij}. Then define

$$\hat{\mu}^0_n := \sum_{nkij \in I^0_n} \left(\frac{1}{n^4} - \frac{1}{n^6}\right) \delta_{x^0_{nkij}}.$$

Repeat the same construction for μ^1, then defining $\hat{\mu}^1_n$.

Step 2.2: We now prove that sequences $\hat{\mu}^0_n, \hat{\mu}^1_n$ defined above, together with μ^0, μ^1, satisfy the hypotheses of Proposition 10.

Define the sequence

$$f_n := e^{LT} \frac{\sqrt{2}}{n},$$

where L is the Lipschitz constant of the vector field v. Then observe that $x^0_{nkij} \in \mathrm{supp}(\mu^0)$ for all $nkij$, hence the fact that μ^0 satisfies the Geometric Condition 1 implies that $\hat{\mu}^0_n$ satisfies it too.

For each n, define the set R_n^0 as the support of $\bar{\mu}_n^0$. This implies that

$$r_n := \mu^0(R_n^0) = \bar{\mu}_n^0(\mathbb{R}^d) = 1 - \tilde{\mu}_n^0(\mathbb{R}^d) = 1 - \left(n^4 - n^3\right)\left(\frac{1}{n^4} - \frac{1}{n^6}\right) \to 0. \tag{45}$$

Moreover, by construction it holds $\hat{\mu}_n^0(\mathbb{R}^d) = \tilde{\mu}_n^0(\mathbb{R}^d) = 1 - r_n$. Finally, observe that each set B_{nkij}^0 has diameter strictly smaller than the diameter of C_{nkij}^0, hence smaller than[3] $\frac{\sqrt{2}}{n} < f_n$. Moreover, it is clear that it holds $(\mu^0)_{|_{(R_n^0)^c}} = \tilde{\mu}_n^0$ by construction. Then, one can estimate $W_\infty((\mu^0)_{|_{(R_n^0)^c}}, \hat{\mu}_n^0)$ by decomposing them in each set B_{nkij}^0, *i.e.*

$$W_\infty((\mu^0)_{|_{(R_n^0)^c}}, \hat{\mu}_n^0) \leq \max_{nkij \in I_n^0} W_\infty(\tilde{\mu}_{nkij}^0, \hat{\mu}_{nkij}^0) \leq$$
$$\max_{nkij \in I_n^0} \operatorname{diam}(\operatorname{supp}(\tilde{\mu}_{nkij}^0) \cup \operatorname{supp}(\hat{\mu}_{nkij}^0)) \leq \max_{nkij \in I_n^0} \operatorname{diam}(B_{nkij}^0) < f_n.$$

Repeat the same procedure for μ^1, defining R_n^1 and proving the same estimates with f_n, r_n. Then, all conditions of (34) are satisfied. Summing up, all hypotheses of Proposition 10 are satisfied.

Since v is uniformly bounded, there exists $R > 0$ independent on n and ε, such that

$$[0,1]^2 \bigcup_{t \in [0,T]} \Phi_t^v(\omega) \subset\subset B_R(0). \tag{46}$$

We now estimate the value of $S_{\varepsilon/4R}(\hat{\mu}_n^0, \hat{\mu}_n^1, \omega_n)$, where ω_n is defined in (33). Observe that it clearly holds

$$S_{\varepsilon/8R}(\mu^0, \mu^1, \omega) \leqslant S(\mu^0, \mu^1, \omega).$$

Apply Proposition 10 for $\delta := s/2$, that gives the following estimate:

$$S_{\varepsilon/4R}(\hat{\mu}_n^0, \hat{\mu}_n^1, \omega_n) + \frac{s}{2} \leqslant S_{\varepsilon/8R}(\mu^0, \mu^1, \omega) + s \leqslant S(\mu^0, \mu^1, \omega) + s = T,$$

for n large enough.

Step 2.3: We now prove existence of a control exactly steering $\hat{\mu}_n^0$ to $\hat{\mu}_n^1$ up to a mass $\varepsilon/4R$. Observe that ω convex implies ω_n convex. Using Proposition 8, it holds

$$T_{e,\varepsilon/4R}(\hat{\mu}_n^0, \hat{\mu}_n^1, \omega_n) = S_{\varepsilon/4R}(\hat{\mu}_n^0, \hat{\mu}_n^1, \omega_n) < T.$$

Then there exists $I_{n,\varepsilon}^l \subset I_n^0$ $(l = 0, 1)$ such that

$$|I_{n,\varepsilon}^l| = (1 - \varepsilon/4R)(n^4 - n^3) \tag{47}$$

(assumed integer for simplicity), a bijection among indexes $\sigma_n : I_{n,\varepsilon}^0 \to I_{n,\varepsilon}^1$ and a control $\mathbb{1}_{\omega_n} \hat{u}_n$ satisfying the Carathéodory condition such that

$$\Phi_T^{v+\mathbb{1}_\omega \hat{u}_n}(x_{nkij}^0) = x_{\sigma_n(nkij)}^1$$

for all $nkij \in I_{n,\varepsilon}^0$. We define

$$\tilde{\mu}_{n,\varepsilon}^l := \sum_{nkij \in I_{n,\varepsilon}^l} \tilde{\mu}_{nkij}^l, \qquad \bar{\mu}_{n,\varepsilon}^l := \mu^l - \tilde{\mu}_{n,\varepsilon}^l, \qquad \hat{\mu}_{n,\varepsilon}^l := \sum_{nkij \in I_{n,\varepsilon}^l} \left(\frac{1}{n^4} - \frac{1}{n^6}\right) \delta_{x_{nkij}^l}.$$

[3] For the space $\mathbb{R}^d$, it is clearly sufficient to replace $\sqrt{2}$ with $\sqrt{d}$.

For simplicity of notation, for each n we assume to re-arrange indexes $nkij$ both for $\hat{\mu}^0_{n,\varepsilon}$ and $\hat{\mu}^1_{n,\varepsilon}$, so that $I^0_{n,\varepsilon} = I^1_{n,\varepsilon}$ and the bijection σ_n is the identity. We will then write from now on

$$\Phi_T^{v+\mathbb{1}_\omega \hat{u}_n}(x^0_{nkij}) = x^1_{nkij}. \tag{48}$$

We also denote by $\hat{x}_{nkij}(t)$ the corresponding trajectory, steering x^0_{nkij} to x^1_{nkij} in time T. Finally, we denote

$$t^0_{nkij} = t^0(x^0_{nkij}, \omega_n), \qquad t^1_{nkij} = t^1(x^1_{nkij}, \omega_n).$$

We also make the following observation: the trajectory steering x^0_{nkij} to x^1_{nkij} is known to enter ω_n at time t^0_{nkij} and to exit it at time $T - t^1_{nkij}$. Due to the construction of the control in the proof of Proposition 7, the different trajectories are straight lines, so all trajectories $\hat{x}_{nkij}(t)$ are contained in ω_n for all times $t \in (t^0_{nkij}, T - t^1_{nkij})$.

Step 3: We now prove approximate controllability from μ^0 to μ^1 in time $T = S(\mu^0, \mu^1, \omega) + s$. The idea is to write a control approximately steering μ^0 to μ^1, based on the controls $\hat{u}_n$ defined in (48) exactly steering $\hat{\mu}^0_{n,\varepsilon}$ to $\hat{\mu}^1_{n,\varepsilon}$ in time T.

Fix a given n. For each $nkij \in I^0_{n,\varepsilon}$ apply the following procedure:

1. the set B^0_{nkij} evolves via the uncontrolled flow Φ^v_t in the time interval $[0, t^0_{nkij}]$; at this final time, the evolved set $B_{nkij}(t^0_{nkij}) := \Phi^v_{t^0_{nkij}}(B^0_{nkij})$ will be completely contained in ω.
2. we then apply a control w_{nkij} concentrating the mass around $\hat{x}_{nkij}(t)$ defined in (48) for the time interval $[t^0_{nkij}, T - t^1_{nkij}]$. Such mass will be then moved to the corresponding evolved set $B_{nkij}(T - t^1_{nkij})$ around $\hat{x}(T - t^1_{nkij})$;
3. we finally let such set evolve via the uncontrolled flow Φ^v_t up to time T; the set $B_{nkij}(T)$ will be centered around $\hat{x}(T) = x^1_{nkij}$.

In the following, we will prove that such strategy provides the desired result. In the first step, we will precisely define the strategy and prove estimates about the evolution of sets. In the second and final step, we will provide estimates about the Wasserstein distance, to prove approximate controllability.

Step 3.1: We now define a control approximately steering μ^0 to μ^1. We first define the control and the desired evolution for each index $nkij \in I^0_{n,\varepsilon}$. We recall that the control $\hat{u}_n$ given in (48) defines the trajectory $\hat{x}_{nkij}(t)$ steering x^0_{nkij} to x^1_{nkij}. We then define an adapted control $\tilde{u}_n(t,x)$ that concentrates mass around such trajectory. We choose

$$\tilde{u}_{nkij}(t,x) = \hat{u}_n(t, \hat{x}_{nkij}(t)) + C_n(\hat{x}_{nkij}(t) - x),$$

where C_n is a positive constant that will be chosen later. It is clearly a linear feedback stabilizing the system around the trajectory $\hat{x}_{nkij}$. We also define a corresponding vector field

$$w_{nkij}(t,x) := v(\hat{x}_{nkij}(t)) + \tilde{u}_{nkij}(t,x). \tag{49}$$

Observe that in this formula the vector field v is evaluated at $\hat{x}_{nkij}(t)$ only, and not at the point x. We then define the evolution of the corresponding evolved cell $B_{nkij}(t)$ as follows:

$$B_{nkij}(t) = \begin{cases} \Phi^v_t(B^0_{nkij}) & \text{if } t \in [0, t^0_{nkij}], \\ \Phi^{w_{nkij}}_t(B_{nkij}(t^0_{nkij})) & \text{if } t \in [t^0_{nkij}, T - t^1_{nkij}], \\ \Phi^v_t(B_{nkij}(T - t^1_{nkij})) & \text{if } t \in [T - t^1_{nkij}, T]. \end{cases} \tag{50}$$

We also give a simple estimation of the diameter of the cell $B_{nkij}(t)$ as follows:

$$\operatorname{diam}(B_{nkij}(t)) < \begin{cases} e^{Lt}\frac{\sqrt{2}}{n} & \text{if } t \in [0, t^0_{nkij}], \\ e^{-C_n(t - t^0_{nkij})} e^{Lt^0_{nkij}} \frac{\sqrt{2}}{n} & \text{if } t \in [t^0_{nkij}, T - t^1_{nkij}], \\ e^{-C_n(t^1_{nkij} - t^0_{nkij})} e^{LT} \frac{\sqrt{2}}{n} & \text{if } t \in [T - t^1_{nkij}, T], \end{cases} \tag{51}$$

where L is the Lipschitz constant of the vector field v. The estimate in the first interval is a classical application of Gronwall lemma, recalling that $\mathrm{diam}(B^0_{nkij}) < \frac{\sqrt{2}}{n}$. For the estimate in the second interval, we use the fact that the term $v(\hat{x}_{nkij}(t)) + \hat{u}_n(t, \hat{x}_{nkij}(t))$ in w_{nkij} is constant with respect to the space variable, while for the term $C_n(\hat{x}_{nkij}(t) - x)$ we again apply the Gronwall lemma. In the third interval, we again apply the Gronwall lemma and estimate $t \leq T$.

We now prove that $B_{nkij}(t)$ is contained in ω for all $t \in [t^0_{nkij}, T - t^1_{nkij}]$. Indeed, it is sufficient to recall from Step 2.3 that one can choose the trajectory $\hat{x}_{nkij}(t)$ to be contained in ω_n for all $t \in [t^0_{nkij}, T - t^1_{nkij}]$. Now observe that for all $t \in [t^0_{nkij}, T - t^1_{nkij}]$ it holds $\hat{x}_{nkij}(t) \in B_{nkij}(t) \cap \omega_n$, that $\mathrm{diam}(B_{nkij}(t)) < e^{LT} \frac{\sqrt{2}}{n}$ and recall the definition (33) of ω_n with $f_n = e^{LT} \frac{\sqrt{2}}{n}$. Then, $B_{nkij}(t)$ is contained in ω.

We now prove that, given a fixed n and choosing a sufficiently large C_n, for each pair of distinct indexes $nkij, nk'i'j' \in I^0_{n,\varepsilon}$ the sets $B_{nkij}(t), B_{nk'i'j'}(t)$ are disjoint. Assume for simplicity that it holds $t^0_{nkij} \leq t^0_{nk'i'j'}$. First observe that at time $t = 0$ the sets $B^0_{nkij}, B^0_{nk'i'j'}$ are disjoint by construction. We then separate the time interval $[0, T]$ in five intervals:

1. Interval $[0, t^0_{nkij}]$: both sets are displaced via the flow defined by the C^1 vector field v, that keeps them disjoint.

2. Interval $[t^0_{nkij}, t^0_{nk'i'j'}]$: observe that the set $B_{nk'i'j'}(t)$ evolves via the flow defined by the C^1 vector field v, hence its diameter can grow of a factor $e^{L(t^0_{nk'i'j'} - t^0_{nkij})}$. For $t \leq t^1_{nkij}$, by choosing a constant
$$C_n > 2e^{LT} > e^{L(t^0_{nk'i'j'} - t^0_{nkij})}$$
sufficiently large, one can reduce the diameter of the set $B_{nkij}(t)$ to have it disjoint with respect to $B_{nk'i'j'}$. For $t > t^1_{nkij}$, it is sufficient to observe that both sets are displaced by the C^1 vector field v, then apply Case 1.

3. Interval $[t^0_{nk'i'j}, \min(t^1_{nkij}, t^1_{nk'i'j'})]$, when non-empty: recall that the trajectories $\hat{x}_{nkij}(t), \hat{x}_{nk'i'j'}(t)$ are disjoint at each time, since the vector field $\widehat{u}_n$ defined in Step 2.3 satisfies the Carathéodory condition. Since in this time interval one can act on both sets with the controls w_{nkij} and $w_{nk'i'j'}$, by choosing C_n sufficiently large one can concentrate the sets in two sufficiently small neighborhoods around disjoint trajectories, then having disjoint sets.

4. Interval $[\min(t^1_{nkij}, t^1_{nk'i'j'}), \max(t^1_{nkij}, t^1_{nk'i'j'})]$ is equivalent to Case 2.

5. Interval $[\max(t^1_{nkij}, t^1_{nk'i'j'}), T]$ is equivalent to Case 1.

Then, given a fixed n, for each pair of distinct indexes $nkij$, $nk'i'j'$ one can choose a sufficiently large C_n ensuring that the sets $B_{nkij}(t), B_{nk'i'j'}(t)$ are disjoint. By observing that the number of pairs are finite, one can choose a sufficiently large C_n for which the property holds for all pairs of indexes.

We are now ready to define the control u_n approximately steering μ^0 to μ^1 in time $T = S(\mu^0, \mu^1, \omega) + s$. For each time $t \in [0, T]$, first define
$$u_n(t, x) := w_{nkij}(t, x) - v(x) \quad \text{ if } \ t \in [t^0_{nkij}, T - t^1_{nkij}], x \in B_{nkij}(t), \tag{52}$$
where w_{nkij} is defined in (49). It is clear that such definition is well-posed, since the $B_{nkij}(t)$ are disjoint at each time. We now complete the definition of u_n as follows:
$$u_n(t, x) := \begin{cases} 0 & \text{for all } \ x \in \omega^c, \\ \begin{array}{l}\text{Lip. spline}\\ \text{in the } x \text{ var.}\end{array} & \text{if } \ t \in [t^0_{nkij}, T - t^1_{nkij}] \text{ for some } nkij \in I^0_{n,\varepsilon}, \\ 0 & \text{for all other } \quad t \in [0, T]. \end{cases} \tag{53}$$

Observe that $t \in [t^0_{nkij}, T - t^1_{nkij}]$ and $x \in B_{nkij}(t)$ imply that $B_{nkij} \in \omega$, *i.e.* that the control is allowed to have non-zero value. Moreover, convexity of ω allow us to choose a Lipschitz spline that is non-zero in ω only. Also observe that $u_n = 0$ on the boundary of ω, that implies that $\mathbb{1}_\omega u_n$ is also Lipschitz for each time. Regularity with respect to time is ensured by the fact that discontinuities are allowed on a finite number of times t^l_{nkij} only. Thus $\mathbb{1}_\omega u_n$ satisfies the Carathéodory condition, hence it is an admissible control.

We finally observe the following key property: when applying the control $\mathbb{1}_\omega u_n$ to the System (2), the dynamics of the sets B^0_{nkij} satisfies (50). We will see in the next step that such property is the key to ensure approximate controllability of μ^0 to μ^1 at time T.

Step 3.2: We now prove that the control u_n defined in (52)-(53) provides approximate controllability of μ^0 to μ^1 at time T. Recall that the solution $\mu_n(t)$ to System (2) with control $\mathbb{1}_\omega u_n$ starting from μ^0 is

$$\mu_n(t) := \Phi^{v+\mathbb{1}_\omega u_n}_t \# \mu^0.$$

We aim to prove that the distance $W_1(\mu_n(T), \mu^1)$ is less than ε for n large enough.

Recall the decomposition $\mu^0 = \tilde{\mu}^0_{n,\varepsilon} + \bar{\mu}^0_{n,\varepsilon}$ introduced in (44), and similarly for μ^1. One can then write $\mu_n(T) = \Phi^{v+\mathbb{1}_\omega u_n}_T \# \tilde{\mu}^0_{n,\varepsilon} + \Phi^{v+\mathbb{1}_\omega u_n}_T \# \bar{\mu}^0_{n,\varepsilon}$. We estimate

$$W_1(\mu_n(T), \mu^1) \le W_1(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \tilde{\mu}^0_{n,\varepsilon}, \tilde{\mu}^1_{n,\varepsilon}) + W_1(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \bar{\mu}^0_{n,\varepsilon}, \bar{\mu}^1_{n,\varepsilon}). \tag{54}$$

For the first term, recall the decomposition $\tilde{\mu}^0_{n,\varepsilon} = \sum\limits_{nkij \in I^0_{n,\varepsilon}} \tilde{\mu}^0_{nkij}$ with the supports $\mathrm{supp}(\tilde{\mu}^0_{nkij}) \subset B^0_{nkij}$ by construction. Then, it holds

$$\Phi^{v+\mathbb{1}_\omega u_n}_T \# \tilde{\mu}^0_{n,\varepsilon} = \sum_{nkij \in I^0_{n,\varepsilon}} \Phi^{v+\mathbb{1}_\omega u_n}_T \# \tilde{\mu}^0_{nkij} \tag{55}$$

and $\mathrm{supp}(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \tilde{\mu}^0_{nkij}) \subset B_{nkij}(T)$. For each $nkij \in I^0_{n,\varepsilon}$, we provide the estimate

$$\begin{aligned} W_1(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \tilde{\mu}^0_{nkij}, \tilde{\mu}^1_{nkij}) &\le W_1\left(\Phi^{v+\mathbb{1}_\omega u_n}_T \# \tilde{\mu}^0_{nkij}, \left(\frac{1}{n^4} - \frac{1}{n^6}\right)\delta_{x^1_{nkij}}\right) \qquad (56)\\ &\quad + W_1\left(\left(\frac{1}{n^4} - \frac{1}{n^6}\right)\delta_{x^1_{nkij}}, \tilde{\mu}^1_{nkij}\right)\\ &\le \left(\frac{1}{n^4} - \frac{1}{n^6}\right)\mathrm{diam}(B_{nkij}(T)) + \left(\frac{1}{n^4} - \frac{1}{n^6}\right)\mathrm{diam}(B^1_{nkij})\\ &< \left(\frac{1}{n^4} - \frac{1}{n^6}\right)(e^{LT}+1)\frac{\sqrt{2}}{n}, \qquad (57) \end{aligned}$$

with the following observation: the point $\hat{x}_{nkij}(T) = x^1_{nkij}$ belongs both to $B_{nkij}(T)$ by construction of such set in (50), and to B^1_{nkij} by construction of such cell in Step 1. Moreover, the mass of both $\tilde{\mu}^0_{nkij}$ and $\tilde{\mu}^1_{nkij}$ is $\frac{1}{n^4} - \frac{1}{n^6}$, then one can apply the triangular inequality (56), then estimate each term by observing that rays in a transference plan sending $\Phi^{v+\mathbb{1}_\omega u_n}_T \# \tilde{\mu}^0_{nkij}$ to $\left(\frac{1}{n^4} - \frac{1}{n^6}\right)\delta_{x^1_{nkij}}$ have length smaller than $\mathrm{diam}(B_{nkij}(T))$, and similarly for the second term. Estimates of the diameters are consequences of (51) and of the construction of B^1_{nkij}.

For the second term of (54), first recall that it holds

$$r_{n,\varepsilon} := \bar{\mu}^1_{n,\varepsilon}(\mathbb{R}^d) = 1 - (1 - \varepsilon/4R)(n^4 - n^3)(1/n^4 - 1/n^6) \underset{n\to\infty}{\longrightarrow} \varepsilon/4R$$

(see (45) and (47)). Take now any point $x^0 \in \mathrm{supp}(\mu^0)$ and study the trajectory given by the flow $\Phi^{v+\mathbb{1}_\omega u_n}_t(x^0)$. Consider the set of times $t \in [0, T]$ for which $\Phi^{v+\mathbb{1}_\omega u_n}_t(x^0) \in \omega$, that is nonempty due to the Geometric Condition 1. Take $\bar{t}$ the supremum of such times and observe that it holds

$$\bar{x} := \Phi^{v+\mathbb{1}_\omega u_n}_{\bar{t}}(x^0) \in \bar{\omega}.$$

Moreover, the control does not act on the time interval $[\bar{t}, T]$, *i.e.* $\Phi_t^{v+\mathbb{1}_\omega u_n}(x^0) = \Phi_t^v(x^0)$. As a consequence, using (46), for n large enough, it holds

$$W_1(\Phi_T^{v+\mathbb{1}_\omega u_n} \# \bar{\mu}_{n,\varepsilon}^0, \bar{\mu}_{n,\varepsilon}^1) \leq 2R\bar{\mu}_{n,\varepsilon}^1(\mathbb{R}^d) = 2Rr_{n,\varepsilon}. \tag{58}$$

Summing up, from (54)-(55)-(57)-(58) it holds

$$W_1(\mu_n(T), \mu^1) \leq (n^4 - n^3)\left(\frac{1}{n^4} - \frac{1}{n^6}\right)(e^{LT}+1)\frac{\sqrt{2}}{n} + 2Rr_{n,\varepsilon} \to \varepsilon/2.$$

For n large enough, $W_1(\mu_n(T), \mu^1)$ is then smaller than ε. Thus System (1) is approximatively controllable for each $T > S(\mu^0, \mu^1, \omega)$.

We now prove **Item (ii)** of Theorem 1.4. To prove it, we first need to recall some results about evoluted sets and the Lebesgue measure of their boundaries, which proofs are postponed to Appendix A.

Lemma 4.2. *Let ω be an open set and v a Lipschitz vector field. For $t > 0$, define the* **evoluted set**

$$\omega^t := \cup_{\tau \in (0,t)} \Phi_\tau^v(\omega).$$

Then, the following statements hold:

1. *If $0 < t_1 < t_2$, then*
$$\omega^{t_1} \subset \omega^{t_2}.$$
2. *For each $t > 0$ it holds $\omega \subset \omega^t$ and $\Phi_t^v(\omega) \subset \omega^t$.*
3. *The set ω^t is open.*
4. *Let μ^0 be a probability measure, with compact support, absolutely continuous with respect to the Lebesgue measure and satisfying the first part of the Geometric Condition 1. Let $x_0 \in \mathrm{supp}(\mu^0)$ and $t_0(x_0)$ the corresponding infimum time to enter ω as defined in (4). Let $\mathbb{1}_\omega u$ be a control satisfying the Carathéodory condition. Then, $t > t_0(x_0)$ implies*
$$\Phi_t^{v+\mathbb{1}_\omega u}(x_0) \in \omega^{t - t_0(x_0)}. \tag{59}$$

Finally, let v be a C^1 vector field and ω an open convex bounded set. It then holds:

5. *For each $t > 0$ the evoluted set ω^t has boundary with zero Lebesgue measure.*

Remark 7. The key interest of estimate (59) is that the flow depends on the choice of the control $\mathbb{1}_\omega u$, but that the set $\omega^{t-t_0(x_0)}$ does not depend on it.

We are now ready to prove **Item (ii)** of Theorem 1.4. Consider

$$T \in (S^*(\mu^0, \mu^1), S(\mu^0, \mu^1)].$$

By definition of $S(\mu^0, \mu^1)$, it exists $m \in [0,1]$ such that

$$T < \mathcal{F}_{\mu^0}^{-1}(m) + \mathcal{B}_{\mu^1}^{-1}(1-m). \tag{60}$$

Define $\bar{t} := \mathcal{F}_{\mu^0}^{-1}(m)$ and the set

$$A := \left\{ x \in \mathrm{supp}(\mu^0) \ \text{ s.t. } \ t_0(x_0) > \bar{t} \right\}.$$

By definition of $\mathcal{F}_{\mu^0}^{-1}$, it holds $\mu^0(A) = 1 - m$.

Fix now any control $\mathbb{1}_\omega u$. By definition of the quantity $S^*(\mu^0, \mu^1)$, for each $x_0 \in \mathrm{supp}(\mu^0)$, it holds $t_0(x_0) < T$. One can then apply Lemma 4.2, statements 1 and 4, in the two following cases:

- if $x_0 \in \mathrm{supp}(\mu^0) \setminus A$, it holds $\Phi_T^{v+\mathbb{1}_\omega u}(x_0) \in \omega^{T-t_0(x_0)} \subset \omega^T$;
- if $x_0 \in A$, it holds $\Phi_T^{v+\mathbb{1}_\omega u}(x_0) \in \omega^{T-t_0(x_0)} \subset \omega^{T-\bar{t}}$. This implies that

$$\Phi_T^{v+\mathbb{1}_\omega u}(A) \subset \omega^{T-\bar{t}}. \tag{61}$$

Consider now the solution $\mu(t)$ to (2) associated to the control $\mathbb{1}_\omega u$. We already recalled that $\mu(t) = \Phi_t^{v+\mathbb{1}_\omega u} \# \mu^0$. By definition of the push-forward and applying (61), we can compute

$$\mu(T)(\omega^{T-\bar{t}}) = \mu^0 \left(\left(\Phi_T^{v+\mathbb{1}_\omega u} \right)^{-1} \left(\omega^{T-\bar{t}} \right) \right) \geq \mu^0(A) = 1 - m. \tag{62}$$

We now aim to prove that $\mu^1(\omega^{T-\bar{t}}) < 1 - m$. Recall that $\omega^{T-\bar{t}}$ is open, by Lemma 4.2, statement 3. Take now $x_1 \in \omega^{T-\bar{t}} \cap \mathrm{supp}(\mu^1)$, and observe that it holds $x_1 \in \Phi_t^v(\omega)$ for some $t \in (0, T-\bar{t})$. This implies that $\Phi_{-t}^v(x_1) \in \omega$, hence $t_1(x_1) \leq t \leq T - \bar{t}$. Since such property holds for any $x_1 \in \omega^{T-\bar{t}} \cap \mathrm{supp}(\mu^1)$, this implies that $\mathcal{B}_{\mu^1}(T-\bar{t}) \geq \mu^1(\omega^{T-\bar{t}})$, by definition of $\mathcal{B}_{\mu^1}$. If $\mu^1(\omega^{T-\bar{t}}) \geq 1-m$, then $\mathcal{B}_{\mu^1}^{-1}(1-m) \leq T - \bar{t}$, hence $T \geq \bar{t} + \mathcal{B}_{\mu^1}^{-1}(1-m)$. This contradicts (60), thus it holds $\mu^1(\omega^{T-\bar{t}}) < 1 - m$.

Recall now that $\partial(\omega^{T-\bar{t}})$ has zero Lebesgue measure (Lemma 4.2, Statement 5) and that μ^1 is absolutely continuous with respect to the Lebesgue measure. Thus, $\mu^1(\overline{\omega^{T-\bar{t}}}) < 1 - m$, hence by standard approximation of measurable sets, there exists an open set $\mathcal{O} \supset \overline{\omega^{T-\bar{t}}}$ such that $\mu^1(\mathcal{O}) < 1 - m$. Observe that such set depends on μ^1 only, thus it does not depend on the choice of the control u. Also observe that

$$D := \inf\{\|x - y\| \ \text{ s.t. } \ x \in \overline{\omega^{T-\bar{t}}}, y \notin \mathcal{O}\}$$

is strictly positive Indeed, consider the set of balls $B_\epsilon(x)$ with $x \in \overline{\omega^{T-\bar{t}}}$ and $B_\epsilon(x) \subset \mathcal{O}$, that is a covering of the compact set $\overline{\omega^{T-\bar{t}}}$. Then, there exists a finite covering $B_{\epsilon_n}(x_n)$ of $\overline{\omega^{T-\bar{t}}}$, thus $D \geq \inf \epsilon_n > 0$.

We now prove that for each control $\mathbb{1}_\omega u$, the solution $\mu(t)$ to (2) associated to the control $\mathbb{1}_\omega u$ satisfies

$$W_1(\mu(T), \mu^1) \geq D(1 - m - \mu^1(\mathcal{O})). \tag{63}$$

The idea is to observe that the $1-m$ mass of $\mu(T)$ in $\omega^{T-\bar{t}}$ cannot be completely transferred to the set $\mathcal{O}$, since $\mu^1(\mathcal{O}) < 1 - m$. Thus, a part of it needs to be transferred to $\mathbb{R}^d \setminus \mathcal{O}$, for which the distance of transfer is larger than D. More formally, consider a transference plan $\pi \in \Pi(\mu(T), \mu^1)$: recall that $\pi(\mathbb{R}^d \times \mathcal{O}) = \mu^1(\mathcal{O}) < 1-m$, thus $\pi(\omega^{T-\bar{t}} \times \mathcal{O}) \leq \mu^1(\mathcal{O}) < 1-m$, while $\pi(\omega^{T-\bar{t}} \times \mathbb{R}^d) = \mu(T)(\omega^{T-\bar{t}}) \geq 1-m$ by (62). Thus it holds $\pi(\omega^{T-\bar{t}} \times (\mathbb{R}^d \setminus \mathcal{O})) \geq 1 - m - \mu^1(\mathcal{O})$. Observe that $(x, y) \in \omega^{T-\bar{t}} \times (\mathbb{R}^d \setminus \mathcal{O})$ implies $\|x - y\| \geq D$. As a consequence, it holds

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\| \, d\pi(x, y) \geq \int_{\omega^{T-\bar{t}} \times (\mathbb{R}^d \setminus \mathcal{O})} D \, d\pi(x, y) = D(1 - m - \mu^1(\mathcal{O})).$$

Passing to the infimum among all transference plans $\pi \in \Pi(\mu(T), \mu^1)$, it holds (63). Observe that neither D nor $1 - m - \mu^1(\mathcal{O})$ depend on the choice of the control $\mathbb{1}_\omega u$, thus there exists no sequence $\mathbb{1}_\omega u_n$ such that the associated solution μ_n satisfies $W_1(\mu_n(T), \mu^1) \to 0$. Hence, the system is not approximately controllable at time T. □ □

Remark 8. We give now two examples in which System (1) is never exactly controllable on $[0, S^*(\mu^0, \mu^1))$ and another where System (1) is exactly controllable at each time $T \in [0, S^*(\mu^0, \mu^1))$:

- Figure 8 (left). Consider $\omega := (-1, 1) \times (-1.5, 1.5)$. The vector field v is $(1, 0)$, thus uncontrolled trajectories are right translations. Define

$$\begin{cases} \mu^0 := \mathbb{1}_{(-2.5,-2)\times(-1,1)} dx, \\ \mu^1 := \mathbb{1}_{(2,2.5)\times(-1,1)} dx. \end{cases}$$

The time $S^*(\mu^0,\mu^1)$ at which we can act on the particles and the minimal time $S(\mu^0,\mu^1)$ are respectively equal to 1.5 and 2.5. We observe that for each time $T\in[0,S^*(\mu^0,\mu^1)]$ System (2) is not approximately controllable. Indeed, each point takes a time $t>2$ to go from $\text{supp}(\mu^0)$ to $\text{supp}(\mu^1)$, hence one cannot expect approximate controllability for smaller times.

- Figure 8 (right). Consider $\omega := (-1,1)\times(-1.5,1.5)$. The vector field v is $(-y,x)$, thus uncontrolled trajectories are rotations with constant angular velocity. Define

$$\mu^0=\mu^1 := \mathbb{1}_{B_1(1,0)\setminus B_{0.5}(1,0)}dx.$$

In this case, both quantities $S^*(\mu^0,\mu^1)$ and $S(\mu^0,\mu^1)$ are equal to π. Since $\Phi_t^v\#\mu^0=\mu^1$ for all $t\geqslant 0$, we remark that System (1) is exactly controllable for all $T\in[0,S^*(\mu^0,\mu^1))$.

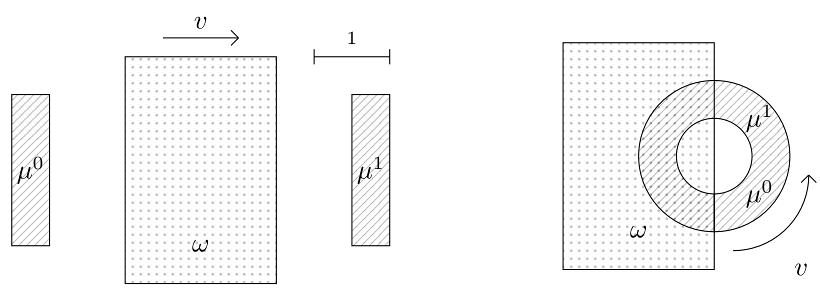


Figure 8: Left : The macroscopic system is not approximately controllable for each $T\in[0,S^*(\mu^0,\mu^1))$. Right : The macroscopic system is approximately controllable for each $T\in[0,S^*(\mu^0,\mu^1))$.

We can adapt the proof of Theorem 1.4 to obtain the minimal time to approximately steer μ^0 to μ^1 up to a mass ε:

Corollary 2. *Let $\mu^0,\mu^1\in\mathcal{P}_c^{ac}(\mathbb{R}^d)$ satisfying the Geometric Condition 1. Then the infimum time $T_{a,\varepsilon}$ to approximately steer μ^0 to μ^1 up to a mass ε is equal to $S_\varepsilon(\mu^0,\mu^1)$ given in (30).*

5 Numerical simulations

In this section, we give some numerical examples of the algorithm developed in the proof of Theorem 1.4 to compute the infimum time and the solution associated to the minimal time problem to approximately steer an AC measure to another. We use a Lagrangian scheme, introduced in [35, 37], for simulations of transport equations.

We recall here the construction of the mesh in Algorithm 1. We then give the numerical method to compute the minimal time in Algorithm 2.

We first give Algorithm 1. To simplify the notations, we assume that the space dimension is $d=2$.

Algorithm 1 Construction of the meshes $\mathcal{T}_n^0$ and $\mathcal{T}_n^1$

Let $n\in\mathbb{N}^*$ and two AC measures μ^0, μ^1 be given. Choose α_0,α_1 such that $\text{supp}(\mu^0)$, $\text{supp}(\mu^1)\subset(\alpha_0,\alpha_1)^2$.

Step 1: Construction, for all $k:=(k_1,k_2)\in\{0,...,n-1\}^2$, of the sets

$$\begin{aligned}C_{n,k} := &\left[\alpha_0+(\alpha_1-\alpha_0)\tfrac{k_1}{n},\alpha_0+(\alpha_1-\alpha_0)\tfrac{k_1+1}{n}\right)\\ &\times\left[\alpha_0+(\alpha_1-\alpha_0)\tfrac{k_2}{n},\alpha_0+(\alpha_1-\alpha_0)\tfrac{k_2+1}{n}\right).\end{aligned}$$

Step 2: Partition of $C_{n,k}$ into subsets $\{C^0_{nki}\}_i$ with $C^0_{nki} = [a^0_i, a^0_{i+1}) \times (\alpha_0, \alpha_1)$ such that $\mu^0_{|C_{n,k}}(C^0_{nki}) = 1/n^3$ (if $\mu^0_{|C_{n,k}}(C^0_{nki}) < 1/n^3$, then we do not partition $C_{n,k}$) and, for each i, partition of C^0_{nki} into some subsets $\{C^0_{nkij}\}_j$ with $C^0_{nkij} = [a^0_i, a^0_{i+1}) \times [a^0_{ij}, a^0_{i(j+1)})$ such that

$$\mu^0(C^0_{nkij}) = 1/n^4.$$

See Figure 6. We define similarly the cells C^1_{nkij}.

Step 3: Construction of $B^0_{nkij} := [b^{0-}_i, b^{0+}_i) \times [b^{0-}_{ij}, b^{0+}_{ij}) \subset\subset C^0_{nkij}$ and $B^1_{nkij} := [b^{1-}_i, b^{1+}_i) \times [b^{1-}_{ij}, b^{1+}_{ij}) \subset\subset C^1_{nkij}$ such that

$$\mu^0(B^0_{nkij}) = \mu^1(B^1_{nkij}) = \frac{1}{n^4} - \frac{1}{n^6}.$$

See Figure 7.

Step 4: Definition of $\mathcal{T}^0_n := \bigcup_{kij \in I^0_n} B^0_{nkij}$ and $\mathcal{T}^1_n := \bigcup_{kij \in I^1_n} B^1_{nkij}$, where, for $l = 0, 1$, I^l_n is the set of kij such that B^l_{nkij} is well defined.

We now recall the algorithm to compute the minimal time.

Algorithm 2 Approximate controllability

Let μ^0 and μ^1 be two AC measures satisfying the Geometric Condition 1.

Step 1: Construction of the meshes $\mathcal{T}^0_n := \bigcup_{kij \in I^0_n} B^0_{nkij}$ and $\mathcal{T}^1_n := \bigcup_{kij \in I^1_n} B^1_{nkij}$ following Algorithm 1.

Step 2: Definition of $X^l := \{x^l_{kij} : kij \in I^l_n\}$ $(l = 0, 1)$ with x^l_{nkij} being a point of $B^l_{kij} \cap \mathrm{supp}(\mu^l)$

Step 3: Definition of

$$\omega_n := \{x \in \mathbb{R}^d : d(x, \omega^c) > e^{LT}\sqrt{2}/n\}.$$

For $t^l_{nkij} := t^l(x^l_{nkij}, \omega_n)$ with $l = 0, 1$, computation of the minimal time to steer X^0 to X^1 up to a mass $\varepsilon/4R$

$$M_{e,\varepsilon/4R}(X^0, X^1, \gamma) := \max_{1 \le i \le n-R} \{t^0_{nkij} + t^1_{nkij+R}\},$$

where $R := \lfloor n\varepsilon\gamma/4R \rfloor$ and the sequences $\{t^0_{nkij}\}_{kij}$, $\{t^1_{nkij}\}_{kij}$ are increasingly and decreasingly ordered, respectively

Step 4: Computation of the optimal permutations σ_0 and σ_1 minimizing (32) to steer X^0 to X^1 up to a mass $\varepsilon/4R$.

Step 5: Concentration of the mass of B^0_{nkij} around x^0_{nkij} in order to obtain no intersection of the cells when they follow the trajectories of $\delta_{x^0_{nkij}}$

Step 6: Computation of the control u_n and the solution μ_n to (2) on $(0, T)$.

5.1 Example 1: the 1-D case

Consider the initial data μ^0 and the target μ^1 defined by

$$\begin{cases} \mu^0 := 0.5 \times \mathbb{1}_{(0,2)} dx, \\ \mu^1 := 0.5 \times \mathbb{1}_{(7,8)\cup(10,11)} dx. \end{cases}$$

Let the velocity field $v := 1$ and the control region $\omega := (5, 6)$ be given. This situation is illustrated in Figure 9. In this case, the infimum time $T_a(\mu^0, \mu^1)$ is 8, which is computed in Step 3 of Algorithm 2. One cannot achieve approximate control at such time, but we aim to control the system at time $T = T_a(\mu^0, \mu^1) + \delta$,

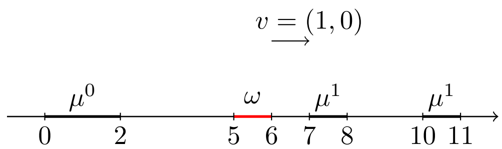


Figure 9: Control set ω, initial configuration μ^0 and final one μ^1 for Example 1.

with $\delta := 0.1$, hence $T = 8.1$. Following Algorithm 2, we obtain the solution presented in Figure 10. The maximal density for this solution is equal to 1.1. It is due to the fact that we concentrate a part of the mass coming from the set $\{x < 5\}$ in the control set, to slow it down. This increase of the maximal density can be seen as a drawback of the method for several key applications, namely for egress problems. Indeed, high concentrations need to be carefully avoided in such settings, since they might induce death by suffocation, that is among the main causes of fatalities in stampedes/crushes (see for instance [27]).

For this reason, in the future we plan to study new control strategies for minimal time problems, in which a constraint on the maximal density is added. Alternatively, we aim to estimate the maximal density value that is reached with optimal strategies.

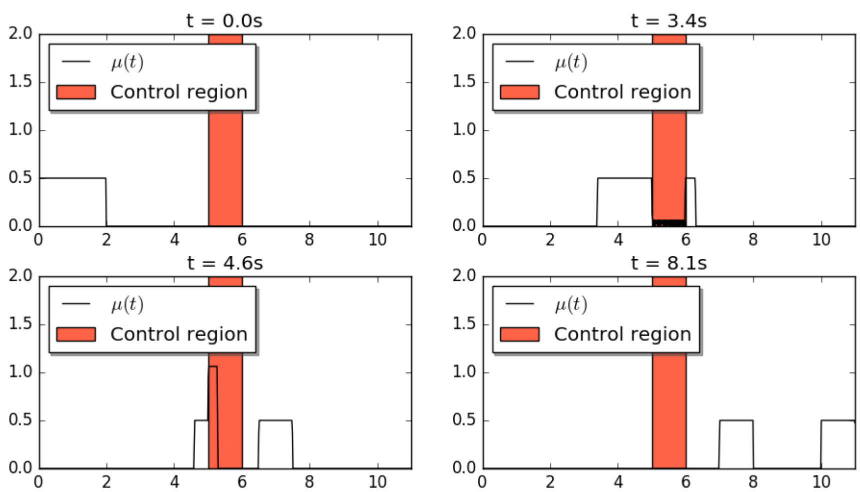


Figure 10: Example 1: solution at time $t = 0$, $t = 3.4$, $t = 4.6$ and $t = T = 8.1$.

5.2 Example 2: the 2D case

We now give an example in the 2D case. Consider the initial data μ^0 and the target μ^1 defined by

$$\begin{cases} \mu^0 := \frac{1}{8} \times \mathbb{1}_{(0,4)\times(1,3)} dx, \\ \mu^1 := \frac{1}{16} \times \mathbb{1}_{(8,14)\times(0,4)\setminus(9,13)\times(1,3)} dx. \end{cases}$$

We fix the velocity field $v := (1,0)$ and the control region $\omega := (5,7) \times (0,4)$. This situation is illustrated in Figure 11. Again, in this case $T_a(\mu^0, \mu^1) = 8$. Since it is not possible to approximatively steer μ^0 to μ^1 at such time, we control the system at time $T = T_a(\mu^0, \mu^1) + \delta$, with $\delta := 0.1$, hence $T = 8.1$. Following Algorithm 2, we present the solution in Figure 12. As in the previous example, we observe a high concentration of the crowd in the control region, in this case with a maximal density equal to 8.

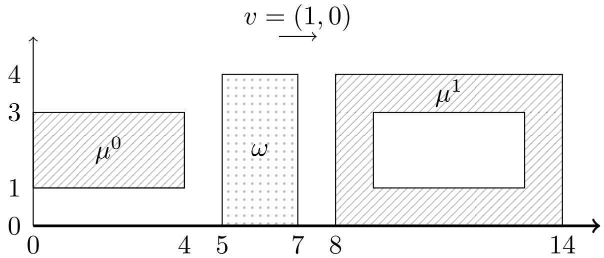


Figure 11: Control set ω, initial configuration μ^0 and final one μ^1 for Example 2.

A Proof of Lemma 4.2

In this section, we prove Lemma 4.2.

Statement 1 directly follows from the definition.

For **Statement 2**, we first prove that $\omega \subset \omega^t$ for a given t. Take $x \in \omega$, that is open; since $t \mapsto \Phi_t^v(x)$ is continuous, there exists τ arbitrarily small such that $\Phi_{-\tau}^v(x) \in \omega$. Choose $\tau < t$ and observe that it holds $x = \Phi_\tau^v(\Phi_{-\tau}^v(x)) \in \omega^t$. Since this property holds for all $x \in \omega$, then $\omega \subset \omega^t$.

To prove that $B := \Phi_t^v(\omega) \subset \omega^t$, consider the vector field $-v$ starting from B. It then holds $\omega^t = \cup_{\tau \in (0,t)} \Phi_\tau^{-v}(B)$. Thus, by applying the previous case, it holds $B \subset \omega^t$.

Statement 3 is a direct consequence of the fact that ω^t is the image of the open set $(0,t) \times \omega$ by the homeomorphism $(t,x) \mapsto \Phi_t^v(x)$.

For **Statement 4**, consider the set

$$A := \left\{ \tau \in [0,t] \text{ s.t. } \Phi_\tau^{v+\mathbb{1}_\omega u}(x_0) \in \omega \right\}.$$

We first prove that $t > t_0(x_0)$ implies that A is non-empty, by contradiction. If it is empty, then the trajectory $\Phi_t^{v+\mathbb{1}_\omega u}(x_0)$ coincides with $\Phi_t^v(x_0)$, since the control does not act outside ω. Then, recall the definition of $t_0(x_0)$ and observe that there exists a sequence of positive times $\tau_n \searrow 0$ such that $\Phi_{t_0(x_0)+\tau_n}^v \in \omega$. Contradiction.

With the same technique, we can also prove that $\tau \in [0, t_0(x_0))$ implies $\tau \notin A$, by definition of $t_0(x_0)$. Since A is nonempty and bounded, it admits $\bar{\tau} := \sup(A)$. By the previous result, it holds $\bar{\tau} \geq t_0(x_0)$.

One can also observe that A is open in the standard topology of $[0,t]$, since ω is open and the flow $\Phi_t^{v+\mathbb{1}_\omega u}(x_0)$ is continuous with respect to the variable t. Then $\bar{\tau} > t_0(x_0)$.

Define

$$\bar{x} := \Phi_{\bar{\tau}}^{v+\mathbb{1}_\omega u}(x_0).$$

We now prove that $\Phi_{-\tau}^v(\bar{x}) \in \omega$ for some $\tau > 0$ sufficiently small, by contradiction. If it is not the case, there exists a $\tilde{\tau} > 0$ such that $\Phi_{-\tau}^v(\bar{x}) \notin \omega$ for all $\tau \in (0, \tilde{\tau})$. Then, adding the control $\mathbb{1}_\omega u$ to this trajectory plays no role, *i.e.* $\Phi_{\bar{\tau}-\tau}^{v+\mathbb{1}_\omega u}(\bar{x}) \notin \omega$ for all $\tau \in (0, \tilde{\tau})$. This is in contradiction with the definition of $\bar{\tau}$. Then, there exists a sequence $\tau_n \searrow 0$ such that $\Phi_{-\tau_n}^v(\bar{x}) \in \omega$.

We are now ready to prove Statement 4, by studying two cases. If $\bar{\tau} < t$, it holds $\Phi_\tau^{v+\mathbb{1}_\omega u}(x_0) \notin \omega$ for all $\tau \in (\bar{\tau}, t]$, by definition of $\bar{\tau}$. Then, the control does not act on the time interval $(\bar{\tau}, t]$. Thus for all $\tau \in (\bar{\tau}, t]$, it holds

$$\Phi_t^{v+\mathbb{1}_\omega u}(x_0) = \Phi_{t-\bar{\tau}}^{v+\mathbb{1}_\omega u}(\bar{x}) = \Phi_{t-\bar{\tau}}^v(\bar{x}) = \Phi_{t-\bar{\tau}+\tau_n}^v(\Phi_{-\tau_n}^v(\bar{x})),$$

where we also used rules of compositions of flows. Take τ_n sufficiently small to have $t - \bar{\tau} + \tau_n > 0$: since $\Phi_{-\tau_n}^v(\bar{x}) \in \omega$ by construction, it holds

$$\Phi_t^{v+\mathbb{1}_\omega u}(x_0) \in \Phi_{t-\bar{\tau}+\tau_n}^v(\omega) \subset \omega^{t-\bar{\tau}+\tau_n}.$$

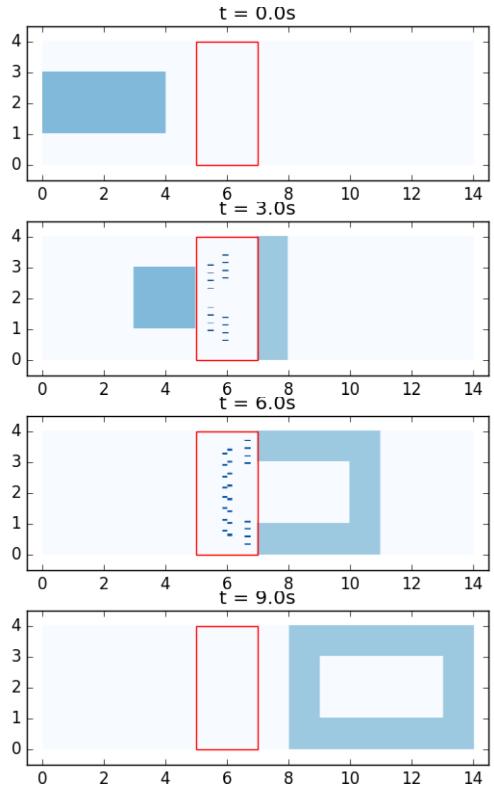


Figure 12: Example 2: solution at time $t = 0$, $t = 3$, $t = 6$ and $t = T = 9$.

We used here Statement 2. Since $\bar{\tau} > t_0(x_0)$ and $\tau_n \searrow 0$, then $t - t_0(x_0) > t - \bar{\tau} + \tau_n$ for some τ_n. Then, by using Statement 1, (59) is proved.

If instead $\bar{\tau} = t$, it holds $\bar{x} = \Phi_t^{v+\mathbb{1}_\omega u}(x_0)$. Then, $\Phi_{-\tau_n}^v(\bar{x}) \in \omega$ implies that $\Phi_t^{v+\mathbb{1}_\omega u}(x_0) \in \omega^{\tau_n}$. By choosing n sufficiently large, it holds $\tau_n \leq t - t_0(x_0)$. Then, by Statement 1 the result is proved.

We now aim to prove Statement 5. We first need to state and prove the following result.

Lemma A.1. *Let*

$$C(x_0, n, \alpha, r) := \left\{ \begin{array}{l} x \in B_r(x_0) \quad s.t. \quad n \cdot (x - x_0) > 0, \\ \qquad \|x - x_0 - n(n \cdot (x - x_0))\| < \alpha\, n \cdot (x - x_0) \end{array} \right\}$$

be a (open) cone centered in x_0, with normal unit vector n, tangent α and radius r.

We say that a set A satisfies the uniform interior cone condition if there exist uniform α, r such that for all $x_0 \in \partial A$ there exists n such that

$$C(x_0, n, \alpha, r) \subset A.$$

Let v be a C^1 vector field, and A be an open bounded set satisfying the uniform interior cone condition. Then, the following statements hold.

1. *The set $\Phi_t^v(A)$ satisfies the uniform interior cone condition;*

2. *For each $t > 0$, the set $A^t = \cup_{\tau \in (0,t)} \Phi_\tau^v(A)$ satisfies the uniform interior cone condition.*

Proof. We prove the first statement. First take the uniform parameter r and observe that the linear operator $\nabla\Phi_t^v(.)$ is continuous on the compact set $\overline{\cup_{x_0\in A} B_r(x_0)}$, hence Φ_v^t admits a Lipschitz constant L on such set. The Gronwall lemma then implies

$$e^{-Lt}\|x-x_1\| \le \|\Phi_t^v(x)-\Phi_t^v(x_1)\| \le e^{Lt}\|x-x_1\|. \tag{64}$$

Take a point $y_0\in\partial\Phi_t^v(A)$ and denote by $M:=\nabla\Phi_{-t}^v(y_0)$, that is seen as a linear isomorphism. Fix the parameters

$$\alpha' = \frac{\min\{1,\frac{\alpha}{4}\}}{2\|M\|\|M^{-1}\|}, \qquad \varepsilon = \frac{\min\{1,\frac{\alpha}{4}\}}{4\|M^{-1}\|\sqrt{1+(\alpha')^2}}. \tag{65}$$

Since v is a C^1 vector field, then $\Phi_{-t}^v(.)$ is C^1 too. Hence, there exists a radius $r'>0$ such that for all $y\in B_{2r'}(y_0)$ it holds

$$\|\nabla\Phi_{-t}^v(y)-M\|<\varepsilon.$$

Eventually reducing such radius, we will also assume $r'<e^{-Lt}r$, where L is the Lipschitz constant in (64) and r is the uniform parameter of the interior cone condition for A.

Take $y_1\in B_{r'}(y_0)\cap\partial\Phi_t^v(A)$ and denote by $x_1=\Phi_{-t}^v(y_1)$. Since Φ_t^v is a diffeomorphism, then $x_1\in\partial A$, hence x_1 admits an interior cone $C(x_1,n,\alpha,r)$. Denote by

$$m:=\frac{M^{-1}n}{\|M^{-1}n\|}.$$

We aim to prove that the cone $C(y_1,m,\alpha',r')$ is interior to $\Phi_t^v(A)$. Since Φ_t^v is an isomorphism, it is sufficient to prove that $\Phi_{-t}^v(C(y_1,m,\alpha',r'))$ is contained in $C(x_1,n,\alpha,r)$, that is in turn contained in A.

With this goal, take $y\in C(y_1,m,\alpha',r')$. We now prove that $x:=\Phi_{-t}^v(y)$ belongs to $C(x_1,n,\alpha,r)$. First observe that $\|y-y_1\|<r'$ implies $\|x-x_1\|<r$ by (64) and the choice of r'. Second, we aim to prove that $n\cdot(x-x_0)>0$. Since $y\in C(y_1,m,\alpha',r')$, it can be written as $y=y_1+\lambda m+\mu a$ where $\lambda>0$, the unitary vector a is orthogonal to m and $|\mu|<\alpha'\lambda$. By the mean value theorem, it holds

$$x-x_1=\nabla\Phi_{-t}^v(\tilde{y})(y-y_1)$$

for some $\tilde{y}\in B_{2r'}(y_0)$, hence

$$\|x-x_1-M(y-y_1)\|<\varepsilon\|y-y_1\|. \tag{66}$$

This implies

$$|n\cdot(x-x_1)-n\cdot M(\lambda m+\mu a)|<\varepsilon\sqrt{\lambda^2+\mu^2}<\varepsilon\lambda\sqrt{1+(\alpha')^2}\le\frac{1}{4}\frac{1}{\|M^{-1}n\|}\lambda. \tag{67}$$

It also holds

$$n\cdot M(\lambda m+\mu a)\ge\lambda\frac{n\cdot n}{\|M^{-1}n\|}-|\mu|\|M\| \tag{68}$$

$$>\frac{\lambda}{\|M^{-1}n\|}-\lambda\frac{1}{2\|M^{-1}\|}\ge\frac{\lambda}{2\|M^{-1}n\|}. \tag{69}$$

Merging (67)-(69), it holds

$$n\cdot(x-x_1)\ge\frac{\lambda}{4\|M^{-1}n\|}>0. \tag{70}$$

We are left to prove that it holds

$$\|x-x_1-n(n\cdot(x-x_1))\|<\alpha\, n\cdot(x-x_1).$$

Since $n(n\cdot(x-x_1))$ is the projection of $x-x_1$ along the direction n, it holds

$$\|x-x_1-n(n\cdot(x-x_1))\|\leq\|x-x_1-n(n\cdot v)\| \tag{71}$$

for any vector v. Choose

$$v=M(y-y_1)=M(\lambda m+\mu a)=\frac{\lambda n}{\|M^{-1}n\|}+\mu Ma$$

and observe that it holds

$$\|v-n(n\cdot v)\|\leq|\mu|\,\|M\|. \tag{72}$$

Apply (71), then (66)-(72) and finally (70) to have

$$\begin{aligned}&\|x-x_1-n(n\cdot(x-x_1))\|\leq\|x-x_1-v\|+\|v-n(n\cdot v)\|\\&<\varepsilon\|y-y_1\|+|\mu|\,\|M\|\\&<\varepsilon\sqrt{\lambda^2+\mu^2}+\alpha'\lambda\|M\|\leq\lambda(\varepsilon\sqrt{1+(\alpha')^2}+\alpha'\|M\|)\\&\leq\Big(4\|M^{-1}n\|(\varepsilon\sqrt{1+(\alpha')^2}+\alpha'\|M\|)\Big)\,n\cdot(x-x_1)\leq\frac{3}{4}\,\alpha\,n\cdot(x-x_1).\end{aligned}$$

We have then proved that $x\in C(x_1,m,\alpha,r)\subset A$, hence $y\in\Phi_t^v(A)$. Since y is a generic element of $C(y_1,m,\alpha',r')$, it holds that $C(y_1,m,\alpha',r')$ is contained in $\Phi_t^v(A)$. We have then proved that, for each $y_0\in\partial\Phi_t^v(A)$ there exists a whole neighbourhood $B_{r'}(y_0)$ and a choice α' such that any $y_1\in B_{r'}(y_0)\cap\partial\Phi_t^v(A)$ admits an internal cone $C(y_1,m,\alpha',r')$.

We now prove that α',r' can be chosen uniformly on $\partial\Phi_t^v(A)$ by a classical compactness argument. For each $y_0\in\partial\Phi_t^v(A)$, consider the corresponding neighbourhood $B_{r'}(y_0)$ and observe that the union of such neighborhoods is a covering of $\partial\Phi_t^v(A)$. Choose then a finite sub-covering $B_{r'(y^i)}(y^i)$ for $i=1,\ldots,N$, for which a corresponding $\alpha'(y^i)$ is also defined. By defining

$$\alpha'':=\min_i\alpha'(y^i),\qquad r'':=\min_i(r'(y^i)),$$

we prove that any $y_1\in\partial\Phi_t^v(A)$ admits an internal cone with parameter α'',r''. Indeed, $y_1\in B_{r'(y^i)}(y^i)$ for some i, then y_1 admits the interior cone $C(y_1,m,\alpha'(y^i),\,r'(y^i))$, that contains $C(y_1,m,\alpha'',r'')$.

We now prove Statement 2. Take $x\in\partial(\cup_{\tau\in(0,t)}\Phi_\tau^v(A))$. By definition, there exists a sequence (τ_k,x_k) with $\tau_k\in(0,t)$ and $x_k\in\Phi_{\tau_k}^v(A)$ such that $x_k\to x$. Since τ_k is bounded, one can pass to a converging subsequence, which limit is $\bar\tau\in[0,t]$. Define $y_k=\Phi_{\bar\tau-\tau_k}^v(x_k)$, that satisfies $y_k\in\Phi_{\bar\tau}^v(A)$ by construction. Since $|x_k-y_k|\leq|\bar\tau-\tau_k|\cdot\|v\|_{L^\infty}$, then it also holds $y_k\to x$, hence $x\in\overline{\Phi_{\bar\tau}^v(A)}$. Since $x\in\partial(\cup_{\tau\in(0,t)}\Phi_\tau^v(A))$, there also exists $z_k\to z$ with $z_k\notin\cup_{\tau\in(0,t)}\Phi_\tau^v(A)$. If $\bar\tau\in(0,t)$, this implies $z_k\notin\Phi_{\bar\tau}^v(A)$. If $\bar\tau=0$ or $\bar\tau=t$, then apply Lemma 4.2, Statement 2 to prove $z_k\notin\Phi_{\bar\tau}^v(A)$.

In both cases, it holds $x\in\overline{\Phi_{\bar\tau}^v(A)}\setminus\Phi_{\bar\tau}^v(A)$, hence $x\in\partial(\Phi_{\bar\tau}^v(A))$. Then, it admits a cone $C(x,m,\alpha,r)$ that is internal to $\Phi_{\bar\tau}^v(A)$, hence that is internal to $\cup_{\tau\in(0,t)}\Phi_\tau^v(A)$. Moreover, by continuity of the parameters α',ε in (65), one has that the parameters can be chosen to satisfy the uniform cone condition. □

We are now ready to prove Statement 5 of Lemma 4.2. One first needs to observe that ω open, bounded and convex implies that it satisfies the uniform interior cone condition. It is sufficient to choose any point $x_0\in\omega$ and a corresponding ball $B_\varepsilon(x_0)$ such that $B_{3\varepsilon}(x_0)\subset\omega$, that exists since ω is open. Since ω is bounded, there also exists a ball $B_R(x_0)$ containing $\partial\omega$.

Take now any point $y\in B_R(x_0)\setminus\overline{B_{2\varepsilon}(x_0)}$, not necessarily in ω, and consider the set

$$\mathcal{A}_y:=\cup_{x\in B_\varepsilon(x_0),\lambda\in(0,1)}\{\lambda y+(1-\lambda)x\}:$$

we aim to prove that such set contains a cone $C(y,m,\alpha',r')$, where parameters α',r' are uniformly bounded from below. By a geometrical rotation argument, one can always assume to have $y=x_0+re_1$, where

$r \in (2\varepsilon, R)$ and e_1 is the first vector of the canonical basis of $\mathbb{R}^d$. Thus, the cone $C(y, -e_1, \alpha', r')$ with $\alpha' = \frac{\varepsilon}{r}$ and $r' = \sqrt{r^2 + \varepsilon^2}$ is contained in $\mathcal{A}_y$. As a consequence, for any $y \in B_R(x_0) \setminus \overline{B_{2\varepsilon}}(x_0)$ there exists a cone $C(y, m, \alpha', r')$ contained in the set $\mathcal{A}_y$, with $\alpha' = \frac{\varepsilon}{R}$ and $r' = \varepsilon$. Observe that such parameters are independent on the point.

Take now any point $x \in \partial\omega$ and a sequence $x_n \to x$. By construction, it holds $x \in B_R(x_0) \setminus \overline{B_{2\varepsilon}}(x_0)$, that is open, hence $x_n \in B_R(x_0) \setminus \overline{B_{2\varepsilon}}(x_0)$ for n sufficiently large. For each of such x_n, consider the set $\mathcal{A}_{x_n}$, that is contained in ω by convexity. It contains a cone $C(x_k, m_k, \alpha', r')$ with uniform parameters a', r' given above. Since m_k is a sequence of unit vectors, then it admits a sub-sequence converging to some m. Then, the union of $C(x_k, m_k, \alpha', r')$ contains the cone $C\left(x, m, \frac{\alpha'}{2}, \frac{r'}{2}\right)$. Thus, such cone is internal. Since such internal cone with uniform parameters $\frac{\alpha'}{2}, \frac{r'}{2}$ exists for any $x \in \partial\omega$, then ω satisfies the uniform interior cone condition.

Since ω satisfies the uniform interior cone condition, by Lemma A.1, Statement 2, it holds that ω^t satisfies the uniform interior cone condition, with uniform parameters α'', r''. We now prove that this implies that $\partial(\omega^t)$ has zero Lebesgue measure. Since ω^t is open and bounded, then it is measurable, thus its characteristic function $\mathbb{1}_{\omega^t}$ is in $L^1(\mathbb{R}^d)$. It is now sufficient to prove that no point of the boundary is a Lebesgue point with respect to the Lebesgue measure. Indeed, observe from one side that for each $x \in \partial(\omega^t)$ it holds $\mathbb{1}_{\omega^t}(x) = 0$ since ω^t is open. On the other side, for $r < r'$ it holds

$$\frac{1}{\mathcal{L}(B_r(x))} \int_{B_r(x)} \mathbb{1}_{\omega^t}(y) d\mathcal{L}(y) \geq \frac{\mathcal{L}(C(x, m, \alpha', r') \cap B_r(x))}{\mathcal{L}(B_r(x))}.$$

By a simple dilation and rototranslation argument, we remark that such term coincides with $\frac{\mathcal{L}(C(0,m,\alpha',1))}{\mathcal{L}(B_1(0))}$, that is a strictly positive constant, independent on r. As a consequence, it holds

$$0 = \mathbb{1}_{\omega^t}(x) \neq \lim_{r \to 0} \frac{1}{\mathcal{L}(B_r(x))} \int_{B_r(x)} \mathbb{1}_{\omega^t}(y) d\mathcal{L}(y),$$

hence x is not a Lebesgue point. Since $x \in \partial(\omega^t)$ is generic, then no point in $\partial(\omega^t)$ is a Lebesgue point. Since the set of points that are not Lebesgue points for a measurable function has zero Lebesgue measure (Lebesgue-Besicovitch differentiation theorem, see e.g. [22, Sec. 1.7]), then $\partial(\omega^t)$ has zero Lebesgue measure.

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